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GLOBAL WEAK SOLUTIONS OF A HAMILTONIAN REGULARISED BURGERS EQUATION

BILLEL GUELMAME, STÉPHANE JUNCA, DIDIER CLAMOND AND ROBERT L. PEGO

Abstract. A nondispersive, conservative regularisation of the inviscid Burgers equation is proposed and studied. Inspired by a related regularisation of the shallow water system recently introduced by Clamond and Dutykh, the new regularisation provides a family of Galilean-invariant interpolants between the inviscid Burgers equation and the Hunter–Saxton equation. It admits weakly singular regularised shocks and cusped traveling-wave weak solutions. The breakdown of local smooth solutions is demonstrated, and the existence of two types of global weak solutions, conserving or dissipating an H^1 energy, is established. Dissipative solutions satisfy an Oleinik inequality like entropy solutions of the inviscid Burgers equation. As the regularisation scale parameter ℓ tends to 0 or ∞, limits of dissipative solutions are shown to satisfy the inviscid Burgers or Hunter–Saxton equation respectively, forced by an unknown remaining term.

AMS Classification: 35B65; 35B44; 35Q35; 35L67.

Key words: Inviscid Burgers equation; regularisation; Hamiltonian; conservative and dissipative solutions; Oleinik inequality.

Contents

1. Introduction

The dispersionless shallow water equations, also called the Saint-Venant equations, admit shock-wave solutions. Recently, a Hamiltonian regularisation of this system (rSV), has been

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proposed which approximates these discontinuous waves by less singular ones [10]. The rSV system can be written

$$h_t + [h u]_x = 0, \tag{1.1a}$$

$$[h u]_t + \left[h u^2 + \tfrac{1}{2} g h^2 + \varepsilon \mathcal{R} h^2\right]_x = 0, \tag{1.1b}$$

$$\mathcal{R} \stackrel{\text{def}}{=} h\left(u_x^2 - u_{xt} - u u_{xx}\right) - g\left(h h_{xx} + \tfrac{1}{2} h_x^2\right), \tag{1.1c}$$

where ε is a small positive parameter, h is the total water depth and u is the velocity. The classical Saint-Venant equations can be obtained letting $\varepsilon \to 0$. This regularisation is Galilean invariant, non-dispersive, non-diffusive, and conserves energy for regular solutions. It also admits regularised shock-wave weak solutions which have the same wave speed and which dissipate energy at the same rate as shocks in the classical Saint-Venant (cSV) equations, [32]. Some mathematical results on rSV were obtained by Pu et al. [32] and Liu et al. [31], but several natural questions remain open, such as the existence of global weak solutions. Inspired by the rSV equations, a more general regularisation of the unidimensional barotropic Euler system has been derived and studied in [20].

In the present work we consider such questions for an analogous but simpler model equation, namely a Hamiltonian regularisation of the inviscid Burgers equation $u_t + uu_x = 0$. Motivated by the rSV and the dispersionless Camassa-Holm [8] equations, in Section 2 we describe a regularised Burgers equation (rB) in the form

$$u_t + u u_x = \ell^2 \left(u_{txx} + 2 u_x u_{xx} + u u_{xxx}\right), \tag{1.2}$$

where $\ell \geqslant 0$ is a parameter. Being a scalar equation, the rB equation is more tractable than the rSV system. An equation mathematically equivalent to (1.2) has previously appeared in [19, Remark 1] together with a Hamiltonian formulation. It can be compared to the well-known Camassa–Holm (CH) equation [8], the Degasperis–Procesi (DP) equation [18] and the Benjamin–Bona–Mahony (BBM) equation [1] (see Section 2 below).

The purpose of the present paper is to establish several basic results for (1.2), including the existence of local smooth solutions, blow-up, global weak solutions, and weakly singular traveling waves, and also to study the limiting cases $\ell \to 0$ and $\ell \to +\infty$.

The local (in time) existence of smooth solutions of the rB equation (1.2) has been established in [35, 36] for a generalised Camassa–Holm equation that covers (1.2) as special case. The existence of global weak solutions of the Camassa–Holm equation in the space H^1 has been widely studied before, we refer to [6, 7, 9, 12, 14, 22, 23, 24, 34]. We also refer to [15, 16, 17] for the existence of solutions of the DP equation.

Our treatment of global weak solutions is analogous to the treatment of the Camassa–Holm equation by Bressan and Constantin in [6, 7]. We rewrite (1.2) into an equivalent semi-linear system, but without asking the initial data to be in H^1. We then prove the existence of a so-called *conservative* global weak solution (Theorem 4.3, cf. [6]), which locally conserves energy. Energy conservation may not be appropriate for approximating shock waves, however. We obtain another type of solution called *dissipative* (see Theorem 5.3 below, cf. [7]), by slightly modifying the equivalent system. Dissipative solutions satisfy an Oleinik inequality of the form

$$u_x(t,x) \leqslant \frac{C}{t}, \qquad t > 0,\ x \in \mathbb{R}. \tag{1.3}$$

This inequality is well known to ensure uniqueness for entropy solutions of the inviscid Burgers equation. However, uniqueness for dissipative solutions of rB remains an open problem.

In order to study the limiting cases $\ell \to 0$ and $\ell \to +\infty$, the equivalent system and the Oleinik inequality (1.3) are used to obtain a uniform BV estimate independent of the parameter ℓ for the dissipative solutions (Lemma 7.1). When $\ell \to 0$, a dissipative solution converges (up to a subsequence) to a function u that satisfies the Burgers equation with a remaining term (see Theorem 7.2 below). If the remaining term is zero, then the entropy solution of Burgers is recovered. We prove that this term is zero for smooth solutions of Burgers equations (see Proposition 7.5). However, the disappearance of the remaining term in general remains an open problem. Similar results are obtained when $\ell \to +\infty$, where the limit is a solution of the Hunter–Saxton equation, at least before the appearance of singularities[1] (Theorem 7.6 and Proposition 7.8 below). The limiting case $\ell \to 0$ of the Camassa–Holm equation is more challenging. Indeed, dissipative solutions of the CH equation satisfy an Oleinik inequality with a constant that depends on ℓ. Thus, the compactness arguments presented in this paper cannot be used for the CH equation. However, the limiting case of the viscous CH equation have been studied in [11, 13, 28] under the condition "ℓ is small enough compared to the viscosity parameter". The authors proved that as the viscosity parameter goes to zero, we recover the unique entropy solution of the scalar conservation law $u_t + (3u^2/2)_x = 0$.

We find below that the rB equation has a great variety of weakly singular traveling wave solutions, solutions which are bounded, continuous and piecewise smooth but which may dissipate (or gain) energy at isolated points where derivatives become infinite. All these waves have analogs for the rSV system (1.1a)–(1.1c). In particular, corresponding to each simple shock-wave entropy solution of the inviscid Burgers equation, there is a monotonic traveling-wave *dissipative* solution of the rB equation, having the same limiting states, shock speed and energy dissipation rate. We also find cusped traveling waves (both periodic and solitary in nature) that are *conservative.* Furthermore, there is a great abundance of composite waveforms that are neither dissipative nor conservative, which were overlooked in [32] but are similar to some of the many types of weak traveling wave solutions of the Camassa-Holm equation found by Lenells [30].

This paper is organised as follows. A heuristic derivation of the rB equation is given in section 2. Section 3 is devoted to study the existence of local smooth solutions. In Section 4, a proof of the existence of global conservative solutions is given. The global dissipative solutions are obtained in Section 5. Weakly singular traveling waves are described in Section 6, including energy-conserving 'cuspons' and energy-dissipating weakly singular shocks. Section 7 studies the limiting cases $\ell \to 0$ and $\ell \to +\infty$ for dissipative solutions. The optimality of the requirement that $u_x \in L^2_{loc}$ for weak solutions is shown in section 8, where we prove in particular that when a smooth solution breaks down, u_x may blow up in L^p_{loc} for all $p > 2$.

2. Heuristic derivation of a regularised Burgers equation

In order to describe a suitable regularisation of the inviscid Burgers equation with similar features as the rSV system (1.1), we note first that the rSV equations yield

$$u_t + u u_x + g h_x + \varepsilon (h \mathcal{R}_x + 2 \mathcal{R} h_x) = 0. \tag{2.1}$$

When h is taken constant, this equation (with the definition of $\mathcal{R}$ given at (1.1c)) becomes

$$u_t + u u_x = \ell^2 [u_{xxt} - u_x u_{xx} + u u_{xxx}], \tag{2.2}$$

[1] "Singularity" is used here to describe the blow-up of derivatives, which corresponds to shocks of the classical Burgers equation. Contrary to the Burgers case, solutions of rB remain continuous at the singularities.

where $\ell \stackrel{\text{def}}{=} h\sqrt{\varepsilon} \geqslant 0$ is a constant characterising a length scale for the regularisation. After the change of independent variables $(t,x) \to (t/\ell, x/\ell)$, which leaves the inviscid Burgers equation invariant, equation (2.2) becomes

$$u_t \;+\; u\,u_x \;=\; u_{xxt} - u_x\,u_{xx} + u\,u_{xxx}. \tag{2.3}$$

Equation (2.3) belongs to a three-parameter family of non-dispersive equations, given by

$$u_t \;-\; u_{xxt} \;=\; a\,u\,u_x \;+\; b\,u_x\,u_{xx} \;+\; c\,u\,u_{xxx}, \qquad \text{for } a,b,c \in \mathbb{R}. \tag{2.4}$$

In this family, we look for an equation that has Galilean invariance and conservation of energy (at least for smooth solutions). The family (2.4) includes a number of famous equations, including the the dispersionless Camassa–Holm equation [8]

$$u_t \;+\; 3\,u\,u_x \;=\; u_{xxt} \;+\; 2\,u_x\,u_{xx} \;+\; u\,u_{xxx}, \tag{2.5}$$

as well as the Degasperis–Procesi equation [18] and the Benjamin–Bona–Mahony equation [1]. None of these equations have the properties we seek, however. E.g., it is well known that the Camassa–Holm equation conserves the H^1 energy [8], but is not Galilean invariant.

In order to obtain a Galilean invariant regularisation of the Burgers equation, one must take $c = -a = 1$ in (2.4). The special case $b = 0$ was studied by Bhat and Fetecau [2, 3, 4], who proved the existence of the solution and the convergence to weak solutions of the Burgers equation in the limit corresponding to $\ell \to 0$. The limit fails to satisfy the entropy condition for the Riemann problem with $u_{\text{left}} < u_{\text{right}}$ [4]. For this regularisation, no energy conservation equation is known.

In the present paper, we consider $c = -a = 1$ (to ensure Galilean invariance, as in [2]) and, in order to maintain conservation of the H^1 norm at least for smooth solutions, we take $b = 2$ (as in the Camassa–Holm equation). With this done, equation (2.4) becomes

$$u_t \;+\; u\,u_x \;=\; u_{txx} \;+\; 2\,u_x\,u_{xx} \;+ u\,u_{xxx}, \tag{2.6}$$

Introducing the scaling $(t,x) \mapsto (\ell\,t, \ell\,x)$, we obtain

$$u_t \;+\; u\,u_x \;=\; \ell^2\,(u_{txx} \;+\; 2\,u_x\,u_{xx} \;+ u\,u_{xxx})\,, \tag{2.7}$$

that is a formal approximation of the Burgers equation for small ℓ. Equation (2.7) is the regularised Burgers (rB) equation studied in this paper. It is Galilean invariant, and smooth solutions of (2.7) satisfy a conservation law for an H^1 energy density, namely

$$\left[\,\tfrac{1}{2}\,u^2 \,+\, \tfrac{1}{2}\,\ell^2\,u_x^2\,\right]_t \;+\left[\,\tfrac{1}{3}\,u^3 \,-\, \ell^2\,u^2\,u_{xx} \,-\, \ell^2\,u\,u_{xt}\,\right]_x \;=\; 0. \tag{2.8}$$

A mathematically equivalent equation was proposed in [19, Remark 1] as a modification of the BBM equation that possesses a Galilean-like invariance property.

