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► **To cite this version:**

Billel Guelmame, Stéphane Junca, Didier Clamond, Robert L Pego. Global weak solutions of a Hamiltonian regularised Burgers equation. 2020. hal-02478872v1

HAL Id: hal-02478872

<https://hal.science/hal-02478872v1>

Preprint submitted on 14 Feb 2020 (v1), last revised 10 Mar 2022 (v2)

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GLOBAL WEAK SOLUTIONS OF A HAMILTONIAN REGULARISED BURGERS EQUATION

BILLEL GUELMAME, STÉPHANE JUNCA, DIDIER CLAMOND AND ROBERT L. PEGO

ABSTRACT. Inspired by a recent nondispersive conservative regularisation of the shallow water equations, a similar regularisation is proposed and studied here for the inviscid Burgers equation. The regularised equation is parametrised by a positive number ℓ , the inviscid Burgers equation corresponding to $\ell = 0$ and the Hunter–Saxton equation being formally obtained letting $\ell \rightarrow \infty$. The breakdown of local smooth solutions is demonstrated. The existence of two types of global weak solutions, conserving or dissipating an H^1 energy, is also studied. The built *dissipative* solution satisfies (uniformly with respect to ℓ) an Oleinik inequality, as do entropy solutions of the inviscid Burgers equation. The limit (up to a subsequence) of the dissipative solution when $\ell \rightarrow 0$ (respectively $\ell \rightarrow \infty$) satisfies the Burgers (resp. Hunter–Saxton) equation forced by an unknown remaining term. At least before the appearance of singularities, the limit satisfies the Burgers (resp. Hunter–Saxton) equation.

AMS Classification: 35B65; 35B44; 35Q35; 35L67.

Key words: Inviscid Burgers equation; regularisation; Hamiltonian; conservative and dissipative solutions; Oleinik inequality.

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Date: February 10, 2020.

1. INTRODUCTION

The dispersionless shallow water equations, also called the Saint-Venant equations, admit shock-wave solutions. Recently, a Hamiltonian regularisation of this system (rSV), has been proposed which approximates these discontinuous waves by less singular ones [10, 11]. The rSV system can be written

$$h_t + [hu]_x = 0, \quad (1a)$$

$$[hu]_t + \left[hu^2 + \frac{1}{2}gh^2 + \epsilon \mathcal{R}h^2 \right]_x = 0, \quad (1b)$$

$$\mathcal{R} \stackrel{\text{def}}{=} h \left(u_x^2 - u_{xt} - uu_{xx} \right) - g \left(hh_{xx} + \frac{1}{2}h_x^2 \right), \quad (1c)$$

where ϵ is a small positive parameter, h is the total water depth and u is the velocity. The classical Saint-Venant equations can be obtained letting $\epsilon \rightarrow 0$. This regularisation is Galilean invariant, non-dispersive, non-diffusive, and conserves energy for regular solutions. It also admits regularised shock-wave weak solutions which have the same wave speed and which dissipate energy at the same rate as shocks in the classical Saint-Venant (cSV) equations, [18].

Some mathematical results on rSV were obtained by Pu et al. [18] and Liu et al. [17], but several natural questions remain open, such as the existence of global weak solutions. In the present work we consider such questions for an analogous but simpler model equation, namely a Hamiltonian regularisation of the inviscid Burgers equation $u_t + uu_x = 0$. Indeed, the Burgers equation being scalar, it is more tractable than the rSV system of equations.

Inspired by the rSV and the dispersionless Camassa-Holm [8] equations, in section 2 we derive a regularised Burgers equation (rB) in the form

$$u_t + uu_x = \ell^2 (u_{txx} + 2u_x u_{xx} + uu_{xxx}), \quad (2)$$

where $\ell \geq 0$ is a parameter. The main purpose of the present paper is to study the existence of local smooth solutions, the blow-up time, global weak solutions of the regularised Burgers equation (2), and also to study the limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow +\infty$.

Some mathematical results on (2) are already known. For a generalisation of the Camassa-Holm equation [21], including rB as special case, the existence of local smooth solutions has been proved ([21], see also Theorem 1 below). The existence of global weak solutions in H^1 has also been proved using vanishing viscosity [20, 9]. Note that Bressan and Constantin [6, 7] have proved global weak solutions of the Camassa-Holm equation (8) in H^1 , using an equivalent semi-linear system.

In this paper, we rewrite (2) into an equivalent system, analogous to the treatment of the Camassa-Holm equation in [6, 7], without asking the initial datum to be in H^1 . We prove the existence of a so-called *conservative* [6] global weak solution (Theorem 3). This conservative solution conserves the energy, but it does not satisfy the Oleinik inequality, which is an important condition for entropic shock waves. To avoid this problem, the equivalent system is slightly modified in order to obtain another type of solutions called *dissipative* [7] (see Theorem 4 below), which satisfies an Oleinik inequality given in (85). This inequality is well known to obtain uniqueness for entropy solutions of the Burgers equation. The dissipative solutions of rB can also be called “entropy solutions” because they satisfy an Oleinik inequality. However, the uniqueness for the dissipative solutions of rB remains an open problem.

The equivalent system, and the Oleinik inequality, are used to obtain a uniform BV estimate independent of the parameter ℓ for the dissipative solutions (Lemma 1), which is a key point

to prove the compactness when $\ell \rightarrow 0$ and $\ell \rightarrow +\infty$. When $\ell \rightarrow 0$, a dissipative solution converges (up to a subsequence) to a function u that satisfies the Burgers equation with a remaining term (103) (see Theorem 5 below). If the remaining term is zero, then the entropy solutions of Burgers is recovered. We prove that this term is zero for smooth solutions of Burgers equations (see Proposition 3). However, the disappearance of the remaining term in general remains an open problem. Similar results are obtained when $\ell \rightarrow +\infty$, where the limit is a solution of the Hunter–Saxton equation, at least before the appearance of singularities¹ (Theorem 6 and Proposition 4 below).

This paper is organised as follow. A heuristic derivation of the rB equation is given in section 2. Section 3 is devoted to study the existence of local smooth solutions. In Section 4, proofs of existence of both global conservative and dissipative solutions are given. Section 5 studies the limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow +\infty$ for dissipative solutions. The optimality of the requirement that $u_x \in L^2_{loc}$ for weak solutions is shown in section 6, where we prove in particular that when a smooth solution breaks down, u_x may blow up in L^p_{loc} for all $p > 2$.

2. HEURISTIC DERIVATION OF A REGULARISED BURGERS EQUATION

In order to introduce a suitable regularisation of the inviscid Burgers equation with similar features as the rSV system (1), we note first that the rSV equations yield the momentum equation

$$u_t + u u_x + g h_x + \epsilon (h \mathcal{R}_x + 2 \mathcal{R} h_x) = 0. \quad (3)$$

When h is constant, this equation (with the definition of \mathcal{R}) becomes

$$u_t + u u_x = \ell^2 [u_{xxt} - u_x u_{xx} + u u_{xxx}], \quad (4)$$

where $\ell \stackrel{\text{def}}{=} h\sqrt{\epsilon} \geq 0$ is a constant characterising a length scale for the regularisation.

After the change of independent variables $(t, x) \rightarrow (t/\ell, x/\ell)$, which leaves the inviscid Burgers equation invariant, equation (4) becomes

$$u_t + u u_x = u_{xxt} - u_x u_{xx} + u u_{xxx}. \quad (5)$$

Equation (5) belongs to the three-parameter family (for $a, b, c \in \mathbb{R}$) of non-dispersive equations

$$u_t - u_{xxt} = a u u_x + b u_x u_{xx} + c u u_{xxx}. \quad (6)$$

In this family, we look for an equation that has Galilean invariance and conservation of energy (at least for smooth solutions). A famous equation in the family (6) is the dispersionless Camassa–Holm (CH) equation [8]

$$u_t + 3u u_x = u_{xxt} + 2u_x u_{xx} + u u_{xxx}. \quad (7)$$

This can be rewritten, by applying the inverse of the Helmholtz operator $1 - \partial_x^2$, as

$$u_t + u u_x + [\mathfrak{G} * (\frac{1}{2} u_x^2 + u^2)]_x = 0, \quad \mathfrak{G}(x) \stackrel{\text{def}}{=} \frac{1}{2} \exp(-|x|), \quad (8)$$

where $*$ denotes the convolution product. The family (6) covers other equations, such as the Degasperis–Procesi (DP) equations [12] and the Benjamin–Bona–Mahony (BBM) equation [1]. It is well known that the Camassa–Holm conserves the H^1 energy [8], but is not Galilean invariant.

In order to obtain a Galilean invariant regularisation of the Burgers equation, one must take $c = -a = 1$ in (6). The special case $b = 0$ was studied by Bhat and Fetecau [2, 3, 4], who

¹ "Singularity" is used here to describe the blow-up of derivatives which corresponds to shocks of the classical Burgers equation. Contrary to the Burgers case, the solutions of rB remains continuous at the singularities.

proved the existence of the solution and the convergence to weak solutions of the Burgers equation in the limit corresponding to $\ell \rightarrow 0$. The limit fails to satisfy the entropy condition for the Riemann problem with $u_{\text{left}} < u_{\text{right}}$ [4]. For this regularisation, no energy conservation equation is known.

In the present paper, we consider $c = -a = 1$ (to ensure Galilean invariance, as in [2]) and, in order to maintain conservation of the H^1 norm at least for smooth solutions, we take $b = 2$ (as in the Camassa–Holm equation). With this done, equation (6) becomes

$$u_t + u u_x = u_{txx} + 2 u_x u_{xx} + u u_{xxx}, \quad (9)$$

Introducing the scaling $(t, x) \mapsto (\ell t, \ell x)$, we obtain

$$u_t + u u_x = \ell^2 (u_{txx} + 2 u_x u_{xx} + u u_{xxx}), \quad (10)$$

that is a formal approximation of the Burgers equation for small ℓ . Equation (10) is the regularised Burgers (rB) equation studied in this paper.

