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ONE-DIMENSIONAL POLYMERS IN RANDOM ENVIRONMENTS: STRETCHING VS. FOLDING

QUENTIN BERGER, CHIEN-HAO HUANG, NICCOLÒ TORRI, AND RAN WEI

ABSTRACT. In this article we study a *non-directed polymer model* on $\mathbb{Z}$, that is a one-dimensional simple random walk placed in a random environment. More precisely, the law of the random walk is modified by the exponential of the sum of "rewards" (or penalities) $\beta\omega_x - h$ sitting on the range of the random walk, where $(\omega_x)_{x\in\mathbb{Z}}$ are i.i.d. random variables (the disorder), and where $\beta \geqslant 0$ (disorder strength) and $h \in \mathbb{R}$ (external field) are two parameters. When $\beta = 0, h > 0$, this corresponds to a random walk penalized by its range; when $\beta > 0, h = 0$, this corresponds to the "standard" polymer model in random environment, except that it is non-directed. In this work, we allow the parameters β, h to vary according to the length of the random walk, and we study in detail the competition between the *stretching effect* of the disorder, the *folding effect* of the external field (if $h \geqslant 0$), and the *entropy cost* of atypical trajectories. We prove a complete description of the (rich) phase diagram. For instance, in the case $\beta > 0, h = 0$ of the non-directed polymer, if ω_x ha a finite second moment, we find a transversal fluctuation exponent $\xi = 2/3$, and we identify the limiting distribution of the rescaled log-partition function.

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1. INTRODUCTION

We study here a simple symmetric random walk on $\mathbb{Z}^d$ placed in a time-independent random environment [20]. The interaction with the environment occurs on the range of the random walk, *i.e.* on the sites visited by the walk. This model can therefore also be seen as a random version of random walks penalized by their range (in the spirit of [16, 10]). One closely related model is the celebrated directed polymer in random environment model (see [13] for a review), which has attracted interests from both mathematical and physical communities over the last forty years, and can be used to describe a polymer chain placed in a solvent with impurities.

1.1. **The model.** Let $S := (S_n)_{n\geqslant 0}$ be a simple symmetric random walk on $\mathbb{Z}^d$, $d \geqslant 1$, starting from 0, whose trajectory represents a (non-directed) polymer. Let $\mathbf{P}$ denote its law. The *random environment* is modeled by a field $\omega := (\omega_x)_{x\in\mathbb{Z}^d}$ of i.i.d. random variables. We let $\mathbb{P}$ denote the law of ω, and $\mathbb{E}$ the expectation with respect to $\mathbb{P}$ (assumptions on the law of ω are detailed in Section 1.2 below).

For $\beta \geqslant 0$ (the disorder strength, or the inverse temperature) and $h \in \mathbb{R}$ (an external field), we define for all $N \in \mathbb{N}$ the following Gibbs transformation of the law $\mathbf{P}$, called the

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polymer measure:

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{N,\beta,h}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z^{\omega}_{N,\beta,h}} \exp\Big(\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - h)\mathbb{1}_{x\in\mathcal{R}_N} \Big), \tag{1.1}$$

where $\mathcal{R}_N = \{S_0, S_1, \dots, S_N\}$ is the range of the random walk up to time N, and

$$Z^{\omega}_{N,\beta,h} := \mathbf{E}\Big[\exp\Big(\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - h)\mathbb{1}_{x\in\mathcal{R}_N} \Big)\Big] = \mathbf{E}\Big[\exp\Big(\beta \sum_{x\in\mathcal{R}_N} \omega_x - h|\mathcal{R}_N| \Big)\Big] \tag{1.2}$$

is the partition function of the model, and is defined so that $\mathbf{P}^{\omega}_{N,\beta,h}$ is a probability measure.

Let us stress the main differences with the standard directed polymer model: (i) here, the random walk does not have a preferred direction; (ii) there is an additional external field $h \in \mathbb{R}$; (iii) the random walk can only pick up one weight $\beta\omega_x - h$ at a site $x \in \mathbb{Z}^d$, and returning to an already visited site does not bring any reward or penalty (in the directed polymer model, the environment is renewed each time).

We now wish to understand the typical behavior of polymer trajectories $(S_0, \dots, S_N)$ under the polymer measure $\mathbf{P}^{\omega}_{N,\beta,h}$. Two important quantities that we are interested in are

- the *end-to-end* exponent ξ, loosely defined as $\mathbb{E}\mathbf{E}^{\omega}_{N,\beta,h}|S_N| \approx N^{\xi}$;
- the *fluctuation* exponent χ, loosely defined as $|\log Z_{N,\beta,h} - \mathbb{E}[\log Z_{N,\beta,h}]| \approx N^{\chi}$.

In view of (1.1), there are several quantities that may influence the behavior of the polymer: the energy collected from the *random environment* ω; the penalty h (or reward depending on its sign) for having a large range; the entropy cost of the exploration of the random walk S. If $\beta = 0$ and $h > 0$, then we recover a random walk penalized by its range. This model is by now quite well understood: the random walk folds itself in a ball of radius $N^{1/(d+2)}$ ($\xi = \frac{1}{d+2}$), see [16, 28, 10, 4, 15] (these works mostly focus on the case of dimension $d \geqslant 2$). If $\beta = 0$ and $h < 0$, then we get a random walk rewarded by its range: the random walk "stretches" to obtain a range of order N. If $\beta > 0$ and $h = 0$, then we obtain a model for a non-directed polymer in the random environment, the environment being seen only once by the random walk (in the same spirit as the excited random walk [3], or more generally the cookie random walk [29]). In general, disorder should have a "stretching" effect, the random walk is trying to reach more favorable regions in the environment. We will see that it is indeed the case in dimension $d = 1$, where we find that the random walk stretches up to a distance $N^{2/3}$ ($\xi = \frac{2}{3}$).

1.2. **Setting of the paper.** In this article, we focus on the case of the dimension $d = 1$: the behavior of the model is already very rich, and we are able to obtain sharp results.

Our main assumption on the environment is that ω_x is in the domain of attraction of some α-stable law, with $\alpha \in (0, 2]$, $\alpha \neq 1$. More precisely, we make the following assumption.

Assumption 1. *If $\alpha = 2$ we assume that $\mathbb{E}[\omega_0] = 0$ and $\mathbb{E}[\omega_0^2] = 1$. If $\alpha \in (0,1) \cup (1,2)$ we assume that $\mathbb{P}(\omega_0 > t) \sim p\, t^{-\alpha}$ and $\mathbb{P}(\omega_0 < -t) \sim q\, t^{-\alpha}$ as $t \to \infty$, with $p + q = 1$ (and $p > 0$). Moreover, if $\alpha \in (1,2)$, we also assume that $\mathbb{E}[\omega_0] = 0$.*

Let us stress that Assumption 1 ensures that:

- if $\alpha = 2$, then ω_i is in the normal domain of attraction, so that $(\frac{1}{\sqrt{n}} \sum_{i=un}^{vn} \omega_i)_{u\leqslant 0\leqslant v}$ converges to a two-sided (standard) Brownian Motion.
- if $\alpha \in (0,1) \cup (1,2)$, then ω_i is in the domain of attraction of some non-Gaussian stable law and $(\frac{1}{n^{1/\alpha}} \sum_{i=un}^{vn} \omega_i)_{u\leqslant 0\leqslant v}$ converges to a two-sided α-stable Lévy process.

We leave the case $\alpha = 1$ aside mostly for simplicity: indeed, to obtain a process convergence as above, a non-zero centering term is in general needed (even in the symmetric case $p = q$, see [18, IX.8], or [5]); however most of our analysis applies in that case.

Henceforth we refer to $(X_t)_{t\in\mathbb{R}}$ as the two-sided Brownian motion if $\alpha = 2$ and as the two-sided Lévy process defined below if $\alpha \in (0,1) \cup (1,2)$.

Definition 1.1. *We denote by $(X_t)_{t\in\mathbb{R}}$ (with $X_0 := 0$) either a two-sided (standard) Brownian motion if $\alpha = 2$ or a two-sided (stable) Lévy process with no drift, no Brownian component and Lévy measure $\nu(\mathrm{d}x) = \alpha(p\mathbb{1}_{\{x>0\}} + q\mathbb{1}_{\{x<0\}})|x|^{-1-\alpha}\,\mathrm{d}x$, if $\alpha \in (0,2)$.*

Remark 1.1. *We couple the discrete environment $(\omega_x)_{x\in\mathbb{Z}}$ with the process $(X_t)_{t\in\mathbb{R}}$ by using an extended version of Skorokhod representation theorem* [21, Corollary 5.12]*, which guarantees that $\frac{1}{n^{1/\alpha}}\sum_{i=un}^{vn}\omega_i$ converges $\mathbb{P}$-a.s. to $X_v - X_u$ for all $u \leqslant v$. This coupling is used to obtain an almost sure convergence in our results.*

In the present paper, we allow β and h to vary with the size of the system, giving rise to a large diversity of possible behaviors. Before we go into these details, let us already state how are results translate in the case of fixed parameters β, h.

Let us define $M_N^+ := \max_{0\leqslant n\leqslant N} S_n \geqslant 0$ and $M_N^- := \min_{0\leqslant n\leqslant N} S_n \leqslant 0$ the right-most and left-most points of the random walk after N steps. In particular, the size of the range is $M_N^+ - M_N^-$. Also, for a doubly indexed process $(Y_{u,v})_{u\leqslant 0\leqslant v}$ such that $Y^* := \sup_{u\leqslant 0\leqslant v} Y_{u,v} < +\infty$, we define for any $\varepsilon > 0$

$$\mathcal{M}_\varepsilon(Y) = \Big\{(u,v) \in \mathbb{R}^- \times \mathbb{R}^+\,,\quad \sup_{\substack{s\leqslant 0\leqslant t\\ |s-u|\leqslant\varepsilon, |t-v|\leqslant\varepsilon}} Y_{s,t} = Y^*\Big\}\,.$$

The set $\mathcal{M}(Y) = \bigcap_{\varepsilon>0}\mathcal{M}_\varepsilon(Y)$ is called the set of *quasi-maximizers* of Y (using the terminology of [24]).

Theorem 1.1. (1) *Case $\alpha \in (1,2]$.*

(a) *If $\beta \geqslant 0$ and $h > 0$. Then, for any $\varepsilon > 0$, we have that*

$$\lim_{N\to\infty} \mathbf{P}^\omega_{N,\beta,h}\Big(\Big|\frac{1}{N^{1/3}}(M_N^+ - M_N^-) - \pi^{\frac{2}{3}}h^{-\frac{1}{3}}\Big| > \varepsilon\Big) = 0\,, \qquad \mathbb{P}\text{-a.s.}$$

(b) *If $\beta > 0$ and $h = 0$. Then considering $Y^{(R_2)}_{u,v} = Y^{\beta,(R_2)}_{u,v} := \beta(X_v - X_u) - I(u,v)$ for $u \leqslant 0 \leqslant v$, with $I(u,v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2$, we have*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{2\alpha-1}}}\log Z^\omega_{N,\beta,h} = \sup_{u\leqslant 0\leqslant v}\Big\{Y^{(R_2)}_{u,v}\Big\} \in (0,+\infty)\,, \qquad \mathbb{P}\text{-a.s.}$$

Additionally, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbf{P}^\omega_{N,\beta,h}\Big(\frac{1}{N^{\frac{\alpha}{2\alpha-1}}}(M_N^-, M_N^+) \notin \mathcal{M}_\varepsilon(Y^{(R_2)})\Big) = 0\,, \qquad \mathbb{P}\text{-a.s.}$$

(c) *If $\beta \geqslant 0$ and $h < 0$. Then for any $\varepsilon > 0$, we have that*

$$\lim_{N\to\infty} \mathbf{P}^\omega_{N,\beta,h}\Big(\Big|\frac{1}{N}|S_N| - \frac{e^{2|h|}-1}{e^{2|h|}+1}\Big| > \varepsilon\Big) = 0\,, \qquad \mathbb{P}\text{-a.s.}$$

(2) *Case $\alpha \in (0,1)$. Let $\beta > 0$ and $h \in \mathbb{R}$. Then considering the doubly indexed process $Y^{(R_3)}_{u,v} = Y^{\beta,(R_3)}_{u,v} = \beta(X_v - X_u) - \infty\mathbb{1}_{\{|u|\wedge|v|+v-u>1\}}$ for $u \leqslant 0 \leqslant v$, we have*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{\alpha}}}\log Z^\omega_{N,\beta,h} = \sup_{u\leqslant 0\leqslant v}\Big\{Y^{(R_3)}_{u,v}\Big\} \in (0,+\infty)\,, \qquad \mathbb{P}\text{-a.s.}$$

Additionally, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbf{P}^{\omega}_{N,\beta,h}\Big(\frac{1}{N}(M_N^-, M_N^+) \notin \mathcal{M}_\varepsilon(Y^{(R_3)})\Big) = 0\,, \qquad \mathbb{P}\text{-}a.s.$$

Remark 1.2. *The sets $\mathcal{M}_\varepsilon(Y^{(R_2)})$ and $\mathcal{M}_\varepsilon(Y^{(R_3)})$ are ε-neighborhoods of the (random) sets of quasi-maximizers of the variational problems $\sup_{u\leqslant 0\leqslant v} Y_{u,v}^{(R_2)}$ and $\sup_{u\leqslant 0\leqslant v} Y_{u,v}^{(R_3)}$ respectively: since the variational problems are almost surely positive and finite (as shown below), the sets $\mathcal{M}_\varepsilon(Y^{(R_2)})$ and $\mathcal{M}_\varepsilon(Y^{(R_3)})$ are (with high probability) bounded away from $(0,0)$ if ε is small enough. When the sets of quasi-mazimizers $\mathcal{M}(Y^{(R_2)})$ and $\mathcal{M}(Y^{(R_3)})$ are reduced to one point (which is a.s. true when $\alpha = 2$, see [25], or when $\alpha \in (1,2)$ with $q = 0$ in Assumption 1, see [24]), this shows that $\mathbb{P}$-a.s., (M_N^-, M_N^+), when properly rescaled, converge in $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$ probability to this (unique) mazimizer.*

In particular, when $\alpha \in (1,2]$, the end-to-end distance[1] of the polymer is:

(a) *of order $N^{1/3}$ if $h > 0$ — folded phase, this is included in Theorem 3.5;*
(b) *of order $N^{\alpha/(2\alpha-1)}$ if $h = 0$, $\beta > 0$ — extended phase, this is included in Theorem 3.2;*
(c) *of order N if $h < 0$ — extended phase, this is included in Theorem 3.8.*

On the other hand, in the case $\alpha \in (0,1)$, the end-to-end distance is always of order N, whatever $h \in \mathbb{R}$ — extended phase, this is included in Theorem 3.3 below.

Let us now turn to a more general setting, where we allow β and h to vary with the size of the system

$$\beta_N := \hat{\beta}\, N^{-\gamma}\,, \quad \text{and} \quad h_N := \hat{h}\, N^{-\zeta}\,, \tag{1.3}$$

where $\gamma, \zeta \in \mathbb{R}$ describe the asymptotic behavior of β_N, h_N, and $\hat{\beta} > 0, \hat{h} \in \mathbb{R}$ are two fixed parameters. Let us stress that we will mostly focus our exposition on the case $h \geqslant 0$: the case $h < 0$ has a less of a rich behavior and is somehow simpler, see Remark 2.1 and Section 3.3 below.

In order to observe a transition between a folded phase ($h > 0, \beta = 0$) and an unfolded phase ($h = 0, \beta > 0$, or $h < 0$), a natural idea is to consider parameters β and h that depend on the size of the system, *i.e.* $\beta := \beta_N$ and $h := h_N$. There are then some sophisticated balances between the energy gain, the range penalty and the entropy cost as we tune β_N and h_N. Our main results identify the different regimes for the behavior of the random walk: we provide a complete (and rich) phase diagram (see Figures 1-2-3 below), and describe each phase precisely (end-to-end and fluctuation exponents, limit of the log-partition function).

Remark 1.3. *We could consider a slightly more general setting, adding some slowly varying function in the asymptotic behavior of β_N or h_N (or $\mathbb{P}(\omega_0 > t)$, if $\alpha < 2$). We chose to stick to the simpler strictly power-law case, to avoid lengthy notation and more technical calculations. It also makes the phase diagram clearer.*

2. Some heuristics: presentation of the phase diagrams

In this section, we only deal with the case $h \geqslant 0$; the case $h < 0$ is considered in Section 3.3. In analogy with the directed polymer model in a heavy-tailed random environment [6, 7], the presence of heavy-tail (Assumption 1) strongly impacts the behavior of the model: there will be different phase diagrams according to whether $\alpha \in (1,2]$, $\alpha \in (\frac{1}{2}, 1)$ and $\alpha \in (0, \frac{1}{2})$.

[1] See Definition 3.1 below for a proper definition.

Let us denote ξ the typical end-to-end fluctuations exponent of the random walk under the polymer measure $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$, namely $\mathbb{E}\mathbf{E}^{\omega}_{N,\beta_N,h_N}[\max_{1\leqslant n\leqslant N}|S_n|] \approx N^{\xi}$ (see Definition 3.1 below for a proper definition), and let us derive some heuristics to try to determine $\xi \in [0,1]$.

First of all, thanks to Lemmas A.1-A.3 in Appendix, we have

$$\log \mathbf{P}(|\mathcal{R}_N| \approx N^{\xi}) \approx \log \mathbf{P}\Big(\max_{1\leqslant n\leqslant N}|S_n| \approx N^{\xi}\Big) \approx \begin{cases} -N^{2\xi-1}, & \text{if } \xi \geqslant \frac{1}{2}, \\ -N^{1-2\xi}, & \text{if } \xi \leqslant \frac{1}{2}. \end{cases} \tag{2.1}$$

If $\xi > 1/2$, this corresponds to a "stretching" of the random walk, whereas when $\xi < 1/2$, this corresponds to a "folding" of the random walk: we will refer to (2.1) as the *entropic cost* of having end-to-end fluctuations N^{ξ}.

Then, if the end-to-end fluctuations are of order N^{ξ} ($|\mathcal{R}_N| \approx N^{\xi}$), we get under Assumption 1, and in view of (1.3), that

$$\beta_N \sum_{x\in\mathcal{R}_N} \omega_x \approx N^{\frac{\xi}{\alpha}-\gamma}, \qquad h_N|\mathcal{R}_N| \approx N^{\xi-\zeta}. \tag{2.2}$$

We refer to the first term as the "energy" term, and to the second one as the "range" term (recall that we focus for now on the case $\hat{h} > 0$ so the "range" term is always with a minus sign). All together, if end-to-end fluctuations are of order N^{ξ}, we have that

$$\log Z^{\omega}_{N,\beta_N,h_N} \approx N^{\frac{\xi}{\alpha}-\gamma} - N^{\xi-\zeta} - \begin{cases} N^{1-2\xi} & \text{if } \xi \leqslant 1/2\,, \\ N^{2\xi-1} & \text{if } \xi \geqslant 1/2\,. \end{cases} \tag{2.3}$$

In (2.3), there is therefore a competition between the "disorde" (first term), the "range" (second term), and the "entropy" (last term). We now discuss how a balance can be achieved between these terms depending on γ and ζ (and how they determine ξ). There are three main possibilities:

(i) there is a "disorder"-"entropy" balance (and the "range" term is negligible);
(ii) there is a "range"-"entropy" balance (and the "energy" term is negligible);
(iii) there is a "range"-"disorder" balance (and the "entropy" term is negligible).

To summarize, all three regimes can occur (depending on γ, ζ) if $\alpha \in (1,2]$; on the other hand, regime (iii) disappears if $\alpha \in (0,1)$, and regime (i) disappears if $\alpha \in (0, \frac{1}{2})$. We now determine for which values of γ, ζ one can observe the different regimes above: we consider the three subcases $\alpha \in (1,2]$, $\alpha \in (\frac{1}{2}, 1)$ and $\alpha \in (0, \frac{1}{2})$ separately.

2.1. **Phase diagram for $\alpha \in (1,2]$.** Instead of looking for "disorder"-"entropy", "range"-"entropy" or "range"-"disorder" balance, we will find conditions to have the "disorder" term much larger, much smaller, or of the order of the "range" term.