We remark that the rB equation (2.7) has variational structure (described further in [21]) that we will not use here and which appears unrelated to H^1 energy conservation. E.g., (2.7) can be obtained as the Euler–Lagrange equation for an action with Lagrangian density

$$\mathcal{L}_\ell \;\stackrel{\text{def}}{=}\; \tfrac{1}{2}\,\phi_x\,\phi_t \;+\; \tfrac{1}{6}\,\phi_x^3 \;+\; \tfrac{1}{2}\,\ell^2\left[\phi_x\,\phi_{xx}^2 \;-\; \phi_{xxx}\,\phi_t\right], \tag{2.9}$$

where ϕ is a velocity potential, i.e., $u = \phi_x$. The rB equation has also a Hamiltonian structure [19, 21], but with Hamiltonian different from the H^1 energy. Indeed, with the Hamiltonian operator and functional

$$\mathscr{D} \;\stackrel{\text{def}}{=}\; \left(1 - \ell^2\,\partial_x^2\right)^{-1}\,\partial_x, \qquad \mathfrak{H} \;\stackrel{\text{def}}{=}\; \int \left[\,\tfrac{1}{6}\,u^3 \,+\, \tfrac{1}{2}\,\ell^2\,u\,u_x^2\,\right] \mathrm{d}x, \tag{2.10}$$

the equation

$$u_t \;=\; -\mathscr{D}\,\delta_u\,\mathfrak{H} \;=\; -\mathscr{D}\left[\,\tfrac{1}{2}\,u^2 \,+\, \tfrac{1}{2}\,\ell^2\,u_x^2 \,-\, \ell^2\,(u u_x)_x\,\right] \tag{2.11}$$

can be rewritten in a form equivalent to the rB equation (2.7), namely

$$u_t \;+\; u\,u_x \;+\; \ell^2\,P_x \;=\; 0, \qquad P \;\stackrel{\text{def}}{=}\; \mathfrak{G} * \tfrac{1}{2}\,u_x^2 \geqslant 0, \qquad \mathfrak{G} \;\stackrel{\text{def}}{=}\; (2\ell)^{-1}\,\exp(-|x|/\ell), \tag{2.12}$$

where $*$ denotes the convolution product. For comparison, the Camassa–Holm equation (2.5) can be rewritten in the form

$$u_t \;+\; u\,u_x \;+\left[\,\mathfrak{G} * \left(\tfrac{1}{2}\,u_x^2 \,+\, u^2\right)\right]_x \;=\; 0, \qquad \mathfrak{G}(x) \;\stackrel{\text{def}}{=}\; \tfrac{1}{2}\,\exp(-|x|). \tag{2.13}$$

Differentiating (2.12) with respect to x, and using that $P - \ell^2 P_{xx} = \frac{1}{2}u_x^2$, one obtains

$$[\,u_t \,+\, u\,u_x\,]_x \;+\; P \;=\tfrac{1}{2}\,u_x^2. \tag{2.14}$$

Note that P goes formally to zero as $\ell \to +\infty$, whence one obtains the Hunter–Saxton (HS) equation [26, 27]

$$[\,u_t \,+\, u\,u_x\,]_x \;=\; \tfrac{1}{2}\,u_x^2. \tag{2.15}$$

Note also that by taking $\ell \to +\infty$ formally in (2.7), we obtain the derivative of (2.15) with respect of x.

In this section, we have provided a heuristic derivation of a regularised Burgers equation by imposing the important physical requirements of Galilean invariance and energy conservation. We have also related this equation with well-known equations. In the rest of the paper, we perform a rigorous mathematical investigation of its solutions.

3. Existence and breakdown of smooth solutions

This section is devoted to show the local existence and breakdown of smooth solutions for the Cauchy problem (2.12) with $u(0,x) = u_0(x)$. The form (2.12) of the regularised Burgers equation is more convenient for studying smooth solutions than (2.7), because it involves fewer derivatives.

Usually, one needs an equation for u_x to study the life span of smooth solutions. Equation (2.14) can be written

$$u_{xt} \;+\; \tfrac{1}{2}\,u_x^2 \;+\; u\,u_{xx} \;+\; P \;=\; 0. \tag{3.1}$$

Multiplying (2.12) by u and multiplying (3.1) by $\ell^2\,u_x$, we obtain

$$\left[\,\tfrac{1}{2}\,u^2\,\right]_t \;+\left[\,\tfrac{1}{3}\,u^3 \,+\, \ell^2\,u\,P\,\right]_x \;=\; \ell^2\,u_x\,P, \tag{3.2}$$

$$\left[\,\tfrac{1}{2}\,\ell^2\,u_x^2\,\right]_t \;+\left[\,\tfrac{1}{2}\,\ell^2\,u\,u_x^2\,\right]_x \;=\; -\,\ell^2\,u_x\,P, \tag{3.3}$$

which imply an energy conservation law for smooth solutions; i.e., we have the (conservative) energy equation

$$\left[\,\tfrac{1}{2}\,u^2 \,+\, \tfrac{1}{2}\,\ell^2\,u_x^2\,\right]_t \;+\left[\,\tfrac{1}{3}\,u^3 \,+\, \ell^2\,u\,P \,+\, \tfrac{1}{2}\,\ell^2\,u\,u_x^2\,\right]_x \;=\; 0. \tag{3.4}$$

For a class of equations including rB as special case, Yin [35, 36] has proven the following local existence result.

Theorem 3.1 (Yin [35, 36])**.** *For an initial datum $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$, there exists a maximal time $T^* > 0$ (independent of s) and a unique solution $u \in \mathcal{C}([0,T^*[\,,H^s)$ of (2.12) such that (blow-up criterium)*

$$T^* < +\infty \qquad \Longrightarrow \qquad \lim_{t\uparrow T^*}\,\|\,u(t,\cdot)\,\|_{H^s} \;=\; +\infty. \tag{3.5}$$

Moreover, if $s \geqslant 3$, then

$$T^* < +\infty \qquad \Longrightarrow \qquad \lim_{t \uparrow T^*} \inf_{x \in \mathbb{R}} u_x(t,x) \; = \; -\infty. \tag{3.6}$$

Furthermore, the solution given in this theorem satisfies the Oleinik inequality:

Proposition 3.2. (Oleinik inequality) *Let $u_0 \in H^s(\mathbb{R})$ with $s \geqslant 2$ and let $M = \sup_{x \in \mathbb{R}} u_0'(x)$. Then, for all $t \in [0, T^*[$ the solution given in Theorem 3.1 satisfies*

$$\sup_{x \in \mathbb{R}} u_x(t,x) \; \leqslant \; \tfrac{2\,M}{M\,t+2} \; \leqslant \; M. \tag{3.7}$$

Proof. Let $x_0 \in \mathbb{R}$ and let the characteristic $\eta(t,x_0)$ be defined as the solution of the Cauchy problem $\eta_t(t,x_0) = u(t,\eta(t,x_0))$, with the initial datum $\eta(0,x_0) = x_0$. With $H(t,x_0) \stackrel{\text{def}}{=} u_x(t,\eta(t,x_0))$, the equation (3.1) can be rewritten

$$H_t \; + \; \tfrac{1}{2}\,H^2 \; + \; P \; = \; 0. \tag{3.8}$$

Since $P \geqslant 0$, it follows that $H_t \leqslant -\frac{1}{2}H^2$ which implies that $H(t,x_0) \leqslant \frac{2H(0,x_0)}{H(0,x_0)t+2} \leqslant \frac{2M}{Mt+2}$. □

Remark 3.3. The Oleinik inequality (3.7) is valid only when the solution u is smooth. In Theorem 5.3 below, we show that this inequality holds for all times also for a certain type of weak solutions (called *dissipative*) such that $u \in H^1$ (and, possibly, for $M = +\infty$).

Unfortunately, the solution given in Theorem 3.1 does not exist globally in time for all non trivial initial data [35]. Since Yin [35] studied a general family of equations including rB, his result is not optimal for rB. In the following proposition, this result is improved with a shorter proof.

Proposition 3.4. (An upper bound on the blow-up time) *Let $u_0 \in H^s(\mathbb{R})$ with $s \geqslant 2$. If there exists $x_0 \in \mathbb{R}$ such that $u_0'(x_0) < 0$, then $T^* \leqslant -2/\inf u_0'$.*

Proof. From the proof of the previous proposition, we have

$$H(t,x_0) \; \leqslant \; \frac{2H(0,x_0)}{tH(0,x_0)+2}, \qquad t < T^*. \tag{3.9}$$

If $T^* > -2/\inf u_0'$ then $H(0,x_0) < 0$ implies

$$\lim_{t \to -2/H(0,x_0)} H(t,x_0) = -\infty,$$

this contradicts $u \in \mathcal{C}([0,T^*[, H^s)$. □

A uniform (with respect to ℓ) lower bound on T^* is needed, in order to prove in section 7 below the convergence of smooth solutions (see Proposition 7.5 and Proposition 7.8).

Theorem 3.5. (A lower bound on the blow-up time) *Let u_0 in H^s be non-trivial with $s \geqslant 2$ and let*

$$m(t) \;\stackrel{\text{def}}{=}\; \inf_{x \in \mathbb{R}} u_x(t,x) \; < \; 0 \; < \; M(t) \;\stackrel{\text{def}}{=}\; \sup_{x \in \mathbb{R}} u_x(t,x), \qquad t \; < \; T^*.$$

If $|m(0)| \geqslant M(0)$ then

$$-\,1\big/\inf u_0' \; \leqslant \; T^* \; \leqslant \; -2\big/\inf u_0'. \tag{3.10}$$

If $|m(0)| < M(0)$ then, there exists t^* such that $0 < t^* \leqslant -m(0)^{-1} - M(0)^{-1}$ and $m(t^*) = -M(t^*)$. Therefore

$$t^* \; + \; 1\big/\sup u_0' \; \leqslant \; T^* \; \leqslant \; -2\big/\inf u_0'. \tag{3.11}$$

Remark 3.6. Note that the blow-up time T^* is uniformly (with respect to ℓ) bounded from below by $1/\sup|u_0'|$.

Proof. Since $u \in H^s$, $u_x \to 0$ when x goes to $\pm\infty$, and u_x is not the zero function, so $m(t) = \min_{x\in\mathbb{R}} H(t,x) < 0 < M(t) = \max_{x\in\mathbb{R}} H(t,x)$. The equation (3.8) implies that m and M are decreasing in time, so $|m| = -m$ is increasing. So, if $|m(t_0)| \geqslant M(t_0)$, then for all $t > t_0$ we have $|m(t)| \geqslant M(t)$.

The inequality (3.9) shows that for $t < T^*$

$$0 \; < \; M(t) \; \leqslant \; \frac{2\,M(0)}{M(0)\,t \; + \; 2}, \qquad m(t) \; \leqslant \; \frac{2\,m(0)}{m(0)\,t \; + \; 2} \; < \; 0, \tag{3.12}$$

which implies that, if $|m(0)| < M(0)$, there exists $t^* \leqslant -(m(0)+M(0))/(m(0)M(0))$ such that $|m(t^*)| = M(t^*)$.

If $\delta > 0$ is small enough, since the function $H(t+\delta,\cdot)$ has a minimum, then there exists x_δ such that $m(t+\delta) = H(t+\delta, x_\delta)$. Inspired by Junca and Lombard [29] one gets

$$\begin{aligned} m(t+\delta) \; &= \; H(t+\delta, x_\delta) \; = \; H(t,x_\delta) \; + \; \int_t^{t+\delta} H_t(s,x_\delta)\,\mathrm{d}s \\ &\geqslant \; m(t) \; - \; \int_t^{t+\delta} \left(\tfrac{1}{2}\,H(s,x_\delta)^2 \; + \; P(s,x_\delta)\right)\mathrm{d}s. \end{aligned} \tag{3.13}$$

Since $m(\cdot) < 0$ and δ is arbitrary small, we have $m(s) \leqslant H(s,x_\delta) \leqslant 0$ then $m(s)^2 \geqslant H(s,x_\delta)^2$, implying that

$$\frac{m(t+\delta) - m(t)}{\delta} \; \geqslant \; -\frac{1}{\delta}\int_t^{t+\delta}\left(\tfrac{1}{2}\,m(s)^2 + \sup_{x\in\mathbb{R}} P(s,x)\right)\mathrm{d}s. \tag{3.14}$$

Defining the generalised derivative

$$\dot{m}(t) \; \stackrel{\text{def}}{=} \; \liminf_{\delta\to 0^+} \frac{m(t+\delta) \; - \; m(t)}{\delta}, \tag{3.15}$$

one can show that

$$\dot{m}(t) \; + \; \tfrac{1}{2}\,m(t)^2 \; \geqslant \; -\sup_{x\in\mathbb{R}} P(t,x). \tag{3.16}$$

Using the definition of P from (2.12) and using that $\|\mathfrak{G}\|_1 = 1$, one obtains

$$\sup_{x\in\mathbb{R}} P(t,x) \; \leqslant \; \tfrac{1}{2}\,\|\,u_x(t,\cdot)\,\|_\infty^2 \; \leqslant \; \max\left\{\tfrac{1}{2}\,M(t)^2, \tfrac{1}{2}\,m(t)^2\right\}. \tag{3.17}$$

and the Riccati-like inequality (3.16) becomes

$$\dot{m}(t) \; + \; m(t)^2 \; \geqslant \; 0 \qquad t > t^*. \tag{3.18}$$

Then $T^* - t^* \geqslant -1/m(t^*) = 1/M(t^*)$, and with (3.12), one obtains

$$T^* \; \geqslant \; t^* \; + \; 1\big/\sup u_0'. \qquad \square$$

4. Global weak solutions: conservative case

Note that Proposition 3.4 shows that, for $s \geqslant 2$, we have $\lim_{t\uparrow T^*} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty$ which implies that

$$\lim_{t\uparrow T^*} \|\,u(t,\cdot)\,\|_{H^s} \; = \; +\infty.$$

Hence the space H^s with $s \geqslant 2$ is not the right space in order to obtain the global existence of the solution.

Bressan and Constantin [6, 7] have proved the existence of two types of global solutions for the Camassa–Holm equation (2.13) in H^1. Using the formal energy equation (3.4), a similar proof (of global existence of conservative and dissipative solutions in H^1) for rB can be done following [6, 7]. Another proof of existence of a dissipative solution, using the vanishing viscosity method, is given by Chen and Tian [9], Xin and Zhang [34].

In this paper, the existence theorem will be developed for solutions not vanishing as $|x| \to \infty$. Note that a major difference between the rB (2.12) and the Camassa–Holm (2.13) equations is that u^2 does not appear in the non-local term of rB. This allows us to get global existence for rB without asking u to be in $L^2(\mathbb{R})$. Moreover, in Theorem 8.2 below, we show that asking $u_x \in L^2$ is optimal.

