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# Invariance, controlled invariance and conditioned invariance in structured systems and applications to disturbance decoupling

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**Abstract.** In this paper, dynamical systems whose structure is defined by means of a simple, directed graph are considered. These objects can be used to model structured systems or, more generally, networks of systems and systems of systems, where the relations between state, input and output variables or, respectively, between agents are known only for being zero or nonzero. Using an approach that is conceptually similar to the geometric approach developed for linear time-invariant systems, suitable notions of invariance, controlled invariance and conditioned invariance are introduced and related to the action of feedbacks. The results are used to provide general solvability conditions for disturbance decoupling problems expressed in graph-theoretic terms.

## 1. Introduction

Simple, directed graphs can be used to describe dynamical systems by associating the vertices to the state variables of the system and the edges to nonzero relationships between them. This representation can be used for modeling uncertain systems, whose equations contain parameters that are unknown except for being equal to or different from 0. Networks of systems and systems of systems can be represented analogously, associating the vertices of the graph to the agents of the network or to the components of a complex system, when the focus is on the influence that each agent or component exerts on the others, rather than on the individual dynamics of each of them. In all cases, we speak of structured systems to highlight the fact that the graph gives information only on those aspects of the dynamical structure of the system that do not depend on specific values of the nonzero parameters or on specific features of the communication lines that link the agents.

The study of linear structured systems was pioneered in [1] and it received contributions by many authors in the subsequent years. Investigation aims mainly at discovering and pointing out which properties of the system are determined by the structure of the graph and, therefore, can be said to be *structural* or, since they hold for any value of the nonzero parameters, *generic*.



Examples are structural (or generic) controllability and structural (or generic) observability. Classical control problems, like noninteractive control and disturbance decoupling, have been studied in that framework, together with problems related to the structure of finite zeros and zeros at infinity. An excellent survey of available results in the area, with an exhaustive list of references, can be found in [2].

More recently, graphs have been used to model the interaction of a plurality of independent agents in complex systems, like networked systems and, more generally, systems of systems (see [3], [4], [5], [6], [7], [8] and the references therein).

The approach to the study of structured systems we develop in this paper is based on geometric concepts and, as such, it is conceptually analogous to the structural geometric approach for classical linear systems of [9] and [10]. To do this, we restrict our attention to a particular class of structured systems, which is slightly less general than that considered e.g. in [2]. More specifically, we assume that, when the system is represented in state space form, each input variable appears in no more than one state equation. In terms of the underlying graph, this means that each input channel involves a single vertex. Structured systems of this kind are those considered in [11], [12], [13], those used in modelling quantum systems in [14] and those appearing in [15] and in examples in [3]. Such feature is crucial to introduce suitable notions of *invariance*, *controlled invariance* and *conditioned invariance* that can be naturally and consistently expressed in graph-theoretic terms. Controlled invariance is then related to and interpreted in terms of a notion that characterizes the (state or output) feedbacks whose action modifies the underlying graph of the structured system, called *essential feedback*. Also the notion of essential feedback can be naturally and simply expressed in graph-theoretic terms.

Invariance and feedback whose action modifies the underlying graph do not have a similar characterization for general structured systems, in which a single input variable may affect more than one state variable. This is probably why, in spite of the existence of a wide literature in this area, they have not, to the best of our knowledge, been previously introduced in graph-theoretic terms as we do here (compare with [2]). The definition of invariance and feedback in graph-theoretic terms for the considered class of structured systems is the main theoretic contribution of this paper.

The approach developed in this way can be applied to the study of a number of problems, which in particular include disturbance decoupling problems by means of state or output feedback. In regard to these problems, we are able to state necessary and sufficient solvability conditions for the considered class of structured systems in Theorem 1 and in Theorem 2. Such conditions are equivalent to those stated in Theorem 7 and 8 of [2] for general structured systems, but, thanks to the notions we introduce here, they are expressed in a more natural and simple way, which formally coincides with the classical formulation of solvability of the analogous problems in the framework of classical linear systems.

The paper is organized as follows. In Section 2, we describe the graph representation of structured systems and we introduce the class of systems we consider. In Section 3, we introduce the fundamental notions of invariance, controlled invariance and conditioned invariance in graph-theoretic terms. Then, we define the notion of essential feedback and we show how it can be used to characterize controlled invariance. In Sections 4 and 5, we study the disturbance decoupling problem by means, respectively, of state feedback and of output feedback and we characterize solvability by means of necessary and sufficient conditions. Proofs of our results are only sketched and they will appear in complete form elsewhere. Section 6 contains conclusions and description of future work.