It can be shown that the equation (10) has Hamiltonian and Lagrangian structure (we omit the details), and that smooth solutions satisfy the following energy conservation law:

$$\left[\frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[\frac{1}{3} u^3 - \ell^2 u^2 u_{xx} - \ell^2 u u_{xt} \right]_x = 0. \quad (11)$$

Applying the inverse Helmholtz operator $(1 - \ell^2 \partial_x^2)^{-1}$, the rB equation (10) can be rewritten

$$u_t + u u_x + \ell^2 P_x = 0, \quad P \stackrel{\text{def}}{=} \mathfrak{G} * \frac{1}{2} u_x^2 \geq 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (12)$$

to be compared with the Camassa–Holm equation in its form (8). Differentiating (12) with respect to x , and using that $P - \ell^2 P_{xx} = \frac{1}{2} u_x^2$, one obtains the Riccati-like equation

$$[u_t + u u_x]_x + P = \frac{1}{2} u_x^2. \quad (13)$$

Note that P goes formally to zero as $\ell \rightarrow +\infty$, whence one obtains the Hunter–Saxton (HS) equation [14, 15]

$$[u_t + u u_x]_x = \frac{1}{2} u_x^2. \quad (14)$$

Note also that by taking $\ell \rightarrow +\infty$ formally in (10), we obtain the derivative of (14) with respect of x .

In this section, we have heuristically derived a regularised Burgers equation, by imposing the important physical requirements of Galilean invariance and energy conservation. We have also related this equation with well-known equations. In the rest of the paper, we perform a rigorous mathematical investigation of solutions of this regularised Burgers equation.

3. EXISTENCE AND BREAKDOWN OF SMOOTH SOLUTIONS

This section is devoted to showing the local existence and breakdown of smooth solutions for the Cauchy problem (12) with $u(0, x) = u_0(x)$. The form (12) of the regularised Burgers equation is more convenient for studying smooth solutions than (10), because it involves fewer derivatives.

Usually, one needs an equation for u_x to study the life span of smooth solutions. Equation (13) can be written

$$u_{xt} + \frac{1}{2} u_x^2 + u u_{xx} + P = 0. \quad (15)$$

Multiplying (12) by u and multiplying (15) by $\ell^2 u_x$, we obtain

$$\left[\frac{1}{2} u^2 \right]_t + \left[\frac{1}{3} u^3 + \ell^2 u P \right]_x = \ell^2 u_x P, \quad (16)$$

$$\left[\frac{1}{2} \ell^2 u_x^2 \right]_t + \left[\frac{1}{2} \ell^2 u u_x^2 \right]_x = -\ell^2 u_x P, \quad (17)$$

which implies an energy conservation law for smooth solutions; i.e., we have the (conservative) energy equation

$$\left[\frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[\frac{1}{3} u^3 + \ell^2 u P + \frac{1}{2} \ell^2 u u_x^2 \right]_x = 0. \quad (18)$$

For a class of equations including rB as special case, Yin [22, 21] has proven the following local existence result.

Theorem 1 (Yin [22, 21]). *For an initial datum $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$, there exists a maximal time $T^* > 0$ (independent of s) and a unique solution $u \in \mathcal{C}([0, T^*[, H^s)$ of (12) such that (blow-up criterium)*

$$T^* < +\infty \quad \implies \quad \lim_{t \uparrow T^*} \|u(t, \cdot)\|_{H^s} = +\infty. \quad (19)$$

Moreover, if $s \geq 3$, then

$$T^* < +\infty \quad \implies \quad \lim_{t \uparrow T^*} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty. \quad (20)$$

Furthermore, the solution given in this theorem satisfies the Oleinik inequality:

Proposition 1. (Oleinik inequality) *Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$ and let $M = \sup_{x \in \mathbb{R}} u'_0(x)$. Then, for all $t \in [0, T^*[$ the solution given in Theorem 1 satisfies*

$$u_x(t, x) \leq \frac{2M}{Mt+2} \leq M. \quad (21)$$

Proof. Let $x_0 \in \mathbb{R}$ and let the characteristic $\eta(t, x_0)$ be defined as the solution of the Cauchy problem $\eta_t(t, x_0) = u(t, \eta(t, x_0))$, with the initial datum $\eta(0, x_0) = x_0$. With $H(t, x_0) \stackrel{\text{def}}{=} u_x(t, \eta(t, x_0))$, the equation (15) can be rewritten

$$H_t + \frac{1}{2} H^2 + P = 0. \quad (22)$$

Since $P \geq 0$, it follows that $H_t \leq -\frac{1}{2} H^2$ which implies that $H(t, x_0) \leq \frac{2H(0, x_0)}{H(0, x_0)t+2} \leq \frac{2M}{Mt+2}$. \square

Remark 1. *The Oleinik inequality (21) is valid only when the solution u is smooth. In the theorem 4 below, we show that this inequality holds for all times also for a certain type of weak solutions (called dissipative) such that $u \in H^1$ (and, possibly, for $M = +\infty$).*

Unfortunately, the solution given in theorem 1 does not exist globally in time for all non trivial initial data [21]. Since Yin [21] studied a general family of equations including rB, his result is not optimal for rB. In the following Proposition, this result is improved with a shorter proof.

Proposition 2. (An upper bound on the blow-up time) *Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$. If there exists $x_0 \in \mathbb{R}$ such that $u'_0(x_0) < 0$, then $T^* \leq -2/\inf u'_0$.*

Proof. From the proof of the previous proposition, we have

$$H(t, x_0) \leq \frac{2H(0, x_0)}{tH(0, x_0) + 2}, \quad t < T^*. \quad (23)$$

If $T^* > -2/\inf u'_0$ then $H(0, x_0) < 0$ implies

$$\lim_{t \rightarrow -2/H(0, x_0)} H(t, x_0) = -\infty,$$

and for a suitable x_0 this contradicts $u \in \mathcal{C}([0, T^*[, H^s)$. \square

A uniform (with respect to ℓ) lower bound on T^* is needed, in order to prove in section 5 below the convergence of smooth solutions (see Proposition 3 and Proposition 4).

Theorem 2. (A lower bound on the blow-up time) *Let u_0 in H^s be non-trivial with $s \geq 2$ and let*

$$m(t) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} u_x(t, x) < 0 < M(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u_x(t, x), \quad t < T^*.$$

If $|m(0)| \geq M(0)$ then

$$-1/\inf u'_0 \leq T^* \leq -2/\inf u'_0. \quad (24)$$

If $|m(0)| < M(0)$ then, there exists t^* such that $0 < t^* \leq -m(0)^{-1} - M(0)^{-1}$ and $m(t^*) = -M(t^*)$. Therefore

$$t^* + 1/\sup u'_0 \leq T^* \leq -2/\inf u'_0. \quad (25)$$

Remark 2. *Note that the blow-up time T^* is uniformly (with respect to ℓ) bounded from below by $1/\sup |u'_0|$.*

Proof. Since $u \in H^s$, $u_x \rightarrow 0$ when x goes to $\pm\infty$, and u_x is not the zero function, so $m(t) = \min_{x \in \mathbb{R}} H(t, x) < 0 < M(t) = \max_{x \in \mathbb{R}} H(t, x)$. The equation (22) implies that m and M are decreasing in time, so $|m| = -m$ is increasing. So, if $|m(t_0)| \geq M(t_0)$, then for all $t > t_0$ we have $|m(t)| \geq M(t)$.

The inequality (23) shows that for $t < T^*$

$$0 < M(t) \leq \frac{2M(0)}{M(0)t + 2}, \quad m(t) \leq \frac{2m(0)}{m(0)t + 2} < 0, \quad (26)$$

which implies that, if $|m(0)| < M(0)$, there exists $t^* \leq -(m(0) + M(0))/(m(0)M(0))$ such that $|m(t^*)| = M(t^*)$.

If $\delta > 0$ is small enough, since the function $H(t + \delta, \cdot)$ has a minimum, then there exists x_δ such that $m(t + \delta) = H(t + \delta, x_\delta)$. Inspired by Junca and Lombard [16] one gets

$$\begin{aligned} m(t + \delta) &= H(t + \delta, x_\delta) = H(t, x_\delta) + \int_t^{t+\delta} H_t(s, x_\delta) ds \\ &\geq m(t) - \int_t^{t+\delta} \left(\frac{1}{2} H(s, x_\delta)^2 + P(s, x_\delta) \right) ds. \end{aligned} \quad (27)$$

Since $m(\cdot) < 0$ and δ is arbitrary small, we have $m(s) \leq H(s, x_\delta) \leq 0$ then $m(s)^2 \geq H(s, x_\delta)^2$, implying that

$$\frac{m(t + \delta) - m(t)}{\delta} \geq -\frac{1}{\delta} \int_t^{t+\delta} \left(\frac{1}{2} m(s)^2 + \sup_{x \in \mathbb{R}} P(s, x) \right) ds. \quad (28)$$

Defining the generalized derivative

$$\dot{m}(t) \stackrel{\text{def}}{=} \liminf_{\delta \rightarrow 0^+} \frac{m(t + \delta) - m(t)}{\delta}, \quad (29)$$

one can show that

$$\dot{m}(t) + \frac{1}{2} m(t)^2 \geq -\sup_{x \in \mathbb{R}} P(t, x). \quad (30)$$

Using the definition of P from (12) and using that $\|\mathfrak{G}\|_1 = 1$, one obtains

$$\sup_{x \in \mathbb{R}} P(t, x) \leq \frac{1}{2} \|u_x(t, \cdot)\|_\infty^2 \leq \max\left\{\frac{1}{2} M(t)^2, \frac{1}{2} m(t)^2\right\}. \quad (31)$$

and the Riccati-like inequality (30) becomes

$$\dot{m}(t) + m(t)^2 \geq 0 \quad t > t^*. \quad (32)$$

Then $T^* - t^* \geq -1/m(t^*) = 1/M(t^*)$, and with (26), one obtains

$$T^* \geq t^* + 1/\sup u'_0. \quad \square$$

4. GLOBAL WEAK SOLUTIONS

Note that Proposition 2 shows that, for $s \geq 2$, we have $\lim_{t \uparrow T^*} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty$ which implies that

$$\lim_{t \uparrow T^*} \|u(t, \cdot)\|_{H^s} = +\infty.$$

Hence the space H^s with $s \geq 2$ is not the right space in order to obtain the global existence of the solution.