These remarks lead us to assert in the following the existence of two types of solutions of rB: conservative and dissipative. We start the analysis in this section by defining a conservative solution.

Definition 4.1. *A function u is called a* conservative *solution of* rB *if*

- *The function u belongs to* $\mathrm{Lip}([0,T], L^2_{\mathrm{loc}})$ *and* $u_x \in L^\infty([0,T], L^2_{\mathrm{loc}})$ *for all* $T > 0$.
- *u satisfies the equation (2.12), with an initial data $u(0,x) = u_0(x)$.*
- *u satisfies (3.4) in the sense of distributions.*

It means that it is a weak solution conserving the energy, as smooth solutions.

Remark 4.2. The regularity $u_x \in L^\infty([0,T], L^2_{\mathrm{loc}})$ ensures that (3.2) is satisfied. Thence, the equalities (3.3) and (3.4) are equivalent.

Introducing the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}) = \{f : \|f'\|_2 < +\infty\}$, we can state the theorem:

Theorem 4.3. *Let $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If there exists a Lipschitz function ϕ such that $\phi' \in L^1(\mathbb{R})$ with $u_0 - \phi \in H^1(\mathbb{R})$, then there exists a global conservative solution u of (2.12), such that $u(t,\cdot) - \phi \in H^1(\mathbb{R})$ for all $t > 0$. In addition, for all $T > 0$*

$$\liminf_{t\uparrow T}\ \inf_{x\in\mathbb{R}} u_x(t,x) \;=\; -\infty \quad\Longrightarrow\quad \limsup_{t\downarrow T}\ \sup_{x\in\mathbb{R}} u_x(t,x) \;=\; +\infty, \tag{4.1}$$

and if $u_0 \in H^1$, then for almost all $t > 0$

$$\int_{\mathbb{R}} \left[\, u(t,x)^2 + \ell^2\ u_x(t,x)^2 \,\right] \mathrm{d}x \;=\; \int_{\mathbb{R}} \left[\, u_0(x)^2 + \ell^2\ u_0'(x)^2 \,\right] \mathrm{d}x. \tag{4.2}$$

Remark 4.4. This theorem covers also some solutions that do not have a limit when $|x| \to \infty$, such as $\phi(x) = u_0(x) = \cos\ln(x^2+1)$.

Remark 4.5. Note that (4.1) implies that the Oleinik inequality (3.7) cannot hold after the appearance of singularities.

Proof of Theorem 4.3. In the special case $u_0 \in H^1(\mathbb{R})$, the proof can be done following Bressan and Constantin [6]. In the general case, the energy is modified as

$$E(t) = \int_{\mathbb{R}} [u(t,x) - \phi(x)]^2 + \ \ell^2\ u_x(t,x)^2\, \mathrm{d}x, \tag{4.3}$$

and the proof is done in several steps:

- In the first step we obtain an energy estimate of the solution in the Eulerian coordinates.

- In the second step we define a mapping from the Eulerian to the Lagrangian coordinates where we obtain an equivalent semi-linear system of rB.
- In the third and the fourth steps we prove the existence of global solutions of the equivalent system in the Lagrangian coordinates.
- In step 5, we rewrite the solution of the equivalent system in the Eulerian coordinates and we show that it is a global conservative solution of rB.

Step 1: Formal energy estimate on the x-variable. Let $\tilde{u}(t,x)=u(t,x)-\phi(x)$. The equation (2.12) can be rewritten

$$u_t\ +\ u\,u_x\ +\ \ell^2\,P_x\ =\ \tilde{u}_t\ +\ u\,u_x\ +\ \ell^2\,P_x\ =\ 0. \tag{4.4}$$

Multiplying (4.4) by $\tilde{u}$, one gets

$$\left[\,\tfrac{1}{2}\,\tilde{u}^2\,\right]_t\ +\left[\,\tfrac{1}{3}\,u^3\ -\ \tfrac{1}{2}\,\phi\,u^2\,\right]_x\ +\ \tfrac{1}{2}\,\phi_x\,u^2\ +\ \ell^2\,u\,P_x\ -\ \ell^2\,\phi\,P_x\ =\ 0. \tag{4.5}$$

Adding (4.5) and (3.3), we obtain

$$\tfrac{1}{2}\left[\tilde{u}^2+\ell^2\,u_x^2\right]_t+\left[\tfrac{1}{3}u^3+\tfrac{1}{6}\phi^3-\tfrac{1}{2}\phi u^2+\tfrac{1}{2}\ell^2uu_x^2+\ell^2uP\right]_x=\ell^2\phi P_x-\tfrac{1}{2}\phi_x\left(\tilde{u}^2+2\phi\tilde{u}\right). \tag{4.6}$$

Integrating over the real line, one gets (exploiting the triangular inequality)

$$\tfrac{1}{2}\,E'(t)\ \leqslant\ \int_{\mathbb{R}}\left(\,\ell^2\,|\phi P_x|\ +\ \tfrac{1}{2}\,|\phi'|\left(\,2\,\tilde{u}^2\ +\ \phi^2\,\right)\right)\mathrm{d}x. \tag{4.7}$$

The Young inequality implies that

$$\|\,P(t)\,\|_p\ \leqslant\ \tfrac{1}{2\,\ell^2}\,\|\,\mathfrak{G}\,\|_p\,E(t)\qquad\forall p\in[1,\infty], \tag{4.8a}$$

$$\|\,P_x(t)\,\|_p\ \leqslant\ \tfrac{1}{2\,\ell^3}\,\|\,\mathfrak{G}\,\|_p\,E(t)\qquad\forall p\in[1,\infty]. \tag{4.8b}$$

Using (4.7) and (4.8b), we obtain

$$E'(t)\ \leqslant\left(\,\ell^{-1}\,\|\,\phi\,\|_\infty\ +\ 2\,\|\,\phi'\,\|_\infty\,\right)E(t)\ +\ \|\,\phi\,\|_\infty^2\,\|\,\phi'\,\|_1. \tag{4.9}$$

Then the Gronwall lemma ensures that $E(t)$ does not blow up in finite time.

Step 2: Equivalent system. As in [6], let $\xi\in\mathbb{R}$ and let $y_0(\xi)$ be defined by

$$\int_0^{y_0(\xi)}\left(1\ +\ {u_0'}^2\right)\mathrm{d}x\ =\ \xi, \tag{4.10}$$

and let $y(t,\xi)$ be the function[2] defined by the equation

$$y_t(t,\xi)\ =\ u(t,y(t,\xi)),\qquad y(0,\xi)\ =\ y_0(\xi). \tag{4.11}$$

Let also $v=v(t,\xi)$ and $q=q(t,\xi)$ be defined as

$$v\ \stackrel{\text{def}}{=}\ 2\arctan(u_x),\qquad q\ \stackrel{\text{def}}{=}\ \left(\,1\ +\ u_x^2\,\right)y_\xi, \tag{4.12}$$

where $u_x(t,\xi)=u_x(t,y(t,\xi))$. Notice that

$$\frac{1}{1+u_x^2}\ =\ \cos^2\!\left(\frac{v}{2}\right),\quad \frac{u_x}{1+u_x^2}\ =\ \frac{\sin(v)}{2},\quad \frac{u_x^2}{1+u_x^2}\ =\ \sin^2\!\left(\frac{v}{2}\right),\quad \frac{\partial y}{\partial\xi}\ =\ q\cos^2\!\left(\frac{v}{2}\right). \tag{4.13}$$

Integrating the last equality in (4.13), one obtains

$$y(t,\xi')\ -\ y(t,\xi)\ =\ \int_\xi^{\xi'}q(t,s)\cos^2\!\left(\frac{v(t,s)}{2}\right)\mathrm{d}s. \tag{4.14}$$

[2]It will turn out that $y(t,\xi)$ is the characteristic of rB corresponding to $y_0(\xi)$, with speed $u(t,y(t,\xi))$.

Using (4.13) and the change of variables $x = y(t,\xi')$, (4.14), P and P_x can be written in the new variables as

$$\begin{aligned} P(t,\xi) &= \frac{1}{4\,\ell}\int_{\mathbb{R}} \exp\!\left(\frac{-|y(t,\xi)-x|}{\ell}\right) u_x^2(t,x)\,\mathrm{d}x \\ &= \frac{1}{4\,\ell}\int_{\mathbb{R}} \exp\!\left(-\frac{1}{\ell}\left|\int_{\xi}^{\xi'} q(t,s)\cos^2\!\left(\frac{v(t,s)}{2}\right)\mathrm{d}s\right|\right) q(t,\xi')\sin^2\!\left(\frac{v(t,\xi')}{2}\right)\mathrm{d}\xi', \end{aligned} \tag{4.15}$$

$$\begin{aligned} P_x(t,\xi) &= \frac{1}{4\,\ell^2}\left(\int_{y(t,\xi)}^{+\infty} - \int_{-\infty}^{y(t,\xi)}\right)\exp(-|y(t,\xi)-x|\ell)\,u_x^2(t,x)\,\mathrm{d}x \\ &= \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)\,\exp\!\left(-\left|\int_{\xi}^{\xi'} q(t,s)\cos^2\!\left(\frac{v(t,s)}{2}\right)\frac{\mathrm{d}s}{\ell}\right|\right) q(t,\xi')\sin^2\!\left(\frac{v(t,\xi')}{2}\right)\frac{\mathrm{d}\xi'}{4\,\ell^2}. \end{aligned} \tag{4.16}$$

Then, a system equivalent to the rB equation is given by

$$y_t = u, \qquad y(0,\xi) = y_0(\xi), \tag{4.17a}$$
$$u_t = -\ell^2\,P_x, \qquad u(0,\xi) = u_0(y_0(\xi)), \tag{4.17b}$$
$$v_t = -P\,(1+\cos(v)) - \sin^2(v/2), \qquad v(0,\xi) = 2\arctan\!\big(u_0'(y_0(\xi))\big), \tag{4.17c}$$
$$q_t = q\left(\tfrac{1}{2}-P\right)\sin(v), \qquad q(0,\xi) = 1. \tag{4.17d}$$

In order to prove Theorem 4.3, we prove first the global existence of the solution of the initial-value problem (4.17), then we infer that this solution yields a conservative solution of rB.

Step 3: Local existence for the new system. Our goal is to prove that the system of equations (4.17) is locally well-posed. The proof given in [6] for the Camassa–Holm equation is slightly simplified here.

We first solve a coupled 2x2 subsystem instead of a 3x3 subsystem in [6]. Let u_0 be a function such that $u_0 - \phi \in H^1$, then y_0 is well defined in (4.10). Note that the right-hand side of (4.17) does not depend on y. Since P and P_x depend only on v and q, the right-hand sides of equations (4.17b), (4.17c) and (4.17d) do not depend on u. Also, the equations (4.17c) and (4.17d) are coupled. Thus, we are left to show that the system of two equations

$$v_t = -P(1+\cos v) - \sin^2\frac{v}{2}, \qquad v(0,\xi) = v_0(\xi) = 2\ \arctan u_0'(y_0(\xi)), \tag{4.18a}$$
$$q_t = q\ \left(\tfrac{1}{2}-P\right)\sin v, \qquad q(0,\xi) = q_0(\xi) = 1 \tag{4.18b}$$

is well defined in the space $X \stackrel{\text{def}}{=} \mathcal{C}([0,T],L^\infty(\mathbb{R},\mathbb{R}^2))$.

Let $U = (v,q)$, and let $\mathfrak{D} \subset X$ be the closed set satisfying $U(0,\xi) = U_0(\xi)$ and

$$1/C \leqslant q(t,\xi) \leqslant C \qquad \forall (t,\xi) \in [0,T]\times\mathbb{R}, \tag{4.19a}$$
$$\left|\left\{\xi,\ \sin^2\tfrac{v(t,\xi)}{2} \geqslant \tfrac{1}{2}\right\}\right| \leqslant C \qquad \forall t \in [0,T], \tag{4.19b}$$

where $C > 0$ is a constant. Then, for $\xi_1 < \xi_2$, from the equations (4.19) we obtain

$$\int_{\xi_1}^{\xi_2} q(\xi)\,\cos^2\frac{v(\xi)}{2}\,\mathrm{d}\xi \geqslant \int_{\left\{\xi\in[\xi_1,\xi_2],\ \sin^2\frac{v(t,\xi)}{2}\leqslant\frac{1}{2}\right\}} \frac{C^{-1}}{2}\,\mathrm{d}\xi \geqslant \left[\frac{\xi_2-\xi_1}{2} - \frac{C}{2}\right]C^{-1}. \tag{4.20}$$

Let Γ be defined as

$$\Gamma(\zeta) = \min\left\{1,\ \exp\left(\frac{1}{2\,\ell} - \frac{|\zeta|}{2\,\ell}C^{-1}\right)\right\}. \tag{4.21}$$

Then, for $(v,q) \in \mathfrak{D}$, the exponential terms in (4.15) and (4.16) are smaller than $\Gamma(\xi - \xi')$.

Let $P(\xi, v, q)$ be defined by (4.15). If $(v,q) \in \mathfrak{D}$ then, using Young inequality, $\partial_v P$ and $\partial_q P$ are bounded, i.e., for $\{U, \tilde{U}\} \in \mathfrak{D}$ we have

$$\|P(\xi, U) \;-\; P(\xi, \tilde{U})\|_X \;\lesssim\; \|U \;-\; \tilde{U}\|_X, \tag{4.22}$$

where the symbol $\lesssim$ means "less or equal" with a constant depending only on C and ℓ. Then, for T small enough, the Picard operator

$$(\mathcal{P}(U))(t,\xi) \;\stackrel{\text{def}}{=}\; U_0 \;+\; \int_0^t \left(-(1+\cos v)\,P \;-\; \sin^2\frac{v}{2}\;,\; q\left(\tfrac{1}{2} - P\right)\sin v\right)\mathrm{d}\tau, \tag{4.23}$$

is a contraction from $\mathfrak{D}$ to $\mathfrak{D}$. The local existence of the solution of the Cauchy problem (4.18) follows at once.