**Notation** Given two sets  $A$  and  $B$ , we will denote by  $A \setminus B$  the set difference of  $A$  and  $B$ , that is the subset of  $A$  defined by  $A \setminus B = \{a \in A, \text{ such that } a \notin B\}$ .

## 2. Preliminaries

Let  $(G, \mathcal{E})$  be a simple directed graph (i.e. a directed graph without multiple edges and without auto-loops) with set of vertices  $G = \{v_1, \dots, v_n\}$  and set of edges  $\mathcal{E} \subseteq G \times G$ . If  $(v_j, v_i) \in \mathcal{E}$ , we say that  $v_j$  is the tail and  $v_i$  is the head of the edge  $(v_j, v_i)$ . A path  $P$  in  $(G, \mathcal{E})$  is an ordered finite sequence of edges  $(e_1, \dots, e_k)$  in which the head of the edge  $e_h$  coincides with the tail of the edge  $e_{h+1}$ . The tail of the first edge in a path  $P$  is called the tail of the path and the head of the last edge is called the head of the path.

Given a graph  $(G, \mathcal{E})$  with  $n$  vertices, we associate to it an  $n \times n$  matrix  $A = [a_{ij}]$  whose entries are real, mutually independent parameters that satisfy the following conditions

- $a_{ij} \neq 0$  for  $i \neq j$  if and only if  $(v_j, v_i) \in \mathcal{E}$  (i.e. there is an edge from  $v_j$  to  $v_i$  in  $(G, \mathcal{E})$ ).

Note that no condition is imposed on  $a_{ij}$  for  $i = j$ .

Letting  $G_{in} = \{v_{i_1}, \dots, v_{i_m}\} \subseteq G$  be a subset of vertices, we associate to the pair  $((G, \mathcal{E}), G_{in})$  an  $n \times m$  matrix  $B = [b_{ij}]$  whose entries are real, mutually independent parameters that satisfy the following conditions

- $b_{ij} = 1$  (or more generally  $b_{ij} \neq 0$ ) if  $v_{i_j} = v_i$  (that is: if the  $j$ -th element of  $G_{in}$  is equal to  $v_i$ )
- $b_{ij} = 0$  otherwise.

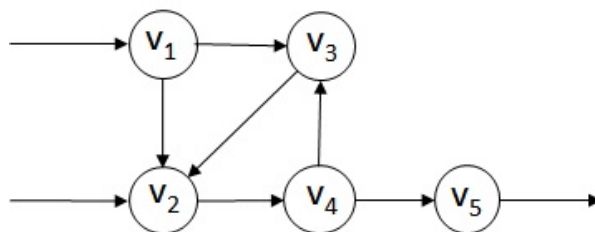
Note that in any column of  $B$  there is just one entry different from 0, while in any row of  $B$  there is at most one entry different from 0.

Letting  $G_{out} = \{v_{i_1}, \dots, v_{i_p}\} \subseteq G$  be a subset of vertices, we associate to the pair  $((G, \mathcal{E}), G_{out})$  a  $p \times n$  matrix  $C = [c_{ij}]$  whose entries are real, mutually independent parameters that satisfy the following conditions

- $c_{ij} = 1$  (or more generally  $c_{ij} \neq 0$ ) if  $v_{i_j} = v_i$  (that is: if the  $j$ -th element of  $G_{out}$  is equal to  $v_i$ )
- $c_{ij} = 0$  otherwise.

Note that in any row of  $C$  there is just one entry different from 0, while in any column of  $C$  there is at most one entry different from 0.

In representing graphically the triple  $((G, \mathcal{E}), G_{in}, G_{out})$ , we use arrows between vertices to indicate edges and we mark the elements of  $G_{in}$  by ingoing arrows and the elements of  $G_{out}$  by outgoing arrows, as in Figure 1.



**Figure 1.** A simple, directed graph with  $G_{in} = \{v_1, v_2\}$  and  $G_{out} = \{v_5\}$ .

The **structured system**  $\Sigma$  associated to the triple  $((G, \mathcal{E}), G_{in}, G_{out})$  is the linear time-invariant system described in parametric state space form by the equation

$$\Sigma \equiv \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \quad (1)$$

with state  $x \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^m$  and output  $y \in \mathbb{R}^p$ .

