



HAL
open science

Fourth order parabolic problem: a complex network approach

Guillaume Cantin, Alexandre Thorel

► **To cite this version:**

Guillaume Cantin, Alexandre Thorel. Fourth order parabolic problem: a complex network approach. 2020. hal-02478426

HAL Id: hal-02478426

<https://hal.science/hal-02478426>

Preprint submitted on 13 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fourth order parabolic problem: a complex network approach

Guillaume Cantin & Alexandre Thorel

Normandie Univ, UNIHAVRE, LMAH, ISCN, FR-CNRS-3335, 76600 Le Havre, France.

email: guillaumecantin@mail.com

email: alexandre.thorel@univ-lehavre.fr

Abstract

In this paper we propose a novel approach for a fourth order differential problem using complex networks of reaction-diffusion equations. We model the biharmonic operator by a network, based on a finite graph, in which the coupling between nodes is linear. To this end, we study the fourth order parabolic problem, establishing results of existence, uniqueness and maximal regularity of the solution *via* operator sums theory and analytic semigroups techniques. We then solve the complex network problem and present sufficient conditions for the solutions of both problems to converge to each other. Finally, we analyze their asymptotic behavior by establishing the existence of a family of exponential attractors.

Keywords. Reaction-diffusion, bilaplacian, analytic semigroup, complex network.

2010 Mathematics Subject Classification: 35A01, 35A02, 35B40, 35K30, 35K57.

1 Introduction

In this article, we propose a novel approach to study parabolic problems. This new methodology aims to establish a relationship between a linear high order parabolic problem, set on a bounded open set and a complex network problem of reduced order with linear couplings, supported by a finite graph.

Establishing a correspondence between a high order parabolic problem and a complex network of reduced order systems can be of great interest for many applications. For instance, the qualitative analysis of neural systems described by complex networks of reaction-diffusion equations could benefit from the knowledge of the biharmonic operator. In return, the biharmonic equation could be studied through the complex network framework and emergent properties are likely to be exhibited. Hence we believe that our method can lead to significant progress in the analysis of parabolic problem of various types.

Here, we consider the following fourth order parabolic initial-value problem

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) &= -\Delta^2 u(x, t) + r\Delta u(x, t) + f(x, t) & (x, t) \in \Omega \times \mathbb{R}_+^*, \\ u(x, t) &= \Delta u(x, t) = 0 & (x, t) \in \partial\Omega \times \mathbb{R}_+^*, \\ u(x, 0) &= u_0(x) & x \in \Omega, \end{cases} \quad (1)$$

where Ω denotes an open bounded set of \mathbb{R}^N with regular boundary, N being a positive integer, $f \in L^p(\Omega, \mathbb{R}_+^*)$, $p \in (1, +\infty)$, models a source term and r is a positive coefficient. In this problem, u models a density of particles (e.g. human beings, animals, dust, *etc.*) living in Ω , subject to external forces modeled by f and to Dirichlet boundary condition. The second

order term Δu models the short range diffusion, whereas the biharmonic term $\Delta^2 u$ models the long range diffusion, which can be interpreted as the diffusion of u in the “neighborhood of the neighborhood” of each point of Ω . Indeed, the Laplace operator, obtained by the simple Fick diffusion modeling fails to reproduce at a refined level spatial effects as, for example, complex cell motion phenomena [8, 11, 30, 36]. However, the Landau-Ginzburg free energy functional approach leads to the superposition of the Laplace operator and the biharmonic operator as in problem (1). In a forthcoming work we aim to focus on non linear parabolic problems with applications in neuroscience.

Many articles have been devoted to studying biharmonic equations by splitting methods, leading to the study of coupled equations of second order [17, 44]. Finite differences schemes have also been studied for numerical computing of the solutions of equations involving biharmonic terms [5, 9, 25, 19]. Many theoretical results of existence, uniqueness and regularity have been obtained for fourth order problems like Cahn-Hilliard equation, see for instance [10, 16, 18, 32, 35]. More recently, problem (1) has been studied in a general Banach space setting for the linear stationary case [27, 28, 46]. Here, we aim to improve and extend the results of the latter papers to the non-stationary case. We present novel results for the parabolic problem (1), with existence, uniqueness and maximal regularity of the solutions in L^p -spaces.

Additionally, we propose to approximate problem (1) by a complex network of reaction-diffusion equations set in L^p -spaces, $p \in (1, +\infty)$. Let us describe the statement of the network problem. First of all, we suppose that the domain Ω can be split into a finite set of non-empty open sub-domains

$$\Omega = \bigcup_{i=1}^n \bar{\omega}_i, \quad \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j, \quad \omega_i \neq \emptyset, \quad 1 \leq i \leq n, \quad (2)$$

as depicted in figure 1.

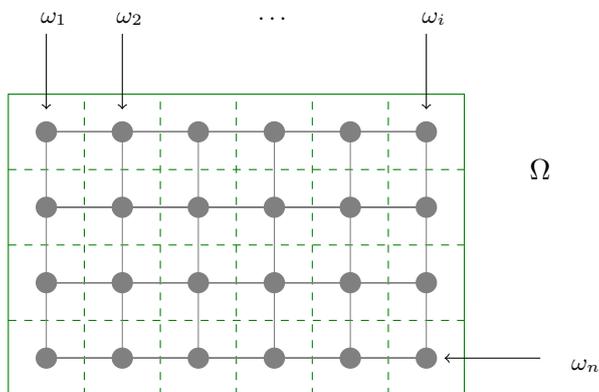


Figure 1: Splitting of a rectangular domain Ω into a grid of sub-domains $\omega_1, \dots, \omega_n$ in the dimension case $N = 2$.

We assume that the open sub-domains ω_i , $1 \leq i \leq n$, admit a regular boundary $\partial\omega_i$; that there exists a family of homeomorphisms h_i , $1 \leq i \leq n$, between ω_i and a generic open domain $\omega \subset \mathbb{R}^N$; and that two distinct sub-domains ω_i and ω_j ($i \neq j$) may share a part of their boundaries. In this way, we have

$$\Omega = \bigcup_{i=1}^n h_i^{-1}(\omega). \quad (3)$$

If Ω admits a rectangular shape, then the homeomorphisms h_i , $1 \leq i \leq n$, are defined by elementary translations. We emphasize that the problem should not be seen as a domain

decomposition problem (see for instance [14, 47] and references therein cited), since we do not impose interface conditions. For each $i \in \{1, \dots, n\}$, we denote by \mathcal{N}_i the subset of indices corresponding to the neighbors of ω_i . The choice of \mathcal{N}_i is not necessarily dictated by the parts of the boundary $\partial\omega_i$ shared with other sub-domains ω_j , $i \neq j$; we shall see in the final section that the relevant choice of the neighbors can be found thanks to a finite differences approach. Finally, we assume that the boundary of ω_i can be split into two parts

$$\partial\omega_i = \Gamma_i^D \cup \Gamma_i^N, \quad \Gamma_i^D \cap \Gamma_i^N = \emptyset, \quad 1 \leq i \leq n, \quad (4)$$

which we shall associate to splitting boundary conditions, with Dirichlet boundary condition on Γ_i^D and Neumann boundary condition on Γ_i^N , $1 \leq i \leq n$. Hence, the complex network of reaction-diffusion equations can be written as follows

$$\begin{cases} \frac{\partial v_i}{\partial t}(\xi, t) = r\Delta v_i(\xi, t) + \delta_i(v_i(\xi, t), \{v_j(\xi, t)\}_{j \in \mathcal{N}_i}) + f_i(\xi, t) & (\xi, t) \in \omega \times \mathbb{R}_+^*, \\ \frac{\partial v_i}{\partial \nu}(\xi, t) = 0 & (\xi, t) \in \gamma_i^N \times \mathbb{R}_+^*, \\ v_i(\xi, t) = 0 & (\xi, t) \in \gamma_i^D \times \mathbb{R}_+^*, \\ v_i(\xi, 0) = v_{i,0}(\xi) & \xi \in \omega, \end{cases} \quad (5)$$

for all $i \in \{1, \dots, n\}$, where $f_i \in (L^p(\omega))^n$ with $p \in (1, +\infty)$, models the source term in ω , ν denotes the outer normal of $\partial\omega$, $v_{i,0}$ are smooth initial conditions and δ_i corresponds to a linear coupling operator, $1 \leq i \leq n$, whose we will precise the form in section 4, where we handle this problem in $Y = (L^2(\omega))^n$ using semigroups methods.

Reaction-diffusion equations or systems have produced a huge literature, partly due to the richness of the dynamics of their solutions, which can for example exhibit traveling waves, Hopf or Turing bifurcations [24, 31, 38, 41, 43]. Complex networks of such equations have been recently analyzed, for instance in [1, 7], where the authors investigate the possibility to extend to infinite dimension the classical problematics of complex networks. Among them, one can quote the synchronization topic or the relationship between the topology of the network, the internal dynamics of its nodes and the global dynamics [2, 3, 6, 20, 40]. However, the approximation of a fourth order parabolic problem by a complex network of reaction-diffusion equations represents a novelty at our knowledge.