Bressan and Constantin [6, 7] have proved the existence of two types of global solutions for the Camassa–Holm equation (8) in H^1 . Using the formal energy equation (18), a similar proof (of global existence of conservative and dissipative solutions in H^1) for rB can be done following [6, 7]. Another proof of existence of a dissipative solution, using the vanishing viscosity method, is given by Chen and Tian [9], Xin and Zhang [20].

In this paper, the existence theorem will be developed for solutions not vanishing as $|x| \rightarrow \infty$. Note that a major difference between the rB (12) and the Camassa–Holm (8) equations is that u^2 does not appear in the non-local term of rB. This allows us to get global existence for rB without asking u to be in $L^2(\mathbb{R})$. Moreover, in Theorem 7 below, we show that asking $u_x \in L^2$ is optimal.

These remarks lead us to assert in the following the existence of two types of solutions of rB: conservative and dissipative.

4.1. Global existence of conservative solutions. We start this subsection by defining a conservative solution.

Definition 4.1. *A function u is called a conservative solution of rB if*

- *The function u belongs to $\text{Lip}([0, T], L^2_{loc}) \cap L^\infty([0, T], \dot{H}^1)$ for all $T > 0$.*
- *u satisfies the equation (12), with an initial data $u(0, x) = u_0(x)$.*
- *u satisfies (18) in the sense of distributions.*

It means that it is a weak solution conserving the energy, as smooth solutions.

Remark 3. *The regularity $u_x \in L^\infty([0, T], L^2)$ ensures that (16) is satisfied. Thence, the equalities (17) and (18) are equivalent.*

Introducing the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}) = \{f : \|f'\|_2 < +\infty\}$, we can state the theorem:

Theorem 3. *Let $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If there exists a Lipschitz function ϕ such that $\phi' \in L^1(\mathbb{R})$ with $u_0 - \phi \in H^1(\mathbb{R})$, then there exists a global conservative solution u of (12), such that $u(t, \cdot) - \phi \in \dot{H}^1(\mathbb{R})$ for all $t > 0$. In addition, for all $T > 0$*

$$\liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty \quad \implies \quad \limsup_{t \downarrow T} \sup_{x \in \mathbb{R}} u_x(t, x) = +\infty, \quad (33)$$

and if $u_0 \in H^1$, then for all $t > 0$

$$\int_{\mathbb{R}} [u(t, x)^2 + \ell^2 u_x(t, x)^2] dx = \int_{\mathbb{R}} [u_0(x)^2 + \ell^2 u_0'(x)^2] dx. \quad (34)$$

Remark 4. This theorem covers also some solutions that do not have a limit when $|x| \rightarrow \infty$, such as $\phi(x) = u_0(x) = \cos \ln(x^2 + 1)$.

Remark 5. Note that (33) implies that the Oleinik inequality (21) cannot hold after the appearance of singularities.

In the special case $u_0 \in H^1(\mathbb{R})$, the proof can be done following Bressan and Constantin [6]. In the general case, the energy is modified as

$$E(t) = \int_{\mathbb{R}} [u(t, x) - \phi(x)]^2 + \ell^2 u_x(t, x)^2 dx, \quad (35)$$

and the proof is done in steps as follows.

Step 1: Formal energy estimate on the x -variable. Let $\tilde{u}(t, x) = u(t, x) - \phi(x)$. The equation (12) can be rewritten

$$u_t + uu_x + \ell^2 P_x = \tilde{u}_t + uu_x + \ell^2 P_x = 0. \quad (36)$$

Multiplying (36) by \tilde{u} , one gets

$$\left[\frac{1}{2} \tilde{u}^2 \right]_t + \left[\frac{1}{3} u^3 - \frac{1}{2} \phi u^2 \right]_x + \frac{1}{2} \phi_x u^2 + \ell^2 u P_x - \ell^2 \phi P_x = 0. \quad (37)$$

Adding (37) and (17), we obtain

$$\frac{1}{2} \left[\tilde{u}^2 + \ell^2 u_x^2 \right]_t + \left[\frac{1}{3} u^3 + \frac{1}{6} \phi^3 - \frac{1}{2} \phi u^2 + \frac{1}{2} \ell^2 uu_x^2 + \ell^2 u P \right]_x = \ell^2 \phi P_x - \frac{1}{2} \phi_x (\tilde{u}^2 + 2\phi\tilde{u}). \quad (38)$$

Integrating over the real line, one gets (exploiting the triangular inequality)

$$\frac{1}{2} E'(t) \leq \int_{\mathbb{R}} \left(\ell^2 |\phi P_x| + \frac{1}{2} |\phi'| (2\tilde{u}^2 + \phi^2) \right) dx. \quad (39)$$

The Young inequality implies that

$$\|P(t)\|_p \leq \frac{1}{2\ell^2} \|\mathfrak{G}\|_p E(t) \quad \forall p \in [1, \infty], \quad (40a)$$

$$\|P_x(t)\|_p \leq \frac{1}{2\ell^3} \|\mathfrak{G}\|_p E(t) \quad \forall p \in [1, \infty]. \quad (40b)$$

Using (39) and (40b), we obtain

$$E'(t) \leq \left(\ell^{-1} \|\phi\|_{\infty} + 2 \|\phi'\|_{\infty} \right) E(t) + \|\phi\|_{\infty}^2 \|\phi'\|_1. \quad (41)$$

Then the Gronwall lemma ensures that $E(t)$ does not blow up in finite time.

Step 2: Equivalent system. As in [6], let $\xi \in \mathbb{R}$ and let $y_0(\xi)$ be defined by

$$\int_0^{y_0(\xi)} \left(1 + u_0'^2 \right) dx = \xi, \quad (42)$$

and let $y(t, \xi)$ be the function² defined by the equation

$$y_t(t, \xi) = u(t, y(t, \xi)), \quad y(0, \xi) = y_0(\xi). \quad (43)$$

Let also $v = v(t, \xi)$ and $q = q(t, \xi)$ be defined as

$$v \stackrel{\text{def}}{=} 2 \arctan(u_x), \quad q \stackrel{\text{def}}{=} \left(1 + u_x^2 \right) y_{\xi}, \quad (44)$$

²It will turn out that $y(t, \xi)$ is the characteristic of rB corresponding to $y_0(\xi)$, with speed $u(t, y(t, \xi))$.

where $u_x(t, \xi) = u_x(t, y(t, \xi))$. Notice that

$$\frac{1}{1+u_x^2} = \cos^2\left(\frac{v}{2}\right), \quad \frac{u_x}{1+u_x^2} = \frac{\sin(v)}{2}, \quad \frac{u_x^2}{1+u_x^2} = \sin^2\left(\frac{v}{2}\right), \quad \frac{\partial y}{\partial \xi} = q \cos^2\left(\frac{v}{2}\right). \quad (45)$$

Integrating the last equality in (45), one obtains

$$y(t, \xi') - y(t, \xi) = \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds. \quad (46)$$

Using (45) and the change of variables $x = y(t, \xi')$, (46), P and P_x can be written in the new variables as

$$\begin{aligned} P(t, \xi) &= \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(-\frac{|y(t, \xi) - x|}{\ell}\right) u_x^2(t, x) dx \\ &= \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(-\frac{1}{\ell} \left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) d\xi', \quad (47) \\ P_x(t, \xi) &= \frac{1}{4\ell^2} \left(\int_{y(t, \xi)}^{+\infty} - \int_{-\infty}^{y(t, \xi)} \right) \exp(-|y(t, \xi) - x|/\ell) u_x^2(t, x) dx \\ &= \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp\left(-\left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) \frac{ds}{\ell} \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) \frac{d\xi'}{4\ell^2}. \quad (48) \end{aligned}$$

Then, a system equivalent to the rB equation is given by

$$y_t = u, \quad y(0, \xi) = y_0(\xi), \quad (49a)$$

$$u_t = -\ell^2 P_x, \quad u(0, \xi) = u_0(y_0(\xi)), \quad (49b)$$

$$v_t = -P(1 + \cos(v)) - \sin^2(v/2), \quad v(0, \xi) = 2 \arctan(u'_0(y_0(\xi))), \quad (49c)$$

$$q_t = q\left(\frac{1}{2} - P\right) \sin(v), \quad q(0, \xi) = 1. \quad (49d)$$

In order to prove Theorem 3, we prove first the global existence of the solution of the initial-value problem (49), then we infer that this solution yields a conservative solution of rB.

Step 3: Local existence of the new system. Our goal is to prove that the system of equations (49) is locally well-posed. The proof given in [6] for the Camassa–Holm equation is slightly simplified here.

We first solve a coupled 2x2 subsystem instead of a 3x3 subsystem in [6]. Let u_0 be a function such that $u_0 - \phi \in H^1$, then y_0 is well defined in (42). Note that the right-hand side of (49) does not depend on y . Since P and P_x depend only on v and q , the right-hand sides of equations (49b), (49c) and (49d) do not depend on u . Also, the equations (49c) and (49d) are coupled. Thus, we are left to show that the system of two equations

$$v_t = -P(1 + \cos v) - \sin^2 \frac{v}{2}, \quad v(0, \xi) = v_0(\xi) = 2 \arctan u'_0(y_0(\xi)), \quad (50a)$$

$$q_t = q\left(\frac{1}{2} - P\right) \sin v, \quad q(0, \xi) = q_0(\xi) = 1. \quad (50b)$$

is well defined in the space $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^2))$.

