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A CATEGORIFICATION OF THE QUANTUM LEFSCHETZ PRINCIPLE

DAVID KERN

ABSTRACT. The quantum Lefschetz formula explains how virtual fundamental classes (or structure sheaves) of moduli stacks of stable maps behave when passing from an ambient target scheme to the zero locus of a section. It is only valid under special assumptions (genus 0, regularity of the section and convexity of the bundle). In this paper, we give a general statement at the geometric level removing these assumptions, using derived geometry. Through a study of the structure sheaves of derived zero loci we deduce a categorification of the formula in the ∞ -categories of quasi-coherent sheaves. We also prove that Manolache’s virtual pullbacks can be constructed as derived pullbacks, and use them to recover the classical Quantum Lefschetz formula when its hypotheses are satisfied.

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1. INTRODUCTION

1.1. The quantum Lefschetz hyperplane principle. Any quasi-smooth derived scheme is Zariski-locally presented as the (derived) zero locus of a section of a vector bundle on some smooth scheme. The Lefschetz hyperplane theorem then gives a way of understanding the cohomology of such a zero locus from the data

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of that of the ambient scheme and of the vector bundle. The quantum Lefschetz principle, similarly, gives the quantum cohomology, that is the Gromov–Witten theory, of the zero locus from that of the ambient scheme and the Euler class of the vector bundle.

Let X be a smooth projective variety and let E be a vector bundle on X , and consider the abelian cone stack $\mathbb{R}^0 p_* \text{ev}^* E$ on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, where $\text{ev}: \overline{\mathcal{C}}_{g,n}(X, \beta) \rightarrow X$ is the canonical evaluation map (corresponding by the isomorphism $\overline{\mathcal{C}}_{g,n}(X, \beta) \simeq \overline{\mathcal{M}}_{g,n+1}(X, \beta)$ to evaluation at the $(n+1)$ th marking) and $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the projection. Let s be a regular section of E and $i: Z \hookrightarrow X$ be its zero locus. An inspection of the moduli problems (see the proof of corollary 3.2.4) reveals that the disjoint union, over all classes $\gamma \in A_1 Z$ mapped by i_* to β , of the moduli stacks of stable maps to Z of degree γ coincides with the zero locus of the induced section $\mathbb{R}^0 p_* \text{ev}^* s$ of $\mathbb{R}^0 p_* \text{ev}^* E$. The natural question, leading to the quantum Lefschetz theorem, is whether this identification remains true at the “virtual” level, which was conjectured by Cox, Katz and Lee in [CKL01, Conjecture 1.1]. It was indeed proved in [KKP03] for Chow homology, and the statement was lifted in [Jos10] to G_0 -theory, that under assumptions on E the Gromov–Witten theory of Z is equivalent to that of X twisted by the Euler class of E , in that the following holds.

Theorem A ([KKP03; Jos10]). *For any $\gamma \in A_1 Z$ such that $i_* \gamma = \beta$, let $u_\gamma: \overline{\mathcal{M}}_{0,n}(Z, \gamma) \hookrightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$ denote the closed immersion. Suppose E is **convex**, that is $\mathbb{R}^1 p_*(C, \mu^* E) = 0$ for any stable map $\mu: C \rightarrow X$ from a rational (i.e. genus-0) stable curve $C \xrightarrow{p} S$ (so that the cone $\mathbb{R}^0 p_* \text{ev}^* E$ is a vector bundle). Then*

$$(1) \quad \sum_{i_* \gamma = \beta} u_{\gamma,*} [\overline{\mathcal{M}}_{0,n}(Z, \gamma)]^{\text{vir}} = [\overline{\mathcal{M}}_{0,n}(X, \beta)]^{\text{vir}} \smile c_{\text{top}}(\mathbb{R}^0 p_* \text{ev}^* E) \in A_\bullet(\overline{\mathcal{M}}_{0,n}(X, \beta)),$$

and

$$(2) \quad \sum_{i_* \gamma = \beta} u_{\gamma,*} [\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(Z, \gamma)}^{\text{vir}}] = [\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(X, \beta)}^{\text{vir}}] \otimes \lambda_{-1}(\mathbb{R}^0 p_* \text{ev}^* E) \in G_0(\overline{\mathcal{M}}_{0,n}(X, \beta)).$$

It was shown in [Coa+12] that the quantum Lefschetz principle as stated in (1) can be false when the vector bundle E is not convex (or as soon as g is greater than 0). The reason for this is that $\mathbb{R}^0 p_* \text{ev}^* E$ no longer equals $\mathbb{R} p_* \text{ev}^* E$ and the twisting Euler class should be corrected by taking into account the term $\mathbb{R}^1 p_* \text{ev}^* E$: in other words, one should use the full derived pushforward and view the induced cone as a *derived* vector bundle $\mathbb{R} p_* \text{ev}^* E$. This will require viewing our moduli stacks through the lens of derived geometry.

In this note, we use this philosophy to undertake the task of simultaneously relaxing the hypotheses on theorem A and lifting it to a categorified (and a geometric) statement, by which we mean that:

- we will give a formula at the level of a derived ∞ -category of quasicoherent sheaves,
- we will not need to fix the genus to 0,
- we will not need to assume that E is convex, or in fact a classical vector bundle (i.e. it can come from any object of the ∞ -category $\mathfrak{Pctf}(\mathcal{O}_X)$),
- we will not need to assume that the section is regular, as we can allow the target to be any derived scheme (or even a 1-algebraic derived stack) rather than a smooth scheme.

We note however that only the categorified form of the formula will hold in full generality, as the usual convexity (and genus) hypotheses are still needed to ensure bounded-coherence conditions so as to decategorify to G-theory.

1.2. Derived moduli stacks and virtual classes. In [MR18], the categorification of Gromov–Witten classes, as a lift from operators between G_0 -theory groups to dg-functors between dg-categories of quasicoherent (or coherent, or perfect) \mathcal{O} -modules, was achieved through the use of derived algebraic geometry. Indeed, this language allows one to interpret the homological corrections appearing in classical algebraic geometry as actual geometric objects; in particular the virtual structure sheaf $\left[\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(X,\beta)}^{\text{vir}}\right]$ was realised as the actual structure sheaf of a derived thickening $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ of the moduli stack, so that applying the $(\infty, 2)$ -functor $\Omega\mathcal{C}oh$ to the appropriate correspondences produces the desired lift of Gromov–Witten theory.

The idea of viewing the virtual fundamental class as a shadow of a higher structure sheaf was introduced in [Kon95], and made more precise first in [CK09] using the language of dg-schemes and in [Toë14, §3.1] *via* derived geometry. The derived moduli stack of stable maps $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ was constructed in [CK02] and [STV15]. Finally, [MR18] showed that the virtual structure sheaf really is given by the structure sheaf of the derived thickening, or rather its image by the isomorphism expressing that G-theory does not detect thickenings. Hence, in order to understand theorem A from a completely geometric point of view, the role of the virtual classes should indeed be played by derived moduli stacks.

We may now state the main result of this note, which addresses the question of similarly understanding the virtual statement of the quantum Lefschetz principle as a derived geometric phenomenon, and of deducing an expression for the “virtual structure sheaf” of $\coprod_{\gamma} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z,\gamma)$, understanding along the way the appearance of the Euler class of the bundle. In the remainder of this introduction, we shall write $\mathbb{R}u: \coprod_{\gamma} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z,\gamma) \hookrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ the canonical closed immersion (beware that $\mathbb{R}u$ is not a right derived functor, but simply a morphism of derived stacks which is a thickening of u).