This paper is organized as follows. In the next section, we recall some basics concerning Sobolev spaces, interpolation spaces and bounded imaginary power operators. In section 3, we present novel results on the study of the fourth order parabolic equation (1), with existence, uniqueness and maximal regularity theorems in a Banach space setting. We then prove the existence and uniqueness of a local in time solution to the complex network problem (5) in section 4, we show how to recover a function defined on the initial domain Ω and we establish energy estimates which imply the global existence of the solutions and the existence of exponential attractors. Finally, we investigate sufficient conditions for the convergence of the approximation problem, from which we deduce the existence of exponential attractors for the semi-flow induced by the initial fourth order problem.

2 Preliminaries

In this section, we present the basic notations and material which we shall use in our paper.

2.1 Functional spaces, interpolation spaces

Throughout this paper, we will use the classical notations for Lebesgue spaces $L^p(\Omega)$, $L^p(a, b, X)$ and Sobolev spaces $W^{k,p}(\Omega)$, where Ω denotes an open bounded domain in \mathbb{R}^N ,

$p \in [1, \infty]$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$, X is a Banach space and $k \in \mathbb{N}$. Those functional spaces are Banach spaces whose norms will be denoted $\|\cdot\|_{L^p(\omega)}$, $\|\cdot\|_{L^p(a,b,X)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ respectively. For $p = 2$, we simply note $H^k(\Omega) = W^{k,2}(\Omega)$; $H^k(\Omega)$ is a Hilbert space whose inner product will be denoted $(\cdot, \cdot)_{H^k(\Omega)}$.

Our reasonings below will use the theory of interpolation spaces, hence we recall the following definition.

Definition 2.1. Let X be a Banach space, and $T : \mathcal{D}(T) \subset X \rightarrow X$ a linear operator such that

$$(0, +\infty) \subset \rho(T) \quad \text{and} \quad \exists C > 0 : \forall t > 0, \quad \|t(T - tI)^{-1}\|_{\mathcal{L}(X)} \leq C. \quad (6)$$

Then, from [22], p. 665, Teorema 3, for $\theta \in (0, 1)$ and $q \in [1, +\infty]$, we can define the real interpolation space

$$(\mathcal{D}(T), X)_{\theta, q} = \left\{ \psi \in X : t \mapsto t^{1-\theta} \|T(T - tI)^{-1}\psi\|_X \in L_*^q(0, +\infty) \right\},$$

where $L_*^q(0, +\infty)$ is defined for instance in [22], p.663. In [48], this space is denoted by $(X, \mathcal{D}(T))_{1-\theta, q}$.

Note that the general situation of the real interpolation space $(X_0, X_1)_{\theta, q}$ with X_0, X_1 two Banach spaces such that $X_0 \hookrightarrow X_1$, is described for instance in [34] or [48].

2.2 Standard inequalities

For convenience, we recall a Gronwall type lemma and the Poincaré inequality [49].

Lemma 2.2 (Gronwall lemma). *Assume that*

$$\phi'(t) + a\phi(t) \leq b, \quad 0 < t \leq T,$$

where ϕ is a continuous function on $[0, T]$, continuously differentiable on $(0, T]$, $a > 0$ and $b > 0$. Then

$$\phi(t) \leq e^{-at}\phi(0) + \frac{b}{a}, \quad 0 < t \leq T.$$

Let γ denote a trace operator defined from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

Theorem 2.3 (Poincaré inequality). *Let Ω denote an open bounded domain with regular boundary $\partial\Omega$, and $\gamma^D \subset \partial\Omega$, $\gamma^D \neq \emptyset$. There exists a positive constant C such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in \dot{H}_D^1(\Omega)$, where

$$\dot{H}_D^1(\Omega) = \{\psi \in H^1(\omega) ; \gamma\psi = 0 \text{ on } \gamma^D\}.$$

2.3 The class of Bounded Imaginary Power operators

We continue with some definitions on sectorial operators, BIP operators and UMD spaces.

Definition 2.4. A closed linear operator T_1 is said to be sectorial of angle $\alpha \in [0, \pi)$, and we write $T_1 \in \text{Sect}(\alpha)$, if

- (i) $\sigma(T_1) \subset \overline{S_\alpha}$,
- (ii) $\forall \alpha' \in (\alpha, \pi), \quad \sup \left\{ \|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}} \right\} < \infty,$

where

$$S_\alpha = \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \alpha\}, & \text{if } \alpha > 0, \\ (0, +\infty), & \text{if } \alpha = 0. \end{cases} \quad (7)$$

The previous definition could be find in [23], p. 19.

It is known that any injective sectorial operator T_1 admits imaginary powers T_1^{is} , $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded, see for instance [26], p. 342.

Definition 2.5. Let $\theta \in [0, \pi)$. We denote by $\text{BIP}(X, \theta)$, the class of sectorial injective operators T_1 such that

- (i) $\overline{\mathcal{D}(T_1)} = \overline{R(T_1)} = X$,
- (ii) $\forall s \in \mathbb{R}, T_1^{is} \in \mathcal{L}(X)$,
- (iii) $\exists C \geq 1, \forall s \in \mathbb{R}, \|T_1^{is}\|_{\mathcal{L}(X)} \leq Ce^{|s|\theta}$.

This definition could be find in [39], p. 430. In this case, it holds that $\overline{\mathcal{D}(T_1) \cap R(T_1)} = X$, see for instance [23], proof of Proposition 3.2.1.

In this paper, we will use the well-known Dore-Venni theorem, see [15] and its generalization in [39], which needs to consider a UMD space X .

Definition 2.6. A Banach space X is a UMD space if and only if for some $p \in (1, +\infty)$ and thus for all p , the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself.

Note that, in our case, we will consider $X = L^p(\Omega)$ with $p \in (1, +\infty)$ thus, from [42], Proposition 3, it holds that X is a UMD space.

2.4 Trace and regularity lemmas

In this section, we recall some known trace and regularity results.

Lemma 2.7 ([28]). *Let T_2 be a linear operator satisfying (6). Let u be such that*

$$u \in W^{n,p}(a_1, b_1; X) \cap L^p(a_1, b_1; \mathcal{D}(T_2)),$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$, $n \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a_1, b_1\}$, we have

$$u^{(j)}(s) \in (\mathcal{D}(T_2), X)_{\frac{j}{n} + \frac{1}{np}, p}.$$

This result is proved in [21], Teorema 2'.

Lemma 2.8 ([12, 48]). *Let $\psi \in X$ and T_3 be a generator of a bounded analytic semigroup in X . Then, the two next properties are equivalent:*

1. $x \mapsto T_3 e^{(x-a)T_3} \psi \in L^p(a, b; X)$,
2. $\psi \in (\mathcal{D}(T_3), X)_{\frac{1}{p}, p}$.

This result is proved in [48], Theorem, p. 96.

Lemma 2.9 ([15]). *Let $T_4 \in BIP(X, \theta)$ with $\theta \in (0, \pi/2)$, and $g \in L^p(a, b; X)$. Then, for almost every $x \in (a, b)$, we have*

$$\int_a^x e^{-(x-s)T_4} g(s) ds \in \mathcal{D}(T_4) \quad \text{and} \quad \int_x^b e^{-(s-x)T_4} g(s) ds \in \mathcal{D}(T_4).$$

Moreover,

$$\begin{aligned} x &\mapsto T_4 \int_a^x e^{-(x-s)T_4} g(s) ds \in L^p(a, b; X) \\ x &\mapsto T_4 \int_x^b e^{-(s-x)T_4} g(s) ds \in L^p(a, b; X). \end{aligned}$$

This result is proved in [15], Theorem 3.2.

3 Analysis of the fourth order diffusion equation

3.1 Operational formulation

We set $X = L^p(\Omega)$, $p \in (1, +\infty)$. Let $r > 0$ and $f \in L^p(\mathbb{R}_+ \times \Omega)$, with $p \in (1, +\infty)$. Our aim in this section is to study the following parabolic problem

$$(P_{pde}) \begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u - r\Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u(x, 0) = u_0(x), & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, t \in \mathbb{R}_+ \\ \Delta u(x, t) = 0, & x \in \partial\Omega, t \in \mathbb{R}_+, \end{cases}$$

where $u_0 \in L^p(\Omega)$, with $p \in (1, +\infty)$.