Case I ("disorder" $\gg$ "range"). This corresponds to having $\xi/\alpha - \gamma > \xi - \zeta$. In that case, the random walk should not feel the penalty for having a large range, so we should have $\xi \geqslant 1/2$. The competition occurs only between energy and entropy, one could achieve a balance if $\xi/\alpha - \gamma = 2\xi - 1$, that is if

$$\xi = \frac{\alpha}{2\alpha-1}(1-\gamma) \quad \text{when} \quad \gamma < \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha}, \tag{2.4}$$

where the condition on γ derives from the fact that $\xi/\alpha - \gamma > \xi - \zeta$ in the regime considered here. However, since $\xi \leqslant 1$, we should have $\xi = 1$ when $\gamma \leqslant -\frac{\alpha-1}{\alpha}$. Thus, we have

$$\xi = 1 \quad \text{when} \quad \gamma \leqslant -\frac{\alpha-1}{\alpha} \text{ and } \gamma < \zeta - \frac{\alpha-1}{\alpha}. \tag{2.5}$$

Also, since $\xi \geqslant 1/2$, we should have $\xi = 1/2$ when $\gamma \geqslant \frac{1}{2\alpha}$. Thus, we have

$$\xi = \frac{1}{2} \quad \text{when} \quad \gamma \geqslant \frac{1}{2\alpha} \text{ and } \gamma < \zeta - \frac{\alpha - 1}{2\alpha}. \tag{2.6}$$

Case II ("disorder" $\ll$ "range"). This corresponds to having $\xi/\alpha - \gamma < \xi - \zeta$. In that case, the random walk feels the penalty for having a large range, and we should have $\xi \leqslant 1/2$. The competition being only between range and entropy, one could achieve a balance if $\xi - \zeta = 1 - 2\xi$, that is if

$$\xi = \frac{1+\zeta}{3} \quad \text{when} \quad \gamma > \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha}, \tag{2.7}$$

where the condition on γ derives from the fact that $\xi/\alpha - \gamma > \xi - \zeta$ in the regime considered here. Since $\xi \in [0, 1/2]$, it is similar to (2.5)-(2.6) that

$$\xi = 0 \quad \text{when} \quad \zeta \leqslant -1 \text{ and } \gamma > \zeta, \tag{2.8}$$

and

$$\xi = \frac{1}{2} \quad \text{when} \quad \zeta \geqslant \frac{1}{2} \text{ and } \gamma > \zeta - \frac{\alpha - 1}{2\alpha}. \tag{2.9}$$

Case III ("disorder" $\approx$ "range" $\gg$ "entropy"). This corresponds to having $\xi/\alpha - \gamma = \xi - \zeta$, that is

$$\xi = \frac{\alpha}{\alpha - 1}(\zeta - \gamma). \tag{2.10}$$

In this regime, the entropy cost should be negligible compared to the disorder gain, and we should therefore have that $\xi/\alpha - \gamma > 1 - 2\xi$ if $\xi \leqslant 1/2$ and $\xi/\alpha - \gamma > 2\xi - 1$ for $\xi \geqslant 1/2$: after some calculation (and using (2.10)), we find the following condition on γ

$$\frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha} < \gamma < \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha}. \tag{2.11}$$

Moreover, since $\xi \in [0, 1]$, we must have

$$\zeta - \frac{\alpha - 1}{\alpha} \leqslant \gamma \leqslant \zeta. \tag{2.12}$$

To summarize, for $\alpha \in (1, 2]$, we have identified six different regimes according to the value of γ, ζ: they are represented in the (ζ, γ)-diagram in Figure 1 below.

To be precise, the different regions are described as follows:

$$\begin{aligned}
R_1 &= \left\{\xi = \tfrac{1}{2},\ \gamma > \tfrac{1}{2\alpha},\ \zeta > \tfrac{1}{2}\right\},\\
R_2 &= \left\{\xi = \tfrac{\alpha}{2\alpha-1}(1-\gamma),\ \tfrac{1-\alpha}{\alpha} < \gamma < \tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \wedge \tfrac{1}{2\alpha}\right\},\\
R_3 &= \left\{\xi = 1,\ \gamma < -\tfrac{\alpha-1}{\alpha},\ \gamma < \zeta - \tfrac{\alpha-1}{\alpha}\right\},\\
R_4 &= \left\{\xi = \tfrac{\alpha}{\alpha-1}(\zeta-\gamma),\ \tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \vee (\zeta - \tfrac{\alpha-1}{\alpha}) < \gamma < \tfrac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha} \wedge \zeta\right\},\\
R_5 &= \left\{\xi = \tfrac{1+\zeta}{3},\ \gamma > \tfrac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha},\ -1 < \zeta < \tfrac{1}{2}\right\},\\
R_6 &= \left\{\xi = 0,\ \gamma > \zeta,\ \zeta < -1\right\}.
\end{aligned}$$

Note that when $\alpha = 1$, the four lines $\gamma = \frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}$, $\gamma = \frac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}$, and $\gamma = \zeta$, $\gamma = \zeta - \frac{\alpha-1}{\alpha}$ all merge to the line $\gamma = \zeta$.

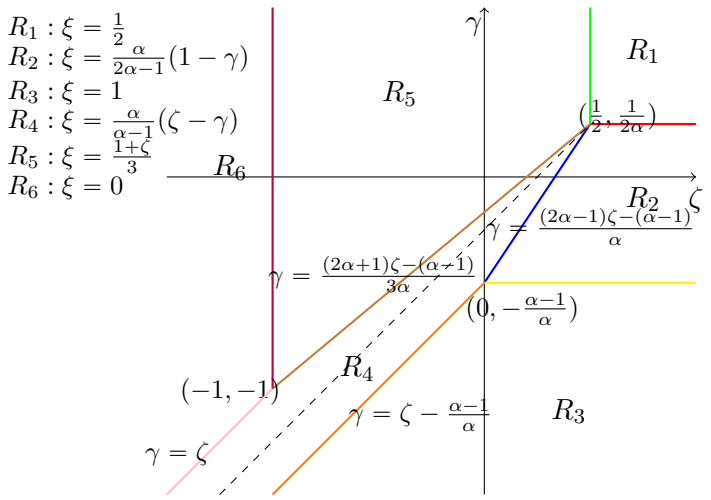

FIGURE 1. Phase diagram in the case $\alpha \in (1,2]$. The region R_1 and the dashed line $\gamma = \zeta - \frac{\alpha-1}{2\alpha}$ are the thresholds that split the regions of super-diffusivity and sub-diffusivity.

2.2. **Phase diagram for $\alpha \in (0,1)$.** Let us simply highlight the main difference with the case $\alpha \in (1,2]$: the region R_4 no longer exists when $\alpha < 1$, and the region R_2 also disappears when $\alpha < 1/2$. Indeed, region R_4 corresponds to the case "disorder"$\approx$"range", in which we have $\xi = \frac{\alpha}{1-\alpha}(\gamma - \zeta)$: it is easy check that for $\alpha \in (0,1)$ there is no γ that can satisfy (2.12), which suggests that there is no "disorder"-"range" balance possible. For the same reason, when $\alpha \in (0, \frac{1}{2})$, there no γ that satisfy $\frac{1-\alpha}{\alpha} < \gamma < \frac{1}{2\alpha}$ (see the definition of R_2 above), which suggests that there is no "disorder"-"entropy" balance possible: region R_2 no longer exists. We also refer to Section 3.2 (Comment 2) for further comments on the reasons why regions R_4 and R_2 disappear precisely for $\alpha < 1$ and $\alpha < 1/2$.

All together, for $\alpha \in (\frac{1}{2}, 1)$ we obtain the (ζ, γ)-diagram presented in Figure 2 below. To be precise, the different regions are described as follows:

$$
\begin{aligned}
R_1 &= \left\{\xi = \tfrac{1}{2},\ \gamma > \tfrac{1}{2\alpha},\ \zeta > \tfrac{1}{2}\right\},\\
R_2 &= \left\{\xi = \tfrac{\alpha}{2\alpha-1}(1-\gamma),\ \tfrac{1-\alpha}{\alpha} < \gamma < \tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \wedge \tfrac{1}{2\alpha}\right\},\\
R_3 &= \left\{\xi = 1,\ \gamma < \tfrac{1-\alpha}{\alpha},\ \gamma < \zeta - \tfrac{\alpha-1}{\alpha}\right\},\\
R_5 &= \left\{\xi = \tfrac{1+\zeta}{3},\ \left(\tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\right) \wedge \left(\zeta - \tfrac{\alpha-1}{\alpha}\right) < \gamma,\ -1 < \zeta < \tfrac{1}{2}\right\},\\
R_6 &= \left\{\xi = 0,\ \gamma > \zeta - \tfrac{\alpha-1}{\alpha},\ \zeta < \tfrac{1}{\alpha} - 2\right\}.
\end{aligned}
$$

On the other hand, for $\alpha \in (0, \frac{1}{2})$ we obtain the (ζ, γ)-diagram presented in Figure 3 below. To be precise, the different regions are described as follows:

$$
\begin{aligned}
R_1 &= \left\{\xi = \tfrac{1}{2},\ \gamma > \tfrac{1-\alpha}{\alpha},\ \zeta > \tfrac{1}{2}\right\},\\
R_3 &= \left\{\xi = 1,\ \gamma < \tfrac{1-\alpha}{\alpha},\ \gamma < \zeta - \tfrac{\alpha-1}{\alpha}\right\},\\
R_5 &= \left\{\xi = \tfrac{1+\zeta}{3},\ \tfrac{1-\alpha}{\alpha} \wedge \left(\zeta - \tfrac{\alpha-1}{\alpha}\right) < \gamma,\ -1 < \zeta < \tfrac{1}{2}\right\},\\
R_6 &= \left\{\xi = 0,\ \gamma > \zeta - \tfrac{\alpha-1}{\alpha},\ \zeta < \tfrac{1}{\alpha} - 2\right\}.
\end{aligned}
$$

Remark 2.1. *In the case $\hat{h} < 0$, one can conduct similar computation as in (2.4)—(2.12) and obtain a different phase diagram than those of Figures 1-2-3, see Figures 4 and 5 below (note that regions R_1, R_2, R_3 are unchanged, since the range term is negligible in these regions). Let us stress that when $\hat{h} < 0$, the "disorder" and "range" terms both play in*

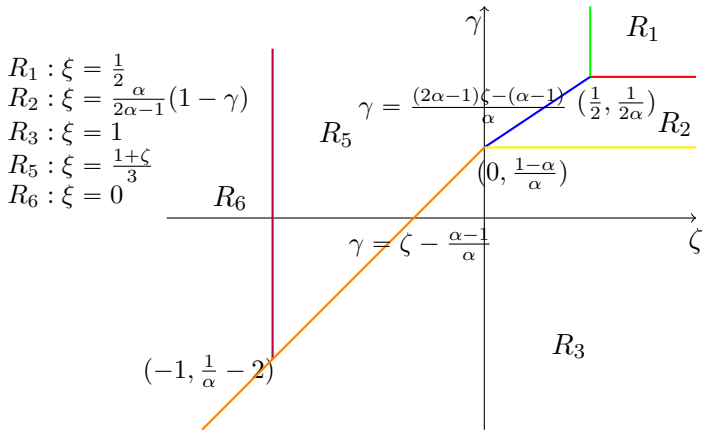

FIGURE 2. Phase diagram in the case $\alpha \in (1/2, 1)$. Compared to Figure 1, the region R_4 no longer exists.

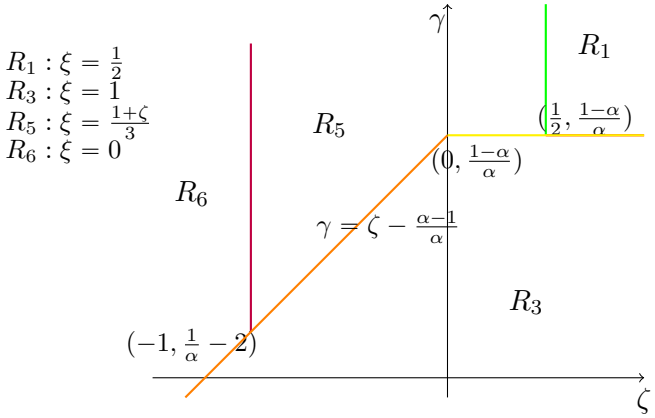

FIGURE 3. Phase diagram in the case $\alpha \in (0, 1/2)$. Compared to Figure 3, the region R_2 no longer exists.

the same direction and encourage exploration, resulting in a much simpler diagram: only end-to-end fluctuations exponents $\xi \geqslant 1/2$ are possible, see Section 3.3 below.

3. Main results

Our main results consist in proving the phase diagrams of Figures 1-2-3, with a precise description of the behavior of the polymer in each region. In order to state our results, let us introduce some definitions.

Definition 3.1. *If $(t_N)_{n\geqslant 0}$ is a sequence of positive real numbers, we say that $(S_n)_{0\leqslant n\leqslant N}$* has end-to-end fluctuations of order t_N under $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$ *if for any $\varepsilon > 0$ there is some η such that for N large enough*

$$\mathbf{P}^{\omega}_{N,\beta_N,h_N}\Big(\max_{1\leqslant n\leqslant N} |S_n| \in [\eta, \tfrac{1}{\eta}]\, t_N\Big) \geqslant 1-\varepsilon \qquad \textit{with } \mathbb{P}\textit{-probability larger than } 1-\varepsilon\,.$$

If $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations of order N^{ξ} under $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$, then we say that the end-to-end exponent is ξ.

3.1. **Statement of the results.** We now prove six different theorems, corresponding to the six possible regions in the phase diagram. We will specify when the results are specific or not to the case $\hat{h} > 0$; the case $\hat{h} < 0$ will be discussed in Section 3.3. Note that the case $h = 0$ or $\beta = 0$ can be recovered by taking $\zeta = +\infty$ or $\gamma = +\infty$ respectively.

Theorem 3.1 (**Region 1**). *Assume that* (1.3) *holds with* $\hat{h} \in \mathbb{R}$ *and*

$$\begin{cases} \gamma > \frac{1}{2\alpha} \quad \text{and} \quad \zeta > \frac{1}{2}, & \text{if} \quad \alpha \in (\frac{1}{2}, 1) \cup (1, 2], \\ \gamma > \frac{1-\alpha}{\alpha} \quad \text{and} \quad \zeta > \frac{1}{2}, & \text{if} \quad \alpha \in (0, \frac{1}{2}). \end{cases}$$

Then, $(S_n)_{0 \leqslant n \leqslant N}$ *has end-to-end fluctuations of order* $\sqrt{N}$ *under* $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$ *(i.e.* $\xi = \frac{1}{2}$*), and we have the following convergence in* $\mathbb{P}$*-probability*

$$Z^{\omega}_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} 1\,. \tag{3.1}$$

Additionally, for every continuous bounded function $F : \mathbb{R} \to \mathbb{R}$*, we have the convergence in probability* $\mathbf{E}^{\omega}_{N,\beta_N,h_N}\big[F\big(\frac{1}{\sqrt{N}} S_N\big)\big] \xrightarrow{\mathbb{P}} \mathbf{E}[F(Z)]$*, where* $Z \sim \mathcal{N}(0,1)$*.*

Theorem 3.2 (**Region 2**). *Assume that* (1.3) *holds with* $\hat{\beta} > 0$*,* $\hat{h} \in \mathbb{R}$ *and*

$$\tfrac{1-\alpha}{\alpha} < \gamma < \tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \wedge \tfrac{1}{2\alpha} \quad \text{and} \quad \alpha \in (\tfrac{1}{2}, 1) \cup (1, 2].$$

Then $(S_n)_{0 \leqslant n \leqslant N}$ *has end-to-end fluctuations of order* N^{ξ} *with* $\xi = \frac{\alpha}{2\alpha-1}(1-\gamma) \in (\frac{1}{2}, 1)$ *under* $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$*, and we have the following convergence*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{\xi}{\alpha}-\gamma}} \log Z^{\omega}_{N,\beta_N,h_N} = \mathcal{W}_{R_2} := \sup_{u \leqslant 0 \leqslant v} \left\{ Y^{(R_2)}_{u,v} \right\} \in (0, +\infty), \qquad \mathbb{P}\text{-}a.s.\,, \tag{3.2}$$

where $Y^{(R_2)}_{u,v} = Y^{\hat{\beta},(R_2)}_{u,v} = \hat{\beta}(X_v - X_u) - I(u,v)$ *is as defined in Theorem 1.1-*(1b)*. Additionally, for any* $\varepsilon > 0$*, we have*

$$\lim_{N\to\infty} \mathbf{P}^{\omega}_{N,\beta_N,h_N}\Big(\frac{1}{N^{\xi}} (M_N^-, M_N^+) \notin \mathcal{M}_{\varepsilon}(Y^{(R_2)}) \Big) = 0\,, \qquad \mathbb{P}\text{-}a.s.$$

Let us stress that the case $\alpha = 2$, $\beta = \beta_N \equiv \beta > 0$ and $h \equiv 0$ corresponds to the case $\gamma = 0$ and $\zeta = +\infty$: we find in that case that the end-to-end fluctuation exponent is $\xi = \frac{2}{3}$. We recall also that the set of quasi-mazimizers $\mathcal{M}(Y^{(R_2)}) = \bigcap_{\varepsilon>0} \mathcal{M}_{\varepsilon}(Y^{(R_2)})$ is reduced to one point when $\alpha = 2$, see [25], or when $\alpha \in (1,2)$ with $q = 0$ in Assumption 1, see [24].

Theorem 3.3 (**Region 3**). *Assume that* (1.3) *holds with* $\hat{\beta} > 0$*,* $\hat{h} \in \mathbb{R}$ *and*

$$\gamma < \big(\zeta - \tfrac{\alpha-1}{\alpha}\big) \wedge \big(\tfrac{1-\alpha}{\alpha}\big) \quad \text{and} \quad \alpha \in (0,1) \cup (1,2].$$

Then $(S_n)_{0 \leqslant n \leqslant N}$ *has end-to-end fluctuations of order* N *under* $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$ *(i.e.* $\xi = 1$*), and we have the following convergence*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{\alpha}-\gamma}} \log Z^{\omega}_{N,\beta_N,h_N} = \mathcal{W}_{R_3} := \sup_{u \leqslant 0 \leqslant v} \left\{ Y^{(R_3)}_{u,v} \right\} \in (0, +\infty)\,, \qquad \mathbb{P}\text{-}a.s. \tag{3.3}$$

where $Y^{(R_3)}_{u,v} = Y^{\hat{\beta},(R_3)}_{u,v} = \hat{\beta}(X_v - X_u) - \infty \mathbb{1}_{\{|u| \wedge |v| + v - u > 1\}}$ *is as defined in Theorem 1.1-*(2)*. Additionally, for any* $\varepsilon > 0$*, we have*

$$\lim_{N\to\infty} \mathbf{P}^{\omega}_{N,\beta,h}\Big(\frac{1}{N} (M_N^-, M_N^+) \notin \mathcal{M}_{\varepsilon}(Y^{(R_3)}) \Big) = 0\,, \qquad \mathbb{P}\text{-}a.s.$$

Also here, the set of quasi-mazimizers $\mathcal{M}(Y^{(R_3)}) = \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(Y^{(R_3)})$ is reduced to one point when $\alpha = 2$, see [25], or when $\alpha \in (1,2)$ with $q = 0$ in Assumption 1, see [24].

Theorem 3.4 (**Region 4**). *Assume that* (1.3) *holds with* $\hat\beta > 0$, $\hat h > 0$ *and*

$$\big(\tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\big) \vee \big(\zeta - \tfrac{\alpha-1}{\alpha}\big) < \gamma < \big(\tfrac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}\big) \wedge \zeta \quad \textit{and} \quad \alpha \in (1,2].$$

Then $(S_n)_{0\leqslant n\leqslant N}$ *has end-to-end fluctuations of order* N^ξ *with* $\xi = \frac{\alpha}{\alpha-1}(\zeta-\gamma) \in (0,1)$ *under* $\mathbf{P}^\omega_{N,\beta_N,h_N}$, *and we have the following convergence*

$$\lim_{N\to\infty} \frac{1}{N^{\xi-\zeta}} \log Z^\omega_{N,\beta_N,h_N} = \mathcal{W}_{R_4} := \sup_{u\leqslant 0\leqslant v} \left\{ Y^{(R_4)}_{u,v} \right\} \in (0,+\infty)\,, \qquad \mathbb{P}\textit{-a.s.}, \tag{3.4}$$

where $Y^{(R_4)}_{u,v} = Y^{\hat\beta,\hat h,(R_4)}_{u,v} := \hat\beta(X_v - X_u) - \hat h(v-u)$. *Additionally, for any* $\varepsilon > 0$, *we have*

$$\lim_{N\to\infty} \mathbf{P}^\omega_{N,\beta,h}\Big(\frac{1}{N^\xi}(M_N^-, M_N^+) \notin \mathcal{M}_\varepsilon(Y^{(R_4)}) \Big) = 0\,, \qquad \mathbb{P}\textit{-a.s.}$$

Also here, the set of quasi-mazimizers $\mathcal{M}(Y^{(R_3)}) = \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(Y^{(R_3)})$ is reduced to one point when $\alpha = 2$, see [25], or when $\alpha \in (1,2)$ with $q = 0$ in Assumption 1, see [24].