Theorem B (Categorified quantum Lefschetz principle, see corollary 3.2.4 and proposition 2.2.2).

Let X be a derived scheme, $\mathcal{E} \in \mathfrak{Pct}^{\geq 0}(\mathcal{O}_X)$ a co-connective perfect module, and s a section of $\mathbb{V}_X(\mathcal{E})$ with zero locus $Z = X \times_{\mathbb{V}_X(\mathcal{E})}^{\mathbb{R}} X$. Write $\mathcal{I}: \mathbb{E}^{\vee} := (\mathbb{R}p_ \text{ev}^* \mathcal{E})^{\vee} \rightarrow \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ the cosection (of modules) corresponding to $\mathbb{R}p_* \text{ev}^* s$. There is an equivalence*

$$(3) \quad (\mathbb{R}u)_* \mathcal{O}_{\coprod_{\gamma} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z,\gamma)} \simeq \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \otimes \text{Sym}(\text{cofib}(\mathcal{I})) / (t-1) = \text{Sym}(\text{cofib}(\mathcal{I})) / (t-1)$$

in $\Omega\mathcal{C}oh(\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta))$, where $\text{cofib}(\mathcal{I})$ denotes the cofibre (or homotopy cokernel) of the linear morphism \mathcal{I} and where $\text{Sym}(\text{cofib}(\mathcal{I}))$ canonically admits an $\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}\{t\}$ -algebra structure.

We first notice that, in this categorified statement and unlike in the G-theoretic one, the Euler class of \mathbb{E}^{\vee} is refined to one taking into account the section s . Nonetheless this is indeed a categorification of theorem A, as we will explain in corollary 2.2.6 and subsection 4.3. When s is the zero section, meaning that \mathcal{I} is the zero morphism, then $\text{Sym}(\text{cofib}(\mathcal{I})) = \text{Sym}(\mathbb{E}^{\vee}[1]) \otimes \mathcal{O}_{\mathbb{A}^1}$, with $\text{Sym}(\mathbb{E}^{\vee}[1]) = \bigwedge^{\bullet}(\mathbb{E}^{\vee})$ so that in that case we do recover a categorified Euler class. In particular,

passing to the G_0 groups will indeed provide an identification of the cofibres of any and all sections, and hence give back eq. (2); this is corollary 4.3.3.

The theorem will in fact come as a corollary of a geometric statement, as a translation of the fact that Euler classes (also known, in the categorified setting, as Koszul complexes) represent zero loci of sections. Indeed, we will show that the moduli stack $\coprod_{\gamma} \overline{\mathcal{M}}_{g,n}(Z, \gamma) = \text{Spec}_{\overline{\mathcal{M}}_{g,n}(X, \beta)} \left((\mathbb{R}u)_* \mathcal{O}_{\coprod_{\gamma} \overline{\mathcal{M}}_{g,n}(Z, \gamma)} \right)$ satisfies the universal property of the zero locus of $\mathbb{R}\mathcal{P}_* \text{ev}^* s$, meaning that (per corollary 3.2.4, the *geometric quantum Lefschetz principle*) it features in the cartesian square

$$(4) \quad \begin{array}{ccc} \coprod_{i_* \gamma = \beta} \overline{\mathcal{M}}_{g,n}(Z, \gamma) & \xrightarrow{\mathbb{R}u_1} & \overline{\mathcal{M}}_{g,n}(X, \beta) \\ \mathbb{R}u_2 \downarrow & \lrcorner & \downarrow \mathbb{R}\mathcal{P}_* \text{ev}^* s \\ \overline{\mathcal{M}}_{g,n}(X, \beta) & \xrightarrow{0_E} & \mathbb{E}|_{\overline{\mathcal{M}}_{g,n}(X, \beta)} \end{array}$$

The formula eq. (3) for its relative function ring will then be a consequence of the general result proposition 2.2.2 describing zero loci of sections of vector bundles.

Remark C. While we have written this introduction with the assumption that the target X is a scheme for simplicity, the geometric and categorified quantum Lefschetz principles are not only valid for (derived) schematic targets, but also for orbifold Gromov–Witten theory, as foreshadowed by [Coa+12, Proposition 5.1]. In fact X and Z can be allowed to be derived algebraic stacks, and $\overline{\mathcal{M}}_{g,n}(X, \beta) \subset \mathbb{R}\text{Map}_{/\mathfrak{M}_{g,n}^{\text{tw}}}(\mathfrak{C}_{g,n}^{\text{tw}}, X \times \mathfrak{M}_{g,n}^{\text{tw}})$ (where $\mathfrak{C}_{g,n}^{\text{tw}} \rightarrow \mathfrak{M}_{g,n}^{\text{tw}}$ denotes the universal twisted curve) can be any open substack corresponding to a quasimap stability condition, as used for example in [CJW19] and detailed in [Ker21, §4.2.1.1].

The original proof of the quantum Lefschetz principle in [KKP03] also consisted of applying an excess intersection formula to a geometric (or homological) statement, here the fact that the embedding u satisfies the compatibility condition implying that Gysin pullback along it preserves the virtual class. The situation was shed light upon in [Man12], where it was shown that, using relative perfect obstruction theories (POTs), one can construct *virtual pullbacks*, which always preserve virtual classes. The embedding u being regular, its own cotangent complex can be used as a POT to construct a virtual pullback, which evidently coincides with the Gysin pullback.

Here we will show (in section 4) that, much in the same way as for the virtual classes, the virtual pullbacks may be understood as coming from derived geometric pullbacks of coherent sheaves, so that our statement for the embedding of derived moduli stacks does imply the quantum Lefschetz formula for the virtual classes (and in fact its standard proof), with the classical convexity hypotheses now appearing as necessary to make decategorification possible.

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1.4. Notations and conventions. We will use freely the language of $(\infty, 1)$ -categories (referred to as ∞ -categories), developed in a model-independent manner in [RV22], and of derived algebraic geometry, as developed for example in [TV08] and [Lur19]. The ∞ -category of ∞ -groupoids, also known as that of spaces in [Lur09], will be denoted $\infty\text{-}\mathcal{G}rpd$, and similarly the ∞ -category of ∞ -categories is $\infty\text{-}\mathcal{C}at$.

We work over a fixed field \mathbb{k} of characteristic 0; hence the ∞ -category of \mathbb{k} -module spectra can be modelled as the localisation of the category of \mathbb{k} -dg-modules along quasi-isomorphisms, in a way compatible with the monoidal structures so that connective $\mathbb{k}\text{-}\mathcal{E}_\infty$ -algebras are modelled by \mathbb{k} -cdgas concentrated in non-positive cohomological degrees. The ∞ -category of derived stacks on the big étale ∞ -site of \mathbb{k} will simply be denoted $\mathcal{D}\mathcal{S}t_{\mathbb{k}}$.

Remark D. The geometric and categorified part of our result, that is section 2 (except remark 2.2.4 and beyond) and section 3 are valid when \mathbb{k} is any \mathcal{E}_∞ -algebra over the sphere spectrum \mathbb{S} . However the formation of free spectral algebras does not have good finiteness properties, so in order to have bounded structure sheaves defining G-theory classes we do need to work over \mathbb{Q} where the free spectral algebras coincide (by [Lur19, Proposition 25.2.6.1]) with the polynomial construction.

