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# ONE-DIMENSIONAL POLYMERS IN RANDOM ENVIRONMENTS: STRETCHING VS. FOLDING

QUENTIN BERGER, CHIEN-HAO HUANG, NICCOLÒ TORRI, AND RAN WEI

ABSTRACT. In this article we study a *non-directed polymer model* on  $\mathbb{Z}$ , that is a one-dimensional simple random walk placed in a random environment. More precisely, the law of the random walk is modified by the exponential of the sum of “rewards” (or penalties)  $\beta\omega_x - h$  sitting on the range of the random walk, where  $(\omega_x)_{x \in \mathbb{Z}}$  are i.i.d. random variables (the disorder), and where  $\beta \geq 0$  (disorder strength) and  $h \in \mathbb{R}$  (external field) are two parameters. When  $\beta = 0, h > 0$ , this corresponds to a random walk penalized by its range; when  $\beta > 0, h = 0$ , this corresponds to the “standard” polymer model in random environment, except that it is non-directed. In this work, we allow the parameters  $\beta, h$  to vary according to the length of the random walk, and we study in detail the competition between the *stretching effect* of the disorder, the *folding effect* of the external field (if  $h \geq 0$ ), and the *entropy cost* of atypical trajectories. We prove a complete description of the (rich) phase diagram. For instance, in the case  $\beta > 0, h = 0$  of the non-directed polymer, if  $\omega_x$  has a finite second moment, we find a transversal fluctuation exponent  $\xi = 2/3$ , and we identify the limiting distribution of the rescaled log-partition function.

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*Keywords:* Random Polymer, Random walk, Range, Heavy-tail distributions, Weak-coupling limit, Super-diffusivity, Sub-diffusivity

## 1. INTRODUCTION

We study here a simple symmetric random walk on  $\mathbb{Z}^d$  placed in a time-independent random environment [19]. The interaction with the environment occurs on the range of the random walk, *i.e.* on the sites visited by the walk. This model can therefore also be seen as a random version of random walks penalized by their range (in the spirit of [15, 9]). One closely related model is the celebrated directed polymer in random environment model (see [12] for a review), which has attracted interests from both mathematical and physical communities over the last forty years, and can be used to describe a polymer chain placed in a solvent with impurities.

**1.1. The model.** Let  $S := (S_n)_{n \geq 0}$  be a simple symmetric random walk on  $\mathbb{Z}^d$ ,  $d \geq 1$ , starting from 0, whose trajectory represents a (non-directed) polymer. Let  $\mathbf{P}$  denotes its law. The *random environment* is modeled by a field  $\omega := (\omega_x)_{x \in \mathbb{Z}^d}$  of i.i.d. random variables. We let  $\mathbb{P}$  denote the law of  $\omega$ , and  $\mathbb{E}$  the expectation with respect to  $\mathbb{P}$  (assumptions on the law of  $\omega$  are detailed in Section 1.2 below).

For  $\beta \geq 0$  (the disorder strength, or the inverse temperature) and  $h \in \mathbb{R}$  (an external field), we define for all  $N \in \mathbb{N}$  the following Gibbs transformation of the law  $\mathbf{P}$ , called the

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*polymer measure:*

$$(1.1) \quad \frac{d\mathbf{P}_{N,\beta,h}^\omega}{d\mathbf{P}}(S) := \frac{1}{Z_{N,\beta,h}^\omega} \exp\left(\sum_{x \in \mathbb{Z}^d} (\beta\omega_x - h)\mathbb{1}_{x \in \mathcal{R}_N}\right),$$

where  $\mathcal{R}_N = \{S_0, S_1, \dots, S_N\}$  is the range of the random walk up to time  $N$ , and

$$(1.2) \quad Z_{N,\beta,h}^\omega := \mathbf{E}\left[\exp\left(\sum_{x \in \mathbb{Z}^d} (\beta\omega_x - h)\mathbb{1}_{x \in \mathcal{R}_N}\right)\right] = \mathbf{E}\left[\exp\left(\beta \sum_{x \in \mathcal{R}_N} \omega_x - h|\mathcal{R}_N|\right)\right]$$

is the partition function of the model, and is defined so that  $\mathbf{P}_{N,\beta,h}^\omega$  is a probability measure.

Let us stress the main differences with the standard directed polymer model: (i) here, the random walk does not have a preferred direction; (ii) there is an additional external field  $h \in \mathbb{R}$ ; (iii) the random walk can only pick up one weight  $\beta\omega_x - h$  at a site  $x \in \mathbb{Z}^d$ , and returning to an already visited site does not bring any reward or penalty (in the directed polymer model, the environment is renewed each time).

We now wish to understand the typical behavior of polymer trajectories  $(S_0, \dots, S_N)$  under the polymer measure  $\mathbf{P}_{N,\beta,h}^\omega$ . Two important quantities that we are interested in are

- the *transversal* exponent  $\xi$ , loosely defined as  $\mathbb{E}\mathbf{E}_{N,\beta,h}^\omega|S_N| \approx N^\xi$ ;
- the *fluctuation* exponent  $\chi$ , loosely defined as  $|\log Z_{N,\beta,h}^\omega - \mathbb{E}[\log Z_{N,\beta,h}^\omega]| \approx N^\chi$ .

In view of (1.1), there are several quantities that may influence the behavior of the polymer: the energy collected from the *random environment*  $\omega$ ; the penalty  $h$  (or reward depending on its sign) for having a large range; the entropy cost of the exploration of the random walk  $S$ . If  $\beta = 0$  and  $h > 0$ , then we recover a random walk penalized by its range. This model is by now quite well understood: the random walk folds itself in a ball of radius  $N^{1/(d+2)}$  ( $\xi = \frac{1}{d+2}$ ), see [15, 23, 9, 4, 14] (these works mostly focus on the case of dimension  $d \geq 2$ ). If  $\beta = 0$  and  $h < 0$ , then we get a random walk rewarded by its range: the random walk “stretches” to obtain a range of order  $N$ . If  $\beta > 0$  and  $h = 0$ , then we obtain a model for a non-directed polymer in the random environment, the environment being seen only once by the random walk (in the same spirit as the excited random walk [3], or more generally the cookie random walk [24]). In general, disorder should have a “stretching” effect, the random walk is trying to reach more favorable regions in the environment. We will see that it is indeed the case in dimension  $d = 1$ , where we find that the random walk stretches up to a distance  $N^{2/3}$  ( $\xi = \frac{2}{3}$ ).

**1.2. Setting of the paper.** In this article, we focus on the case of the dimension  $d = 1$ : the behavior of the model is already very rich, and we are able to obtain sharp results. Also, we focus on the case  $h \geq 0$ : the case  $h < 0$ , which has a less of a rich behavior, and is somehow simpler, see Remark 2.1 below.

In order to observe a transition between a folded phase ( $h > 0, \beta = 0$ ) and supposedly an unfolded phase ( $h = 0, \beta > 0$ ), a natural idea is to consider parameters  $\beta$  and  $h$  that depend on the size of the system, *i.e.*  $\beta := \beta_N$  and  $h := h_N$ . There are then some sophisticated balances between the energy gain, the range penalty and the entropy cost as we tune  $\beta_N$  and  $h_N$ . Our main results identify the different regimes for the behavior of the random walk: we provide a complete (and rich) phase diagram (see Figures 1-2-3 below), and describe each phase precisely (transversal and fluctuation exponents, limiting distribution of the log-partition function).

Our main assumption on the environment is that  $\omega_x$  is in the domain of attraction of some  $\alpha$ -stable law, with  $\alpha \in (0, 2]$ ,  $\alpha \neq 1$ . More precisely, we make the following assumption.

**Assumption 1.** *If  $\alpha = 2$  we assume that  $\mathbb{E}[\omega_0] = 0$  and  $\mathbb{E}[\omega_0^2] = 1$ . If  $\alpha \in (0, 1) \cup (1, 2)$  we assume that  $\mathbb{P}(\omega_0 > t) \sim pt^{-\alpha}$  and  $\mathbb{P}(\omega_0 < -t) \sim qt^{-\alpha}$  as  $t \rightarrow \infty$ , with  $p + q = 1$  (and  $p > 0$ ). Moreover, if  $\alpha \in (1, 2)$ , we also assume that  $\mathbb{E}[\omega_0] = 0$ .*

Let us stress that Assumption 1 ensures that:

- if  $\alpha = 2$ , then  $\omega_i$  is in the normal domain of attraction, so that  $(\frac{1}{\sqrt{n}} \sum_{i=un}^{vn} \omega_i)_{u \leq 0 \leq v}$  converges to a two-sided (standard) Brownian Motion.
- if  $\alpha \in (0, 1) \cup (1, 2)$ , then  $\omega_i$  is in the domain of attraction of some non-Gaussian stable law and  $(\frac{1}{n^{1/\alpha}} \sum_{i=un}^{vn} \omega_i)_{u \leq 0 \leq v}$  converges to a two-sided  $\alpha$ -stable Lévy process (given in Definition 3.2 below).

Henceforth we refer to  $(X_x)_{x \in \mathbb{R}}$  as the two-sided Brownian motion if  $\alpha = 2$  and as the two-sided Lévy process of Definition 3.2 if  $\alpha \in (0, 1) \cup (1, 2)$ . We leave the case  $\alpha = 1$  aside essentially for simplicity: indeed, to obtain a process convergence as above, a non-zero centering term is in general needed (even in the symmetric case  $p = q$ , see [17, IX.8], or [5]); however most of our analysis applies in that case.

We set

$$(1.3) \quad \beta_N := \hat{\beta} N^{-\gamma}, \quad \text{and} \quad h_N := \hat{h} N^{-\zeta},$$

where  $\gamma, \zeta \in \mathbb{R}$  describe the asymptotic behavior of  $\beta_N, h_N$ , and  $\hat{\beta}, \hat{h} > 0$  are two fixed parameters.

**Remark 1.1.** *We could consider a slightly more general setting, adding some slowly varying function in the asymptotic behavior of  $\beta_N$  or  $h_N$  (or  $\mathbb{P}(\omega_0 > t)$ , if  $\alpha < 2$ ). We chose to stick to the simpler strictly power-law case, to avoid lengthy notation and more technical calculations. It also makes the phase diagram clearer.*

## 2. SOME HEURISTICS: THE PRESENTATION OF THE PHASE DIAGRAMS

In analogy with the directed polymer model in a heavy-tailed random environment [6, 7], the presence of heavy-tail (Assumption 1) strongly impacts the behavior of the model: there will be different phase diagrams according to whether  $\alpha \in (1, 2]$ ,  $\alpha \in (\frac{1}{2}, 1)$  and  $\alpha \in (0, \frac{1}{2})$ .

Let us denote  $\xi$  the typical transversal fluctuations exponent of the random walk under the polymer measure  $\mathbf{P}_{N, \beta_N, h_N}^\omega$ , namely  $\mathbb{E} \mathbf{E}_{N, \beta_N, h_N}^\omega [\max_{1 \leq n \leq N} |S_n|] \approx N^\xi$ , and let us derive some heuristics to try to determine  $\xi \in [0, 1]$ .

First of all, thanks to Lemmas A.1-A.3 in Appendix, we have

$$(2.1) \quad \log \mathbf{P}(|\mathcal{R}_N| \approx N^\xi) \approx \log \mathbf{P} \left( \max_{1 \leq n \leq N} |S_n| \approx N^\xi \right) \approx \begin{cases} -N^{2\xi-1}, & \text{if } \xi \geq \frac{1}{2}, \\ -N^{1-2\xi}, & \text{if } \xi \leq \frac{1}{2}. \end{cases}$$

If  $\xi > 1/2$ , this corresponds to a “stretching” of the random walk, whereas when  $\xi < 1/2$ , this corresponds to a “folding” of the random walk: we will refer to (2.1) as the *entropic cost* of having transversal fluctuations  $N^\xi$ .

Then, if the transversal fluctuations are of order  $N^\xi$  ( $|\mathcal{R}_N| \approx N^\xi$ ), we get under Assumption 1, and in view of (1.3), that

$$(2.2) \quad \beta_N \sum_{x \in \mathcal{R}_N} \omega_x \approx N^{\frac{\xi}{\alpha} - \gamma}, \quad h_N |\mathcal{R}_N| \approx N^{\xi - \zeta}.$$

We refer to the first term as the “energy” term, and to the second one as the “range” term. All together, if transversal fluctuations are of order  $N^\xi$ , we have that

$$(2.3) \quad \log Z_{N,\beta_N,h_N}^\omega \approx N^{\frac{\xi}{\alpha}-\gamma} - N^{\xi-\zeta} - \begin{cases} N^{1-2\xi} & \text{if } \xi \leq 1/2, \\ N^{2\xi-1} & \text{if } \xi \geq 1/2. \end{cases}$$

In (2.3), there is therefore a competition between the “disorder” (first term), the “range” (second term), and the “entropy” (last term). We now discuss how a balance can be achieved between these terms depending on  $\gamma$  and  $\zeta$  (and how they determine  $\xi$ ). There are three main possibilities:

- (i) there is a “disorder”-“entropy” balance (and the “range” term is negligible);
- (ii) there is a “range”-“entropy” balance (and the “energy” term is negligible);
- (iii) there is a “range”-“disorder” balance (and the “entropy” term is negligible).

To summarize, all three regimes can occur (depending on  $\gamma, \zeta$ ) if  $\alpha \in (1, 2]$ ; on the other hand, regime (iii) disappears if  $\alpha \in (0, 1)$ , and regime (i) disappears if  $\alpha \in (0, \frac{1}{2})$ . We now determine for which values of  $\gamma, \zeta$  one can observe the different regimes above: we consider the three subcases  $\alpha \in (1, 2]$ ,  $\alpha \in (\frac{1}{2}, 1)$  and  $\alpha \in (0, \frac{1}{2})$  separately.

**2.1. Phase diagram for  $\alpha \in (1, 2]$ .** Instead of looking for “disorder”-“entropy”, “range”-“entropy” or “range”-“disorder” balance, we will find conditions to have the “disorder” term much larger, much smaller, or of the order of the “range” term.