Step 4: Global existence for the equivalent system. After proving the local existence of the solution of system (4.18), an estimate of the quantity

$$\|\,q(t)\,\|_\infty \;+\; \|\,1\,/\,q(t)\,\|_\infty \;+\; \left\|\sin^2(v(t)\,/\,2)\right\|_1 \;+\; \|\,v(t)\,\|_\infty, \tag{4.24}$$

is needed to ensure that the solutions exist for all time. Let u be defined as

$$u(t,\xi) \;\stackrel{\text{def}}{=}\; u_0(y_0(\xi)) \;-\; \int_0^t \ell^2\, P_x(s,\xi)\,\mathrm{d}s, \tag{4.25}$$

and let y be the family of characteristics

$$y(t,\xi) \;\stackrel{\text{def}}{=}\; y_0(\xi) \;+\; \int_0^t u(s,\xi)\,\mathrm{d}s, \tag{4.26}$$

and, finally, let $\phi(t,\xi) \stackrel{\text{def}}{=} \phi(y(t,\xi))$. Our task here is to show that the modified energy

$$\tilde{E}(t) \;=\; \int_{\mathbb{R}} \left[(u-\phi)^2\cos^2\frac{v}{2} \;+\; \ell^2\sin^2\frac{v}{2}\right] q\,\mathrm{d}\xi \tag{4.27}$$

does not blow-up in finite time.

The system (4.17) implies that

$$\left(q\,\cos^2\frac{v}{2}\right)_t \;=\; \tfrac{1}{2}\,q\,\sin v, \qquad \left(q\,\sin^2\frac{v}{2}\right)_t \;=\; q_t \;-\; \tfrac{1}{2}\,q\,\sin v \;=\; -\,q\,P\,\sin v, \tag{4.28}$$

while the equations (4.15) and (4.16) imply that

$$P_\xi \;=\; q\,P_x\cos^2\frac{v}{2}, \qquad \ell^2\,(P_x)_\xi \;=\; q\,P\cos^2\frac{v}{2} \;-\; \tfrac{1}{2}\,q\,\sin^2\frac{v}{2}. \tag{4.29}$$

From (4.17), (4.25) and (4.29), we have

$$\left(u_\xi \;-\; \tfrac{1}{2}\,q\,\sin v\right)_t \;=\; 0,$$

and, for $t = 0$, we have from (4.12) and (4.13)

$$u_\xi \;-\; \tfrac{1}{2}\,q\,\sin v \;=\; u_x\,\frac{\partial y}{\partial \xi} \;-\; \tfrac{1}{2}\,\sin v \;=\; 0.$$

Thus, as long as the solution of (4.17) is defined, the equality

$$u_\xi \;=\; \tfrac{1}{2}\,q\,\sin v \tag{4.30}$$

holds. Therefore, the equations (4.28), (4.29), (4.30) yield

$$\left[\left(u^2\cos^2\frac{v}{2} + \ell^2\sin^2\frac{v}{2}\right)q\right]_t + \left[2\,\ell^2\,u\,P - \tfrac{1}{3}\,u^3\right]_\xi = 0, \tag{4.31}$$

which expresses conservation of energy in the (t,ξ)-variables when $u_+ = u_- = 0$, i.e., for $\phi = 0$.

From (4.28), (4.30) and (4.26), we have

$$\left(q\cos^2\frac{v}{2}\right)_t = u_\xi = \left(\frac{\partial y}{\partial \xi}\right)_t,$$

implying that the equality

$$\frac{\partial y}{\partial \xi} = q\cos^2\frac{v}{2} \tag{4.32}$$

holds for the (t,ξ)-variables (note that the equality is true for $t = 0$ from (4.13)). Then, using (4.26) and (4.32), we get

$$\phi_t(t,\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\,\phi(y(t,\xi)) = u\,\phi', \qquad \phi_\xi(t,\xi) = q\cos^2\!\left(\frac{v}{2}\right)\phi', \tag{4.33}$$

so, using (4.25), (4.17), (4.33) and (4.28), we obtain

$$\begin{aligned}
\left[(\phi^2 - 2\,u\,\phi)\,q\cos^2\frac{v}{2}\right]_t &+ \left[\tfrac{1}{2}\,\phi\,u^2 + \tfrac{1}{2}\,\phi\,(u-\phi)^2 - \tfrac{1}{6}\,\phi^3\right]_\xi = \\
&2\,\ell^2\,P_x\,\phi\,q\cos^2\frac{v}{2} - 2\,u\,\phi_\xi\,(u-\phi) + \tfrac{1}{2}\,\phi_\xi\,u^2 + (\phi^2 - 2\,u\,\phi)u_\xi \\
&+ \phi\,u\,u_\xi + \tfrac{1}{2}\,\phi_\xi\,(u-\phi)^2 + \phi\,(u-\phi)_\xi(u-\phi) - \tfrac{1}{2}\,\phi^2\phi_\xi \\
&= -\,\phi_\xi(u-\phi)^2 + 2\,\ell^2\,P_x\,\phi\,q\cos^2\frac{v}{2} - 2\,\phi\,\phi_\xi(u-\phi).
\end{aligned} \tag{4.34}$$

Adding (4.31) and (4.34), with the trivial relation $2\,\phi\,(u - \phi) \leqslant \phi^2 + (u-\phi)^2$, then integrating the result with respect of ξ, we get

$$\tilde{E}'(t) \leqslant \int_{\mathbb{R}}\left(2\,\ell^2\,|\phi\,P_x|\,q\cos^2\frac{v}{2} + |\phi_\xi|\,\left(2\,(u-\phi)^2 + \phi^2\right)\right)\mathrm{d}\xi. \tag{4.35}$$

Using (4.32) and (4.33) with the change of variables $x = y(t,\xi)$, then expoiting (4.8), one obtains

$$\begin{aligned}
\tilde{E}'(t) &\leqslant \int_{\{\xi,\cos v\neq -1\}}\left(2\,\ell^2\,|\phi\,P_x| + |\phi'|\,\left(2\,(u-\phi)^2 + \phi^2\right)\right)q\cos^2\frac{v}{2}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}}\left(2\,\ell^2\,|\phi\,P_x| + |\phi'|\,\left(2\,(u-\phi)^2 + \phi^2\right)\right)\mathrm{d}x \\
&\leqslant \left(\ell^{-1}\,\|\,\phi\,\|_\infty + 2\,\|\,\phi'\,\|_\infty\right)E(t) + \|\,\phi\,\|_\infty^2\,\|\,\phi'\,\|_1,
\end{aligned}$$

where P_x in the second equation is defined as $P_x = \tfrac{1}{2}\,\mathfrak{G}_x * u_x^2$.

From (4.3) and (4.27), and using the change of variables $x = y(t,\xi)$, one can show easily that

$$E(t) = \int_{\{\xi,\cos v\neq -1\}}\left[(u-\phi)^2\cos^2\frac{v}{2} + \ell^2\sin^2\frac{v}{2}\right]q\,\mathrm{d}\xi \leqslant \tilde{E}(t). \tag{4.36}$$

Thence, the uniform estimate of $\tilde{E}(t)$ on any bounded interval $[0,T]$ follows by using Gronwall lemma.

We can show now that the quantity (4.24) does not blow up in finite time. Using Young inequality, (4.15), (4.16) and (4.21), one obtains

$$\|P(t)\|_p \leqslant \tfrac{1}{4\ell^3}\,\|\Gamma\|_p\,\tilde{E}(t) \qquad \forall p\in[1,\infty], \tag{4.37a}$$

$$\|P_x(t)\|_p \leqslant \tfrac{1}{4\ell^4}\,\|\Gamma\|_p\,\tilde{E}(t) \qquad \forall p\in[1,\infty]. \tag{4.37b}$$

The inequalities (4.37) are the identical estimates as (4.8), but in the (t,ξ)-variables. Using (4.17d) and (4.37), we get

$$|q_t| \leqslant \left(\tfrac{1}{2} + \tfrac{1}{4\ell^3}\,E(t)\right) q, \tag{4.38}$$

implying that $\|q(t)\|_\infty + \|1/q(t)\|_\infty$ does not blow-up in finite time. The equation (4.17c) and (4.37) imply that $\|v(t)\|_\infty$ remains bounded on any finite interval $[0,T]$. Also, the boundedness of the energy $\tilde{E}(t)$ and $\|1/q(t)\|_\infty$ implies that $\left\|\cos^2(v(t)/2)\right\|_1$ remains bounded on any interval $[0,T]$. This completes the proof of the global existence.

Step 5: Global existence of a conservative solution. Here, we show that the global solution of the equivalent system (4.17) yields a global solution of the rB equation.

Let u and y be defined by (4.25) and (4.26), respectively. We claim that the solution of rB can be written as

$$u(t,x) = u(t,\xi), \qquad y(t,\xi) = x. \tag{4.39}$$

Using (4.30), (4.33) and the change of variables $x = y(t,\xi)$ with (4.32), one obtains

$$\begin{aligned}
|u(t,\xi) - \phi(t,\xi)|^2 &\leqslant 2\int_{\mathbb{R}} |u-\phi|\,|u_\xi - \phi_\xi|\,\mathrm{d}\xi \\
&\leqslant 2\int_{\mathbb{R}} |u-\phi|\,q\left(\sin\frac{v}{2}\cos\frac{v}{2} + \phi'\cos^2\frac{v}{2}\right)\mathrm{d}\xi \\
&\leqslant 2\,E(t) + \|\phi'\|_2^2,
\end{aligned}$$

implying that $\|u(t)\|_\infty$ is uniformly bounded on any bounded interval $[0,T]$. Therefore, from (4.26), we get

$$y_0(\xi) - \|u(t)\|_\infty\, t \leqslant y(t,\xi) \leqslant y_0(\xi) + \|u(t)\|_\infty\, t, \tag{4.40}$$

and thus

$$\lim_{\xi\to\pm\infty} y_0(t,\xi) = \pm\infty. \tag{4.41}$$

The equation (4.32) implies that the mapping $\xi\mapsto y(t,\xi)$ is non-decreasing and, if for $\xi < \xi'$ we have $y(t,\xi) = y(t,\xi')$, then $\sin(v) = 2\cos(v/2)\sin(v/2) = 0$ between ξ and ξ' (see eq. 4.32). Integrating (4.30) with respect to ξ, one obtains that $u(t,\xi) = u(t,\xi')$, so u is well-defined in (4.39).

Proceeding as in [6, section 4], we can prove that for each interval $[t_1,t_2]$ there exists a constant $C = C(\ell,t_2)$ such that, $\forall t\in[t_1,t_2-h]$,

$$\int_{\mathbb{R}} |u(t+h,x) - u(t,x)|^2\mathrm{d}x \leqslant C\,h^2, \tag{4.42}$$

and then u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,u(t,y(t,\xi)) = -P_x(t,\xi). \tag{4.43}$$

The inequality (4.42) implies that u belongs to $Lip([0,T],L^2_{loc})$. Straightforward calculations show that, for $x = y(t,\xi)$ and for $\cos(v(t,\xi))\neq 1$, we have

$$u_x(t,x) = \tan\!\left(\frac{v(t,\xi)}{2}\right) = \frac{\sin(v(t,\xi))}{1+\cos(v(t,\xi))}. \tag{4.44}$$

Using the change of variables $x = y(t,\xi)$ with (4.32), one can show that u is a global solution of rB.

In order to prove (3.3), let ψ be a test function and let $\tilde{\psi}(t,\xi) = \psi(t, y(t,\xi))$. Multiplying (4.28$b$) by $\tilde{\psi}$ and integrating the result with respect to ξ, one obtains

$$\begin{aligned}
0 \ &= \ \int_0^{+\infty}\int_{\mathbb{R}} \left[\left(q\,\sin^2 v/2\right)_t \, + \, q\,P\,\sin v\right]\tilde{\psi}\,\mathrm{d}t\,\mathrm{d}\xi, \\
&= \ \int_0^{+\infty}\int_{\mathbb{R}} \left[-\tilde{\psi}_t\,q\,\sin^2 v/2 \, + \, \tilde{\psi}\,q\,P\,\sin v\right]\mathrm{d}t\,\mathrm{d}\xi \ + \ \int_{\mathbb{R}} \tilde{\psi}(0,x)\,\sin^2 v(0,\xi)/2\,\mathrm{d}\xi, \\
&= \ \int\int_{\{\cos v > -1\}} \left[-\tilde{\psi}_t\,q\,\sin^2 v/2 \, + \, \tilde{\psi}\,q\,P\,\sin v\right]\mathrm{d}t\,\mathrm{d}\xi \ + \ \int_{\{v_0 > -\pi\}} \tilde{\psi}(0,x)\,\sin^2 v(0,\xi)/2\,\mathrm{d}\xi, \\
&\quad + \ \int\int_{\{\cos v = -1\}} -q\,\tilde{\psi}_t\,\mathrm{d}t\,\mathrm{d}\xi \ + \ \int_{\{v_0 = -\pi\}} \tilde{\psi}(0,x)\,\mathrm{d}\xi.
\end{aligned} \tag{4.45}$$

It is clear from (4.17c) that

$$|\{\xi, \cos v(t,\xi) = -1\}| = 0 \quad \text{for almost all } t \geqslant 0. \tag{4.46}$$

Using that $\tilde{\psi}_t = \psi_t + u\psi_x$ and the change of variables $x = y(t,\xi)$, the equation (3.3) follows in the sense of distributions.

