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# Multiplicity-induced-dominancy for delay-differential equations of retarded type

Guilherme Mazanti<sup>\*†</sup>, Islam Boussaada<sup>\*†</sup>, Silviu-Iulian Niculescu<sup>\*</sup>

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## Abstract

An important question of ongoing interest for linear time-delay systems is to provide conditions on its parameters guaranteeing exponential stability of solutions. Recent works have explored spectral techniques to show that, for some delay-differential equations of retarded type of low order, spectral values of maximal multiplicity are dominant, and hence determine the asymptotic behavior of the system, a property known as multiplicity-induced-dominancy. This work further explores such a property and shows its validity for general linear delay-differential equations of retarded type and arbitrary order with a single delay, exploiting for that purpose an interesting link between characteristic functions with a root of maximal multiplicity and Kummer's confluent hypergeometric functions.

**Keywords.** Time-delay equations, multiplicity-induced-dominancy, stability analysis, confluent hypergeometric functions, spectral methods, root assignment.

**Notation.** In this paper,  $\mathbb{N}^*$  denotes the set of positive integers and  $\mathbb{N} = \mathbb{N}^* \cup \{0\}$ . The set of all integers is denoted by  $\mathbb{Z}$  and, for  $a, b \in \mathbb{R}$ , we denote  $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ , with the convention that  $[a, b] = \emptyset$  if  $a > b$ . For a complex number  $s$ ,  $\operatorname{Re} s$  and  $\operatorname{Im} s$  denote its real and imaginary parts, respectively. Given  $k, n \in \mathbb{N}$  with  $k \leq n$ , the binomial coefficient  $\binom{n}{k}$  is defined as  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  and this notation is extended to  $k, n \in \mathbb{Z}$  by setting  $\binom{n}{k} = 0$  when  $n < 0$ ,  $k < 0$ , or  $k > n$ .

We find it useful in this paper to consider that the indices of rows and columns of matrices start from 0. More precisely, given  $n, m \in \mathbb{N}^*$ , an  $n \times m$  matrix  $A$  is described by its coefficients  $a_{ij}$  for integers  $i, j$  with  $0 \leq i < n$  and  $0 \leq j < m$ . This non-standard convention simplifies several expressions in the paper.

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# 1 Introduction

This paper is interested in the asymptotic behavior of the generic delay differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0y(t) + \alpha_{n-1}y^{(n-1)}(t - \tau) + \cdots + \alpha_0y(t - \tau) = 0, \quad (1.1)$$

where the unknown function  $y$  is real-valued,  $n$  is a positive integer,  $a_k, \alpha_k \in \mathbb{R}$  for  $k \in \{0, \dots, n-1\}$  are constant coefficients, and  $\tau > 0$  is a delay. Equation (1.1) is a delay differential equation of *retarded* type since the derivative of highest order appears only in the non-delayed term  $y^{(n)}(t)$ .

Systems and equations with time delays have found numerous applications in a wide range of scientific and technological domains, such as in biology, chemistry, economics, physics, or engineering, in which time delays are often used as simplified models for finite-speed propagation of mass, energy, or information. Due to their applications and the challenging mathematical problems arising in their analysis, they have been extensively studied in the scientific literature by researchers from several fields, in particular since the 1950s and 1960s. We refer to [3, 18, 19, 25, 32, 34] for more details on time-delay systems and their applications.

A motivation for considering (1.1) comes from the study of linear control systems with a delayed feedback under the form

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0y(t) = u(t - \tau), \quad (1.2)$$

where  $u$  is the control input, typically chosen in such a way that (1.2) behaves in some prescribed manner. In the absence of the delay  $\tau$ , an usual choice is  $u(t) = -\alpha_{n-1}y^{(n-1)}(t) - \cdots - \alpha_0y(t)$ , in which case (1.2) becomes

$$y^{(n)}(t) + (a_{n-1} + \alpha_{n-1})y^{(n-1)}(t) + \cdots + (a_0 + \alpha_0)y(t) = 0, \quad (1.3)$$

and hence, by a suitable choice of the coefficients  $\alpha_0, \dots, \alpha_{n-1}$ , one may choose the roots of the characteristic equation of (1.3) and hence the asymptotic behavior of its solutions. However, control systems often operate in the presence of delays, primarily due to the time it takes to acquire the information needed for decision-making, to create control decisions and to execute these decisions [32]. Equation (1.1) can be seen as the counterpart of (1.3) in the presence of the delay  $\tau$ .

The stability analysis of time-delay systems has attracted much research effort and is an active field [1, 13, 14, 18, 20, 21, 25, 32]. One of its main difficulties is that, contrarily to the delay-free situation where Routh–Hurwitz stability criterion is available [2], there is no simple known criterion for determining the asymptotic stability of a general linear time-delay system based only on its coefficients and delays. The investigation of conditions on coefficients and delays guaranteeing asymptotic stability of solutions is a question of ongoing interest, see for instance [18, 23].

In the absence of delays, stability of linear systems and equations such as (1.3) can be addressed by spectral methods by considering the corresponding characteristic polynomial, whose complex roots determine the asymptotic behavior of solutions of the system. For systems with delays, spectral methods can also be used to understand the asymptotic behavior of solutions by considering the roots of some characteristic function (see, e.g., [3, 12, 15, 19, 25, 32, 34, 37]) which, for (1.1), is the function  $\Delta : \mathbb{C} \rightarrow \mathbb{C}$  defined for  $s \in \mathbb{C}$  by

$$\Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^{n-1} \alpha_k s^k. \quad (1.4)$$

More precisely, the exponential behavior of solutions of (1.1) is given by the real number  $\gamma_0 = \sup\{\operatorname{Re} s \mid s \in \mathbb{C}, \Delta(s) = 0\}$ , called the *spectral abscissa* of  $\Delta$ , in the sense that, for every  $\varepsilon > 0$ , there exists  $C > 0$  such that, for every solution  $y$  of (1.1), one has  $|y(t)| \leq Ce^{(\gamma_0 + \varepsilon)t} \max_{\theta \in [-\tau, 0]} |y(\theta)|$  [19, Chapter 1, Theorem 6.2]. Moreover, all solutions of (1.1) converge exponentially to 0 if and only if  $\gamma_0 < 0$ . An important difficulty in the analysis of the asymptotic behavior of (1.1) is that, contrarily to the situation for (1.3), the corresponding characteristic function  $\Delta$  has infinitely many roots.

The function  $\Delta$  is a particular case of a *quasipolynomial*. Quasipolynomials have been extensively studied due to their importance in the spectral analysis of time-delay systems [6, 17, 22, 26, 30, 33, 36]. The precise definition of a quasipolynomial is recalled in Section 2.1, in which we also provide some useful classical properties of this class of functions, including the fact that the multiplicity of a root of a quasipolynomial is bounded by some integer, called the *degree* of the quasipolynomial. In particular, according to Definition 2.2, the degree of  $\Delta$  is  $2n$ . Recent works such as [5, 6] have provided characterizations of multiple roots of quasipolynomials using approaches based on Birkhoff and Vandermonde matrices.