*Case I (“disorder”  $\gg$  “range”).* This corresponds to having  $\xi/\alpha - \gamma > \xi - \zeta$ . In that case, the random walk should not feel the penalty for having a large range, so we should have  $\xi \geq 1/2$ . The competition occurs only between energy and entropy, one could achieve a balance if  $\xi/\alpha - \gamma = 2\xi - 1$ , that is if

$$(2.4) \quad \xi = \frac{\alpha}{2\alpha - 1}(1 - \gamma) \quad \text{when} \quad \gamma < \frac{(2\alpha - 1)\zeta - (\alpha - 1)}{\alpha},$$

where the condition on  $\gamma$  derives from the fact that  $\xi/\alpha - \gamma > \xi - \zeta$  in the regime considered here. However, since  $\xi \leq 1$ , we should have  $\xi = 1$  when  $\gamma \leq -\frac{\alpha-1}{\alpha}$ . Thus, we have

$$(2.5) \quad \xi = 1 \quad \text{when} \quad \gamma \leq -\frac{\alpha - 1}{\alpha} \quad \text{and} \quad \gamma < \zeta - \frac{\alpha - 1}{\alpha}.$$

Also, since  $\xi \geq 1/2$ , we should have  $\xi = 1/2$  when  $\gamma \geq \frac{1}{2\alpha}$ . Thus, we have

$$(2.6) \quad \xi = \frac{1}{2} \quad \text{when} \quad \gamma \geq \frac{1}{2\alpha} \quad \text{and} \quad \gamma < \zeta - \frac{\alpha - 1}{2\alpha}.$$

*Case II (“disorder”  $\ll$  “range”).* This corresponds to having  $\xi/\alpha - \gamma < \xi - \zeta$ . In that case, the random walk feels the penalty for having a large range, and we should have  $\xi \leq 1/2$ . The competition being only between range and entropy, one could achieve a balance if  $\xi - \zeta = 1 - 2\xi$ , that is if

$$(2.7) \quad \xi = \frac{1 + \zeta}{3} \quad \text{when} \quad \gamma > \frac{(2\alpha + 1)\zeta - (\alpha - 1)}{3\alpha},$$

where the condition on  $\gamma$  derives from the fact that  $\xi/\alpha - \gamma > \xi - \zeta$  in the regime considered here. Since  $\xi \in [0, 1/2]$ , it is similar to (2.5)-(2.6) that

$$(2.8) \quad \xi = 0 \quad \text{when} \quad \zeta \leq -1 \quad \text{and} \quad \gamma > \zeta,$$

and

$$(2.9) \quad \xi = \frac{1}{2} \quad \text{when} \quad \zeta \geq \frac{1}{2} \quad \text{and} \quad \gamma > \zeta - \frac{\alpha - 1}{2\alpha}.$$

*Case III* (“disorder”  $\approx$  “range”  $\gg$  “entropy”). This corresponds to having  $\xi/\alpha - \gamma = \xi - \zeta$ , that is

$$(2.10) \quad \xi = \frac{\alpha}{\alpha - 1}(\zeta - \gamma).$$

In this regime, the entropy cost should be negligible compared to the disorder gain, and we should therefore have that  $\xi/\alpha - \gamma > 1 - 2\xi$  if  $\xi \leq 1/2$  and  $\xi/\alpha - \gamma > 2\xi - 1$  for  $\xi \geq 1/2$ : after some calculation (and using (2.10)), we find the following condition on  $\gamma$

$$(2.11) \quad \frac{(2\alpha - 1)\zeta - (\alpha - 1)}{\alpha} < \gamma < \frac{(2\alpha + 1)\zeta - (\alpha - 1)}{3\alpha}.$$

Moreover, since  $\xi \in [0, 1]$ , we must have

$$(2.12) \quad \zeta - \frac{\alpha - 1}{\alpha} \leq \gamma \leq \zeta.$$

To summarize, for  $\alpha \in (1, 2]$ , we have identified six different regimes according to the value of  $\gamma, \zeta$ : they are represented in the  $(\zeta, \gamma)$ -diagram in Figure 1 below.

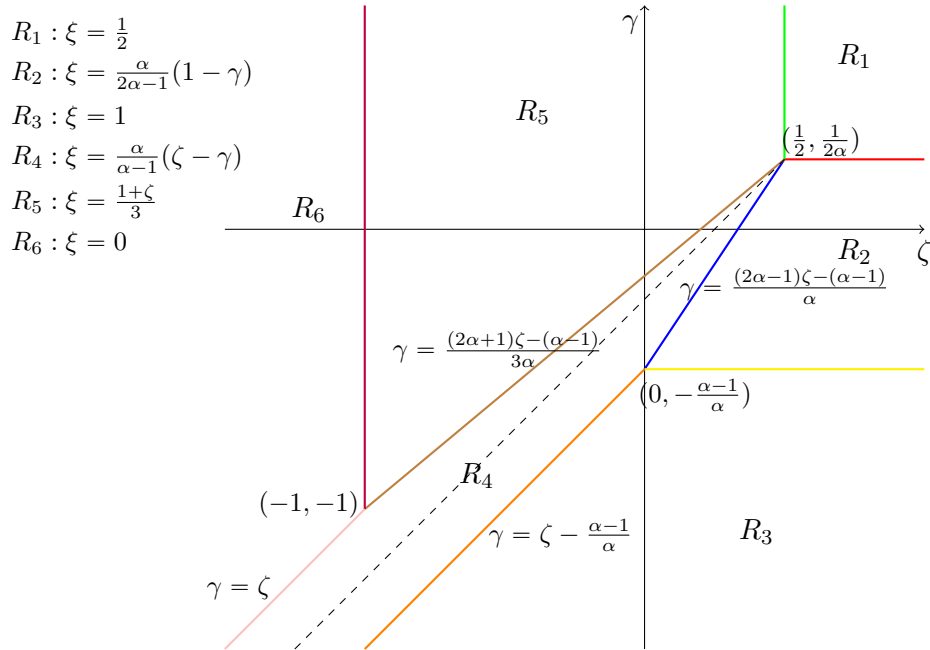


FIGURE 1. Phase diagram in the case  $\alpha \in (1, 2]$ . The region  $R_1$  and the dashed line  $\gamma = \zeta - \frac{\alpha-1}{2\alpha}$  are the thresholds that split the regions of super-diffusivity and sub-diffusivity.

To be precise, the different regions are described as follows:

$$\begin{aligned}
 R_1 &= \left\{ \xi = \frac{1}{2}, \gamma > \frac{1}{2\alpha}, \zeta > \frac{1}{2} \right\}, \\
 R_2 &= \left\{ \xi = \frac{\alpha}{2\alpha-1}(1-\gamma), \frac{1-\alpha}{\alpha} < \gamma < \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha} \wedge \frac{1}{2\alpha} \right\}, \\
 R_3 &= \left\{ \xi = 1, \gamma < -\frac{\alpha-1}{\alpha}, \gamma < \zeta - \frac{\alpha-1}{\alpha} \right\},
 \end{aligned}$$

$$\begin{aligned}
R_4 &= \left\{ \xi = \frac{\alpha}{\alpha-1}(\zeta - \gamma), \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha} \vee \left(\zeta - \frac{\alpha-1}{\alpha}\right) < \gamma < \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha} \wedge \zeta \right\}, \\
R_5 &= \left\{ \xi = \frac{1+\zeta}{3}, \gamma > \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha}, -1 < \zeta < \frac{1}{2} \right\}, \\
R_6 &= \left\{ \xi = 0, \gamma > \zeta, \zeta < -1 \right\}.
\end{aligned}$$

Note that when  $\alpha = 1$ , the four lines  $\gamma = \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha}$ ,  $\gamma = \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha}$ , and  $\gamma = \zeta$ ,  $\gamma = \zeta - \frac{\alpha-1}{\alpha}$  all merge to the line  $\gamma = \zeta$ .

**2.2. Phase diagram for  $\alpha \in (0, 1)$ .** Let us simply highlight the main difference with the case  $\alpha \in (1, 2]$ : the region  $R_4$  no longer exists when  $\alpha < 1$ , and the region  $R_2$  also disappears when  $\alpha < 1/2$ . Indeed, region  $R_4$  corresponds to the case “disorder”  $\approx$  “range”, in which we have  $\xi = \frac{\alpha}{1-\alpha}(\gamma - \zeta)$ : it is easy check that for  $\alpha \in (0, 1)$  there is no  $\gamma$  that can satisfy (2.12), which suggests that there is no “disorder”-“range” balance possible. For the same reason, when  $\alpha \in (0, \frac{1}{2})$ , there no  $\gamma$  that satisfy  $\frac{1-\alpha}{\alpha} < \gamma < \frac{1}{2\alpha}$  (see the definition of  $R_2$  above), which suggests that there is no “disorder”-“entropy” balance possible: region  $R_2$  no longer exists. We also refer to Section 3.2 (Comment 2) for further comments on the reasons why regions  $R_4$  and  $R_2$  disappear precisely for  $\alpha < 1$  and  $\alpha < 1/2$ .

All together, for  $\alpha \in (\frac{1}{2}, 1)$  we obtain the  $(\zeta, \gamma)$ -diagram presented in Figure 2 below. To be precise, the different regions are described as follows:

$$\begin{aligned}
R_1 &= \left\{ \xi = \frac{1}{2}, \gamma > \frac{1}{2\alpha}, \zeta > \frac{1}{2} \right\}, \\
R_2 &= \left\{ \xi = \frac{\alpha}{2\alpha-1}(1 - \gamma), \frac{1-\alpha}{\alpha} < \gamma < \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha} \wedge \frac{1}{2\alpha} \right\}, \\
R_3 &= \left\{ \xi = 1, \gamma < \frac{1-\alpha}{\alpha}, \gamma < \zeta - \frac{\alpha-1}{\alpha} \right\}, \\
R_5 &= \left\{ \xi = \frac{1+\zeta}{3}, \left(\frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha}\right) \wedge \left(\zeta - \frac{\alpha-1}{\alpha}\right) < \gamma, -1 < \zeta < \frac{1}{2} \right\}, \\
R_6 &= \left\{ \xi = 0, \gamma > \zeta - \frac{\alpha-1}{\alpha}, \zeta < \frac{1}{\alpha} - 2 \right\}.
\end{aligned}$$

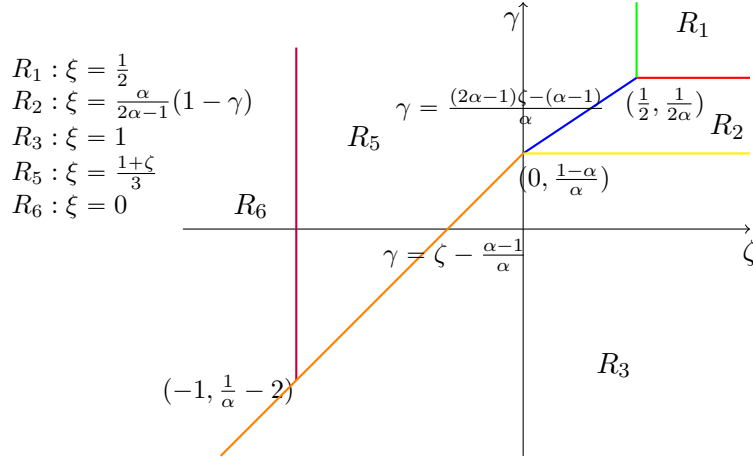


FIGURE 2. Phase diagram in the case  $\alpha \in (1/2, 1)$ . Compared to Figure 1, the region  $R_4$  no longer exists.

On the other hand, for  $\alpha \in (0, \frac{1}{2})$  we obtain the  $(\zeta, \gamma)$ -diagram presented in Figure 3 below. To be precise, the different regions are described as follows:

$$\begin{aligned}
R_1 &= \left\{ \xi = \frac{1}{2}, \gamma > \frac{1-\alpha}{\alpha}, \zeta > \frac{1}{2} \right\}, \\
R_3 &= \left\{ \xi = 1, \gamma < \frac{1-\alpha}{\alpha}, \gamma < \zeta - \frac{\alpha-1}{\alpha} \right\},
\end{aligned}$$

$$R_5 = \left\{ \xi = \frac{1+\zeta}{3}, \frac{1-\alpha}{\alpha} \wedge \left( \zeta - \frac{\alpha-1}{\alpha} \right) < \gamma, -1 < \zeta < \frac{1}{2} \right\},$$

$$R_6 = \left\{ \xi = 0, \gamma > \zeta - \frac{\alpha-1}{\alpha}, \zeta < \frac{1}{\alpha} - 2 \right\}.$$

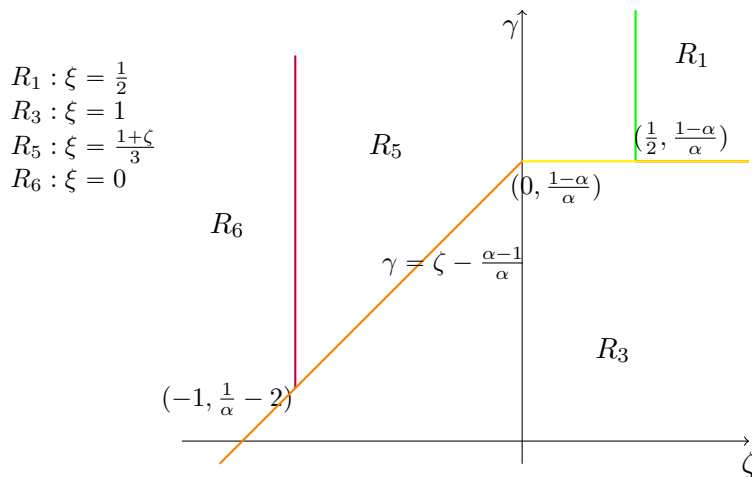


FIGURE 3. Phase diagram in the case  $\alpha \in (0, 1/2)$ . Compared to Figure 3, the region  $R_2$  no longer exists.

**Remark 2.1.** *Let us stress that in the case  $\hat{h} < 0$ , one can conduct similar computation as in (2.4)–(2.12) and obtain a different phase diagram than those of Figures 1-2-3. In that case, both “disorder” and “range” encourage exploration: only transversal fluctuations exponents  $\xi \geq 1/2$  are possible, resulting in a much simpler diagram.*

### 3. MAIN RESULTS

Our main results consist in proving the phase diagrams of Figures 1-2-3, with a precise description of the behavior of the polymer in each region. In order to state our results, let us introduce some definitions.

**Definition 3.1.** *If  $(t_N)_{n \geq 0}$  is a sequence of positive real numbers, we say that  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $t_N$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$  if for any  $\varepsilon > 0$  there is some  $\eta$  such that for  $N$  large enough*

$$\mathbf{P}_{N, \beta_N, h_N}^\omega \left( \max_{1 \leq n \leq N} |S_n| \in [\eta, \frac{1}{\eta}] t_N \right) \geq 1 - \varepsilon \quad \text{with } \mathbb{P}\text{-probability larger than } 1 - \varepsilon.$$

**Definition 3.2.** *We denote by  $(X_x)_{x \in \mathbb{R}}$  (with  $X_0 := 0$ ) either a two-sided (standard) Brownian motion if  $\alpha = 2$  or a two-sided (stable) Lévy process with no drift, no Brownian component and Lévy measure  $\nu(dx) = \alpha(p\mathbb{1}_{\{x>0\}} + q\mathbb{1}_{\{x<0\}})|x|^{-1-\alpha} dx$ , if  $\alpha \in (0, 2)$ . Since it will not draw any confusion we denote its law by  $\mathbb{P}$ .*

**3.1. Statement of the results.** We now prove six different theorems, corresponding to the six possible regions in the phase diagram.

**Theorem 3.1 (Region 1).** *Assume that (1.3) holds with*

$$\begin{cases} \gamma > \frac{1}{2\alpha} \text{ and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (\frac{1}{2}, 1) \cup (1, 2], \\ \gamma > \frac{1-\alpha}{\alpha} \text{ and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (0, \frac{1}{2}). \end{cases}$$



Then,  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $\sqrt{N}$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$  (i.e.  $\xi = \frac{1}{2}$ ), and we have the following convergence in probability

$$(3.1) \quad Z_{N, \beta_N, h_N}^\omega \xrightarrow{\mathbb{P}} 1.$$

**Theorem 3.2 (Region 2).** Assume that (1.3) holds with

$$\frac{1-\alpha}{\alpha} < \gamma < \frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha} \wedge \frac{1}{2\alpha} \quad \text{and} \quad \alpha \in \left(\frac{1}{2}, 1\right) \cup (1, 2].$$

Then  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N^\xi$  with  $\xi = \frac{\alpha}{2\alpha-1}(1-\gamma) \in (\frac{1}{2}, 1)$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$ , and we have the following convergence in distribution

$$(3.2) \quad \frac{1}{N^{\frac{\xi}{\alpha}-\gamma}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \mathcal{W}_{R_2} := \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - I(u, v) \right\},$$

where  $I(u, v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2$  for  $u \leq 0 \leq v$ . Moreover, we have that  $\mathcal{W}_{R_2} \in (0, +\infty)$ ,  $\mathbb{P}$ -almost surely.