Theorem 3.5 (**Region 5**). *Assume that* (1.3) *holds with* $\hat\beta > 0$, $\hat h > 0$ *and*

$$\begin{cases} \gamma > \frac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}, \quad \textit{and} \quad -1 < \zeta < \frac12, & \textit{if} \ \ \alpha \in (1,2]\,, \\ \gamma > \big(\frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\big) \wedge \big(\zeta - \frac{\alpha-1}{\alpha}\big) \quad \textit{and} \quad -1 < \zeta < \frac12, & \textit{if} \ \ \alpha \in (\frac12, 1)\,, \\ \gamma > \big(\frac{1-\alpha}{\alpha}\big) \wedge \big(\zeta - \frac{\alpha-1}{\alpha}\big) \quad \textit{and} \quad -1 < \zeta < \frac12, & \textit{if} \ \ \alpha \in (0, \frac12)\,. \end{cases}$$

Then $(S_n)_{0\leqslant n\leqslant N}$ *has end-to-end fluctuations of order* N^ξ *with* $\xi = \frac{1+\zeta}{3} \in (0, \frac12)$ *under* $\mathbf{P}^\omega_{N,\beta_N,h_N}$, *and we have the following convergence in probability*

$$\frac{1}{N^{\xi-\zeta}} \log Z^\omega_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} -\frac32(\hat h\pi)^{2/3} = \sup_{r\geqslant 0}\left\{ -\hat h r - \frac{\pi^2}{2r^2} \right\}. \tag{3.5}$$

Additionally, for every $\varepsilon > 0$, *we have* $\mathbf{P}^\omega_{N,\beta_N,h_N}\big(\big| \frac{1}{N^\xi}(M_N^+ - M_N^-) - \pi^{\frac23}\hat h^{-\frac13} \big| > \varepsilon \big) \to 0$ *in* $\mathbb{P}$*-probability.*

Theorem 3.6 (**Region 6**). *Assume that* (1.3) *holds with* $\hat\beta > 0$, $\hat h > 0$ *and*

$$\begin{cases} \gamma > \zeta, \quad \textit{and} \quad \zeta < -1, & \textit{if} \ \ \alpha \in (1,2], \\ \gamma > \zeta - \frac{\alpha-1}{\alpha}, \quad \textit{and} \quad \zeta < -1, & \textit{if} \ \ \alpha \in (0,1). \end{cases}$$

Then we have the following convergences in probability (implying $\xi = 0$*)*

$$\mathbf{P}^\omega_{N,\beta_N,h_N}(|\mathcal{R}_n| = 2) \xrightarrow[N\to\infty]{\mathbb{P}} 1, \quad \textit{and} \quad N^\zeta \log Z^\omega_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} -2\hat h\,. \tag{3.6}$$

3.2. **Some comments on the results (case $\hat h > 0$).** Let us now make some observations on our results.

Comment 1. Our results describe the transition from *folded* trajectories ($\xi < 1/2$) to *stretched* trajectories ($\xi > 1/2$), this transition being induced by the presence of disorder. Let us illustrate this fact, in the case $\alpha \in (1,2]$ for simplicity: we refer to the phase diagram of Figure 1. If $\beta_N = \hat\beta > 0$ and $h_N = \hat h > 0$, that is $\gamma = \zeta = 0$, we find that the trajectories are folded, with end-to-end exponent $\xi = 1/3$.

Now, if we keep $h_N = \hat{h} > 0$ (*i.e.* $\zeta = 0$) fixed, and increase the strength of disorder, that is decrease γ (take $\gamma < 0$), we realize that we have transitions between the following regimes:

(i) if $\gamma > \frac{1-\alpha}{3\alpha}$, the random walk is folded with end-to-end exponent $\xi = 1/3$ (disorder is not strong enough);
(ii) if $\frac{1-\alpha}{3\alpha} > \gamma > \frac{1-\alpha}{2\alpha}$, then the random walk is still folded, with end-to-end exponent $1/3 < \xi = \gamma\alpha/(1-\alpha) < 1/2$ (disorder makes the random walk less folded);
(iii) if $\frac{1-\alpha}{2\alpha} > \gamma > \frac{1-\alpha}{\alpha}$, then the random walk is stretched, with end-to-end exponent $1/2 < \xi = \gamma\alpha/(1-\alpha) < 1$ (disorder is strong enough to stretch the random walk);
(iv) if $\gamma < \frac{1-\alpha}{\alpha}$, then the random walk is completely unfolded, and has end-to-end exponent $\xi = 1$.

Analogously, if we keep $\beta_N = \hat{\beta} > 0$ (*i.e.* $\gamma = 0$) fixed, and decrease the penalty for the range, that is increase ζ (take $\zeta > 0$), we have transitions between the following regimes:

(i) if $0 < \zeta < \frac{\alpha-1}{2\alpha+1}$, then the random walk is still folded with end-to-end exponent $1/3 < \xi = (1+\zeta)/3 < \alpha/(2\alpha+1) (< 1/2)$ (and it does not feel the disorder);
(ii) if $\frac{\alpha-1}{2\alpha+1} < \zeta < \frac{\alpha-1}{2\alpha}$, then the random walk is still folded with end-to-end exponent $\alpha/(2\alpha+1) < \xi = \zeta\alpha/(\alpha-1) < 1/2$ (and disorder plays a role);
(iii) if $\frac{\alpha-1}{2\alpha} < \zeta < \frac{\alpha-1}{2\alpha-1}$, then the random walk is stretched, with end-to-end exponent $1/2 < \xi = \gamma\alpha/(1-\alpha) < \alpha/(2\alpha-1)$ (< 1);
(iv) if $\zeta > \frac{\alpha-1}{2\alpha-1}$, then the random walk is stretched and has end-to-end exponent $2/3 \leqslant \xi = \alpha/(2\alpha-1) < 1$ (it does not feel the penalty for the range anymore).

Comment 2. Let us now comment on the limiting distributions for the log-partition function in regions R_2, R_3, R_4. For simplicity, we will restrict ourselves to the case where $u = 0$ in the variational problems (3.2)-(3.3)-(3.4) (which corresponds to considering the case of a random walk constrained to stay non-negative): the variational problems become, respectively

$$\widetilde{\mathcal{W}}_{R_2} := \sup_{v \geqslant 0} \left\{ \hat{\beta} X_v - \tfrac{1}{2} v^2 \right\}, \quad \widetilde{\mathcal{W}}_{R_3} := \hat{\beta} \sup_{v \in [0,1]} \{X_v\}, \quad \widetilde{\mathcal{W}}_{R_4} := \sup_{v \geqslant 0} \left\{ \hat{\beta} X_v - \hat{h} v \right\}. \tag{3.7}$$

a) The variational problem $\widetilde{\mathcal{W}}_{R_3}$ is clearly always finite. In the case $\alpha = 2$, $(X_t)_{t \geqslant 0}$ is a Brownian motion, and it is standard to get that $\widetilde{\mathcal{W}}_{R_3}$ has the distribution of $\hat{\beta}|Z|$, with $Z \sim \mathcal{N}(0,1)$. In the case $\alpha \in (0,2)$, $(X_t)_{t \geqslant 0}$ is a stable Lévy process, and we get that $\widetilde{\mathcal{W}}_{R_3}$ is a postitive α-stable random variable (see [8, Ch. VIII], and also [22]).

b) The variational problem $\widetilde{\mathcal{W}}_{R_4}$ is finite only when $\alpha > 1$: when $\alpha \in (0,1)$, then X_v grows typically as $v^{1/\alpha} \gg v$ as $v \to \infty$, and we therefore have $\mathcal{W}_{R_4} = +\infty$. This explains in particular why there is no energy-range balance possible if $\alpha \in (0,1)$, and why region R_4 no longer exists in that case. If $\alpha = 2$, $(X_t)_{t \geqslant 0}$ is a Brownian motion, and it is standard to get that $\widetilde{\mathcal{W}}_{R_4}$ is an exponential random variable (here with parameter $2\hat{h}/\beta^2$). If $\alpha \in (1,2)$, $(X_t)_{t \geqslant 0}$ is a stable Lévy process, and $(\hat{\beta} X_t - \hat{h} t)_{t \geqslant 0}$ is also a Lévy process: the distribution of its supremum $\widetilde{\mathcal{W}}_{R_4}$ has been studied intensively, going back to [2], but the exact distribution does not appear to be known (we refer to the recent papers [12, 23]).

c) The variational problem $\widetilde{\mathcal{W}}_{R_2}$ is finite only when $\alpha > 1/2$: when $\alpha \in (0, 1/2)$, then X_v grows typically as $v^{1/\alpha} \gg v^2$ as $v \to \infty$, and we therefore have $\mathcal{W}_{R_2} = +\infty$. This explains in particular why there is no "energy"-"entropy" balance possible if $\alpha \in (0, \frac{1}{2})$, and why region R_2 no longer exists in that case. In the case $\alpha = 2$, that is when $(X_t)_{t \geqslant 0}$ is a standard

Brownian motion, then $\widetilde{\mathcal{W}}_{R_4}$ has appeared in various contexts, and its density is known (its Fourier transform is expressed in terms of Airy function, see for instance [14, 19]). In the case $\alpha \in (\frac{1}{2}, 2)$, we are not aware whether the distribution of $\widetilde{\mathcal{W}}_{R_4}$ has been studied.

Comment 3. We chose in this paper not to treat the cases of the boundaries between different regions of the phase diagrams, mostly to keep the paper lighter. These boundary regions do not really hide anything deep: features of both regions should appear in the limit, and "disorder", "range" and "entropy" may all compete at the same (exponential) scale. Let us state for instance the limiting variational problems that one should find in some the most interesting boundary cases, in the case $\alpha \in (1, 2]$ (we refer to the phase diagram of Figure 1):

• Line between regions R_2 and R_4: $\gamma = \frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}$ and $\zeta \in (0, \frac{1}{2})$. Then one should have $\xi = \frac{\alpha(1-\gamma)}{2\alpha-1}$ and

$$\frac{1}{N^{2\xi-1}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \sup_{u \leqslant 0 \leqslant v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v-u) - I(u,v) \right\}.$$

• Line between regions R_4 and R_5: $\gamma = \frac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}$ and $\zeta \in (-1, \frac{1}{2})$. Then one should have $\xi = \frac{1+\zeta}{3}$ and

$$\frac{1}{N^{1-2\xi}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \sup_{u \leqslant 0 \leqslant v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v-u) - \frac{\pi}{2(v-u)^2} \right\}.$$

where the last term inside the supremum comes from the entropic cost of "folding" the random walk in the interval $[uN^{\xi}, vN^{\xi}]$ (see Lemma A.3).

• Line between regions R_2 and R_3: $\gamma = -(\alpha-1)/\alpha$ and $\zeta > 0$. Then one should have $\xi = 1$ and

$$\frac{1}{N^{2\xi-1}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \sup_{u \leqslant 0 \leqslant v} \left\{ \hat{\beta}(X_v - X_u) - \kappa(|u| \wedge |v| + v - u) \right\},$$

where $\kappa(t) := \frac{1}{2}(1+t)\log(1+t) + \frac{1}{2}(1-t)\log(1-t)$ for $t \in [0,1]$ ($\kappa(t) = +\infty$ for $t > 1$) is the rate function for the large deviations of the simple random walk, see Lemma A.2

• Line between regions R_3 and R_4: $\gamma = \zeta - (\alpha-1)/\alpha$ and $\zeta < 0$. Then one should have $\xi = 1$ and

$$\frac{1}{N^{2\xi-1}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \sup_{u \leqslant 0 \leqslant v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v-u) - \kappa(|u| \wedge |v| + v - u) \right\}.$$

Comment 4. In region R_5, the disorder term does not appear in the variational formula. In the case $\beta = 0$ and $h > 0$ (*i.e.* $\gamma = \infty$, $\zeta = 0$) corresponding to the random walk penalized by its range with no disorder term, the behavior of the random walk is well understood: it is confined in a segment of length $(\pi^{\frac{2}{3}}\hat{h}^{-\frac{1}{3}})N^{1/3}$ with a random center, see [27] for the continuum limit of the process. In our model, we have shown that trajectories are still confined in a segment of length $(\pi^{\frac{2}{3}}\hat{h}^{-\frac{1}{3}})N^{1/3}$. However, disorder should appear in the fluctuations of the log-partition function and in particular we believe that, depending on the strength β_N of the disorder interaction, the center of this segment should be determined so as to maximize the amount of potentials in that segment; in particular, it should not be random anymore (under $\mathbf{P}^{\omega}_{N,\beta_N,h_N}$, for almost every realization of ω). We leave this as an open problem.

3.3. Complements on the results: the case $\hat{h} < 0$.

3.3.1. *The phase diagram.* In the case $\hat{h} < 0$, the same type of "energy" vs. "range" vs. "entropy" heuristics as in Section 2 can be carried out. The main difference is that the "range" term is now a reward rather than a penalty, and thus plays in the same direction as the "disorder" term and encourages stretching: the end-to-end exponent will always be $\xi \geqslant 1/2$. Recall that for a polymer with typical end-to-end distance N^ξ, the "range" term is of order $N^{\xi-\zeta}$, the "disorder" term is of order $N^{\xi/\alpha-\gamma}$ and the entropy term is $N^{2\xi-1}$ (since $\xi \geqslant 1/2$). In a similar fashion than in Section 2, we find that there are two cases that need to be considered.

Case I ("disorder" $\gg$ "range"). As mentioned in Remark 2.1, regions R_1, R_2, R_3 are unchanged when $h < 0$: we refer to (2.4)-(2.5)-(2.6) for the determination of ξ in these three regions.

Case II ("disorder" $\ll$ "range"). The balance between range and entropy is achieved if $\xi - \zeta = 2\xi - 1$ (with $\xi \in [\frac{1}{2}, 1]$), which gives $\xi = 1 - \zeta$ when $\gamma > \frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}$. Also, we have $\xi = 1$ when $\zeta \leqslant 0$ and $\gamma > \zeta - \frac{\alpha-1}{\alpha}$, and we have $\xi = 1/2$ when $\zeta \geqslant 1/2$ and $\gamma > \frac{1}{2\alpha}$.

To summarize, we can identify different regimes according to the values of γ, ζ: there are five regimes when $\alpha \in (\frac{1}{2}, 2]$, see Figure 4 below; there are four regimes when $\alpha \in (0, \frac{1}{2})$, see Figure 5 below.

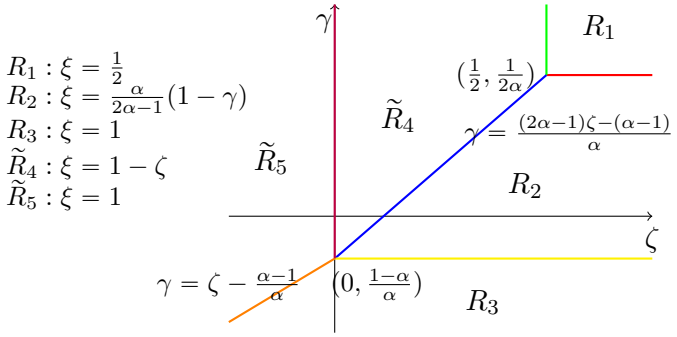

FIGURE 4. Phase diagram for $\hat{h} < 0$, in the case $\alpha \in (1/2, 2]$.

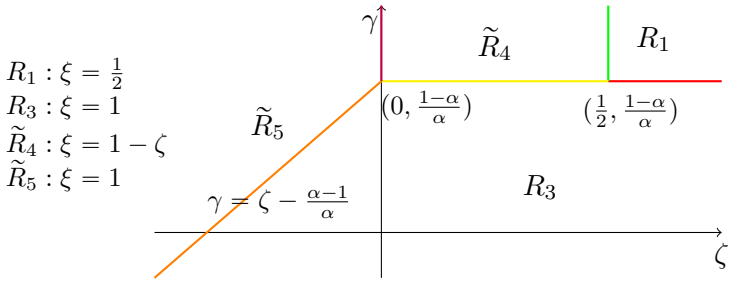

FIGURE 5. Phase diagram for $\hat{h} < 0$, in the case $\alpha \in (0, 1/2)$.

3.3.2. *Statement of the results.* We only state the results in regions $\widetilde{R}_4$ and $\widetilde{R}_5$, since the regions R_1, R_2 and R_3 are treated in Section 3.1, see Theorems 3.1, 3.2 and 3.3 (respectively).

Theorem 3.7 (Region $\widetilde{R}_4$)**.** *Assume that* (1.3) *holds with* $\hat{\beta} > 0$, $\hat{h} < 0$ *and*

$$\gamma > \frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \vee \frac{1-\alpha}{\alpha}, \qquad \zeta \in (0, \frac{1}{2})$$

Then $(S_n)_{0\leqslant n\leqslant N}$ *has end-to-end fluctuations of order* N^ξ *with* $\xi = 1-\zeta \in (\frac{1}{2},1)$ *under* $\mathbf{P}^\omega_{N,\beta_N,h_N}$*, and we have the following convergence in probability*

$$\frac{1}{N^{\xi-\zeta}} \log Z^\omega_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} \frac{1}{2}\hat{h}^2 = \sup_{u\leqslant 0\leqslant v} \left\{ |\hat{h}|(v-u) - I(u,v) \right\}. \tag{3.8}$$

Additionally, for every $\varepsilon > 0$*, we have* $\mathbf{P}^\omega_{N,\beta_N,h_N}\big(\big|\frac{1}{N^\xi}|S_N| - |\hat{h}|\big| > \varepsilon\big) \to 0$ *in* $\mathbb{P}$*-probability.*

Before we state the result in region $\widetilde{R}_5$ (which is somehow degenerate), let us state a result in the case $\zeta = 0$, that is at the boundary of regions $\widetilde{R}_4$ and $\widetilde{R}_5$.

Theorem 3.8 (Boundary $\widetilde{R}_4$—$\widetilde{R}_5$)**.** *Assume that* (1.3) *holds with* $\hat{\beta} > 0$*,* $\hat{h} < 0$ *and with* $\zeta = 0$*,* $\gamma > -\frac{\alpha-1}{\alpha}$*. Then we have the following convergence in probability*

$$\frac{1}{N} \log Z^\omega_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} \log\big(\tfrac{e^{2|\hat{h}|}-1}{2}\big) - |\hat{h}| = \sup_{u\leqslant 0\leqslant v} \left\{ |\hat{h}|(v-u) - \kappa(|u|\wedge|v| + v - u) \right\}. \tag{3.9}$$

Additionally, for every $\varepsilon > 0$*, we have* $\mathbf{P}^\omega_{N,\beta_N,h_N}\big(\big|\frac{1}{N}|S_N| - \frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1}\big| > \varepsilon\big) \to 0$ *in* $\mathbb{P}$*-probability.*

To conclude, we state the result in region $\widetilde{R}_5$.

Theorem 3.9 (Region $\widetilde{R}_5$)**.** *Assume that* (1.3) *holds with* $\hat{\beta} > 0$*,* $\hat{h} < 0$ *and* $\zeta < 0$*,* $\gamma > \zeta - \frac{\alpha-1}{\alpha}$*. Then*

$$\mathbf{P}^\omega_{N,\beta_N,h_N}\big(|S_N| = N\big) \xrightarrow[N\to\infty]{\mathbb{P}} 1\,, \qquad \textit{and} \qquad N^{\zeta-1}\log Z^\omega_{N,\beta_N,h_N} \xrightarrow[N\to\infty]{\mathbb{P}} |\hat{h}|\,.$$

Comment 5. Notice that in Theorems 3.7, 3.8 and 3.9, the disorder term disappears in the limiting variational problems, and the displacement of S_N under $\mathbf{P}^\omega_{N,\beta_N,h_N}$ is given by a (non-random) law of large number. Analogously to our Comment 4 above, disorder should appear in the fluctuations of the log-partition function and in the second order term for the displacement of S_N. For simplicity, let us comment further the case where $\beta > 0$, $h < 0$ are fixed (namely $\zeta = 0, \gamma = 0$) and $\alpha \in (1,2]$, *i.e.* Case 1.(c) of Theorem 1.1 (or Theorem 3.8). In that case, the polymer has a (non-random) velocity $v_h := \frac{e^{2|h|}-1}{e^{2|h|}+1}$. But randomness should have the effect of stretching (or at least moving) further the polymer. One can check that moving the random walk further from $v_h N$ by tN^ξ (with $t \in \mathbb{R}$) has an additional entropic cost of roughly $\frac{1}{2}\kappa''(v_h)t^2N^{2\xi-1}$ (the range variation $-\hat{h}tN^\xi$ is canceled by $\kappa'(v_h)tN^\xi$), whereas the energy gain is roughly $(tN^\xi)^{1/\alpha}$. This suggests that under $\mathbf{P}^\omega_{N,\beta,h}$ one should have $|S_N| \approx v_h N + \mathcal{V}N^\xi$ with $\xi = \frac{\alpha}{2\alpha-1}$, and where $\mathcal{V}$ is the location of the maximum of the variational problem $\sup_{t\in\mathbb{R}}\{\beta X_t - \frac{1}{2}\kappa''(v_h)t^2\}$. This goes beyond the scope of this article and we leave this as an open problem.

3.4. **Organisation of the proof, and useful notation.** We prove our results for Regions R_1 to R_6 (and $\widetilde{R}_4$ to $\widetilde{R}_6$) in the order listed above, by making our heuristic analysis for (2.3) rigorous. The results in Regions R_2, R_4 and R_5 (and $\widetilde{R}_4$) involve competitions between "energy", "range" or "entropy" (but all have the same scheme of proof), while Regions R_1, R_3 and R_6 (and $\widetilde{R}_5$) are extreme cases where only one factor is significant and hence are much simpler.

In the rest of the paper, to lighten the notations, we will drop the dependence on β_N and h_N: we write $\mathbf{P}^\omega_N$ instead of $\mathbf{P}^\omega_{N,\beta_N,h_N}$ and Z^ω_N instead of Z^ω_{N,β_N,h_N}. We also use the

convenient notation $Z_N^\omega(E)$ for the partition function restricted to trajectories $(S_n)_{n\in\mathbb{N}}$ in E; more precisely,

$$Z_N^\omega(E) := \mathbf{E}\Big[\exp\Big(\sum_{x\in\mathbb{Z}^d} (\beta_N\omega_x - h_N)\mathbb{1}_{x\in\mathcal{R}_N}\Big)\mathbb{1}_E\Big]. \tag{3.10}$$

This way, we have that $\mathbf{P}_N^\omega(E) = Z_N^\omega(E)/Z_N^\omega$.