The spectral abscissa of  $\Delta$  is related to the notion of *dominant* roots, i.e., roots with the largest real part (see Definition 2.5). Generally speaking, dominant roots may not exist for a given function of a complex variable, but they always exist for functions of the form (1.4), as a consequence, for instance, of the fact that  $\Delta$  has finitely many roots on any vertical strip in the complex plane [19, Chapter 1, Lemma 4.1]. Notice also that exponential stability of (1.1) is equivalent to the dominant root of  $\Delta$  having negative real part.

It turns out that, for quasipolynomials, real roots of maximal multiplicity are often dominant, a property known as *multiplicity-induced-dominancy* (MID for short). MID has been shown to hold, for instance, in the case  $n = 1$ , proving dominance by introducing a factorization of  $\Delta$  in terms of an integral expression when it admits a root of multiplicity 2 [9]; in the case  $n = 2$  and  $\alpha_1 = 0$ , using also the same factorization technique [8]; and in the case  $n = 2$  and  $\alpha_1 \neq 0$ , using Cauchy's argument principle to prove dominance of the multiple root [7].

Another motivation for studying roots of high multiplicity is that, for delay-free systems, if the spectral abscissa admits a minimizer among a class of polynomials with an affine constraint on their coefficients, then one such a minimizer is a polynomial with a single root of maximal multiplicity (see [4, 11]). Similar properties have also been obtained for some time-delay systems in [24, 29, 35]. Hence, the interest in investigating multiple roots does not rely on the multiplicity itself, but rather on its connection with dominance of this root and the corresponding consequences for stability.

The aim of this paper is to extend previous results on multiplicity-induced-dominancy for low order single-delay systems from [7–9] to the general setting of linear single-delay differential equations of arbitrary order (1.1) by exploiting the integral factorization introduced in [9]. Our main result, Theorem 3.1, states that, given any  $s_0 \in \mathbb{R}$ , there exists a unique choice of  $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$  such that  $s_0$  is a root of multiplicity  $2n$  of  $\Delta$ , and that, under this choice,  $s_0$  is a strictly dominant root of  $\Delta$ , determining thus the asymptotic behavior of solutions of (1.1).

The strategy of our proof of Theorem 3.1 starts by a suitable classical change of variables allowing to treat only the case  $s_0 = 0$  and  $\tau = 1$  (see, for instance, [7]). The coefficients  $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$  ensuring that 0 is a root of multiplicity  $2n$  of  $\Delta$  are characterized as solutions to a linear system, which allows to prove their existence and uniqueness (note that this characterization can be seen as a particular case of that of [5]). The key part of the proof, concerning the dominance of the multiple root at 0, makes use

of a suitable factorization of  $\Delta$  in terms of an integral expression that turns out to be a particular confluent hypergeometric function.

The paper is organized as follows. Section 2 presents some preliminary material on quasipolynomials, confluent hypergeometric functions and binomial coefficients that shall be of use in the sequel of the paper. The main result of the paper is stated in Section 3, which also contains some of its consequences. The proof of the main result is carried out in Section 4.

## 2 Preliminaries and prerequisites

This section contains some preliminary results on quasipolynomials (Section 2.1), confluent hypergeometric functions (Section 2.2), and binomial coefficients (Section 2.3) which are used in the sequel of the paper. Before turning to the core of this section, we present the following result on the integral of the product of a polynomial and an exponential, which is rather simple but of crucial importance in the proof of our main result.

**Proposition 2.1.** *Let  $d \in \mathbb{N}$  and  $p$  be a polynomial of degree  $d$ . Then, for every  $z \in \mathbb{C} \setminus \{0\}$ ,*

$$\int_0^1 p(t)e^{-zt} dt = \sum_{k=0}^d \frac{p^{(k)}(0) - p^{(k)}(1)e^{-z}}{z^{k+1}}. \quad (2.1)$$

*Proof.* The proof is done by induction on  $d$ . If  $d = 0$ , then  $p$  is constant and one immediately verifies that (2.1) holds. Let now  $d \in \mathbb{N}$  be such that (2.1) holds and let  $p$  be a polynomial of degree  $d+1$ . Integrating by parts and using the fact that  $p'$  is a polynomial of degree  $d$ , one gets

$$\begin{aligned} \int_0^1 p(t)e^{-zt} dt &= \frac{p(0) - p(1)e^{-z}}{z} + \frac{1}{z} \int_0^1 p'(t)e^{-zt} dt \\ &= \frac{p(0) - p(1)e^{-z}}{z} + \frac{1}{z} \sum_{k=0}^d \frac{p^{(k+1)}(0) - p^{(k+1)}(1)e^{-z}}{z^{k+1}} \\ &= \sum_{k=0}^{d+1} \frac{p^{(k)}(0) - p^{(k)}(1)e^{-z}}{z^{k+1}}, \end{aligned}$$

establishing thus (2.1) by induction. □

### 2.1 Quasipolynomials

Let us start by recalling the classical definition of a quasipolynomial and its degree.

**Definition 2.2.** A *quasipolynomial*  $Q$  is an entire function  $Q : \mathbb{C} \rightarrow \mathbb{C}$  which can be written under the form

$$Q(s) = \sum_{k=0}^{\ell} P_k(s)e^{\lambda_k s}, \quad (2.2)$$

where  $\ell$  is a nonnegative integer,  $\lambda_0, \dots, \lambda_{\ell}$  are pairwise distinct real numbers, and, for  $k \in \{0, \dots, \ell\}$ ,  $P_k$  is a non-zero polynomial with complex coefficients of degree  $d_k \geq 0$ . The integer  $D = \ell + \sum_{k=0}^{\ell} d_k$  is called the *degree* of  $Q$ .

The main motivation for the study of quasipolynomials is that, when  $\lambda_0 = 0$  and  $\lambda_k < 0$  for  $k \in \{1, \dots, \ell\}$  in the above definition,  $Q$  is the characteristic function of a linear time-delay system with delays  $-\lambda_1, \dots, -\lambda_\ell$ . Contrarily to the case of polynomials, the degree of a quasipolynomial does not determine the number of its roots, which is infinite except in trivial cases. However, there does exist a link between the degree of a quasipolynomial and the number of its roots in horizontal strips of the complex plane, thanks to a classical result known as *Pólya–Szegő bound* (see [28, Problem 206.2]), which we state in the next proposition.

**Proposition 2.3.** *Let  $Q$  be a quasipolynomial of degree  $D$  given under the form (2.2),  $\alpha, \beta \in \mathbb{R}$  be such that  $\alpha \leq \beta$ , and  $\lambda_\delta = \max_{j,k \in \{0, \dots, \ell\}} \lambda_j - \lambda_k$ . Let  $m_{\alpha, \beta}$  denote the number of roots of  $Q$  contained in the set  $\{s \in \mathbb{C} \mid \alpha \leq \operatorname{Im} s \leq \beta\}$  counting multiplicities. Then*

$$\frac{\lambda_\delta(\beta - \alpha)}{2\pi} - D \leq m_{\alpha, \beta} \leq \frac{\lambda_\delta(\beta - \alpha)}{2\pi} + D.$$

Given a root  $s_0 \in \mathbb{C}$  of a quasipolynomial  $Q$ , one immediately obtains, by letting  $\beta = \alpha = \operatorname{Im} s_0$  in the statement of Proposition 2.3, the following link between the multiplicity of  $s_0$  and the degree of  $Q$ .