Finally, let $u_0 \in H^1$. The equation (4.31) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(u^2\,\cos^2\tfrac{v(t,\xi)}{2} \ + \ \ell^2\,\sin^2\tfrac{v(t,\xi)}{2}\right) q(t,\xi)\,\mathrm{d}\xi \ = \ 0, \tag{4.47}$$

hence $\tilde{E}(t) = \tilde{E}(0)$. In addition, using the change of variables $x = y(t,\xi)$ with (4.32) and (4.44), one obtains

$$\int_{\mathbb{R}} u(t,x)^2 \ + \ \ell^2\,u_x(t,x)^2\,\mathrm{d}x \ = \ \int_{\{\xi,\ \cos v(t,\xi) > -1\}} \left(u^2\,\cos^2\tfrac{v(t,\xi)}{2} \ + \ \ell^2\,\sin^2\tfrac{v(t,\xi)}{2}\right) q(t,\xi)\,\mathrm{d}\xi. \tag{4.48}$$

Using (4.46), the conservation of the energy (4.2) follows.

We end this demonstration with the proof of the property (4.1). The equation (4.17c) implies that v is decreasing in time. Further, if $v(T,\xi) = -\pi$ (corresponding to an infinite value of u_x, see (4.44) above) then $v_t(T,\xi) = -1$, meaning that the value of $v(t,\xi)$ crosses $-\pi$ and $v(t,\xi) < -\pi$ for all $t > T$. Then, (4.1) follows using (4.44). $\square$

5. Global weak solutions: dissipative case

We start this section by defining *dissipative* solutions, this kind of solution being very important for applications. We note in passing that when ℓ goes to zero, we expect to recover the entropy solution of the Burgers equation. However, in Section 7, we show that the limit (up to a subsequence) is a solution of the Burgers equation with a remaining forcing term.

Definition 5.1. *A function u is called a* dissipative *solution of* rB *if*

- *The function u belongs to* $\mathrm{Lip}([0,T], L^2_{\mathrm{loc}})$ *and* $u_x \in L^\infty([0,T], L^2_{\mathrm{loc}})$ *for all* $T > 0$.*;*
- *u satisfies the equation (2.12), with an initial data $u(0,x) = u_0(x)$;*
- *u satisfies the inequality*

$$\left[\tfrac{1}{2}\,u^2 \, + \, \tfrac{1}{2}\,\ell^2\,u_x^2\right]_t \ + \left[\tfrac{1}{3}\,u^3 \, + \, \ell^2\,u\,P \, + \, \tfrac{1}{2}\,\ell^2\,u\,u_x^2\right]_x \ \leqslant \ 0, \tag{5.1}$$

in the sense of distributions.

- *There exists a constant C such that u satisfies the Oleinik inequality*

$$u_x(t,x) \leqslant C/t \qquad \forall t,x.$$

Remark 5.2. Following [7], we construct in Theorem 5.3 a dissipative solution of rB with $C=2$. The entropy solutions of the classical Burgers equation satisfy the Oleinik inequality with $C=1$.

As mentioned above, when v crosses the value $-\pi$, u_x jumps from $-\infty$ to $+\infty$, which means that the Oleinik inequality cannot be satisfied. Thus, to enforce the Oleinik inequality, the value of v is not allowed to leave the interval $[-\pi,\pi[$. For that purpose, the system (4.18) is modified (as in [7]) to become

$$u_t = -\ell^2 P_x, \tag{5.2a}$$

$$v_t = \begin{cases} -P\,(1+\cos v) - \sin^2(v/2), & v > -\pi, \\ 0 & v \leqslant -\pi, \end{cases} \tag{5.2b}$$

$$q_t = \begin{cases} q\,(\tfrac{1}{2} - P)\sin(v), & v > -\pi \\ 0 & v \leqslant -\pi. \end{cases} \tag{5.2c}$$

and P and P_x are also modified as

$$P(t,\xi) = \frac{1}{4\,\ell}\int_{\mathbb{R}} \exp\left\{-\frac{1}{\ell}\left|\int_{\xi}^{\xi'} \bar{q}(t,s)\cos^2\frac{v(t,s)}{2}\,\mathrm{d}s\right|\right\} \bar{q}(t,\xi')\sin^2\frac{v(t,\xi')}{2}\,\mathrm{d}\xi', \tag{5.3}$$

$$P_x(t,\xi) = \frac{1}{4\,\ell^2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{-\frac{1}{\ell}\left|\int_{\xi}^{\xi'} \bar{q}(t,s)\cos^2\frac{v(t,s)}{2}\,\mathrm{d}s\right|\right\} \bar{q}(t,\xi')\sin^2\frac{v(t,\xi')}{2}\,\mathrm{d}\xi', \tag{5.4}$$

where $\bar{q}(t,\xi) = q(t,\xi)$ if $v(t,\xi) > -\pi$ and $\bar{q}(t,\xi) = 0$ if $v(t,\xi) \leqslant -\pi$. The system (5.2) is the key tool to prove the following theorem.

Theorem 5.3. *Let $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If there exist a Lipschitz function ϕ such that $\phi' \in L^1(\mathbb{R})$ and with $u_0 - \phi \in H^1(\mathbb{R})$, then there exists a global dissipative solution u of the equation* (2.12)*, such that $u(t,\cdot) - \phi \in H^1(\mathbb{R})$ for all $t > 0$. In addition, for all $t > 0$*

$$u_x(t,x) \leqslant 2/t \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{5.5}$$

and if $u_0 \in H^1$, then for almost all $t > 0$

$$\int_{\mathbb{R}} \left[\, u(t,x)^2 + \ell^2\, u_x(t,x)^2 \,\right] \mathrm{d}x \leqslant \int_{\mathbb{R}} \left[\, u_0(x)^2 + \ell^2\, u_0'(x)^2 \,\right] \mathrm{d}x. \tag{5.6}$$

Remark 5.4. Due to the loss of the Oleinik inequality (cf. Remark 4.5), the system (4.17) is slightly modified to (5.2) in order to obtain *dissipative* solutions of rB that satisfies the Oleinik inequality (5.5).

Remark 5.5. In general, if the initial datum satisfies $u_0' \leqslant M \in \mathbb{R} \cup \{+\infty\}$, then the Oleinik inequality (5.5) can be improved as

$$u_x(t,x) \leqslant 2\,M\,/\,(Mt+2) \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{5.7}$$

as shown in (5.17) below.

Proof of Theorem 5.3. The idea of the proof is similar to Theorem 4.3 above and it is done in the following steps:

- In the first step we prove the existence of the global solution as in Theorem 4.3.
- In the second step we prove the dissipation of the energy and the Oleinik inequality.

Step 1: Existence of a solution. As in the proof of Theorem 4.3, it suffices to show that (5.2b) and (5.2c) are locally well posed in the domain $\mathfrak{D} \subset X$, $\mathfrak{D}$ being defined below and $X \stackrel{\text{def}}{=} \mathcal{C}([0,T], L^\infty(\mathbb{R},\mathbb{R}^2))$.

Note that if v is near $-\pi$ the right-hand side of (5.2b) is discontinuous. To avoid this discontinuity, the system (5.2) is replaced, as in [7], by

$$U_t(t,\xi) = F(U(t,\xi)) + G(\xi, U(t,\cdot)), \qquad U = (v,q), \tag{5.8}$$

with

$$F(U) \stackrel{\text{def}}{=} \begin{cases} (-\sin^2\frac{v}{2}, \frac{1}{2}\, q \sin v) & v > -\pi, \\ (-1, 0) & v \leqslant -\pi, \end{cases} \quad G(U) \stackrel{\text{def}}{=} \begin{cases} (-P\,(1 + \cos v), -P\, q \sin v) & v > -\pi, \\ (0,0) & v \leqslant -\pi, \end{cases}$$

Note also that, as long as the solution to (5.8) is well defined, replacing v by $\max\{-\pi, v\}$ gives a solution of the equations (5.2b) and (5.2c). In the rest of this step, our aim is to show that the system (5.8) is locally well-posed. Let $\delta \in]0, \frac{2\pi}{3}]$ and let Λ be defined by

$$\Lambda \stackrel{\text{def}}{=} \big\{ \xi,\ v_0(\xi) \in]-\pi,\ \delta - \pi] \big\}. \tag{5.9}$$

The equation (5.8) implies that, if $v \in]-\pi,\ \delta-\pi] \ \subset\ ]-\pi,\ -\frac{\pi}{3}]$, then $v_t \leqslant -\frac{1}{2}$. Let $\mathfrak{D} \subset X$ satisfy $U(0,\xi) = U_0(\xi)$ and

$$1/C \leqslant q(t,\xi) \leqslant C \qquad \forall (t,\xi) \in [0,T]\times\mathbb{R}, \tag{5.10a}$$

$$\left|\left\{ \xi,\ \sin^2(v(t,\xi)/2) \geqslant \tfrac{1}{2}\right\}\right| \leqslant C \qquad \forall t \in [0,T], \tag{5.10b}$$

$$\| U(t) - U(s) \|_\infty \leqslant C\,|t-s| \qquad \forall t,s \in [0,T], \tag{5.10c}$$

$$v(t,\xi) - v(s,\xi) \leqslant -\tfrac{t-s}{2} \qquad \forall \xi \in \Lambda,\ 0 \leqslant s \leqslant t \leqslant T. \tag{5.10d}$$

Taking $(v,q) \in \mathfrak{D}$ and using (4.37), one gets that the right-hand sides of (5.2b) and (5.2c) are bounded. However, the inequality (4.22) is no longer true and we have instead

$$\left\| P(U) - P(\tilde{U}) \right\|_\infty \lesssim \left\| U - \tilde{U} \right\|_\infty + |\{\xi,\ (v(\xi)+\pi)(\tilde{v}(\xi)+\pi) < 0\}|, \tag{5.11}$$

which implies that

$$\left\| F(U) - F(\tilde{U})) \right\|_\infty \lesssim \left\| U - \tilde{U} \right\|_\infty, \tag{5.12}$$

$$\left\| G(U) - G(\tilde{U})) \right\|_\infty \lesssim \left\| U - \tilde{U} \right\|_\infty + \left| \left\{ \xi,\ (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0 \right\} \right|. \tag{5.13}$$

In order to estimate the second term of the right-hand side of the last equation, the crossing time τ is defined as

$$\tau(\xi) \stackrel{\text{def}}{=} \sup\, \{t \in [0,T],\ v(t,\xi) > -\pi\}. \tag{5.14}$$

Note that the equation (5.10c) implies that $|v(t,\xi) - v_0(\xi)| \leqslant C\,t$. So, if $\xi \notin \Lambda$ then

$$\min\{\tau(\xi),\ \tilde{\tau}(\xi)\} \geqslant \delta / C.$$

Taking T small enough ($T < \delta/C$) and using (5.10d), one obtains

$$\begin{aligned} \int_0^T \left| \left\{ \xi,\ (v(\tau,\xi) + \pi)(\tilde{v}(\tau,\xi) + \pi) < 0 \right\} \right| \mathrm{d}\tau &\leqslant \int_\Lambda |\tau(\xi) - \tilde{\tau}(\xi)|\, \mathrm{d}\xi \\ &\leqslant 2\,|\Lambda|\, \left\| U - \tilde{U} \right\|_\infty. \end{aligned}$$

Now, the Picard operator

$$(\mathcal{P}(U))(t,\xi) = U_0 + \int_0^t [F(U) + G(U)]\, \mathrm{d}\tau, \tag{5.15}$$

satisfies

$$\left\| \mathcal{P}(U) - \mathcal{P}(\tilde{U}) \right\|_\infty \leqslant K\,(T + |\Lambda|)\, \left\| U - \tilde{U} \right\|_\infty, \tag{5.16}$$

where K depends only on C and ℓ. Since $\sin^2 \frac{v_0}{2} \in L^1$, by choosing $\delta > 0$ small enough, one can make $|\Lambda|$ arbitrary small. Choosing also T small enough, one obtains the local existence of the solution of the system (5.8), yielding a solution of (5.2). The rest of the proof of the existence can be done following the proof of Theorem 4.3.

Step 2: Oleinik inequality and the dissipation of the energy. The equation (5.2b) implies that if $v(0,\xi) \leqslant 0$, then for all $t \geqslant 0$ $v(t,\xi)$ remains in $[-\pi, 0]$. If $v_0(\xi) \in]0,\pi[$ then, as long as v is positive, the following inequality holds

$$\left[\arctan \frac{v}{2} \right]_t \leqslant -\tfrac{1}{2} \arctan^2 \frac{v}{2}.$$

This implies that, if $\arctan \frac{v_0(\xi)}{2} \leqslant M$, then

$$u_x = \arctan \frac{v(t,\xi)}{2} \leqslant \frac{2M}{Mt+2}. \tag{5.17}$$

The Oleinik inequality (5.5) follows taking $M = +\infty$ and using (4.44).