Let $U = (v, q)$, and let $\mathfrak{D} \subset X$ be the closed set satisfying $U(0, \xi) = U_0(\xi)$ and

$$1/C \leq q(t, \xi) \leq C \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (51a)$$

$$\left| \left\{ \xi, \sin^2 \frac{v(t, \xi)}{2} \geq \frac{1}{2} \right\} \right| \leq C \quad \forall t \in [0, T], \quad (51b)$$

where $C > 0$ is a constant. Then, for $\xi_1 < \xi_2$, we get from equations (51)

$$\int_{\xi_1}^{\xi_2} q(\xi) \cos^2 \frac{v(\xi)}{2} d\xi \geq \int_{\left\{ \xi \in [\xi_1, \xi_2], \sin^2 \frac{v(t, \xi)}{2} \leq \frac{1}{2} \right\}} \frac{C^{-1}}{2} d\xi \geq \left[\frac{\xi_2 - \xi_1}{2} - \frac{C}{2} \right] C^{-1}. \quad (52)$$

Let Γ be defined as

$$\Gamma(\zeta) = \min \left\{ 1, \exp \left(\frac{1}{2\ell} - \frac{|\zeta|}{2\ell} C^{-1} \right) \right\}. \quad (53)$$

Then, for $(v, q) \in \mathfrak{D}$, the exponential terms in (47) and (48) are smaller than $\Gamma(\xi - \xi')$.

Let $P(\xi, v, q)$ be defined by (47). If $(v, q) \in \mathfrak{D}$ then, using Young inequality, $\partial_v P$ and $\partial_q P$ are bounded, i.e., for $\{U, \tilde{U}\} \in \mathfrak{D}$ we have

$$\|P(\xi, U) - P(\xi, \tilde{U})\|_X \lesssim \|U - \tilde{U}\|_X, \quad (54)$$

where the symbol \lesssim means “less or equal” with a constant depending only on C and ℓ . Then, for T small enough, the Picard operator

$$(\mathcal{P}(U))(t, \xi) \stackrel{\text{def}}{=} U_0 + \int_0^t \left(-(1 + \cos v) P - \sin^2 \frac{v}{2}, q \left(\frac{1}{2} - P \right) \sin v \right) d\tau, \quad (55)$$

is a contraction from \mathfrak{D} to \mathfrak{D} . The local existence of the solution of the Cauchy problem (50) follows at once.

Step 4: Global existence for the equivalent system. After proving the local existence of the solution of system (50), an estimate of the quantity

$$\|q(t)\|_\infty + \|1/q(t)\|_\infty + \|\sin^2(v(t)/2)\|_1 + \|v(t)\|_\infty, \quad (56)$$

is needed to ensure that the solutions exist for all time. Let u be defined as

$$u(t, \xi) \stackrel{\text{def}}{=} u_0(y_0(\xi)) - \int_0^t \ell^2 P_x(s, \xi) ds, \quad (57)$$

and let y be the family of characteristics

$$y(t, \xi) \stackrel{\text{def}}{=} y_0(\xi) + \int_0^t u(s, \xi) ds, \quad (58)$$

and, finally, let $\phi(t, \xi) \stackrel{\text{def}}{=} \phi(y(t, \xi))$. Our task here is to show that the modified energy

$$\tilde{E}(t) = \int_{\mathbb{R}} \left[(u - \phi)^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2} \right] q d\xi \quad (59)$$

does not blow-up in finite time.

The system (49) implies that

$$\left(q \cos^2 \frac{v}{2} \right)_t = \frac{1}{2} q \sin v, \quad \left(q \sin^2 \frac{v}{2} \right)_t = q_t - \frac{1}{2} q \sin v = -q P \sin v, \quad (60)$$

while the equations (47) and (48) imply that

$$P_\xi = q P_x \cos^2 \frac{v}{2}, \quad \ell^2 (P_x)_\xi = q P \cos^2 \frac{v}{2} - \frac{1}{2} q \sin^2 \frac{v}{2}. \quad (61)$$

From (49), (57) and (61), we have

$$\left(u_\xi - \frac{1}{2}q \sin v\right)_t = 0,$$

and, for $t = 0$, we have from (44) and (45)

$$u_\xi - \frac{1}{2}q \sin v = u_x \frac{\partial y}{\partial \xi} - \frac{1}{2} \sin v = 0.$$

Thus, as long as the solution of (49) is defined, the equality

$$u_\xi = \frac{1}{2}q \sin v \quad (62)$$

holds. Therefore, the equations (60), (61), (62) yield

$$\left[\left(u^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2}\right) q\right]_t + \left[2\ell^2 u P - \frac{1}{3}u^3\right]_\xi = 0, \quad (63)$$

which expresses conservation of energy in the (t, ξ) -variables when $u_+ = u_- = 0$, i.e., for $\phi = 0$.

From (60), (62) and (58), we have

$$\left(q \cos^2 \frac{v}{2}\right)_t = u_\xi = \left(\frac{\partial y}{\partial \xi}\right)_t,$$

implying that the equality

$$\frac{\partial y}{\partial \xi} = q \cos^2 \frac{v}{2}, \quad (64)$$

holds for the (t, ξ) -variables (note that the equality is true for $t = 0$ from (45)). Then, using (58) and (64), we get

$$\phi_t(t, \xi) = \frac{d}{dt} \phi(y(t, \xi)) = u \phi', \quad \phi_\xi(t, \xi) = q \cos^2 \left(\frac{v}{2}\right) \phi', \quad (65)$$

so, using (57), (49), (65) and (60), we obtain

$$\begin{aligned} & \left[(\phi^2 - 2u\phi)q \cos^2 \frac{v}{2}\right]_t + \left[\frac{1}{2}\phi u^2 + \frac{1}{2}\phi(u - \phi)^2 - \frac{1}{6}\phi^3\right]_\xi = \\ & \quad 2\ell^2 P_x \phi q \cos^2 \frac{v}{2} - 2u\phi_\xi(u - \phi) + \frac{1}{2}\phi_\xi u^2 + (\phi^2 - 2u\phi)u_\xi \\ & \quad + \phi u u_\xi + \frac{1}{2}\phi_\xi(u - \phi)^2 + \phi(u - \phi)_\xi(u - \phi) - \frac{1}{2}\phi^2 \phi_\xi \\ & = -\phi_\xi(u - \phi)^2 + 2\ell^2 P_x \phi q \cos^2 \frac{v}{2} - 2\phi\phi_\xi(u - \phi). \end{aligned} \quad (66)$$

Adding (63) and (66), with the trivial relation $2\phi(u - \phi) \leq \phi^2 + (u - \phi)^2$, then integrating the result with respect of ξ , we get

$$\tilde{E}'(t) \leq \int_{\mathbb{R}} \left(2\ell^2 |\phi P_x| q \cos^2 \frac{v}{2} + |\phi_\xi| (2(u - \phi)^2 + \phi^2)\right) d\xi. \quad (67)$$

Using (64) and (65) with the change of variables $x = y(t, \xi)$, then exploiting (40), one obtains

$$\begin{aligned} \tilde{E}'(t) & \leq \int_{\{\xi, \cos v \neq -1\}} (2\ell^2 |\phi P_x| + |\phi'| (2(u - \phi)^2 + \phi^2)) q \cos^2 \frac{v}{2} d\xi \\ & = \int_{\mathbb{R}} (2\ell^2 |\phi P_x| + |\phi'| (2(u - \phi)^2 + \phi^2)) dx \\ & \leq (\ell^{-1} \|\phi\|_\infty + 2\|\phi'\|_\infty) E(t) + \|\phi\|_\infty^2 \|\phi'\|_1, \end{aligned}$$

where P_x in the second equation is defined as $P_x = \frac{1}{2} \mathfrak{G}_x * u_x^2$.

From (35) and (59), and using the change of variables $x = y(t, \xi)$, one can show easily that

$$E(t) = \int_{\{\xi, \cos v \neq -1\}} \left[(u - \phi)^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2} \right] q \, d\xi \leq \tilde{E}(t). \quad (68)$$

Thence, the uniform estimate of $\tilde{E}(t)$ on any bounded interval $[0, T]$ follows by using Gronwall lemma.

We can show now that the quantity (56) does not blow up in finite time. Using Young inequality, (47), (48) and (53), one obtains

$$\|P(t)\|_p \leq \frac{1}{4\ell^3} \|\Gamma\|_p \tilde{E}(t) \quad \forall p \in [1, \infty], \quad (69a)$$

$$\|P_x(t)\|_p \leq \frac{1}{4\ell^4} \|\Gamma\|_p \tilde{E}(t) \quad \forall p \in [1, \infty]. \quad (69b)$$

The inequalities (69) are the identical estimates as (40), but in the (t, ξ) -variables. Using (49d) and (69), we get

$$|qt| \leq \left(\frac{1}{2} + \frac{1}{4\ell^3} E(t) \right) q, \quad (70)$$

implying that $\|q(t)\|_\infty + \|1/q(t)\|_\infty$ does not blow-up in finite time. The equation (49c) and (69) imply that $\|v(t)\|_\infty$ remains bounded on any finite interval $[0, T]$. Also, the boundedness of the energy $\tilde{E}(t)$ and $\|1/q(t)\|_\infty$ implies that $\|\cos^2(v(t)/2)\|_1$ remains bounded on any interval $[0, T]$. This completes the proof of the global existence.

Step 5: Global existence of a conservative solution. Here, we show that the global solution of the equivalent system (49) yields a global solution of the rB equation.