Structured systems are used to model families of linear systems whose elements are characterized by specific values of the parameters appearing in (1), or uncertain systems where parameters account for unknown relations between variables. In this case, the vertex  $v_i \in G$  corresponds to the component  $x_i$  of the state vector  $x = (x_1, \dots, x_n)^\top$  of  $\Sigma$ . The elements of  $G_{in}$  identify the components of the state vector whose dynamics is directly influenced by the input, while the elements of  $G_{out}$  identify the component of the state that are directly measurable from the output. One can also interpret  $\Sigma$  as the model of a network of systems, or dynamical agents, or a complex system of systems. In this case, the vertices of the graph represent systems, or agents, in the network and the edges represent monodirectional communication lines, whose weights are expressed by the parameters, that account for the way in which agents influence each other. Vertices in  $G_{in}$  correspond to agents that can be influenced by inputs coming from the external environment and vertices in  $G_{out}$  corresponds to agents that send outputs to the external environment.

In all the above mentioned interpretations, the topology of the underlying graph can be exploited to obtain information on dynamical properties that are shared by all the elements in the family, i.e. that are valid for all the values of the uncertain parameters, or that characterize the overall properties of the network.

Note that, as pointed out above, the graph gives no information about the diagonal elements  $a_{ii}$  of  $A$  since it does not contain auto-loops. Actually, the fact that an element of that kind is equal to or different from 0 does not affect the dynamical properties we will deal with in the sequel.

**Remark 1** *It is important to remark that the class of structured systems we consider is less general than that considered in [2]. The particular structure of the matrix  $B$  in our framework, in fact, means that each input variable in (1) appears in only one state equation or, in other words, that it influences directly the dynamics of only one state variable. This restriction is relevant for assuring validity and consistency to the notions that will be introduced in the following section and for deriving the results that follow. Structured systems of this class are explicitly considered in [11], [12], [13], they are used in modelling quantum systems [14] and example appears in [15], [3]. Similarly, the particular structure of the matrix  $C$  means that each output variable coincides, except for a scale factor, with one state variable.*

**Notation** We denote the structured system associated, as described above, to the triple  $((G, \mathcal{E}), G_{in}, G_{out})$  by  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  or simply by  $\Sigma$ , if no confusion arises.

### 3. Invariance and controlled invariance

Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , we call *dynamically closed* a subset of vertices  $V \subseteq G$  if and only if any path  $P$  whose tail and head are in  $V$  consists of edges whose tails and heads are in  $V$ .

A subsystem of  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  is a structured system  $\Sigma((V, \mathcal{E}|_V), V_{in}, V_{out})$  such that

- $V$  is a dynamically closed subset of  $G$ ,
- $E|_V = \{(v_j, v_i) \in \mathcal{E}, \text{ such that } v_j \in V \text{ and } v_i \in V\}$ ,
- $V_{in} = V \cap G_{in}$ ,
- $V_{out} = V \cap G_{out}$ .

In accordance, from a conceptual point of view, with the geometric approach developed for linear systems in [9] and [10], we can introduce in our framework the notions of *invariance* and *controlled invariance* as follows.

**Definition 1** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , a subset  $V \subseteq G$  of vertices is said to be **invariant** for  $\Sigma$  if  $(v_j, v_i) \in \mathcal{E}$  with  $v_j \in V$  implies  $v_i \in V$ .

If  $V \subseteq G$  is invariant, then it is dynamically closed and  $\Sigma((V, \mathcal{E}|_V), V_{in}, V_{out})$  is a subsystem of  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ .

**Definition 2** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , a subset  $V \subseteq G$  of vertices is said to be **controlled invariant** for  $\Sigma$  if  $(v_j, v_i) \in \mathcal{E}$  with  $v_j \in V$  implies  $v_i \in V \cup G_{in}$ .

Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , we are interested in considering state feedbacks that modify the dynamics expressed by the underlying graph. Remarking that the action of a state feedback may only involve the relationship between any component of the state (or any agent) and the components (agents) whose dynamics is directly affected by the inputs, we fix our attention on the subset  $\mathcal{E}_{in} \subseteq \mathcal{E}$  defined by

$$\mathcal{E}_{in} = \{(v_j, v_i) \in \mathcal{E}, \text{ such that } v_i \in G_{in}\}$$

and we introduce the following definition.