Let us stress that the case  $\alpha = 2$ ,  $\beta = \beta_N \equiv \beta > 0$  and  $h \equiv 0$  corresponds to the case  $\gamma = 0$  and  $\zeta = +\infty$ : we find in that case that the transversal fluctuation exponent is  $\xi = \frac{2}{3}$ .

**Theorem 3.3 (Region 3).** Assume that (1.3) holds with

$$\gamma < \left(\zeta - \frac{\alpha-1}{\alpha}\right) \wedge \left(\frac{1-\alpha}{\alpha}\right) \quad \text{and} \quad \alpha \in (0, 1) \cup (1, 2].$$

Then  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$  (i.e.  $\xi = 1$ ), and we have the following convergence in distribution

$$(3.3) \quad \frac{1}{N^{\frac{1}{\alpha}-\gamma}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \mathcal{W}_{R_3} := \sup_{\substack{u \leq 0 \leq v \\ |u| \wedge |v| + v - u \leq 1}} \left\{ \hat{\beta}(X_v - X_u) \right\}.$$

Moreover, we have that  $\mathcal{W}_{R_3} \in (0, +\infty)$ ,  $\mathbb{P}$ -almost surely.

**Theorem 3.4 (Region 4).** Assume that (1.3) holds with

$$\left(\frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\right) \vee \left(\zeta - \frac{\alpha-1}{\alpha}\right) < \gamma < \left(\frac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}\right) \wedge \zeta \quad \text{and} \quad \alpha \in (1, 2].$$

Then  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N^\xi$  with  $\xi = \frac{\alpha}{\alpha-1}(\zeta - \gamma) \in (0, 1)$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$ , and we have the following convergence in distribution

$$(3.4) \quad \frac{1}{N^{\xi-\zeta}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \mathcal{W}_{R_4} := \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) \right\}.$$

Moreover, we have that  $\mathcal{W}_{R_4} \in (0, +\infty)$   $\mathbb{P}$ -almost surely.

**Theorem 3.5 (Region 5).** Assume that (1.3) holds with

$$\begin{cases} \gamma > \frac{(2\alpha+1)\zeta-(\alpha-1)}{3\alpha}, \quad \text{and} \quad -1 < \zeta < \frac{1}{2}, & \text{if } \alpha \in \left(\frac{1}{2}, 2\right], \\ \gamma > \left(\frac{(2\alpha-1)\zeta-(\alpha-1)}{\alpha}\right) \wedge \left(\zeta - \frac{\alpha-1}{\alpha}\right) \quad \text{and} \quad -1 < \zeta < \frac{1}{2}, & \text{if } \alpha \in \left(\frac{1}{2}, 1\right), \\ \gamma > \left(\frac{1-\alpha}{\alpha}\right) \wedge \left(\zeta - \frac{\alpha-1}{\alpha}\right) \quad \text{and} \quad -1 < \zeta < \frac{1}{2}, & \text{if } \alpha \in \left(0, \frac{1}{2}\right). \end{cases}$$

Then  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N^\xi$  with  $\xi = \frac{1+\zeta}{3} \in (0, \frac{1}{2})$  under  $\mathbf{P}_{N, \beta_N, h_N}^\omega$ , and we have the following convergence in probability

$$(3.5) \quad \frac{1}{N^{\xi-\zeta}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{\mathbb{P}} -\frac{3(4\hat{h}\pi)^{2/3}}{8} = \sup_{r \geq 0} \left\{ -\hat{h}r - \frac{\pi^2}{8r^2} \right\}.$$

**Theorem 3.6 (Region 6).** *Assume that (1.3) holds with*

$$\begin{cases} \gamma > \zeta, & \text{and } \zeta < -1, & \text{if } \alpha \in (1, 2], \\ \gamma > \zeta - \frac{\alpha-1}{\alpha}, & \text{and } \zeta < -1, & \text{if } \alpha \in (0, 1). \end{cases}$$

*Then we have the following convergences in probability (i.e.  $\xi = 0$ )*

$$(3.6) \quad \mathbf{P}_{N, \beta_N, h_N}^\omega(|\mathcal{R}_n| = 2) \xrightarrow{\mathbb{P}} 1, \quad \text{and} \quad N^\zeta \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{\mathbb{P}} -2\hat{h}.$$

**3.2. Some comments on the results.** Let us now make some observations on our results.

**Comment 1.** Our results describe the transition from *folded* trajectories ( $\xi < 1/2$ ) to *stretched* trajectories ( $\xi > 1/2$ ), this transition being induced by the presence of disorder. Let us illustrate this fact, in the case  $\alpha \in (1, 2]$  for simplicity: we refer to the phase diagram of Figure 1. If  $\beta_N = \hat{\beta} > 0$  and  $h_N = \hat{h} > 0$ , that is  $\gamma = \zeta = 0$ , we find that the trajectories are folded, with transversal exponent  $\xi = 1/3$ .

Now, if we keep  $h_N = \hat{h} > 0$  (i.e.  $\zeta = 0$ ) fixed, and increase the strength of disorder, that is decrease  $\gamma$  (take  $\gamma < 0$ ), we realize that we have transitions between the following regimes:

- (i) if  $\gamma > \frac{1-\alpha}{3\alpha}$ , the random walk is folded with transversal exponent  $\xi = 1/3$  (disorder is not strong enough);
- (ii) if  $\frac{1-\alpha}{3\alpha} > \gamma > \frac{1-\alpha}{2\alpha}$ , then the random walk is still folded, with transversal exponent  $1/3 < \xi = \gamma\alpha/(1-\alpha) < 1/2$  (disorder makes the random walk less folded);
- (iii) if  $\frac{1-\alpha}{2\alpha} > \gamma > \frac{1-\alpha}{\alpha}$ , then the random walk is stretched, with transversal exponent  $1/2 < \xi = \gamma\alpha/(1-\alpha) < 1$  (disorder is strong enough to stretch the random walk);
- (iv) if  $\gamma < \frac{1-\alpha}{\alpha}$ , then the random walk is completely unfolded, and has transversal exponent  $\xi = 1$ .

Analogously, if we keep  $\beta_N = \hat{\beta} > 0$  (i.e.  $\gamma = 0$ ) fixed, and decrease the penalty for the range, that is increase  $\zeta$  (take  $\zeta > 0$ ), we have transitions between the following regimes:

- (i) if  $0 < \zeta < \frac{\alpha-1}{2\alpha+1}$ , then the random walk is still folded with transversal exponent  $1/3 < \xi = (1+\zeta)/3 < \alpha/(2\alpha+1) (< 1/2)$  (and it does not feel the disorder);
- (ii) if  $\frac{\alpha-1}{2\alpha+1} < \zeta < \frac{\alpha-1}{2\alpha}$ , then the random walk is still folded with transversal exponent  $\alpha/(2\alpha+1) < \xi = \zeta\alpha/(\alpha-1) < 1/2$  (and disorder plays a role);
- (iii) if  $\frac{\alpha-1}{2\alpha} < \zeta < \frac{\alpha-1}{2\alpha-1}$ , then the random walk is stretched, with transversal exponent  $1/2 < \xi = \gamma\alpha/(1-\alpha) < 2/3$ ;
- (iv) if  $\zeta > \frac{\alpha-1}{2\alpha-1}$ , then the random walk is stretched and has transversal exponent  $\xi = 2/3$  (it does not feel the penalty for the range anymore).

**Comment 2.** Let us now comment on the limiting distributions for the log-partition function in regions  $R_2, R_3, R_4$ . For simplicity, we will restrict ourselves to the case where  $u = 0$  in the variational problems (3.2)-(3.3)-(3.4) (which corresponds to considering the case of a random walk constrained to stay non-negative): the variational problems become, respectively

$$(3.7) \quad \widetilde{\mathcal{W}}_{R_2} := \sup_{v \geq 0} \left\{ \hat{\beta} X_v - \frac{1}{2} v^2 \right\}, \quad \widetilde{\mathcal{W}}_{R_3} := \hat{\beta} \sup_{v \in [0, 1]} \{ X_v \}, \quad \widetilde{\mathcal{W}}_{R_4} := \sup_{v \geq 0} \left\{ \hat{\beta} X_v - \hat{h} v \right\}.$$

**a)** The variational problem  $\widetilde{\mathcal{W}}_{R_3}$  is clearly always finite. In the case  $\alpha = 2$ ,  $(X_t)_{t \geq 0}$  is a Brownian motion, and it is standard to get that  $\widetilde{\mathcal{W}}_{R_3}$  has the distribution of  $\hat{\beta}|Z|$ , with

$Z \sim \mathcal{N}(0, \hat{\beta}^2)$ . In the case  $\alpha \in (0, 2)$ ,  $(X_t)_{t \geq 0}$  is a stable Lévy process, and we get that  $\widetilde{\mathcal{W}}_{R_3}$  is a positive  $\alpha$ -stable random variable (see [8, Ch. VIII], and also [20]).

**b)** The variational problem  $\widetilde{\mathcal{W}}_{R_4}$  is finite only when  $\alpha > 1$ : when  $\alpha \in (0, 1)$ , then  $X_v$  grows typically as  $v^{1/\alpha} \gg v$  as  $v \rightarrow \infty$ , and we therefore have  $\mathcal{W}_{R_4} = +\infty$ . This explains in particular why there is no energy-range balance possible if  $\alpha \in (0, 1)$ , and why region  $R_4$  no longer exists in that case. If  $\alpha = 2$ ,  $(X_t)_{t \geq 0}$  is a Brownian motion, and it is standard to get that  $\widetilde{\mathcal{W}}_{R_4}$  is an exponential random variable (here with parameter  $2\hat{h}/\beta^2$ ). If  $\alpha \in (1, 2)$ ,  $(X_t)_{t \geq 0}$  is a stable Lévy process, and  $(\hat{\beta}X_t - \hat{h}t)_{t \geq 0}$  is also a Lévy process: the distribution of its supremum  $\widetilde{\mathcal{W}}_{R_4}$  has been studied intensively, going back to [2], but the exact distribution does not appear to be known (we refer to the recent papers [10, 21]).

**c)** The variational problem  $\widetilde{\mathcal{W}}_{R_2}$  is finite only when  $\alpha > 1/2$ : when  $\alpha \in (0, 1/2)$ , then  $X_v$  grows typically as  $v^{1/\alpha} \gg v^2$  as  $v \rightarrow \infty$ , and we therefore have  $\mathcal{W}_{R_2} = +\infty$ . This explains in particular why there is no “energy”-“entropy” balance possible if  $\alpha \in (0, \frac{1}{2})$ , and why region  $R_2$  no longer exists in that case. In the case  $\alpha = 2$ , that is when  $(X_t)_{t \geq 0}$  is a standard Brownian motion, then  $\widetilde{\mathcal{W}}_{R_4}$  has appeared in various contexts, and its density is known (its Fourier transform is expressed in terms of Airy function, see for instance [13, 18]). In the case  $\alpha \in (\frac{1}{2}, 2)$ , we are not aware whether the distribution of  $\widetilde{\mathcal{W}}_{R_4}$  has been studied.

**Comment 3.** We chose in this paper not to treat the cases of the boundaries between different regions of the phase diagrams, mostly to keep the paper lighter. These boundary regions do not really hide anything deep: features of both regions should appear in the limit, and “disorder”, “range” and “entropy” may all compete at the same (exponential) scale. Let us state for instance the limiting variational problems that one should find in some of the most interesting boundary cases, in the case  $\alpha \in (1, 2]$  (we refer to the phase diagram of Figure 1):

- Line between regions  $R_2$  and  $R_4$ :  $\gamma = \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha}$  and  $\zeta \in (0, \frac{1}{2})$ . Then one should have  $\xi = \frac{\alpha(1-\gamma)}{2\alpha-1}$  and

$$\frac{1}{N^{2\xi-1}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) - \frac{1}{2}(|u| \wedge |v| + v - u)^2 \right\}.$$

- Line between regions  $R_4$  and  $R_5$ :  $\gamma = \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha}$  and  $\zeta \in (-1, \frac{1}{2})$ . Then one should have  $\xi = \frac{1+\zeta}{3}$  and

$$\frac{1}{N^{1-2\xi}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) - \frac{\pi}{2(v - u)^2} \right\}.$$

where the last term inside the supremum comes from the entropic cost of “folding” the random walk in the interval  $[uN^\xi, vN^\xi]$  (see Lemma A.3).

- Line between regions  $R_2$  and  $R_3$ :  $\gamma = -(\alpha - 1)/\alpha$  and  $\zeta > 0$ . Then one should have  $\xi = 1$  and

$$\frac{1}{N^{2\xi-1}} \log Z_{N, \beta_N, h_N}^\omega \xrightarrow{d} \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - J(|u| \wedge |v| + v - u) \right\},$$

where  $J(t) := \frac{1}{2}(1+t) \log(1+t) + \frac{1}{2}(1-t) \log(1-t)$  for  $t \in [0, 1]$  ( $J(t) = +\infty$  for  $t > 1$ ) is the rate function for the large deviations of the simple random walk, see Lemma A.2

• Line between regions  $R_3$  and  $R_4$ :  $\gamma = \zeta - (\alpha - 1)/\alpha$  and  $\zeta < 0$ . Then one should have  $\xi = 1$  and

$$\frac{1}{N^{2\xi-1}} \log Z_{N,\beta_N,h_N}^\omega \xrightarrow{d} \sup_{u \leq 0 \leq v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) - \mathbf{J}(|u| \wedge |v| + v - u) \right\}.$$

**3.3. Organisation of the proof, and useful notation.** We prove our results for Regions  $R_1$  to  $R_6$  in the order listed above, by making our heuristic analysis for (2.3) rigorous. The results in Regions  $R_2$ ,  $R_4$  and  $R_5$  involve competitions between “energy”, “range” or “entropy” (but all have the same scheme of proof), while Region  $R_1$ ,  $R_3$  and  $R_6$  are extreme cases where only one factor is significant and hence is much simpler.

In the rest of the paper, to lighten the notations, we will drop the dependence on  $\beta_N$  and  $h_N$ : we write  $\mathbf{P}_N^\omega$  instead of  $\mathbf{P}_{N,\beta_N,h_N}^\omega$  and  $Z_N^\omega$  instead of  $Z_{N,\beta_N,h_N}^\omega$ . We also use the convenient notation  $Z_N^\omega(E)$  for the partition function restricted to trajectories  $(S_n)_{n \in \mathbb{N}}$  in  $E$ ; more precisely,

$$(3.8) \quad Z_N^\omega(E) := \mathbf{E} \left[ \exp \left( \sum_{x \in \mathbb{Z}^d} (\beta_N \omega_x - h_N) \mathbb{1}_{x \in \mathcal{R}_N} \right) \mathbb{1}_E \right].$$

This way, we have that  $\mathbf{P}_N^\omega(E) = Z_N^\omega(E)/Z_N^\omega$ .