We handle the latter problem in $(0, T] \times \Omega$, with $T > 0$ instead of $(0, +\infty) \times \Omega$. To this end, let us define the following linear operator in \mathbb{R}^n :

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{\psi \in W^{4,p}(\Omega) : \psi = \Delta\psi = 0 \text{ on } \partial\Omega\} \\ \forall \psi \in \mathcal{D}(\mathcal{A}), \quad \mathcal{A}\psi = \Delta^2\psi - r\Delta\psi. \end{cases} \quad (8)$$

Thus, using operator \mathcal{A} , problem (PL_{pde}) can be rewritten

$$(P) \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in (0, T] \\ u(0) = u_0 \end{cases}$$

where $f \in L^p(0, T; L^p(\Omega))$ and $p \in (1, +\infty)$, with $u(t) = u(t, \cdot)$ and $f(t) = f(t, \cdot)$. We will search a classical solution of problem (P) , that is, a solution u such that

$$u \in W^{1,p}(0, T; X) \cap L^p(0, T; \mathcal{D}(\mathcal{A})).$$

3.2 Existence, uniqueness and optimal regularity

Let us introduce the linear operators A_i , for all $i \in \{1, \dots, n\}$, by setting

$$\begin{cases} \mathcal{D}(A_i) = \{\varphi \in W^{2,p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\} \\ A_i\varphi = -\frac{\partial^2 \varphi}{\partial x_i^2}, \quad \varphi \in \mathcal{D}(A_i). \end{cases} \quad (9)$$

Moreover, we also define the linear operator A by

$$\begin{cases} \mathcal{D}(A) = \{\varphi \in W^{2,p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\} \\ A\varphi = -\Delta\varphi, \quad \varphi \in \mathcal{D}(A). \end{cases} \quad (10)$$

Proposition 3.1. *For all $i \in \{1, \dots, n\}$ and all $n \in \mathbb{N}$, the linear operators A_i , defined by (9), and A defined by (10), satisfy the following properties:*

1. $A_i \in \text{BIP}(X, \theta_i)$, for all $\theta_i \in (0, \pi)$,
2. $A^n \in \text{BIP}(X, \theta)$, for all $n \in \mathbb{N}$ and all $\theta \in (0, \pi)$.

Proof.

1. It is well-known that, for all $i \in \{1, \dots, n\}$, $A_i \in \text{Sect}(0)$ and $0 \in \rho(A_i)$. Moreover, since X is reflexive and from [29], Proposition 3.1, p. 191, for all $\theta_i > 0$, we obtain

$$A_i \in \text{BIP}(X, \theta_i).$$

In particular, without loss of generality, we could take each θ_i as small as we want.

2. When $n = 0$, we have $A^n = A^0 = I$, where I denotes the identity operator. It is known that $I \in \text{Sect}(0)$ with $\sigma(I) = \{1\}$. Thus, since $I \in \mathcal{L}(X)$, we can deduce that $I \in \text{BIP}(X, \theta)$, for all $\theta > 0$.

Let $j \in \{1, \dots, n\}$ such that $j \neq i$. Since the actions of A_i and A_j are independent and since A_i and A_j have the same domain, then we have

$$A_i A_j = A_j A_i. \quad (11)$$

Then, from (11) and Theorem 5 in [39], we obtain that

$$A = \sum_{i=1}^n A_i \in \text{BIP}(X, \theta_A), \quad (12)$$

where $\theta_A = \max_{i \in \{1, \dots, n\}} \theta_i$. Thus, since each θ_i can be as small as we want, then $\theta_A > 0$ can be as small as we want.

Let $n \in \mathbb{N} \setminus \{0\}$. Assume that $A^n \in \text{BIP}(X, \theta)$, for all $\theta > 0$, then since $0 \in \rho(A)$, from (12) and [39], Corollary 3, p. 444, one has

$$A^{n+1} = A A^n \in \text{BIP}(X, \theta_A + \theta).$$

Moreover, since θ_A and θ can be as small as we want, we deduce that $\theta_A + \theta > 0$ can be as small as we want. Then, by recurrence, we obtain the result. □

Proposition 3.2. *The linear operator \mathcal{A} defined by (8) satisfies the two following properties:*

1. $0 \in \rho(\mathcal{A})$,
2. $\mathcal{A} \in \text{BIP}(X, \theta)$, for all $\theta \in (0, \pi)$.

Proof.

1. The result follows from [28], Theorem 2.2, p. 355 and Remark 2.6, p. 357.

2. Let $r > 0$. From Proposition 3.1 and [39], Corollary 1, p. 435, for all $\theta_A > 0$, we have

$$rA \in \text{BIP}(X, \theta_A).$$

From Proposition 3.1, for all $\theta > 0$, we deduce

$$A^2 \in \text{BIP}(X, \theta).$$

Note that, from [39], Corollary 3, p. 444, we obtain $\theta = 2\theta_A > \theta_A$. Finally, from [39], Theorem 5, p. 443, we obtain

$$\mathcal{A} = A^2 + rA \in \text{BIP}(X, \theta).$$

□

Theorem 3.3. *Let $f \in L^p(0, T; X)$, $p \in (1, +\infty)$. There exists a unique classical solution given by*

$$u(t) = e^{-t\mathcal{A}}u_0 + \int_0^t e^{-(t-s)\mathcal{A}}f(s) ds, \quad (13)$$

of problem:

$$(P) \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in (0, T] \\ u(0) = u_0, \end{cases}$$

if and only if

$$u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}. \quad (14)$$

Proof. From Proposition 3.2, we deduce that for all $\theta \in (0, \pi)$, $\mathcal{A} \in \text{BIP}(X, \theta)$. In particular we could take $\theta < \pi/2$. Thus, $-\mathcal{A}$ generate an analytic semigroup $(e^{-t\mathcal{A}})_{t \geq 0}$. Then, since $f \in L^p(0, T; X)$, $p \in (1, +\infty)$, from [13], Lemma 2.1, p. 208 there exists a unique solution of problem (P) given by (13):

$$u(t) = e^{-t\mathcal{A}}u_0 + \int_0^t e^{-(t-s)\mathcal{A}}f(s) ds.$$

If u , given by (13), is a classical solution of problem (P), then

$$u \in W^{1,p}(0, T; X) \cap L^p(0, T; \mathcal{D}(\mathcal{A})).$$

It follows from Lemma 2.7 that

$$u(0) = u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}.$$

If $u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}$, since the unique solution u is given by (13), it remains to show that u is a classical solution. From Lemma 2.7, we deduce that

$$t \mapsto e^{-t\mathcal{A}}u_0 \in W^{1,p}(0, T; X) \cap L^p(0, T; \mathcal{D}(\mathcal{A})).$$

Finally, from Lemma 2.9, we obtain that

$$t \mapsto \int_0^t e^{-(t-s)\mathcal{A}}f(s) ds \in W^{1,p}(0, T; X) \cap L^p(0, T; \mathcal{D}(\mathcal{A})).$$

□

As a consequence of Theorem 3.3, we deduce the following result for problem (P_{pde}) .

Corollary 3.4. *Assume that Ω is a bounded open set of \mathbb{R}^n where $p > \frac{n}{4} + 1$ with C^2 -boundary. Let $f \in L^p(\mathbb{R}_+ \times \Omega)$ with $p \in (1, +\infty)$; let $r \in \mathbb{R}_+ \setminus \{0\}$. Then, there exists a unique solution u of (P_{pde}) , such that*

$$u \in W^{1,p}(\mathbb{R}_+, L^p(\Omega)) \cap L^p(\mathbb{R}_+, W^{4,p}(\Omega)),$$

if and only if

$$u_0 \in \{\varphi \in W^{4-\frac{4}{p},p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\}.$$

Remark 3.5. Taking into account the result of Theorem 3.3, we can also obtain anisotropic result by considering $f \in L^p(0, T; L^q(\Omega))$ with $p, q \in (1, +\infty)$.

Proof. Let $(t, x) \in (0, +\infty) \times \Omega$. Set $X = L^p(\Omega)$. Using \mathcal{A} the linear operator defined by (8), we obtain that problem (P_{pde}) becomes problem (P) when $t \in (0, T]$. From Theorem 3.3, there exists a unique classical solution of problem (P) if and only if (14) holds.