We implicitly embed stacks into derived stacks; as such all construction are derived by default. In particular the symbol \times will refer to the (homotopical) fibre product of derived stacks; the truncated (*i.e.* strict, or underived) fibre product of classical stacks will be denoted \times^t , that is $X \times_Y^t Z = \tau_0(X \times_Y Z)$ for X, Y and Z classical.

We shall always use cohomological indexing. By a dg-category (over \mathbb{k}) we will mean a \mathbb{k} -linear stable ∞ -category. For any derived stack X , one defines its G_0 -theory group $G_0(X)$ as the zeroth homotopy group of the K-theory spectrum of the dg-category $\mathcal{C}oh^b(X)$.

2. ZERO LOCI OF SECTIONS OF DERIVED VECTOR BUNDLES

2.0. A spicilege of derived geometry for Gromov–Witten theory. The main import of derived geometry in Gromov–Witten theory is to make the homological objects which appear to correct defaults of smoothness more natural (and geometric) by incorporating them from the start as the basic blocks of the theory. Since the complexes intervening can only be considered up to quasi-isomorphism, this amounts in essence to replacing the category of \mathbb{k} -modules with the *derived* category of such as the place in which to define \mathbb{k} -algebras. Furthermore, in order to work properly with morphisms between derived \mathbb{k} -algebras, it is necessary to take the derived category not just as a homotopy category, but as a full ∞ -categorical localisation.

In this note, working over a base field \mathbb{k} containing \mathbb{Q} , we will take the view that the chain complexes of (classical) \mathbb{k} -modules, seen as objects of the derived ∞ -category, are nothing more than models presenting a more intrinsic notion of “derived” (sometimes also called “animated”) \mathbb{k} -modules¹. In other words, rather than constructing ∞ -categorical objects from classical ones, we will take the ∞ -categorical language as the more primitive one. As such, for any (possibly derived) \mathbb{k} -algebra A , we will simply call **A -modules** the objects of the derived $(\infty, 1)$ -category of A -modules, which in Gromov–Witten theory are usually rather seen as *complexes* of truncated A -modules. Our only exception to this terminology, for historical reasons as for example in [Lur17], will be for the following important example:

Example 2.0.1 (Cotangent complex). If A is a truncated \mathbb{k} -algebra and $A \rightarrow B$ is an A -algebra which is truncated as well, its cotangent complex $\mathbb{L}_{B/A}$ can be seen as enhancing the cotangent module $\Omega_{B/A}^1$ with homological corrections² carrying deformation-theoretic information; this is the role it plays in the construction of virtual classes. Returning now to the case where A is any general (*i.e.* derived) \mathbb{k} -algebra, the **cotangent complex** of an A -algebra $A \rightarrow B$, denoted $\mathbb{L}_{B/A}$, can be characterised as representing $(\infty$ -categorical) A -derivations from B , so now plays in higher algebra the exact same role that the cotangent module plays in classical algebra.

The ideas sketched above provide the notion of affine derived \mathbb{k} -schemes, as the objects of the opposite ∞ -category to that of derived \mathbb{k} -algebras. Since our interest is in enumerative geometry, we shall use the definition of derived \mathbb{k} -stacks as moduli problems, *i.e.* given by their ∞ -functors of points, ∞ -functors $\mathcal{A}ff_{\mathbb{k}}^{\text{op}} = \mathcal{A}lg_{\mathbb{k}} \rightarrow \infty\text{-}\mathcal{G}rpd$ satisfying descent conditions for the étale topology on $\mathcal{A}ff_{\mathbb{k}}$.

Example 2.0.2 (Quasicoherent modules). The assignment to $\text{Spec } A \in \mathcal{A}ff_{\mathbb{k}}^{\text{op}}$ of (the maximal ∞ -groupoid of) the ∞ -category $\mathcal{Q}coh(\text{Spec } A)$ of A -modules defines by [TV08, Theorem 1.3.7.2] a derived stack (of ∞ -categories) denoted $\mathcal{Q}coh$, whose groupoidal core is viewed as the classifying stack for quasicoherent modules. Then, for any derived stack X , the ∞ -category of quasicoherent sheaves on X is

$$(5) \quad \mathcal{Q}coh(X) = \text{hom}(X, \mathcal{Q}coh) \simeq \varprojlim_{\text{Spec } A \rightarrow X} \mathcal{Q}coh(\text{Spec } A).$$

That is, a quasicoherent \mathcal{O}_X -module \mathcal{M} is given by an A -module \mathcal{M}_x for every $x: \text{Spec } A \rightarrow X$, and base-change isomorphisms $f^* \mathcal{M}_{x'} \xrightarrow{\sim} \mathcal{M}_x$ for every morphism $f: \text{Spec } A \rightarrow \text{Spec } A'$ of X -schemes (along with higher compatibilities).

Every derived stack X has its **truncation** $\tau_0 X$, a classical (higher) stack obtained by restricting the functor of points X along the inclusion of truncated (or classical) algebras in all (derived) algebras. The truncation ∞ -functor $\tau_0: \mathcal{D}\mathcal{G}t_{\mathbb{k}} \rightarrow \mathcal{G}t_{\mathbb{k}}$ is right-adjoint to an ∞ -functor $\iota: \mathcal{G}t_{\mathbb{k}} \rightarrow \mathcal{D}\mathcal{G}t_{\mathbb{k}}$ which is fully faithful (providing an embedding of classical higher stacks into derived stacks, by viewing them as trivially derived) and will be omitted from notation. This is in keeping with our principle of implicitly embedding stacks into derived stacks; as such all construction

¹Defined more formally as modules over the Eilenberg–MacLane spectrum of \mathbb{k} in the ∞ -category of spectra.

²In practice, it can be constructed, as a left-derived functor of $\Omega_{-/A}^1$, by taking a semi-free resolution of B and applying $\Omega_{-/A}^1$ degreeewise.

are derived by default. In particular the symbol \times will refer to the fibre product of derived stacks (given on affines by the “derived” tensor product of algebras); the truncated (*i.e.* strict, or underived) fibre product of classical stacks will be denoted \times^ℓ , that is $X \times_Y^\ell Z = \ell_0(X \times_Y Z)$ for X, Y and Z classical.

The counit of the adjunction $i \dashv \ell_0$ will be denoted j ; its components $j_X: \ell_0 X \hookrightarrow X$ are closed immersions and will play an important role in the construction of virtual pullbacks in subsection 4.1.

In subsection 3.1, we will recall in more details the relevance of derived geometry, and in particular the role played by the cotangent complex, in Gromov–Witten theory.

2.1. Vector bundles in derived geometry.

Definition 2.1.1 (Total space of a quasicoherent module). Let X be a derived Artin \mathbb{k} -stack. For any quasicoherent \mathcal{O}_X -module \mathcal{M} , the linear derived stack $\mathbb{V}_X(\mathcal{M})$ is described by the ∞ -functor of points mapping an X -derived stack $\phi: T \rightarrow X$ to the ∞ -groupoid

$$(6) \quad \mathrm{hom}_{\mathrm{QCoh}(T)}(\mathcal{O}_T, \phi^* \mathcal{M}).$$

We call **abelian cone** over X any X -stack equivalent to the total space $\mathbb{V}_X(\mathcal{M})$ of a quasicoherent \mathcal{O}_X -module \mathcal{M} . We shall say that $\mathbb{V}_X(\mathcal{M})$ is a **perfect cone** if \mathcal{M} is perfect (equivalently, dualisable), and a **vector bundle** if \mathcal{M} is locally free of finite rank (as defined in [Lur19, Notation 2.9.3.1]).