**Definition 3** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , an **essential state feedback** consists of a subset  $\mathcal{F} \subset G \times G_{in}$ . The action of an essential state feedback  $\mathcal{F}$  on the structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  gives rise to the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$ , where  $\mathcal{E}^{\mathcal{F}} = (\mathcal{E} \cup \mathcal{F}) \setminus (\mathcal{E} \cap \mathcal{F}) = \{(v_j, v_i) \in \mathcal{E} \cup \mathcal{F} \text{ such that } (v_j, v_i) \notin \mathcal{E} \cap \mathcal{F}\}$ .

Note that applying an essential state feedback to  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  means to modify  $\mathcal{E}_{in}$ , either by adding new elements of the form  $(v_j, v_i)$  with  $v_j \in G$  and  $v_i \in G_{in}$  to it or by removing elements of the form  $(v_j, v_i)$  with  $v_i \in G_{in}$ , if present, from it. We can associate to the essential state feedback  $\mathcal{F}$  the relation  $u = Fx$ , where  $F = [f_{ij}]$  is an  $m \times n$  matrix whose entries are real, mutually independent parameters. To describe how to do this, let us first remark that if  $(v_j, v_i)$  belongs to  $\mathcal{E} \cap \mathcal{F}$ , then we have that  $v_i$  is, for some  $k$ , the  $k$ -element of  $G_{in}$ , i.e.  $v_i = v_{i_k} \in G_{in}$ , and, hence,  $b_{i_k} \neq 0$  in the matrix  $B$ . Now, let us take the parameters  $f_{ij}$  in such a way that they satisfy the following conditions

- $f_{ij} \neq 0$  if and only if  $(v_j, v_i) \in \mathcal{F}$
- $f_{ij} = -a_{ij}/b_{i_k}$  if  $(v_j, v_i) \in \mathcal{F}$  with  $v_i = v_{i_k} \in G_{in}$  and  $a_{i_k} \neq 0$ .

Note that no condition is imposed on  $f_{ij}$  if  $i = j$  and that only the first of the two conditions above applies if, for  $i \neq j$  and  $(v_j, v_i) \in \mathcal{F}$  with  $v_i = v_{i_k} \in G_{in}$ , one has  $a_{i_k} = 0$ .

With the above choice, the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$  turns out to be defined in parametric form by the following set of equations

$$\Sigma^{\mathcal{F}} \equiv \begin{cases} \dot{x}(t) &= (A + BF)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \quad (2)$$

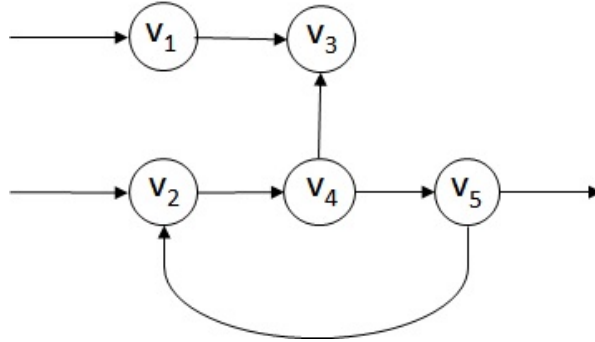
**Example 1** Let us consider the graph of Figure 1 and the associated structured system  $\Sigma = ((G, \mathcal{E}), G_{in}, G_{out})$ , which is described in parametric state space form by the following equations:

$$\Sigma \equiv \begin{cases} \dot{x}_1(t) &= a_{11}x_1 + b_{11}u_1 \\ \dot{x}_2(t) &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{22}u_2 \\ \dot{x}_3(t) &= a_{31}x_1 + a_{33}x_3 + a_{34}x_4 \\ \dot{x}_4(t) &= a_{42}x_2 + a_{44}x_4 \\ \dot{x}_5(t) &= a_{54}x_4 + a_{55}x_5 \\ y(t) &= c_{15}x_5(t) \end{cases} \quad (3)$$