Denote also

$$(3.9) \quad \Omega_j^+ := \sum_{x=0}^j \omega_x, \quad \Omega_j^- := \sum_{x=-j}^{-1} \omega_x \quad \text{for } j \geq 0.$$

(with the convention that  $\Omega_0^- = 0$ ), and let

$$(3.10) \quad \Omega_\ell^* := \sup_{0 \leq j \leq \ell} |\Omega_j^-| + \sup_{0 \leq j \leq \ell} |\Omega_j^+|.$$

We also set  $M_N^+ := \max_{0 \leq n \leq N} S_n \geq 0$  and  $M_N^- := \min_{0 \leq n \leq N} S_n \leq 0$  the right-most and left-most points of the random walk after  $N$  steps; denote also  $M_N^* := \max_{0 \leq n \leq N} |S_n| = \max(M_N^+, -M_N^-)$ . With these notations, notice that we have  $\sum_{x \in \mathcal{R}_N} \omega_x = \Omega_{M_N^+}^+ + \Omega_{M_N^-}^-$ . Let us state the following (standard) lemma, that we prove in Appendix A.2 for completeness.

**Lemma 3.7.** *Let  $\Omega_\ell^*$  defined as in (3.10). Then, under Assumption 1 ( $\alpha \in (0, 1) \cup (1, 2]$ ), there exists a constant  $c \in (1, +\infty)$  such that for any  $\mathsf{T} > 0$  we have*

$$(3.11) \quad \mathbb{P}(\Omega_\ell^* > \mathsf{T}) \leq c \ell \mathsf{T}^{-\alpha}.$$

#### 4. PROOF OF THE MAIN RESULTS

**4.1. Region R1: Proof of Theorem 3.1.** Recall that in Region  $R_1$  we have

$$\begin{cases} \gamma > \frac{1}{2\alpha} & \text{and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (\frac{1}{2}, 1) \cup (1, 2], \\ \gamma > -\frac{\alpha-1}{\alpha} & \text{and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (0, \frac{1}{2}). \end{cases}$$

Let us note that we always have  $\gamma > \frac{1}{2\alpha}$ , since  $\frac{1-\alpha}{\alpha} > \frac{1}{2\alpha}$  when  $\alpha > 1/2$ .

**Convergence in probability.** Fix  $A$  (large), and split the partition function in the following way

$$(4.1) \quad Z_N^\omega = Z_N^\omega(M_N^* \leq A\sqrt{N}) + Z_N^\omega(M_N^* > A\sqrt{N}).$$

*Upper bound.* It is easy to see that, recalling the definition (3.8) of the restricted partition function, since  $h_N \geq 0$  we have

$$(4.2) \quad Z_N^\omega(M_N^* \leq A\sqrt{N}) \leq \exp\left(\hat{\beta}N^{-\gamma}\Omega_{A\sqrt{N}}^*\right) \mathbf{P}\left(M_N^* \leq A\sqrt{N}\right) \leq \exp\left(\hat{\beta}N^{-\gamma}\Omega_{A\sqrt{N}}^*\right)$$

By Lemma 3.7, we get that

$$(4.3) \quad \mathbb{P}\left(Z_N^\omega(M_N^* \leq A\sqrt{N}) \geq e^\varepsilon\right) \leq \mathbb{P}\left(\hat{\beta}N^{-\gamma}\Omega_{A\sqrt{N}}^* \geq \varepsilon\right) \leq c\varepsilon^{-\alpha}\hat{\beta}^\alpha AN^{(\frac{1}{2}-\alpha\gamma)}.$$

Therefore, since in Region 1 we have that  $\gamma > \frac{1}{2\alpha}$ , we get that (4.2) is bounded above by  $e^\varepsilon$ , with  $\mathbb{P}$ -probability going to 1 as  $N$  goes to infinity.

It remains to show that the second term in (4.1) is small, with high  $\mathbb{P}$ -probability. In this case the computation for the case  $\alpha \in (0, 1/2)$  and  $\alpha \in (1/2, 1) \cup (1, 2]$  are different and we present them separately.

*Case  $\alpha \in (1/2, 1) \cup (1, 2]$ .* We have the following upper bound

$$(4.4) \quad \begin{aligned} Z_N^\omega(M_N^* > A\sqrt{N}) &\leq \sum_{k=1}^{\infty} Z_N^\omega(M_N^* \in (2^{k-1}A\sqrt{N}, 2^k A\sqrt{N}]) \\ &\leq \sum_{k=1}^{\infty} \exp\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A\sqrt{N}}^*\right) \mathbf{P}\left(M_N^* \geq 2^{k-1}A\sqrt{N}\right) \end{aligned}$$

Then, it is standard to get that  $\mathbb{P}(M_N^* > x) \leq \exp(-\frac{x^2}{2N})$  for any  $x > 0$  and  $N \in \mathbb{N}$  (see e.g. Feller [16]), so that

$$(4.5) \quad \mathbf{P}(M_N^* > 2^{k-1}A\sqrt{N}) \leq 2 \exp\left(-2^{2k-3}A^2\right),$$

and

$$Z_N^\omega(M_N^* > A\sqrt{N}) \leq \sum_{k=1}^{\infty} 2 \exp\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A\sqrt{N}}^*\right) e^{-2^{2k-3}A^2}.$$

We therefore get, by a union bound, that

$$(4.6) \quad \begin{aligned} \mathbb{P}\left(Z_N^\omega(M_N^* > A\sqrt{N}) > e^{-A}\right) &\leq \sum_{k=1}^{\infty} \mathbb{P}\left(2 \exp\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A\sqrt{N}}^*\right) e^{-2^{2k-3}A^2} > \frac{e^{-A}}{2^k}\right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A\sqrt{N}}^* > 2^{2k-4}A^2\right) \end{aligned}$$

where for the last inequality  $A$  is chosen large enough so that  $e^{-A}2^{-k} \times \frac{1}{2} \exp(2^{2k-3}A^2) \geq \exp(2^{2k-4}A^2)$  for all  $k \geq 1$ . Using Lemma 3.7 we get that

$$(4.7) \quad \mathbb{P}\left(\Omega_{2^k A\sqrt{N}}^* > \hat{\beta}^{-1}2^{2k-4}A^2N^\gamma\right) \leq c\hat{\beta}^\alpha 2^{-k(2\alpha-1)}N^{(\frac{1}{2}-\alpha\gamma)}A^{1-2\alpha}.$$

Summing over  $k$ , and using that  $\alpha > \frac{1}{2}$ , we finally obtain that for  $A$  sufficiently large,

$$\mathbb{P}\left(Z_N^\omega(M_N^* > A\sqrt{N}) > e^{-A}\right) \leq c\hat{\beta}^\alpha A^{1-2\alpha}N^{(\frac{1}{2}-\alpha\gamma)}.$$

Note that this goes to 0 as  $N \rightarrow \infty$ , since  $\gamma > \frac{1}{2\alpha}$ .

*Case  $\alpha \in (0, 1/2)$ .* Let us consider the following decomposition

$$(4.8) \quad Z_N^\omega(M_N^* > A\sqrt{N}) = Z_N^\omega(M_N^* \in (A\sqrt{N}, N^{3/4})) + Z_N^\omega(M_N^* \in (N^{3/4}, N]).$$

We then bound the first term as above, see (4.4):

$$(4.9) \quad Z_N^\omega \left( M_N^* \in (A\sqrt{N}, N^{3/4}) \right) \leq \sum_{k=1}^{\log_2 N^{1/4}} \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k A\sqrt{N}}^* \right) \mathbf{P} \left( M_N^* \geq 2^{k-1} A\sqrt{N} \right)$$

and the second one similarly:

$$(4.10) \quad \begin{aligned} Z_N^\omega \left( M_N^* \in (N^{3/4}, N] \right) &= \sum_{k=0}^{\log_2 N^{1/4}+1} Z_N^\omega \left( M_N^* \in (2^{-k-1}N, 2^{-k}N] \right) \\ &\leq \sum_{k=0}^{\log_2 N^{1/4}+1} \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^{-k}N}^* \right) \mathbf{P} \left( M_N^* \geq 2^{-k-1}N \right). \end{aligned}$$

Let us observe that as in (4.5), we get that

$$\mathbf{P} \left( M_N^* > 2^{k-1} A\sqrt{N} \right) \leq 2 \exp \left( -2^{2k-3} A^2 \right), \quad \mathbf{P} \left( M_N^* \geq 2^{-k-1} N \right) \leq 2 \exp \left( -2^{-2k-3} N \right).$$

Combining (4.8) to (4.10), we get that, analogously to (4.6), by a union bound

$$\begin{aligned} \mathbb{P} \left( Z_N^\omega \left( M_N^* > A\sqrt{N} \right) > e^{-A} \right) &\leq \sum_{k=1}^{\log_2 N^{1/4}} \mathbb{P} \left( e^{\hat{\beta} N^{-\gamma} \Omega_{2^k A\sqrt{N}}^*} > \frac{1}{2^{k+1}} e^{-A+2^{2k-3} A^2} \right) \\ &\quad + \sum_{k=1}^{\log_2 N^{1/4}+1} \mathbb{P} \left( e^{\hat{\beta} N^{-\gamma} \Omega_{2^{-k}N}^*} > \frac{1}{2^{k+1}} e^{-A+2^{-2k-3} N} \right). \end{aligned}$$

Using again Lemma 3.7 as in (4.7) (with  $A$  fixed sufficiently large and  $N$  large enough), we get that the above is bounded by a constant times

$$\sum_{k=1}^{\log_2 N^{1/4}} \hat{\beta}^\alpha 2^{-k(2\alpha-1)} N^{(\frac{1}{2}-\alpha\gamma)} A^{1-2\alpha} + \sum_{k=1}^{\log_2 N^{1/4}+1} \hat{\beta}^\alpha 2^{k(2\alpha-1)} N^{1-\alpha(1+\gamma)}.$$

Since  $\alpha < 1/2$ , we therefore get that

$$\mathbb{P} \left( Z_N^\omega \left( M_N^* > A\sqrt{N} \right) > e^{-A} \right) \leq c_{A, \hat{\beta}, \alpha} N^{(\frac{3}{4}-\frac{\alpha}{2}-\alpha\gamma)} + c'_{\hat{\beta}, \alpha} N^{1-\alpha(1+\gamma)},$$

and both terms go to 0 as  $N$  goes to infinity, using that  $\gamma > \frac{1-\alpha}{\alpha}$  and  $\alpha < \frac{1}{2}$  for the first term, and  $\gamma > \frac{1-\alpha}{\alpha}$  for the second term.

All together, we have proved that in both cases  $\alpha \in (\frac{1}{2}, 1) \cup (1, 2]$  and  $\alpha \in (0, \frac{1}{2})$ , for any  $\varepsilon > 0$  and  $A$  sufficiently large, with  $\mathbb{P}$ -probability going to 1, we have  $Z_N^\omega \leq e^\varepsilon + e^{-A}$ .

*Lower bound.* To achieve the lower bound, we use that

$$(4.11) \quad Z_N^\omega \geq Z_N^\omega \left( M_N^* \leq A\sqrt{N} \right) \geq \exp \left( -\hat{\beta} N^{-\gamma} \Omega_{A\sqrt{N}}^* - 2A\hat{h}N^{\frac{1}{2}-\zeta} \right) \mathbf{P} \left( M_N^* \leq A\sqrt{N} \right).$$

Using that  $\mathbf{P} \left( M_N^* \leq A\sqrt{N} \right) \geq (1 - 2e^{-A^2/2})$  and the last upper bound in (4.3), we get that with  $\mathbb{P}$ -probability going to 1 as  $N$  goes to infinity, the right-hand-side of (4.11) is larger than

$$e^{-\varepsilon-2A\hat{h}N^{1/2-\zeta}} (1 - 2e^{-A^2/2}) \geq e^{-2\varepsilon} (1 - 2e^{-A^2/2}),$$

the last inequality being valid for  $N$  large enough, using that  $\zeta > \frac{1}{2}$ . We then get that

$$\mathbb{P} \left( Z_N^\omega \leq e^{-2\varepsilon} (1 - 2e^{-A^2/2}) \right) \xrightarrow{N \rightarrow \infty} 0.$$

Combined with the upper bound, this concludes the proof, since  $\varepsilon$  and  $A$  are arbitrary.

**Transversal fluctuations.** From the calculation above, we directly have that, for  $A$  large,  $\mathbf{P}_N^\omega(M_N^* > A\sqrt{N}) = Z_N^\omega(M_N^* > A\sqrt{N})/Z_N^\omega$  is bounded by a constant times  $e^{-A}$ , with  $\mathbb{P}$ -probability going to 1 as  $N$  goes to infinity. It remains to show that if  $\eta$  is small then  $\mathbf{P}_N^\omega(M_N^* < \eta\sqrt{N})$  is small, with high  $\mathbb{P}$ -probability. Since  $Z_N^\omega \xrightarrow{\mathbb{P}} 1$ , it is sufficient to show that  $Z_N^\omega(M_N^* \leq \eta\sqrt{N})$  is small, with high  $\mathbb{P}$ -probability. But we have the following upper bound, identical to (4.2):

$$(4.12) \quad Z_N^\omega(M_N^* \leq \eta\sqrt{N}) \leq \exp\left(\hat{\beta}N^{-\gamma}\Omega_{\eta\sqrt{N}}^*\right) \mathbf{P}\left(M_N^* \leq \eta\sqrt{N}\right).$$

Then, standard arguments (in the spirit of Lemma A.3) show that  $\mathbf{P}(M_N^* \leq \eta\sqrt{N}) \leq e^{-c/\eta^2}$ . On the other hand, using again Lemma 3.7, we get that  $\mathbb{P}(\hat{\beta}N^{-\gamma}\Omega_{\eta\sqrt{N}}^* \geq 1)$  goes to 0 as  $N \rightarrow \infty$  (using again that  $\gamma > \frac{1}{2\alpha}$ ). This shows that  $Z_N^\omega(M_N^* \leq \eta\sqrt{N}) \leq e^{1-c/\eta^2}$  with probability going to 1 as  $N \rightarrow \infty$ .

All together, this shows that  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $\sqrt{N}$  under  $\mathbf{P}_N^\omega$ .

**4.2. Region R2: proof of Theorem 3.2.** We prove that in region  $R_2$  the transversal fluctuations of order  $N^\xi$  with  $\xi = \frac{\alpha}{2\alpha-1}(1-\gamma)$ . Recall that in Region  $R_2$  we have

$$2\xi - 1 = \frac{\xi}{\alpha} - \gamma > \xi - \zeta \quad \text{with } \alpha \in \left(\frac{1}{2}, 1\right) \cup (1, 2],$$

and that region  $R_2$  does not exist when  $\alpha < 1/2$ .

**Convergence in distribution.** We fix some  $A$  large, and split the partition function as

$$(4.13) \quad Z_N^\omega = Z_N^\omega\left(M_N^* \leq AN^\xi\right) + Z_N^\omega\left(M_N^* > AN^\xi\right).$$

The proof of the convergence in distribution is divided into three steps: (1) we show that after taking logarithm and dividing by  $N^{2\xi-1}$ , the first term weakly converges to some random variable  $\mathcal{W}_{R_2}^A$  as  $N \rightarrow \infty$ ; (2) we show that the second term is small compared to the first one; (3) we let  $A \rightarrow \infty$ , and observe that  $\mathcal{W}_{R_2}^A$  weakly converges to  $\mathcal{W}_{R_2}$ .