**Corollary 2.4.** *Let  $Q$  be a quasipolynomial of degree  $D$ . Then any root  $s_0 \in \mathbb{C}$  of  $Q$  has multiplicity at most  $D$ .*

Note that, since the quasipolynomial  $\Delta$  from (1.4) has degree  $2n$ , any of its roots has multiplicity at most  $2n$ . The main result of this paper, Theorem 3.1, proves that roots of the quasipolynomial  $\Delta$  from (1.4) with maximal multiplicity are necessarily dominant, in the sense of the following definition.

**Definition 2.5.** Let  $Q : \mathbb{C} \rightarrow \mathbb{C}$  and  $s_0 \in \mathbb{C}$  be such that  $Q(s_0) = 0$ . We say that  $s_0$  is a *dominant* (respectively, *strictly dominant*) root of  $Q$  if, for every  $s \in \mathbb{C} \setminus \{s_0\}$  such that  $Q(s) = 0$ , one has  $\operatorname{Re} s \leq \operatorname{Re} s_0$  (respectively,  $\operatorname{Re} s < \operatorname{Re} s_0$ ).

## 2.2 Confluent hypergeometric functions

As it will be proved in Section 4.3, when the quasipolynomial  $\Delta$  from (1.4) admits a root of maximal multiplicity  $2n$ , it can be factorized in terms of a confluent hypergeometric function. This family of special functions has been extensively studied in the literature (see, e.g., [10], [16, Chapter VI], [27, Chapter 13]). This section provides a brief presentation of the results that shall be of use in the sequel. We start by recalling the definition of Kummer's confluent hypergeometric functions used in this paper.

**Definition 2.6.** Let  $a, b \in \mathbb{C}$  and assume that  $b$  is not a nonpositive integer. *Kummer's confluent hypergeometric function*  $M(a, b, \cdot) : \mathbb{C} \rightarrow \mathbb{C}$  is the entire function defined for  $z \in \mathbb{C}$  by the series

$$M(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \quad (2.3)$$

where, for  $\alpha \in \mathbb{C}$  and  $k \in \mathbb{N}$ ,  $(\alpha)_k$  is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as  $(\alpha)_0 = 1$  and  $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$  for  $k \in \mathbb{N}$ .

**Remark 2.7.** Note that the series in (2.3) converges for every  $z \in \mathbb{C}$ . The function  $M(a, b, \cdot)$  satisfies *Kummer's differential equation*

$$z \frac{\partial^2 M}{\partial z^2}(a, b, z) + (b - z) \frac{\partial M}{\partial z}(a, b, z) - aM(a, b, z) = 0. \quad (2.4)$$

Other solutions of (2.4) are usually also called Kummer's confluent hypergeometric functions, but they shall not be used in this paper.

We shall need the following classical integral representation of  $M$ , which can be found, for instance, in [10, 16, 27].

**Proposition 2.8.** *Let  $a, b \in \mathbb{C}$  and assume that  $\operatorname{Re} b > \operatorname{Re} a > 0$ . Then*

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt, \quad (2.5)$$

where  $\Gamma$  denotes the Gamma function.

The main result on confluent hypergeometric functions used in this paper is the following one, proved in [38], on the location of the roots of some particular functions.

**Proposition 2.9.** *Let  $a \in \mathbb{R}$  be such that  $a > -\frac{1}{2}$ .*

- (a) *If  $z \in \mathbb{C}$  is such that  $M(a, 2a+1, z) = 0$ , then  $\operatorname{Re} z > 0$ .*
- (b) *If  $z \in \mathbb{C}$  is such that  $M(a+1, 2a+1, z) = 0$ , then  $\operatorname{Re} z < 0$ .*

## 2.3 Binomial coefficients

We present in this section some properties of binomial coefficients used in the sequel of the paper. Even though the proofs of most of such properties are straightforward, we present them in this section for the sake of completeness. We recall that, with the convention that  $\binom{k}{j} = 0$  whenever  $k < 0$ ,  $j < 0$ , or  $j > k$ , binomial coefficients satisfy the relation  $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$  for every  $(k, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ . The first property on binomial coefficients needed in this paper is the following identity.

**Proposition 2.10.** *Let  $j, k, \ell \in \mathbb{N}$  be such that  $j \leq k \leq \ell$ . Then*

$$\binom{k}{j} \binom{\ell}{k} = \binom{\ell}{j} \binom{\ell-j}{k-j} = \binom{\ell-k+j}{j} \binom{\ell}{\ell-k+j}.$$

*Proof.* One immediately computes

$$\binom{k}{j} \binom{\ell}{k} = \frac{k!\ell!}{(k-j)!(\ell-k)!j!k!} = \frac{\ell!(\ell-j)!}{(k-j)!(\ell-k)!j!(\ell-j)!} = \binom{\ell}{j} \binom{\ell-j}{\ell-k+j}$$

and

$$\begin{aligned} \binom{k}{j} \binom{\ell}{k} &= \frac{k!\ell!}{(k-j)!(\ell-k)!j!k!} \\ &= \frac{\ell!(\ell-k+j)!}{(k-j)!(\ell-k)!j!(\ell-k+j)!} = \binom{\ell-k+j}{j} \binom{\ell}{\ell-k+j}. \quad \square \end{aligned}$$

Using Proposition 2.10, one can prove the following summation formula for products of binomial coefficients with alternating signs.

**Proposition 2.11.** *Let  $j, k \in \mathbb{N}$  be such that  $j \leq k$ . Then*

$$\sum_{\ell=j}^k (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The case  $j = k$  follows from a straightforward computation. If  $j < k$ , then, using Proposition 2.10, we have

$$\begin{aligned} \sum_{\ell=j}^k (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} &= \binom{k}{j} \sum_{\ell=j}^k (-1)^{\ell-j} \binom{k-j}{\ell-j} \\ &= \binom{k}{j} \sum_{\ell=0}^{k-j} (-1)^{\ell} \binom{k-j}{\ell} = \binom{k}{j} (1 + (-1))^{k-j} = 0. \quad \square \end{aligned}$$

Another auxiliary result we shall need in this paper is the following, concerning the sum of part of a row of binomial coefficients with alternating signs.