Let u and y be defined by (57) and (58), respectively. We claim that the solution of rB can be written as

$$u(t, x) = u(t, \xi), \quad y(t, \xi) = x. \quad (71)$$

Using (62), (65) and the change of variables $x = y(t, \xi)$ with (64), one obtains

$$\begin{aligned} |u(t, \xi) - \phi(t, \xi)|^2 &\leq 2 \int_{\mathbb{R}} |u - \phi| |u_\xi - \phi_\xi| \, d\xi \\ &\leq 2 \int_{\mathbb{R}} |u - \phi| q \left(\sin \frac{v}{2} \cos \frac{v}{2} + \phi' \cos^2 \frac{v}{2} \right) \, d\xi \\ &\leq 2E(t) + \|\phi'\|_2^2, \end{aligned}$$

implying that $\|u(t)\|_\infty$ is uniformly bounded on any bounded interval $[0, T]$. Therefore, from (58), we get

$$y_0(\xi) - \|u(t)\|_\infty t \leq y(t, \xi) \leq y_0(\xi) + \|u(t)\|_\infty t, \quad (72)$$

and thus

$$\lim_{\xi \rightarrow \pm\infty} y_0(t, \xi) = \pm\infty. \quad (73)$$

The equation (64) implies that the mapping $\xi \mapsto y(t, \xi)$ is non-decreasing and, if for $\xi < \xi'$ we have $y(t, \xi) = y(t, \xi')$, then $\sin(v) = 2 \cos(v/2) \sin(v/2) = 0$ between ξ and ξ' (see eq. 64). Integrating (62) with respect to ξ , one obtains that $u(t, \xi) = u(t, \xi')$, so u is well-defined in (71).

Proceeding as in [6, section 4], we can prove that for each interval $[t_1, t_2]$ there exists a constant $C = C(\ell, t_2)$ such that, $\forall t \in [t_1, t_2 - h]$,

$$\int_{\mathbb{R}} |u(t+h, x) - u(t, x)|^2 \, dx \leq C h^2, \quad (74)$$

and then u satisfies

$$\frac{d}{dt} u(t, y(t, \xi)) = -P_x(t, \xi). \quad (75)$$

The inequality (74) implies that u belongs to $Lip([0, T], L_{loc}^2)$. Straightforward calculations show that, for $x = y(t, \xi)$ and for $\cos(v(t, \xi)) \neq 1$, we have

$$u_x(t, x) = \tan\left(\frac{v(t, \xi)}{2}\right) = \frac{\sin(v(t, \xi))}{1 + \cos(v(t, \xi))}. \quad (76)$$

Using the change of variables $x = y(t, \xi)$ with (64), one can show that u is a global solution of rB.

In order to prove (17), let ψ be a test function and let $\tilde{\psi}(t, \xi) = \psi(t, y(t, \xi))$. Multiplying (60b) by $\tilde{\psi}$ and integrating the result with respect to ξ , one obtains

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathbb{R}} [(q \sin^2 v/2)_t + q P \sin v] \tilde{\psi} dt d\xi, \\ &= \int_0^{+\infty} \int_{\mathbb{R}} [-\tilde{\psi}_t q \sin^2 v/2 + \tilde{\psi} q P \sin v] dt d\xi + \int_{\mathbb{R}} \tilde{\psi}(0, x) \sin^2 v(0, \xi)/2 d\xi, \\ &= \int \int_{\{\cos v > -1\}} [-\tilde{\psi}_t q \sin^2 v/2 + \tilde{\psi} q P \sin v] dt d\xi + \int_{\{v_0 > -\pi\}} \tilde{\psi}(0, x) \sin^2 v(0, \xi)/2 d\xi, \\ &\quad + \int \int_{\{\cos v = -1\}} -q \tilde{\psi}_t dt d\xi + \int_{\{v_0 = -\pi\}} \tilde{\psi}(0, x) d\xi. \end{aligned} \quad (77)$$

It is clear from (49c) that

$$|\{\xi, \cos v(t, \xi) = -1\}| = 0 \quad \text{for almost all } t \geq 0. \quad (78)$$

Using that $\tilde{\psi}_t = \psi_t + u\psi_x$ and the change of variables $x = y(t, \xi)$, the equation (17) follows in the sense of distributions.

Finally, let $u_0 \in H^1$. The equation (63) implies that

$$\frac{d}{dt} \int_{\mathbb{R}} \left(u^2 \cos^2 \frac{v(t, \xi)}{2} + \ell^2 \sin^2 \frac{v(t, \xi)}{2} \right) q(t, \xi) d\xi = 0, \quad (79)$$

hence $\tilde{E}(t) = \tilde{E}(0)$. In addition, using the change of variables $x = y(t, \xi)$ with (64) and (76), one obtains

$$\int_{\mathbb{R}} u(t, x)^2 + \ell^2 u_x(t, x)^2 dx = \int_{\{\xi, \cos v(t, \xi) > -1\}} \left(u^2 \cos^2 \frac{v(t, \xi)}{2} + \ell^2 \sin^2 \frac{v(t, \xi)}{2} \right) q(t, \xi) d\xi. \quad (80)$$

Using (78), the conservation of the energy (34) follows.

We end this demonstration with the proof of the property (33). The equation (49c) implies that v is decreasing in time. Further, if $v(T, \xi) = -\pi$ (corresponding to an infinite value of u_x , see (76) above) then $v_t(T, \xi) = -1$, meaning that the value of $v(t, \xi)$ crosses $-\pi$ and $v(t, \xi) < -\pi$ for all $t > T$. Then, (33) follows using (76). \square

4.2. Global existence of dissipative solutions. We start this subsection by defining *dissipative* solutions, this kind of solutions being very important for applications. We note in passing that when ℓ goes to zero, we expect to recover the entropy solution of the Burgers equation. However, in section 5, we show that the limit (up to a subsequence) is a solution of the Burgers equation with a remaining forcing term.

Definition 4.2. A function u is called a dissipative solution of rB if

- The function u belongs to $\text{Lip}([0, T], L^2_{loc}) \cap L^\infty([0, T], \dot{H}^1)$ for all $T > 0$;
- u satisfies the equation (12), with an initial data $u(0, x) = u_0(x)$;
- u satisfies the inequality

$$\left[\frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[\frac{1}{3} u^3 + \ell^2 u P + \frac{1}{2} \ell^2 u u_x^2 \right]_x \leq 0, \quad (81)$$

in the sense of distributions.

- There exists a constant C such that u satisfies the Oleinik inequality

$$u_x(t, x) \leq C/t \quad \forall t, x.$$

Remark 6. Following [7], we construct in Theorem 4 a dissipative solution of rB with $C = 2$. The entropy solutions of the classical Burgers equation satisfy the Oleinik inequality with $C = 1$.

As mentioned above, when v crosses the value $-\pi$, u_x jumps from $-\infty$ to $+\infty$, which means that the Oleinik inequality cannot be satisfied. Thus, to enforce the Oleinik inequality, the value of v is not allowed to leave the interval $[-\pi, \pi[$. For that purpose, the system (50) is modified (as in [7]) to become

$$u_t = -\ell^2 P_x, \quad (82a)$$

$$v_t = \begin{cases} -P(1 + \cos v) - \sin^2(v/2), & v > -\pi, \\ 0 & v \leq -\pi, \end{cases} \quad (82b)$$

$$q_t = \begin{cases} q(\frac{1}{2} - P) \sin(v), & v > -\pi \\ 0 & v \leq -\pi. \end{cases} \quad (82c)$$

and P and P_x are also modified as

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} d\xi', \quad (83)$$

$$P_x(t, \xi) = \frac{1}{4\ell^2} \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} d\xi', \quad (84)$$

where $\bar{q}(t, \xi) = q(t, \xi)$ if $v(t, \xi) > -\pi$ and $\bar{q}(t, \xi) = 0$ if $v(t, \xi) \leq -\pi$. The system (82) is the key tool to prove the following theorem.

Theorem 4. Let $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If there exist a Lipschitz function ϕ such that $\phi' \in L^1(\mathbb{R})$ and with $u_0 - \phi \in H^1(\mathbb{R})$, then there exists a global dissipative solution u of the equation (12), such that $u(t, \cdot) - \phi \in H^1(\mathbb{R})$ for all $t > 0$. In addition, for all $t > 0$

$$u_x(t, x) \leq 2/t \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (85)$$

and if $u_0 \in H^1$, then for all $t > 0$

$$\int_{\mathbb{R}} [u(t, x)^2 + \ell^2 u_x(t, x)^2] dx \leq \int_{\mathbb{R}} [u_0(x)^2 + \ell^2 u_0'(x)^2] dx. \quad (86)$$

Remark 7. Due to the loss of the Oleinik inequality (cf. Remark 5), the system (49) is slightly modified to (82) in order to obtain dissipative solutions of rB that satisfies the Oleinik inequality (85).

Remark 8. In general, if the initial datum satisfies $u'_0 \leq M \in \mathbb{R} \cup \{+\infty\}$, then the Oleinik inequality (85) can be improved as

$$u_x(t, x) \leq 2M/(Mt + 2) \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (87)$$

as shown in (97) below.

The idea of the proof is similar to Theorem 3 above and it is done in several steps:

Step 1: Existence of a solution. As in the proof of Theorem 3, it suffices to show that (82b) and (82c) are locally well posed in the domain $\mathfrak{D} \subset X$, \mathfrak{D} being defined below and $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^2))$.