To obtain a classical solution on \mathbb{R}_+ , we have to show that $\|u(t)\|_X$ is bounded on \mathbb{R}_+ . From [37], equation (6.5), p. 70, since $0 \in \rho(\mathcal{A})$, see Proposition 3.2, there exists $M, \delta > 0$ such that

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \leq Me^{-\delta t}.$$

Thus, we have

$$\begin{aligned} \|u(t)\|_X &= \left\| e^{-t\mathcal{A}}u_0 + \int_0^t e^{-(t-s)\mathcal{A}}f(s) ds \right\|_X \\ &\leq \|e^{-t\mathcal{A}}\|_{\mathcal{L}(X)}\|u_0\|_X + \left\| \int_0^t e^{-(t-s)\mathcal{A}}f(s) ds \right\|_X \\ &\leq Me^{-\delta t}\|u_0\|_X + \int_0^t \|e^{-(t-s)\mathcal{A}}\|_{\mathcal{L}(X)}\|f(s)\|_X ds \\ &\leq Me^{-\delta t}\|u_0\|_X + \int_0^t Me^{-\delta(t-s)}\|f(s)\|_X ds \\ &\leq Me^{-\delta t}\|u_0\|_X + M \int_0^t e^{-\delta(t-s)}\|f(s)\|_X ds \\ &\leq Me^{-\delta t}\|u_0\|_X + M \left(\int_0^t e^{-q\delta(t-s)} ds \right)^{1/q} \left(\int_0^t \|f(s)\|_X^p ds \right)^{1/p} \\ &\leq Me^{-\delta t}\|u_0\|_X + \frac{M}{(q\delta)^{1/q}}(1 - e^{-q\delta t})^{1/q}\|f\|_{L^p(0,+\infty;X)}. \end{aligned}$$

Now, it remains to show that if u_0 satisfy (14), then the classical solution u satisfies

$$u \in W^{1,p}(\mathbb{R}_+, L^p(\Omega)) \cap L^p(\mathbb{R}_+, W^{4,p}(\Omega)).$$

To this end, we will make explicit the interpolation space that appear in (14). We have

$$(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p},p} = \left(\{\varphi \in W^{4,p}(\Omega) : \varphi = \Delta\varphi = 0 \text{ on } \partial\Omega\}, L^p(\Omega) \right)_{\frac{1}{p},p},$$

and from [21], p. 683, proposizione 3 and p. 681, 1.10, and [48], p. 317, Theorem 1, since $4 - \frac{1}{p} > 1$ is never integer, we have

$$\left(W^{4,p}(\Omega), L^p(\Omega) \right)_{\frac{1}{p},p} = B_{p,p}^{4(1-\frac{1}{p})}(\Omega) = B_{p,p}^{4-\frac{4}{p}}(\Omega) = W^{4-\frac{4}{p},p}(\Omega). \quad (15)$$

Set $\nu = 4 - \frac{4}{p} - \frac{n}{p} = 4 - \frac{n+4}{p}$. Since $p > \frac{n}{4} + 1$, we have $\nu > 0$. From the Sobolev embedding theorem, see [48], section 4.6.1, (e), p. 327-328, we have:

$$W^{4-\frac{4}{p},p}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Thus, we keep the traces of the elements of the space described in (15) that make sense and which are well defined. From [22], Proposition 5.9, p. 334 and [48], section 4.3.3, Theorem, p. 321, we deduce that

$$(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p},p} = \left\{ \varphi \in W^{4-\frac{4}{p},p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega \right\}.$$

□

4 Analysis of the complex network problem

In this section, our aim is to study the complex network problem of reaction-diffusion equations approximating the fourth order parabolic equation (1).

4.1 Setting of the complex network problem

As mentioned in our introduction, we suppose that the domain Ω can be split into a finite set of non-empty open sub-domains

$$\Omega = \bigcup_{i=1}^n \overline{\omega_i}, \quad \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j, \quad \omega_i \neq \emptyset, \quad 1 \leq i \leq n. \quad (16)$$

We assume that the open sub-domains ω_i , $1 \leq i \leq n$, admit a regular boundary $\partial\omega_i$, and that two distinct sub-domains ω_i and ω_j ($i \neq j$) may share a part of their boundaries $\partial\omega_i$ and $\partial\omega_j$. For each $i \in \{1, \dots, n\}$, we denote by \mathcal{N}_i a subset of indices corresponding to the neighbors of ω_i . We recall that the choice of \mathcal{N}_i is not necessarily dictated by the parts of the boundary $\partial\omega_i$ shared with other sub-domains ω_j , $i \neq j$. The sub-domains ω_i and the sets \mathcal{N}_i , $1 \leq i \leq n$, generate a graph \mathcal{G} , that underly the complex network of reaction-diffusion systems that we construct. Furthermore, we assume that there exist an open bounded domain $\omega \subset \mathbb{R}^N$ and a family of homeomorphisms h_i , $1 \leq i \leq n$, defined on ω_i with values in ω . We denote by h_i^{-1} the inverse homeomorphism of h_i . The relationship between Ω and ω has been given previously (see equation (3)). Every function ϕ defined on $\Omega \times J$, $J \subset \mathbb{R}$, with values in \mathbb{R} determines a function (ϕ_1, \dots, ϕ_n) defined on $\omega \times J$ with values in \mathbb{R}^n , such that

$$\phi_i(\xi, t) = \phi(h_i^{-1}(\xi), t), \quad (\xi, t) \in \omega \times J, \quad 1 \leq i \leq n.$$

In particular, the source function f defined on $\Omega \times (0, T]$ involved in problem (1), determines a function (f_1, \dots, f_n) , as well as the initial condition u_0 of problem (1) induces a function $(u_{0,1}, \dots, u_{0,n})$. Conversely, every continuous function (ϕ_1, \dots, ϕ_n) defined on $\omega \times J$, $J \subset \mathbb{R}$, with values in \mathbb{R}^n determines a function ϕ defined on $\Omega \times J$ with

$$\phi(x, t) = \sum_{i=1}^n \phi_i(h_i(x), t) \mathbb{1}_{\omega_i}(x),$$

where $\mathbb{1}_{\omega_i}$ denotes the indicator function of ω_i , with the convention that if $x \in \bigcap_{i \in I} \omega_i$ with $I \subset \{1, \dots, n\}$, then $\phi(x, t) = \frac{1}{|I|} \sum_{i \in I} \phi_i(h_i(x), t)$. In the rest of the paper, we will identify

each function ϕ defined on $\Omega \times J$, with the corresponding function (ϕ_1, \dots, ϕ_n) defined on $\omega \times J$.

Finally, we assume that the boundary of ω_i can be split into two parts

$$\partial\omega_i = \Gamma_i^D \cup \Gamma_i^N, \quad \Gamma_i^D \cap \Gamma_i^N = \emptyset, \quad 1 \leq i \leq n, \quad (17)$$

and that the homeomorphisms h_i , $1 \leq i \leq n$, preserve this cutting, that is

$$\gamma_i^D = h_i(\Gamma_i^D), \quad \gamma_i^N = h_i(\Gamma_i^N), \quad \partial\omega = \gamma_i^D \cup \gamma_i^N, \quad \gamma_i^D \cap \gamma_i^N = \emptyset, \quad 1 \leq i \leq n. \quad (18)$$

Now we consider the following complex network of reaction-diffusion systems

$$\begin{cases} \frac{\partial v_i}{\partial t}(\xi, t) = r\Delta v_i(\xi, t) + \delta_i(v_i(\xi, t), \{v_j(\xi, t)\}_{j \in \mathcal{N}_i}) + f_i(\xi, t) & (\xi, t) \in \omega \times \mathbb{R}_+^*, \\ \frac{\partial v_i}{\partial \nu}(\xi, t) = 0 & (\xi, t) \in \gamma_i^N \times \mathbb{R}_+^*, \\ v_i(\xi, t) = 0 & (\xi, t) \in \gamma_i^D \times \mathbb{R}_+^*, \\ v_i(\xi, 0) = v_{i,0}(\xi) & \xi \in \omega, \end{cases} \quad (19)$$

for all $i \in \{1, \dots, n\}$, where ν denotes the outer normal of $\partial\omega$, $u_0 = (u_{i,0})_{1 \leq i \leq n}$ is an initial condition.

Let us finally precise the form of the coupling operator $\delta = (\delta_1, \dots, \delta_n)$. We suppose that there exists a symmetric matrix of order n denoted by $L = (L_{ij})$, such that

$$L_{ji} > 0, \quad j \in \mathcal{N}_i, \quad (20)$$

and

$$L_{ii} \leq - \sum_{j \in \mathcal{N}_i} L_{ji} < 0, \quad (21)$$

and we set

$$\delta_i(v_i(\xi, t), \{v_j(\xi, t)\}_{j \in \mathcal{N}_i}) = L_{ii}v_i + \sum_{j \in \mathcal{N}_i} L_{ji}v_j.$$

4.2 Abstract formulation of the complex network problem

Next, we give an abstract formulation of the complex network problem (19) in order to establish existence and uniqueness results via semigroups techniques. Denote by γ the trace operator that associates to any sufficiently regular function ψ defined on ω its trace defined on $\partial\omega$. For each $i \in \{1, \dots, n\}$, we introduce the space

$$\mathring{H}_{D,i}^1(\omega) = \{\psi \in H^1(\omega) ; \gamma\psi = 0 \text{ on } \gamma_i^D\}, \quad (22)$$

whose dual space is $H_D^{-1}(\omega)$. We then consider the triplet of Hilbert spaces $Z \subset Y \subset Z^*$ where

$$Z = \prod_{i=1}^n \mathring{H}_{D,i}^1(\omega), \quad Y = (L^2(\omega))^n, \quad Z^* = (H_D^{-1}(\omega))^n,$$

and the diagonal operator B defined by

$$B = \text{diag}(B_1, \dots, B_n),$$

where B_i is the operator defined in $H_D^{-1}(\omega)$, associated to the sesquilinear form given by

$$a_i(u, v) = r \sum_{k=1}^n \int_{\omega} \frac{\partial u}{\partial \xi_k} \frac{\partial \bar{v}}{\partial \xi_k} d\xi - L_{ii} \int_{\omega} u \bar{v} d\xi, \quad u, v \in \mathring{H}_{D,i}^1(\omega). \quad (23)$$

In this way, B_i is the realization of the operator $-r\Delta - L_{ii}$ in $H_D^{-1}(\omega)$, under the splitting boundary conditions (Dirichlet boundary condition on γ_i^D and Neumann boundary condition on γ_i^N). It is known that the operators B_i , $1 \leq i \leq n$, and their parts in $L^2(\omega)$ and $\dot{H}_{D,i}^1(\omega)$, are sectorial operators of $H_D^{-1}(\omega)$, $L^2(\omega)$ and $\dot{H}_{D,i}^1(\omega)$ respectively, with angles strictly lesser than $\frac{\pi}{2}$, see [49], Theorem 2.5.