Denote also

$$\Omega_j^+ := \sum_{x=0}^{j} \omega_x\,, \qquad \Omega_j^- := \sum_{x=-j}^{-1} \omega_x \quad \text{for } j \geqslant 0\,. \tag{3.11}$$

(with the convention that $\Omega_0^- = 0$), and let

$$\Omega_\ell^* := \sup_{0\leqslant j\leqslant \ell} |\Omega_j^-| + \sup_{0\leqslant j\leqslant \ell} |\Omega_j^+|\,. \tag{3.12}$$

Recall that we set $M_N^+ := \max_{0\leqslant n\leqslant N} S_n \geqslant 0$ and $M_N^- := \min_{0\leqslant n\leqslant N} S_n \leqslant 0$ the right-most and left-most points of the random walk after N steps; denote also $M_N^* := \max_{0\leqslant n\leqslant N} |S_n| = \max(M_N^+, -M_N^-)$. With these notations, notice that we have $\sum_{x\in\mathcal{R}_N}\omega_x = \Omega^+_{M_N^+} + \Omega^-_{-M_N^-}$. Let us state the following (standard) lemma, that we prove in Appendix A.2 for completeness.

Lemma 3.10. *Let Ω_ℓ^* defined as in* (3.12)*. Then, under Assumption 1 ($\alpha \in (0,1)\cup(1,2]$), there exists a constant $c \in (1,+\infty)$ such that for any $\mathsf{T} > 0$ and any ℓ we have*

$$\mathbb{P}\big(\Omega_\ell^* > \mathsf{T}\big) \leqslant c\,\ell\,\mathsf{T}^{-\alpha}\,. \tag{3.13}$$

4. Proof of the main results

4.1. **Region R1: Proof of Theorem 3.1.** Recall that in Region R_1 we have

$$\begin{cases} \gamma > \frac{1}{2\alpha} \quad \text{and} \quad \zeta > \frac{1}{2}, \quad \text{if} \quad \alpha \in (\frac{1}{2}, 1)\cup(1,2], \\ \gamma > -\frac{\alpha-1}{\alpha} \quad \text{and} \quad \zeta > \frac{1}{2}, \quad \text{if} \quad \alpha \in (0, \frac{1}{2}). \end{cases}$$

Let us note that we always have $\gamma > \frac{1}{2\alpha}$, since $\frac{1-\alpha}{\alpha} > \frac{1}{2\alpha}$ when $\alpha < 1/2$.

Convergence of the partition function. Fix A (large), and split the partition function in the following way

$$Z_N^\omega = Z_N^\omega\big(M_N^* \leqslant A\sqrt{N}\big) + Z_N^\omega\big(M_N^* > A\sqrt{N}\big). \tag{4.1}$$

Upper bound. It is easy to see that, recalling the definition (3.10) of the restricted partition function, since $h_N \geqslant 0$ we have

$$Z_N^\omega\big(M_N^* \leqslant A\sqrt{N}\big) \leqslant \exp\Big(\hat\beta N^{-\gamma}\Omega^*_{A\sqrt{N}}\Big)\,\mathbf{P}\Big(M_N^* \leqslant A\sqrt{N}\Big) \leqslant \exp\Big(\hat\beta N^{-\gamma}\Omega^*_{A\sqrt{N}}\Big) \tag{4.2}$$

By Lemma 3.10, we get that

$$\mathbb{P}\Big(Z_N^\omega\big(M_N^* \leqslant A\sqrt{N}\big) \geqslant e^\varepsilon\Big) \leqslant \mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{A\sqrt{N}} \geqslant \varepsilon\right) \leqslant c\varepsilon^{-\alpha}\hat\beta^\alpha A N^{(\frac{1}{2}-\alpha\gamma)}. \tag{4.3}$$

Therefore, since in Region 1 we have that $\gamma > \frac{1}{2\alpha}$, we get that (4.2) is bounded above by e^ε, with $\mathbb{P}$-probability going to 1 as N goes to infinity.

It remains to show that the second term in (4.1) is small, with high $\mathbb{P}$-probability. In this case the computations for the case $\alpha \in (0,1/2)$ and $\alpha \in (1/2,1)\cup(1,2]$ are different and we present them separately.

Case $\alpha \in (1/2, 1) \cup (1, 2]$. We have the following upper bound

$$\begin{aligned} Z_N^\omega\Big(M_N^* > A\sqrt{N}\Big) &\leqslant \sum_{k=1}^{\infty} Z_N^\omega\Big(M_N^* \in (2^{k-1}A\sqrt{N}, 2^k A\sqrt{N}]\Big) \\ &\leqslant \sum_{k=1}^{\infty} \exp\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A\sqrt{N}}\Big)\mathbf{P}\Big(M_N^* \geqslant 2^{k-1}A\sqrt{N}\Big) \end{aligned} \tag{4.4}$$

Then, it is standard to get that $\mathbb{P}(M_N^* > x) \leqslant 2\exp(-\frac{x^2}{2N})$ for any $x > 0$ and $N \in \mathbb{N}$ (thanks to Lévy's inequality and a standard Chernov bound), so that

$$\mathbf{P}\big(M_N^* > 2^{k-1}A\sqrt{N}\big) \leqslant 2\exp\Big(-2^{2k-3}A^2\Big)\,, \tag{4.5}$$

and

$$Z_N^\omega\Big(M_N^* > A\sqrt{N}\Big) \leqslant \sum_{k=1}^{\infty} 2\exp\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A\sqrt{N}}\Big)e^{-2^{2k-3}A^2}\,.$$

We therefore get, by a union bound, that

$$\begin{aligned} \mathbb{P}\Big(Z_N^\omega\big(M_N^* > A\sqrt{N}\big) > e^{-A}\Big) &\leqslant \sum_{k=1}^{\infty} \mathbb{P}\Big(2\exp\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A\sqrt{N}}\Big)e^{-2^{2k-3}A^2} > \frac{e^{-A}}{2^k}\Big) \\ &\leqslant \sum_{k=1}^{\infty} \mathbb{P}\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A\sqrt{N}} > 2^{2k-4}A^2\Big) \end{aligned} \tag{4.6}$$

where for the last inequality A is chosen large enough so that $e^{-A}2^{-k} \times \frac{1}{2}\exp\big(2^{2k-3}A^2\big) \geqslant \exp(2^{2k-4}A^2)$ for all $k \geqslant 1$. Using Lemma 3.10 we get that

$$\mathbb{P}\Big(\Omega^*_{2^k A\sqrt{N}} > \hat{\beta}^{-1}2^{2k-4}A^2N^\gamma\Big) \leqslant c\hat{\beta}^\alpha 2^{-k(2\alpha-1)}N^{(\frac{1}{2}-\alpha\gamma)}A^{1-2\alpha}\,. \tag{4.7}$$

Summing over k, and using that $\alpha > \frac{1}{2}$, we finally obtain that for A sufficiently large,

$$\mathbb{P}\Big(Z_N^\omega\big(M_N^* > A\sqrt{N}\big) > e^{-A}\Big) \leqslant c\hat{\beta}^\alpha A^{1-2\alpha}N^{(\frac{1}{2}-\alpha\gamma)}\,. \tag{4.8}$$

Note that this goes to 0 as $N \to \infty$, since $\gamma > \frac{1}{2\alpha}$.

Case $\alpha \in (0, 1/2)$. Let us consider the following decomposition

$$Z_N^\omega\Big(M_N^* > A\sqrt{N}\Big) = Z_N^\omega\Big(M_N^* \in (A\sqrt{N}, N^{3/4})\Big) + Z_N^\omega\Big(M_N^* \in (N^{3/4}, N]\Big)\,. \tag{4.9}$$

We then bound the first term as above, see (4.4):

$$Z_N^\omega\Big(M_N^* \in (A\sqrt{N}, N^{3/4})\Big) \leqslant \sum_{k=1}^{\log_2 N^{1/4}} \exp\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A\sqrt{N}}\Big)\mathbf{P}\Big(M_N^* \geqslant 2^{k-1}A\sqrt{N}\Big) \tag{4.10}$$

and the second one similarly:

$$\begin{aligned} Z_N^\omega\Big(M_N^* \in (N^{3/4}, N]\Big) &= \sum_{k=0}^{\log_2 N^{1/4}+1} Z_N^\omega\Big(M_N^* \in (2^{-k-1}N, 2^{-k}N]\Big) \\ &\leqslant \sum_{k=0}^{\log_2 N^{1/4}+1} \exp\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^{-k}N}\Big)\mathbf{P}\Big(M_N^* \geqslant 2^{-k-1}N\Big)\,. \end{aligned} \tag{4.11}$$

Let us observe that as in (4.5), we get that

$$\mathbf{P}\big(M_N^* > 2^{k-1}A\sqrt{N}\big) \leqslant 2\exp\Big(-2^{2k-3}A^2\Big)\ , \quad \mathbf{P}\big(M_N^* \geqslant 2^{-k-1}N\big) \leqslant 2\exp\Big(-2^{-2k-3}N\Big)\ .$$

Combining (4.9) to (4.11), we get that, analogously to (4.6), by a union bound

$$\mathbb{P}\Big(Z_N^\omega\big(M_N^* > A\sqrt{N}\big) > e^{-A}\Big) \leqslant \sum_{k=1}^{\log_2 N^{1/4}} \mathbb{P}\Big(e^{\hat\beta N^{-\gamma}\Omega^*_{2^k A\sqrt{N}}} > \frac{1}{2^{k+1}}e^{-A+2^{2k-3}A^2}\Big) + \sum_{k=1}^{\log_2 N^{1/4}+1} \mathbb{P}\Big(e^{\hat\beta N^{-\gamma}\Omega^*_{2^{-k}N}} > \frac{1}{2^{k+1}}e^{-A+2^{-2k-3}N}\Big)\,.$$

Using again Lemma 3.10 as in (4.7) (with A fixed sufficiently large and N large enough), we get that the above is bounded by a constant times

$$\sum_{k=1}^{\log_2 N^{1/4}} \hat\beta^\alpha 2^{-k(2\alpha-1)}N^{(\frac12-\alpha\gamma)}A^{1-2\alpha} + \sum_{k=1}^{\log_2 N^{1/4}+1} \hat\beta^\alpha 2^{k(2\alpha-1)}N^{1-\alpha(1+\gamma)}\,.$$

Since $\alpha < 1/2$, we therefore get that

$$\mathbb{P}\Big(Z_N^\omega\big(M_N^* > A\sqrt{N}\big) > e^{-A}\Big) \leqslant c_{A,\hat\beta,\alpha}N^{(\frac34-\frac{\alpha}{2}-\alpha\gamma)} + c'_{\hat\beta,\alpha}N^{1-\alpha(1+\gamma)}\,, \tag{4.12}$$

and both terms go to 0 as N goes to infinity, using that $\gamma > \frac{1-\alpha}{\alpha}$ and $\alpha < \frac12$ for the first term, and $\gamma > \frac{1-\alpha}{\alpha}$ for the second term.

All together, we have proved that in both cases $\alpha \in (\frac12, 1) \cup (1, 2]$ and $\alpha \in (0, \frac12)$, for any $\varepsilon > 0$ and A sufficiently large, with $\mathbb{P}$-probability going to 1, we have $Z_N^\omega \leqslant e^\varepsilon + e^{-A}$.

Lower bound. To achieve the lower bound, we use that

$$Z_N^\omega \geqslant Z_N^\omega(M_N^* \leqslant A\sqrt{N}) \geqslant \exp\Big(-\hat\beta N^{-\gamma}\Omega^*_{A\sqrt{N}} - 2A\hat h N^{\frac12-\zeta}\Big)\,\mathbf{P}\Big(M_N^* \leqslant A\sqrt{N}\Big)\,. \tag{4.13}$$

Using that $\mathbf{P}(M_N^* \leqslant A\sqrt{N}) \geqslant (1 - 2e^{-A^2/2})$ and the last upper bound in (4.3), we get that with $\mathbb{P}$-probability going to 1 as N goes to infinity, the right-hand-side of (4.13) is larger than

$$e^{-\varepsilon-2A\hat h N^{1/2-\zeta}}(1 - 2e^{-A^2/2}) \geqslant e^{-2\varepsilon}(1 - 2e^{-A^2/2})\,,$$

the last inequality being valid for N large enough, using that $\zeta > \frac12$. We then get that

$$\mathbb{P}\Big(Z_N^\omega \leqslant e^{-2\varepsilon}(1 - 2e^{-A^2/2})\Big) \xrightarrow[N\to\infty]{} 0\,.$$

Combined with the upper bound, this concludes the proof, since ε and A are arbitrary.

End-to-end fluctuations. From the calculation above, we directly have that, for A large, $\mathbf{P}_N^\omega(M_N^* > A\sqrt{N}) = Z_N^\omega(M_N^* > A\sqrt{N})/Z_N^\omega$ is bounded by a constant times e^{-A}, with $\mathbb{P}$-probability going to 1 as N goes to infinity. It remains to show that if η is small then $\mathbf{P}_N^\omega(M_N^* < \eta\sqrt{N})$ is small, with high $\mathbb{P}$-probability. Since $Z_N^\omega \xrightarrow{\mathbb{P}} 1$, it is sufficient to show that $Z_N^\omega(M_N^* \leqslant \eta\sqrt{N})$ is small, with high $\mathbb{P}$-probability. But we have the following upper bound, identical to (4.2):

$$Z_N^\omega(M_N^* \leqslant \eta\sqrt{N}) \leqslant \exp\Big(\hat\beta N^{-\gamma}\Omega^*_{\eta\sqrt{N}}\Big)\,\mathbf{P}\Big(M_N^* \leqslant \eta\sqrt{N}\Big). \tag{4.14}$$

Then, standard arguments (in the spirit of Lemma A.3) show that $\mathbf{P}(M_N^* \leqslant \eta\sqrt{N}) \leqslant e^{-c/\eta^2}$. On the other hand, using again Lemma 3.10, we get that $\mathbb{P}(\hat{\beta}N^{-\gamma}\Omega^*_{\eta\sqrt{N}} \geqslant 1)$ goes to 0 as $N \to \infty$ (using again that $\gamma > \frac{1}{2\alpha}$). This shows that $Z_N^\omega(M_N^* \leqslant \eta\sqrt{N}) \leqslant e^{1-c/\eta^2}$ with probability going to 1 as $N \to \infty$.

All together, this shows that $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations of order $\sqrt{N}$ under $\mathbf{P}_N^\omega$.

Convergence in distribution of $\frac{1}{\sqrt{N}}S_N$. In this regime the presence of the disorder and the range disappear and the central limit theorem that we get is the same as a simple random walk $(S, \mathbf{P})$.

We bound the difference between the characteristic functions: let $A > 0$, then

$$\begin{aligned}\Big|\mathbf{E}_N^\omega\big[e^{it\frac{S_N}{\sqrt{N}}}\big] - \mathbf{E}\big[e^{it\frac{S_N}{\sqrt{N}}}\big]\Big| &\leqslant \mathbf{E}\Big[\Big|\frac{1}{Z_N^\omega}e^{\sum_{x\in\mathcal{R}_N}\beta_N\omega_x - h_N|\mathcal{R}_N|} - 1\Big|\mathbb{1}_{\{M_N^*\leqslant A\sqrt{N}\}}\Big] \\ &\quad + \mathbf{P}\big(M_N^* > A\sqrt{N}\big) + \frac{Z_N^\omega(M_N^* > A\sqrt{N})}{Z_N^\omega}.\end{aligned} \tag{4.15}$$

As in (4.5), we have $\mathbf{P}\big(M_N^* > A\sqrt{N}\big) \leqslant e^{-cA^2}$ and by (4.8) and (4.12), for A sufficiently large we have

$$\mathbb{P}\Big(\frac{Z_N^\omega(M_N^* > A\sqrt{N})}{Z_N^\omega} \leqslant e^{-A}\Big) \xrightarrow[N\to\infty]{} 0.$$

Moreover, for any $\varepsilon \in (0,1)$ we let

$$\mathcal{A}_{N,\varepsilon}^\omega := \Big\{\beta_N\Omega^*_{A\sqrt{N}} \leqslant \varepsilon,\, 1-\varepsilon \leqslant Z_N^\omega \leqslant 1+\varepsilon\Big\}.$$

Then, by (3.1) and (4.3), $\mathbb{P}(\mathcal{A}_{N,\varepsilon}^\omega) \to 1$ as $N \to +\infty$. Note that on the event $\{M_N^* \leqslant A\sqrt{N}\}$ we have that $h_N|\mathcal{R}_N| \leqslant 2Ah_N\sqrt{N} \to 0$ because $\zeta > \frac{1}{2}$, so that on $\mathcal{A}_{N,\varepsilon}^\omega$ it holds that

$$\mathbf{E}\Big[\Big|\frac{1}{Z_N^\omega}e^{\sum_{x\in\mathcal{R}_N}\beta_N\omega_x - h_N|\mathcal{R}_N|} - 1\Big|\mathbb{1}_{\{M_N^*\leqslant A\sqrt{N}\}}\Big] \leqslant c\varepsilon\,.$$

Therefore, for any $\varepsilon \in (0,1)$ and A sufficiently large, with $\mathbb{P}$-probability going to 1, (4.15) is smaller than $c\varepsilon + 2e^{-A}$. Hence, the left-hand side of (4.15) converges to 0 as $N \to \infty$, in $\mathbb{P}$-probability. This concludes the proof since $\mathbf{E}[\exp(it\frac{1}{\sqrt{N}}S_N)]$ converges to $\mathbf{E}[\exp(itZ)]$ with $Z \sim \mathcal{N}(0,1)$, by the central limit theorem.

4.2. **Region R2: proof of Theorem 3.2.** We prove that in region R_2 the end-to-end fluctuations are of order N^ξ with $\xi = \frac{\alpha}{2\alpha-1}(1-\gamma)$. Recall that in Region R_2 we have

$$2\xi - 1 = \frac{\xi}{\alpha} - \gamma > \xi - \zeta \quad \text{with} \quad \alpha \in (\tfrac{1}{2}, 1) \cup (1, 2]\,,$$

and that region R_2 does not exist when $\alpha < 1/2$.

Convergence of the log-partition function. We fix some A large, and split the partition function as

$$Z_N^\omega = Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) + Z_N^\omega\left(M_N^* > AN^\xi\right). \tag{4.16}$$

The proof of the convergence is divided into three steps: (1) we show that after taking logarithm and dividing by $N^{2\xi-1}$, the first term converges to some random variable $\mathcal{W}_{R_2}^A$ as

$N \to \infty$; (2) we show that the second term is small compared to the first one; (3) we let $A \to \infty$, and observe that $\mathcal{W}^A_{R_2}$ converges to $\mathcal{W}_{R_2}$.
Step 1. We prove the following lemma.

Lemma 4.1. *In Region R_2, we have that $\mathbb{P}$-a.s., for any $A \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N^{2\xi-1}} \log Z^\omega_N \left(M^*_N \leqslant AN^\xi \right) = \mathcal{W}^A_{R_2} := \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A} \left\{ \hat\beta(X_v - X_u) - I(u,v) \right\},$$

with $(X_t)_{t\in\mathbb{R}}$ from Definition 1.1, and $I(u,v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2$.

