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**A SUFFICIENT CONDITION FOR $(\theta^N)_N$ TO HAVE A
DISTRIBUTION MODULO ONE, WHEN θ IS IN $\mathbb{F}_2(X)$**

J-M. Deshouillers

*Université de Bordeaux, Bordeaux INP et CNRS, UMR 5251, 33405 TALENCE
Cedex, France*

`jean-marc.deshouillers@math.u-bordeaux.fr`

R. Thangadurai

*Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhansi, Allahbad,
211019, India*

`thanga@hri.res.in`

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Abstract

Let θ be a given element in $\mathbb{F}_2(X)$. In this article, we give a sufficient condition for the sequence $(\theta^n)_{n \geq 0}$ to have a distribution modulo 1.

1. Introduction

Many number theoretic problems have natural counterparts in the domain of function fields. We are concerned here with the question of the distribution modulo 1 of the powers of an element $\theta \in \mathbb{F}_q(X)$, the counterpart of the question of the distribution modulo 1 of $(3/2)^n$. The reader will notice that the method and result of this note can easily be extended to the case of an algebraic element over $\mathbb{F}_q(X)$; since our result is only partial, we see no interest in stating it in a more general form, as long as generalisation does not bring a better understanding.

Let us start by giving some definition. We denote $\mathbb{F}_q((X))$ by the set of all the Laurent expansions

$$\eta = \sum_{k \geq -k_0} \varepsilon_k(\eta) X^k, \quad k_0 \in \mathbb{N} \text{ and } \varepsilon_k(\eta) \in \mathbb{F}_q.$$

It is a field which contains $\mathbb{F}_q(X)$.

Definition 1 (Densities). *Let $\theta \in \mathbb{F}_q((X))$. We say that the sequence $(\theta^n)_{n \geq 0}$ has a distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence*

$$\mathcal{N}(\theta, b_L) = \{n \in \mathbb{N} : (\varepsilon_1(\theta^n), \dots, \varepsilon_L(\theta^n)) = b_L\} \tag{1}$$

has an asymptotic density, i.e., if the following limit

$$\lim_{x \rightarrow \infty} \frac{1}{x} \text{Card} \{n \leq x : n \in \mathcal{N}(\theta, b_L)\} \tag{2}$$

exists.

Similarly, we say that the sequence $(\theta^n)_{n \geq 0}$ has a logarithmic distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence $\mathcal{N}(\theta, b_L)$ has a logarithmic density, i.e., if the following limit

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \in \mathcal{N}(\theta, b_L), n \leq x} \frac{1}{n} \tag{3}$$

exists.

Houndonougbo proved in [5] the existence of the distribution modulo 1 of the sequence $(\theta^n)_{n \geq 0}$, where $\theta = P(X)^\mu + 1/P(X)^\nu$ for positive integers μ and ν and P a non constant polynomial in $\mathbb{F}_q[X]$; he indeed showed more, namely that the sequence $\mathcal{N}(\theta, (0, 0, \dots, 0))$ has density 1. Deshouillers proved in [4] that the sequence $(\theta^n)_{n \geq 0}$ also has a distribution modulo 1 when $\theta = P(X)/X^\nu$, i.e. when the Laurent expansion of θ is finite: he showed that for any b_L the sequence $\mathcal{N}(\theta, b_L)$ is q -automatic and that it has a density. Allouche and Deshouillers proved in [1] that for any θ algebraic over $\mathbb{F}_q(X)$, the sequence $\mathcal{N}(\theta, b_L)$ is q -automatic; by a general result of Cobham [3], this implies that the sequence $(\theta^n)_{n \geq 0}$ has a logarithmic distribution modulo 1, but the existence of a distribution modulo 1 is still an open question.

Our aim is to provide a criterion which is sufficient to prove the existence of the distribution modulo 1 of $(\theta^n)_{n \geq 0}$. We made some ten hand numerical experiments on θ with an infinite Laurent expansion; in the cases we considered, this criterion turned out to be satisfied and indeed led to a limit distribution which is the Dirac measure at 0.

From now on, we assume that $q = 2$ and that $\theta \in \mathbb{F}_2(X)$. In order to describe the 2-automata which generate the sequences $\mathcal{N}(\theta, b_L)$ we follow [1] and first introduce some definition.