Step 1. We prove the following lemma.

**Lemma 4.1.** *In Region  $R_2$ , we have that for any  $A > 0$ , as  $N \rightarrow \infty$*

$$\frac{1}{N^{2\xi-1}} \log Z_N^\omega\left(M_N^* \leq AN^\xi\right) \xrightarrow{d} \mathcal{W}_{R_2}^A := \sup_{-A \leq u \leq 0 \leq v \leq A} \left\{ \hat{\beta}(X_v - X_u) - I(u, v) \right\},$$

with  $(X_x)_{x \in \mathbb{R}}$  defined in Definition 3.2, and  $I(u, v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2$ .

*Proof.* Let us fix  $\delta > 0$ , and write

$$(4.14) \quad Z_N^{\omega, \leq} := Z_N^\omega(M_N^* \leq AN^\xi) = \sum_{k_1=0}^{\lfloor A/\delta \rfloor} \sum_{k_2=0}^{\lfloor A/\delta \rfloor} Z_N^\omega(k_1, k_2, \delta),$$

where we define

$$(4.15) \quad Z_N^\omega(k_1, k_2, \delta) := Z_N^\omega\left(M_N^- \in (-(k_1+1)\delta N^\xi, -k_1\delta N^\xi], M_N^+ \in [k_2\delta N^\xi, (k_2+1)\delta N^\xi)\right)$$

(recall the definitions  $M_N^- := \min_{0 \leq n \leq N} S_n$  and  $M_N^+ := \max_{0 \leq n \leq N} S_n$ ). Since there are at most  $(A/\delta)^2$  terms in the sum, we easily get that

$$(4.16) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \log Z_N^\omega(k_1, k_2, \delta) \leq \log Z_N^{\omega, \leq} \leq 2 \log(A/\delta) + \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \log Z_N^\omega(k_1, k_2, \delta).$$

*Upper bound.* As an upper bound on  $\log Z_N^\omega(k_1, k_2, \delta)$ , since  $h_N \geq 0$ , we have

$$(4.17) \quad \log Z_N^\omega(k_1, k_2, \delta) \leq \beta_N \left( \Omega_{[k_1 \delta N^\xi]}^- + \Omega_{[k_2 \delta N^\xi]}^+ \right) + \beta_N R_N^\delta(k_1 \delta, k_2 \delta) + p_N^\delta(k_1 \delta, k_2 \delta),$$

where for  $u, v \geq 0$  we defined

$$(4.18) \quad R_N^\delta(u, v) := \max_{u N^\xi + 1 \leq j \leq (u + \delta) N^\xi - 1} \left| \Omega_j^- - \Omega_{[u N^\xi]}^- \right| + \max_{v N^\xi \leq j \leq (v + \delta) N^\xi - 1} \left| \Omega_j^+ - \Omega_{[v N^\xi]}^+ \right|,$$

and

$$(4.19) \quad p_N^\delta(u, v) := \log \mathbf{P} \left( M_N^- \in (-(u + \delta) N^\xi, -u N^\xi], M_N^+ \in [v \delta N^\xi, (v + \delta) N^\xi] \right).$$

Write  $u = k_1 \delta, v = k_2 \delta$  and set  $U_\delta = \{0, \delta, 2\delta, \dots, A\}$ : using that  $2\xi - 1 = \xi/\alpha - \gamma$ , we get that

$$(4.20) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1, k_2, \delta)}{N^{2\xi - 1}} \leq \max_{u, v \in U_\delta} \left\{ \hat{\beta} N^{-\frac{\xi}{\alpha}} \left( \Omega_{[u N^\xi]}^- + \Omega_{[v N^\xi]}^+ \right) + \hat{\beta} N^{-\frac{\xi}{\alpha}} R_N^\delta(u, v) + N^{-(2\xi - 1)} p_N^\delta(u, v) \right\}.$$

Note that we have that  $(N^{-\xi/\alpha} \Omega_{[u N^\xi]}^-)_{u \in [0, A + \delta]}$  and  $(N^{-\xi/\alpha} \Omega_{[v N^\xi]}^+)_{v \in [0, A + \delta]}$  converge to two independent Lévy processes  $(X_u^{(1)})_{u \in [0, A + \delta]}$  and  $(X_v^{(2)})_{v \in [0, A + \delta]}$  (with no drift, no Brownian component and Lévy measure  $\nu(dx) = \alpha(p\mathbb{1}_{\{x > 0\}} + q\mathbb{1}_{\{x < 0\}}) |x|^{-1-\alpha} dx$  if  $\alpha \in (0, 1) \cup (1, 2)$ , or standard Brownian motions if  $\alpha = 2$ ). Note also that thanks to Lemma A.1 (see (A.2)) we have

$$\lim_{N \rightarrow \infty} N^{-(2\xi - 1)} p_N(u, v, \delta) = -J(u, v), \quad \text{with } J(u, v) := \frac{1}{2}(u \wedge v + u + v)^2, \quad u, v \geq 0.$$

Since the maximum is over a finite set (and recall the definition (4.18) of  $R_N^\delta$ ), we readily have that the upper bound in (4.20) converges in distribution to

$$(4.21) \quad \widehat{\mathcal{W}}_{R_2}^{A, \delta} := \max_{u, v \in U_\delta} \left\{ \hat{\beta} (X_u^{(1)} + X_v^{(2)}) + \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{u+t}^{(1)} - X_u^{(1)}| + \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{v+t}^{(2)} - X_v^{(2)}| - J(u, v) \right\}.$$

*Lower bound.* On the other hand, we have the following lower bound on  $\log Z_N^\omega(k_1, k_2, \delta)$ :

$$\beta_N \left( \Omega_{[k_1 \delta N^\xi]}^- + \Omega_{[k_2 \delta N^\xi]}^+ \right) - \beta_N R_N^\delta(k_1 \delta, k_2 \delta) - h_N(k_2 - k_1 + 2)\delta N^\xi + p_N^\delta(k_1 \delta, k_2 \delta).$$

and thus, setting  $u = k_1 \delta, v = k_2 \delta$  and  $U_\delta = \{0, \delta, \dots, A\}$  as above, we obtain

$$(4.22) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1, k_2, \delta)}{N^{2\xi - 1}} \geq \max_{u, v \in U_\delta} \left\{ \hat{\beta} N^{-\frac{\xi}{\alpha}} \left( \Omega_{[u N^\xi]}^- + \Omega_{[v N^\xi]}^+ \right) - \hat{\beta} N^{-\frac{\xi}{\alpha}} R_N^\delta(u, v) - \hat{h}(u + v + 2\delta) N^{(\xi - \zeta) - (2\xi - 1)} + N^{-(2\xi - 1)} p_N^\delta(u, v) \right\}.$$



Since in Region 2 we have  $\xi - \zeta < 2\xi - 1$ , the third term in the maximum goes to 0, and we get as above that the lower bound in (4.22) converges in distribution towards

$$(4.23) \quad \widetilde{\mathcal{W}}_{R_2}^{A,\delta} := \max_{u,v \in U_\delta} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) - \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{u+t}^{(1)} - X_u^{(1)}| - \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{v+t}^{(2)} - X_v^{(2)}| - J(u, v) \right\}.$$

*Conclusion.* By Skorohod's representation theorem, the upper bound (4.21) and the lower bound (4.23) can be realized on the same probability space, and be a.s. upper and lower bounds for  $\limsup N^{-(2\xi-1)} \log Z_N^{\omega, \leq}$  and  $\liminf N^{-(2\xi-1)} \log Z_N^{\omega, \leq}$  respectively. Notice that, by a.s. càd-làg structure of trajectories of Lévy process (continuity in the case of Brownian motion), we clearly have that

$$\lim_{\delta \downarrow 0} \widetilde{\mathcal{W}}_{R_2}^{A,\delta} = \lim_{\delta \downarrow 0} \widehat{\mathcal{W}}_{R_2}^{A,\delta} = \sup_{u,v \in [0,A]} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) - J(u, v) \right\},$$

which is exactly  $\mathcal{W}_{R_2}^A$ , defining  $X_x = -X_{-x}^{(1)}$  if  $x \leq 0$  and  $X_x = X_x^{(2)}$  if  $x \geq 0$ , and  $I(u, v) = J(-u, v)$ . Letting  $N \rightarrow \infty$  and then  $\delta \downarrow 0$ , this concludes the proof of Lemma 4.1.  $\square$

Step 2. Next, we prove the following result.

**Lemma 4.2.** *In region  $R_2$ , there is some  $A_0 > 0$  and some constant  $C = C_{\hat{\beta}}$  such that, for all  $A \geq A_0$ ,*

$$\mathbb{P} \left( \frac{1}{N^{2\xi-1}} \log Z_N^\omega \left( M_N^* > AN^\xi \right) \geq -1 \right) \leq CA^{1-2\alpha}.$$

(Recall that  $\alpha > 1/2$  in region  $R_2$ .)

This proves that with probability at least  $1 - CA^{1-2\alpha}$ , the second term in (4.13) is small compared to the first one, thanks to Lemma 4.1, using also that  $\mathcal{W}_{R_2}^A \geq 0$  (by taking  $u = 0 = v$ ). Note also that since  $\alpha > 1/2$ , this probability can be made arbitrarily close to 1 by taking  $A$  large.

*Proof.* Let us write

$$(4.24) \quad \begin{aligned} Z_N^{\omega, >} &:= Z_N^\omega \left( M_N^* > AN^\xi \right) = \sum_{k=1}^{\infty} Z_N^\omega \left( M_N^* \in (2^{k-1} AN^\xi, 2^k AN^\xi] \right) \\ &\leq \sum_{k=1}^{\infty} \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \right) \mathbf{P} \left( M_N^* \geq 2^{k-1} AN^\xi \right), \end{aligned}$$

and note that  $\mathbf{P} \left( M_N^* \geq 2^{k-1} AN^\xi \right) \leq 2 \exp \left( -2^{2k-3} A^2 N^{2\xi-1} \right)$ . Therefore, thanks to a union bound, we get

$$(4.25) \quad \begin{aligned} \mathbb{P} \left( Z_N^{\omega, >} \geq e^{-N^{2\xi-1}} \right) &\leq \sum_{k=1}^{\infty} \mathbb{P} \left( \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \right) 2e^{-2^{2k-3} A^2 N^{2\xi-1}} \geq \frac{1}{2^{k+1}} e^{-N^{2\xi-1}} \right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P} \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \geq 2^{2k-4} A^2 N^{2\xi-1} \right), \end{aligned}$$

where the last inequality holds provided that  $A$  is large enough. Then, using Lemma 3.7 and the fact that  $2\xi - 1 + \gamma = \xi/\alpha$ , we get that

$$\mathbb{P} \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \geq 2^{2k-4} A^2 N^{2\xi-1} \right) \leq c \hat{\beta}^\alpha 2^{(1-2\alpha)k} A^{1-2\alpha}.$$

Summing this inequality over  $k$ , since  $\alpha > 1/2$ , we conclude the proof of Lemma 4.2.  $\square$

Step 3. Let us note that, by monotonicity in  $A$ , we have that  $\mathcal{W}_{R_2} = \lim_{A \uparrow \infty} \mathcal{W}_{R_2}^A$  is well defined (possibly infinite) and non negative. We prove the following lemma:

**Lemma 4.3.** *If  $\alpha \in (\frac{1}{2}, 2]$ , we have that  $\mathcal{W}_{R_2} := \sup_{u \leq 0 \leq v} \{\hat{\beta}(X_v - X_u) - I(u, v)\}$  is  $\mathbb{P}$ -a.s. positive and finite.*

Combined with Lemmas 4.1-4.2, this readily proves that  $N^{-(2\xi-1)} \log Z_N^\omega$  converges in distribution to  $\mathcal{W}_{R_2}$  as  $N \rightarrow \infty$ .

*Proof.* To show that  $\mathcal{W}_{R_2} > 0$  almost surely, notice that taking  $u = 0$  we have

$$\mathcal{W}_{R_2} \geq \sup_{v \geq 0} \left\{ \hat{\beta} X_v - \frac{1}{2} v^2 \right\}.$$

Then, a.s. we can find some sequence  $v_n \downarrow 0$  such that  $X_{v_n} \geq v_n^{1/\alpha}$  for all  $n$  (cf. [1, Th. 2.1]): we then get that  $\mathcal{W}_{R_2} \geq \sup_{n \geq 0} \left\{ \hat{\beta} v_n^{1/\alpha} - \frac{1}{2} v_n^2 \right\} > 0$  since  $\alpha > 1/2$ .

To show that  $\mathcal{W}_{R_2} < +\infty$  a.s., notice that  $I(u, v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2 \geq \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2$ : we therefore get that

$$\mathcal{W}_{R_2} \leq \sup_{u \leq 0} \left\{ \hat{\beta} X_u - \frac{1}{2} u^2 \right\} + \sup_{v \geq 0} \left\{ \hat{\beta} X_v - \frac{1}{2} v^2 \right\}.$$

Each term is clearly a.s. finite. Indeed, if we consider the second term we have that almost surely,  $\hat{\beta} X_v - \frac{1}{2} v^2 \leq 0$  for  $v$  large enough: this is a consequence of the fact that for any  $\varepsilon > 0$ , a.s.  $X_v \leq v^{(1+\varepsilon)/\alpha}$  for  $v$  large enough, cf. [22, Sec. 3]. Therefore,  $\hat{\beta} X_v - \frac{1}{2} v^2 \leq \hat{\beta} v^{(1+\varepsilon)/\alpha} - \frac{1}{2} v^2 \leq 0$  for all  $v$  large enough, provided that  $\varepsilon$  is small enough so that  $(1 + \varepsilon)/\alpha < 2$ . Similarly a.s.  $\hat{\beta} X_u - \frac{1}{2} u^2 \leq 0$  for all  $u$  large enough, which concludes the proof.  $\square$

**Transversal fluctuations.** Notice that Lemma 4.2 shows that the polymer  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations *at most* of order  $N^\xi$  under  $\mathbf{P}_N^\omega$ . We can actually deduce from Lemmas 4.1 and 4.3 that it also has transversal fluctuations *at least* of order  $N^\xi$ . Indeed, we can show that  $\mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi)$  is small, with high  $\mathbb{P}$ -probability: we have that

$$\log \mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi) = \log Z_N^\omega(M_N^* \leq \eta N^\xi) - \log Z_N^\omega,$$

and so, by Lemma 4.1 we get that  $N^{-(2\xi-1)} \log \mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi)$  converges in distribution toward  $\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2}$  (again, we use Skorohod representation theorem so that  $\mathcal{W}_{R_2}^\eta$  and  $\mathcal{W}_{R_2}$  are defined with the same realization of  $(X_x)_{x \in \mathbb{R}}$ ). We therefore get that  $\mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi)$  goes to 0 if  $\log \mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi) \rightarrow -\infty$ , which happens with probability  $\mathbb{P}(\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2} < 0)$ . Since  $\mathcal{W}_\eta$  goes to 0 almost surely as  $\eta \downarrow 0$  (both  $X_v - X_u$  and  $I(u, v)$  tend to 0), we get that  $\mathbb{P}(\mathcal{W}_{R_2}^\eta - \mathcal{W}_{R_2} < 0)$  goes to 1 as  $\eta \downarrow 0$ . This concludes the proof that the polymer  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N^\xi$  under  $\mathbf{P}_N^\omega$ .  $\square$

**4.3. Region R3: proof of Theorem 3.3.** We show that in Region  $R_3$ , we have  $\xi = 1$ . Recall that in this region

$$\gamma < \zeta - \frac{\alpha-1}{\alpha} \quad \text{and} \quad \gamma < \frac{1-\alpha}{\alpha},$$

with  $\alpha \in (0, 1) \cup (1, 2]$ . First, we prove the convergence in distribution in (3.3).