In order to prove the dissipation of the energy (5.1), let ψ be a non-negative test function, then we follow the same computations in the proof of Theorem 4.3. Since (4.46) is no longer true for the system (5.2), one can obtain from (4.45) that

$$\iint\limits_{[0,+\infty[\times\mathbb{R}} \left[-u_x^2 \psi_t - u u_x^2 \psi_x + u_x P \right] \mathrm{d}t\,\mathrm{d}x - \int\limits_{\mathbb{R}} u_0'^2(x) \psi(0,x)\,\mathrm{d}x$$
$$= - \int\limits_{\{\tau(\xi)<+\infty\}} q(\tau(\xi),\xi)\tilde{\psi}(\tau(\xi),\xi)\,\mathrm{d}\xi \leqslant 0,$$

where $\tau(\xi)$ is the crossing time defined as $\tau(\xi) \stackrel{\text{def}}{=} \sup\,\{t \geqslant 0, v(t,\xi) > -\pi\}$. Since (3.2) is satisfied (see Remark 4.2), the dissipation of the energy (5.1) follows.

If $u_0 \in H^1$, as in the last step of the proof of Theorem 4.3, one can show that (4.47) and (4.48) hold for the solution of (5.2), while the measure in (4.46) is not always zero. Then, the dissipation of the energy (5.6) follows. □

6. Traveling waves of permanent form

The rB equation (1.2) can be written in the conservative form

$$\left[\, u - \ell^2 u_{xx} \,\right]_t + \left[\, \tfrac{1}{2} u^2 - \ell^2 u\, u_{xx} - \tfrac{1}{2} \ell^2 u_x^2 \,\right]_x = 0. \tag{6.1}$$

In this section, we describe traveling waves of permanent form, i.e., we seek bounded weak solutions of (6.1) having the form $u = u(x - ct)$. We find a great variety of such waves, roughly comprising a subset of the plethora of traveling wave solutions of the Camassa-Holm equation found by Lenells [30]. For CH, Lenells found peakons, cuspons, stumpons, smooth periodic waves, monotone waves, and composite waves of various kinds. When we consider the requirements imposed by energy conservation or dissipation, however, many composite constructions are eliminated. In particular, the only traveling waves of rB we find that are dissipative in the sense of Definition 5.1 are monotone weakly singular shock layers that correspond to entropy-satisfying shocks of the inviscid Burgers equation in the limit $\ell \to 0$. The rB equation has the nice property that such shock layers exist for every entropy-satisfying shock. This is the scalar analog of the property found by Pu et al. [32] for the nondispersively regularized shallow water system (1.1a)–(1.1c).

Since the rB equation is Galilean invariant, we can work in the frame of reference moving with the wave, where the motion is steady. Moreover, by the rescaling $x/\ell \to x$ we can presume $\ell = 1$. I.e., we look for (weak) solutions such that $u = u(x/\ell)$.

For steady motions with $\ell = 1$, the momentum flux S, given by

$$S \stackrel{\text{def}}{=} \tfrac{1}{2}\,u^2 - u\,u_{xx} - \tfrac{1}{2}\,u_x^2 \;=\; \tfrac{1}{2}\,u^2 + P, \tag{6.2}$$

is constant, cf. (2.14). In any open set where $u \neq 0$, equation (6.2) is an ODE and any weak solution must be smooth. The energy conservation law (2.8) then implies that the energy flux F, given by

$$F(x) \stackrel{\text{def}}{=} \tfrac{1}{3}\,u^3 - u^2\,u_{xx} \;=\; \tfrac{1}{3}\,u^3 + u\,P + \tfrac{1}{2}\,u\,u_x^2, \tag{6.3}$$

is locally constant, cf. (3.4). Eliminating u_{xx} between (6.2) and (6.3), one obtains the first-order differential equation

$$\tfrac{1}{2}\,u\,u_x^2 \;=\; F \;-\; S\,u \;+\; \tfrac{1}{6}u^3. \tag{6.4}$$

6.1. **Local analysis of weak solutions.** As it turns out, the energy flux F may be discontinuous at points where $u = 0$. Consider the possibility of a singularity at $x = x_0$, which can exist only if $u(x_0) = 0$. Equation (6.4) then yields the following asymptotic behavior: As $x \to x_0$ (from either the right or the left),

$$u\,u_x^2 \;\sim\; 2\,F \qquad \text{if } Fu \;>\; 0, \tag{6.5}$$

$$u_x^2 \;\sim\; -2\,S \qquad \text{if } F \;=\; 0 \text{ and } S \;<\; 0, \tag{6.6}$$

implying that, as $x \to x_0$,

$$|u| \;\sim\; \left(\tfrac{3}{2}\sqrt{|2F|}\,|x - x_0|\right)^{2/3} \qquad \text{if } Fu \;>\; 0, \tag{6.7}$$

$$|u| \;\sim\; \sqrt{|2S|}\;|x - x_0| \qquad \text{if } F \;=\; 0 \text{ and } S \;<\; 0. \tag{6.8}$$

By consequence, the quantity $u_x^2 = O(|x - x_0|^{-2/3})$ is locally integrable. Hence by (6.2), one can construct a valid weak solution of (6.1) by solving (6.4) separately for $x < x_0$ and $x > x_0$, allowing F to jump between any two real values from left to right across x_0,

6.2. **Cuspons and periodic cuspons.** In the simplest case, F is globally a nonzero constant. This is necessary for the wave to be a conservative solution according to Definition 4.1, since (3.4) requires $F_x = 0$. Then we obtain a family of waves with cusps having the behavior in (6.7) for x_0 either at a single point or at any point in a periodic grid.

Note that $u(-x)$ is a solution of (6.4) whenever $u(x)$ is, and $-u$ is a solution for $-F$ in place of F. Without loss, then, we may consider the case when $u > 0$ for $x > 0 = x_0$, and $F > 0$. Necessarily, if u is to be bounded, the cubic polynomial on the right-hand side of (6.4) must have two positive roots at points $u_1 \geqslant u_0 > 0$ and a negative root at $-u_1 - u_0$, related to F and S via

$$F = \tfrac{1}{6}u_0u_1(u_0 + u_1), \qquad S = \tfrac{1}{6}(u_0^2 + u_0u_1 + u_1^2). \tag{6.9}$$

By consequence, we can separate variables in (6.4) and find that $u = \eta(x)$ where η is determined implicitly by x from the relation

$$H(\eta) \stackrel{\text{def}}{=} \int_0^{\eta} \left(\frac{3v}{(u_0 - v)(u_1 - v)(u_0 + u_1 + v)}\right)^{1/2} dv = x, \qquad 0 < x < x_* = H(u_0) \leq \infty. \tag{6.10}$$

We obtain a *periodic cuspon* solution whenever $u_0 < u_1$. In this case the integral converges at $\eta = u_0$ to a finite value, and $u(x)$ can be defined by reflection about $x_0 = 0$ as

$$u(x) = \eta(|x|/\ell), \qquad |x| \leqslant x_*\ell, \tag{6.11}$$

then extended as smooth and periodic with period $2x_*\ell$ and maximum value $u_0 = u(x_*\ell)$. Here we have put back the scale parameter ℓ to obtain a stationary weak solution of (6.1) valid for any $\ell > 0$.

We find a *cuspon* with $u(x) \to u_0$ as $x \to \pm\infty$, provided by the same formulas in the case $u_0 = u_1$, when we find $x_* = H(u_0) = +\infty$. This has a single singular point at $x_0 = 0$. In this case,

$$F = \tfrac{1}{3}u_0^3, \qquad S = \tfrac{1}{2}u_0^2, \qquad H(\eta) = \int_0^\eta \left(\frac{3v}{2u_0 + v}\right)^{1/2} \frac{dv}{u_0 - v} = |x|. \tag{6.12}$$

A similar analysis provides negative cuspons $u < 0$ with reversed signs $F < 0$ and $u_1 \leqslant u_0 < 0$. Necessarily $S > 0$ in this case also. In case $F = 0$ globally, one has only the trivial solution $u \equiv 0$, for there are no other bounded solutions, since $u_{xx} = \frac{1}{3}u$ for $u \neq 0$ by (6.3). Finally, also we note that

In summary we can state the following, noting that $\min_{u>0}(|F| - Su + \frac{1}{6}u^3) \le 0$ in all cases.

Proposition 6.1. *For globally constant energy flux F, a bounded nonzero stationary solution u of (6.1) exists, having the form*

$$u(x) = \eta(|x|/\ell)\operatorname{sgn} F, \qquad \text{for } |x| \leqslant x_*\ell, \tag{6.13}$$

provided

$$0 < 3|F| \leqslant (2S)^{3/2}, \tag{6.14}$$

where we have equality for cupsons, strict inequality for periodic cuspons. All these cuspons and periodic cuspons are conservative solutions according to Definition 4.1.

6.3. **Shock layers.** For a stationary solution u to be *dissipative* according to Definition 5.1, we require $u_x \le 0$ to satisfy the Oleinik inequality (1.3). The energy flux $F(x)$ can be discontinuous, but it must be *decreasing* across singularities in order to satisfy the energy dissipation inequality (5.1).

Thus we require $u > 0$ and $F = F_- > 0$ for $x < 0 = x_0$, while $u < 0$ and $F = F_+ < 0$ for $x > 0$. Since $u(x)$ is required to be bounded and monotone we should have

$$u(x) \to \begin{cases} u_- & \text{as } \ x \to -\infty, \\ u_+ & \text{as } \ x \to +\infty, \end{cases} \tag{6.15}$$

where necessarily $S > 0$ and $u_- = -u_+ = \sqrt{2S}$ by taking $x \to \pm\infty$ in (6.2). Since we must solve the same ODE as in the previous subsection, to obtain a global monotonic solution it is necessary that

$$|u_\pm| = (2S)^{1/2} = (3|F_\pm|)^{1/3}. \tag{6.16}$$

We obtain such solutions by taking $u_0 = u_-$ in the formulas in (6.12), and taking the odd extension of the left half of the cuspon, setting

$$u(x) = \begin{cases} \eta(-x/\ell), & x < 0, \\ -\eta(x/\ell), & x > 0. \end{cases} \tag{6.17}$$

As before this yields a solution of (6.1) for any $\ell > 0$. In the limit $\ell \to 0$ we evidently obtain any stationary entropy-satifying shock for the inviscid Burgers equation, which must take two

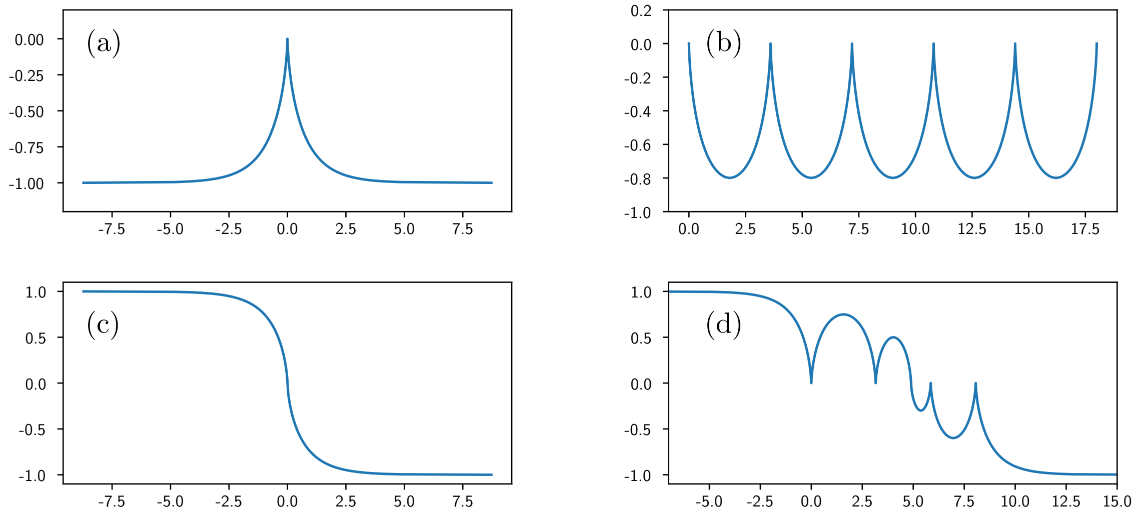


FIGURE 1. Types of weakly singular stationary waves: (a) cuspon; (b) periodic cuspon; (c) shock layer; (d) composite wave

values $u_- > u_+$ with $u_-^2 = u_+^2$. By Galilean boosts we obtain traveling weakly singular shock layer solutions which converge in the limit $\ell \to 0$ to any arbitrary entropy-satisfying simple shock for the inviscid Burgers equation. Thereby we obtain the following.

Proposition 6.2. *Corresponding to any entropy-satisfying simple shock for the inviscid Burgers equation, taking constant values $u_- > u_+$ respectively for $x < ct$ and $x > ct$ with $c = \frac{1}{2}(u_- + u_+)$, the regularized Burgers equation (6.1) admits a weakly singular shock layer solution satsifying*

$$u(x,t) = c - \eta(|x - ct|/\ell)\,\mathrm{sgn}(x - ct) \to \begin{cases} u_- & x \to -\infty, \\ u_+ & x \to +\infty, \end{cases} \tag{6.18}$$

where $\eta(x)$ is determined from (6.12) with $u_0 = \frac{1}{2}(u_- - u_+)$.