Remark 2.1.2. If \mathcal{M} is a locally free \mathcal{O}_X -module, by [Lur19, Proposition 2.9.2.3] we may take a Zariski open cover $\coprod_i U_i \rightarrow X$ with $\mathcal{M}|_{U_i}$ free of rank r_i . We deduce from this (or from [Lur17, Remark 7.2.4.22] and [Lur19, Remark 2.9.1.2]) that any locally free module has Tor-amplitude concentrated in degree 0, and it will follow from proposition 2.1.10 that any vector bundle is smooth over its base.

Remark 2.1.3. If \mathcal{M} is dualisable, with dual \mathcal{M}^\vee , then as pullbacks commute with taking duals we have for any $\phi: T \rightarrow X$

$$(7) \quad \begin{aligned} \mathbb{V}_X(\mathcal{M})(\phi) &= \mathrm{hom}_{\mathrm{QCoh}(T)}(\phi^* \mathcal{M}^\vee, \mathcal{O}_T) \\ &= \mathrm{hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{M}^\vee), \phi_* \mathcal{O}_T) = \mathrm{Spec}_X^{\mathrm{nc}}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{M}^\vee))(\phi) \end{aligned}$$

where $\mathrm{Spec}_X^{\mathrm{nc}}$ denotes the non-connective relative spectrum ∞ -functor. Hence the restriction of \mathbb{V}_X to $\mathfrak{Perf}(\mathcal{O}_X)$ is naturally equivalent to the composite $\mathrm{Spec}_X^{\mathrm{nc}} \circ \mathrm{Sym}_{\mathcal{O}_X} \circ (-)^\vee$. In particular, if \mathcal{M} is a connective module then $\mathbb{V}_X(\mathcal{M})$ is a relatively coaffine stack, while if \mathcal{M} is co-connective, so that $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{M}^\vee)$ is a connective algebra, $\mathbb{V}_X(\mathcal{M})$ is an affine derived X -scheme.

Note however that the ∞ -functor $\mathrm{Spec}_X^{\mathrm{nc}}$ only becomes fully faithful when restricted to connective \mathcal{O}_X -algebras (as this restriction is equivalent to the Yoneda embedding thereof) but not when acting on general \mathcal{O}_X -algebras in degrees of arbitrary positivity (see for example [Mon21] for a counterexample, as well as details on the full-faithfulness of the ∞ -functor $\mathbb{V}_X(-^\vee)$).

Warning 2.1.4 (Terminology). Note that our convention for derived perfect cones is dual to that used in (among others) [Toë14] (and dating back to EGA2), which defines the total space of a quasicoherent \mathcal{O}_X -module \mathcal{M} as the X -stack whose sheaf of sections is \mathcal{M}^\vee , *i.e.* what we denote $\mathbb{V}_X(\mathcal{M}^\vee)$.

- Example 2.1.5.* i. If X is a classical Deligne–Mumford stack and \mathcal{M} is of perfect amplitude in $[-1, 0]$, the truncation $\mathcal{E}_0(\mathbb{V}_X(\mathcal{M}[1]^\vee))$ is the abelian cone Picard stack $\mathcal{H}^1/\mathcal{H}^0(\mathcal{M}^\vee)$ of [BF97, Proposition 2.4].
- ii. By [TV08, Proposition 1.4.1.6], $\mathbb{V}_X(\mathbb{T}_X) = \mathbb{T}X \simeq \mathbb{R}\mathrm{Map}(\mathbb{k}[\varepsilon], X)$ is the tangent bundle stack of X . More generally, using $\mathbb{k}[\varepsilon_n]$ where ε_n is of cohomological degree $-n$ (so of homotopical degree n) we have the shifted tangent bundle $\mathbb{T}[-n]X \simeq \mathbb{V}_X(\mathbb{T}_X[-n])$. Dually, one also defines the shifted cotangent stack $\mathbb{T}^\vee[n]X = \mathbb{V}_X(\mathbb{L}_X[n])$.

Lemma 2.1.6 ([TV07, Sub-lemma 3.9], [AG14, Theorem 5.2]³). *Suppose \mathcal{M} is of perfect Tor-amplitude contained in $[a, b]$ (where $a, b \in \mathbb{Z}$). Then the derived stack $\mathbb{V}_X(\mathcal{M})$ is $(-a)$ -geometric and strongly of finite presentation.*

Construction 2.1.7. For any derived stack X , the ∞ -functor \mathbb{V}_X gives a link between two functorial (in X) constructions. On the one hand we have the ∞ -functor $(-)_\text{ét}: \mathfrak{d}\mathfrak{S}\mathfrak{t}_{\mathbb{k}} \rightarrow \infty\text{-}\mathfrak{C}\mathfrak{a}\mathfrak{t}$ mapping a derived \mathbb{k} -stack X to its étale ∞ -topos $X_\text{ét}$ and a map of derived stacks $f: X \rightarrow Y$ to the direct image f_* of the induced geometric morphism, mapping a sheaf \mathcal{F} on $\mathfrak{d}\mathfrak{S}\mathfrak{t}_{\mathbb{k},/X}$ to the sheaf $f_*\mathcal{F}: (\mathcal{U} \rightarrow Y) \mapsto \mathcal{F}(\mathcal{U} \times_Y X \rightarrow X)$.

On the other hand, we have the ∞ -functor $\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(-)$ mapping a derived \mathbb{k} -stack X to the underlying ∞ -category of the dg-category $\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X)$, and a map $f: X \rightarrow Y$ to $\mathcal{M} \mapsto f_*\mathcal{M}$ (where the direct image sheaf is considered an \mathcal{O}_Y -module through $f^\sharp: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$). Then for any $\mathcal{M} \in \mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X)$, we obtain the functor of points of its total space, $\mathbb{V}_X(\mathcal{M})$, which is an étale sheaf on $\mathfrak{d}\mathfrak{S}\mathfrak{t}_{\mathbb{k},/X}$.

Lemma 2.1.8. *Let $\mathfrak{d}\mathfrak{S}\mathfrak{t}_{\mathbb{k}}^{(\text{f.coh.d.})}$ denote the wide and 2-full sub- ∞ -category whose 1-arrows are the morphisms of finite cohomological dimension (see [HP14, Definition A.1.4, Lemma A.1.6]). The ∞ -functors $\mathbb{V}_X: \mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X) \rightarrow X_\text{ét}$ assemble into a natural transformation $\mathbb{V}: \mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(-) \Rightarrow (-)_\text{ét}$ of ∞ -functors $\mathfrak{d}\mathfrak{S}\mathfrak{t}_{\mathbb{k}}^{(\text{f.coh.d.})} \rightarrow \infty\text{-}\mathfrak{C}\mathfrak{a}\mathfrak{t}$.*

Proof. We must construct, for any $f: X \rightarrow Y$ and any $\mathcal{M} \in \mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X)$, an equivalence $f_*(\mathbb{V}_X(\mathcal{M})) = \mathbb{V}_Y(f_*\mathcal{M})$. For any $\phi: \mathcal{U} \rightarrow Y$, the base change along f will take place in the cartesian square

$$(8) \quad \begin{array}{ccc} X \times_Y \mathcal{U} & \xrightarrow{X \times_Y \phi} & X \\ \downarrow f \times_Y \mathcal{U} & \lrcorner & \downarrow f \\ \mathcal{U} & \xrightarrow{\phi} & Y \end{array}$$