**Proposition 2.12.** *Let  $k \in \mathbb{N}^*$  and  $\ell \in \llbracket 0, k \rrbracket$ . Then*

$$\sum_{j=0}^{\ell} (-1)^j \binom{k}{j} = (-1)^{\ell} \binom{k-1}{\ell}. \quad (2.6)$$

*Proof.* Let  $k \in \mathbb{N}^*$ . We prove (2.6) for  $\ell \in \llbracket 0, k \rrbracket$  by induction on  $\ell$ . Clearly, (2.6) is satisfied for  $\ell = 0$ , since both left- and right-hand sides of (2.6) are equal to 1 in this case. Assume now that  $\ell \in \llbracket 0, k-1 \rrbracket$  is such that (2.6) is satisfied. Then

$$\begin{aligned} \sum_{j=0}^{\ell+1} (-1)^j \binom{k}{j} &= (-1)^{\ell} \binom{k-1}{\ell} + (-1)^{\ell+1} \binom{k}{\ell+1} = (-1)^{\ell+1} \left[ \binom{k}{\ell+1} - \binom{k-1}{\ell} \right] \\ &= (-1)^{\ell+1} \binom{k-1}{\ell+1}, \end{aligned}$$

showing that (2.6) is satisfied with  $\ell$  replaced by  $\ell + 1$ . Hence, by induction, (2.6) holds for every  $\ell \in \llbracket 0, k \rrbracket$ .  $\square$

The following identity is also useful.

**Proposition 2.13.** *For  $j, k, n \in \mathbb{N}$ , let*

$$S_{j,k}^n = \sum_{\ell=0}^k (-1)^{\ell} \binom{n+k-j}{\ell} \binom{n+k-\ell}{n}. \quad (2.7)$$

*Then, for every  $k, n \in \mathbb{N}$  and  $j \in \llbracket 0, n \rrbracket$ , one has*

$$S_{j,k}^n = \binom{j}{k}. \quad (2.8)$$

*Proof.* If  $j \neq n + k + 1$ , then

$$\begin{aligned} S_{j,k}^n - S_{j-1,k}^n &= \sum_{\ell=0}^k (-1)^{\ell} \left[ \binom{n+k-j}{\ell} - \binom{n+k-j+1}{\ell} \right] \binom{n+k-\ell}{n} \\ &= \sum_{\ell=1}^k (-1)^{\ell-1} \binom{n+k-j}{\ell-1} \binom{n+k-\ell}{n} \\ &= \sum_{\ell=0}^{k-1} (-1)^{\ell} \binom{n+k-j}{\ell} \binom{n+k-\ell-1}{n} = S_{j-1,k-1}^n. \end{aligned}$$

For  $j \geq 1$ , using Proposition 2.10, we have

$$S_{j,j}^n = \sum_{\ell=0}^j (-1)^\ell \binom{n}{\ell} \binom{n+j-\ell}{n} = \sum_{\ell=0}^j (-1)^\ell \binom{j}{\ell} \binom{n+j-\ell}{j}.$$

For  $\ell \in \llbracket 0, j \rrbracket$ , let  $a_\ell = (-1)^\ell \binom{j}{\ell}$ ,  $b_\ell = \binom{n+j-\ell}{j}$ , and  $A_\ell = \sum_{m=0}^\ell a_m$ . By Proposition 2.12, one has  $A_\ell = (-1)^\ell \binom{j-1}{\ell}$ , and one immediately computes  $b_\ell - b_{\ell+1} = \binom{n+j-\ell-1}{j-1}$  for  $\ell \in \llbracket 0, j-1 \rrbracket$ . Noticing that  $S_{j,j}^n = \sum_{\ell=0}^j a_\ell b_\ell$ , one computes, using summation by parts (see, e.g., [31, Theorem 3.41]), that

$$S_{j,j}^n = \sum_{\ell=0}^{j-1} A_\ell (b_\ell - b_{\ell+1}) = \sum_{\ell=0}^{j-1} (-1)^\ell \binom{j-1}{\ell} \binom{n+j-\ell-1}{j-1} = S_{j-1,j-1}^n.$$

We also compute, for  $j \in \llbracket 0, n \rrbracket$ , that

$$S_{j,0}^n = \binom{n-j}{0} \binom{n}{n} = 1.$$

We have thus shown that

$$S_{j,k}^n = S_{j-1,k-1}^n + S_{j-1,k}^n, \quad \text{for } j \neq n+k+1, \quad (2.9a)$$

$$S_{j,j}^n = S_{j-1,j-1}^n, \quad \text{for } j \geq 1, \quad (2.9b)$$

$$S_{j,0}^n = 1, \quad \text{for } j \in \llbracket 0, n \rrbracket. \quad (2.9c)$$

In particular, from (2.9b) and (2.9c), one obtains that  $S_{j,j}^n = 1$  for every  $j, n \in \mathbb{N}$ . Together with (2.9a), one obtains that  $S_{j,j+1}^n = 0$  for every  $j, n \in \mathbb{N}$  and, using an immediate inductive argument and (2.9a), one obtains that  $S_{j,k}^n = 0$  for every  $j, k, n \in \mathbb{N}$  with  $k > j$ . Moreover, it also follows from (2.9b) and (2.9c) that  $S_{j,k}^n = \binom{j}{k}$  whenever  $n \in \mathbb{N}$ ,  $j \in \llbracket 0, n \rrbracket$ , and  $k \in \{0, j\}$ , and using (2.9a) and an immediate inductive argument, one obtains that this equality also holds for  $k \in \llbracket 0, j \rrbracket$ .  $\square$

The last identity we provide is the following sum of products of some binomial coefficients.

**Proposition 2.14.** *Let  $j, k \in \mathbb{N}$ . Then, for every  $\ell \in \llbracket 0, k \rrbracket$ , one has*

$$\binom{k}{j} = \sum_{m=0}^\ell \binom{\ell}{m} \binom{k-\ell}{j-m}. \quad (2.10)$$

*Proof.* The proof is done by induction on  $\ell$ . For  $\ell = 0$ , (2.10) holds trivially. Assume that  $\ell \in \llbracket 0, k-1 \rrbracket$  is such that (2.10) holds. Then

$$\begin{aligned} \binom{k}{j} &= \sum_{m=0}^\ell \binom{\ell}{m} \binom{k-\ell}{j-m} = \sum_{m=0}^\ell \binom{\ell}{m} \left[ \binom{k-\ell-1}{j-m-1} + \binom{k-\ell-1}{j-m} \right] \\ &= \sum_{m=0}^\ell \binom{\ell}{m} \binom{k-\ell-1}{j-m-1} + \sum_{m=0}^\ell \binom{\ell}{m} \binom{k-\ell-1}{j-m} \\ &= \sum_{m=1}^{\ell+1} \binom{\ell}{m-1} \binom{k-\ell-1}{j-m} + \sum_{m=0}^\ell \binom{\ell}{m} \binom{k-\ell-1}{j-m} \end{aligned}$$

$$= \sum_{m=0}^{\ell+1} \left[ \binom{\ell}{m-1} + \binom{\ell}{m} \right] \binom{k-\ell-1}{j-m} = \sum_{m=0}^{\ell+1} \binom{\ell+1}{m} \binom{k-(\ell+1)}{j-m},$$

showing that (2.10) also holds with  $\ell$  replaced by  $\ell + 1$ . Hence the result is established by induction.  $\square$

### 3 Statement of the main result

The main result we prove in this paper is the following characterization of roots of maximal multiplicity of  $\Delta$  and their dominance and the corresponding consequences for the stability of the trivial solution of (1.1).