For $n \geq 0$, we consider the Laurent expansions

$$\theta^n = \sum_{k \geq -k_0(n)} \varepsilon_k(\theta^n) X^k.$$

Since θ is rational, its expansion is ultimately periodic and the following definition makes sense.

Definition 2 (Parameter). *The parameter of an element θ in $\mathbb{F}_2(X)$ is the smallest even positive integer T satisfying*

$$\varepsilon_{-h}(\theta) = 0 \text{ if } h \geq T$$

and

$$\varepsilon_{h+T}(\theta) = \varepsilon_h(\theta) \text{ if } h \geq T.$$

From now on we denote T as the parameter of θ . For $n \geq 0$ and $K, L \geq 0$, we define

$$\mathcal{B}(n, K, L) = (\varepsilon_{-K}(\theta^n), \varepsilon_{-K+1}(\theta^n), \dots, \varepsilon_{L-1}(\theta^n), \varepsilon_L(\theta^n)) \in \mathbb{F}_2^{L+1+T}, \tag{4}$$

$$\mathcal{B}(n, L) = \mathcal{B}(n, T, L) \tag{5}$$

and

$$\mathcal{M}(n) = (m(n, 0), \dots, m(n, T-1)) \in \mathbb{F}_2^T, \tag{6}$$

$$\text{where, for } t \in \mathbb{Z}: m(n, t) = \sum_{h=0}^{\infty} \varepsilon_{-t-hT}(\theta^n) \in \mathbb{F}_2,$$

which is well defined since this sum contains only a finite number of non-zero elements.

The key ingredient in [1] is the fact that, for $L \geq T$, the two $(2T + L + 1)$ -tuples $(\mathcal{M}(2n), \mathcal{B}(2n, L))$ and $(\mathcal{M}(2n + 1), \mathcal{B}(2n + 1, L))$ only depend on $(\mathcal{M}(n), \mathcal{B}(n, L))$. Since [1] is not easily available, we give here a proof of this fact.

Proposition 1. *Let $L \geq T$; there exist two maps ρ and τ from \mathbb{F}_2^{2T+L+1} into itself such that for every $n \geq 0$ one has,*

$$(\mathcal{M}(2n), \mathcal{B}(2n, L)) = \rho((\mathcal{M}(n), \mathcal{B}(n, L))), \tag{7}$$

$$(\mathcal{M}(2n + 1), \mathcal{B}(2n + 1, L)) = \tau((\mathcal{M}(n), \mathcal{B}(n, L))). \tag{8}$$

Proof. We first observe that

$$\forall k \in \mathbb{Z}: \varepsilon_{2k}(\theta^{2n}) = \varepsilon_k(\theta^n), \tag{9}$$

$$\forall k \in \mathbb{Z}: \varepsilon_{2k+1}(\theta^{2n}) = 0, \tag{10}$$

$$\text{For } t \text{ even in } [0, T]: m(2n, t) = m(n, t/2) + m(n, t/2 + T/2), \tag{11}$$

$$\text{For } t \text{ odd in } [0, T]: m(2n, t) = 0. \tag{12}$$

This implies that as soon as one knows $\mathcal{B}(n, L)$, all the coefficients of θ^{2n} with indices between $-2T - 1$ and $2L + 1$ are known: so are $\mathcal{B}(2n, 2T + 1, 2L + 1)$ and *a fortiori* $\mathcal{B}(2n, L)$. Similarly, the knowledge of $\mathcal{M}(n)$ implies that of $\mathcal{M}(2n)$. This implies (7).

We noticed that the knowledge of $\mathcal{B}(n, L)$ gives us that of $\mathcal{B}(2n, 2T + 1, 2L + 1)$. Let us show that the knowledge of $\mathcal{B}(n, L)$ and of $\mathcal{M}(n)$ gives us the knowledge of

$$m(2n, t) = \sum_{h=0}^{\infty} \varepsilon_{-t-hT}(\theta^{2n}) \text{ for } t \in [-2L - T - 1, 3T + 1]. \tag{13}$$

Indeed, if $t \in [0, T - 1]$, then $m(2n, t)$ is an element of $\mathcal{M}(2n)$; Otherwise $m(2n, t)$ is an element of $\mathcal{M}(2n)$ which is modified by a few terms which belong to $\mathcal{B}(2n, 2T + 1, 2L + T + 1)$, *e.g.* $m(2n, -2) = \varepsilon_2(\theta^{2n}) + m(2n, T - 2)$, $m(2n, T) = -\varepsilon_0(\theta^{2n}) + m(2n, 0)$.