Then we have $\mathbb{V}_Y(f_*\mathcal{M})(\mathcal{U}) = \mathrm{hom}_{\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(\mathcal{U})}(\mathcal{O}_{\mathcal{U}}, \phi^*f_*\mathcal{M})$ while

$$(9) \quad \begin{aligned} f_*(\mathbb{V}_X(\mathcal{M}))(\mathcal{U}) &= \mathrm{hom}_{\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X \times_Y \mathcal{U})}(\mathcal{O}_{X \times_Y \mathcal{U}}, (X \times_Y \phi)^*\mathcal{M}) \\ &\simeq \mathrm{hom}_{\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(X \times_Y \mathcal{U})}((f \times_Y \mathcal{U})^*\mathcal{O}_{\mathcal{U}}, (X \times_Y \phi)^*\mathcal{M}) \\ &\simeq \mathrm{hom}_{\mathfrak{Q}\mathfrak{C}\mathfrak{o}\mathfrak{h}(\mathcal{U})}(\mathcal{O}_{\mathcal{U}}, (f \times_Y \mathcal{U})_*(X \times_Y \phi)^*\mathcal{M}). \end{aligned}$$

By the base-change property of [HP14, Proposition A.1.5 (3)] the two coincide.

³The grading convention used in [AG14] is homotopical, in opposition to our cohomological convention.

Since the isomorphisms appearing in eq. (9) and the base-change map are defined from adjunctions, they come equipped with functoriality property which furnish the higher naturality coherences. \square

Remark 2.1.9. By [Toë12, Theorem 2.1], if $f: X \rightarrow Y$ is quasi-smooth and proper then f_* sends perfect \mathcal{O}_X -modules to perfect \mathcal{O}_Y -modules.

Finally, we shall use the following well-known description of the cotangent complex of a perfect cone.

Proposition 2.1.10 ([AG14, Theorem 5.2]). *Let \mathcal{M} be a perfect \mathcal{O}_X -module, and write $\pi: \mathbb{V}_X(\mathcal{M}) \rightarrow X$ the structure morphism. Then $\mathbb{L}_{\pi}: \mathbb{V}_X(\mathcal{M})/X \simeq \pi^* \mathcal{M}^\vee$.*

Proof. The equivalence is established fibrewise in [Lur17, Proposition 7.4.3.14]. \square

2.2. Excess intersection formula. In this subsection, we work with a derived stack M and the closed embedding $u: T \hookrightarrow M$ of derived stacks defined as the zero locus of a section $s = \text{Spec}_M s^\sharp$ of a (relatively affine) perfect cone $\text{Spec}_M \text{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee)$ on M : we fix a co-connective (for the relative affineness) perfect \mathcal{O}_M -module \mathcal{F} and a morphism of \mathcal{O}_M -algebras $s^\sharp: \text{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee) \rightarrow \mathcal{O}_M$, corresponding (by the left-adjoint property of $\text{Sym}_{\mathcal{O}_M}$) to the cosection $\tilde{s}: \mathcal{F}^\vee \rightarrow \mathcal{O}_M$ of the module \mathcal{F}^\vee .

Remark 2.2.1 (Notation, derived versus spectral symmetric powers). In spectral algebraic geometry, over an \mathcal{E}_∞ -ring spectrum \mathcal{O} , the construction of polynomial algebras, usually denoted $\mathcal{O}[t_1, \dots, t_m]$, differs from that of free symmetric algebras, denoted $\mathcal{O}\{t_1, \dots, t_m\} = \text{Sym}_{\mathcal{O}}(\mathcal{O}^{\oplus m})$. Working as we do in characteristic zero, the difference between the two vanishes; however, as we wish to emphasise that the left-adjoint property of the symmetric algebra ∞ -functor is the one that matters for us, making the main result of this subsection valid over not just over our base \mathbb{k} but over a general ring spectrum, we shall use the spectral notation. In particular, the affine line over M is $\mathbb{A}_M^1 = \text{Spec}_M(\mathcal{O}_M\{t\}) = \text{Spec}_M \text{Sym}_{\mathcal{O}_M} \mathcal{O}_M^{\oplus 1}$.

Proposition 2.2.2. *The derived M -stack T may be recovered as the fibre*

$$(10) \quad T \simeq \mathbb{V}_M(\text{cofib}(\tilde{s})^\vee) \times_{\mathbb{A}_M^1} \{1\}_M$$

for a certain structure of stack over $\mathbb{A}_M^1 = \text{Spec}_M(\mathcal{O}_M\{t\})$ on $\mathbb{V}_M(\text{cofib}(\tilde{s})^\vee)$, that is T is the relative spectrum of the quotient \mathcal{O}_M -algebra

$$(11) \quad u_* \mathcal{O}_T = \text{Sym}_{\mathcal{O}_M}(\text{cofib}(\tilde{s})) / (t - 1),$$

where the structure map $e_{\mathcal{O}_M}: \mathcal{O}_M\{t\} \rightarrow \mathcal{O}_M$ is the quotient arrow $\mathcal{O}_M\{t\} \rightarrow \mathcal{O}_M\{t\}/(t-1) \simeq \mathcal{O}_M$ mapping t to 1 (i.e. corresponding to the identity morphism of \mathcal{O}_M -modules $\mathbb{1}_{\mathcal{O}_M}: \mathcal{O}_M \rightarrow \mathcal{O}_M$).

More generally, the monad $u_* u^*$ on $\mathfrak{QCoh}(M)$ identifies with tensoring by the algebra $\text{Sym}_{\mathcal{O}_M}(\text{cofib}(\tilde{s})) / (t - 1)$.

Proof. From the canonical fibre sequence $\mathcal{F}^\vee \xrightarrow{\tilde{s}} \mathcal{O}_M \rightarrow \text{cofib}(\tilde{s})$ we obtain, by application of the $(\infty, 1)$ -functor $\text{Sym}_{\mathcal{O}_M}$, an $\mathcal{O}_M\{t\}$ -algebra structure $\mathcal{O}_M\{t\} := \text{Sym}_{\mathcal{O}_M}(\mathcal{O}_M) \rightarrow \text{Sym}_{\mathcal{O}_M}(\text{cofib}(\tilde{s}))$. As $\text{Sym}_{\mathcal{O}_M}$ is a left-adjoint it preserves colimits (by [RV22, Theorem 2.4.2]) whence the latter term, image by $\text{Sym}_{\mathcal{O}_M}$ of the \mathcal{O}_M -module $\mathcal{O} \oplus_{\mathcal{F}^\vee} \mathcal{O}_M^{\oplus 1} =: \text{cofib}(\tilde{s})$, is the pushout of algebras (so by [Lur17, Proposition 3.2.4.7] the tensor product) $\mathcal{O}_M \otimes_{\text{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee)} \mathcal{O}_M\{t\}$.

By definition, the algebra $\mathrm{Sym}_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s})) \otimes_{\mathcal{O}_M\{t\}} \mathcal{O}_M$ under consideration fits in the left pushout square in the diagram

$$(12) \quad \begin{array}{ccccc} \mathrm{Sym}_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s}))/ (t-1) & \longleftarrow & \mathrm{Sym}_{\mathcal{O}_M}(\mathrm{cofib}(\tilde{s})) & \longleftarrow & \mathcal{O}_M \\ \uparrow & \lrcorner & \uparrow & \lrcorner & \uparrow \\ \mathcal{O}_M & \xleftarrow{\epsilon_{\mathcal{O}_M}} & \mathcal{O}_M\{t\} & \xleftarrow{\mathrm{Sym}(\tilde{s})} & \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee). \end{array}$$

From the previous discussion the right square is also cocartesian, so that the bigger diagram is also a pushout square. We now observe that the lower composite identifies with s^\sharp (since the map $\epsilon_{\mathcal{O}_M}$ is the counit of the adjunction $\mathrm{Sym}_{\mathcal{O}_M} \dashv \mathrm{fret}$), so that the big pushout square computes the function \mathcal{O}_M -algebra of the zero locus of s .