A generic state feedback  $u = Fx$  is given by the equations

$$\begin{cases} u_1(t) = f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{14}x_4 + f_{15}x_5 \\ u_2(t) = f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 + f_{25}x_5 \end{cases} \quad (4)$$

where the terms  $f_{ij}$  are real, independent parameters. Applying such state feedback to  $\Sigma$ , the resulting compensated system turns out to be described by a triple  $((G, \mathcal{E}'), G_{in}, G_{out})$  that may differ from  $((G, \mathcal{E}), G_{in}, G_{out})$  only because edges have been added to or removed from  $\mathcal{E}$ . More precisely, edges are added if some of the parameters  $f_{12}, f_{13}, f_{14}, f_{15}, f_{24}, f_{25}$  is different from 0, while edges are removed if  $f_{21} = -a_{21}/b_{22}$  or  $f_{23} = -a_{23}/b_{22}$ . The values of the remaining parameters in (4) do not affect  $\mathcal{E}$ . Note that, for instance, if  $\mathcal{F}$  is the essential state feedback defined by  $\mathcal{F} = \{(v_1, v_2), (v_3, v_2), (v_5, v_2)\}$ , its action on  $\Sigma = ((G, \mathcal{E}), G_{in}, G_{out})$  produces the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$  that is described by the graph in Figure 2.



**Figure 2.** The structured compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$ .

We can now give in our framework a simple but fundamental characterization of controlled invariance that is analogous to the one given for the analogous concept in the classical linear framework (compare with [9], [10]).

**Proposition 1** *Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , a subset  $V \subseteq G$  of vertices is controlled invariant for  $\Sigma$  if and only if there exists an essential state feedback  $\mathcal{F}$  such that  $V$  is invariant for the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$ .*

**Hint of proof.** Assume that  $V \subseteq G$  is controlled invariant for  $\Sigma$  and take  $\mathcal{F} = \{(v_j, v_i) \in \mathcal{E}, \text{ such that } v_j \in V \text{ and } v_i \in G_{in}\}$ . We have that  $\mathcal{E}^{\mathcal{F}} = (\mathcal{E} \cup \mathcal{F}) \setminus (\mathcal{E} \cap \mathcal{F}) = \mathcal{E} \setminus \mathcal{F}$  and, therefore,  $(v_j, v_i) \in \mathcal{E}^{\mathcal{F}}$  with  $v_j \in V$  implies  $v_i \in V$ . Conversely, if  $V$  is invariant for  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$  for some feedback  $\mathcal{F}$ , we have that  $(v_j, v_i) \in \mathcal{E}$  with  $v_j \in V$  implies that either  $v_i \in V$  or  $(v_j, v_i) \in \mathcal{F}$ . Therefore,  $(v_j, v_i) \in \mathcal{E}$  with  $v_j \in V$  implies that either  $v_i \in V$  or  $v_i \in G_{in}$ .

The essential state feedback  $\mathcal{F}$  constructed in the proof of Proposition 1 is not the only one that makes  $V$  invariant in the compensated system. Let  $\mathcal{E}_1 \subseteq \mathcal{E}$  be defined by

$$\mathcal{E}_1 = \{(v_j, v_i) \in \mathcal{E}, \text{ such that } v_i \in G_{in} \text{ and } v_j \in V \text{ implies } v_i \in V \cap G_{in}\}.$$

Any feedback  $\mathcal{F}'$  that satisfies the condition

$$\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{E}_1 \quad (5)$$

makes  $V$  invariant in the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}'}), G_{in}, G_{out})$ . Any feedback  $\mathcal{F}$  that has the property of making  $V$  invariant in  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$  is called a *friend* of  $V$ .

**Proposition 2** *Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  and a subset  $K \subseteq G$ , there exists a maximal subset of vertices  $V$  such that  $V \subseteq K$  and  $V$  is controlled invariant for  $\Sigma$ . We denote such subset by  $V^*(\mathcal{E}, G_{in}, K)$  or simply by  $V^*$  if no confusion arises.*

Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  and a subset  $K \subseteq G$ , it is possible to construct  $V^*(\mathcal{E}, G_{in}, K)$  by considering the sequence of subset  $V_k \subseteq G$  defined recursively by

$$\begin{aligned} V_0 &= K \\ V_{k+1} &= V_k \setminus \{v_j, \text{ such that } (v_j, v_i) \in \mathcal{E} \text{ and } v_i \notin V_k \cup G_{in}\}. \end{aligned} \quad (6)$$

Clearly,  $V_k$  converges to  $V^*(\mathcal{E}, G_{in}, K)$  (which may be empty) in at most  $r$  steps, where  $r$  is the cardinality of  $K$ .