Proof. Let us fix $\delta > 0$, and write

$$Z^{\omega,\leqslant}_N := Z^\omega_N\big(M^*_N \leqslant AN^\xi\big) = \sum_{k_1=0}^{\lfloor A/\delta\rfloor} \sum_{k_2=0}^{\lfloor A/\delta\rfloor} Z^\omega_N(k_1,k_2,\delta), \tag{4.17}$$

where we define

$$Z^\omega_N(k_1,k_2,\delta) := Z^\omega_N \left(M^-_N \in (-(k_1+1)\delta N^\xi, -k_1\delta N^\xi], M^+_N \in [k_2\delta N^\xi, (k_2+1)\delta N^\xi) \right) \tag{4.18}$$

(recall the definitions $M^-_N := \min_{0\leqslant n\leqslant N} S_n$ and $M^+_N := \max_{0\leqslant n\leqslant N} S_n$). Since there are at most $(A/\delta)^2$ terms in the sum, we easily get that

$$\max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \log Z^\omega_N(k_1,k_2,\delta) \leqslant \log Z^{\omega,\leqslant}_N \leqslant 2\log(A/\delta) + \max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \log Z^\omega_N(k_1,k_2,\delta)\,. \tag{4.19}$$

Upper bound. As an upper bound on $\log Z^\omega_N(k_1,k_2,\delta)$, we have

$$\begin{aligned} \log Z^\omega_N(k_1,k_2,\delta) \leqslant & \beta_N \left(\Omega^-_{\lfloor k_1\delta N^\xi\rfloor} + \Omega^+_{\lfloor k_2\delta N^\xi\rfloor} \right) + \beta_N R^\delta_N(k_1\delta, k_2\delta) \\ & + |\hat h|(k_1+k_2+2)\delta N^{\xi-\zeta} + p^\delta_N(k_1\delta, k_2\delta), \end{aligned} \tag{4.20}$$

where for $u, v \geqslant 0$ we defined

$$R^\delta_N(u,v) := \max_{uN^\xi+1\leqslant j\leqslant (u+\delta)N^\xi-1} \left| \Omega^-_j - \Omega^-_{\lfloor uN^\xi\rfloor} \right| + \max_{vN^\xi\leqslant j\leqslant (v+\delta)N^\xi-1} \left| \Omega^+_j - \Omega^+_{\lfloor vN^\xi\rfloor} \right|, \tag{4.21}$$

and

$$p^\delta_N(u,v) := \log \mathbf{P} \left(M^-_N \in \big(-(u+\delta)N^\xi, -uN^\xi \big], M^+_N \in \big[vN^\xi, (v+\delta)N^\xi \big) \right). \tag{4.22}$$

Write $u = k_1\delta, v = k_2\delta$ and set $U_\delta = \{0, \delta, 2\delta, \dots, A\}$: using that $2\xi - 1 = \xi/\alpha - \gamma$, we get that

$$\begin{aligned} \max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \frac{\log Z^\omega_N(k_1,k_2,\delta)}{N^{2\xi-1}} \leqslant \max_{u,v\in U_\delta} \Big\{ & \hat\beta N^{-\frac{\xi}{\alpha}} \big(\Omega^-_{\lfloor uN^\xi\rfloor} + \Omega^+_{\lfloor vN^\xi\rfloor} \big) + \hat\beta N^{-\frac{\xi}{\alpha}} R^\delta_N(u,v) \\ & + |\hat h|(u+v+2\delta) N^{(\xi-\zeta)-(2\xi-1)} + N^{-(2\xi-1)} p^\delta_N(u,v) \Big\}. \end{aligned} \tag{4.23}$$

It is easy to see that the third term in the maximum goes to 0 uniformly in u, v, since $u + v + 2\delta < 3A$ and we have $\xi - \zeta < 2\xi - 1$ in Region 2. Note that we have that $(N^{-\xi/\alpha}\Omega^-_{\lfloor uN^\xi\rfloor})_{u\in[0,A+\delta]}$ and $(N^{-\xi/\alpha}\Omega^+_{\lfloor vN^\xi\rfloor})_{v\in[0,A+\delta]}$ converge to two independent Lévy processes $(X^{(1)}_u)_{u\in[0,A+\delta]}$ and $(X^{(2)}_v)_{v\in[0,A+\delta]}$ (with no drift, no Brownian component and Lévy measure $\nu(\mathrm{d}x) = \alpha(p\mathbb{1}_{\{x>0\}} + q\mathbb{1}_{\{x<0\}})\,|x|^{-1-\alpha}\,\mathrm{d}x$ if $\alpha \in (0,1)\cup(1,2)$, or standard Brownian motions if $\alpha = 2$).

Note also that thanks to Lemma A.1 (see (A.2)) we have

$$\lim_{N\to\infty} N^{-(2\xi-1)} p_N(u,v,\delta) = -J(u,v)\,, \quad \text{with } J(u,v) := \frac{1}{2}(u\wedge v + u + v)^2\,, \quad u,v\geqslant 0\,.$$

Since the maximum is over a finite set (and recall the definition (4.21) of R_N^δ), we readily have that the upper bound in (4.23) converges to

(4.24)
$$\widehat{\mathcal{W}}_{R_2}^{A,\delta} := \max_{u,v\in U_\delta} \Big\{ \hat\beta(X_u^{(1)} + X_v^{(2)}) + \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{u+t}^{(1)} - X_u^{(1)}| + \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{v+t}^{(2)} - X_v^{(2)}| - J(u,v) \Big\}.$$

Lower bound. On the other hand, we have the following lower bound on $\log Z_N^\omega(k_1,k_2,\delta)$:

$$\beta_N\Big(\Omega^-_{\lfloor k_1\delta N^\xi\rfloor} + \Omega^+_{\lfloor k_2\delta N^\xi\rfloor}\Big) - \beta_N R_N^\delta(k_1\delta,k_2\delta) - |h_N|(k_2+k_1+2)\delta N^\xi + p_N^\delta(k_1\delta,k_2\delta).$$

and thus, setting $u=k_1\delta, v=k_2\delta$ and $U_\delta=\{0,\delta,\dots,A\}$ as above, we obtain

(4.25)
$$\max_{0\leqslant k_1,k_2\leqslant\frac{A}{\delta}} \frac{\log Z_N^\omega(k_1,k_2,\delta)}{N^{2\xi-1}} \geqslant \max_{u,v\in U_\delta} \Big\{ \hat\beta N^{-\frac{\xi}{\alpha}}\big(\Omega^-_{\lfloor uN^\xi\rfloor} + \Omega^+_{\lfloor vN^\xi\rfloor}\big) - \hat\beta N^{-\frac{\xi}{\alpha}} R_N^\delta(u,v) \\ - |\hat h|(u+v+2\delta)N^{(\xi-\zeta)-(2\xi-1)} + N^{-(2\xi-1)} p_N^\delta(u,v) \Big\}\,.$$

We get as above that the lower bound in (4.25) converges towards

(4.26)
$$\widetilde{\mathcal{W}}_{R_2}^{A,\delta} := \max_{u,v\in U_\delta} \Big\{ \hat\beta(X_u^{(1)} + X_v^{(2)}) - \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{u+t}^{(1)} - X_u^{(1)}| - \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{v+t}^{(2)} - X_v^{(2)}| - J(u,v) \Big\}.$$

Conclusion. By Skorohod's representation theorem, we have realized the upper bound (4.24) and the lower bound (4.26) on the same probability space, and they are a.s. upper and lower bounds for $\limsup N^{-(2\xi-1)}\log Z_N^{\omega,\leqslant}$ and $\liminf N^{-(2\xi-1)}\log Z_N^{\omega,\leqslant}$ respectively. Notice that, by a.s. càd-làg structure of trajectories of Lévy process (continuity in the case of Brownian motion), we clearly have that

$$\lim_{\delta\downarrow 0} \widetilde{\mathcal{W}}_{R_2}^{A,\delta} = \lim_{\delta\downarrow 0} \widehat{\mathcal{W}}_{R_2}^{A,\delta} = \sup_{u,v\in[0,A]} \Big\{ \hat\beta(X_u^{(1)} + X_v^{(2)}) - J(u,v) \Big\}\,,$$

which is exactly $\mathcal{W}_{R_2}^A$, defining $X_t = -X_{-t}^{(1)}$ if $t\leqslant 0$ and $X_t = X_t^{(2)}$ if $t\geqslant 0$, and $I(u,v) = J(-u,v)$. Letting $N\to\infty$ and then $\delta\downarrow 0$, this concludes the proof of Lemma 4.1. □

Step 2. Next, we prove the following result.

Lemma 4.2. *In region R_2, there is some $A_0>0$ and some constant $C=C_{\hat\beta}$ such that, for all $A\geqslant A_0$,*

$$\mathbb{P}\Big(\frac{1}{N^{2\xi-1}}\log Z_N^\omega\Big(M_N^* > AN^\xi\Big) \geqslant -1\Big) \leqslant CA^{1-2\alpha}\,.$$

(Recall that $\alpha>1/2$ in region R_2.)

Since $\alpha>1/2$, this proves that almost surely, there exists some $A=A(\omega)$ such that the second term in (4.16) is small compared to the first one, thanks to Lemma 4.1, using also that $\mathcal{W}_{R_2}^A \geqslant 0$ (by taking $u=0=v$).

Proof. Let us write

$$Z_N^{\omega,>} := Z_N^\omega\left(M_N^* > AN^\xi\right) = \sum_{k=1}^\infty Z_N^\omega\left(M_N^* \in (2^{k-1}AN^\xi, 2^kAN^\xi]\right)$$

$$\leqslant \sum_{k=1}^\infty \exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi} + |\hat h|N^{-\zeta}2^{k+1}N^\xi\right)\mathbf{P}\left(M_N^* \geqslant 2^{k-1}AN^\xi\right), \tag{4.27}$$

and note that $\mathbf{P}\left(M_N^* \geqslant 2^{k-1}AN^\xi\right) \leqslant 2\exp\left(-2^{2k-3}A^2N^{2\xi-1}\right)$. Therefore, thanks to a union bound, we get

$$\mathbb{P}\left(Z_N^{\omega,>} \geqslant e^{-N^{2\xi-1}}\right) \leqslant \sum_{k=1}^\infty \mathbb{P}\left(\exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi}\right)2e^{-2^{2k-4}A^2N^{2\xi-1}} \geqslant \frac{1}{2^{k+1}}e^{-N^{2\xi-1}}\right)$$

$$\leqslant \sum_{k=1}^\infty \mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi} \geqslant 2^{2k-5}A^2N^{2\xi-1}\right), \tag{4.28}$$

where for the first inequality, we use the fact that $2^{k+1}|\hat h|N^{\xi-\zeta} \leqslant 2^{2k-4}A^2N^{2\xi-1}$ for large enough N since $\xi-\zeta < 2\xi-1$ in Region 2; the last inequality holds provided that A is large enough. Then, using Lemma 3.10 and the fact that $2\xi-1+\gamma = \xi/\alpha$, we get that

$$\mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi} \geqslant 2^{2k-4}A^2N^{2\xi-1}\right) \leqslant c\hat\beta^\alpha 2^{(1-2\alpha)k}A^{1-2\alpha}.$$

Summing this inequality over k, since $\alpha > 1/2$, we conclude the proof of Lemma 4.2. □

Step 3. Let us note that, by monotonicity in A, we have that $\mathcal{W}_{R_2} = \lim_{A\uparrow\infty}\mathcal{W}^A_{R_2}$ is well defined (possibly infinite) and non-negative. We prove the following lemma:

Lemma 4.3. *If $\alpha \in (\frac12, 2]$, we have that $\mathcal{W}_{R_2} := \sup_{u\leqslant 0\leqslant v}\{\hat\beta(X_v - X_u) - I(u,v)\}$ is $\mathbb{P}$-a.s. positive and finite.*

Combined with Lemmas 4.1-4.2, this readily proves that $N^{-(2\xi-1)}\log Z_N^\omega$ converges almost surely to $\mathcal{W}_{R_2}$ as $N\to\infty$.

Proof. To show that $\mathcal{W}_{R_2} > 0$ almost surely, notice that taking $u = 0$ we have

$$\mathcal{W}_{R_2} \geqslant \sup_{v\geqslant 0}\left\{\hat\beta X_v - \tfrac12 v^2\right\}.$$

Then, almost surely, we can find some sequence $v_n \downarrow 0$ such that $X_{v_n} \geqslant v_n^{1/\alpha}$ for all n (cf. [1, Th. 2.1]): we then get that $\mathcal{W}_{R_2} \geqslant \sup_{n\geqslant 0}\{\hat\beta v_n^{1/\alpha} - \frac12 v_n^2\} > 0$ since $\alpha > 1/2$.

To show that $\mathcal{W}_{R_2} < +\infty$ a.s., notice that $I(u,v) = \frac12(|u|\wedge|v| + v - u)^2 \geqslant \frac12|u|^2 + \frac12|v|^2$: we therefore get that

$$\mathcal{W}_{R_2} \leqslant \sup_{u\leqslant 0}\left\{\hat\beta X_u - \tfrac12 u^2\right\} + \sup_{v\geqslant 0}\left\{\hat\beta X_v - \tfrac12 v^2\right\}.$$

Each term is clearly a.s. finite. Indeed, if we consider the second term we have that almost surely, $\hat\beta X_v - \frac12 v^2 \leqslant 0$ for v large enough: this is a consequence of the fact that for any $\varepsilon > 0$, a.s. $X_v \leqslant v^{(1+\varepsilon)/\alpha}$ for v large enough, cf. [26, Sec. 3]. Therefore, $\hat\beta X_v - \frac12 v^2 \leqslant \hat\beta v^{(1+\varepsilon)/\alpha} - \frac12 v^2 \leqslant 0$ for all v large enough, provided that ε is small enough so that $(1+\varepsilon)/\alpha < 2$. Similarly a.s. $\hat\beta X_u - \frac12 u^2 \leqslant 0$ for all u large enough, which concludes the proof. □

End-to-end fluctuations. Notice that Lemma 4.2 shows that the polymer $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations *at most* of order N^ξ under $\mathbf{P}_N^\omega$. We can actually deduce from Lemmas 4.1 and 4.3 that it also has end-to-end fluctuations *at least* of order N^ξ. Indeed, we can show that $\mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi)$ is small, with high $\mathbb{P}$-probability: we have that

$$\log \mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi) = \log Z_N^\omega(M_N^* \leqslant \eta N^\xi) - \log Z_N^\omega\,,$$

and so, by Lemma 4.1 we get that $N^{-(2\xi-1)} \log \mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi)$ converges a.s. toward $\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2}$. We therefore get that $\mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi)$ goes to 0 if $\log \mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi) \to -\infty$, which happens with probability $\mathbb{P}(\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2} < 0)$. Since $\mathcal{W}_\eta$ goes to 0 almost surely as $\eta \downarrow 0$ (both $X_v - X_u$ and $I(u,v)$ tend to 0), we get that $\mathbb{P}(\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2} < 0)$ goes to 1 as $\eta \downarrow 0$. This concludes the proof that the polymer $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations of order N^ξ under $\mathbf{P}_N^\omega$. □

Convergence of (M_N^-, M_N^+). Let us define, for $\varepsilon, \varepsilon' \in (0,1)$

$$\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'} = \Big\{(u,v) \in \mathbb{R}^- \times \mathbb{R}^+ \colon \sup_{(s,t)\in B_\varepsilon(u,v)} \big\{\hat\beta(X_t - X_s) - I(s,t)\big\} \geqslant \mathcal{W}_{R_2} - \varepsilon'\Big\},$$

where $B_\varepsilon(u,v)$ is the closed ball of center (u,v) and of radius $\varepsilon > 0$. Let us observe that $\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}$ nay not be closed, but it is bounded: we know that a.s. the supremum outside a compact $[-A(\omega), 0] \times [0, A(\omega)]$ is smaller than $-1 \leqslant \mathcal{W}_{R_2} - \varepsilon'$, see Lemma 4.3. We now prove that for any $\varepsilon, \varepsilon' \in (0,1)$, we have $\lim_{N\to\infty} \mathbf{P}_N^\omega(\frac{1}{N^\xi}(M_N^-, M_N^+) \in \mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}) = 1$ almost surely.

To simplify the notation, we denote the event $\{\frac{1}{N^\xi}(M_N^-, M_N^+) \notin \mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}\}$ by $\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}$. We have

$$\log \mathbf{P}_N^\omega(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}) = \log Z_N^\omega(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}) - \log Z_N^\omega\,.$$

From above, we have that $N^{-(2\xi-1)} \log Z_N^\omega$ converges almost surely to $\mathcal{W}_{R_2}$, so we only have to prove that $\limsup_{N\to\infty} N^{-(2\xi-1)} \log Z_N^\omega(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}) < \mathcal{W}_{R_2}$ a.s. Thanks to Lemma 4.2 we only need to estimate $Z_N^\omega(M_N^* \leqslant AN^\xi; \mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'})$. For any $\delta > 0$, we perform a similar decomposition as (4.17) to get

$$Z_N^{\omega,\leqslant}(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}) := Z_N^\omega\big(M_N^* \leqslant AN^\xi; \mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}\big) = \sum_{k_1=0}^{\lfloor A/\delta\rfloor} \sum_{k_2=0}^{\lfloor A/\delta\rfloor} Z_N^\omega(k_1,k_2,\delta; \mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}), \tag{4.29}$$

where we defined $Z_N^\omega(k_1,k_2,\delta; \mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'})$ as

$$Z_N^\omega\left(M_N^- \in (-(k_1+1)\delta N^\xi, -k_1\delta N^\xi], M_N^+ \in [k_2\delta N^\xi, (k_2+1)\delta N^\xi); \mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}\right). \tag{4.30}$$

By definition of $\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}$, we get that

$$Z_N^{\omega,\leqslant}(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'}) \leqslant \left(\frac{A}{\delta}\right)^2 \max_{(k_1,k_2)\in U_{\delta,R_2}^{\varepsilon,\varepsilon'}} Z_N^\omega(k_1,k_2,\delta), \tag{4.31}$$

where $U_{\delta,R_2}^{\varepsilon,\varepsilon'} := \big\{(k_1,k_2) : k_1\delta, k_2\delta \in U_\delta, (-(k_1+1)\delta, -k_1\delta] \times [k_2\delta, (k_2+1)\delta) \not\subset \mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}\big\}$, with $U_\delta = \{0, \delta, 2\delta, \dots, A\}$, and the maximum is 0 if $U_\delta^{\varepsilon,\varepsilon'}$ is empty.

Now, by the same argument as in **Step 1**, we have that

(4.32)
$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^{2\xi-1}}\log Z_N^{\omega,\leqslant}(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'})\leqslant \sup_{(u,v)\notin\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}}\Big\{\hat\beta(X_v-X_u)-I(u,v)\Big\}\leqslant \mathcal{W}_{R_2}-\varepsilon'\,,$$

by definition of $\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}$. This concludes the proof that $\mathbf{P}_N^\omega(\mathcal{A}_{N,R_2}^{\varepsilon,\varepsilon'})\to 0$ a.s.

At this stage, we show that for any $\varepsilon>0$, there is a.s. some ε' (that depends on ω) such that $\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}$ is included in $\mathring{\mathcal{M}}_{2\varepsilon}$, the interior of $\mathcal{M}_{2\varepsilon}:=\mathcal{M}_{2\varepsilon}(Y^{(R_2)})$. Then, since $\mathring{\mathcal{M}}_{2\varepsilon}\subset\mathcal{M}_{2\varepsilon}$ we have a.s.

$$\lim_{N\to\infty}\mathbf{P}_N^\omega\Big(\frac{1}{N^\xi}(M_N^-,M_N^+)\notin\mathcal{M}_{2\varepsilon}\Big)=0.$$

Let us proceed by contradiction: if for any $\varepsilon'>0$, $\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}$ is not contained in $\mathring{\mathcal{M}}_{2\varepsilon}$ we can select a subsequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k\in\mathcal{U}_{R_2}^{\varepsilon,1/k}\backslash\mathcal{M}_{2\varepsilon}$. Then, by compactness, we can suppose that the sequence $(x_k)_{k\in\mathbb{N}}$ converges to a limit x_*. We claim that $x_*\in\bigcap_{\varepsilon'>0}\mathcal{U}_{R_2}^{3\varepsilon/2,\varepsilon'}=\mathcal{M}_{3\varepsilon/2}\subset\mathring{\mathcal{M}}_{2\varepsilon}$: indeed, for any k large enough we have that $\|x_k-x_*\|<\varepsilon/8$, so that $B_\varepsilon(x_k)\subset B_{3\varepsilon/2}(x_*)$. But by assumption, since $\mathring{\mathcal{M}}_{2\varepsilon}$ is open, we have $x_*\notin\mathring{\mathcal{M}}_{2\varepsilon}$, which is a contradiction.

4.3. **Region R3: proof of Theorem 3.3.** We show that in Region R_3, we have $\xi=1$. Recall that in this region

$$\gamma<\zeta-\tfrac{\alpha-1}{\alpha}\quad\text{and}\quad\gamma<\tfrac{1-\alpha}{\alpha}\,,$$

with $\alpha\in(0,1)\cup(1,2]$. First, we prove the convergence in (3.3).

Convergence of the log-partition function. First of all, notice that we can reduce to the case $h_N\equiv 0$. Indeed, we have the bounds

$$Z^\omega_{N,\beta_N,h_N=0}\times e^{-|h_N|N}\leqslant Z^\omega_{N,\beta_N,h_N}\leqslant Z^\omega_{N,\beta_N,h_N=0}\times e^{|h_N|N}\,.$$

Since $h_N\sim\hat h N^{-\zeta}$ with $\zeta>\gamma+\frac{\alpha-1}{\alpha}$, we have that $N^{-(\frac1\alpha-\gamma)}|h_N|N\to 0$. In the following, we therefore focus on the convergence of $N^{-(\frac1\alpha-\gamma)}\log Z^\omega_{N,\beta_N,h_N=0}$. We write for simplicity Z_N^ω for $Z^\omega_{N,\beta_N,h_N=0}$.

For any $\delta>0$, we can write

$$Z_N^\omega=\sum_{k_1=0}^{\lfloor1/\delta\rfloor}\sum_{k_2=0}^{\lfloor1/\delta\rfloor}Z_N^\omega(k_1,k_2,\delta),\tag{4.33}$$

with $Z_N^\omega(k_1,k_2,\delta)$ as in (4.18) with $\xi=1$. Let us stress right away that since there are at most N steps for the random walk, we can have $M_N^-\leqslant -k_1\delta N$ and $M_N^+\geqslant k_2\delta N$ only if $\delta(k_1\wedge k_2+k_1+k_2)\leqslant 1$.