For any k we have

$$\begin{aligned} \varepsilon_k(\theta^{2n+1}) &= \sum_{r=-\infty}^{+\infty} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) = \sum_{r=0}^{+\infty} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) \\ &= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) + \sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) \\ &= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) + \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{\substack{r \geq 2T \\ r \equiv \nu \pmod T}} \varepsilon_{k+T-r}(\theta^{2n}) \\ &= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta) \varepsilon_{k+T-r}(\theta^{2n}) + \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) m(2n, T + \nu - k). \end{aligned}$$

The last relation shows that as soon as one knows $\mathcal{B}(n, L)$ and $\mathcal{M}(n, L)$ (and the digits of θ with indices between $-T$ and $2T$ which are our initial data), we have enough information to determine $\mathcal{B}(2n + 1, L)$ (cf. (13) and the fact that for $k \in [-T, L]$ we have $T + \nu - k \in [T - L, 3T - 1] \subset [-2L - T - 1, 3T + 1]$).

We finally study $\mathcal{M}(2n + 1)$. Let $t \in [0, T - 1]$. Reasoning as above, we have.

$$\begin{aligned} m(2n + 1, t) &= \sum_{h=0}^{\infty} \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta) \varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\ &\quad + \sum_{h=0}^{\infty} \sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta) \varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\ &= S_1 + S_2, \text{ say.} \end{aligned}$$

By interchanging the sums in the first term on the right hand side, we see that it

is equal to

$$S_1 = \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta) m(2n, t+r-T). \tag{14}$$

Since $r \in [0, 2T - 1]$ and $t \in [0, T - 1]$, we have $-T \leq t+r-T \leq 2T-2$ and thus the term in (14) is known as soon as $\mathcal{M}(n)$ is known. Let us look at the second term. We have

$$\begin{aligned} S_2 &= \sum_{h=0}^{\infty} \sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta) \varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\ &= \sum_{h=0}^{\infty} \sum_{s=T}^{\infty} \varepsilon_s(\theta) \varepsilon_{-t-s-hT}(\theta^{2n}) \\ &= \sum_{h=0}^{\infty} \sum_{\nu=0}^{T-1} \sum_{\substack{r>T \\ s \equiv \nu \pmod T}} \varepsilon_s(\theta) \varepsilon_{-t-s-hT}(\theta^{2n}). \end{aligned}$$

We use the periodicity of the digits of θ and write $s = \nu + T + kT$. We have

$$\begin{aligned} S_2 &= \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \varepsilon_{-t-\nu-T-(h+k)T}(\theta^{2n}) \\ &= \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{\ell=0}^{\infty} \left(\sum_{\substack{h \geq 0, k \geq 0 \\ h+k=\ell}} 1 \right) \varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}). \end{aligned}$$

It is enough to consider each inside sum over ℓ . We notice that if $t + \nu$ is odd, then all the terms $\varepsilon_{-t-\nu-T-\ell T}(\theta^{2n})$ are zero and so is the sum of those terms over ℓ . We also notice that the sum $\sum_{\substack{h \geq 0, k \geq 0 \\ h+k=\ell}} 1$ is equal to 1 when ℓ is even and to 0 when ℓ is odd. Combining those two remarks and writing $\ell = 2\lambda$, we have, when $t + \nu$ is even

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \left(\sum_{\substack{h \geq 0, k \geq 0 \\ h+k=\ell}} 1 \right) \varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}) = \sum_{\lambda=0}^{\infty} \varepsilon_{-(\nu+t+T)/2-\lambda T}(\theta^n) \\ &= m(n, (\nu+t+T)/2); \end{aligned}$$

when $\nu + t + T \leq 2T$, then $m(n, (\nu + t + T)/2)$ is an element in $\mathcal{M}(n)$; otherwise, we write $m(n, (\nu + t + T)/2) = m(n, (\nu + t - T)/2) - \varepsilon_{(T-\nu-t)/2}(\theta^n)$, which the difference of an element of $\mathcal{M}(n)$ and an element of $\mathcal{B}(n, L)$.