**Theorem 3.1.** *Consider the quasipolynomial  $\Delta$  given by (1.4) and let  $s_0 \in \mathbb{R}$ .*

(a) *The number  $s_0$  is a root of multiplicity  $2n$  of  $\Delta$  if and only if, for every  $k \in \llbracket 0, n-1 \rrbracket$ ,*

$$\begin{cases} a_k = \binom{n}{k} (-s_0)^{n-k} + (-1)^{n-k} n! \sum_{j=k}^{n-1} \binom{j}{k} \binom{2n-j-1}{n-1} \frac{s_0^{j-k}}{j! \tau^{n-j}}, \\ \alpha_k = (-1)^{n-1} e^{s_0 \tau} \sum_{j=k}^{n-1} \frac{(-1)^{j-k} (2n-j-1)!}{k! (j-k)! (n-j-1)!} \frac{s_0^{j-k}}{\tau^{n-j}}. \end{cases} \quad (3.1)$$

(b) *If (3.1) is satisfied, then  $s_0$  is a strictly dominant root of  $\Delta$ .*

(c) *If (3.1) is satisfied, then the trivial solution of (1.1) is exponentially stable if and only if  $a_{n-1} > -\frac{n^2}{\tau}$ .*

**Remark 3.2.** Let  $s_0 \in \mathbb{R}$ ,  $\Delta$  be the quasipolynomial given by (1.4), and assume that the coefficients of  $\Delta$  are given by (3.1). Then, by considering the first equation in (3.1) with  $k = n-1$ , one obtains the simple relation between  $s_0$ ,  $\tau$ , and  $a_{n-1}$  given by

$$s_0 = -\frac{a_{n-1}}{n} - \frac{n}{\tau}. \quad (3.2)$$

**Corollary 3.3.** *Let  $s_0 \in \mathbb{R}$ ,  $\Delta$  be the quasipolynomial given by (1.4), and assume that the coefficients of  $\Delta$  are given by (3.1). Then*

$$s_0 = -\frac{a_{n-1}}{n} - \frac{n}{\tau}.$$

### 4 Proof of the main result

The proof of Theorem 3.1 consists in three steps: the normalization of the quasipolynomial  $\Delta$ , the establishment of the necessary and sufficient conditions guaranteeing the maximal multiplicity, and the proof of dominance of the multiple root with respect to the remaining spectrum.

## 4.1 Normalization of $\Delta$

The first step of the proof is to perform an affine change of variable in  $\Delta$  in order to write it in a normalized form, in which the desired multiple root  $s_0$  becomes 0 and the delay  $\tau$  becomes 1. The next lemma provides relations between the coefficients of  $\Delta$  and those of the quasipolynomial  $\tilde{\Delta}$  obtained after the change of variables.

**Lemma 4.1.** *Let  $s_0 \in \mathbb{R}$  and consider the quasipolynomial  $\tilde{\Delta} : \mathbb{C} \rightarrow \mathbb{C}$  obtained from  $\Delta$  by the change of variables  $z = \tau(s - s_0)$  and multiplication by  $\tau^n$ , i.e.,*

$$\tilde{\Delta}(z) = \tau^n \Delta\left(s_0 + \frac{z}{\tau}\right). \quad (4.1)$$

Then

$$\tilde{\Delta}(z) = z^n + \sum_{k=0}^{n-1} b_k z^k + e^{-z} \sum_{k=0}^{n-1} \beta_k z^k, \quad (4.2)$$

where, for  $k \in \llbracket 0, n-1 \rrbracket$ ,

$$\begin{cases} b_k = \binom{n}{k} \tau^{n-k} s_0^{n-k} + \tau^{n-k} \sum_{j=k}^{n-1} \binom{j}{k} s_0^{j-k} a_j, \\ \beta_k = \tau^{n-k} e^{-s_0 \tau} \sum_{j=k}^{n-1} \binom{j}{k} s_0^{j-k} \alpha_j. \end{cases} \quad (4.3)$$

*Proof.* To simplify the notations, let us define  $a_n = 1$ . Then the first equation in (4.3) can be written in a more compact manner as

$$b_k = \tau^{n-k} \sum_{j=k}^n \binom{j}{k} s_0^{j-k} a_j.$$

By (4.1), one has

$$\begin{aligned} \tilde{\Delta}(z) &= \tau^n \sum_{j=0}^n a_j \left(s_0 + \frac{z}{\tau}\right)^j + e^{-(s_0 + \frac{z}{\tau})\tau} \tau^n \sum_{j=0}^{n-1} \alpha_j \left(s_0 + \frac{z}{\tau}\right)^j \\ &= \sum_{j=0}^n a_j \tau^{n-j} (s_0 \tau + z)^j + e^{-s_0 \tau} e^{-z} \sum_{j=0}^{n-1} \alpha_j \tau^{n-j} (s_0 \tau + z)^j \\ &= \sum_{j=0}^n a_j \tau^{n-j} \sum_{k=0}^j \binom{j}{k} s_0^{j-k} \tau^{j-k} z^k + e^{-s_0 \tau} e^{-z} \sum_{j=0}^{n-1} \alpha_j \tau^{n-j} \sum_{k=0}^j \binom{j}{k} s_0^{j-k} \tau^{j-k} z^k \\ &= \sum_{k=0}^n \left( \tau^{n-k} \sum_{j=k}^n \binom{j}{k} s_0^{j-k} a_j \right) z^k + e^{-z} \sum_{k=0}^{n-1} \left( \tau^{n-k} e^{-s_0 \tau} \sum_{j=k}^{n-1} \binom{j}{k} s_0^{j-k} \alpha_j \right) z^k \\ &= z^n + \sum_{k=0}^{n-1} \left( \tau^{n-k} \sum_{j=k}^n \binom{j}{k} s_0^{j-k} a_j \right) z^k + e^{-z} \sum_{k=0}^{n-1} \left( \tau^{n-k} e^{-s_0 \tau} \sum_{j=k}^{n-1} \binom{j}{k} s_0^{j-k} \alpha_j \right) z^k, \end{aligned}$$

which is precisely (4.2) with coefficients given by (4.3).  $\square$

The relations between the coefficients  $b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_{n-1}$  and  $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_{n-1}$  can be expressed under matrix form as

$$b = Ta + v, \quad \beta = e^{-s_0 \tau} T\alpha,$$

where

$$b = \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix}, \quad a = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}, \quad v = \begin{pmatrix} \binom{n}{0} \tau^n s_0^n \\ \binom{n}{1} \tau^{n-1} s_0^{n-1} \\ \vdots \\ \binom{n}{n-1} \tau s_0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} \binom{0}{0} \tau^n & \binom{1}{0} \tau^n s_0 & \binom{2}{0} \tau^n s_0^2 & \cdots & \binom{n-2}{0} \tau^n s_0^{n-2} & \binom{n-1}{0} \tau^n s_0^{n-1} \\ 0 & \binom{1}{1} \tau^{n-1} & \binom{2}{1} \tau^{n-1} s_0 & \cdots & \binom{n-2}{1} \tau^{n-1} s_0^{n-3} & \binom{n-1}{1} \tau^{n-1} s_0^{n-2} \\ 0 & 0 & \binom{2}{2} \tau^{n-2} & \cdots & \binom{n-2}{2} \tau^{n-2} s_0^{n-4} & \binom{n-1}{2} \tau^{n-2} s_0^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-2}{n-2} \tau^2 & \binom{n-1}{n-2} \tau^2 s_0 \\ 0 & 0 & 0 & \cdots & 0 & \binom{n-1}{n-1} \tau \end{pmatrix}. \quad (4.4)$$

Noticing that  $T$  is invertible, one may thus express  $a$  and  $\alpha$  in terms of  $b$  and  $\beta$  as

$$a = T^{-1}(b - v), \quad \alpha = e^{s_0 \tau} T^{-1} \beta. \quad (4.5)$$

Our next result provides explicit expressions for (4.5).