Finally, both u_* and u^* are left-adjoints, so by the homotopical Eilenberg–Watts theorem of [Hov15] (see also [GR17a, Chapter 4, Corollary 3.3.5]) their composite u_*u^* is equivalent to tensoring by $u_*u^*\mathcal{O}_M$. This can also be seen as a projection formula (proved for example in [Lur19, Remark 3.4.2.6]) between $u_*(u^* - \otimes_{\mathcal{O}_T})$ and $- \otimes u_*\mathcal{O}_T$, or indeed, more tautologically, as the definition of the direct and inverse image functors from the point of view of derived stacks as ringed ∞ -topoi (from which the further identification of the monad structures follows readily). \square

Remark 2.2.2.1 (Geometric interpretation). Let $\tilde{s}: \mathbb{A}_M^1 \rightarrow \mathbb{V}_M(\mathcal{F})$ be the linearisation of s , obtained as the image of \tilde{s} by \mathbb{V}_M . The zero locus of \tilde{s} is $\mathbb{A}_M^1|_T \cup \mathbb{A}_{M \setminus T}^0$, so taking the fibre at any non-zero element λ of \mathbb{A}_M^1 recovers $T \times \{\lambda\} \cup \emptyset \simeq T$.

Example 2.2.2.2 (Koszul complexes). Suppose \mathcal{F} is locally free. Then, passing to a Zariski open cover $\coprod U_i \rightarrow M$, we may assume as in remark 2.1.2 that $\mathcal{F}|_{U_i}$ is free of rank r_i . Write $\tilde{s}|_{U_i} = (s_\ell)_{1 \leq \ell \leq r}$ in coordinates. Then we recover the Koszul complex $\bigotimes_{\ell=1}^r \mathrm{cofib}(s_\ell)$, as studied for instance in [KR19, §2.3.1] or [Vez11].

Recall that the exterior algebra of the quasicoherent \mathcal{O}_M -module \mathcal{F}^\vee is $\bigwedge^\bullet \mathcal{F}^\vee := \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee[1]) = \bigoplus_{n \geq 0} (\bigwedge^n \mathcal{F}^\vee)[n]$.

Corollary 2.2.3 (Excess intersection formula). *For any quasicoherent \mathcal{O}_T -module \mathcal{M} that is the restriction (along u^*) of an \mathcal{O}_M -module, there is an equivalence*

$$(13) \quad u^*u_*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_T} \bigwedge^\bullet \mathcal{F}^\vee|_T.$$

Proof. The ∞ -functor u^* is a left-adjoint so it preserves colimits, among which in particular cofibres. By definition, we are given an equivalence $u^*\tilde{s} \simeq u^*0 = 0$, so the image by u^* of eq. (11) takes the form $\mathrm{Sym}(\mathrm{cofib} 0)/(t-1)$. By definition of the zero morphism, we may decompose this pushout as the composite of two amalgamated sums:

$$(14) \quad \begin{array}{ccccc} \mathcal{F}^\vee[1] \oplus \mathcal{O}_M & \longleftarrow & \mathcal{F}^\vee[1] & \xleftarrow{!} & 0 \\ \uparrow & \lrcorner & \uparrow & \lrcorner & \uparrow ! \\ \mathcal{O}_M & \xleftarrow{!} & 0 & \xleftarrow{!} & \mathcal{F}^\vee \\ & \searrow & \text{0} & \swarrow & \end{array}$$

so that $\mathrm{Sym}_{\mathcal{O}_M}(\mathrm{cofib}\, 0) = \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{F}^\vee[1] \oplus \mathcal{O}_M) = \mathrm{Sym}(\mathcal{F}[-1]^\vee) \otimes_{\mathcal{O}_M} \mathcal{O}_M\{t\}$. As u^* has a structure of monoidal ∞ -functor, this extends to any \mathcal{O}_T -module \mathcal{M} in the image of u^* .

Of course, this can also be obtained more directly from the fact that, when s is restricted to zero, the leftmost diagram below is the image by $\mathrm{Spec}_M \mathrm{Sym}_{\mathcal{O}_M}$ of the rightmost one:

$$(15) \quad \begin{array}{ccc} T & \xrightarrow{u} & M \\ u \downarrow & \lrcorner & \downarrow \mathcal{O}_{V_M(\mathcal{F})} \\ M & \xrightarrow{\mathcal{O}_{V_M(\mathcal{F})}} & \mathbb{V}_M(\mathcal{F}) \end{array} \quad \begin{array}{ccc} u_* \mathcal{O}_T & \longleftarrow & 0 \\ \uparrow & \lrcorner & \uparrow ! \\ 0 & \longleftarrow & \mathcal{F}^\vee \end{array} .$$

□

Remark 2.2.4 (Lie-theoretic interpretation). The excess intersection formula can also be seen as coming from the study of the \mathcal{L}_∞ -algebroid associated with the closed embedding u . Indeed, we are studying the geometry of a closed sub-derived stack $T \subset M$, which can be understood through that of its formal neighbourhood $\widehat{M}_T = M \times_{M_{\mathrm{dR}}} T_{\mathrm{dR}}$. This is a formally algebraic derived stack (see [CG18, section 4.1] or [GR17b, Chapter 1, Definition 7.1.2] for details) which is a formal thickening of T . By [GR17b, Chapter 5, Theorem 2.3.2], the ∞ -category of formal thickenings of T is equivalent to that of groupoid objects in formally algebraic derived stacks over T (via the ∞ -functor sending a thickening $T \rightarrow \mathcal{F}$ to its simplicial kernel, or Čech nerve), and following the philosophy of formal moduli problems it can be considered as a model for the ∞ -category of \mathcal{L}_∞ -algebroids.

We have the sequence of adjunctions $u^* \dashv u_* \dashv u^!$, implying that the comonad $u^* u_*$ is left-adjoint to the monad $u^! u_*$ (on $\mathrm{Ind}(\mathcal{Coh}^b(T))$, only $u^* u_*$ restricting to a comonad on $\mathcal{Coh}^b(T)$ when u is quasi-smooth by [GR17a, Chapter 4., Lemma 3.1.3]). Let us write $T \xrightarrow{\widehat{u}} \widehat{M}_T \xrightarrow{p} M$ the factorisation of u , so that $u^! u_* = \widehat{u}^! p^! p_* \widehat{u}_*$. Note that $p: M \times_{M_{\mathrm{dR}}} T_{\mathrm{dR}} \rightarrow M$ is the canonical projection, and as both T_{dR} and M_{dR} are étale over $\mathrm{Spec} \mathbb{k}$ it is also an étale morphism, and we recover $\widehat{u}^! \widehat{u}_*$. Following [GR17b, Chapter 8, 4.1.2], the monad $u^! u_*$ becomes the universal enveloping algebra of the \mathcal{L}_∞ -algebroid associated with u , endowed with the Poincaré–Birkhoff–Witt filtration. As the ∞ -functor of associated graded is conservative when restricted to (co)connective filtrations, we only need an expression for the associated graded of the PBW filtration. The result is then nothing but the PBW isomorphism of [GR17b, Chapter 9, Theorem 6.1.2] stating that for any regular embedding of derived stacks $u: T \hookrightarrow M$, the monad $\widehat{u}^! \widehat{u}_*$ on $\mathrm{Ind}(\mathcal{Coh}^b(T))$ is equivalent to tensoring by $\mathrm{Sym}_{\mathcal{O}_T}(\mathbb{T}_{\widehat{u}})$, and $\mathbb{T}_{\widehat{u}} = \mathbb{T}_u$ since p is étale. Passing back to the adjoint, we do obtain that $u^* u_*$ is equivalent to tensoring with $\mathrm{Sym}_{\mathcal{O}_T}(\mathbb{T}_u^\vee)$.