**Convergence in distribution.** For any  $\delta > 0$ , we can write

$$(4.26) \quad Z_N^\omega = \sum_{k_1=0}^{\lfloor 1/\delta \rfloor} \sum_{k_2=0}^{\lfloor 1/\delta \rfloor} Z_N^\omega(k_1, k_2, \delta),$$

with  $Z_N^\omega(k_1, k_2, \delta)$  as in (4.15) with  $\xi = 1$ . Let us stress right away that since there are at most  $N$  steps for the random walk, we can have  $M_N^- \leq -k_1\delta N$  and  $M_N^+ \geq k_2\delta N$  only if  $\delta(k_1 \wedge k_2 + k_1 + k_2) \leq 1$ .

Hence, writing  $u = k_1\delta$ , and  $v = k_2\delta$ , and  $U_\delta = \{0, \delta, 2\delta, \dots, 1\}$  we have

$$(4.27) \quad \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \log Z_N^\omega\left(\frac{u}{\delta}, \frac{v}{\delta}, \delta\right) \leq \log Z_N^\omega \leq -2 \log \delta + \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \log Z_N^\omega\left(\frac{u}{\delta}, \frac{v}{\delta}, \delta\right).$$

For the upper bound, we have

$$(4.28) \quad \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} N^{\gamma - \frac{1}{\alpha}} \log Z_N^\omega\left(\frac{u}{\delta}, \frac{v}{\delta}, \delta\right) \leq \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \hat{\beta} \left( N^{-\frac{1}{\alpha}} \Omega_{[uN]}^- + N^{-\frac{1}{\alpha}} \Omega_{[vN]}^+ + N^{-\frac{1}{\alpha}} R_N^\delta(u, v) \right),$$

where, analogously to (4.18), we set

$$(4.29) \quad R_N^\delta(u, v) := \max_{uN+1 \leq j \leq (u+\delta)N-1} \left| \Omega_j^- - \Omega_{[uN]}^- \right| + \max_{vN+1 \leq j \leq (v+\delta)N-1} \left| \Omega_j^+ - \Omega_{[vN]}^+ \right|.$$

As in the previous section, we get that the right-hand side in (4.28) converges in distribution toward

$$(4.30) \quad \widehat{\mathcal{W}}_{R_3}^\delta := \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) + \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{u+t}^{(1)} - X_u^{(1)}| + \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{v+t}^{(2)} - X_v^{(2)}| \right\}.$$

For the lower bound, we have

$$(4.31) \quad \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} N^{\gamma - \frac{1}{\alpha}} \log Z_N^\omega\left(\frac{u}{\delta}, \frac{v}{\delta}, \delta\right) \geq \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \left\{ \hat{\beta} \left( N^{-\frac{1}{\alpha}} \Omega_{[uN]}^- + N^{-\frac{1}{\alpha}} \Omega_{[vN]}^+ \right) - N^{-\frac{1}{\alpha}} R_N^\delta(u, v) - \hat{h} N^{1-\zeta+\gamma-\frac{1}{\alpha}} - N^{1+\gamma-\frac{1}{\alpha}} \log 2 \right\},$$

where we used that any non-empty event of  $(S_n)_{0 \leq n \leq N}$  has probability at least  $2^{-N}$ . Now, since  $\gamma < \zeta + \frac{1}{\alpha} - 1$  and  $\gamma < \frac{1}{\alpha} - 1$ , the last two terms go to 0: we get that the right-hand side of (4.31) converges in distribution toward

$$(4.32) \quad \widetilde{\mathcal{W}}_{R_3}^\delta := \max_{\substack{u, v \in U_\delta \\ u \wedge v + u + v \leq 1}} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) - \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{u+t}^{(1)} - X_u^{(1)}| - \hat{\beta} \sup_{0 \leq t \leq \delta} |X_{v+t}^{(2)} - X_v^{(2)}| \right\}.$$

Then, we can conclude by Skorohod's representation theorem, in the same manner as in the proof of Lemma 4.1: the upper and lower bound can be realized on the same space, so that letting  $N \rightarrow \infty$  and then  $\delta \downarrow 0$ , we get that  $N^{\gamma - \frac{1}{2}} \log Z_N^\omega$  converges in distribution to

$$\lim_{\delta \downarrow 0} \widehat{\mathcal{W}}_{R_3}^\delta = \lim_{\delta \downarrow 0} \widetilde{\mathcal{W}}_{R_3}^\delta = \sup_{\substack{u, v \in [0, 1] \\ u \wedge v + u + v \leq 1}} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) \right\},$$

where the limit holds thanks to the a.s. càdlàg property of trajectories of the Lévy process (or the a.s. continuity of the Brownian motion), and is exactly the variational problem  $\mathcal{W}_{R_3}$  defined in Theorem 3.3 (by setting  $X_x = -X_{-x}^{(1)}$  if  $x \leq 0$  and  $X_x = X_x^{(2)}$  if  $x \geq 0$ ). Together with the (trivial) fact that  $\mathcal{W}_{R_3} \in (0, +\infty)$  a.s., this concludes the proof of Theorem 3.3.

**Transversal fluctuations.** It remains to show that  $(S_n)_{0 \leq n \leq N}$  has fluctuations of order at least  $N$  under  $\mathbf{P}_N^\omega$ , since we already know that  $\max_{0 \leq n \leq N} |S_n| \leq N$ . We proceed as in the previous section. For  $\eta > 0$  we can write

$$\log \mathbf{P}_N^\omega(M_N^* \leq \eta N) = \log Z_N^\omega(M_N^* \leq \eta N) - \log Z_N^\omega,$$

and so, by Lemma 4.1 we get that  $N^{\gamma - \frac{1}{\alpha}} \log \mathbf{P}_N^\omega(M_N^* \leq \eta N)$  (by a straightforward adaptation of the above proof) converges in distribution toward  $\mathcal{W}_{R_3}^\eta - \mathcal{W}_{R_3}$ , where

$$\mathcal{W}_{R_3}^\eta := \sup_{\substack{-\eta \leq u, v \leq \eta \\ u \wedge v + u + v \leq 1}} \left\{ \hat{\beta}(X_u^{(1)} + X_v^{(2)}) \right\}.$$

We therefore get that  $\mathbf{P}_N^\omega(M_N^* \leq \eta N)$  goes to 0 with probability  $\mathbb{P}(\mathcal{W}_\eta - \mathcal{W} < 0)$ . Since  $\mathcal{W}_{R_3}^\eta$  goes to 0 almost surely as  $\eta \downarrow 0$ , we get that  $\mathbb{P}(\mathcal{W}_{R_3}^\eta - \mathcal{W}_{R_3} < 0)$  goes to 1 as  $\eta \downarrow 0$ . This concludes the proof that the polymer  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N$  under  $\mathbf{P}_N^\omega$ .  $\square$

**4.4. Region R4: proof of Theorem 3.4.** We prove that in Region  $R_4$ , we have  $\xi = \frac{\alpha}{\alpha-1}(\gamma - \zeta)$ . Recall that in region  $R_4$  we have

$$\left( \frac{(2\alpha-1)\zeta - (\alpha-1)}{\alpha} \right) \vee \left( \zeta - \frac{\alpha-1}{\alpha} \right) < \gamma < \left( \frac{(2\alpha+1)\zeta - (\alpha-1)}{3\alpha} \right) \wedge \zeta, \quad \text{with } \alpha \in (1, 2],$$

and that  $\xi - \zeta = \xi/\alpha - \gamma > |2\xi - 1|$  ( $\xi \in (0, 1)$ ). Recall also that region  $R_4$  does not exist if  $\alpha < 1$ .

**Convergence in distribution.** For any  $A > 0$ , we first write

$$(4.33) \quad Z_N^{\omega, \leq} = Z_N^\omega \left( M_N^* \leq AN^\xi \right) + Z_N^\omega \left( M_N^* > AN^\xi \right).$$

The strategy is similar to that in Region  $R_2$ , and we use analogous notation. We proceed in three steps: (1) after taking logarithm and dividing by  $N^{\xi - \zeta}$ , we show that the first term weakly converges to some limit  $\mathcal{W}_{R_4}^A$  when  $N \rightarrow \infty$ ; (2) we show that the second term above is small compared to the first one; (3) we show that  $\mathcal{W}_{R_4}^A \rightarrow \mathcal{W}_{R_4}$  as  $A \rightarrow \infty$ , with  $\mathcal{W}_{R_4} \in (0, +\infty)$  almost surely.

Step 1. We prove the following lemma.

**Lemma 4.4.** *In Region  $R_4$ , for any  $A > 0$ , we have that for  $N \rightarrow \infty$ ,*

$$\frac{1}{N^{\xi - \zeta}} \log Z_N^\omega \left( M_N^* \leq AN^\xi \right) \xrightarrow{d} \mathcal{W}_{R_4}^A := \sup_{-A \leq u \leq 0 \leq v \leq A} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) \right\},$$

with  $(X_x)_{x \in \mathbb{R}}$  defined in Definition 3.2.

*Proof.* For fixed  $\delta > 0$ , we write (cf. (4.15))

$$(4.34) \quad Z_N^{\omega, \leq} := Z_N^\omega \left( M_N^* \leq AN^\xi \right) = \sum_{k_1=0}^{\lfloor A/\delta \rfloor} \sum_{k_2=0}^{\lfloor A/\delta \rfloor} Z_N^\omega(k_1, k_2, \delta).$$

Since the number of summands above is finite, we have

$$(4.35) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \log Z_N^\omega(k_1, k_2, \delta) \leq \log Z_N^{\omega, \leq} \leq 2 \log(A/\delta) + \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \log Z_N^\omega(k_1, k_2, \delta).$$

*Upper bound.* We write  $u = k_1\delta, v = k_2\delta$  and set  $U_\delta = \{0, \delta, 2\delta, \dots, A\}$ . Recall that  $\xi - \zeta = \xi/\alpha - \gamma$ , so we get that

$$(4.36) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1, k_2, \delta)}{N^{\xi-\zeta}} \leq \max_{u, v \in U_\delta} \left\{ \hat{\beta} N^{-\frac{\xi}{\alpha}} (\Omega_{[uN^\xi]}^- + \Omega_{[vN^\xi]}^+) + \hat{\beta} N^{-\frac{\xi}{\alpha}} R_N^\delta(u, v) \right. \\ \left. - \hat{h}(v+u) + N^{\xi-\zeta} p_N^\delta(u, v) \right\}.$$

Note that  $(N^{-\xi/\alpha} \Omega_{[uN^\xi]}^-)_{u \in [0, A+\delta]}$  and  $(N^{-\xi/\alpha} \Omega_{[vN^\xi]}^+)_{v \in [0, A+\delta]}$  converge to two independent Lévy processes  $(X_u^{(1)})_{u \in [0, A+\delta]}$  and  $(X_v^{(2)})_{v \in [0, A+\delta]}$  (with no drift, no Brownian component and Lévy measure  $\nu(dx) = \alpha(p\mathbb{1}_{\{x>0\}} + q\mathbb{1}_{\{x<0\}}) |x|^{-1-\alpha} dx$ , if  $\alpha \in (1, 2)$ , or standard Brownian motions, if  $\alpha = 2$ ). Also note that  $\lim_{N \rightarrow \infty} N^{\xi-\zeta} p_N^\delta(u, v) = 0$ , thanks to Lemma A.1 and Lemma A.3, since we have  $\xi - \zeta > |2\xi - 1|$ . Since the maximum is over finite terms, then by the continuous mapping theorem, the upper bound in (4.36) converges in distribution to

$$(4.37) \quad \widehat{\mathcal{W}}_{R_4}^{A, \delta} := \max_{u, v \in U_\delta} \left\{ \hat{\beta} \left( X_u^{(1)} + X_v^{(2)} + \sup_{t \in [0, \delta]} |X_{u+t}^{(1)} - X_u^{(1)}| + \sup_{t \in [0, \delta]} |X_{v+t}^{(2)} - X_v^{(2)}| \right) - \hat{h}(v+u) \right\}.$$

*Lower bound.* On the other hand, we may bound  $\log Z_N^\omega(k_1, k_2, \delta)$  from below by

$$\beta_N \left( \Omega_{[k_1\delta N^\xi]}^- + \Omega_{[k_2\delta N^\xi]}^+ \right) - \beta_N R_N^\delta(k_1\delta, k_2\delta) - h_N(k_2 - k_1 + 2)\delta N^\xi + p_N^\delta(k_1\delta, k_2\delta).$$

Thus, setting  $u = k_1\delta, v = k_2\delta$  and  $U_\delta = \{0, \delta, \dots, A\}$  as above, we obtain

$$(4.38) \quad \max_{0 \leq k_1, k_2 \leq \frac{A}{\delta}} \frac{\log Z_N^\omega(k_1, k_2, \delta)}{N^{\xi-\zeta}} \geq \max_{u, v \in U_\delta} \left\{ \hat{\beta} N^{-\frac{\xi}{\alpha}} (\Omega_{[uN^\xi]}^- + \Omega_{[uN^\xi]}^+) - \hat{\beta} N^{-\frac{\xi}{\alpha}} R_N^\delta(u, v) \right. \\ \left. - \hat{h}(u+v+2\delta) + N^{\xi-\zeta} p_N^\delta(u, v) \right\}.$$

Hence, analogously to (4.37), the lower bound in (4.38) converges in distribution to

$$(4.39) \quad \widetilde{\mathcal{W}}_{R_4}^{A, \delta} := \max_{u, v \in U_\delta} \left\{ \hat{\beta} \left( X_u^{(1)} + X_v^{(2)} - \sup_{t \in [0, \delta]} |X_{u+t}^{(1)} - X_u^{(1)}| - \sup_{t \in [0, \delta]} |X_{v+t}^{(2)} - X_v^{(2)}| \right) - \hat{h}(v+u+2\delta) \right\}.$$

*Conclusion.* By Skorohod's representation theorem, the upper and lower bounds (4.37)-(4.38) can be realized on the same probability space and hence become the a.s. upper and lower bound for  $\limsup N^{\zeta-\xi} \log Z_N^{\omega, \leq}$  and  $\liminf N^{\zeta-\xi} \log Z_N^{\omega, \leq}$ . By the a.s. càd-làg property of trajectories of Lévy processes (or continuity of the Brownian motion), we have

$$(4.40) \quad \lim_{\delta \downarrow 0} \widehat{\mathcal{W}}_{R_4}^{A, \delta} = \lim_{\delta \downarrow 0} \widetilde{\mathcal{W}}_{R_4}^{A, \delta} = \sup_{u, v \in [0, A]} \left\{ \hat{\beta} \left( X_u^{(1)} + X_v^{(2)} \right) - \hat{h}(v+u) \right\},$$

which is exactly  $\mathcal{W}_{R_4}^{A, \delta}$ , defining  $X_x = -X_{-x}^{(1)}$  if  $x \leq 0$  and  $X_x = X_x^{(2)}$  if  $x \geq 0$ . The weak convergence in Lemma 4.4 is therefore achieved by letting  $N \rightarrow \infty$  and then  $\delta \rightarrow 0$ .  $\square$

Step 2. Next, we prove the following lemma.