#### 4. Disturbance decoupling by state feedback

A structured system subject to a disturbance is a system  $\Sigma_D((G, \mathcal{E}), G_{in}, G_{out})$  in which  $G_{in}$  is partitioned as  $G_{in} = G_c \cup G_d$  (possibly with  $G_c \cap G_d = \emptyset$ ) and it is assumed that the disturbance inputs act directly on the dynamics of the variables (or agents) corresponding to the vertices in  $G_d$ , while the control input act directly on the dynamics of the variables (or agents) corresponding to the vertices in  $G_c$ . In that situation, we write  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$  and we can consider the following problem.

**Problem 1** *Given a disturbed structured system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$ , the **Disturbance Decoupling Problem by State Feedback (DDPSF)** consists in finding an essential state feedback  $\mathcal{F}$ , if any exists, such that in the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), G_{out})$  there is no path in  $(G, \mathcal{E}^{\mathcal{F}})$  with tail in  $G_d$  and head in  $G_{out}$  (this means that the output is not influenced by the disturbance).*

Applying the procedure that has been used to derive the system of equations (1), we get the following representation in parametric terms of the disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$

$$\Sigma_D \equiv \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) \\ y(t) &= Cx(t) \end{cases} \quad (7)$$

where, the matrices  $B = [b_{ij}]$  and  $D = [d_{ij}]$  have dimensions, respectively,  $n \times m_1$  and  $n \times m_2$ , with  $m_1 = \text{card}(G_c)$  and  $m_2 = \text{card}(G_d)$  and, letting  $G_c = \{v_{c_1}, \dots, v_{c_{m_1}}\}$  and  $G_d = \{v_{d_1}, \dots, v_{d_{m_2}}\}$ , their entries satisfies the following conditions

- $b_{ij} = 1$  (or more generally  $b_{ij} \neq 0$ ) if  $v_{c_j} = v_i \in$  (that is: if the  $j$ -th element of  $G_c$  is equal to  $v_i$ )
- $d_{ij} = 1$  (or more generally  $d_{ij} \neq 0$ ) if  $v_{d_j} = v_i \in$  (that is: if the  $j$ -th element of  $G_d$  is equal to  $v_i$ )
- $b_{ij} = 0$  and  $d_{ij} = 0$  otherwise

with control input  $u \in \mathbb{R}^{m_1}$  and disturbance input  $d \in \mathbb{R}^{m_2}$ .

Solvability of the DDPSF stated above means solvability of the same problem for all values of the parameters which appear in (7). We can say that solvability of the DDPSF for  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$  means **structural solvability** of the problem. The following theorem gives necessary and sufficient condition for the solvability of the DDPSF and it also indicates how to construct a feedback  $\mathcal{F}$  that solves it, if any exists.

**Theorem 1** *Given a disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$ , let  $K = G \setminus G_{out}$ . Then, the associated DDPSF is solvable if and only if the condition*

$$G_d \subseteq V^*(\mathcal{E}, G_c, K). \quad (8)$$

*is satisfied.*

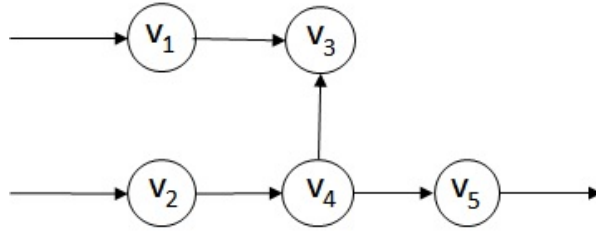


**Hint of proof** For sufficiency, let  $\mathcal{F}$  be a friend of  $V^*(\mathcal{E}, G_c, K)$ , so that  $V^*(\mathcal{E}, G_c, K)$  is invariant in  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), G_{out})$ . Condition (8) implies that any edge in  $\mathcal{E}^{\mathcal{F}}$  with tail in  $G_d$  has its head in  $V^*(\mathcal{E}, G_c, K)$  and, therefore, in  $K$ . This implies that there are no paths in  $(G, \mathcal{E}^{\mathcal{F}})$  with tail in  $G_d$  and head in  $G_{out}$ .

For necessity, let  $\mathcal{F}$  be a solution and consider the largest invariant  $V \subseteq K$  for the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), G_{out})$ . Clearly,  $G_d \subseteq V$  and  $V$  is controlled invariant for  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$ . The conclusion follows by maximality of  $V^*(\mathcal{E}, G_c, K)$

**Remark 2** Comparing the conditions of Theorem 1 with those stated in Theorem 7 of [2] for general structured systems, it appears clearly that the notions of controlled invariance and essential feedback introduced in graph theoretic terms in the previous section simplify the situation and allow a quite natural solution of the DDPSF.