A similar equivalence between the Hopf comonad $u^* u_*$ and tensoring by the jet algebra (the dual of the universal enveloping algebra) of \mathbb{T}_u was established in [CCT14, Theorem 1.3] using the model of dg-Lie algebroids for \mathcal{L}_∞ -algebroids (see [CG18, Proposition 4.3, Theorem 4.11] for a precise statement of the equivalence between dg-Lie algebroids and formally algebraic derived stacks as models for \mathcal{L}_∞ -algebroids). However this approach does not provide the PBW theorem needed to identify the jet algebra of \mathbb{T}_u with $\mathrm{Sym}(\mathbb{L}_u)$.

Finally, it is easy to see from proposition 2.1.10 that the base-change property of cotangent complexes and the fibre sequence associated with the composition $\omega \circ s = 1$ imply $\mathbb{L}_u = u^* \mathbb{L}_s = u^* s^* \mathbb{L}_\omega[1] = u^* \mathcal{F}^\vee[1]$.

Then, conservativity of the restriction of $u^!$ to the ∞ -category $\text{Ind}(\mathcal{Coh}^b(M)_T) \simeq \text{Ind}(\mathcal{Coh}^b(\widehat{M}_T))$ of coherent sheaves with support (by [GR17a, Chapter 4, Proposition 6.1.3 (c)]) gives another reason for the equivalence $u_* \mathcal{O}_Z \simeq (\text{cofib}(\tilde{s})) / (t - 1)$.

Although it is not possible to directly relate s and the zero section at the geometric level and to obtain an expression of $u_* \mathcal{O}_T$ in terms of the Euler class of \mathcal{F}^\vee , passing to G-theory a homotopy between the maps they induce always does exist, and hence we recover the classical formulation of the quantum Lefschetz hyperplane formula.

Warning 2.2.5. When T is quasi-smooth, \mathcal{O}_T belongs to $\mathcal{Coh}^b(T)$ so by [GR17a, Chapter 4, Lemma 5.1.4] $u_* \mathcal{O}_T$ is in $\mathcal{Coh}^b(M)$ and thus defines a class in $G_0(M)$.

However this is no longer the case if T is not quasi-smooth (or, more generally, when the embedding $u: T \hookrightarrow M$ is not quasi-smooth even if T itself is); for example when $s = 0$, $\bigwedge^\bullet \mathcal{F}^\vee$ will fail to be bounded if \mathcal{F} does not have Tor-amplitude concentrated in degree 0.

We recall the notation of the G-theoretic Euler class of a locally free \mathcal{O}_M -module \mathcal{G} of finite rank: $\lambda_{-1}(\mathcal{G}) := [\bigwedge^\bullet \mathcal{G}] = \sum_{i \geq 0} [\bigwedge^i \mathcal{G}[i]] = \sum_i (-1)^i [\bigwedge^i \mathcal{G}] \in G_0(M)$.

Corollary 2.2.6 ([Kha21, Lemma 2.1]). *Suppose \mathcal{F} is a vector bundle. There is an equivalence of G-theory operators*

$$(16) \quad u_* u^* \simeq (-) \otimes \lambda_{-1}(\mathcal{F}^\vee): G(M) \rightarrow G(M).$$

Proof. We first note that, by definition, \mathcal{F} being locally free of finite rank means that it is (flat-locally) almost perfect, which makes it bounded, and flat, which makes it of Tor-amplitude concentrated in $[0]$ and implies that $\mathcal{F}^\vee[1]$ has Tor-amplitude in $[-1, 0]$ so that its symmetric algebra is still bounded and thus in $\mathcal{Coh}^b(M)$, defining an element of $G_0(M)$.

By [Kha21, Lemma 1.3], the fibre sequence $\mathcal{O}_M \rightarrow \text{cofib}(\tilde{s}) \rightarrow \mathcal{F}^\vee[1]$ implies that $[\text{Sym}_{\mathcal{O}_M}^n(\text{cofib}(\tilde{s}))] = \bigoplus_{i=0}^n [\text{Sym}^{n-i}(\mathcal{O}_M) \otimes \text{Sym}^i(\mathcal{F}^\vee[1])]$ for all $n \geq 0$. By the \mathbb{A}^1 -invariance of G-theory we may remove the symmetric algebra of \mathcal{O}_M , which gives the result. \square

3. THE GEOMETRIC LEFSCHETZ PRINCIPLE

3.1. Review of the derived moduli stack of stable maps. Let X be a target derived 1-Artin stack. We denote $\pi_{g,n}: \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$ — omitting mention of the twisted structure — the universal curve over the moduli stack of prestable stacky curves of genus g with n markings (and arbitrary orders of isotropy groups at the markings).

Remark 3.1.1. Note that we can allow X to be derived — although it is still required to be only 1-algebraic, as otherwise twisted curves will not be enough to ensure properness of the stack of stable maps to it — without any change to the usual theories of stable maps to X , as the moduli problem for prestable curves parametrises *flat* families, whose fibres over a derived stack must still be classical. More precisely, [Lur04, Theorem 8.1.3] shows (see also [PY20, Proposition 4.5] for a precise proof of the non-archimedean analogue) that the obvious extension of the moduli

problem for prestable curves to a derived moduli problem is representable by a classical DM stack $\mathfrak{M}_{g,n}$.

The ∞ -category of derived stacks (or any of its slices), as an ∞ -topos, is also cartesian closed, with internal hom denoted $\mathbb{R}\mathrm{Map}(-, -)$; the property of being right-adjoint to the cartesian product imposes that, as a functor of points, for any base B and B -stacks M and N , the B -stack $\mathbb{R}\mathrm{Map}_{/B}(M, N)$ be given by

$$(17) \quad \mathbb{R}\mathrm{Map}_{/B}(M, N): (T \rightarrow B) \mapsto \mathrm{hom}_{\mathfrak{d}\mathfrak{S}t_{/B}}(M \times_B T, N).$$

Proposition 3.1.2 ([MR18, (4.3.4)], [HP14, Proposition 5.1.10], [Lur19, Proposition 19.1.4.1 (2)]). *Let M be a base derived stack and $C \rightarrow M$ and $D \rightarrow M$ be two M -derived stacks. Then*

$$(18) \quad \mathbb{T}_{\mathbb{R}\mathrm{Map}_{/M}(C,D)/M} = \varpi_* \mathrm{ev}^* \mathbb{T}_{D/M}$$

where $\varpi: C \times_M \mathbb{R}\mathrm{Map}_{/M}(C, D) \rightarrow \mathbb{R}\mathrm{Map}_{/M}(C, D)$ is the projection and $\mathrm{ev}: \mathbb{R}\mathrm{Map}_{/M}(C, D) \rightarrow D$ is the evaluation map.