**Lemma 4.5.** *In region  $R_4$ , there is some  $A_0 > 0$  and some constant  $C = C_{\hat{\beta}, \hat{h}}$ , such that for  $A \geq A_0$ ,*

$$\mathbb{P} \left( \frac{1}{N^{\xi-\zeta}} \log Z_N^\omega(M_N^* > AN^\xi) > -1 \right) \leq CA^{1-\alpha}.$$

(Recall that  $\alpha > 1$  in region  $R_4$ .)

This lemma proves that with probability at least  $1 - CA^{1-\alpha}$ , the second term in (4.33) is small compared to the first one, thanks to Lemma 4.4, using also that  $\mathcal{W}_{R_4}^A \geq 0$ .

*Proof.* First, let us write

$$\begin{aligned} Z_N^{\omega, >} &:= Z_N^\omega \left( M_N^* > AN^\xi \right) = \sum_{k=1}^{\infty} Z_N^\omega \left( M_N^* \in (2^{k-1}AN^\xi, 2^kAN^\xi] \right) \\ &\leq \sum_{k=1}^{\infty} \exp \left( \hat{\beta}N^{-\gamma}\Omega_{2^kAN^\xi}^* - \hat{h}N^{-\zeta}2^{k-1}AN^\xi \right). \end{aligned}$$

By a union bound, we therefore get that

$$\begin{aligned} \mathbb{P} \left( Z_N^{\omega, >} \geq e^{-N^{\xi-\zeta}} \right) &\leq \sum_{k=1}^{\infty} \mathbb{P} \left( e^{\hat{\beta}N^{-\gamma}\Omega_{2^kAN^\xi}^* - \hat{h}N^{-\zeta}2^{k-1}AN^\xi} \geq \frac{1}{2^{k+1}} e^{-N^{\xi-\zeta}} \right) \\ (4.41) \quad &\leq \sum_{k=1}^{\infty} \mathbb{P} \left( \hat{\beta}N^{-\gamma}\Omega_{2^kAN^\xi}^* \geq \hat{h}2^{k-2}AN^{\frac{\xi}{\alpha}-\gamma} \right), \end{aligned}$$

where the last inequality holds provided that  $A$  has been fixed large enough (we also used that  $\xi - \zeta = \frac{\xi}{\alpha} - \gamma$ ). Then Lemma 3.7 gives that

$$\mathbb{P} \left( \hat{\beta}\Omega_{2^kAN^\xi}^* \geq \hat{h}2^{k-2}AN^{\frac{\xi}{\alpha}} \right) \leq c\hat{\beta}^\alpha \hat{h}^{-\alpha} 2^{k(1-\alpha)} A^{1-\alpha}.$$

Summing this inequality over  $k$ , since  $\alpha > 1$ , this concludes the proof of Lemma 4.5.  $\square$

Step 3. By monotone convergence,  $\mathcal{W}_{R_4}^A$  converges a.s. to  $\mathcal{W}_{R_4}$ : we only need to show that  $\mathcal{W}_{R_4}$  is positive and finite. Combining this with Lemma 4.4 and Lemma 4.5, this completes the proof Theorem 3.4.

**Lemma 4.6.** *If  $\alpha \in (1, 2]$ , we have  $\mathcal{W}_{R_4} := \sup_{u \leq 0 \leq v} \{ \hat{\beta}(X_v - X_u) - \hat{h}(v - u) \}$  is a.s. positive and finite.*

*Proof.* The proof is analogous to the proof of Lemma 4.3. To show that  $\mathcal{W}_{R_4} > 0$ , we use that  $\mathcal{W}_{R_4} \geq \sup_{v \geq 0} \{ \hat{\beta}X_v - \hat{h}v \}$ . By [1, Th 2.1]), there is a.s. a sequence  $v_n \downarrow 0$ , such that  $X_{v_n} \geq v_n^{1/\alpha}$  for all  $n$ . Hence, for large enough  $n$ ,  $\mathcal{W}_{R_4} \geq \hat{\beta}v_n^{1/\alpha} - \hat{h}v_n > 0$ , since  $\alpha > 1$ .

To show that  $\mathcal{W}_{R_4} < \infty$ , we use  $\mathcal{W}_{R_4} \leq \sup_{u \leq 0} \{ \hat{\beta}X_u + \hat{h}u \} + \sup_{v \geq 0} \{ \hat{\beta}X_v - \hat{h}v \}$ . By [22], we have that for any  $\varepsilon > 0$ , a.s.  $X_v \leq v^{(1+\varepsilon)/\alpha}$  for  $v$  large enough. Therefore, if  $\varepsilon$  is sufficiently small so that  $(1+\varepsilon)/\alpha < 1$  (recall  $\alpha > 1$ ), we get that  $\hat{\beta}X_v - \hat{h}v \leq \hat{\beta}v^{(1+\varepsilon)/\alpha} - \hat{h}v \leq 0$  for all  $v$  sufficiently large. Similarly we also have that  $\hat{\beta}X_u + \hat{h}u \leq 0$  for all  $u$  large enough. This concludes the proof.  $\square$

**Transversal fluctuations.** We prove that the transversal fluctuations are of order  $N^\xi$ . Lemma 4.5 already shows that the transversal fluctuations are *at most*  $N^\xi$ . On the other hand, the fact that  $\mathcal{W}_{R_4} > 0$  a.s. ensures that the transversal fluctuations are *at least*  $N^\xi$ . Indeed, we have thanks to Lemma 4.4 that for  $\eta > 0$ ,

$$\begin{aligned} &\frac{1}{N^{\xi-\zeta}} \log \mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi) \\ &= \frac{1}{N^{\xi-\zeta}} \log Z_N^\omega(M_N^* \leq \eta N^\xi) - \frac{1}{N^{\xi-\zeta}} \log Z_N^\omega \xrightarrow{d} \mathcal{W}_{R_4}^\eta - \mathcal{W}_{R_4}. \end{aligned}$$

Hence,  $\mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi) \rightarrow 0$  with probability  $\mathbb{P}(\mathcal{W}_{R_4}^\eta - \mathcal{W}_{R_4} < 0)$ : since  $\mathcal{W}_{R_4}^\eta$  goes to 0 as  $\eta \downarrow 0$ , we can make this probability arbitrarily close to 1 by choosing  $\eta$  small. This concludes the proof that  $(S_n)_{0 \leq n \leq N}$  has transversal fluctuations of order  $N$  under  $\mathbf{P}_N^\omega$ .  $\square$

**4.5. Region R5: proof of Theorem 3.5.** In this region, we prove that the transversal fluctuations of order  $N^\xi$  with  $\xi = \frac{1+\zeta}{3} \in (0, 1/2)$ . Note that in Region 5, we have

$$1 - 2\xi = \xi - \zeta > \frac{\xi}{\alpha} - \gamma \quad \text{and} \quad -1 < \zeta < \frac{1}{2}.$$

**Convergence in probability.** We fix some constant  $A > 0$  (large), and we split the partition function as

$$(4.42) \quad Z_N^\omega = Z_N^\omega(M_N^* \leq AN^\xi) + Z_N^\omega(M_N^* > AN^\xi).$$

The strategy of proof is similar to that in Region  $R_2$ , but with only two steps: (1) we show that for  $A$  large enough, after taking logarithm and dividing by  $N^{1-2\xi}$ , the first term converges to some constant (independent of  $A$  if  $A$  is large enough) in probability; (2) we show that if  $A$  is large enough, the second term is negligible compared to the first one.

Step 1. We prove the following lemma.

**Lemma 4.7.** *In Region 5, we have that for any  $A > 0$ , as  $N \rightarrow \infty$ ,*

$$\frac{1}{N^{1-2\xi}} \log Z_N^\omega(M_N^* \leq AN^\xi) \xrightarrow{\mathbb{P}} \sup_{-A \leq u \leq 0 \leq v \leq A} \left\{ -\hat{h}(v-u) - \bar{I}(u, v) \right\},$$

where  $\bar{I}(u, v) = \frac{\pi^2}{2}(v-u)^{-2}$  for  $u \leq 0 \leq v$ . By a simple calculation, the supremum is  $\frac{3}{2}(\hat{h}\pi)^{\frac{2}{3}}$  for any  $A \geq \frac{1}{2}\pi^{\frac{2}{3}}\hat{h}^{-\frac{1}{3}}$ , since it is achieved at  $v-u = \pi^{\frac{2}{3}}\hat{h}^{-\frac{1}{3}}$ .

*Proof.* For any fixed  $A$ , we have the following upper and lower bounds

$$(4.43) \quad \log \hat{Z}_N^A - \hat{\beta}N^{-\gamma}\Omega_{AN^\xi}^* \leq \log Z_N^\omega(M_N^* \leq AN^\xi) \leq \log \hat{Z}_N^A + \hat{\beta}N^{-\gamma}\Omega_{AN^\xi}^*,$$

where  $\hat{Z}_N^A := \mathbf{E}[\exp(-h_N|\mathcal{R}_N|)\mathbb{1}_{\{M_N^* \leq AN^\xi\}}]$ .

Since in Region 5 we have  $1 - 2\xi > \frac{\xi}{\alpha} - \gamma$ , we get that  $N^{-(1-2\xi)} \times \hat{\beta}N^{-\gamma}\Omega_{AN^\xi}^*$  goes to 0 in probability. Hence, we only need to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N^{1-2\xi}} \log \hat{Z}_N^A = \sup_{-A \leq u \leq 0 \leq v \leq A} \left\{ -\hat{h}(v-u) - \bar{I}(u, v) \right\}$$

(there is no disorder anymore). But this is quite standard, since we have by Lemma A.3 that  $\bar{I}(u, v)$  is the rate function for the LDP for  $(N^{-\xi}M_N^-, N^{-\xi}M_N^+)$ , more precisely

$$(4.44) \quad -\bar{I}(u, v) = \lim_{N \rightarrow \infty} \frac{1}{N^{1-2\xi}} \log \mathbf{P}(M_N^- \geq uN^\xi; M_N^+ \leq vN^\xi).$$

This is enough to conclude thanks to Varadhan's lemma.  $\square$

Step 2. Next, we prove the following lemma.

**Lemma 4.8.** *In region  $R_5$ , for any constant  $T > 0$ , there is some  $A_T > 0$  such that for  $A \geq A_T$*

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\frac{1}{N^{1-2\xi}} \log Z_N^\omega(M_N^* > AN^\xi) > -T\right) = 0.$$

Combining this result with Lemma 4.7 readily yields Theorem 3.5.

*Proof.* We consider two cases: (i)  $\alpha \in (1, 2]$  and  $\zeta \in (-1, 1/2)$  or  $\alpha \in (0, 1)$  and  $\zeta \in (-1, 0]$ ; (ii)  $\alpha \in (0, 1)$  and  $\zeta \in (0, 1/2)$ . The strategy of proof is different for each case and we present them separately.

*Case (i).* We write  $Z_N^{\omega, >} := Z_N^\omega(M_N^* > AN^\xi)$ . Then

$$(4.45) \quad \begin{aligned} Z_N^{\omega, >} &\leq \sum_{k=1}^{\log_2 N^{1-\xi}} Z_N^\omega \left( M_N^* \in (2^{k-1} AN^\xi, 2^k AN^\xi] \right) \\ &\leq \sum_{k=1}^{\log_2 N^{1-\xi}} \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* - \hat{h} 2^{k-1} AN^{\xi-\zeta} \right). \end{aligned}$$

Recall that in Region  $R_5$  we have that  $\xi - \zeta = 1 - 2\xi$ . By union bound we obtain that

$$\begin{aligned} \mathbb{P} \left( Z_N^{\omega, >} \geq e^{-TN^{1-2\xi}} \right) &\leq \sum_{k=1}^{\log_2 N^{1-\xi}} \mathbb{P} \left( \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \right) e^{-\hat{h} 2^{k-1} AN^{\xi-\zeta}} \geq \frac{1}{2^k} e^{-TN^{\xi-\zeta}} \right) \\ &\leq \sum_{k=1}^{\log_2 N^{1-\xi}} \mathbb{P} \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \geq 2^{k-2} \hat{h} AN^{\xi-\zeta} \right), \end{aligned}$$

where the last inequality holds for sufficient large  $A$  (depending on  $\hat{h}, T$ ). Then by Lemma 3.7, we get that

$$\mathbb{P} \left( Z_N^{\omega, >} \geq e^{-TN^{1-2\xi}} \right) \leq C \sum_{k=1}^{\log_2 N^{1-\xi}} 2^{k(1-\alpha)} N^{\xi-\alpha(\xi-\zeta+\gamma)} \leq \begin{cases} C' N^{\xi-\alpha(\xi-\zeta+\gamma)} & \text{if } \alpha \in (1, 2], \\ C' N^{\alpha(\frac{1-\alpha}{\alpha}+\zeta-\gamma)} & \text{if } \alpha \in (0, 1). \end{cases}$$

To get that this upper bound goes to 0 when  $N \rightarrow \infty$ , we use that  $\xi - \zeta > \frac{\xi}{\alpha} - \gamma$  in the case  $\alpha \in (1, 2]$ , and that  $\gamma > \zeta + \frac{1-\alpha}{\alpha}$  for  $\zeta \leq 0$  in the case  $\alpha \in (0, 1)$ .

*Case (ii).* In that case, we have  $\zeta \in (0, \frac{1}{2})$  and  $\xi \in (0, \frac{1}{2})$ . Hence, we can write

$$(4.46) \quad Z_N^\omega(M_N^* > AN^\xi) = Z_N^\omega(M_N^* \in (AN^\xi, N^{1-\zeta}]) + Z_N^\omega(M_N^* \in (N^{1-\zeta}, N]).$$

For the first term on the right-hand side of (4.46), using that  $\xi - \zeta = 1 - 2\xi$ , we get by a union bound that

$$(4.47) \quad \begin{aligned} &\mathbb{P} \left( Z_N^\omega(M_N^* \in (AN^\xi, N^{1-\zeta}]) > \frac{1}{2} e^{-TN^{1-2\xi}} \right) \\ &\leq \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P} \left( Z_N^\omega(M_N^* \in (2^{k-1} AN^\xi, 2^k AN^\xi]) > \frac{1}{2^{k+1}} e^{-TN^{\xi-\zeta}} \right) \\ &\leq \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P} \left( \exp \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \right) e^{-\hat{h} 2^{k-1} AN^{\xi-\zeta}} \geq \frac{1}{2^{k+1}} e^{-TN^{\xi-\zeta}} \right) \\ &\leq \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} \mathbb{P} \left( \hat{\beta} N^{-\gamma} \Omega_{2^k AN^\xi}^* \geq \hat{h} 2^{k-2} AN^{\xi-\zeta} \right) \end{aligned}$$



where the last inequality holds for sufficient large  $A$  (depending on  $\hat{h}, T$ ). Using Lemma 3.7, we therefore get that the left-hand side term of (4.47) is bounded by a constant times

$$(4.48) \quad \sum_{k=1}^{\log_2 N^{1-\xi-\zeta}} 2^{k(1-\alpha)} N^{\xi(1-\alpha)+\alpha(\zeta-\gamma)} \leq C N^{1-\alpha+(2\alpha-1)\zeta-\alpha\gamma},$$

Since we have  $\gamma > \frac{(2\alpha-1)\zeta+1-\alpha}{\alpha}$  for  $\zeta \in (0, \frac{1}{2})$ , we get that (4.47) goes to 0 as  $N \rightarrow \infty$ .