Remark 6.3. We note from (5.1) that the rate of energy dissipation for the stationary shock layer is

$$F_- - F_+ = \tfrac{2}{3}u_-^3 = \tfrac{1}{12}(u_- - u_+)^3,$$

and all of the energy dissipation occurs at the location of the weak singularity. This rate is exactly the same as for the corresponding inviscid Burgers shock, which is famously cubic in the amplitude of the shock.

6.4. **Composite waves.** If one discards the criteria we have imposed to find conservative or dissipative solutions, a great many more stationary weak solutions of (6.1) can be constructed, by joining together segments of (periodic) cuspons between consecutive zeros, while allowing the energy flux $F(x)$ to jump discontinuously at the zeros, in an essentially arbitrary way subject only to the inequality (6.14).

We will not describe these solutions in any further detail, since we have already described the only such solutions that are conservative or dissipative according to Definitions 4.1 and 5.1.

At any isolated singular point x_0 where the left-to-right jump $F(x_0+) - F(x_0-)$ is negative, energy is *dissipated*, while energy is *generated* if this jump is positive. One can construct composite solutions having multiple zeros that all dissipate energy, by taking $F(x)$ piecewise constant and non-increasing. One such composite solution is plotted in Figure 1(d). These are not dissipative solutions in the sense of Definition 5.1, however, since they are not monotonic hence violate the Oleinik inequality.

6.5. **Non-existing waves.** Some of the kinds of waves found by Lenells for CH are not possible for rB. In particular, we find that rB does not admit smooth periodic waves, peakons with bounded derivatives, or "stumpons": Such waves can be made stationary by a Galilean transformation, then must have a single sign, which may be assumed nonnegative. If $\min u = 0$ we must have $S > 0$ and the wave must be a cuspon, as found above. And if $\min u > 0$, then the cubic polynomial in (6.4) must take positive values between two zeros $u_0 = \min u < \max u = u_1$. But this is not possible since $-u_0 - u_1$ is the only other zero.

7. The limiting cases $\ell \to 0$ and $\ell \to +\infty$ for dissipative solutions

Taking formally $\ell = 0$, the rB equation becomes the classical Burgers equation, and letting $\ell \to +\infty$ it becomes the Hunter–Saxton equation. In this section, we study the compactness of the dissipative solutions when taking $\ell \to 0$ and $\ell \to +\infty$.

Let the initial datum u_0 be taken in H^1, with $u_0' \in L^1$ and $M \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} u_0'(x) < +\infty$. Let also u^ℓ be the dissipative solution of the rB equation given in Theorem 5.3. In order to take the limit, an estimate on the total variation of u^ℓ is needed. For that purpose, the following Lemma is given

Lemma 7.1. *[BV estimate] If u_0' satisfies the conditions of Theorem 5.3 with $u_0' \in L^1$ and $u_0'(x) \leqslant M\ \ \forall x$, then for all $t \in \mathbb{R}^+$*

$$\mathrm{TV}u^\ell(t,\cdot) \ = \ \left\|u_x^\ell(t,\cdot)\right\|_1 \ \leqslant \ \left\|u_0'\right\|_1 \left(\frac{Mt+2}{2}\right)^2. \tag{7.1}$$

Proof. For $v \in]-\pi,\pi[$, the equation (4.30) implies

$$\tilde{s} \stackrel{\text{def}}{=} \operatorname{sgn}(u_\xi^\ell) \ = \ \operatorname{sgn}(\sin(v^\ell)) \ = \ \operatorname{sgn}(\sin(\frac{v^\ell}{2})), \qquad \cos(\frac{v^\ell}{2}) \ \geqslant \ 0. \tag{7.2}$$

Note that $\tan(v^\ell/2) \leqslant \tan(v_0/2) = 2M/(Mt+2)$ from (5.17). Differentiating (5.2a) w.r.t ξ, multiplying by $\tilde{s}$ — and using (4.29), (4.30) and $\sin v = 2\sin(v/2)\cos(v/2)$ — one gets

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u_\xi^\ell|\,\mathrm{d}\xi \ &= \ -\ell^2 \int_{\mathbb{R}} \tilde{s}\ (P_x)_\xi\ \mathrm{d}\xi = \ -\ell^2 \int_{\{\tilde{s}\,>\,0\}} (P_x)_\xi\ \mathrm{d}\xi \ + \ \ell^2 \int_{\{\tilde{s}\,<\,0\}} (P_x)_\xi\ \mathrm{d}\xi \\
&= \ -2\,\ell^2 \int_{\{\tilde{s}\,>\,0\}} (P_x)_\xi\ \mathrm{d}\xi \ + \ \ell^2 \int_{\mathbb{R}} (P_x)_\xi\ \mathrm{d}\xi \\
&= \ -2 \int_{\{\tilde{s}\,>\,0\}} \left(q^\ell\, P \cos^2\frac{v^\ell}{2} \ - \ \tfrac{1}{2} q^\ell \sin^2\frac{v^\ell}{2} \right) \mathrm{d}\xi \\
&\leqslant \ \int_{\{\tilde{s}\,>\,0\}} q^\ell \sin\frac{v^\ell}{2} \cos\frac{v^\ell}{2} \tan\frac{v^\ell}{2}\ \mathrm{d}\xi \\
&\leqslant \ \frac{2M}{Mt+2} \int_{\mathbb{R}} |u_\xi^\ell|\ \mathrm{d}\xi.
\end{aligned} \tag{7.3}$$

Gronwall lemma then implies that

$$\|u_\xi\|_1 \leqslant \|(u_0)_\xi\|_1 \left(\frac{Mt+2}{2}\right)^2. \tag{7.4}$$

Note that the last inequality is on the ξ−variable. Using that the application $\xi \mapsto y(t,\xi)$ is not decreasing for all t and using that $\mathrm{TV} f = \|f'\|_1$ for smooth solutions ($f \in W^{1,1}_{loc}$), the result follows. □

7.1. **The limiting case $\ell \to 0$.** The goal of this subsection is to show that when $\ell \to 0$, the dissipative solution u^ℓ converges (up to a subsequence) to a function u satisfying the Burgers equation with a source term:

$$u_t + \tfrac{1}{2}\left[u^2\right]_x = -\mu_x, \tag{7.5}$$

where μ is a measure such that $0 \leqslant \mu \in L^\infty([0,+\infty[,\mathcal{M}^1)$. In Proposition 7.5 below, we show that the measure μ is zero before the appearance of singularities. The question whether or not μ is zero after singularities is open, in general. The following theorem can be stated

Theorem 7.2. *Let $u_0 \in H^1$, such that $u_0' \in L^1$ and $u_0'(x) \leqslant M$ $\forall x$, then there exists $u \in L^\infty([0,T],BV(\mathbb{R}))$ for all $T>0$, such that there exists a subsequence of u^ℓ (also noted u^ℓ) and for all interval $\mathcal{I} \subset \mathbb{R}$ we have*

$$u^\ell \xrightarrow{\ell \to 0} u \text{ in } \mathcal{C}([0,T],L^1(\mathcal{I})), \tag{7.6}$$

and u satisfies the equation (7.5). Moreover, u satisfies the Oleinik inequality

$$u_x(t,x) \leqslant \frac{2M}{Mt+2} \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{7.7}$$

Remark 7.3. If $\mu = 0$ then, due to the Oleinik inequality, u is the unique entropy solution of the Burgers equation.

In order to prove Theorem 7.2, the following definition and lemma are needed:
Let $\mathcal{I} \subset \mathbb{R}$ be a bounded interval and let

$$W(\mathcal{I}) \stackrel{\text{def}}{=} \left\{f \in \mathcal{D}'(\mathcal{I}),\ \exists F \in L^1(\mathcal{I}) \text{ such that } F' = f\right\}, \tag{7.8}$$

where the norm of the space $W(\mathcal{I})$ is given by

$$\|f\|_{W(\mathcal{I})} \stackrel{\text{def}}{=} \inf_{c\in\mathbb{R}} \|F+c\|_{L^1(\mathcal{I})} = \min_{c\in\mathbb{R}} \|F+c\|_{L^1(\mathcal{I})}. \tag{7.9}$$

Lemma 7.4. *The space $W(\mathcal{I})$ is a Banach space and the embedding*

$$L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \tag{7.10}$$

is continuous.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W(\mathcal{I})$ and let F_n be a primitive of f_n. From the definition of the norm (7.9), there exists a constant c_n such that $(\tilde{F}_n - c_n)_{n\in\mathbb{N}}$ (where $\tilde{F}_n = F_n + c_n$) is a Cauchy sequence in $L^1(\mathcal{I})$. Let $\tilde{F}$ be the limit of $\tilde{F}_n$ in $L^1(\mathcal{I})$. Then

$$\|f_n - \tilde{F}'\|_{W(\mathcal{I})} \leqslant \|\tilde{F}_n - \tilde{F}\|_{L^1(\mathcal{I})}, \tag{7.11}$$

implying that $W(\mathcal{I})$ is a Banach space. Now, the continuous embedding can be proved.
If $f \in L^1(\mathcal{I})$, then $F(x) - F(a) = \int_a^x f(y)\,\mathrm{d}y$ for almost all $x,a \in \mathcal{I}$. Therefore,

$$\|f\|_{W(\mathcal{I})} \leqslant \int_{\mathcal{I}} |F(x) - F(a)|\,\mathrm{d}x \leqslant |\mathcal{I}| \int_{\mathcal{I}} |f(y)|\,\mathrm{d}y, \tag{7.12}$$

which ends the proof. □

The previous lemma and Helly's selection theorem imply that

$$W^{1,1}(\mathcal{I}) \hookrightarrow L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \tag{7.13}$$

where the first embedding is compact and the second is continuous.

Proof of Theorem 7.2: Let the compact set $[0,T]\times\mathcal{I}\subset\mathbb{R}^+\times\mathbb{R}$. Supposing that $\ell\leqslant 1$ then, from (5.6), the dissipative solutions of rB satisfies

$$\|u^\ell\|_2^2 \leqslant \|u_0\|_{H^1}^2, \qquad \ell^2\|P\|_1 = \tfrac{1}{2}\ell^2\|u_x^\ell\|_2^2 \leqslant \tfrac{1}{2}\|u_0\|_{H^1}^2, \tag{7.14}$$

implying that u^ℓ is uniformly bounded on $L^\infty([0,T],L^2(\mathbb{R}))$. Subsequently, it is also uniformly bounded on $L^\infty([0,T],L^1(\mathcal{I}))$. Because Lemma 7.1 yields that u^ℓ is bounded on $L^\infty([0,T],W^{1,1}(\mathcal{I}))$, and the equation (7.14) implies that $\frac{1}{2}{u^\ell}^2+\ell^2P$ is uniformly bounded on $L^\infty([0,T],L^1(\mathcal{I}))$, then since $u_t^\ell=-\left(\frac{1}{2}{u^\ell}^2+\ell^2P\right)_x$, (7.9) implies that u_t^ℓ is bounded on $L^\infty([0,T],W(\mathcal{I}))$. Then, using the Aubin theorem (see Corollary 4 in [33]), the compactness follows.

The quantity ℓ^2P is non-negative and bounded in $L^\infty([0,+\infty[,L^1(\mathbb{R}))$, implying the existence of a non-negative measure $\mu\in L^\infty([0,+\infty[,\mathcal{M}^1(\mathbb{R}))$ such that ℓ^2P converges (up to a subsequence) weakly to μ. The equation (7.5) follows taking the limit $\ell\to 0$ in the weak formulation of (2.12). Finally, taking the limit in the weak formulation of (5.7), we can prove that $u_x(t,x)\leqslant 2M/(Mt+2)$. □

The question whether or not $\mu=0$ is open. The following proposition shows that when $\ell\to 0$ for smooth solutions (i.e., before appearance of singularities), u^ℓ converges to the unique solution u of the classical Burgers equation.

Proposition 7.5. *If u_0 is in $H^s\cap BV$ with $s\geqslant 3$, then for $t<1/\sup_x|u_0'(x)|$ we have*

$$\mu = 0. \tag{7.15}$$

Proof. From Theorem 3.5 and Remark 3.6, we can find a uniform upper bound on u_x^ℓ in the space $L^\infty([0,T],L^\infty(\mathbb{R}))$ with $T<1/\sup_x|u_0'(x)|$, which implies that $\ell^2P\to 0$. □

7.2. **The limiting case $\ell\to+\infty$.** The goal of this subsection is to show that, when $\ell\to+\infty$, the dissipative solution u^ℓ converges (up to a subsequence) to a function u that satisfies:

$$\left[u_t+\tfrac{1}{2}(u^2)_x\right]_x = \nu, \tag{7.16}$$

where $0\leqslant\nu\in L^\infty([0,+\infty[,\mathcal{M}^1)$. In Proposition 7.8 below, we show that before the appearance of singularities, the measure $\nu=u_x^2/2$. The question whether or not $\nu=u_x^2/2$ in general is posed. We have the following theorem:

Theorem 7.6. *Let $u_0\in H^1$ such that $u_0'\in L^1$ and $u_0'(x)\leqslant M$ $\forall x$, then there exists $u\in L^\infty([0,T],BV(\mathbb{R}))$ for all $T>0$, such that there exists a subsequence of u^ℓ (noted also u^ℓ) and for all interval $\mathcal{I}\subset\mathbb{R}$ we have*

$$u^\ell \xrightarrow{\ell\to+\infty} u \text{ in } \mathcal{C}([0,T],L^1(\mathcal{I})), \tag{7.17}$$

and u satisfies the equation (7.16). Moreover, u satisfies the Oleinik inequality

$$u_x(t,x)\leqslant\frac{2M}{Mt+2} \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{7.18}$$

Remark 7.7. If $\nu = \frac{1}{2}u_x^2$ then u is a dissipative solution of the Hunter–Saxton equation [5].