**Example 2** Let us consider the structured disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$ , where  $(G, \mathcal{E})$  is the graph of Figure 1 and  $G_c = \{v_2\}$ ,  $G_d = \{v_1\}$ . We have that  $K = \{v_1, v_2, v_3, v_4\}$  and computation shows that  $V^*(\mathcal{E}, G_c, K) = \{v_1, v_3\}$ . Applying the essential state feedback  $\mathcal{F}$  defined by  $\mathcal{F} = \{(v_1, v_2), (v_3, v_2)\}$ , which is a friend of  $V^*$ , we get the compensated system  $\Sigma((V, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), G_{out})$  described by the graph of Figure 3, whose output is not influenced by the disturbance that acts on the dynamics of the component (agent) corresponding to  $v_1$ .



**Figure 3.** The compensated system  $\Sigma((V, \mathcal{E}^{\mathcal{F}}), G_c \cup G_d, G_{out})$ .

## 5. Disturbance decoupling by output feedback

Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , together with that of essential state feedback, we can consider a notion of essential output feedback. In order to state it formally, let us consider the subset  $\mathcal{E}_{oi} \subseteq \mathcal{E}$  defined by

$$\mathcal{E}_{oi} = \{(v_j, v_i) \in \mathcal{E}, \text{ such that } v_j \in G_{out} \text{ and } v_i \in G_{in}\}.$$

Then, we introduce the following definition.

**Definition 4** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , an **essential output feedback** consists of a subset  $\mathcal{F} \subset G_{out} \times G_{in}$ . The action of an essential output feedback  $\mathcal{F}$  on the structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  gives rise to the compensated structured system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), G_{in}, G_{out})$ , where  $\mathcal{E}^{\mathcal{F}} = (\mathcal{E} \cup \mathcal{F}) \setminus (\mathcal{E} \cap \mathcal{F}) = \{(v_j, v_i) \in \mathcal{E} \cup \mathcal{F} \text{ such that } (v_j, v_i) \notin \mathcal{E} \cap \mathcal{F}\}$ .

In the present framework, output feedbacks form a restricted class of state feedbacks. Note that applying an essential output feedback to  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  means to modify  $\mathcal{E}_{oi}$ , which is contained in  $\mathcal{E}_i$ , either by adding new elements of the form  $(v_j, v_i)$  with  $v_j \in G_{out}$  and  $v_i \in G_{in}$  to it or by removing elements of the form  $(v_j, v_i)$  with  $v_i \in G_{in}$ , if present, from it.

Now, given a disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), G_{out})$ , in which  $G_{out}$  is partitioned as

$G_{out} = G_{co} \cup G_{mo}$ , we can consider the problem of decoupling the disturbance from the (controlled) output variables corresponding to  $G_{co}$  by means of an output feedback that exploits the (measured) output variables corresponding to  $G_{mo}$ .

**Problem 2** Given a disturbed structured system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$ , the **Disturbance Decoupling Problem by Output Feedback (DDPOF)** consists in finding a essential output feedback  $\mathcal{F} \subseteq G_{co} \times G_c$ , if any exists, such that in the structured system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$  there is no path in  $(G, \mathcal{E}^{\mathcal{F}})$  with tail in  $G_d$  and head in  $G_{co}$  (this means that the controlled output is not influenced by the disturbance).

In order to provide solvability conditions for the DDPOF, it is convenient to introduce in our framework the notion of conditioned invariance. This is done as follows.

**Definition 5** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$ , a subset  $S \subseteq G$  of vertices is said to be **conditioned invariant** for  $\Sigma$  if  $(v_j, v_i) \in \mathcal{E}$  with  $v_j \in S \cap G \setminus G_{out}$  implies  $v_i \in S$ .

Conditioned invariant subsets have a number of structural properties that are relevant for our study.

**Proposition 3** Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  and a subset  $U \subseteq G$ , there exists a minimal subset of vertices  $S$  such that  $S \supseteq U$  and  $S$  is conditioned invariant for  $\Sigma$ . We denote such subset by  $S^*(\mathcal{E}, G_{out}, U)$  or simply by  $S^*$  if no confusion arises.