**Lemma 4.2.** *Let  $\tau > 0$ ,  $s_0 \in \mathbb{R}$ , and  $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_{n-1}, b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_{n-1}$  be real numbers satisfying (4.3) for every  $k \in \llbracket 0, n-1 \rrbracket$ . Then, for  $k \in \llbracket 0, n-1 \rrbracket$ ,*

$$\begin{cases} a_k = \binom{n}{k} (-s_0)^{n-k} + \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{s_0^{j-k}}{\tau^{n-j}} b_j, \\ \alpha_k = e^{s_0 \tau} \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{s_0^{j-k}}{\tau^{n-j}} \beta_j. \end{cases} \quad (4.6)$$

*Proof.* Let  $T = (T_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$  be the matrix defined in (4.4) and  $S = (S_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$  be the matrix whose coefficients are given, for  $j, k \in \llbracket 0, n-1 \rrbracket$ , by

$$S_{j,k} = \begin{cases} 0 & \text{if } j > k, \\ (-1)^{k-j} \binom{k}{j} \frac{1}{\tau^{n-k}} s_0^{k-j} & \text{if } j \leq k. \end{cases} \quad (4.7)$$

We claim that  $S = T^{-1}$ . Indeed, let  $M = TS$  and write  $M = (M_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$ . Hence, for  $j, k \in \llbracket 0, n-1 \rrbracket$ , the coefficient  $M_{j,k}$  is given by  $M_{j,k} = \sum_{\ell=0}^{n-1} T_{j,\ell} S_{\ell,k}$ . Since  $T_{j,\ell} = 0$  for  $\ell < j$  and  $S_{\ell,k} = 0$  for  $\ell > k$ , one immediately obtains that  $M_{j,k} = 0$  for  $k < j$ . For  $k \geq j$ , one has

$$\begin{aligned} M_{j,k} &= \sum_{\ell=j}^k T_{j,\ell} S_{\ell,k} = \sum_{\ell=j}^k \binom{\ell}{j} \tau^{n-j} s_0^{\ell-j} (-1)^{k-\ell} \binom{k}{\ell} \frac{1}{\tau^{n-k}} s_0^{k-\ell} \\ &= (-\tau s_0)^{k-j} \sum_{\ell=j}^k (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} \end{aligned}$$

and it follows from Proposition 2.11 that  $M_{j,k} = 1$  if  $j = k$  and  $M_{j,k} = 0$  for  $k > j$ . Hence  $M = \text{Id}$ , proving that  $S = T^{-1}$ .

The expression for  $\alpha_k$  in (4.6) follows immediately from (4.5) and (4.7). Concerning the expression for  $a_k$  in (4.6), one obtains from (4.5) and (4.7), using also Proposition 2.11, that, for  $k \in \llbracket 0, n-1 \rrbracket$ ,

$$\begin{aligned}
a_k &= \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{1}{\tau^{n-j}} s_0^{j-k} \left( b_j - \binom{n}{j} \tau^{n-j} s_0^{n-j} \right) \\
&= \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{s_0^{j-k}}{\tau^{n-j}} b_j - s_0^{n-k} \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \binom{n}{j} \\
&= \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{s_0^{j-k}}{\tau^{n-j}} b_j - s_0^{n-k} \left[ \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} \binom{n}{j} - (-1)^{n-k} \binom{n}{k} \right] \\
&= \binom{n}{k} (-s_0)^{n-k} + \sum_{j=k}^{n-1} (-1)^{j-k} \binom{j}{k} \frac{s_0^{j-k}}{\tau^{n-j}} b_j. \quad \square
\end{aligned}$$

## 4.2 Root of maximal multiplicity

Now that we have established by Lemmas 4.1 and 4.2 a correspondence between the coefficients of  $\Delta$  and the normalized quasipolynomial  $\tilde{\Delta}$ , we provide necessary and sufficient conditions on the coefficients of  $\tilde{\Delta}$  in order for 0 to be a root of maximal multiplicity  $2n$ .

**Lemma 4.3.** *Let  $n \in \mathbb{N}^*$ ,  $b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{R}$ , and  $\tilde{\Delta}$  be the quasipolynomial given by (4.2). Then 0 is a root of multiplicity  $2n$  of  $\tilde{\Delta}$  if and only if, for every  $k \in \llbracket 0, n-1 \rrbracket$ , one has*

$$\begin{cases} b_k = (-1)^{n-k} \frac{n!}{k!} \binom{2n-k-1}{n-1}, \\ \beta_k = (-1)^{n-1} \frac{(2n-k-1)!}{k!(n-k-1)!}. \end{cases} \quad (4.8)$$

*Proof.* Since the degree of the quasipolynomial  $\tilde{\Delta}$  is  $2n$ , 0 is a root of multiplicity  $2n$  of  $\tilde{\Delta}$  if and only if  $\tilde{\Delta}^{(k)}(0) = 0$  for every  $k \in \llbracket 0, 2n-1 \rrbracket$ . Let  $P, Q : \mathbb{C} \rightarrow \mathbb{C}$  be the polynomials defined by

$$P(z) = z^n + \sum_{k=0}^{n-1} b_k z^k, \quad Q(z) = \sum_{k=0}^{n-1} \beta_k z^k.$$

Then  $\tilde{\Delta}(z) = P(z) + e^{-z} Q(z)$  for every  $z \in \mathbb{C}$ . Then, by an immediate inductive argument, one computes

$$\tilde{\Delta}^{(k)}(z) = P^{(k)}(z) + e^{-z} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} Q^{(j)}(z), \quad \forall z \in \mathbb{C}. \quad (4.9)$$

Using (4.9) and the fact that  $P$  and  $Q$  are polynomials of degree  $n$  and  $n-1$ , respectively,

with  $P^{(n)}(0) = n!$ , one obtains that 0 is a root of multiplicity  $2n$  of  $\tilde{\Delta}$  if and only if

$$\begin{cases} P^{(k)}(0) + \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} Q^{(j)}(0) = 0, & \forall k \in \llbracket 0, n-1 \rrbracket, \\ \sum_{j=0}^{n-1} (-1)^{n-j} \binom{n}{j} Q^{(j)}(0) = -n!, \\ \sum_{j=0}^{n-1} (-1)^{k-j} \binom{k}{j} Q^{(j)}(0) = 0, & \forall k \in \llbracket n+1, 2n-1 \rrbracket. \end{cases} \quad (4.10)$$