Thus S_2 is also known as soon as $\mathcal{B}(n, L)$ and $\mathcal{M}(n)$ are known. This ends the proof of Proposition 1. □

This permits to build a directed graph Γ_L with edges indexed by 0 or 1 as follows. We first consider the set of vertices

$$\mathcal{R}_L = \{(\mathcal{M}(n), \mathcal{B}(n, L)) : n \geq 0\}. \tag{15}$$

We then build two edges starting from $r_L \in \mathcal{R}_L$ depending on $\varepsilon = 0$ or $\varepsilon = 1$ in the following way: since $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$ for some integer $n \geq 0$, for each $\varepsilon \in \{0, 1\}$, we define the state

$$\delta_L(r_L, \varepsilon) := (\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L)) \tag{16}$$

and the edge $(r_L, \delta_L(r_L, \varepsilon))$, which are well defined. The above-mentioned observation of [1] implies that our definition is indeed independent of the choice of n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$. The reader who needs to refresh her/his knowledge on *Automatic Sequences* is strongly recommended to visit [2]¹, specially subsections 4.1 and 5.1. The refreshment being performed, it is not difficult to see that the sequence $\mathcal{N}(\theta, b_L)$ is 2-automatic: it is recognized by the automaton $\mathcal{A}(b_L) = \{\mathcal{R}_L, \{0, 1\}, \delta_L, r_{0,L}, F(b_L)\}$, where \mathcal{R}_L and δ_L have already been defined,

$$r_{0,L} = (\mathcal{M}(0), \mathcal{B}(0, L))$$

and

$F(b_L)$ is the set of those $r \in \mathcal{R}_L$, the last L components of which are b_L .

It will be convenient to extend the function δ_L to a new function still called δ_L , defined over the words w on $\{0, 1\}$, satisfying

$$\forall r \in \mathcal{R}_L, \forall \varepsilon \in \{0, 1\}, \forall w \in \{0, 1\}^*: \delta_L(r, \emptyset) = r, \delta_L(r, \varepsilon w) = \delta_L(\delta_L(r, \varepsilon), w).$$

Let us recall a criterion on the graph Γ_L which insures that the sequence $\mathcal{N}(\theta, b_L)$ has a density.

In the directed graph Γ_L , we say that two vertices r and s are *equivalent* if there is a directed path leading from r to s and a directed path leading from s to r ; this permits to consider *equivalent classes*, which form a tree, which leads to the notion of *final class*; we finally say that an equivalent class is *regular* if there exists an integer ℓ such that for any pair (r, s) , there is a directed path of length ℓ leading from r to s . We have the following criterion.

Proposition 2. *Let $L \geq T$ be a given integer and $b_L \in \mathbb{F}_2^L$ be a given vector. If the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class and if this class is regular, then, the sequence $\mathcal{N}(\theta, b_L)$ has an asymptotic density.*

¹Thanks, Jeff, for this invaluable monography... and for the rest!

Theorem 8.4.7, p. 272 of [2] deals with the special case where all the states constitute a single final regular equivalence class. Proposition 2 is a mere extension of this result where the key tool is Perron-Frobenius theorem.

Back to our question, we notice that if $u \leq v$, then $\mathcal{N}(\theta, b_u)$ is a finite union of sequences $\mathcal{N}(\theta, b_v)$, and it is thus enough for our purpose to consider the sequences $\mathcal{N}(\theta, b_L)$ for all sufficiently large values of L .

We wish to prove here that if the criterion applies to an automaton $\mathcal{A}(b_L)$, it also applies to the automaton $\mathcal{A}(b_{L+1})$, which leads to the following

Theorem 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single final class and if this final class is regular, then the sequence $(\theta^n)_{n \geq 0}$ has a distribution modulo 1.*

We remark that in the statement of Theorem 1, the automaton $\mathcal{A}(b_T)$ only occurs through its graph Γ_T , which itself depends on $\{\mathcal{R}_T, \delta_T, r_{0,T}\}$ but not on $F(b_T)$.