Given a structured system  $\Sigma((G, \mathcal{E}), G_{in}, G_{out})$  and a subset  $U \subseteq G$ , it is possible to construct  $S^*(\mathcal{E}, G_{out}, U)$  by considering the sequence of subset  $S_k \subseteq G$  defined recursively by

$$\begin{aligned} S_0 &= U \\ S_{k+1} &= S_k \cup \{v_i, \text{ such that } (v_j, v_i) \in E \text{ and } v_j \in S_k \cap G_{out}\}. \end{aligned} \quad (9)$$

Clearly,  $S_k$  converges to  $S^*(\mathcal{E}, G_{out}, K)$  (which may coincide with  $G$ ) in at most  $n - r$  steps, where  $r$  is the cardinality of  $U$ .

Now, we can state the following result.

**Theorem 2** Given a disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$ , let  $K = V \setminus G_{co}$ . Then, the associated DDPOF is solvable if and only if the condition

$$S^*(\mathcal{E}, G_{mo}, G_d) \subseteq V^*(\mathcal{E}, G_c, K). \quad (10)$$

is satisfied.

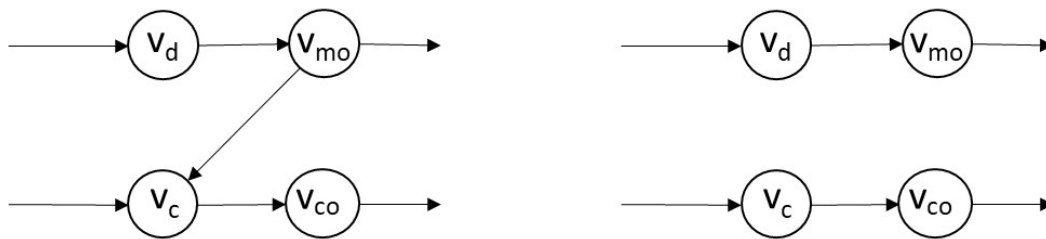
**Hint of proof** Condition (10) is equivalent to the fact that any path in  $(G, \mathcal{E})$  with tail in  $G_d$  and head in  $G_{co}$  contains at least one edge  $(v_j, v_i)$  with tail  $v_j \in G_{mo}$  and head  $v_i \in V_i$ . Then, the solutions of the problem are the essential output feedbacks that remove all edges of that kind.

**Remark 3** Also in this case, as in that of the DDPSF, comparing the conditions of Theorem 2 with those stated in Theorem 8 of [2] for general structured systems, it appears clearly that the notions of controlled invariance and essential feedback introduced in graph theoretic terms in the Section 3 simplify the situation and allow a quite natural solution of the DDPOF.

**Example 3** Consider the very simple example provided by the structured, disturbed system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$  described by the graph of Figure 4 on the left, where  $G_c = \{v_c\}$ ,  $G_d = \{v_d\}$ ,  $G_{co} = \{v_{co}\}$ ,  $G_{mo} = \{v_{mo}\}$  and the associated DDPOF. Clearly,

$$S^*(\mathcal{E}, G_{mo}, G_d) = G_d = \{v_d\} \subseteq \{v_d, v_{mo}\} = V^*(\mathcal{E}, G_c, K)$$

and condition (10) is satisfied. The essential output feedback  $\mathcal{F} = \{(v_{mo}, v_c)\}$  gives rise to the compensated system  $\Sigma((G, \mathcal{E}^{\mathcal{F}}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$ , whose underlying graph is illustrated in Figure 4 on the right, and it solves the DDPOF.



**Figure 4.** The structured system  $\Sigma((G, \mathcal{E}), (G_c \cup G_d), (G_{co} \cup G_{mo}))$  (on the left) and the compensated system  $\Sigma((G, \mathcal{E}^F), (G_c \cup G_d), (G_{co} \cup G_{mo}))$  (on the right).

## 6. Conclusions

A novel approach to the study of a class of structured systems has been developed introducing and using novel graph-theoretic notions of invariance, controlled invariance, conditioned invariance and essential feedback. It has been shown that this approach provide a natural interpretation of relevant properties of the dynamics and that it can be efficiently used to characterize, in a new simple way, solvability of classical disturbance decoupling problems. Future work will aim at exploiting this approach in other noninteracting control problems for structured systems and in developing specific applications to networked systems and systems of systems.

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