Thus the diagonal operator B defined in Z^* , and its parts in Y and Z are sectorial operators of Z^* , Y and Z respectively, with angles strictly lesser than $\frac{\pi}{2}$. Furthermore, it is known that $B|_Y$ is a positive and self-adjoint operator in Y , and that the domain of $B|_Y$ is such that

$$\begin{aligned} \mathcal{D}(B|_Y) &\subset (W^{1,p_0}(\omega))^n, \quad p_0 > 2, \\ \|v\|_{(W^{1,p_0}(\omega))^n} &\leq C \left(\|B|_Y v\|_Y + \|v\|_Y \right), \quad v \in \mathcal{D}(B|_Y), \end{aligned} \quad (24)$$

if the following assumptions are satisfied:

$$\begin{aligned} |B(\xi_0, \rho) \cap \gamma_i^D| &\geq C\rho^{N-1}, \quad \xi_0 \in \gamma_i^D, \\ |B(\xi_0, \rho) \cap \gamma_i^N| &\geq C\rho^{N-1}, \quad B(\xi_0, \rho) \cap \gamma_i^D = \emptyset, \quad \xi_0 \in \gamma_i^N, \end{aligned}$$

with positive constants ρ, C , where $B(\xi, \rho)$ denotes the open ball in \mathbb{R}^N with center $\xi \in \partial\omega$ and radius ρ , see [4], Theorem 2.2.

The operator B admits fractional powers which are defined by means of the Dunford-Riesz integral, whose domains are characterized by the interpolation spaces, see [49], Theorem 2.35, as follows

$$\mathcal{D}(B^\theta) = (Z, Y)_{2\theta-1}, \quad \frac{1}{2} \leq \theta \leq 1. \quad (25)$$

Note that the interpolation space $(Z, Y)_{2\theta-1}$ is defined for instance in [48], Theorem 1.9.2. Consequently, we have

$$\mathcal{D}(B^{\frac{1}{2}}) = Y, \quad \mathcal{D}(B^\theta) \subset (H^{2\theta-1}(\omega))^n, \quad \frac{1}{2} < \theta \leq 1. \quad (26)$$

Finally, we suppose that $p_0 > N$ and we define the non-linear operator F by setting

$$F(v) = \left(\sum_{j \in \mathcal{N}_i} L_{ji} v_j, \dots, \sum_{j \in \mathcal{N}_n} L_{ji} v_j \right)^T, \quad (27)$$

for all $v \in \mathcal{D}(F) = \left(H^{\frac{N}{p_0}}(\omega) \right)^n$.

The abstract formulation of the complex network problem (19) now reads

$$\begin{cases} v'(t) + Bv(t) = F(v(t)) + f(t), & t \in (0, T], \\ v(0) = u_0. \end{cases} \quad (28)$$

The latter equation is seen to belong to the class of semilinear parabolic abstract equations. In the next section, we show that this problem admits a unique local solution.

4.3 Local solutions for the complex network problem

Let us introduce $\eta = \frac{1}{2} + \frac{N}{2p_0}$, where $p_0 > N$ satisfies equation (24) above. By virtue of (26), we have

$$\mathcal{D}(B^\eta) \subset (H^{2\eta-1}(\omega))^n.$$

Since $2\eta - 1 = \frac{N}{p_0}$, it holds that $\mathcal{D}(B^\eta) \subset \mathcal{D}(F)$.

For two exponents β and σ such that $0 < \sigma < \beta \leq 1$, the space of weighted Hölder continuous functions defined on $(0, T]$, with values in Z^* , is denoted by $\mathcal{F}^{\beta, \sigma}((0, T], Z^*)$, see [49], section 1.2.4.

Theorem 4.1. *For any initial condition $u_0 \in Y$, and any function $f \in \mathcal{F}^{\frac{1}{2}, \sigma}((0, T], Z^*)$ with exponent σ such that $0 < \sigma < 1 - \eta$, the complex network problem (28) admits a unique solution v in the function space*

$$\mathcal{C}((0, T_{u_0, f}], \mathcal{D}(B)) \cap \mathcal{C}([0, T_{u_0, f}], Y) \cap \mathcal{C}^1((0, T_{u_0, f}], Z^*), \quad (29)$$

where $T_{u_0, f}$ is a positive final time depending on u_0 and f . Furthermore, v is necessarily given by the representation formula

$$v(t) = v(0)e^{-tB} + \int_0^t e^{-(t-s)B} [F(v(s)) + f(s)] ds, \quad 0 \leq t \leq T_{u_0, f}. \quad (30)$$

Proof. As noticed previously, we have $\mathcal{D}(B^\eta) \subset \mathcal{D}(F)$. We claim that

$$\|F(\phi) - F(\psi)\|_{Z^*} \leq \|B^\eta(\phi - \psi)\|_{Z^*},$$

for all ϕ, ψ in $\mathcal{D}(B^\eta)$. To prove this assertion, let us consider $\phi = (\phi_1, \dots, \phi_n)$, $\psi = (\psi_1, \dots, \psi_n)$ in $\mathcal{D}(B^\eta)$. For each $i \in \{1, \dots, n\}$, we have, by virtue of the definition of the dual norm:

$$\begin{aligned} \|\phi_i - \psi_i\|_{H_D^{-1}(\omega)} &= \sup_{\|z\|_{\dot{H}_D^1, i} \leq 1} \left| \int_\omega (\phi_i - \psi_i) z d\xi \right| \\ &\leq C \sup_{\|z\|_{\dot{H}_D^1, i} \leq 1} \|\phi_i - \psi_i\|_{L^2(\omega)} \times \|z\|_{L^2(\omega)}, \end{aligned}$$

thanks to the Hölder inequality, where C denotes any positive constant. The Sobolev embeddings, see [49], Theorem 1.36, guaranty that

$$H^1(\omega) \subset L^2(\omega), \quad H^{\frac{N}{p_0}}(\omega) \subset L^2(\omega),$$

which leads to

$$\begin{aligned} \|\phi_i - \psi_i\|_{H_D^{-1}(\omega)} &\leq C \|\phi_i - \psi_i\|_{L^2(\omega)} \\ &\leq C \|\phi_i - \psi_i\|_{H^{\frac{N}{p_0}}(\omega)}, \end{aligned}$$

form which we deduce

$$\|\phi - \psi\|_{Z^*} \leq C \|B^\eta(\phi - \psi)\|_{Z^*}.$$

The conclusion follows from Theorem 4.1 (with $\beta = \frac{1}{2}$) in [49]. \square

Remark 4.2. It is possible to apply the latter proof to a more general class of coupling, defined by the superposition of linear couplings and quadratic couplings. However, we focus on this paper on linear couplings, which we will choose in order to guaranty the convergence of the solution of the complex network (19) towards the solution of the fourth order problem (1).

4.4 Global existence and asymptotic behavior

In this section, we investigate sufficient conditions for which the local solutions of the complex network problem (19) are global in time. Sufficient conditions for the solutions of reaction-diffusion equations or systems to be global are well-known for various boundary conditions, such as Dirichlet, Neumann or Robin boundary conditions. It is also observed that the solutions of such equations can explode in finite time if the non-linearities present an over-polynomial growth, see for instance [33] or [38]. However, since the complex network problem (19) involves mixed boundary conditions, we propose another method to prove that the solutions of problem (19) are global in time. Thus we first prove the non-negativity of the solutions stemming from non-negative initial data, under the assumption that the source term f is itself non-negative and we establish an estimation of the solution in L^2 thanks to Poincaré inequality.