Corollary 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single equivalence class, then the sequence $(\theta^n)_{n \geq 0}$ has a distribution modulo 1.*

2. Connection between the automata $\mathcal{A}(b_L)$ and $\mathcal{A}(b_{L+1})$

Our key tool to understand the connection between the automata $\mathcal{A}(b_{L+1})$ and $\mathcal{A}(b_L)$ is a natural map from \mathcal{R}_{L+1} onto \mathcal{R}_L ; we define it and give its main properties in the following proposition.

Proposition 3. *Let θ and T be as in Theorem 1 and let $L \geq T$. The map σ_L from \mathcal{R}_{L+1} to \mathbb{F}_2^{2T+L+1} , defined by suppressing the last component of an element, has the following properties*

$$\sigma_L(\mathcal{R}_{L+1}) = \mathcal{R}_L, \tag{17}$$

$$\forall r \in \mathcal{R}_L, \text{ at most two elements } s \in \mathcal{R}_{L+1} \text{ such that } \sigma_L(s) = r, \tag{18}$$

$$\sigma_L(r_{0,L+1}) = r_{0,L}, \tag{19}$$

$$\forall r \in \mathcal{R}_{L+1}, \text{ and } \varepsilon \in \{0, 1\}, \text{ we have } : \sigma_L(\delta_{L+1}(r, \varepsilon)) = \delta_L(\sigma_L(r), \varepsilon). \tag{20}$$

Proof. By definition, cf. (15), for a state r_{L+1} in \mathcal{R}_{L+1} , there exists an integer n such that r_{L+1} is the $(2T+L+2)$ -tuple $(\mathcal{M}(n), \mathcal{B}(n, L+1))$. If we suppress its last component we get the $(2T+L+1)$ -tuple $(\mathcal{M}(n), \mathcal{B}(n, L))$, which is an element of \mathcal{R}_L . In the other direction, if we start with an element r_L in \mathcal{R}_L , there exists an n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$, and for this n we have $r_L = \sigma_L((\mathcal{M}(n), \mathcal{B}(n, L+1)))$.

Thus, the suppression of the last component defines the map σ_L which satisfies the Property (17). Property (18) comes from the fact that the last component of an element of \mathcal{R}_{L+1} belongs to $\{0, 1\}$. We have $\sigma_L(r_{0,L+1}) = \sigma_L((\mathcal{M}(0), \mathcal{B}(0, L+1))) = (\mathcal{M}(0), \mathcal{B}(0, L)) = r_{0,L}$, which proves Property (19). Finally, let $r \in \mathcal{R}_{L+1}$, there exists an integer n such that $r = (\mathcal{M}(n), \mathcal{B}(n, L+1))$; for $\varepsilon \in \{0, 1\}$, we have, cf. (16), $\delta_{L+1}(r, \varepsilon) := (\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L+1))$ and so $\sigma_L(\delta_{L+1}(r, \varepsilon)) = \sigma_L((\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L+1))) = (\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L)) = \delta_L(\sigma_L(r), \varepsilon)$, which proves (20). \square

We say that $r_{L+1} \in \mathcal{R}_{L+1}$ is *sitting above* r_L for some $r_L \in \mathcal{R}_L$, if $\sigma(r_{L+1}) = r_L$.

Claim 1. *Assume that \mathcal{C}_L is the unique final class in Γ_L and let \mathcal{C}_{L+1} be one of the final classes of Γ_{L+1} . Then any element of \mathcal{C}_{L+1} is sitting above some element of \mathcal{C}_L .*

Proof. Let $r_{L+1} \in \mathcal{C}_{L+1}$ be a given element. Then we have, $\sigma_L(r_{L+1}) \in \mathcal{R}_L$. If $\sigma_L(r_{L+1}) \in \mathcal{C}_L$, then we are done. If not, then there exists a word $w \in \{0, 1\}^*$ such that $\delta_L(\sigma_L(r_{L+1}), w) \in \mathcal{C}_L$. Since r_{L+1} belongs to \mathcal{C}_{L+1} which is a final class, the state $\delta_{L+1}(r_{L+1}, w)$ belongs to \mathcal{C}_{L+1} . Therefore, there exists a word $w' \in \{0, 1\}^*$ such that $\delta_{L+1}(r_{L+1}, ww') = r_{L+1}$. Thus, by the definition of \mathcal{C}_L , we see that $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. Since r_{L+1} is sitting above $\sigma_L(r_{L+1})$, by (20) we conclude that $r_{L+1} = \delta_{L+1}(r_{L+1}, ww')$ is sitting above $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. \square

Now, we look at a converse of Claim 1.