Hence, writing $u=k_1\delta$, and $v=k_2\delta$, and $U_\delta=\{0,\delta,2\delta,\dots,1\}$ we have

$$\max_{\substack{u,v\in U_\delta\\ u\wedge v+u+v\leqslant1}}\log Z_N^\omega(\tfrac{u}{\delta},\tfrac{v}{\delta},\delta)\leqslant\log Z_N^\omega\leqslant -2\log\delta+\max_{\substack{u,v\in U_\delta\\ u\wedge v+u+v\leqslant1}}\log Z_N^\omega(\tfrac{u}{\delta},\tfrac{v}{\delta},\delta).\tag{4.34}$$

For the upper bound, we have

(4.35)
$$\max_{\substack{u,v\in U_\delta\\ u\wedge v+u+v\leqslant1}}N^{\gamma-\frac1\alpha}\log Z_N^\omega(\tfrac{u}{\delta},\tfrac{v}{\delta},\delta)\leqslant\max_{\substack{u,v\in U_\delta\\ u\wedge v+u+v\leqslant1}}\hat\beta\left(N^{-\frac1\alpha}\Omega^-_{\lfloor uN\rfloor}+N^{-\frac1\alpha}\Omega^+_{\lfloor vN\rfloor}+N^{-\frac1\alpha}R_N^\delta(u,v)\right),$$

where, analogously to (4.21), we set

$$R_N^\delta(u,v) := \max_{uN+1\leqslant j\leqslant (u+\delta)N-1} \left|\Omega_j^- - \Omega_{\lfloor uN\rfloor}^-\right| + \max_{vN+1\leqslant j\leqslant (v+\delta)N-1} \left|\Omega_j^+ - \Omega_{\lfloor vN\rfloor}^+\right|. \tag{4.36}$$

As in the previous section, we get that the right-hand side in (4.35) converges toward

$$\widehat{\mathcal{W}}_{R_3}^\delta := \max_{\substack{u,v\in U_\delta \\ u\wedge v+u+v\leqslant 1}} \left\{\hat\beta(X_u^{(1)}+X_v^{(2)}) + \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{u+t}^{(1)} - X_u^{(1)}| + \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{v+t}^{(2)} - X_v^{(2)})|\right\}. \tag{4.37}$$

For the lower bound, we have

$$\begin{aligned}\max_{\substack{u,v\in U_\delta \\ u\wedge v+u+v\leqslant 1}} N^{\gamma-\frac{1}{\alpha}} \log Z_N^\omega(\tfrac{u}{\delta},\tfrac{v}{\delta},\delta) \geqslant \max_{\substack{u,v\in U_\delta \\ u\wedge v+u+v\leqslant 1}} \Big\{ &\hat\beta\left(N^{-\frac{1}{\alpha}}\Omega_{\lfloor uN\rfloor}^- + N^{-\frac{1}{\alpha}}\Omega_{\lfloor vN\rfloor}^+\right) \\ &- N^{-\frac{1}{\alpha}} R_N^\delta(u,v) - \hat h N^{1-\zeta+\gamma-\frac{1}{\alpha}} - N^{1+\gamma-\frac{1}{\alpha}}\log 2\Big\},\end{aligned} \tag{4.38}$$

where we used that any non-empty event of $(S_n)_{0\leqslant n\leqslant N}$ has probability at least 2^{-N}. Now, since $\gamma < \zeta + \frac{1}{\alpha} - 1$ and $\gamma < \frac{1}{\alpha} - 1$, the last two terms go to 0: we get that the right-hand side of (4.38) converges toward

$$\widetilde{\mathcal{W}}_{R_3}^\delta := \max_{\substack{u,v\in U_\delta \\ u\wedge v+u+v\leqslant 1}} \left\{\hat\beta(X_u^{(1)}+X_v^{(2)}) - \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{u+t}^{(1)} - X_u^{(1)}| - \hat\beta \sup_{0\leqslant t\leqslant\delta} |X_{v+t}^{(2)} - X_v^{(2)}|\right\}. \tag{4.39}$$

Then, we can conclude in the same manner as in the proof of Lemma 4.1: letting $N\to\infty$ and then $\delta\downarrow 0$, we get that $N^{\gamma-\frac{1}{2}}\log Z_N^\omega$ converges almost surely to

$$\lim_{\delta\downarrow 0} \widehat{\mathcal{W}}_{R_3}^\delta = \lim_{\delta\downarrow 0} \widetilde{\mathcal{W}}_{R_3}^\delta = \sup_{\substack{u,v\in[0,1] \\ u\wedge v+u+v\leqslant 1}} \left\{\hat\beta(X_u^{(1)}+X_v^{(2)})\right\},$$

where the limit holds thanks to the a.s. càdlàg property of trajectories of the Lévy process (or the a.s. continuity of the Brownian motion), and is exactly the variational problem $\mathcal{W}_{R_3}$ defined in Theorem 3.3 (by setting $X_t = -X_{-t}^{(1)}$ if $t\leqslant 0$ and $X_t = X_t^{(2)}$ if $t\geqslant 0$). Together with the (trivial) fact that $\mathcal{W}_{R_3}\in(0,+\infty)$ a.s., this concludes the proof of Theorem 3.3.

End-to-end fluctuations. It remains to show that $(S_n)_{0\leqslant n\leqslant N}$ has fluctuations of order *at least* N under $\mathbf{P}_N^\omega$, since we already know that $\max_{0\leqslant n\leqslant N}|S_n|\leqslant N$. we proceed as in the previous section. For $\eta>0$ we can write

$$\log \mathbf{P}_N^\omega(M_N^*\leqslant \eta N) = \log Z_N^\omega(M_N^*\leqslant\eta N) - \log Z_N^\omega,$$

and so, by Lemma 4.1 we get that $N^{\gamma-\frac{1}{\alpha}}\log\mathbf{P}_N^\omega(M_N^*\leqslant\eta N)$ (by a straightforward adaptation of the above proof) converges toward $\mathcal{W}_{R_3}^\eta - \mathcal{W}_{R_3}$, where

$$\mathcal{W}_{R_3}^\eta := \sup_{\substack{-\eta\leqslant u,v\leqslant\eta \\ u\wedge v+u+v\leqslant 1}} \left\{\hat\beta(X_u^{(1)}+X_v^{(2)})\right\}.$$

We therefore get that $\mathbf{P}_N^\omega(M_N^*\leqslant\eta N)$ goes to 0 with probability $\mathbb{P}(\mathcal{W}_\eta - \mathcal{W} < 0)$. Since $\mathcal{W}_{R_3}^\eta$ goes to 0 almost surely as $\eta\downarrow 0$, we get that $\mathbb{P}(\mathcal{W}_{R_3}^\eta - \mathcal{W}_{R_3} < 0)$ goes to 1 as $\eta\downarrow 0$. This concludes the proof that the polymer $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations of order N under $\mathbf{P}_N^\omega$. □

Convergence of (M_N^-, M_N^+). The proof follows the same strategy as in Region 2, so we only give a sketch. Let us define the counterpart of $\mathcal{U}_{R_2}^{\varepsilon,\varepsilon'}$ in Region 3 by

$$\mathcal{U}_{R_3}^{\varepsilon,\varepsilon'} = \Big\{(u,v) \in \mathbb{R}^- \times \mathbb{R}^+; |u| \wedge |v| + v - u \leqslant 1, \sup_{\substack{s\leqslant 0\leqslant t,(s,t)\in B_\varepsilon(u,v)\\ |s|\wedge|t|+t-s\leqslant 1}} \{\hat\beta(X_t - X_s)\} \geqslant \mathcal{W}_{R_3} - \varepsilon'\Big\}.$$

Then we denote the event $\{\frac{1}{N}(M_N^-, M_N^+) \notin \mathcal{U}_{R_3}^{\varepsilon,\varepsilon'}\}$ by $\mathcal{A}_{N,R_3}^{\varepsilon,\varepsilon'}$. By the same procedure as in Region 2, we can first show that a.s.

$$\limsup_{N\to\infty} \frac{1}{N^{\frac{1}{\alpha}-\gamma}} \log \mathbf{Z}_N^\omega(\mathcal{A}_{N,R_3}^{\varepsilon,\varepsilon'}) < \mathcal{W}_{R_3} \quad \text{ and so } \quad \limsup_{N\to\infty} \frac{1}{N^{\frac{1}{\alpha}-\gamma}} \log \mathbf{P}_N^\omega(\mathcal{A}_{N,R_3}^{\varepsilon,\varepsilon'}) < 0\,.$$

We then deduce as done in Region 2 that $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty} \mathbf{P}_N^\omega\Big(\frac{1}{N}(M_N^-, M_N^+) \in \mathcal{M}_{2\varepsilon}\Big) = 1, \tag{4.40}$$

which completes the proof.

4.4. **Region R4: proof of Theorem 3.4.** We prove that in Region R_4, we have $\xi = \frac{\alpha}{\alpha-1}(\zeta-\gamma)$. Recall that in region R_4 we have

$$\big(\tfrac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\big) \vee \big(\zeta - \tfrac{\alpha-1}{\alpha}\big) < \gamma < \big(\tfrac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}\big) \wedge \zeta\,, \quad \text{with } \alpha \in (1,2],$$

and that $\xi - \zeta = \xi/\alpha - \gamma > |2\xi - 1|$ ($\xi \in (0,1)$). Recall also that region R_4 does not exist if $\alpha < 1$.

Convergence of the log-partition function. For any $A > 0$, we first write

$$Z_N^\omega = Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) + Z_N^\omega\left(M_N^* > AN^\xi\right). \tag{4.41}$$

The strategy is similar to that in Region R_2, and we use analogous notation. We proceed in three steps: (1) after taking logarithm and dividing by $N^{\xi-\zeta}$, we show that the first term converges to some limit $\mathcal{W}_{R_4}^A$ when $N \to \infty$; (2) we show that the second term above is small compared to the first one; (3) we show that $\mathcal{W}_{R_4}^A \to \mathcal{W}_{R_4}$ as $A \to \infty$, with $\mathcal{W}_{R_4} \in (0, +\infty)$ almost surely.

Step 1. We prove the following lemma.

Lemma 4.4. *In Region R_4, we have that $\mathbb{P}$-a.s., for any $A \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N^{\xi-\zeta}} \log Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) = \mathcal{W}_{R_4}^A := \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A} \left\{\hat\beta(X_v - X_u) - \hat h(v-u)\right\},$$

with $(X_t)_{t\in\mathbb{R}}$ from Definition 1.1.

Proof. For fixed $\delta > 0$, we write (cf. (4.18))

$$Z_N^{\omega,\leqslant} := Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) = \sum_{k_1=0}^{\lfloor A/\delta\rfloor} \sum_{k_2=0}^{\lfloor A/\delta\rfloor} Z_N^\omega(k_1,k_2,\delta). \tag{4.42}$$

Since the number of summands above is finite, we have

$$\max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \log Z_N^\omega(k_1,k_2,\delta) \leqslant \log Z_N^{\omega,\leqslant} \leqslant 2\log(A/\delta) + \max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \log Z_N^\omega(k_1,k_2,\delta)\,. \tag{4.43}$$

Upper bound. We write $u = k_1\delta, v = k_2\delta$ and set $U_\delta = \{0, \delta, 2\delta, \dots, A\}$. Recall that $\xi - \zeta = \xi/\alpha - \gamma$, so we get that

$$\max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1,k_2,\delta)}{N^{\xi-\zeta}} \leqslant \max_{u,v\in U_\delta} \Big\{ \hat\beta N^{-\frac{\xi}{\alpha}} \big(\Omega^-_{\lfloor uN^\xi\rfloor} + \Omega^+_{\lfloor vN^\xi\rfloor}\big) + \hat\beta N^{-\frac{\xi}{\alpha}} R_N^\delta(u,v) - \hat h(v+u) + N^{\xi-\zeta} p_N^\delta(u,v) \Big\}. \tag{4.44}$$

Note that $(N^{-\xi/\alpha}\Omega^-_{\lfloor uN^\xi\rfloor})_{u\in[0,A+\delta]}$ and $(N^{-\xi/\alpha}\Omega^+_{\lfloor vN^\xi\rfloor})_{v\in[0,A+\delta]}$ converge to two independent Lévy processes $(X^{(1)}_u)_{u\in[0,A+\delta]}$ and $(X^{(2)}_v)_{v\in[0,A+\delta]}$ (with no drift, no Brownian component and Lévy measure $\nu(\mathrm{d}x) = \alpha(p\mathbb{1}_{\{x>0\}} + q\mathbb{1}_{\{x<0\}})\ |x|^{-1-\alpha}\ \mathrm{d}x$, if $\alpha \in (1,2)$, or standard Brownian motions, if $\alpha = 2$). Also note that $\lim_{N\to\infty} N^{\xi-\zeta} p_N^{(\delta)}(u,v) = 0$, thanks to Lemma A.1 and Lemma A.3, since we have $\xi - \zeta > |2\xi - 1|$. Since the maximum is over finite terms, then the upper bound in (4.44) converges a.s. to

$$\widehat{\mathcal{W}}^{A,\delta}_{R_4} := \max_{u,v\in U_\delta} \Big\{ \hat\beta \Big(X^{(1)}_u + X^{(2)}_v + \sup_{t\in[0,\delta]} |X^{(1)}_{u+t} - X^{(1)}_u| + \sup_{t\in[0,\delta]} |X^{(2)}_{v+t} - X^{(2)}_v| \Big) - \hat h(v+u) \Big\}. \tag{4.45}$$

Lower bound. On the other hand, we may bound $\log Z_N^\omega(k_1,k_2,\delta)$ from below by

$$\beta_N \Big(\Omega^-_{\lfloor k_1\delta N^\xi\rfloor} + \Omega^+_{\lfloor k_2\delta N^\xi\rfloor} \Big) - \beta_N R_N^\delta(k_1\delta, k_2\delta) - h_N(k_2 - k_1 + 2)\delta N^\xi + p_N^\delta(k_1\delta, k_2\delta).$$

Thus, setting $u = k_1\delta, v = k_2\delta$ and $U_\delta = \{0, \delta, \dots, A\}$ as above, we obtain

$$\max_{0\leqslant k_1,k_2\leqslant \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1,k_2,\delta)}{N^{\xi-\zeta}} \geqslant \max_{u,v\in U_\delta} \Big\{ \hat\beta N^{-\frac{\xi}{\alpha}} \big(\Omega^-_{\lfloor uN^\xi\rfloor} + \Omega^+_{\lfloor uN^\xi\rfloor}\big) - \hat\beta N^{-\frac{\xi}{\alpha}} R_N^\delta(u,v) - \hat h(u+v+2\delta) + N^{\xi-\zeta} p_N^\delta(u,v) \Big\}. \tag{4.46}$$

Hence, analogously to (4.45), the lower bound in (4.46) converges a.s. to

$$\widetilde{\mathcal{W}}^{A,\delta}_{R_4} := \max_{u,v\in U_\delta} \Big\{ \hat\beta \Big(X^{(1)}_u + X^{(2)}_v - \sup_{t\in[0,\delta]} |X^{(1)}_{u+t} - X^{(1)}_u| - \sup_{t\in[0,\delta]} |X^{(2)}_{v+t} - X^{(2)}_v| \Big) - \hat h(v+u+2\delta) \Big\}. \tag{4.47}$$

Conclusion. The upper and lower bounds (4.45)-(4.46) are almoste sure upper and lower bound for $\limsup N^{\zeta-\xi}\log Z_N^{\omega,\leqslant}$ and $\liminf N^{\zeta-\xi}\log Z_N^{\omega,\leqslant}$. By the a.s. càd-làg property of trajectories of Lévy processes (or continuity of the Brownian motion), we have

$$\lim_{\delta\downarrow 0} \widehat{\mathcal{W}}^{A,\delta}_{R_4} = \lim_{\delta\downarrow 0} \widetilde{\mathcal{W}}^{A,\delta}_{R_4} = \sup_{u,v\in[0,A]} \Big\{ \hat\beta \Big(X^{(1)}_u + X^{(2)}_v \Big) - \hat h(v+u) \Big\}, \tag{4.48}$$

which is exactly $\mathcal{W}^{A,\delta}_{R_4}$, defining $X_x = -X^{(1)}_{-x}$ if $x \leqslant 0$ and $X_x = X^{(2)}_x$ if $x \geqslant 0$. The convergence in Lemma 4.4 is therefore achieved by letting $N \to \infty$ and then $\delta \to 0$. □

Step 2. Next, we prove the following lemma.

Lemma 4.5. *In region R_4, there is some $A_0 > 0$ and some constant $C = C_{\hat\beta,\hat h}$, such that for $A \geqslant A_0$,*

$$\mathbb{P}\left(\frac{1}{N^{\xi-\zeta}} \log Z_N^\omega\big(M_N^* > AN^\xi\big) > -1 \right) \leqslant CA^{1-\alpha}.$$

(Recall that $\alpha > 1$ in region R_4.)

Since $\alpha > 1$, this proves that almost surely, there exists some $A = A(\omega)$ such that the second term in (4.16) is small compared to the first one, thanks to Lemma 4.1, using also that $\mathcal{W}_{R_4}^A \geqslant 0$.

Proof. First, let us write

$$Z_N^{\omega,>} := Z_N^\omega \left(M_N^* > AN^\xi \right) = \sum_{k=1}^\infty Z_N^\omega \left(M_N^* \in (2^{k-1}AN^\xi, 2^k AN^\xi] \right)$$
$$\leqslant \sum_{k=1}^\infty \exp\left(\hat\beta N^{-\gamma} \Omega^*_{2^k AN^\xi} - \hat h N^{-\zeta} 2^{k-1} AN^\xi \right).$$

By a union bound, we therefore get that

$$\mathbb{P}\left(Z_N^{\omega,>} \geqslant e^{-N^{\xi-\zeta}} \right) \leqslant \sum_{k=1}^\infty \mathbb{P}\left(e^{\hat\beta N^{-\gamma}\Omega^*_{2^k AN^\xi} - \hat h N^{-\zeta} 2^{k-1} AN^\xi} \geqslant \frac{1}{2^{k+1}} e^{-N^{\xi-\zeta}} \right)$$
$$\leqslant \sum_{k=1}^\infty \mathbb{P}\left(\hat\beta N^{-\gamma} \Omega^*_{2^k AN^\xi} \geqslant \hat h 2^{k-2} A N^{\frac{\xi}{\alpha}-\gamma} \right), \tag{4.49}$$

where the last inequality holds provided that A has been fixed large enough (we also used that $\xi - \zeta = \frac{\xi}{\alpha} - \gamma$). Then Lemma 3.10 gives that

$$\mathbb{P}\left(\hat\beta \Omega^*_{2^k AN^\xi} \geqslant \hat h 2^{k-2} AN^{\frac{\xi}{\alpha}} \right) \leqslant c\hat\beta^\alpha \hat h^{-\alpha} 2^{k(1-\alpha)} A^{1-\alpha}.$$

Summing this inequality over k, since $\alpha > 1$, this concludes the proof of Lemma 4.5. □

Step 3. By monotone convergence, $\mathcal{W}_{R_4}^A$ converges a.s. to $\mathcal{W}_{R_4}$: we only need to show that $\mathcal{W}_{R_4}$ is positive and finite. Combining this with Lemma 4.4 and Lemma 4.5, this completes the proof Theorem 3.4.

Lemma 4.6. *If $\alpha \in (1,2]$, we have $\mathcal{W}_{R_4} := \sup_{u\leqslant 0\leqslant v}\{\hat\beta(X_v - X_u) - \hat h(v-u)\}$ is a.s. positive and finite.*

Proof. The proof is analogous to the proof of Lemma 4.3. To show that $\mathcal{W}_{R_4} > 0$, we use that $\mathcal{W}_{R_4} \geqslant \sup_{v\geqslant 0}\{\hat\beta X_v - \hat h v\}$. By [1, Th 2.1]), there is a.s. a sequence $v_n \downarrow 0$, such that $X_{v_n} \geqslant v_n^{1/\alpha}$ for all n. Hence, for large enough n, $\mathcal{W}_{R_4} \geqslant \hat\beta v_n^{1/\alpha} - \hat h v_n > 0$, since $\alpha > 1$.