Note that if v is near $-\pi$ the right-hand side of (82b) is discontinuous. To avoid this discontinuity, the system (82) is replaced, as in [7], by

$$U_t(t, \xi) = F(U(t, \xi)) + G(\xi, U(t, \cdot)), \quad U = (v, q), \quad (88)$$

with

$$F(U) \stackrel{\text{def}}{=} \begin{cases} (-\sin^2 \frac{v}{2}, \frac{1}{2} q \sin v) & v > -\pi \\ (-1, 0) & v \leq -\pi. \end{cases}, \quad G(U) \stackrel{\text{def}}{=} \begin{cases} (-P(1 + \cos v), -Pq \sin v) & v > -\pi \\ (0, 0) & v \leq -\pi. \end{cases}.$$

Note also that, as long as the solution to (88) is well defined, replacing v by $\max\{-\pi, v\}$ gives a solution of the equations (82b) and (82c). In the rest of this step, our aim is to show that the system (88) is locally well-posed. Let $\delta \in]0, \frac{2\pi}{3}]$ and let Λ be defined by

$$\Lambda \stackrel{\text{def}}{=} \{ \xi, v_0(\xi) \in]-\pi, \delta - \pi] \}. \quad (89)$$

The equation (88) implies that, if $v \in]-\pi, \delta - \pi] \subset]-\pi, -\frac{\pi}{3}]$, then $v_t \leq -\frac{1}{2}$. Let $\mathfrak{D} \subset X$ satisfy $U(0, \xi) = U_0(\xi)$ and

$$1/C \leq q(t, \xi) \leq C \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (90a)$$

$$|\{ \xi, \sin^2(v(t, \xi)/2) \geq \frac{1}{2} \}| \leq C \quad \forall t \in [0, T], \quad (90b)$$

$$\|U(t) - U(s)\|_\infty \leq C|t - s| \quad \forall t, s \in [0, T], \quad (90c)$$

$$v(t, \xi) - v(s, \xi) \leq -\frac{t-s}{2} \quad \forall \xi \in \Lambda, 0 \leq s \leq t \leq T, \quad (90d)$$

Taking $(v, q) \in \mathfrak{D}$ and using (69), one gets that the right-hand sides of (82b) and (82c) are bounded. However, the inequality (54) is no longer true and we have instead

$$\|P(U) - P(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty + |\{ \xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0 \}|, \quad (91)$$

which implies that

$$\|F(U) - F(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty, \quad (92)$$

$$\|G(U) - G(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty + |\{ \xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0 \}|. \quad (93)$$

In order to estimate the second term of the right-hand side of the last equation, the crossing time τ is defined as

$$\tau(\xi) \stackrel{\text{def}}{=} \sup \{ t \in [0, T], v(t, \xi) > -\pi \}. \quad (94)$$

Note that the equation (90c) implies that $|v(t, \xi) - v_0(\xi)| \leq Ct$. So, if $\xi \notin \Lambda$ then

$$\min \{ \tau(\xi), \tilde{\tau}(\xi) \} \geq \delta/C.$$

Taking T small enough ($T < \delta/C$) and using (90d), one obtains

$$\begin{aligned} \int_0^T |\{ \xi, (v(\tau, \xi) + \pi)(\tilde{v}(\tau, \xi) + \pi) < 0 \}| d\tau &\leq \int_\Lambda |\tau(\xi) - \tilde{\tau}(\xi)| d\xi \\ &\leq 2|\Lambda| \|U - \tilde{U}\|_\infty. \end{aligned}$$

Now, the Picard operator

$$(\mathcal{P}(U))(t, \xi) = U_0 + \int_0^t [F(U) + G(U)] d\tau, \quad (95)$$

satisfies

$$\|\mathcal{P}(U) - \mathcal{P}(\tilde{U})\|_\infty \leq K(T + |\Lambda|) \|U - \tilde{U}\|_\infty, \quad (96)$$

where K depends only on C and ℓ . Since $\sin^2 \frac{v_0}{2} \in L^1$, by choosing $\delta > 0$ small enough, one can make $|\Lambda|$ arbitrary small. Choosing also T small enough, one obtains the local existence of the solution of the system (88), yielding a solution of (82). The rest of the proof of the existence can be done following the proof of Theorem 3.

Step 2: Oleinik inequality and the dissipation of the energy. The equation (82b) implies that if $v(0, \xi) \leq 0$, then for all $t \geq 0$ $v(t, \xi)$ remains in $[-\pi, 0]$. If $v_0(\xi) \in]0, \pi[$ then, as long as v is positive, the following inequality holds

$$\left[\arctan \frac{v}{2} \right]_t \leq -\frac{1}{2} \arctan^2 \frac{v}{2}.$$

This implies that, if $\arctan \frac{v_0(\xi)}{2} \leq M$, then

$$u_x = \arctan \frac{v(t, \xi)}{2} \leq \frac{2M}{Mt + 2}. \quad (97)$$

The Oleinik inequality (85) follows taking $M = +\infty$ and using (76).

In order to prove the dissipation of the energy (81), let ψ be a non-negative test function, then we follow the same computations in the proof of Theorem 3. Since (78) is no longer true for the system (82), one can obtain from (77) that

$$\begin{aligned} & \iint_{[0, +\infty[\times \mathbb{R}} [-u_x^2 \psi_t - uu_x^2 \psi_x + u_x P] dt dx - \int_{\mathbb{R}} u_0'^2(x) \psi(0, x) dx \\ &= - \int_{\{\tau(\xi) < +\infty\}} q(\tau(\xi), \xi) \tilde{\psi}(\tau(\xi), \xi) d\xi \leq 0, \end{aligned}$$

where $\tau(\xi)$ is the crossing time defined as $\tau(\xi) \stackrel{\text{def}}{=} \sup \{t \geq 0, v(t, \xi) > -\pi\}$. Since (16) is satisfied (see Remark 3), the dissipation of the energy (81) follows.

If $u_0 \in H^1$, as in the last step of the proof of Theorem 3, one can show that (79) and (80) hold for the solution of (82), while the measure in (78) is not always zero. Then, the dissipation of the energy (86) follows. \square

5. THE LIMITING CASES $\ell \rightarrow 0$ AND $\ell \rightarrow +\infty$ FOR DISSIPATIVE SOLUTIONS

Taking formally $\ell = 0$, the rB equation becomes the classical Burgers equation, and letting $\ell \rightarrow +\infty$ it becomes the Hunter–Saxton equation. In this section, we study the compactness of the dissipative solutions when taking $\ell \rightarrow 0$ and $\ell \rightarrow +\infty$.

Let the initial datum u_0 be taken in H^1 , with $u_0' \in L^1$ and $M \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u_0'(x) < +\infty$. Let also u^ℓ be the dissipative solution of the rB equation given in Theorem 4. In order to take the limit, an estimate on the total variation of u^ℓ is needed. For that purpose, the following Lemma is given

Lemma 1. [BV estimate] *If u'_0 satisfies the conditions of Theorem 4 with $u'_0 \in L^1$ and $u'_0(x) \leq M \forall x$, then for all $t \in \mathbb{R}^+$*

$$\text{TV}u^\ell(t, \cdot) = \|u_x^\ell(t, \cdot)\|_1 \leq \|u'_0\|_1 \left(\frac{Mt+2}{2} \right)^2. \quad (98)$$

Proof. We commence this proof by a formal computation on $\|u_x^\ell(t, \cdot)\|_1$. Multiplying (15) by $s \stackrel{\text{def}}{=} \text{sgn}(u_x^\ell)$ we have $|u_x^\ell|_t + (u^\ell u_x^\ell)_x s = \left(u_x^{\ell 2} / 2 - P \right) s$. Due to Saks' lemma [19], the integral $\int_{\mathbb{R}} (u^\ell u_x^\ell)_x s dx$ equals to zero. Using that $P - u_x^{\ell 2} / 2 = \ell^2 P_{xx}$ and the Oleinik inequality (85) one obtains

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |u_x^\ell| dx &= \int_{\mathbb{R}} \left(\frac{1}{2} u_x^{\ell 2} - P \right) s dx \\ &= \int_{\{u_x^\ell \geq 0\}} \left(\frac{1}{2} u_x^{\ell 2} - P \right) dx + \int_{\{u_x^\ell < 0\}} \left(P - \frac{1}{2} u_x^{\ell 2} \right) dx \\ &= 2 \int_{\{u_x^\ell \geq 0\}} \left(\frac{1}{2} u_x^{\ell 2} - P \right) dx + \ell^2 \int_{\mathbb{R}} P_{xx} dx \\ &\leq \frac{2M}{Mt+2} \int_{\mathbb{R}} |u_x^\ell| dx. \end{aligned} \quad (99)$$

Thus, the result follows by Gronwall's lemma.

The Saks lemma is used for smooth solutions. If u is not smooth enough, the same estimates can be done on the ξ -variable in the system (82). For $v \in]-\pi, \pi[$, the equation (62) implies

$$\tilde{s} \stackrel{\text{def}}{=} \text{sgn}(u_\xi^\ell) = \text{sgn}(\sin(v^\ell)) = \text{sgn}\left(\sin\left(\frac{v^\ell}{2}\right)\right), \quad \cos\left(\frac{v^\ell}{2}\right) \geq 0. \quad (100)$$

Note that $\tan(v^\ell/2) \leq \tan(v_0/2) = 2M/(Mt+2)$ from (97). Differentiating (82a) w.r.t ξ , multiplying by \tilde{s} — and using (61), (62) and $\sin v = 2 \sin(v/2) \cos(v/2)$ — one gets

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |u_\xi^\ell| d\xi &= -\ell^2 \int_{\mathbb{R}} \tilde{s} (P_x)_\xi d\xi \\ &= -\ell^2 \int_{\{\tilde{s} > 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\{\tilde{s} < 0\}} (P_x)_\xi d\xi \\ &= -2\ell^2 \int_{\{\tilde{s} > 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\mathbb{R}} (P_x)_\xi d\xi \\ &= -2 \int_{\{\tilde{s} > 0\}} \left(q^\ell P \cos^2 \frac{v^\ell}{2} - \frac{1}{2} q^\ell \sin^2 \frac{v^\ell}{2} \right) d\xi \\ &\leq \int_{\{\tilde{s} > 0\}} q^\ell \sin \frac{v^\ell}{2} \cos \frac{v^\ell}{2} \tan \frac{v^\ell}{2} d\xi \\ &\leq \frac{2M}{Mt+2} \int_{\mathbb{R}} |u_\xi^\ell| d\xi. \end{aligned} \quad (101)$$