For the second term on the right-hand side of (4.46), we use a union bound to get that

$$(4.49) \quad \begin{aligned} & \mathbb{P}\left(Z_N^\omega(M_N^* \in (N^{1-\zeta}, N]) > \frac{1}{2}e^{-TN^{\xi-\zeta}}\right) \\ & \leq \sum_{k=1}^{\log_2 N^\zeta} \mathbb{P}\left(Z_N^\omega(M_N^* \in (2^{-k}N, 2^{-k+1}N]) > \frac{1}{2^{k+1}}e^{-TN^{\xi-\zeta}}\right) \\ & \leq \sum_{k=1}^{\log_2 N^\zeta} \mathbb{P}\left(\exp\left(\hat{\beta}N^{-\gamma}\Omega_{2^{-k+1}N}^*\right)e^{-2^{-2k-1}N} > \frac{1}{2^{k+3}}e^{-TN^{\xi-\zeta}}\right), \end{aligned}$$

where we have use that  $\mathbf{P}(M_N^* > x) \leq 2\exp(x^2/2N)$  for the simple random walk. Now, since for all  $k \leq \log_2 N^\zeta$  we have  $2^{-2k}N \geq N^{1-2\zeta}$  and that  $1 - 2\zeta > \xi - \zeta$ , we get that for  $N$  large enough, the left-hand side of (4.49) is bounded by

$$\sum_{k=1}^{\log_2 N^\zeta} \mathbb{P}\left(\hat{\beta}N^{-\gamma}\Omega_{2^{-k+1}N}^* \geq 2^{-2k-2}N\right) \leq c\hat{\beta}^\alpha \sum_{k=1}^{\log_2 N^\zeta} 2^{k(2\alpha-1)}N^{1-\alpha(1+\gamma)},$$

where we used Lemma 3.7 for the last inequality. If  $\alpha \in (0, \frac{1}{2}]$ , this is bounded above by a constant times  $(\log N)N^{1-\alpha(1+\gamma)}$ , and this goes to 0 as  $N \rightarrow \infty$ , since  $\gamma > \frac{1-\alpha}{\alpha}$ . If  $\alpha \in (\frac{1}{2}, 1)$ , this is bounded above by a constant times  $N^{\zeta(2\alpha-1)+1-\alpha-\alpha\gamma}$ , which goes to 0 as  $N \rightarrow \infty$  since  $\gamma > \frac{(2\alpha-1)\zeta+1-\alpha}{\alpha}$  for  $\zeta \in (0, \frac{1}{2})$ . This concludes the proof of the lemma.  $\square$

**Transversal fluctuations.** Lemma 4.8 already shows that transversal fluctuations are at most of order  $N^\xi$ . On the other hand, Lemma 4.7 shows that for any  $\eta > 0$  we have that  $N^{1-2\xi} \log Z_N^\omega(M_N^* \leq \eta N^\xi)$  converges to  $\sup_{-\eta \leq u \leq 0 \leq v \leq \eta} \{-\hat{h}(v-u) - \bar{I}(u, v)\}$  in probability. Since the supremum is strictly smaller than  $(\hat{h}\pi)^{\frac{2}{3}} = \lim_{N \rightarrow \infty} N^{2\xi-1} \log Z_N^\omega$  if  $\eta < \frac{1}{2}\pi^{\frac{2}{3}}\hat{h}^{-\frac{1}{3}}$ , we get that  $\mathbf{P}_N^\omega(M_N^* \leq \eta N^\xi) \rightarrow 0$  for such  $\eta$ , which shows that transversal fluctuations are at least of order  $N^\xi$ .

**4.6. Region R6: proof of Theorem 3.6.** Note that in Region  $R_6$ , we have  $\zeta < (-1) \wedge \gamma$  if  $\alpha \in (1, 2]$  and  $\zeta < (-1) \wedge (\gamma + \frac{\alpha-1}{\alpha})$  if  $\alpha \in (0, 1)$ . Let us note that in all cases,  $\gamma > \zeta$ . We split  $Z_N^\omega$  in two parts

$$(4.50) \quad Z_N^\omega = Z_N^\omega(|\mathcal{R}_N| = 2) + Z_N^\omega(|\mathcal{R}_N| \geq 3).$$

It is clear that

$$Z_N^\omega(|\mathcal{R}_N| = 2) = e^{-2\hat{h}N^{-\zeta}} \left( e^{\hat{\beta}N^{-\gamma}(\omega_0+\omega_1)} 2^{-N} + e^{\hat{\beta}N^{-\gamma}(\omega_0+\omega_{-1})} 2^{-N} \right),$$

so that  $N^\zeta \log Z_N^\omega(|\mathcal{R}_N| = 2)$  converges in probability to  $-2\hat{h}$  (we use here that  $\zeta < \gamma$  and  $\zeta < -1$ ).

We now prove that  $N^\zeta \log Z_N^\omega (|\mathcal{R}_N| \geq 3)$  is strictly smaller than  $-2\hat{h}$  with  $\mathbb{P}$ -probability going to 1: this will imply that the second term in (4.50) is negligible compared to the first one, and as a by-product prove that  $\mathbf{P}_N^\omega (|\mathcal{R}_N| = 2)$  converges to 1 in probability.

We fix  $A$  large, and split

$$(4.51) \quad Z_N^\omega (|\mathcal{R}_N| \geq 3) = Z_N^\omega (|\mathcal{R}_N| \geq 3, M_N^* \leq A) + Z_N^\omega (M_N^* > A).$$

For the first term, we simply use the upper bound

$$Z_N^\omega (|\mathcal{R}_N| \geq 3, M_N^* \leq A) \leq e^{-3\hat{h}N^{-\zeta}} \exp\left(\hat{\beta}N^{-\gamma}\Omega_A^*\right).$$

Using the fact that  $\gamma > \zeta$ , we get that  $N^{\zeta-\gamma}\Omega_A^*$  goes to 0 in probability, and thus  $N^\zeta \log Z_N^\omega (|\mathcal{R}_N| \geq 3, M_N^* \leq A) \leq -\frac{5}{2}\hat{h}$  with probability going to 1.

For the second term in (4.51), we have

$$Z_N^\omega (M_N^* > A) = \sum_{k=1}^{\log_2 N} Z_N^\omega (M_N^* \in (2^{k-1}A, 2^k A]) \leq \sum_{k=1}^{\log_2 N} e^{-2^{k-1}A\hat{h}} \exp\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A}^*\right).$$

Then, as in the proof of Lemma 4.8, (see the proof of Case (i)), a union bound gives that for any  $T > 0$  (we will pick  $T = \frac{5}{2}\hat{h}$ ) we can choose  $A_T$  so that for any  $A > A_T$ ,

$$(4.52) \quad \begin{aligned} \mathbb{P}\left(Z_N(M_N^* > A) \geq e^{-TN^{-\zeta}}\right) &\leq \sum_{k=1}^{\log_2 N} \mathbb{P}\left(\hat{\beta}N^{-\gamma}\Omega_{2^k A}^* \geq 2^{k-2}\hat{h}AN^{-\zeta}\right) \\ &\leq c \sum_{k=1}^{\log_2 N} \hat{\beta}^\alpha \hat{h}^{-\alpha} A^{1-\alpha/2} 2^{k(1-\alpha)} N^{\alpha(\zeta-\gamma)}, \end{aligned}$$

where we used Lemma 3.7 for the last inequality. If  $\alpha \in (1, 2]$  this is bounded above by a constant times  $N^{\alpha(\zeta-\gamma)}$ : this goes to 0 as  $N \rightarrow \infty$ , since  $\gamma > \zeta$ . When  $\alpha \in (0, 1)$ , this is bounded above by a constant times  $N^{1-\alpha+\alpha(\zeta-\gamma)}$ : this goes to 0 as  $N \rightarrow \infty$ , since  $\zeta < \gamma + \frac{\alpha-1}{\alpha}$ . Therefore, the right-hand-side of (4.52) converges to zero as  $N \rightarrow \infty$ , which shows that  $N^\zeta \log Z_N^\omega (M_N^* > A) \leq -T$  with probability going to 1, and concludes the proof.  $\square$

## APPENDIX A. TECHNICAL ESTIMATES

**A.1. Estimates on deviation probabilities.** Let us present here some results on large deviation probabilities for the simple random walk that are needed throughout the paper. Recall the notations  $M_N^- := \min_{0 \leq n \leq N} S_n$  and  $M_N^+ := \max_{0 \leq n \leq N} S_n$ .

*Stretching.* Our first lemma deals with the super-diffusive case: we estimate the probability that  $M_N^+ \geq vN^\xi$  and  $M_N^- \leq uN^\xi$  when  $\xi \in (\frac{1}{2}, 1)$ , for  $u \leq 0 \leq v$ . The one-sided large deviation are classical, see e.g. [16, Ch.6]: we get that if  $\xi \in (\frac{1}{2}, 1)$

$$\lim_{N \rightarrow \infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P}(M_N^+ \geq vN^\xi) = \frac{1}{2}v^2.$$

The case where both the minimum and maximum are required to have large deviations is an easy extension of the result, and we omit its proof (one simply needs to observe that the random walk must cover a distance at least  $(|u| \wedge |v| + v - u)N^\xi$ ).

**Lemma A.1.** *If  $\frac{1}{2} < \xi < 1$ , then for any  $u \leq 0 \leq v$  we have that*

$$(A.1) \quad \lim_{N \rightarrow \infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P} \left( M_N^- \leq uN^\xi; M_N^+ \geq vN^\xi \right) = \frac{1}{2} (|u| \wedge |v| + v - u)^2.$$

As an easy consequence of this lemma, we get that for any  $\delta > 0$ , for any  $u \leq 0 \leq v$ ,

$$(A.2) \quad \lim_{N \rightarrow \infty} -\frac{1}{N^{2\xi-1}} \log \mathbf{P} \left( M_N^- \in [u - \delta, u]N^\xi; M_N^+ \in [v, v + \delta]N^\xi \right) = \frac{1}{2} (|u| \wedge |v| + v - u)^2.$$

We also state the large deviation result in the case  $\xi = 1$  (it is not needed in this paper).

**Lemma A.2.** *For any  $u \leq 0 \leq v$ , we have that*

$$\lim_{N \rightarrow \infty} -\frac{1}{N} \log \mathbf{P} \left( M_N^- \leq uN; M_N^+ \geq vN \right) = J(|u| \wedge |v| + v - u),$$

where  $J : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is the LDP rate function for the simple random walk, that is  $J(t) = \frac{1}{2}(1+t) \log(1+t) + \frac{1}{2}(1-t) \log(1-t)$  if  $0 \leq t \leq 1$  and  $J(t) = +\infty$  if  $t > 1$ .

*Folding.* Our second lemma deals with the sub-diffusive case: we estimate the probability that  $M_N^+ \leq vN^\xi$  and  $M_N^- \geq uN^\xi$  when  $\xi \in (\frac{1}{2}, 1)$ , for  $u \leq 0 \leq v$ . The result follows from Donsker-Varadhan theory [15] (see also [11] in the case  $|u| = |v|$ ; Feller [16, Ch.6] gives an explicit formula)

**Lemma A.3.** *If  $0 < \xi < \frac{1}{2}$ , then for any  $u \leq 0 \leq v$  we have that*

$$(A.3) \quad \lim_{N \rightarrow \infty} -\frac{1}{N^{1-2\xi}} \log \mathbf{P} \left( M_N^- \geq uN^\xi; M_N^+ \leq vN^\xi \right) = \frac{\pi^2}{2} (v - u)^{-2}.$$

As an easy consequence of this lemma, we get that for any  $\delta > 0$  and any  $u \leq 0 \leq v$ ,

$$(A.4) \quad \lim_{N \rightarrow \infty} -\frac{1}{N^{1-2\xi}} \log \mathbf{P} \left( M_N^- \in [u, u + \delta]N^\xi; M_N^+ \in [v - \delta, v]N^\xi \right) = \frac{\pi^2}{2} (v - u)^{-2}.$$

**A.2. Proof of Lemma 3.7.** First of all, notice that the bound is trivial if  $\ell T^{-\alpha} > 1$ : we assume that  $\ell T^{-\alpha} \leq 1$ . Using Etemadi's inequality we get that

$$\mathbb{P}(\Omega_\ell^* > T) \leq 3 \max_{k \in \{1, \dots, \ell\}} \mathbb{P}(|\Omega_k^+| > \frac{1}{6}T) + 3 \max_{k \in \{1, \dots, \ell\}} \mathbb{P}(|\Omega_k^-| > \frac{1}{6}T).$$

Let us detail the bound for  $\mathbb{P}(|\Omega_k^+| > \frac{1}{6}T)$ , the same bound holds for  $\mathbb{P}(|\Omega_k^-| > \frac{1}{6}T)$ . The case  $\alpha = 2$  is a consequence of Kolmogorov's maximal inequality, and the case  $\alpha \in (0, 2)$  ( $\alpha \neq 1$ ) follows from the so-called *big-jump* (or one-jump) behavior. Let us give an easy proof: define  $\bar{\omega}_x := \omega_x \mathbb{1}_{\{|\omega_x| \leq T\}}$ , so that

$$\begin{aligned} \mathbb{P}(|\Omega_k^+| > \frac{1}{6}T) &\leq \mathbb{P}(\exists 0 \leq x \leq k, |\omega_x| > T) + \mathbb{P}\left(\left|\sum_{x=0}^k \bar{\omega}_x\right| > \frac{1}{6}T\right) \\ &\leq (k+1)\mathbb{P}(|\omega_0| > T) + \frac{36}{T^2} \left( (k+1)\mathbb{E}[(\bar{\omega}_0)^2] + k(k+1)\mathbb{E}[\bar{\omega}_0^2] \right), \end{aligned}$$

where we used a union bound for the first term and Markov's inequality (applied to  $(\sum_{x=0}^k \bar{\omega}_x)^2$ ) for the second. Now, the first term is clearly bounded by a constant times  $kT^{-\alpha}$  thanks to Assumption 1. For the second term, we use again Assumption 1, to get that if  $\alpha \in (0, 1) \cup (1, 2)$ ,  $\mathbb{E}[(\bar{\omega}_0)^2] \leq cT^{2-\alpha}$  and  $\mathbb{E}[\bar{\omega}_0] \leq cT^{1-\alpha}$  (when  $\alpha \in (1, 2)$  we use for this last inequality that  $\mathbb{E}[\omega_0] = 0$ ). Therefore, we end up with the bound

$$\mathbb{P}(|\Omega_\ell^+| > \frac{1}{6}T) \leq c\ell T^{-\alpha} + c\ell^2 T^{-2\alpha} \leq 2c\ell T^{-\alpha}.$$

where we used that  $\ell T^{-\alpha} \leq 1$ . □

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