To show that $\mathcal{W}_{R_4} < \infty$, we use $\mathcal{W}_{R_4} \leqslant \sup_{u\leqslant 0}\{\hat\beta X_u + \hat h u\} + \sup_{v\geqslant 0}\{\hat\beta X_v - \hat h v\}$. By [26], we have that for any $\varepsilon > 0$, a.s. $X_v \leqslant v^{(1+\varepsilon)/\alpha}$ for v large enough. Therefore, if ε is sufficiently small so that $(1+\varepsilon)/\alpha < 1$ (recall $\alpha > 1$), we get that $\hat\beta X_v - \hat h v \leqslant \hat\beta v^{(1+\varepsilon)/a} - \hat h v \leqslant 0$ for all v sufficiently large. Similarly we also have that $\hat\beta X_u + \hat h u \leqslant 0$ for all u large enough. This concludes the proof. □

End-to-end fluctuations. We prove that the end-to-end fluctuations are of order N^ξ. Lemma 4.5 already shows that the end-to-end fluctuations are *at most* N^ξ. On the other hand, the fact that $\mathcal{W}_{R_4} > 0$ a.s. ensures that the end-to-end fluctuations are *at least* N^ξ. Indeed, we have thanks to Lemma 4.4 that for $\eta > 0$,

$$\lim_{N\to\infty} \frac{1}{N^{\xi-\zeta}} \log \mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi) = \mathcal{W}_{R_4}^\eta - \mathcal{W}_{R_4}, \qquad \mathbb{P}\text{-a.s.}$$

Hence, $\mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi) \to 0$ with probability $\mathbb{P}(\mathcal{W}_{R_4}^\eta - \mathcal{W}_{R_4} < 0)$: since $\mathcal{W}_{R_4}^\eta$ goes to 0 as $\eta \downarrow 0$, we can make this probability arbitrarily close to 1 by choosing η small. This concludes the proof that $(S_n)_{0\leqslant n\leqslant N}$ has end-to-end fluctuations of order N under $\mathbf{P}_N^\omega$. □

Convergence of (M_N^-, M_N^+). As in previous sections, we define

$$\mathcal{U}_{R_4}^{\varepsilon,\varepsilon'} = \Big\{(u,v) \in \mathbb{R}^- \times \mathbb{R}^+ : \sup_{(s,t)\in B_\varepsilon(u,v)} \{\hat\beta(X_t - X_s) - \hat h(t-s)\} \geqslant \mathcal{W}_{R_4} - \varepsilon'\Big\},$$

and the event $\mathcal{A}_{N,R_4}^{\varepsilon,\varepsilon'} = \{\frac{1}{N^\xi}(M_N^-, M_N^+) \notin \mathcal{U}_{R_4}^{\varepsilon,\varepsilon'}\}$. Then, in an identical manner as in Regions 2 and 3, we have that

$$\limsup_{N\to\infty} \frac{1}{N^{\xi-\zeta}} \log \mathbf{Z}_N^\omega(\mathcal{A}_{N,R_4}^{\varepsilon,\varepsilon'}) < \mathcal{W}_{R_4} \quad \text{ and so } \quad \limsup_{N\to\infty} \frac{1}{N^{\xi-\zeta}} \log \mathbf{P}_N^\omega(\mathcal{A}_{N,R_4}^{\varepsilon,\varepsilon'}) < 0\,,$$

from which one deduces that

$$\lim_{N\to\infty} \mathbf{P}_N^\omega\Big(\frac{1}{N^\xi}(M_N^-, M_N^+) \in \mathcal{M}_{2\varepsilon}\Big) = 1\,, \qquad \mathbb{P}\text{-a.s.} \tag{4.50}$$

4.5. **Region R5: proof of Theorem 3.5.** In this region, we prove that the end-to-end fluctuations of order N^ξ with $\xi = \frac{1+\zeta}{3} \in (0, 1/2)$. Note that in Region 5, we have

$$1 - 2\xi = \xi - \zeta > \frac{\xi}{\alpha} - \gamma \quad \text{ and } \quad -1 < \zeta < \frac{1}{2}\,.$$

Convergence of the log-partition function. We fix some constant $A > 0$ (large), and we split the partition function as

$$Z_N^\omega = Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) + Z_N^\omega\left(M_N^* > AN^\xi\right). \tag{4.51}$$

The strategy of proof is similar to that in Region R_2, but with only two steps: (1) we show that for A large enough, after taking logarithm and dividing by $N^{1-2\xi}$, the first term converges to some constant (independent of A if A is large enough) in probability; (2) we show that if A is large enough, the second term is negligible compared to the first one.
Step 1. We prove the following lemma.

Lemma 4.7. *In Region 5, we have that for any $A > 0$, as $N \to \infty$,*

$$\frac{1}{N^{1-2\xi}} \log Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) \xrightarrow[N\to\infty]{\mathbb{P}} \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A} \Big\{-\hat h(v-u) - \bar I(u,v)\Big\}\,,$$

where $\bar I(u,v) = \frac{\pi^2}{2}(v-u)^{-2}$ for $u \leqslant 0 \leqslant v$. By a simple calculation, the supremum is $\frac{3}{2}(\hat h\pi)^{\frac{2}{3}}$ for any $A \geqslant 2\pi^{\frac{2}{3}}\hat h^{-\frac{1}{3}}$, since it is achieved at $v - u = \pi^{\frac{2}{3}}\hat h^{-\frac{1}{3}}$.

Proof. For any fixed A, we have the following upper and lower bounds

$$\log \widehat Z_N^A - \hat\beta N^{-\gamma}\Omega_{AN^\xi}^* \leqslant \log Z_N^\omega\left(M_N^* \leqslant AN^\xi\right) \leqslant \log \widehat Z_N^A + \hat\beta N^{-\gamma}\Omega_{AN^\xi}^*\,, \tag{4.52}$$

where $\widehat Z_N^A := \mathbf{E}\big[\exp(-h_N|\mathcal{R}_N|)\mathbb{1}_{\{M_N^*\leqslant AN^\xi\}}\big]$.

Since in Region 5 we have $1 - 2\xi > \frac{\xi}{\alpha} - \gamma$, we get that $N^{-(1-2\xi)} \times \hat\beta N^{-\gamma}\Omega_{AN^\xi}^*$ goes to 0 in probability. Hence, we only need to prove that

$$\lim_{N\to\infty} \frac{1}{N^{1-2\xi}} \log \widehat Z_N^A = \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A} \Big\{-\hat h(v-u) - \bar I(u,v)\Big\}$$

(there is no disorder anymore). But this is quite standard, since we have by Lemma A.3 that $\bar I(u,v)$ is the rate function for the LDP for $(N^{-\xi}M_N^-,\, N^{-\xi}M_N^+)$, more precisely

$$-\bar I(u,v) = \lim_{N\to\infty} \frac{1}{N^{1-2\xi}} \log \mathbf{P}\Big(M_N^- \geqslant uN^\xi; M_N^+ \leqslant vN^\xi\Big)\,. \tag{4.53}$$

This is enough to conclude thanks to Varadhan's lemma. □

Step 2. Next, we prove the following lemma.

Lemma 4.8. *In region R_5, for any constant $T > 0$, there is some $A_T > 0$ such that for $A \geqslant A_T$*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{N^{1-2\xi}} \log Z_N^\omega \left(M_N^* > AN^\xi\right) > -T\right) = 0\,.$$

Combining this result with Lemma 4.7 readily yields Theorem 3.5.

Proof. We consider two cases: (i) $\alpha \in (1,2]$ and $\zeta \in (-1, 1/2)$ or $\alpha \in (0,1)$ and $\zeta \in (-1, 0]$; (ii) $\alpha \in (0,1)$ and $\zeta \in (0, 1/2)$. The strategy of proof is different for each case and we present them separately.

Case (i). We write $Z_N^{\omega,>} := Z_N^\omega(M_N^* > AN^\xi)$. Then

$$\begin{aligned} Z_N^{\omega,>} &\leqslant \sum_{k=1}^{\log_2 N^{1-\xi}} Z_N^\omega\left(M_N^* \in (2^{k-1}AN^\xi, 2^k AN^\xi]\right) \\ &\leqslant \sum_{k=1}^{\log_2 N^{1-\xi}} \exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^k AN^\xi} - \hat h 2^{k-1} AN^{\xi-\zeta}\right). \end{aligned} \tag{4.54}$$

Recall that in Region R_5 we have that $\xi - \zeta = 1 - 2\xi$. By union bound we obtain that

$$\begin{aligned} \mathbb{P}\left(Z_N^{\omega,>} \geqslant e^{-TN^{1-2\xi}}\right) &\leqslant \sum_{k=1}^{\log_2 N^{1-\xi}} \mathbb{P}\left(\exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^k AN^\xi}\right) e^{-\hat h 2^{k-1} AN^{\xi-\zeta}} \geqslant \frac{1}{2^k} e^{-TN^{\xi-\zeta}}\right) \\ &\leqslant \sum_{k=1}^{\log_2 N^{1-\xi}} \mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{2^k AN^\xi} \geqslant 2^{k-2}\hat h AN^{\xi-\zeta}\right), \end{aligned}$$

where the last inequality holds for sufficient large A (depending on $\hat h, T$). Then by Lemma 3.10, we get that

$$\mathbb{P}\left(Z_N^{\omega,>} \geqslant e^{-TN^{1-2\xi}}\right) \leqslant C \sum_{k=1}^{\log_2 N^{1-\xi}} 2^{k(1-\alpha)} N^{\xi-\alpha(\xi-\zeta+\gamma)} \leqslant \begin{cases} C'\, N^{\xi-\alpha(\xi-\zeta+\gamma)} & \text{if } \alpha \in (1,2]\,, \\ C' N^{\alpha(\frac{1-\alpha}{\alpha}+\zeta-\gamma)} & \text{if } \alpha \in (0,1)\,. \end{cases}$$

To get that this upper bound goes to 0 when $N \to \infty$, we use that $\xi - \zeta > \frac{\xi}{\alpha} - \gamma$ in the case $\alpha \in (1,2]$, and that $\gamma > \zeta + \frac{1-\alpha}{\alpha}$ for $\zeta \leqslant 0$ in the case $\alpha \in (0,1)$.

Case (ii). In that case, we have $\zeta \in (0, \frac{1}{2})$ and $\xi \in (0, \frac{1}{2})$. Hence, we can write

$$Z_N^\omega\big(M_N^* > AN^\xi\big) = Z_N^\omega\big(M_N^* \in (AN^\xi, N^{1-\zeta}]\big) + Z_N^\omega\big(M_N^* \in (N^{1-\zeta}, N]\big). \tag{4.55}$$

For the first term on the right-hand side of (4.55), using that $\xi - \zeta = 1 - 2\xi$, we get by a union bound that

$$\begin{aligned}
&\mathbb{P}\Big(Z_N^\omega\big(M_N^* \in (AN^\xi, N^{1-\zeta}]\big) > \tfrac{1}{2}e^{-TN^{1-2\xi}}\Big) \\
&\quad\leqslant \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P}\left(Z_N^\omega\big(M_N^* \in (2^{k-1}AN^\xi, 2^kAN^\xi]\big) > \tfrac{1}{2^{k+1}}e^{-TN^{\xi-\zeta}}\right) \\
&\quad\leqslant \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P}\left(\exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi}\right)e^{-\hat h 2^{k-1}AN^{\xi-\zeta}} \geqslant \tfrac{1}{2^{k+1}}e^{-TN^{\xi-\zeta}}\right). \\
&\quad\leqslant \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{2^kAN^\xi} \geqslant \hat h 2^{k-2}AN^{\xi-\zeta}\right)
\end{aligned} \tag{4.56}$$

where the last inequality holds for sufficient large A (depending on $\hat h, T$). Using Lemma 3.10, we therefore get that the left-hand side term of (4.56) is bounded by a constant times

$$\sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} 2^{k(1-\alpha)}N^{\xi(1-\alpha)+\alpha(\zeta-\gamma)} \leqslant CN^{1-\alpha+(2\alpha-1)\zeta-\alpha\gamma}, \tag{4.57}$$

Since we have $\gamma > \frac{(2\alpha-1)\zeta+1-\alpha}{\alpha}$ for $\zeta \in (0, \frac{1}{2})$, we get that (4.56) goes to 0 as $N \to \infty$.

For the second term on the right-hand side of (4.55), we use a union bound to get that

$$\begin{aligned}
&\mathbb{P}\Big(Z_N^\omega(M_N^* \in (N^{1-\zeta}, N]) > \tfrac{1}{2}e^{-TN^{\xi-\zeta}}\Big) \\
&\quad\leqslant \sum_{k=1}^{\log_2 N^{\zeta}} \mathbb{P}\Big(Z_N^\omega(M_N^* \in (2^{-k}N, 2^{-k+1}N]) > \tfrac{1}{2^{k+1}}e^{-TN^{\xi-\zeta}}\Big) \\
&\quad\leqslant \sum_{k=1}^{\log_2 N^{\zeta}} \mathbb{P}\left(\exp\left(\hat\beta N^{-\gamma}\Omega^*_{2^{-k+1}N}\right)e^{-2^{-2k-1}N} > \tfrac{1}{2^{k+3}}e^{-TN^{\xi-\zeta}}\right),
\end{aligned} \tag{4.58}$$

where we have use that $\mathbf{P}(M_N^* > x) \leqslant 2\exp(-x^2/2N)$ for the simple random walk. Now, since for all $k \leqslant \log_2 N^\zeta$ we have $2^{-2k}N \geqslant N^{1-2\zeta}$ and that $1 - 2\zeta > \xi - \zeta$, we get that for N large enough, the left-hand side of (4.58) is bounded by

$$\sum_{k=1}^{\log_2 N^{\zeta}} \mathbb{P}\left(\hat\beta N^{-\gamma}\Omega^*_{2^{-k+1}N} \geqslant 2^{-2k-2}N\right) \leqslant c\hat\beta^\alpha \sum_{k=1}^{\log_2 N^{\zeta}} 2^{k(2\alpha-1)}N^{1-\alpha(1+\gamma)},$$

where we used Lemma 3.10 for the last inequality. If $\alpha \in (0, \frac{1}{2}]$, this is bounded above by a constant times $(\log N)N^{1-\alpha(1+\gamma)}$, and this goes to 0 as $N \to \infty$, since $\gamma > \frac{1-\alpha}{\alpha}$. If $\alpha \in (\frac{1}{2}, 1)$, this is bounded above by a constant times $N^{\zeta(2\alpha-1)+1-\alpha-\alpha\gamma}$, which goes to 0 as $N \to \infty$ since $\gamma > \frac{(2\alpha-1)\zeta+1-\alpha}{\alpha}$ for $\zeta \in (0, \frac{1}{2})$. This concludes the proof of the lemma. □

End-to-end fluctuations. Lemma 4.8 already shows that end-to-end fluctuations are at most of order N^ξ. On the other hand, Lemma 4.7 shows that for any $\eta > 0$ we have that $N^{1-2\xi}\log Z_N^\omega(M_N^* \leqslant \eta N^\xi)$ converges to $\sup_{-\eta\leqslant u\leqslant 0\leqslant v\leqslant\eta}\{-\hat h(v-u) - \bar I(u,v)\}$ in probability. Since the supremum is strictly smaller than $(\hat h\pi)^{\frac{2}{3}} = \lim_{N\to\infty} N^{2\xi-1}\log Z_N^\omega$ if $\eta < \frac{1}{2}\pi^{\frac{2}{3}}\hat h^{-\frac{1}{3}}$,

we get that $\mathbf{P}_N^\omega(M_N^* \leqslant \eta N^\xi) \to 0$ for such η, which shows that end-to-end fluctuations are at least of order N^ξ.

Convergence of $M_N^+ - M_N^-$. Let us define $c_{\hat{h}} := \pi^{\frac{2}{3}} \hat{h}^{-\frac{1}{3}}$, set $\varepsilon > 0$, and define the event

$$\mathcal{A}_{N,R_5}^\varepsilon = \Big\{ \Big| \frac{1}{N^\xi}(M_N^+ - M_N^-) - c_{\hat{h}} \Big| > \varepsilon \Big\}.$$

As in the previous sections, since $\log \mathbf{P}_N^\omega(\mathcal{A}_{N,R_5}^\varepsilon) = \log Z_N^\omega(\mathcal{A}_{N,R_5}^\varepsilon) - \log Z_N^\omega$, using the convergence (3.5) we simply need to show that there is some $\delta_\varepsilon > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}\Big(\frac{1}{N^{1-2\xi}} \log Z_N^\omega(\mathcal{A}_{N,R_5}^\varepsilon) < -\frac{3}{2}(\hat{h}\pi)^{2/3} - \delta_\varepsilon \Big) = 1 \tag{4.59}$$

But this is simply due to the fact that analogously to Lemma 4.7, we have that

$$\frac{1}{N^{1-2\xi}} \log Z_N^\omega(\mathcal{A}_{R_5}^\varepsilon) \xrightarrow[N\to\infty]{\mathbb{P}} \sup_{u\leqslant 0\leqslant v, |v-u-c_{\hat{h}}|>\varepsilon} \Big\{ -\hat{h}(v-u) - \bar{I}(u,v) \Big\} < -\frac{3}{2}(\hat{h}\pi)^{2/3},$$

where the inequality is strict since the supremum in Lemma 4.7 is attained for $v - u = c_{\hat{h}}$.

4.6. **Region R6: proof of Theorem 3.6.** Note that in Region R_6, we have $\zeta < (-1) \wedge \gamma$ if $\alpha \in (1, 2]$ and $\zeta < (-1) \wedge (\gamma + \frac{\alpha-1}{\alpha})$ if $\alpha \in (0, 1)$. Let us note that in all cases, $\gamma > \zeta$. We split Z_N^ω in two parts

$$Z_N^\omega = Z_N^\omega\left(|\mathcal{R}_N| = 2\right) + Z_N^\omega\left(|\mathcal{R}_N| \geqslant 3\right). \tag{4.60}$$

It is clear that

$$Z_N^\omega\left(|\mathcal{R}_N| = 2\right) = e^{-2\hat{h}N^{-\zeta}} \Big(e^{\hat{\beta}N^{-\gamma}(\omega_0+\omega_1)} 2^{-N} + e^{\hat{\beta}N^{-\gamma}(\omega_0+\omega_{-1})} 2^{-N} \Big),$$

so that $N^\zeta \log Z_N^\omega\left(|\mathcal{R}_N| = 2\right)$ converges in probability to $-2\hat{h}$ (we use here that $\zeta < \gamma$ and $\zeta < -1$).

We now prove that $N^\zeta \log Z_N^\omega\left(|\mathcal{R}_N| \geqslant 3\right)$ is stricly smaller than $-2\hat{h}$ with $\mathbb{P}$-probability going to 1: this will imply that the second term in (4.60) is negligible compared to the first one, and as a by-product prove that $\mathbf{P}_N^\omega(|\mathcal{R}_N| = 2)$ converges to 1 in probability.

We fix A large, and split

$$Z_N^\omega\left(|\mathcal{R}_N| \geqslant 3\right) = Z_N^\omega\left(|\mathcal{R}_N| \geqslant 3, M_N^* \leqslant A\right) + Z_N^\omega\left(M_N^* > A\right). \tag{4.61}$$

For the first term, we simply use the upper bound

$$Z_N^\omega\left(|\mathcal{R}_N| \geqslant 3, M_N^* \leqslant A\right) \leqslant e^{-3\hat{h}N^{-\zeta}} \exp\Big(\hat{\beta} N^{-\gamma} \Omega_A^* \Big).$$

Using the fact that $\gamma > \zeta$, we get that $N^{\zeta-\gamma}\Omega_A^*$ goes to 0 in probability, and thus $N^\zeta \log Z_N^\omega(|\mathcal{R}_N| \geqslant 3, M_N^* \leqslant A) \leqslant -\frac{5}{2}\hat{h}$ with probability going to 1.

For the second term in (4.61), we have

$$Z_N^\omega\left(M_N^* > A\right) = \sum_{k=1}^{\log_2 N} Z_N^\omega\Big(M_N^* \in (2^{k-1}A, 2^k A] \Big) \leqslant \sum_{k=1}^{\log_2 N} e^{-2^{k-1}A\hat{h}N^{-\zeta}} \exp\Big(\hat{\beta} N^{-\gamma} \Omega_{2^k A}^* \Big).$$

Then, as in the proof of Lemma 4.8, (see the proof of Case (i)), a union bound gives that for any $T > 0$ (we will pick $T = \frac{5}{2}\hat{h}$) we can choose A_T so that for any $A > A_T$,

$$\begin{aligned} \mathbb{P}\Big(Z_N\big(M_N^* > A\big) \geqslant e^{-TN^{-\zeta}}\Big) &\leqslant \sum_{k=1}^{\log_2 N} \mathbb{P}\Big(\hat{\beta}N^{-\gamma}\Omega^*_{2^k A} \geqslant 2^{k-2}\hat{h}AN^{-\zeta}\Big) \\ &\leqslant c\sum_{k=1}^{\log_2 N} \hat{\beta}^{\alpha}\hat{h}^{-\alpha}A^{1-\alpha}2^{k(1-\alpha)}N^{\alpha(\zeta-\gamma)}\,, \end{aligned} \tag{4.62}$$

where we used Lemma 3.10 for the last inequality. If $\alpha \in (1,2]$ this is bounded above by a constant times $N^{\alpha(\zeta-\gamma)}$: this goes to 0 as $N \to \infty$, since $\gamma > \zeta$. When $\alpha \in (0,1)$, this is bounded above by a constant times $N^{1-\alpha+\alpha(\zeta-\gamma)}$: this goes to 0 as $N \to \infty$, since $\zeta < \gamma + \frac{\alpha-1}{\alpha}$. Therefore, the right-hand-side of (4.62) converges to zero as $N \to \infty$, which shows that $N^{\zeta}\log Z_N^{\omega}(M_N^* > A) \leqslant -T$ with probability going to 1, and concludes the proof. □

4.7. **Region $\widetilde{R}_4$: Theorem 3.7.** In this region, we prove that the end-to-end fluctuations of order N^{ξ} with $\xi = 1-\zeta \in (1/2, 1)$. Note that in this region we have

$$2\xi - 1 = \xi - \zeta > \frac{\xi}{\alpha} - \gamma \quad \text{and} \quad 0 < \zeta < \frac{1}{2}\,.$$

The proofs are identical to what is done in regions R_5-R_6, so we give much less detail. We fix some constant $A > 0$ (large), and we split the partition function as

$$Z_N^{\omega} = Z_N^{\omega}\left(M_N^* \leqslant AN^{\xi}\right) + Z_N^{\omega}\left(M_N^* > AN^{\xi}\right). \tag{4.63}$$

Step 1. We have the following lemma, analogous to Lemma 4.7.