Gronwall lemma then implies that

$$\|u_\xi\|_1 \leq \|(u_0)_\xi\|_1 \left(\frac{Mt+2}{2} \right)^2. \quad (102)$$

Note that the last inequality is on the ξ -variable. Using that the application $\xi \mapsto y(t, \xi)$ is not decreasing for all t and using that $\text{TV}f = \|f'\|_1$ for smooth solutions ($f \in W_{loc}^{1,1}$), the result follows. \square

5.1. The limiting case $\ell \rightarrow 0$. The goal of this subsection is to show that when $\ell \rightarrow 0$, the dissipative solution u^ℓ converges (up to a subsequence) to a function u satisfying the Burgers equation with a source term:

$$u_t + \frac{1}{2} [u^2]_x = -\mu_x, \quad (103)$$

where μ is a measure such that $0 \leq \mu \in L^\infty([0, +\infty[, \mathcal{M}^1)$. In the proposition 3 below, we show that the measure μ is zero before the appearance of singularities. The question whether or not μ is zero after singularities is open, in general. The following theorem can be stated

Theorem 5. *Let $u_0 \in H^1$, such that $u'_0 \in L^1$ and $u'_0(x) \leq M \forall x$, then there exists $u \in L^\infty([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$ for all $T > 0$, where $\mathcal{I} \Subset \mathbb{R}$, such that there exists a subsequence of u^ℓ (also noted u^ℓ) such that*

$$u^\ell \xrightarrow{\ell \rightarrow 0} u \text{ in } L^\infty([0, T], L^1(\mathcal{I})), \quad (104)$$

u satisfying the equation (103). Moreover, u satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{2M}{Mt + 2} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (105)$$

Remark 9. *If $\mu = 0$ then, due to the Oleinik inequality, u is the unique entropy solution of the Burgers equation.*

In order to prove Theorem 5, the following definition and lemma are needed:
Let $\mathcal{I} \subset \mathbb{R}$ be a bounded interval and let

$$W(\mathcal{I}) \stackrel{\text{def}}{=} \{f \in \mathcal{D}'(\mathcal{I}), \exists F \in L^1(\mathcal{I}) \text{ such that } F' = f\}, \quad (106)$$

where the norm of the space $W(\mathcal{I})$ is given by

$$\|f\|_{W(\mathcal{I})} \stackrel{\text{def}}{=} \inf_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})} = \min_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})}. \quad (107)$$

Lemma 2. *The space $W(\mathcal{I})$ is a Banach space and the embedding*

$$L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (108)$$

is continuous.

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $W(\mathcal{I})$ and let F_n be a primitive of f_n . From the definition of the norm (107), there exists a constant c_n such that $(\tilde{F}_n - c_n)_{n \in \mathbb{N}}$ (where $\tilde{F}_n = F_n + c_n$) is a Cauchy sequence in $L^1(\mathcal{I})$. Let \tilde{F} be the limit of \tilde{F}_n in $L^1(\mathcal{I})$. Then

$$\|f_n - \tilde{F}'\|_{W(\mathcal{I})} \leq \|\tilde{F}_n - \tilde{F}\|_{L^1(\mathcal{I})}, \quad (109)$$

implying that $W(\mathcal{I})$ is a Banach space. Now, the continuous embedding can be proved.

If $f \in L^1(\mathcal{I})$, then $F(x) - F(a) = \int_a^x f(y) dy$ for almost all $x, a \in \mathcal{I}$. Therefore,

$$\|f\|_{W(\mathcal{I})} \leq \int_{\mathcal{I}} |F(x) - F(a)| dx \leq |\mathcal{I}| \int_{\mathcal{I}} |f(y)| dy, \quad (110)$$

which ends the proof. \square

The previous lemma and Helly's selection theorem imply that

$$W^{1,1}(\mathcal{I}) \hookrightarrow L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (111)$$

where the first embedding is compact and the second is continuous.

Proof of Theorem 5: Let the compact set $[0, T] \times \mathcal{I} \subset \mathbb{R}^+ \times \mathbb{R}$. Supposing that $\ell \leq 1$ then, from (86), the dissipative solutions of rB satisfies

$$\|u^\ell\|_2^2 \leq \|u_0\|_{H^1}^2, \quad \ell^2 \|P\|_1 = \frac{1}{2} \ell^2 \|u_x^\ell\|_2^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2, \quad (112)$$

implying that u^ℓ is uniformly bounded on $L^\infty([0, T], L^2(\mathbb{R}))$. Subsequently, it is also uniformly bounded on $L^\infty([0, T], L^1(\mathcal{I}))$. Because Lemma 1 yields that u^ℓ is bounded on $L^\infty([0, T], W^{1,1}(\mathcal{I}))$, and the equation (112) implies that $\frac{1}{2}u^{\ell^2} + \ell^2 P$ is uniformly bounded on $L^\infty([0, T], L^1(\mathcal{I}))$, then since $u_t^\ell = -\left(\frac{1}{2}u^{\ell^2} + \ell^2 P\right)_x$, (107) implies that u_t^ℓ is bounded on $L^\infty([0, T], W(\mathcal{I}))$. Then, using the Aubin theorem, the compactness follows.

The quantity $\ell^2 P$ is non-negative and bounded in $L^\infty([0, +\infty[, L^1(\mathbb{R}))$, implying the existence of a non-negative measure $\mu \in L^\infty([0, +\infty[, \mathcal{M}^1(\mathbb{R}))$ such that $\ell^2 P$ converges (up to a subsequence) weakly to μ . The equation (103) follows taking the limit $\ell \rightarrow 0$ in the weak formulation of (12). Finally, taking the limit in the weak formulation of (87), we can prove that $u_x(t, x) \leq 2M/(Mt + 2)$. \square

The question whether or not $\mu = 0$ is open. The following proposition shows that when $\ell \rightarrow 0$ for smooth solutions (i.e., before appearance of singularities), u^ℓ converges to the unique solution u of the classical Burgers equation.

Proposition 3. *If u_0 is in $H^s \cap BV$ with $s \geq 3$, then for $t < 1/\sup_x |u_0'(x)|$ we have*

$$\mu = 0. \quad (113)$$

Proof. From Theorem 2 and Remark 2, we can find a uniform upper bound on u_x^ℓ in the space $L^\infty([0, T], L^\infty(\mathbb{R}))$ with $T < T^*$, which implies that $\ell^2 P \rightarrow 0$. \square

5.2. The limiting case $\ell \rightarrow +\infty$. The goal of this subsection is to show that, when $\ell \rightarrow +\infty$, the dissipative solution u^ℓ converges (up to a subsequence) to a function u that satisfies:

$$\left[u_t + \frac{1}{2}(u^2)_x \right]_x = \nu, \quad (114)$$

where $0 \leq \nu \in L^\infty([0, +\infty[, \mathcal{M}^1)$. In proposition 4 below, we show that before the appearance of singularities, the measure $\nu = u_x^2/2$. The question whether or not $\nu = u_x^2/2$ in general is posed. We have the following theorem:

Theorem 6. *Let $u_0 \in H^1$ such that $u_0' \in L^1$ and $u_0'(x) \leq M \forall x$, then there exists $u \in \mathcal{C}([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$ for all $T > 0$, where $\mathcal{I} \Subset \mathbb{R}$, such that there exists a subsequence of u^ℓ (noted also u^ℓ) and such that*

$$u^\ell \xrightarrow{\ell \rightarrow +\infty} u \text{ in } \mathcal{C}([0, T], L^1(\mathcal{I})), \quad (115)$$

u satisfying the equation (114). Moreover, u satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{2M}{Mt + 2} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (116)$$

Remark 10. *If $\nu = \frac{1}{2}u_x^2$ then u is a dissipative solution of the Hunter–Saxton equation [5].*

Proof. Let the compact set $[0, T] \times \mathcal{I} \subset \mathbb{R}^+ \times \mathbb{R}$. Supposing that $\ell \geq 1$ then, from (86), the dissipative solution of rB satisfies

$$\|P\|_1 = \frac{1}{2} \|u_x^\ell\|_2^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2. \quad (117)$$

Using Lemma 1, one gets that u^ℓ is bounded in $L^\infty([0, T] \times \mathbb{R})$ and

$$\int_{\mathbb{R}} |u^\ell(t, x+h) - u^\ell(t, x)| dx \leq \|u'_0\|_1 \left(\frac{MT+2}{2} \right)^2 |h|. \quad (118)$$

Integrating (12) between t_1 and t_2 , one obtains

$$u^\ell(t_1, x) - u^\ell(t_2, x) = \int_{t_1}^{t_2} (u^\ell u_x^\ell + \ell^2 P_x) dt. \quad (119)$$

Using Lemma 1, inequality (117) and

$$\|P_x\|_\infty \leq \frac{1}{4} \ell^{-2} \|u_x^\ell\|_2^2,$$

we can show that there exists $B = B(T, \mathcal{I})$ such that

$$\int_{\mathcal{I}} |u^\ell(t_2, x) - u^\ell(t_1, x)| dx \leq B |t_2 - t_1|. \quad (120)$$

The compactness follows using Theorem A.8 in [13].

The quantity $\frac{1}{2} u_x^\ell{}^2$ is non-negative and bounded in $L^\infty([0, +\infty[, L^1(\mathbb{R}))$, which implies that there exists a non-negative measure $\nu \in L^\infty([0, +\infty[, \mathcal{M}^1(\mathbb{R}))$ such that P converges (up to a subsequence) weakly to ν . The equation (114) follows by taking the limit $\ell \rightarrow +\infty$, in the weak formulation of (15). Finally, taking the limit in the weak formulation of (87), we can prove that $u_x(t, x) \leq \frac{2M}{Mt+2}$. \square

The question whether or not the equality always holds $\nu = u_x^2/2$ is open. The following proposition shows that, when $\ell \rightarrow +\infty$ for smooth solutions (before appearance of singularities), u^ℓ converges to a dissipative solution u of the Hunter–Saxton equation [5].