Claim 2. *Assume that \mathcal{C}_L is the unique final class in Γ_L and let \mathcal{C}_{L+1} be one of the final classes of Γ_{L+1} . For any element $r \in \mathcal{C}_L$, there is an element $s \in \mathcal{C}_{L+1}$ which is sitting above r .*

Proof. Let r be an element in \mathcal{C}_L and t an element in \mathcal{C}_{L+1} . There exists a word $w \in \{0, 1\}^*$ such that $\delta_L(\sigma_L(t), w) \in \mathcal{C}_L$. Since $r \in \mathcal{C}_L$, there exists a word w' such that $\delta_L(\sigma_L(t), ww') = r$. Now, we look at $\delta_{L+1}(t, ww')$. Since $t \in \mathcal{C}_{L+1}$, we conclude that $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$. Since t is sitting above $\sigma_L(t)$, by (20), we have $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$ is sitting above r . \square

Claim 3. *Assume that \mathcal{C}_L is the unique final class in Γ_L . If \mathcal{C}_{L+1} is a final class in Γ_{L+1} , then $|\mathcal{C}_{L+1}| \geq |\mathcal{C}_L|$.*

Proof. If $r \neq s$ are two different elements in \mathcal{C}_L , then, by Claim 2, there exist two elements r', s' of \mathcal{C}_{L+1} such that r' is sitting above r and s' is sitting above s . Since $r \neq s$, by the definition, we see that $r' \neq s'$. Hence the claim. \square

Claim 4. *Assume that \mathcal{C}_L is the unique final class in Γ_L . Then there can be at most two distinct final classes in Γ_{L+1} .*

Proof. We first remark that two classes are distinct if and only if they are disjoint; now, the claim simply follows from Claim 3 and (18): each final class has at least $|\mathcal{C}_L|$ elements and for any element r in \mathcal{C}_L we can have at most two elements of \mathcal{R}_{L+1} sitting above r . \square

Claim 5. *Assume that \mathcal{C}_L is the unique final class in Γ_L . Suppose that $\mathcal{C}_{L+1}^{(1)}$ and $\mathcal{C}_{L+1}^{(2)}$ are two distinct final classes in Γ_{L+1} sitting above \mathcal{C}_L . They are disjoint and for every element $r \in \mathcal{C}_L$, there is exactly one element $r_1 \in \mathcal{C}_{L+1}^{(1)}$ and one element $r_2 \in \mathcal{C}_{L+1}^{(2)}$ which are sitting above r . Moreover, for any $(r_1, r_2) \in \mathcal{C}_{L+1}^{(1)} \times \mathcal{C}_{L+1}^{(2)}$ and for any word $w \in \{0, 1\}^*$, we have*

$$\delta_{L+1}(r_1, w) \neq \delta_{L+1}(r_2, w). \tag{21}$$

Proof. Since the two classes $\mathcal{C}_{L+1}^{(1)}$ and $\mathcal{C}_{L+1}^{(2)}$ are distinct, they are disjoint. By Claim 2, above each element $r \in \mathcal{C}_L$, there exist $r_1 \in \mathcal{C}_{L+1}^{(1)}$ and $r_2 \in \mathcal{C}_{L+1}^{(2)}$ sitting above r ; since the two classes are disjoint, we have $r_1 \neq r_2$. By (18), this implies that above r , there can be only one element from each of the $\mathcal{C}_{L+1}^{(i)}$. The last assertion follows the fact that for any word w , the element $\delta_{L+1}(r_i, w)$ is in $\mathcal{C}_{L+1}^{(i)}$. \square

3. Proof of Theorem 1

By Proposition 2, it is enough to prove that for any $L \geq T$ and any b_L , the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class, and this class is regular. Let us recall that Γ_L is independent of b_L . We shall prove our assertion by induction on L .

The assumption of Theorem 1 is simply the case $L = T$: the graph Γ_T has a single final class and this class is regular.

Let us assume that for some $L \geq T$, the graph Γ_L has a single final class and this class is regular; let \mathcal{C}_L be this class.