Lemma 4.9. *In Region $\widetilde{R}_4$, we have that for any $A > 0$ the following convergence in probability*

$$\frac{1}{N^{2\xi-1}}\log Z_N^{\omega}\left(M_N^* \leqslant AN^{\xi}\right) \xrightarrow[N\to\infty]{\mathbb{P}} \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A}\Big\{|\hat{h}|(v-u) - I(u,v)\Big\}\,,$$

By a simple calculation, the supremum is $\frac{1}{2}\hat{h}^2$ for any $A \geqslant |\hat{h}|$, and it is attained at $(u,v) = (0,|\hat{h}|)$ or $(u,v) = (-|\hat{h}|, 0)$.

Proof. Since in Region $\widetilde{R}4$ we have $2\xi - 1 > \frac{\xi}{\alpha} - \gamma$, for any fixed A we get that $N^{-(2\xi-1)} \times \hat{\beta}N^{-\gamma}\Omega^*_{AN^{\xi}}$ goes to 0 in probability. Therefore we only need to prove that

$$\lim_{N\to\infty}\frac{1}{N^{2\xi-1}}\log Z_N^{\omega}\left(M_N^* \leqslant AN^{\xi}\right) = \sup_{-A\leqslant u\leqslant 0\leqslant v\leqslant A}\Big\{|\hat{h}|(v-u) - I(u,v)\Big\}$$

This follows by Varadhan's lemma, since we have by Lemma A.1 that $I(u,v)$ is the rate function for the LDP for $(N^{-\xi}M_N^-,\, N^{-\xi}M_N^+)$. □

Step 2. To conclude the proof of the convergence (3.8), it remains to show the following.

Lemma 4.10. *In region $\widetilde{R}_4$, for any constant $T > 0$, there is some $A_T > 0$ such that for $A \geqslant A_T$*

$$\lim_{N\to\infty}\mathbb{P}\Big(\frac{1}{N^{2\xi-1}}\log Z_N^{\omega}\left(M_N^* > AN^{\xi}\right) > -T\Big) = 0\,.$$

The steps are identical as in the proof of Lemma 4.8 and the calculations are very similar: the main difference is that in (4.54) we have to take into account the entropy contribution of the random walk $\mathbf{P}(M_N^* \geqslant 2^{k-1}AN^\xi)$. We omit the details.
Convergence of the trajectories. For $\varepsilon > 0$, denote the events

$$\mathcal{A}^{\varepsilon,+}_{N,\widetilde{R}_4} = \Big\{ \frac{1}{N^\xi}(M_N^-, M_N^+) \in [-\varepsilon,\varepsilon] \times [|\hat{h}| - \varepsilon, |\hat{h}| + \varepsilon] \Big\},$$
$$\mathcal{A}^{\varepsilon,-}_{N,\widetilde{R}_4} = \Big\{ \frac{1}{N^\xi}(M_N^-, M_N^+) \in [-|\hat{h}| - \varepsilon, -|\hat{h}| + \varepsilon] \times [-\varepsilon,\varepsilon] \Big\}.$$

As in the previous sections, since the supremum in in the variational problem of (3.8) is achieved at $(u,v) = (0,|\hat{h}|)$ or $(u,v) = (-|\hat{h}|, 0)$, one can easily prove that for any $\varepsilon > 0$, we have $\lim_{N\to]infty} \mathbf{P}_N^\omega(\mathcal{A}^{\varepsilon,+}_{N,\widetilde{R}_4} \cup \mathcal{A}^{\varepsilon,-}_{N,\widetilde{R}_4}) = 1$ in $\mathbb{P}$-probability.

Then, since $\mathcal{A}^{\varepsilon,+}_{N,\widetilde{R}_4}$ and $\mathcal{A}^{\varepsilon,-}_{N,\widetilde{R}_4}$ are disjoint for sufficiently small $\varepsilon > 0$, one can also prove that $\mathbf{P}_N^\omega(\mathcal{A}^{\varepsilon,+}_{N,\widetilde{R}_4} \cap \{\frac{1}{N^\xi} S_N > |\hat{h}| - 2\varepsilon\}) \to \frac{1}{2}$ in $\mathbb{P}$-probability by symmetry (it has an extra entropic cost for the random walk to backtrack), and analogously for $\mathcal{A}^{\varepsilon,-}_{N,\widetilde{R}_4}$. All together, one concludes that $\mathbf{P}_N^\omega(\frac{1}{N^\xi}|S_N| \in [|\hat{h}| - 2\varepsilon, |\hat{h}| + 2\varepsilon]) \to 1$ in $\mathbb{P}$-probability.

4.8. **Boundary region $\widetilde{R}_4$—$\widetilde{R}_5$: Theorem 3.8.** The proof is similar to that for region $\widetilde{R}_4$: one even only needs the analogous to Lemma 4.9. Replacing the rate function $I(u,v)$ for $(N^{-\xi}M_N^-, N^{-\xi}M_N^+)$ by the rate function $\kappa(|u| \wedge |v| + v - u)$ of $(N^{-1}M_N^-, N^{-1}M_N^+)$, see Lemma A.2, we get that

$$\frac{1}{N}\log Z_N^\omega \xrightarrow[N\to\infty]{\mathbb{P}} \sup_{-1\leqslant u\leqslant 0\leqslant v\leqslant 1} \big\{ |\hat{h}|(v-u) - \kappa(|u| \wedge |v| + v - u) \big\}.$$

Then, using that $\kappa(t) = \frac{1}{2}(1+t)\log(1+t) + \frac{1}{2}(1-t)\log(1-t)$ if $0 \leqslant t \leqslant 1$ and $\kappa(t) = +\infty$ if $t > 1$, a straightforward calculation finds that the supremum is attained at $(u,v) = (0, \frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1})$ and $(u,v) = (-\frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1}, 0)$, and equals $\log(\frac{e^{2|\hat{h}|-1}}{2}) - |\hat{h}|$.

Then, as above, one can easily deduce that $N^{-1}(M_N^-, M_N^+)$ is with high $\mathbf{P}_N^\omega$-probability (and high $\mathbb{P}$-probability) close to one of these maximizers, *i.e.* $(0, \frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1})$ or $(-\frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1}, 0)$ One can then deduce from it that $\mathbf{P}(|\frac{1}{N}|S_N| - \frac{e^{2|\hat{h}|}-1}{e^{2|\hat{h}|}+1}| > \varepsilon) \to 0$ in $\mathbb{P}$-probability. Details are left to the reader.

4.9. **Region $\widetilde{R}_5$: Theorem 3.9.** We observe that $\{|S_N| = N\} = \{|\mathcal{R}_N| = N\}$, and thus

$$e^{-\hat{\beta}N^{-\gamma}\Omega_N^* - \hat{h}N^{1-\zeta}} 2^{-N} \leqslant Z_N^\omega(|S_N| = N) \leqslant e^{\hat{\beta}N^{-\gamma}\Omega_N^* - \hat{h}N^{1-\zeta}}. \tag{4.64}$$

Now, we have $\beta N^{\zeta-1-\gamma}\Omega_N^* \xrightarrow{\mathbb{P}} 0$ since $\gamma > \zeta - \frac{\alpha-1}{\alpha}$: using also that $\zeta < 0$ we get that $N^{\zeta-1}\log Z_N^\omega(|S_N| = N)$ converges to $-\hat{h}$.

It therefore remains to prove that $\mathbf{P}_N^\omega(|S_N| = N) \to 1$ in $\mathbb{P}$-probability. Analogously to (4.64) above, we clearly have that $\log Z_N^\omega(|\mathcal{R}_N| \leqslant \frac{3}{4}N) \leqslant -(\frac{3}{4} + \varepsilon)\hat{h}N^{1-\zeta}$ with $\mathbb{P}$-probability going to 1 as $N \to \infty$. Hence, we get that

$$\mathbf{P}_N^\omega\Big(|\mathcal{R}_N| \leqslant \frac{3}{4}N\Big) \leqslant \frac{Z_N^\omega(|\mathcal{R}_N| \leqslant \frac{3}{4}N)}{Z_N^\omega(|\mathcal{R}_N| = N)} \to 0$$

in probability. Now, we can split the event $|\mathcal{R}_N| > \frac{3}{4}N$ according to whether $M_N^+ > \frac{1}{2}N$ or $M_N^- < -\frac{1}{2}N$. Hence, we only have to prove that $\mathbf{P}_N^\omega(\frac{1}{2}N < M_N^+ \leqslant N-1) \to 0$ in probability, and similarly for M_N^-. The proof will therefore be complete if we show that

$$\frac{Z_N^\omega(\frac{1}{2}N < M_N^+ \leqslant N-1)}{Z_N^\omega(|S_N| = N)} \leqslant C \exp\Big(\frac{1}{4}\hat{h}N^{-\zeta}\Big) \tag{4.65}$$

with $\mathbb{P}$-probability going to 1. Notice that $\frac{1}{4}\hat{h}N^{-\zeta} \to -\infty$ since $\zeta < 0$ and $\hat{h} < 0$, and that the order $N^{-\zeta}$ comes after many cancellations between the numerator and denominator. Notice that $Z_N^\omega(S_N = N) = 2^{-N}e^{\beta_N\Omega_N^+ - h_N N}$. Hence, using that if $M_N^+ = k$ we have $M_N^- \geqslant -\frac{1}{2}(N-k)$ so $|\mathcal{R}_N| \leqslant \frac{1}{2}(N+k)$, we get that for $\frac{1}{2}N < k \leqslant N-1$, after simplifications of the numerator and denominator,

$$\frac{Z_N^\omega(M_N^+ = k)}{Z_N^\omega(S_N = N)} \leqslant \exp\Big(\beta_N \sum_{i=k+1}^{N} \omega_i + \beta_N \Omega^*_{\frac{1}{2}(N-k)} + h_N \frac{1}{2}(N-k)\Big) 2^N \mathbf{P}(M_N^+ = k)\,.$$

Denoting $\widetilde{\Omega}_k^* = \Omega_k^* + \sup_{j=0}^k |\sum_{i=j+1}^N \omega_i|$, we therefore get that for $j \in \{1, \dots, \log_2 N - 1\}$

$$\frac{Z_N^\omega(N - M_N^+ \in [2^{j-1}, 2^j))}{Z_N^\omega(S_N = N)} \leqslant e^{\hat{\beta}N^{-\gamma}\widetilde{\Omega}^*_{2^j} + \hat{h}N^{-\zeta}2^{j-1}} 2^{N+1}\mathbf{P}(S_N \geqslant N - 2^j)$$

where we used that $\mathbf{P}(M_N^+ \geqslant N - 2^j) = 2\mathbf{P}(S_N \geqslant N - 2^j) - 1$ by the reflection principle. Then $\mathbf{P}(S_N \geqslant N - 2^j) \leqslant c2^j(\mathbf{P}(S_N = N - 2^j) + \mathbf{P}(S_N = N - 1 - 2^j)) \leqslant c2^{j-N}\binom{N}{2^j}\binom{2^j}{2^{j-1}}$, by a simple counting argument. Note that $2^j\binom{N}{2^j}\binom{2^j}{2^{j-1}} \leqslant 2^j(CN)^{2^j} \leqslant \exp(2^{j-2}|\hat{h}|N^{-\zeta})$ for N large enough (uniformly for $j \geqslant 1$), so we end up with

$$\frac{Z_N^\omega(N - M_N^+ \in [2^{j-1}, 2^j))}{Z_N^\omega(S_N = N)} \leqslant \exp\Big(\hat{\beta}N^{-\gamma}\widetilde{\Omega}^*_{2^j} + \hat{h}N^{-\zeta}2^{j-2}\Big)\,. \tag{4.66}$$

Summing this over j on the event that $\hat{\beta}N^{-\gamma}\widetilde{\Omega}^*_{2^j} < 2^{j-3}|\hat{h}|N^{-\zeta}$ for all $1 \leqslant j \leqslant \log_2 N$, we therefore get that (3.9) holds. It remains to show that this event has a probability going to 1: by a union bound, its complement has a probability smaller than

$$\sum_{j=1}^{\log_2 N} \mathbb{P}\Big(\hat{\beta}N^{-\gamma}\widetilde{\Omega}^*_{2^j} \geqslant 2^{j-3}|\hat{h}|N^{-\zeta}\Big) \leqslant \sum_{j=1}^{\log_2 N} c_{\hat{\beta},\hat{h}} 2^{j(1-\alpha)} N^{-(\gamma-\zeta)\alpha}\,,$$

where we used Lemma 3.10. This is bounded by: a constant times $N^{-(\gamma-\zeta)\alpha}$ (which goes to 0 as $N \to \infty$) if $\alpha \in (1,2]$; a constant times $N^{(1-\alpha)-(\gamma-\zeta)\alpha}$ (which goes to 0 as $N \to \infty$, recall $\gamma > \zeta - \frac{\alpha-1}{\alpha}$) if $\alpha \in (0,1)$. Hence, we conclude that (4.65) holds and hence $\mathbf{P}_N^\omega(|\mathcal{R}_N| = N) \to 1$ with $\mathbb{P}$-probability going to 1.

Appendix A. Technical estimates

A.1. **Estimates on deviation probabilities.** Let us present here some results on large deviation probabilities for the simple random walk that are needed throughout the paper. Recall the notations $M_N^- := \min_{0\leqslant n\leqslant N} S_n$ and $M_N^+ := \max_{0\leqslant n\leqslant N} S_n$.

Stretching. Our first lemma deals with the super-diffusive case: we estimate the probability that $M_N^+ \geqslant vN^\xi$ and $M_N^- \leqslant uN^\xi$ when $\xi \in (\frac{1}{2}, 1)$, for $u \leqslant 0 \leqslant v$. The one-sided large deviation are classical, using e.g. explicit calculations for the simple random walk (see [17, Ch. III.7]): we get that if $\xi \in (\frac{1}{2}, 1)$

$$\lim_{N\to\infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P}\big(M_N^+ \geqslant vN^\xi\big) = \lim_{N\to\infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P}\big(S_N \geqslant vN^\xi\big) = \frac{1}{2}v^2 \,.$$

The case where both the minimum and maximum are required to have large deviations is an easy extension of the result, and we omit its proof. Let us simply mention that the best strategy for the random walk to have $M_N^- \leqslant uN^\xi$ ($u \leqslant 0$) and $M_N^+ \geqslant vN^\xi$ ($v \geqslant 0$) consists in traveling to either uN^ξ or vN^ξ (whichever is the closest) and then go in the other direction to reach vN^ξ or uN^ξ. In other words, the random walk must travel a distance at least $(|u| \wedge |v| + v - u)N^\xi$, and we thus have $\log \mathbf{P}(M_N^- \leqslant uN^\xi; M_N^+ \geqslant vN^\xi) \sim \log \mathbf{P}(M_N^+ \geqslant (|u| \wedge |v| + v - u)N^\xi)$.

Lemma A.1. *If $\frac{1}{2} < \xi < 1$, then for any $u \leqslant 0 \leqslant v$ we have that*

$$\lim_{N\to\infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P}\Big(M_N^- \leqslant uN^\xi; M_N^+ \geqslant vN^\xi\Big) = \frac{1}{2}(|u| \wedge |v| + v - u)^2 \,. \tag{A.1}$$

As an easy consequence of this lemma, we get for that for any $\delta > 0$, for any $u \leqslant 0 \leqslant v$,

$$\lim_{N\to\infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P}\Big(M_N^- \in [u-\delta, u]N^\xi; M_N^+ \in [v, v+\delta]N^\xi\Big) = \frac{1}{2}\big(|u| \wedge |v| + v - u\big)^2. \tag{A.2}$$

We also state the large deviation result in the case $\xi = 1$ (it is not needed in this paper). We do not give the complete proof of the following statement, but as above, it derives from the fact that $\log \mathbf{P}(M_N^- \leqslant uN; M_N^+ \geqslant vN) \sim \log \mathbf{P}(M_N^+ \geqslant (|u| \wedge |v| + v - u)N)$.

Lemma A.2. *For any $u \leqslant 0 \leqslant v$, we have that*

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathbf{P}\Big(M_N^- \leqslant uN; M_N^+ \geqslant vN\Big) = \kappa\big(|u| \wedge |v| + v - u\big) \,,$$

where $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is the LDP rate function for the simple random walk, that is $\kappa(t) = \frac{1}{2}(1+t)\log(1+t) + \frac{1}{2}(1-t)\log(1-t)$ if $0 \leqslant t \leqslant 1$ and $\kappa(t) = +\infty$ if $t > 1$.

Folding. Our second lemma deals with the sub-diffusive case: we estimate the probability that $M_N^+ \leqslant vN^\xi$ and $M_N^- \geqslant uN^\xi$ when $\xi \in (0, \frac{1}{2})$, for $u \leqslant 0 \leqslant v$. The result follows from classical random walk calculations, leading to explicit expressions of ruin probabilities (see Eq. (5.8) in [17, Ch. XIV]); one may refer to [11, Lem. 2.1] and its proof for the following statement.

Lemma A.3. *If $0 < \xi < \frac{1}{2}$, then for any $u \leqslant 0 \leqslant v$ we have that*

$$\lim_{N\to\infty} -\frac{1}{N^{1-2\xi}} \log \mathbf{P}\Big(M_N^- \geqslant uN^\xi; M_N^+ \leqslant vN^\xi\Big) = \frac{\pi^2}{2(v-u)^2} \,. \tag{A.3}$$

As an easy consequence of this lemma, we get that for any $\delta > 0$ and any $u \leqslant 0 \leqslant v$,

$$\lim_{N\to\infty} -\frac{1}{N^{1-2\xi}} \log \mathbf{P}\Big(M_N^- \in [u, u+\delta]N^\xi; M_N^+ \in [v-\delta, v]N^\xi\Big) = \frac{\pi^2}{2(v-u)^2} \,. \tag{A.4}$$

A.2. **Proof of Lemma 3.10.** First of all, notice that the bound is trivial if $\ell \mathsf{T}^{-\alpha} > 1$: we assume that $\ell \mathsf{T}^{-\alpha} \leqslant 1$. Using Etemadi's inequality (see [9, Thm. 2.2.5]) we get that

$$\mathbb{P}\big(\Omega_\ell^* > \mathsf{T}\big) \leqslant 3 \max_{k\in\{1,\dots,\ell\}} \mathbb{P}\big(|\Omega_k^+| > \tfrac{1}{6}\mathsf{T}\big) + 3 \max_{k\in\{1,\dots,\ell\}} \mathbb{P}\big(|\Omega_k^-| > \tfrac{1}{6}\mathsf{T}\big)\,.$$

Let us detail the bound for $\mathbb{P}\big(|\Omega_k^+| > \frac{1}{6}\mathsf{T}\big)$, the same bound holds for $\mathbb{P}\big(|\Omega_k^-| > \frac{1}{6}\mathsf{T}\big)$. The case $\alpha = 2$ is a consequence of Kolmogorov's maximal inequality, and the case $\alpha \in (0,2)$ ($\alpha \neq 1$) follows from the so-called *big-jump* (or one-jump) behavior. Let us give an easy proof: define $\bar\omega_x := \omega_x \mathbb{1}_{\{|\omega_x|\leqslant \mathsf{T}\}}$, so that

$$\begin{aligned}\mathbb{P}\big(|\Omega_k^+| > \tfrac{1}{6}\mathsf{T}\big) &\leqslant \mathbb{P}\big(\exists\, 0 \leqslant x \leqslant k\,, |\omega_x| > \mathsf{T}\big) + \mathbb{P}\Big(\Big|\sum_{x=0}^k \bar\omega_x\Big| > \tfrac{1}{6}\mathsf{T}\Big)\\ &\leqslant (k+1)\mathbb{P}\big(|\omega_0| > \mathsf{T}\big) + \frac{36}{\mathsf{T}^2}\Big((k+1)\mathbb{E}\big[(\bar\omega_0)^2\big] + k(k+1)\mathbb{E}[\bar\omega_0]^2\Big)\,,\end{aligned}$$

where we used a union bound for the first term and Markov's inequality (applied to $(\sum_{x=0}^k \bar\omega_x)^2$) for the second. Now, the first term is clearly bounded by a constant times $k\,\mathsf{T}^{-\alpha}$ thanks to Assumption 1. For the second term, we use again Assumption 1, to get that if $\alpha \in (0,1)\cup(1,2)$, $\mathbb{E}[(\bar\omega_0)^2] \leqslant c\mathsf{T}^{2-\alpha}$ and $\mathbb{E}[\bar\omega_0] \leqslant c\mathsf{T}^{1-\alpha}$ (when $\alpha \in (1,2)$ we use for this last inequality that $\mathbb{E}[\omega_0] = 0$). Therefore, we end up with the bound

$$\mathbb{P}\big(|\Omega_\ell^+| > \tfrac{1}{6}\mathsf{T}\big) \leqslant c\ell\mathsf{T}^{-\alpha} + c\ell^2\mathsf{T}^{-2\alpha} \leqslant 2c\ell\mathsf{T}^{-\alpha}\,.$$

where we used that $\ell\mathsf{T}^{-\alpha} \leqslant 1$. □

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