Lemma 4.3. *Let $u_0 \in Y$, and $f \in \mathcal{F}^{\frac{1}{2},\sigma}((0, T], Z^*)$ with $0 < \sigma < 1 - \eta$, be such that $u_{0,i}(\xi) \geq 0$, $f_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in (0, T]$ and $1 \leq i \leq n$. Denote by $v = (v_i)_{1 \leq i \leq n}$ the unique solution of problem (28) stemming from u_0 , in function space (29). Then we have $v_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in [0, T_{u_0, f}]$ and $1 \leq i \leq n$.*

Proof. We introduce the auxiliary problem given by

$$\frac{\partial \tilde{v}_i}{\partial t} = r\Delta \tilde{v}_i + L_{ii}\tilde{v}_i + \sum_{j \in \mathcal{N}_i} L_{ji} |\tilde{v}_j| + f_i, \quad 1 \leq i \leq n,$$

with the same initial condition u_0 , and the same boundary condition as in (19). We can apply the same method as in the proof of Theorem 4.1 in order to prove that the auxiliary problem admits a unique local solution \tilde{v} defined on some interval $[0, \tilde{T}_{u_0, f}]$ with $\tilde{T}_{u_0, f} > 0$. Furthermore, we consider the truncation function χ defined on \mathbb{R} by

$$\chi(s) = \begin{cases} 0 & \text{if } s \geq 0, \\ s^2 & \text{else.} \end{cases}$$

It is easy to see that χ is continuously differentiable on \mathbb{R} and satisfies the properties

$$\chi(s) \geq 0, \quad \chi'(s) \leq 0, \quad s\chi'(s) \geq 0, \quad t \in \mathbb{R}. \quad (31)$$

Next we define for each $i \in \{1, \dots, n\}$ the function ρ_i by setting

$$\rho_i(t) = \int_{\omega} \chi(\tilde{v}_i(\xi, t)) d\xi, \quad t \in [0, \tilde{T}_{u_0, f}].$$

We have $\rho_i(t) \geq 0$ for all $t \in [0, \tilde{T}_{u_0, f}]$ by construction and $\rho_i(0) = 0$ since u_0 admits non-negative components. Moreover, ρ_i is continuously differentiable on $[0, \tilde{T}_{u_0, f}]$, with

$$\begin{aligned} \rho_i'(t) &= \int_{\omega} \frac{\partial \tilde{v}_i}{\partial t} \chi'(\tilde{v}_i) d\xi \\ &= \int_{\omega} \left[r\Delta \tilde{v}_i + L_{ii}\tilde{v}_i + \sum_{j \in \mathcal{N}_i} L_{ji} |\tilde{v}_j| + f_i \right] \chi'(\tilde{v}_i) d\xi \\ &= r \int_{\omega} \Delta \tilde{v}_i \chi'(\tilde{v}_i) d\xi + L_{ii} \int_{\omega} \tilde{v}_i \chi'(\tilde{v}_i) d\xi + \sum_{j \in \mathcal{N}_i} L_{ji} \int_{\omega} (|\tilde{v}_j| + f_i) \chi'(\tilde{v}_i) d\xi \\ &= -r \int_{\omega} |\nabla \chi'(\tilde{v}_i)|^2 d\xi + L_{ii} \int_{\omega} \tilde{v}_i \chi'(\tilde{v}_i) d\xi + \sum_{j \in \mathcal{N}_i} L_{ji} \int_{\omega} (|\tilde{v}_j| + f_i) \chi'(\tilde{v}_i) d\xi, \end{aligned}$$

where we omit the variables ξ and t in order to lighten our notations. It follows from (21) and (31) that $\rho'_i(t) \leq 0$ for all $t \in [0, \tilde{T}_{u_0, f}]$. We can deduce that $\rho_i \equiv 0$ on $[0, \tilde{T}_{u_0, f}]$, which means that $\tilde{v}_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in [0, \tilde{T}_{u_0, f}]$ and $1 \leq i \leq n$. Thus \tilde{v} is also a solution of the initial complex network problem (28). By uniqueness, we obtain $\tilde{v} = v$ on $[0, T_{u_0, f}] \cap [0, \tilde{T}_{u_0, f}]$. Finally, we easily prove that $T_{u_0, f} = \tilde{T}_{u_0, f}$ and this achieves the proof. \square

The next proposition establishes an energy estimate of the solutions of the complex network problem (19).

Proposition 4.4. *Assume that $u_0 \in Y$ and $f \in L^\infty(\omega \times [0, T])$ satisfy $u_{0,i}(\xi) \geq 0$, $f_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in (0, T]$ and $1 \leq i \leq n$. Let $v = (v_i)_{1 \leq i \leq n}$ denote the unique solution of the complex network problem (19) stemming from u_0 , defined on $[0, T_{u_0, f}]$. Then we have:*

$$\sum_{i=1}^n \|v_i(t)\|_{L^2(\omega)}^2 \leq e^{-Kt} \sum_{i=1}^n \|u_{0,i}\|_{L^2(\omega)}^2 + \frac{4n|\omega| \|f\|_{L^\infty(\omega \times [0, T])}}{K^2}, \quad 0 < t \leq T_{u_0, f}, \quad (32)$$

for some positive constant K .

Proof. The duality product in $H_D^{-1}(\omega) \times \dot{H}_{D,i}^1(\omega)$ between the equation giving the state of node v_i in system (19) and v_i leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\omega} v_i^2(\xi, t) d\xi + r \int_{\omega} |\nabla v_i(\xi, t)|^2 d\xi \leq \int_{\omega} f_i(\xi, t) v_i(\xi, t) d\xi, \quad 0 < t \leq T_{u_0, f}.$$

By virtue of Poincaré inequality 2.3, we have

$$\|v_i\|_{L^2(\omega)} \leq C_i \|\nabla v_i\|_{L^2(\omega)},$$

for some positive constants C_i , $1 \leq i \leq n$, since $v_i \in \dot{H}_{D,i}^1(\omega)$. It follows that

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi + K \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi \leq \sum_{i=1}^n \int_{\omega} f_i(\xi, t) v_i(\xi, t) d\xi, \quad 0 < t \leq T_{u_0, f},$$

for some positive constant $K > 0$. By assumption, we have $f \in L^\infty(\omega \times [0, T])$, from which we deduce

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi + K \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi \leq \|f\|_{L^\infty(\omega \times [0, T])} \sum_{i=1}^n \int_{\omega} v_i(\xi, t) d\xi, \quad 0 < t \leq T_{u_0, f}.$$

Next, we use the elementary inequality

$$s \leq \theta s^2 + \frac{1}{\theta}, \quad \forall s \in \mathbb{R},$$

with $\theta > 0$ arbitrarily small. We obtain

$$\|f\|_{L^\infty(\omega \times [0, T])} \sum_{i=1}^n \int_{\omega} v_i(\xi, t) d\xi \leq \theta \|f\|_{L^\infty(\omega \times [0, T])} \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi + \frac{n|\omega|}{\theta},$$

which leads to

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi + \frac{K}{2} \sum_{i=1}^n \int_{\omega} v_i^2(\xi, t) d\xi \leq \frac{2n|\omega| \|f\|_{L^\infty(\omega \times [0, T])}}{K},$$

by choosing $\theta = \frac{K}{2\|f\|_{L^\infty(\omega \times [0, T])}}$. Applying Gronwall lemma 2.2 yields the conclusion. \square

As a first corollary of the latter proposition, we directly obtain the global in time existence of the solutions of the complex network problem (19).

Corollary 4.5. *Under the assumptions of proposition 4.4, the solution v of the complex network (19) stemming from u_0 is global, that is $T_{u_0, f} = T$. Furthermore, in the case where $f \in L^\infty(\Omega \times (0, +\infty))$, then $T = +\infty$.*

The energy estimate (32) also guarantees that the complex network problem (28) determines a continuous dynamical system $(S(t), \Phi, Y)$ defined in Y , with phase space

$$\Phi = \{u_0 \in Y ; u_{0,i}(\xi) \geq 0 \text{ in } \omega, 1 \leq i \leq n\},$$

where $S(t)$ is a semigroup of non-linear operators defined by

$$\begin{aligned} S(t) : \quad \Phi \times (0, +\infty) &\longrightarrow \Phi \\ (u_0, t) &\longmapsto v(t, u_0), \end{aligned}$$

where $v(t, u_0)$ denotes the unique solution of problem (19) stemming from u_0 . By virtue of Theorem 6.15 in [49], we have the following corollary.

Corollary 4.6. *The continuous dynamical system $(S(t), \Phi, Y)$ defined above admits a family (\mathcal{M}) of exponential attractors.*

We recall that an exponential attractor is a compact subset $\mathfrak{M} \subset \Phi$ which is positively invariant and attracts the bounded subsets of Φ at an exponential rate.

5 Sufficient conditions of convergence

In this section, our aim is to establish sufficient conditions for the solution of the complex network problem (19) to converge to the solution of the fourth order equation (1), when the number n of nodes in the network tends to infinity. Our approach consists in constructing an intermediate finite differences scheme which links the solution of the fourth order problem (1) to the solution of the complex network (19).