The  $2n$  equations in (4.10) form a linear system on the  $2n$  variables  $P^{(k)}(0), Q^{(k)}(0)$ ,  $k \in \llbracket 0, n-1 \rrbracket$ , which can be written in matrix form as

$$\begin{cases} p + Aq = 0, \\ Bq = f, \end{cases} \quad (4.11)$$

where

$$p = \begin{pmatrix} P^{(0)}(0) \\ \vdots \\ P^{(n-1)}(0) \end{pmatrix}, \quad q = \begin{pmatrix} Q^{(0)}(0) \\ \vdots \\ Q^{(n-1)}(0) \end{pmatrix}, \quad f = \begin{pmatrix} -n! \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ -\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-1} \binom{n-1}{0} & (-1)^{n-2} \binom{n-1}{1} & (-1)^{n-3} \binom{n-1}{2} & \cdots & \binom{n-1}{n-1} \end{pmatrix},$$

$$B = \begin{pmatrix} (-1)^n \binom{n}{0} & (-1)^{n-1} \binom{n}{1} & (-1)^{n-2} \binom{n}{2} & \cdots & -\binom{n}{n-1} \\ (-1)^{n+1} \binom{n+1}{0} & (-1)^n \binom{n+1}{1} & (-1)^{n-1} \binom{n+1}{2} & \cdots & \binom{n+1}{n-1} \\ (-1)^{n+2} \binom{n+2}{0} & (-1)^{n+1} \binom{n+2}{1} & (-1)^n \binom{n+2}{2} & \cdots & -\binom{n+2}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\binom{2n-1}{0} & \binom{2n-1}{1} & -\binom{2n-1}{2} & \cdots & (-1)^n \binom{2n-1}{n-1} \end{pmatrix}.$$

One has  $B = AC$ , where  $C = (C_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$  and  $C_{j,k} = (-1)^{n-k+j} \binom{n}{k-j}$  for  $j, k \in \llbracket 0, n-1 \rrbracket$ . Indeed, writing  $A = (A_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$  and  $B = (B_{j,k})_{j,k \in \llbracket 0, n-1 \rrbracket}$ , one computes, for  $j, k \in \llbracket 0, n-1 \rrbracket$ ,

$$\begin{aligned} \sum_{\ell=0}^{n-1} A_{j,\ell} C_{\ell,k} &= \sum_{\ell=0}^{n-1} (-1)^{j-\ell} \binom{j}{\ell} (-1)^{n-k+\ell} \binom{n}{k-\ell} \\ &= (-1)^{n-k+j} \sum_{\ell=0}^{n-1} \binom{j}{\ell} \binom{n}{k-\ell} = (-1)^{n-k+j} \binom{n+j}{k} = B_{j,k}, \end{aligned}$$

where we use Proposition 2.14. Notice that the factorization  $B = AC$  corresponds to the LU factorization of  $B$ . As a consequence of this factorization, one also obtains that  $\det B = (-1)^n$  and, in particular,  $B$  is invertible. Hence, (4.11) admits a unique solution  $(p, q) \in \mathbb{R}^{2n}$ .

For  $j', k', n' \in \mathbb{N}$ , let  $S_{j', k'}^{n'}$  be defined by (2.7). Solving the second equation in (4.11), one gets

$$Q^{(k)}(0) = (-1)^{n-1} n! \binom{2n-k-1}{n-k-1}, \quad \forall k \in \llbracket 0, n-1 \rrbracket. \quad (4.12)$$

Indeed, let  $\tilde{q} = (q_k)_{k \in \llbracket 0, n-1 \rrbracket}$  be defined by  $q_k = (-1)^{n-1} n! \binom{2n-k-1}{n-k-1}$  for  $k \in \llbracket 0, n-1 \rrbracket$ . Letting  $x = B\tilde{q}$  and writing  $x = (x_j)_{j \in \llbracket 0, n-1 \rrbracket}$ , we have, for  $j \in \llbracket 0, n-1 \rrbracket$ ,

$$\begin{aligned} x_j &= \sum_{k=0}^{n-1} B_{j,k} q_k = \sum_{k=0}^{n-1} (-1)^{n+j-k} \binom{n+j}{k} (-1)^{n-1} n! \binom{2n-k-1}{n-k-1} \\ &= -n! \sum_{k=0}^{n-1} (-1)^{j-k} \binom{n+j}{k} \binom{2n-k-1}{n-k-1} = (-1)^{j+1} n! S_{n-j-1, n-1}^n. \end{aligned}$$

Hence, by Proposition 2.13,

$$x_j = (-1)^{j+1} n! \binom{n-j-1}{n-j},$$

and thus  $x_0 = -n!$  and  $x_j = 0$  for  $j \in \llbracket 1, n-1 \rrbracket$ , from where one obtains that  $x = f$ . Hence  $q = \tilde{q}$  is a solution of the second equation of (4.11) and, since this equation admits a unique solution, one deduces that (4.12) holds.

One may now compute  $P^{(k)}(0)$  for  $k \in \llbracket 0, n-1 \rrbracket$  using the first equation of (4.11). We have

$$P^{(k)}(0) = - \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} Q^{(j)}(0) = (-1)^{n-k} n! \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{2n-j-1}{n}.$$

Using Propositions 2.14 and 2.13, we obtain

$$\begin{aligned} P^{(k)}(0) &= (-1)^{n-k} n! \sum_{j=0}^k (-1)^j \binom{k}{j} \sum_{\ell=0}^{k-j} \binom{k-j}{\ell} \binom{2n-k-1}{n-\ell} \\ &= (-1)^{n-k} n! \sum_{\ell=0}^k \left[ \sum_{j=0}^{k-\ell} (-1)^j \binom{k}{j} \binom{k-j}{\ell} \right] \binom{2n-k-1}{n-\ell} \\ &= (-1)^{n-k} n! \sum_{\ell=0}^k S_{0, k-\ell}^\ell \binom{2n-k-1}{n-\ell} = (-1)^{n-k} n! \binom{2n-k-1}{n-k}. \end{aligned} \quad (4.13)$$

Finally, (4.8) follows from (4.12) and (4.13) by noticing that  $P^{(k)}(0) = k! b_k$  and  $Q^{(k)}(0) = k! \beta_k$  for  $k \in \llbracket 0, n-1 \rrbracket$ .  $\square$

### 4.3 Factorization of the characteristic quasipolynomial and dominance of the multiple root

Conditions (4.8) from Lemma 4.3 characterize the fact that 0 is a root of multiplicity  $2n$  of the quasipolynomial  $\tilde{\Delta}$  defined by (4.2). It turns out that, under (4.8),  $\tilde{\Delta}$  can be factorized as the product of  $z^{2n}$  and an entire function expressed as an integral.