We first prove that the graph Γ_{L+1} has a single final class. By Claim 1, any single final class of Γ_{L+1} is sitting above \mathcal{C}_L ; thus by Claim 4, there are at most two final classes in Γ_{L+1} . If we have indeed two distinct final classes, then we can apply Claim 5: let r be an element in \mathcal{C}_L : there exist two elements r_1 and r_2 which are sitting above r and which belong to the two different classes above \mathcal{C}_L . Choose an integer h such that $2^h > L + 1$ and consider a word w consisting of h zeroes. By (20), the elements $\delta_{L+1}(r_i, w)$ are sitting above $\delta_L(r, w)$ for $i = 1$ and 2 . This means that they differ at most by their last digit. Let n_1 and n_2 be two integers such that

$$r_i = (\mathcal{M}(n_i), \mathcal{B}(n_i, L + 1)) \text{ for } i = 1, 2.$$

Then, we have

$$\begin{aligned} \delta_{L+1}(r_i, w) &= (\mathcal{M}(n_i 2^h), \mathcal{B}(n_i 2^h, L+1)) \\ &= (\mathcal{M}(n_i 2^h), \mathcal{B}(n_i 2^h, L) \circ \varepsilon_{L+1}(\theta^{n_i 2^h})) \end{aligned}$$

for $i = 1, 2$, where the symbol \circ represents the concatenation. Since $2^h > L + 1$, we have

$$\varepsilon_{L+1}(\theta^{n_1 2^h}) = \varepsilon_{L+1}(\theta^{n_2 2^h}) = 0,$$

which implies

$$\delta_{L+1}(r_1, w) = \delta_{L+1}(r_2, w),$$

a contradiction to Claim 5. Hence, there is only one final class in the graph Γ_{T+1} .

Let \mathcal{C}_{L+1} be the unique final class in Γ_{L+1} . It remains to show that this class is regular.

If $|\mathcal{C}_{L+1}| = |\mathcal{C}_L|$, then, by Claim 2, for every $r_L \in \mathcal{C}_L$, there is exactly only one $r_{L+1} \in \mathcal{C}_{L+1}$ such that r_{L+1} is sitting above r_L and conversely. Therefore, by (20), $\delta_{L+1}(r_{L+1}, w)$ is sitting above $\delta_L(r_L, w)$ for every word w . Since \mathcal{C}_L is regular, this implies that \mathcal{C}_{L+1} is also regular.

Suppose that $|\mathcal{C}_{L+1}| > |\mathcal{C}_L|$. Then, there exists r in \mathcal{C}_L such that the two elements $r \circ 0$ and $r \circ 1$ are in \mathcal{C}_{L+1} . Choose an integer h such that $2^h > L + 1$ and let w be the word consisting of h zeroes. By the above argument, we have

$$\delta_{L+1}(r \circ 0, w) = \delta_{L+1}(r \circ 1, w),$$

and we denote this element by s . Since \mathcal{C}_L is regular, there exists an integer K such that for any $k \geq K$, there is a word w_k of length k satisfying

$$\delta_L(\sigma_L(s), w_k) = r;$$

thus $\delta_{L+1}(s, w_k)$ is either $r \circ 0$ or $r \circ 1$. But in either case, we have $\delta_{L+1}(s, w_k w) = s$, so that for any $\ell \geq K + h$ there is a path of length ℓ which connects s to s . Since \mathcal{C}_{L+1} is an equivalent class, any element u can be connected to any element v by a path of length exactly $2|\mathcal{C}_{L+1}| + K + h$, which implies that \mathcal{C}_{L+1} is regular. Theorem 1 is proved.

□

Proof of Corollary 1. Since the graph of $\mathcal{A}(b_T)$ has unique class, say, \mathcal{C}_T , it is the final class. Choose an integer h such that $2^h > T$. Let w be a word consisting only h zeroes. Then $\delta_T(r_{0,T}, w) = (\mathcal{M}(2^h), \mathcal{B}(2^h, T)) = r_h$, say. Therefore there exists a word w' such that $\delta_T(r_h, w') = r_{0,T}$. Thus, we have

$$\delta_T(r_{0,T}, ww') = r_{0,T} \text{ with } |ww'| = K, \text{ say.}$$

Since it is an equivalent class, any element u can be connected to any other element v by a path of length $K + 2|\mathcal{C}_T|$. Hence it is regular. Therefore, by Theorem 1, we get the corollary. \square

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