5.1 Consistency and stability assumptions

First of all, we define a double discretization in space and time of Ω and $[0, T]$, where T is a given positive final time. For each $i \in \{1, \dots, n\}$, we choose $\bar{x}_i \in \omega_i$, and we set

$$\delta x = \max_{1 \leq i \leq n} \sup_{y, z \in \omega_i} \|y - z\|_{\mathbb{R}}.$$

The coefficient δx is regarded as the size of the space discretization. We also introduce a discretization of the time interval $[0, T]$ of step $\delta t > 0$:

$$0 < \delta t < 2\delta t < \dots < k\delta t < \dots < m\delta t, \quad 0 \leq k \leq m, \quad (33)$$

where m is a positive integer. For each function ϕ defined in $\Omega \times [0, T]$, we set

$$\begin{aligned} \phi_i^k &= \phi(\bar{x}_i, k\delta t), \quad 1 \leq i \leq n, \quad 0 \leq k \leq m, \\ \phi^k &= (\phi_1^k, \dots, \phi_n^k)^T, \quad 0 \leq k \leq m. \end{aligned} \quad (34)$$

Additionally, we assume that one can choose two matrices Q_1 and Q_2 of order n , such that the following explicit finite differences scheme

$$\frac{w^{k+1} - w^k}{\delta t} = Q_1 w^k + f^k, \quad (35)$$

approximates the heat equation

$$\frac{\partial y}{\partial t} = r\Delta y + f, \quad (36)$$

while the second following scheme

$$\frac{w^{k+1} - w^k}{\delta t} = Q_2 w^k, \quad (37)$$

approximates the biharmonic equation

$$\frac{\partial z}{\partial t} = -\Delta^2 z. \quad (38)$$

More precisely, we assume that there exists a norm $\|\cdot\|_\star$ in \mathbb{R}^n such that the following assumptions are satisfied.

- (H_1) [consistency of scheme (35)]: the first scheme (35) is consistent for the norm $\|\cdot\|_\star$, with accuracy $d_1 > 0$ in space and $e_1 > 0$ in time, that is

$$\frac{y^{k+1} - y^k}{\delta t} = Q_1 y^k + f^k + \tau_1, \quad \|\tau_1\|_\star = O((\delta x)^{d_1}) + O((\delta t)^{e_1}),$$

where y denotes a solution of the heat equation (36) in Ω with fixed boundary condition.

- (H_2) [consistency of scheme (37)]: the second scheme (37) is consistent for the norm $\|\cdot\|_\star$, with accuracy $d_2 > 0$ in space and $e_2 > 0$ in time, that is

$$\frac{z^{k+1} - z^k}{\delta t} = Q_2 z^k + \tau_2, \quad \|\tau_2\|_\star = O((\delta x)^{d_2}) + O((\delta t)^{e_2}),$$

where z denotes a solution of the biharmonic equation (38) in Ω with fixed boundary condition.

- (H_3) [conditional stability of schemes (35) and (37)]: the matrices Q_1 and Q_2 satisfy

$$\|Q_i u_0\|_\star \leq K_i \|u_0\|_\star, \quad \forall n \geq 0, \quad \forall u_0 \in \mathbb{R}^n, \quad i \in \{1, 2\},$$

with positive constants K_1 and K_2 , provided δx and δt fulfill a common stability condition of the type

$$h(\delta x, \delta t) \leq 0,$$

where h denotes a polynomial of degree 2.

Before we state our convergence result, we show that the above assumptions can easily be satisfied at least for space dimensions $N = 1$ and $N = 2$ (see for instance [50] and references therein cited for the three-dimensional case).

First, suppose that $N = 1$ and $\Omega = (0, 1)$. The discretization of Ω can be defined by setting $\delta x = \frac{1}{n}$ and $\omega_i = (i\delta x, (i+1)\delta x)$ for $0 \leq i \leq n-1$. It is known, see for instance [5] or [45], that scheme (35) can be defined stemming from the expression

$$\Delta u(x) = \frac{u(x - \delta x) - 2u(x) + u(x + \delta x)}{(\delta x)^2} + O((\delta x)^2),$$

which determines the *stencil* of the scheme:

$$\frac{1}{(\delta x)^2} \begin{bmatrix} & & & & \\ & & & & \\ & & 1 & & \\ & & & -2 & \\ & & & & 1 \\ & & & & & \end{bmatrix}.$$

In parallel, scheme (37) can be determined by the 5 points stencil

$$\frac{1}{(\delta x)^2} \begin{bmatrix} -1 & 4 & -6 & 4 & -1 \end{bmatrix}.$$

In this way, it is verified that schemes (35) and (37) are consistent for the L^2 norm defined by

$$\|u\|_* = \sqrt{\sum_{j=1}^n |u_j|^2 \delta x}, \quad u \in \mathbb{R}^n,$$

and that those schemes are conditionally stable in this norm, under the Courant-Friedrichs-Lewy condition (CFL)

$$\delta t - K(\delta x)^2 \leq 0,$$

for some positive constant K .

Now suppose that the space dimension is $N = 2$, and assume that $\Omega = (0, 1)^2$. The discretization of Ω can be defined by setting $\delta x = \frac{1}{\sqrt{n}}$ and

$$\Omega = \bigcup_{1 \leq i_1, i_2 \leq \sqrt{n}} \bar{\omega}_{i_1, i_2}, \quad \omega_{i_1, i_2} = (i_1 \delta x, (i_1 + 1) \delta x) \times (i_2 \delta x, (i_2 + 1) \delta x), \quad 1 \leq i_1, i_2 \leq \sqrt{n},$$

provided n is chosen as a squared integer. Scheme (35) can be defined by the 5 points stencil

$$\frac{1}{(\delta x)^2} \begin{bmatrix} & & 1 & & \\ & 1 & -4 & 1 & \\ & & 1 & & \end{bmatrix},$$

whereas scheme (37) can be defined by the 13 points stencil

$$\frac{1}{(\delta x)^2} \begin{bmatrix} & & & -1 & & & \\ & & -2 & 8 & -2 & & \\ & -1 & 8 & -20 & 8 & -1 & \\ & & -2 & 8 & -2 & & \\ & & & -1 & & & \end{bmatrix}.$$

Once again, it is known that those expressions determine L^2 consistent schemes, which are conditionally stable in the L^2 norm under the CFL condition.

Remark 5.1. It is worth emphasizing that the stencils corresponding to the approximation of the biharmonic equation (38) involve a greater number of points than the stencils corresponding to the approximation of the heat equation (36), thus we recover the principle of diffusion “in the neighborhood of the neighborhood” mentioned in our introduction.

Furthermore, it is observed that the central coefficients of the stencils presented above are negative and that the sums of the coefficients of the stencils always equal zero. Consequently, we can choose to define the couplings of the complex network problem (19) by setting

$$L = Q_2. \tag{39}$$

In this way, we easily verify that conditions (20) and (21) hold for matrix L .

Obviously, many other approximation schemes could be considered equivalently, see [45], but we are not focusing on this point in this paper.

5.2 Convergence result

Here, we finally establish the convergence of the solution of the complex network problem (19) towards the solution of the fourth order problem (1), under sufficient conditions which guarantee a relevant choice of the connectivity matrix L .

Proposition 5.2. *Assume that hypotheses (H_1) , (H_2) and (H_3) are fulfilled, and furthermore, that the connectivity matrix L is defined by (39). Let f be a source term in $L^\infty(\Omega \times [0, T])$, u_0 an initial condition in $(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{2}, 2} \cap \Phi$, and denote by u and v the global solutions of problems (1) and (19) respectively. Then it holds that*

$$\sup_{0 \leq k \leq m} \|u^k - v^k\|_* = O((\delta x)^{d_3}) + O((\delta t)^{e_3}),$$

with $d_3 = \max(d_1, d_2)$ and $e_3 = \max(e_1, e_2)$.

Proof. Let us introduce the following approximation scheme

$$\frac{w^{k+1} - w^k}{\delta t} = (Q_1 + L)w^k + f^k, \quad 0 \leq k \leq m. \quad (40)$$

First, we show that the solution v of the complex network problem (19) converges to w . To that aim, we compute for each $k \in \{0, \dots, m\}$

$$\frac{v^{k+1} - v^k}{\delta t} - \frac{w^{k+1} - w^k}{\delta t} = Q_1 v^k + L v^k + f^k + \tau_1 - [(Q_1 + L)w^k + f^k] = \tau_1,$$

from which we can deduce that

$$v^{k+1} - w^{k+1} = [I_n + \delta t(Q_1 + L)](v^k - w^k).$$

We can deduce from the conditional stability of matrices Q_1 and L , which holds by virtue of assumption (H_3) , that matrix $Q_1 + L$ is also conditionally stable. Using classical methods of Lax theorem, see [45], it follows that v converges towards w , with accuracy d_1 in space and e_1 in time.

Using similar arguments, we show that the solution u of the fourth order problem (1) also converges to w , with accuracy d_2 in space and e_2 in time.

The conclusion follows from the triangular inequality. \square

As a direct consequence, we obtain the following corollary.

Corollary 5.3. *Each solution of the fourth order problem (1) is attracted at an exponential rate by the attractors (\mathcal{M}) of the complex network problem.*

Conclusion and perspectives

In this paper, we have analyzed a fourth order parabolic problem modeling generalized diffusion in population dynamics. Using sectorial operators and analytic semigroups techniques, we have proved new existence and regularity results. Additionally, we have proposed a novel approximation of our fourth order problem by a complex network of reaction-diffusion equations. In a restricted set of reasonable assumptions, we have obtained a first convergence result, from which we have deduced the existence of exponential attractors for the fourth order problem.