Proposition 4. *If u_0 is in $H^s \cap BV$ with $s \geq 3$, then for $t < 1/\sup_x |u'_0(x)|$ we have*

$$\nu = \frac{1}{2} u_x^2. \quad (121)$$

Proof. From Theorem 2 and Remark 2, we can find a uniform upper bound on u_x^ℓ in the space $L^\infty([0, T], L^\infty(\mathbb{R}))$ with $T < T^*$, which implies that the convergence u_x^ℓ to u_x is strong. Thus, $u_x^\ell{}^2 \rightarrow u_x^2$. \square

6. OPTIMALITY OF THE \dot{H}_{loc} SPACE

In the previous sections (see Proposition 2, Theorem 2 and Theorem 4), we have shown, on the one side, that even if the initial datum u_0 is smooth, there exists a blow-up finite time $T^* > 0$ such that

$$\inf_{x \in \mathbb{R}} u_x(t, x) > -\infty \quad \forall t < T^*, \quad \inf_{x \in \mathbb{R}} u_x(T^*, x) = -\infty. \quad (122)$$

On the other side, the Oleinik inequality (85) shows that, even if the initial datum is not Lipschitz, the derivative of the solution becomes instantly bounded from above, i.e.

$$\sup_{x \in \mathbb{R}} u'_0(x) = +\infty, \quad \sup_{x \in \mathbb{R}} u_x(t, x) < +\infty \quad \forall t > 0. \quad (123)$$

Remark 11. *If the derivative of the initial datum is bounded from below and not from above, it will be instantly bounded from both sides³ and, after T^* , it will be bounded from above and not from below.*

³Note that the gain of regularity (123) is instantaneous, while the loss of regularity (122) needs some time.

This remark is important to prove that the space \dot{H}_{loc}^1 is the best space to obtain global (in time) solutions, the optimality being in the following sense:

Theorem 7. *Let g be a non-negative and locally bounded function, such that $g(u) \rightarrow +\infty$ when $|u| \rightarrow +\infty$. Then there exist $u_0 \in H^1 \cap W^{1,\infty}$, $T > 0$ and a compact \mathcal{K} , such that there exists a solution u of (12) satisfying*

$$\int_{\mathbb{R}} u_0'(x)^2 g(u_0'(x)) dx < +\infty, \quad \int_{\mathcal{K}} u_x(T, x)^2 g(u_x(T, x)) dx = +\infty. \quad (124)$$

Thus, we cannot expect that u belongs to $W^{1,p}$ for $p > 2$ for all time. In other words, the space $H^1 = W^{1,2}$ is optimal for the equation (12).

Before proving Theorem 7, let $u_0 \in H^s$ with s big enough, and let u be a solution of rB with $u(0, x) = u_0(x)$. The main quantity is the following integral

$$\int_{\mathcal{K}} u_x^2(T, x) g(u_x(T, x)) dx, \quad (125)$$

where $T > 0$ and \mathcal{K} is a compact set. Using the change of variable $x = y(T, \xi)$, one gets

$$\int_{\mathcal{K}} u_x^2(T, x) g(u_x(T, x)) dx = \int_{\mathcal{K}'} q \sin^2(v/2) g(\tan(v/2)) d\xi, \quad (126)$$

where \mathcal{K}' is another compact. From previous sections, the quantity q is always bounded, which implies that if g is bounded then (125) is bounded. If g is not bounded (see the conditions of Theorem 7), then the quantity (125) depends on the behaviour of the derivative u_x at time T . The proof of Theorem 7 is done by building $u(T, \cdot)$ as a function of g , such that the quantity (125) is infinite. Then, we use a backward system to go back in time and find a Lipschitz initial datum.

Proof of Theorem 7: Let g satisfy the conditions of Theorem 7 and let $x_0 \in \mathbb{R}$, then there exists a function $\bar{u} \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R}/\{x_0\})$ such that

$$\int_{\mathcal{V}(x_0)} \bar{u}'(x)^2 g(\bar{u}'(x)) dx = +\infty, \quad \bar{u}'(x) \leq C, \quad (127)$$

where $\mathcal{V}(x_0)$ denotes a neighbourhood of x_0 .

The idea of the proof is to use a backward (in time) system such that $u(T, x) = \bar{u}(x)$. The initial datum u_0 is the unknown. To simplify the presentation, the conservative system (49) is used. With this system, we will obtain a local (in time) Oleinik inequality, which is enough for our construction. A similar proof can be used with the dissipative system (50) with a global Oleinik inequality. The built solution in the interval $[0, T[$ is Lipschitz, so both systems (49), (50) yield the same solution.

In order to build u_0 , we use the forward existence proof given in Section 4. One can use the change of variable $t \rightarrow -t$. The conservative system (49) becomes then

$$y_t = -u, \quad y(-T, \xi) = \bar{y}(\xi), \quad (128a)$$

$$u_t = \ell^2 P_x, \quad u(-T, \xi) = \bar{u}(\bar{y}(\xi)), \quad (128b)$$

$$v_t = P(1 + \cos(v)) + \sin^2(v/2), \quad v(-T, \xi) = 2 \arctan(\bar{u}'(\bar{y}(\xi))), \quad (128c)$$

$$q_t = -q\left(\frac{1}{2} - P\right) \sin(v), \quad q(-T, \xi) = 1, \quad (128d)$$

where $t \in [-T, 0]$ and \bar{y} is defined as in (42), replacing u_0 by \bar{u} .

The proof of a local existence of solutions can be done as in Section 4. Due to the change of variable $t \rightarrow -t$, the Oleinik inequality becomes

$$u_x(t, x) \geq 2/(t+T) \quad (129)$$

for $t > -T$ and t close enough to $-T$. The proof of this Oleinik inequality proceeds as in Section 4 using the equation (128c), which implies that the derivative of the solution is bounded from below. As in Remark 11, since $\bar{u}' = u_x(-T, \cdot) \leq C$, the derivative of the solution remains bounded from above for $t > -T$ and t close enough to $-T$. Taking $T > 0$ small so the solution is Lipschitz until $t = 0$, and thus

$$\int_{\mathbb{R}} u_x(0, x)^2 g(u_x(0, x)') dx < +\infty.$$

The result follows directly by using the change of variable $t \rightarrow -t$. \square

7. CONCLUSION

In this paper, we have studied a regularisation of the inviscid Burgers equation (12). For a smooth initial datum, the regularised equation (12) has a unique smooth solution locally in time. After the blow-up time, the solution is no longer unique, nor smooth. At least two types of solutions exist: conservative and dissipative solutions. We find that the built dissipative solutions are more interesting because they satisfy an Oleinik inequality (85), which plays an important role in showing that solutions converge (up to a subsequence) when $\ell \rightarrow 0$ and when $\ell \rightarrow \infty$ (ℓ the regularising positive parameter). Before the appearance of singularities, the limit when $\ell \rightarrow 0$ (respectively $\ell \rightarrow \infty$) is a smooth solution of the inviscid Burgers (resp. the Hunter–Saxton) equation. After the breakdown time, it remains open to determine whether the Burgers (resp. the Hunter–Saxton) equation holds in the limit without a remaining forcing term.

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation under Grant No. DMS 1812609 (RLP).

APPENDIX A. COMMENTS ON THE CONSERVATIVE AND DISSIPATIVE SOLUTIONS

As shown above, the major difference between the conservative system (49) and the dissipative system (82) is that the system (49) allows v to cross the value $-\pi$, causing a jump of u_x from $-\infty$ to $+\infty$ (see eq. (76)), which implies (33), thence the loss of the Oleinik inequality (Remark 5). But, the value $v = -\pi$ is a barrier that cannot be crossed for the system (82). It follows that if $v(t, \xi_0) = -\pi$ at a time t , then $v(\tau, \xi_0) = -\pi$ for all times $\tau \geq t$ (see figure 1). This property is important to obtain the Oleinik inequality (85), which yields the dissipation of the energy (86).

The figure 1 shows the domains where $v = -\pi$ for the systems (49) and (82).

Note that the dissipative solution of rB has similar properties (85) and (86) as the entropy solution of the classical inviscid Burgers equation that are

$$u_x(t, x) \leq 1/t, \quad \|u\|_2 \leq \|u_0\|_2. \quad (130)$$

Because of this similarity, the dissipative solutions of rB are more likely to converge to the entropic solution of the Burgers equation when $\ell \rightarrow 0$, but this result remains to be proven.

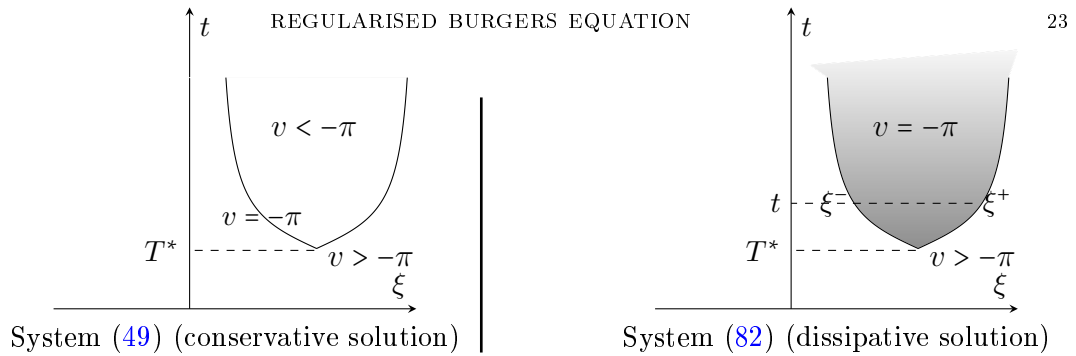


FIGURE 1. Regions where $v = -\pi$.

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