Proof. Let the compact set $[0,T] \times \mathcal{I} \subset \mathbb{R}^+ \times \mathbb{R}$. Supposing that $\ell \geqslant 1$ then, from (5.6), the dissipative solution of rB satisfies

$$\|P\|_1 \;=\; \tfrac{1}{2}\,\|u_x^\ell\|_2^2 \;\leqslant\; \tfrac{1}{2}\,\|u_0\|_{H^1}^2. \tag{7.19}$$

Using Lemma 7.1, one gets that u^ℓ is bounded in $L^\infty([0,T]\times\mathbb{R})$ and

$$\int_{\mathbb{R}} |u^\ell(t,x+h) \;-\; u^\ell(t,x)|\,\mathrm{d}x \;\leqslant\; \|u_0'\|_1 \left(\frac{MT+2}{2}\right)^2 |h|. \tag{7.20}$$

Integrating (2.12) between t_1 and t_2, one obtains

$$u^\ell(t_1,x) \;-\; u^\ell(t_2,x) \;=\; \int_{t_1}^{t_2} \left(u^\ell\, u_x^\ell \;+\; \ell^2\, P_x\right) \mathrm{d}t. \tag{7.21}$$

Using Lemma 7.1, inequality (7.19) and

$$\|P_x\|_\infty \;\leqslant\; \tfrac{1}{4}\,\ell^{-2}\,\|u_x^\ell\|_2^2,$$

we can show that there exists $B = B(T,\mathcal{I})$ such that

$$\int_{\mathcal{I}} |u^\ell(t_2,x) \;-\; u^\ell(t_1,x)|\,\mathrm{d}x \;\leqslant\; B\,|t_2 - t_1|. \tag{7.22}$$

The compactness follows using Theorem A.8 in [25].

The quantity $\frac{1}{2}\,{u_x^\ell}^2$ is non-negative and bounded in $L^\infty([0,+\infty[, L^1(\mathbb{R}))$, which implies that there exists a non-negative measure $\nu \in L^\infty([0,+\infty[, \mathcal{M}^1(\mathbb{R}))$ such that P converges (up to a subsequence) weakly to ν. The equation (7.16) follows by taking the limit $\ell \to +\infty$, in the weak formulation of (3.1). Finally, taking the limit in the weak formulation of (5.7), we can prove that $u_x(t,x) \leqslant \frac{2M}{Mt+2}$. □

The question whether or not the equality always holds $\nu = u_x^2/2$ is open. The following proposition shows that, when $\ell \to +\infty$ for smooth solutions (before appearance of singularities), u^ℓ converges to a dissipative solution u of the Hunter–Saxton equation [5].

Proposition 7.8. *If u_0 is in $H^s \cap BV$ with $s \geqslant 3$, then for $t < 1/\sup_x |u_0'(x)|$ we have*

$$\nu \;=\; \tfrac{1}{2}\,u_x^2. \tag{7.23}$$

Proof. From Theorem 3.5 and Remark 3.6, we can find a uniform upper bound on u_x^ℓ in the space $L^\infty([0,T], L^\infty(\mathbb{R}))$ with $T < 1/\sup_x |u_0'(x)|$, which implies that the convergence u_x^ℓ to u_x is strong. Thus, ${u_x^\ell}^2 \to u_x^2$. □

8. Optimality of the $\dot{H}_{loc}$ space

In the previous sections (see Proposition 3.4, Theorem 3.5 and Theorem 5.3), we have shown, on one side, that even if the initial datum u_0 is smooth, there exists a finite blow-up time $T^* > 0$ such that

$$\inf_{x\in\mathbb{R}} u_x(t,x) \;>\; -\infty \;\; \forall t < T^*, \qquad \inf_{x\in\mathbb{R}} u_x(T^*,x) \;=\; -\infty. \tag{8.1}$$

On the other side, the Oleinik inequality (5.5) shows that, even if the initial datum is not Lipschitz, the derivative of the solution becomes instantly bounded from above, i.e.

$$\sup_{x\in\mathbb{R}} u_0'(x) \;=\; +\infty, \qquad \sup_{x\in\mathbb{R}} u_x(t,x) \;<\; +\infty \quad \forall t > 0. \tag{8.2}$$

Remark 8.1. If the derivative of the initial datum is bounded from below and not from above, it will be instantly bounded from both sides [3] and, after T^*, it will be bounded from above and not from below.

This remark is important to prove that the space $\dot{H}^1_{loc}$ is the best space to obtain global (in time) solutions, the optimality being in the following sense.

Theorem 8.2. *Let $\delta > 0$ and $g(h) \stackrel{def}{=} [\ln|h|]^\delta$, then there exist $u_0 \in H^1 \cap W^{1,\infty}$, $T > 0$ and a compact set $\mathcal{K}$, such that there exists a solution u of (2.12) satisfying*

$$\int_{\mathbb{R}} u_0'(x)^2\, g(u_0'(x))\, \mathrm{d}x \;<\; +\infty, \qquad\qquad \int_{\mathcal{K}} u_x(T,x)^2\, g(u_x(T,x))\, \mathrm{d}x \;=\; +\infty. \tag{8.3}$$

Thus, we cannot expect that u belongs to $W^{1,p}$ for $p > 2$ for all time. In other words, the space $H^1 = W^{1,2}$ is optimal for the equation (2.12).

Before proving Theorem 8.2, let $u_0 \in H^s$ with s big enough, and let u be a solution of rB with $u(0,x) = u_0(x)$. The main quantity is the following integral

$$\int_{\mathcal{K}} u_x^2(T,x)\, g(u_x(T,x))\, \mathrm{d}x, \tag{8.4}$$

where $T > 0$ and $\mathcal{K}$ is a compact set. Using the change of variable $x = y(T,\xi)$, one gets

$$\int_{\mathcal{K}} u_x^2(T,x)\, g(u_x(T,x))\, \mathrm{d}x \;=\; \int_{\mathcal{K}'} q\, \sin^2(v/2)\, g(\tan(v/2))\, \mathrm{d}\xi, \tag{8.5}$$

where $\mathcal{K}'$ is another compact set. From previous sections, the quantity q is always bounded, which implies that if g is bounded then (8.4) is bounded. If g is not bounded (see Theorem 8.2), then the quantity (8.4) depends on the behaviour of the derivative u_x at time T. The proof of Theorem 8.2 is done by building $u(T,\cdot)$, such that the quantity (8.4) is infinite. Then, we use a backward system to go back in time and find a Lipschitz initial datum.

Proof of Theorem 8.2: Let $g(h) \stackrel{\text{def}}{=} [\ln|h|]^\delta$ for $\delta > 0$ and let $\bar{u}$ be a compactly supported odd function such that $\bar{u} \in \mathcal{C}^\infty(\mathbb{R}/\{0\})$ and for all $x \in]0, \frac{1}{2}[$ we have

$$\bar{u}'(x) \;\stackrel{\text{def}}{=}\; -\frac{1}{\sqrt{x}}\, (-\ln(x))^{-\frac{1+\delta}{2}}.$$

It is clear that $\bar{u} \in H^1(\mathbb{R})$ and

$$\int_{\mathcal{V}(0)} \bar{u}'(x)^2\, g(\bar{u}'(x))\, \mathrm{d}x \;=\; +\infty, \qquad \bar{u}'(x) \;\leqslant C, \tag{8.6}$$

where $\mathcal{V}(0)$ denotes a neighbourhood of 0.

The idea of the proof is to use a backward (in time) system such that $u(T,x) = \bar{u}(x)$. The initial datum u_0 is the unknown. To simplify the presentation, the conservative system (4.17) is used. With this system, we will obtain a local (in time) Oleinik inequality, which is enough for our construction. A similar proof can be used with the dissipative system (4.18) with a global Oleinik inequality. The built solution in the interval $[0,T[$ is Lipchitz, so both systems (4.17), (4.18) yield the same solution.

[3] Note that the gain of regularity (8.2) is instantaneous, while the loss of regularity (8.1) needs some time.

In order to build u_0, we use the forward existence proof given in Section 4. One can use the change of variable $t \to -t$. The conservative system (4.17) becomes then

$$y_t = -u, \qquad y(-T,\xi) = \bar{y}(\xi), \tag{8.7a}$$
$$u_t = \ell^2 P_x, \qquad u(-T,\xi) = \bar{u}(\bar{y}(\xi)), \tag{8.7b}$$
$$v_t = P\,(1+\cos(v)) + \sin^2(v/2), \qquad v(-T,\xi) = 2\arctan\bigl(\bar{u}'(\bar{y}(\xi))\bigr), \tag{8.7c}$$
$$q_t = -q\left(\tfrac{1}{2} - P\right)\sin(v), \qquad q(-T,\xi) = 1, \tag{8.7d}$$

where $t \in [-T, 0]$ and $\bar{y}$ is defined as in (4.10), replacing u_0 by $\bar{u}$.

The proof of a local existence of solutions can be done as in Section 4. Due to the change of variable $t \to -t$, the Oleinik inequality becomes

$$u_x(t,x) \geqslant -2/(t+T) \tag{8.8}$$

for $t > -T$ and t close enough to $-T$. The proof of this Oleinik inequality proceeds as in Section 4 using the equation (8.7c), which implies that the derivative of the solution is bounded from below. As in Remark 8.1, since $\bar{u}' = u_x(-T,\cdot) \leqslant C$, the derivative of the solution remains bounded from above for $t > -T$ and t close enough to $-T$. Taking $T > 0$ small so the solution is Lipschitz until $t = 0$, and thus

$$\int_{\mathbb{R}} u_x(0,x)^2\, g(u_x(0,x)')\,\mathrm{d}x < +\infty.$$

The result follows directly by using the change of variable $t \to -t$. $\square$

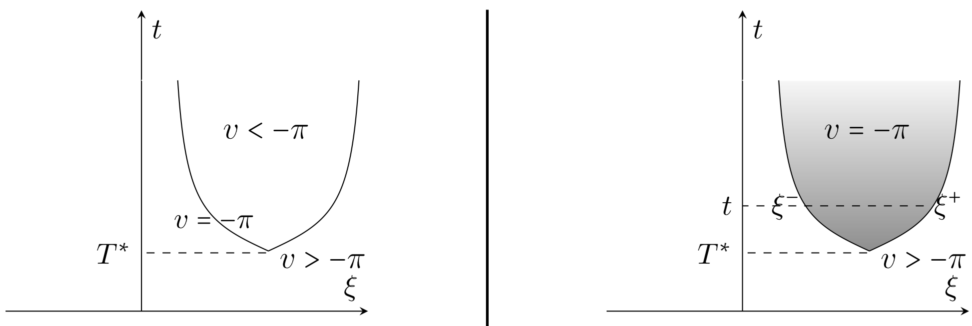
Remark 8.3.
1. The optimality given in Theorem 8.2 is also true for the Camassa–Holm equation.
2. Xin and Zhang [34] have proved that the Camassa–Holm equation admits dissipative solutions that satisfy
$$\int_0^T \int_{|x|\leqslant R} |u_x(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t < +\infty \qquad \forall T>0, R>0, p<3. \tag{8.9}$$
This result can also be proven for the rB equation.
3. Theorem 8.2 does not contradict with (8.9). Theorem 8.2 shows that the function
$$t \mapsto \int_{|x|\leqslant R} |u_x(t,x)|^p\,\mathrm{d}x$$
does not necessarily belong to $L^\infty_{loc}([0,+\infty))$. However, the inequality (8.9) shows that this function belongs to $L^1_{loc}([0,+\infty))$ if $p<3$.

9. Conclusion and discussion

In this paper, we have studied a regularisation of the inviscid Burgers equation (2.12). For a smooth initial datum, the regularised equation (2.12) has a unique smooth solution locally in time. After the blow-up time, the solution is no longer unique, nor smooth. At least two types of solutions exist: conservative and dissipative solutions. We find that the built dissipative solutions are more interesting because they satisfy an Oleinik inequality (5.5), which plays an important role in showing that solutions converge (up to a subsequence) when $\ell \to 0$ and when $\ell \to \infty$ (ℓ the regularising positive parameter). Before the appearance of singularities, the limit when $\ell \to 0$ (respectively $\ell \to \infty$) is a smooth solution of the inviscid Burgers (resp. the Hunter–Saxton) equation. After the breakdown time, it remains open to determine whether the Burgers (resp. the Hunter–Saxton) equation holds in the limit without a remaining forcing term.

As shown above, the major difference between the conservative system (4.17) and the dissipative system (5.2) is that the system (4.17) allows v to cross the value $-\pi$, causing a jump of u_x from $-\infty$ to $+\infty$ (see eq. (4.44)), which implies (4.1), thence the loss of the Oleinik inequality (Remark 4.5). But, the value $v=-\pi$ is a barrier that cannot be crossed for the system (5.2). It follows that if $v(t,\xi_0)=-\pi$ at a time t, then $v(\tau,\xi_0)=-\pi$ for all times $\tau\geqslant t$ (see figure 2). This property is important to obtain the Oleinik inequality (5.5), which yields the dissipation of the energy (5.6).

The figure 2 shows the domains where $v=-\pi$ for the systems (4.17) and (5.2).



System (4.17) (conservative solution) System (5.2) (dissipative solution)

FIGURE 2. Regions where $v=-\pi$.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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