**Lemma 4.4.** *Let  $n \in \mathbb{N}^*$ ,  $b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{R}$  be given by (4.8), and  $\tilde{\Delta}$  be the quasipolynomial given by (4.2). Then, for every  $z \in \mathbb{C}$ ,*

$$\tilde{\Delta}(z) = \frac{z^{2n}}{(n-1)!} \int_0^1 t^{n-1} (1-t)^n e^{-zt} dt. \quad (4.14)$$

*Proof.* For  $z \in \mathbb{C} \setminus \{0\}$ , one has

$$\begin{aligned}
\tilde{\Delta}(z) &= z^n + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{n!}{k!} \binom{2n-k-1}{n-1} z^k + (-1)^{n-1} e^{-z} \sum_{k=0}^{n-1} \frac{(2n-k-1)!}{k!(n-k-1)!} z^k \\
&= \frac{z^{2n}}{(n-1)!} \left[ \frac{(n-1)!}{z^n} + \sum_{k=0}^{n-1} (-1)^{n-k} (n-1)! \frac{n!}{k!} \binom{2n-k-1}{n-1} \frac{1}{z^{2n-k}} \right. \\
&\quad \left. + (-1)^{n-1} e^{-z} \sum_{k=0}^{n-1} (n-1)! \frac{(2n-k-1)!}{k!(n-k-1)!} \frac{1}{z^{2n-k}} \right] \\
&= \frac{z^{2n}}{(n-1)!} \left[ \frac{(n-1)!}{z^n} + \sum_{k=n}^{2n-1} (-1)^{k-n+1} (n-1)! \frac{n!}{(2n-k-1)!} \binom{k}{n-1} \frac{1}{z^{k+1}} \right. \\
&\quad \left. + (-1)^{n-1} e^{-z} \sum_{k=n}^{2n-1} (n-1)! \frac{k!}{(2n-k-1)!(k-n)!} \frac{1}{z^{k+1}} \right] \\
&= \frac{z^{2n}}{(n-1)!} \left[ \sum_{k=n-1}^{2n-1} \frac{(-1)^{k-n+1} k! n!}{(2n-k-1)!(k-n+1)!} \frac{1}{z^{k+1}} \right. \\
&\quad \left. + \sum_{k=n}^{2n-1} \frac{(-1)^{n-1} k! (n-1)!}{(2n-k-1)!(k-n)!} \frac{e^{-z}}{z^{k+1}} \right]. \tag{4.15}
\end{aligned}$$

Let  $p$  be the polynomial given by  $p(t) = t^{n-1}(1-t)^n$ . One computes

$$\begin{aligned}
p(t) &= t^{n-1} \sum_{k=0}^n (-1)^k \binom{n}{k} t^k = \sum_{k=n-1}^{2n-1} (-1)^{k-n+1} \binom{n}{k-n+1} t^k \\
&= \sum_{k=n-1}^{2n-1} \frac{(-1)^{k-n+1} k! n!}{(2n-k-1)!(k-n+1)!} \frac{t^k}{k!},
\end{aligned}$$

and thus

$$p^{(k)}(0) = \begin{cases} \frac{(-1)^{k-n+1} k! n!}{(2n-k-1)!(k-n+1)!}, & \text{if } k \in \llbracket n-1, 2n-1 \rrbracket, \\ 0, & \text{otherwise.} \end{cases} \tag{4.16}$$

Similarly, one computes

$$\begin{aligned}
p(t) &= (-1)^n (t-1)^n ((t-1)+1)^{n-1} = (-1)^n (t-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (t-1)^k \\
&= (-1)^n \sum_{k=n}^{2n-1} \binom{n-1}{k-n} (t-1)^k = \sum_{k=n}^{2n-1} \frac{(-1)^n k! (n-1)!}{(2n-k-1)!(k-n)!} \frac{(t-1)^k}{k!},
\end{aligned}$$

and thus

$$p^{(k)}(1) = \begin{cases} \frac{(-1)^n k! (n-1)!}{(2n-k-1)!(k-n)!}, & \text{if } k \in \llbracket n, 2n-1 \rrbracket, \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}$$

Combining (4.15) with (4.16) and (4.17) and using Proposition 2.1, one gets, for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$\tilde{\Delta}(z) = \frac{z^{2n}}{(n-1)!} \sum_{k=0}^{2n-1} \frac{p^{(k)}(0) - p^{(k)}(1)e^{-z}}{z^{k+1}} = \frac{z^{2n}}{(n-1)!} \int_0^1 t^{n-1} (1-t)^n e^{-zt} dt.$$

Since (4.14) trivially holds for  $z = 0$ , one finally deduces that (4.14) holds for every  $z \in \mathbb{C}$ .  $\square$

The factorization (4.14) can also be written, thanks to (2.5), as

$$\tilde{\Delta}(z) = \frac{n!}{(2n)!} z^{2n} M(n, 2n+1, -z), \quad (4.18)$$

where  $M$  is Kummer's confluent hypergeometric function defined in (2.3). The next lemma uses properties of the roots of  $M$  in order to deduce that 0 is a dominant root of  $\tilde{\Delta}$ .

**Lemma 4.5.** *Let  $n \in \mathbb{N}^*$ ,  $b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{R}$  be given by (4.8), and  $\tilde{\Delta}$  be the quasipolynomial given by (4.2). Let  $z$  be a root of  $\tilde{\Delta}$  with  $z \neq 0$ . Then  $\operatorname{Re} z < 0$ .*

*Proof.* By Lemma 4.4,  $\tilde{\Delta}$  admits the factorization (4.18). Hence, if  $z$  is a root of  $\tilde{\Delta}$  with  $z \neq 0$ , then  $-z$  must be a root of  $M(n, 2n+1, \cdot)$ . It follows from Proposition 2.9 that  $\operatorname{Re}(-z) > 0$ , and thus  $\operatorname{Re}(z) < 0$ .  $\square$

## 4.4 Conclusion of the proof of Theorem 3.1

We may now use Lemmas 4.2, 4.3, and 4.5 to conclude the proof of Theorem 3.1.

*Proof of Theorem 3.1.* To prove (a), define  $\tilde{\Delta}$  from  $\Delta$  as in (4.1). One immediately verifies that  $s_0$  is a root of multiplicity  $2n$  of  $\Delta$  if and only if 0 is a root of multiplicity  $2n$  of  $\tilde{\Delta}$ . The result then follows from Lemmas 4.2 and 4.3.

Part (b) can be shown by noticing that, if  $s$  is a root of  $\Delta$  with  $s \neq s_0$ , then, by (4.1),  $z = \tau(s - s_0)$  is a root of  $\Delta$  with  $z \neq 0$ . Hence, by Lemma 4.5,  $\operatorname{Re}(\tau(s - s_0)) < 0$ , showing, since  $\tau > 0$ , that  $\operatorname{Re} s < \operatorname{Re} s_0$ .

Finally, (c) follows from (b), (3.2), and the fact that the trivial solution of (1.1) is exponentially stable if and only if its spectral abscissa is negative.  $\square$

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