In a future work, we aim to enlarge our framework, by considering other boundary conditions for the initial fourth order problem and by adding non-linearities in the source term. Furthermore, we believe that the convergence results can also be extended to a more general functional spaces setting.

Acknowledgment

The authors wish to sincerely thank Professor Aziz-Alaoui from University of Le Havre Normandie for his useful comments and encouragements on this work.

References

- [1] B. Ambrosio, M. Aziz-Alaoui, and V. Phan. Large time behaviour and synchronization of complex networks of reaction–diffusion systems of Fitzhugh–Nagumo type. *IMA Journal of Applied Mathematics*, 2019.
- [2] M. Aziz-Alaoui. Synchronization of chaos. *Encyclopedia of Mathematical Physics, Elsevier*, 5:213–226, 2006.
- [3] I. Belykh, M. Hasler, M. Lauret, and H. Nijmeijer. Synchronization and graph topology. *International Journal of Bifurcation and Chaos*, 15(11):3423–3433, 2005.
- [4] A. Bensoussan and J. Frehse. *Regularity results for nonlinear elliptic systems and applications*, volume 151. Springer Science & Business Media, 2013.
- [5] B. Bialecki. A fourth order finite difference method for the Dirichlet biharmonic problem. *Numerical Algorithms*, 61(3):351–375, Nov 2012.
- [6] G. Cantin. Non identical coupled networks with a geographical model for human behaviors during catastrophic events. *International Journal of Bifurcation and Chaos*, 27(14):1750213, 2017.
- [7] G. Cantin, N. Verdière, and M. Aziz-Alaoui. Large time dynamics in complex networks of reaction-diffusion systems applied to a panic model. *IMA Journal of Applied Mathematics*, 9 2019.
- [8] R. S. Cantrell and C. Cosner. *Spatial ecology via reaction-diffusion equations*. John Wiley & Sons, 2004.
- [9] G. Chen, Z. Li, and P. Lin. A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible Stokes flow. *Advances in Computational Mathematics*, 29(2):113–133, Aug 2008.
- [10] L. Cherfils, A. Miranville, and S. Zelik. On a generalized Cahn-Hilliard equation with biological applications. *DCDS-B*, 19(7):2013–2026, 2014.
- [11] D. S. Cohen and J. D. Murray. A generalized diffusion model for growth and dispersal in a population. *Journal of Mathematical Biology*, 12(2):237–249, 1981.
- [12] G. Da Prato. Sommes d’opérateurs linéaires et équations différentielles opérationnelles. *J. Math. Pures Appl.*(9), 54:305–387, 1975.
- [13] L. De Simon. Un’applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rendiconti del Seminario Matematico della Università di Padova*, 34:205–223, 1964.
- [14] V. Dolean, P. Jolivet, and F. Nataf. *An Introduction to Domain Decomposition Methods*. Society for Industrial and Applied Mathematics, 2015.
- [15] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Mathematische Zeitschrift*, 196(2):189–201, 1987.
- [16] M. Ebenbeck and H. Garcke. On a cahn–hilliard–brinkman model for tumor growth and its singular limits. *SIAM Journal on Mathematical Analysis*, 51(3):1868–1912, 2019.
- [17] L. Ehrlich. Solving the biharmonic equation as coupled finite difference equations. *SIAM Journal on Numerical Analysis*, 8(2):278–287, 1971.

- [18] A. Giorgini, A. Miranville, and R. Temam. Uniqueness and regularity for the navier–stokes–cahn–hilliard system. *SIAM Journal on Mathematical Analysis*, 51(3):2535–2574, 2019.
- [19] R. Glowinski and O. Pironneau. Numerical methods for the first biharmonic equation and for the two-dimensional stokes problem. *SIAM Review*, 21(2):167–212, 1979.
- [20] M. Golubitsky and I. Stewart. Nonlinear dynamics of networks: the groupoid formalism. *Bulletin of the american mathematical society*, 43(3):305–364, 2006.
- [21] P. Grisvard. Équations différentielles abstraites. In *Annales scientifiques de l’Ecole Normale Supérieure*, volume 2, pages 311–395, 1969.
- [22] P. Grisvard. Spazi di tracce e applicazioni. *Rend. Mat.*, 5:657–729, 1972.
- [23] M. Haase. The functional calculus for sectorial operators. In *The Functional Calculus for Sectorial Operators*, pages 19–60. Springer, 2006.
- [24] D. Hilhorst, M. Mimura, and R. Weidenfeld. On a reaction-diffusion system for a population of hunters and farmers. In *Free Boundary Problems*, pages 189–196. Springer, 2003.
- [25] N. Katzourakis and T. Pryer. On the numerical approximation of p -biharmonic and ∞ -biharmonic functions. *Numerical Methods for Partial Differential Equations*, 35(1):155–180, 2019.
- [26] H. Komatsu. Fractional powers of operators. *Pacific Journal of Mathematics*, 19(2):285–346, 1966.
- [27] R. Labbas, K. Lemrabet, S. Maingot, and A. Thorel. Generalized linear models for population dynamics in two juxtaposed habitats. *Discrete & Continuous Dynamical Systems - A*, 39:2933, 2019.
- [28] R. Labbas, S. Maingot, D. Manceau, and A. Thorel. On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain. *Journal of Mathematical Analysis and Applications*, 450(1):351 – 376, 2017.
- [29] R. Labbas and M. Moussaoui. On the resolution of the heat equation with discontinuous coefficients. In *Semigroup Forum*, volume 60, pages 187–201. Springer, 2000.
- [30] F. Lara Ochoa. A generalized reaction diffusion model for spatial structure formed by motile cells. *Biosystems*, 17(1):35 – 50, 1984.
- [31] H. Matano. Asymptotic behavior of solutions of semilinear heat equations on S^1 . In *Nonlinear Diffusion Equations and Their Equilibrium States II*, pages 139–162, New York, NY, 1988. Springer US.
- [32] D. Matthes and J. Zinsl. Existence of solutions for a class of fourth order cross-diffusion systems of gradient flow type. *Nonlinear Analysis*, 159:316–338, 2017.
- [33] J. Morgan. Global existence for semilinear parabolic systems. *SIAM journal on mathematical analysis*, 20(5):1128–1144, 1989.
- [34] A. Novick-Cohen. Sur une classe d’espaces d’interpolation. *Publications mathématiques de l’I.H.É.S.*, 19:5–68, 1964.
- [35] A. Novick-Cohen. On Cahn-Hilliard type equations. *Nonlinear Analysis: Theory, Methods & Applications*, 15(9):797 – 814, 1990.
- [36] A. Okubo. Diffusion and ecological problems: mathematical models. *Biomath*, 10, 1980.
- [37] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer Verlag, 1983.
- [38] M. Pierre. Global existence in reaction-diffusion systems with control of mass: a survey. *Milan Journal of Mathematics*, 78(2):417–455, 2010.
- [39] J. Prüss and H. Sohr. On operators with bounded imaginary powers in banach spaces. *Mathematische Zeitschrift*, 203:429–452, 1990.

- [40] B. Rink and J. Sanders. Coupled cell networks: semigroups, Lie algebras and normal forms. *Transactions of the American Mathematical Society*, 367(5):3509–3548, 2015.
- [41] W. Ruan. Wavefront solutions of degenerate quasilinear reaction–diffusion systems with mixed quasi-monotonicity. *Nonlinear Analysis*, 182:75–96, 2019.
- [42] J. Rubio de Francia. Martingale and integral transforms of banach space valued functions. *Lecture Notes in Math.*, 1221:195–22, 1985.
- [43] J. Smoller. *Shock waves and reaction-diffusion equations*, volume 258. Springer Science & Business Media, 1994.
- [44] J. Stephenson. Single cell discretizations of order two and four for biharmonic problems. *Journal of Computational Physics*, 55(1):65 – 80, 1984.
- [45] J. Thomas. *Numerical partial differential equations: finite difference methods*, volume 22. Springer Science & Business Media, 2013.
- [46] A. Thorel. Operational approach for biharmonic equations in L^p -spaces. *Journal of Evolution Equations*, 9 2019. Accepted.
- [47] A. Toselli and O. Widlund. *Domain Decomposition Methods - Algorithms and Theory*. Springer Series in Computational Mathematics. Springer Berlin Heidelberg, 2004.
- [48] H. Triebel. *Interpolation theory, function Spaces, differential operators*. North-Holland publishing company Amsterdam New York Oxford, 1978.
- [49] A. Yagi. *Abstract parabolic evolution equations and their applications*. Springer Science & Business Media, 2009.
- [50] J. Zhang. An explicit fourth-order compact finite difference scheme for three-dimensional convection–diffusion equation. *Communications in numerical methods in engineering*, 14(3):209–218, 1998.