Remark 3.1.3. In the case of the open moduli substack of stable maps (recall that a Zariski-open immersion, like any étale map, has vanishing relative cotangent complex), we recognise in eq. (18) the formula defining the perfect obstruction theory used to define the virtual fundamental class in Gromov–Witten theory, cf [MR18, Proposition 4.3.1].

Corollary 3.1.4. *If X is smooth (resp. smooth with convex tangent bundle), then the derived stack $\mathbb{R}\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ is a quasi-smooth (resp. smooth). \square*

Remark 3.1.5. For any classical scheme $T \rightarrow \mathfrak{M}_{g,n}$, we can compute that

$$\begin{aligned} (19) \quad & (\mathfrak{t}_0 \mathbb{R}\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n}))(T \rightarrow \mathfrak{M}_{g,n}) \\ &= \mathrm{hom}_{\mathfrak{d}\mathfrak{S}t}(\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}} T, X) \\ &\simeq \mathrm{hom}_{\mathfrak{d}\mathfrak{S}t}(\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}}^{\mathfrak{t}} T, X) \\ &\simeq \mathrm{hom}_{\mathfrak{S}t}(\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}}^{\mathfrak{t}} T, \mathfrak{t}_0 X) \\ &= (\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, \mathfrak{t}_0 X \times \mathfrak{M}_{g,n}))(T \rightarrow \mathfrak{M}_{g,n}) \end{aligned}$$

where the first isomorphism is because $\mathfrak{C}_{g,n} \rightarrow \mathfrak{M}_{g,n}$ is flat and the second from the right-adjoint property of \mathfrak{t}_0 . This shows that $\mathbb{R}\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ is a derived thickening of the classical mapping stack $\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ (see also [TV08, Theorem 2.2.6.11, hypothesis (1)]).

We can view $\mathbb{R}\mathrm{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ as a (derived) moduli stack for families of *prestable* maps to X ; in particular, it is only Artin and not Deligne–Mumford. Since our reasoning for proving the quantum Lefschetz principle is purely formal, it will mainly work at the level of these general mapping stacks. However, we are interested in a more geometric subclass of maps, which only have finite automorphisms and define a Deligne–Mumford substack: this comes down to imposing a stability condition on the maps.

Since our result holds for stacky targets as well as schematic ones, we will use an adapted stability condition, inspired by the quasimap stability condition of [CCK15] for global quotient orbifolds, and laid out for this generality in some more detail in [Ker21, §4.2.1.1].

Construction 3.1.6 (Stability condition). Let $\mathcal{L} = \mathcal{L}_0 \otimes \varepsilon \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a line bundle with ε positive.

Say, following [Hei18] as a simplified version of the criterion of [Hal18], that a point x of X is \mathcal{L}_0 -stable (or equivalently, since $\varepsilon > 0$, \mathcal{L} -stable) if for any map $f: [\mathbb{A}^1/\mathbb{G}_m] \rightarrow X$ such that $f(0) \neq f(1)$, the weight of the \mathbb{G}_m -action on $f(0)^*\mathcal{L}_0$ (a quasicoherent sheaf on $\{0\} \simeq [*/\mathbb{G}_m] \subset [\mathbb{A}^1/\mathbb{G}_m]$, viewed as a \mathbb{G}_m -equivariant module) is negative. We will also require (as in [Hei18]) that the stable points have finite isotropy groups, so that the \mathcal{L} -stable locus defines a Deligne–Mumford substack $X^{\mathcal{L}\text{-st}} \subset X$.

If $(C; \Sigma_1, \dots, \Sigma_n)$ is an n -pointed stacky curve, a representable morphism $C \rightarrow X$ is said to be **pre- \mathcal{L} -quasistable** if it maps the generic point of any irreducible component of C to $X^{\mathcal{L}\text{-st}}$ (so that it has only a finite number of basepoints), and its basepoints are disjoint from the special points of C .

Since $X^{\mathcal{L}\text{-st}}$ is a DM stack, it makes sense to require in addition that the restriction of \mathcal{L}_0 be ample, and we do so. We can now further say that $f: C \rightarrow X$ is **\mathcal{L} -quasistable** if

- (1) letting e denote the least common multiple of the $\text{ord}(\text{Aut}(x))$ for x points of X and $|C|$ the coarse moduli space of C , we have that

$$(20) \quad \omega_{|C|} \left(\sum_{i=1}^n |\Sigma_i| \right) \otimes (f^*\mathcal{L}_0^e)^{\varepsilon/e}$$

is ample;

- (2) for any point p of C , $\varepsilon \cdot \text{lgth}_f(p) \leq 1$, where $\text{lgth}_f(p)$ is the order of contact of f with the unstable locus of X at p .

The quasistability condition for a map from a stable curve to X , that is a point of $\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$, is open, and thus defines, for any target class β on X , an open substack which we denote $\overline{\mathcal{M}}_{g,n}(X, \beta)$ — leaving the polarisation \mathcal{L} implicit since it will not play a role in the results of this paper.

Now, by [TV08, Corollary 2.2.2.10] the (small) Zariski ∞ -sites of a derived stack M and of its truncation $\tau_0 M$ are equivalent (and in particular 1-sites). It ensues as in [STV15, Proposition 2.1] that any open substack U of $\tau_0 M$ lifts uniquely to an open sub-derived stack $\mathbb{R}U \subset M$ such that $\mathbb{R}U \times_M \tau_0 M = U$ (so in particular $\mathbb{R}U$ is a derived thickening of U).

Definition 3.1.7. The **derived moduli stack** $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$ of genus- g , n -pointed **stable quasimaps** to X of class β is the open sub-derived stack of $\mathbb{R}\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ corresponding to the open substack $\overline{\mathcal{M}}_{g,n}(X, \beta) \subset \text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$.

By remark 3.1.3 and corollary 3.1.4, when X^{st} is smooth, $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$ enhances to a (quasi-smooth) derived geometric object the data of the moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and its perfect obstruction theory.

3.2. Identification of the derived moduli stacks. Let X be an Artin derived stack and $\mathcal{E} \in \mathfrak{P}erf(\mathcal{O}_X)$ a perfect \mathcal{O}_X -module, giving the perfect cone $E = \mathbb{V}_X(\mathcal{E})$. Let s be a section of E , and denote

$$(21) \quad Z = X \times_{s, E, 0} X \subset X$$

its (derived) zero locus.

For a fixed morphism $\pi: \mathfrak{C} \rightarrow \mathfrak{M}$ of derived \mathbb{k} -stacks *proper and of finite cohomological dimension*, we consider the universal map from a base-change of \mathfrak{C} over the derived mapping \mathfrak{M} -stack $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$:

$$(22) \quad \begin{array}{ccc} \mathfrak{C} \times_{\mathfrak{M}} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \xrightarrow{\text{ev}} & X \\ \rho \downarrow & & \\ \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & & \end{array} .$$

Let $\mathbb{E} := \rho_* \text{ev}^* E = \mathbb{V}_{\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\rho_* \text{ev}^* \mathcal{E})$ be the induced abelian (and perfect by remark 2.1.9 if ρ is quasi-smooth) cone over $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$, and $\sigma := \rho_* \text{ev}^* s$ its induced section. Write also $0_{\mathbb{E}}: \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \rightarrow \mathbb{E}$ for the zero section.

Theorem 3.2.1. *There is an equivalence of $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ -derived stacks*

$$(23) \quad \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) \simeq \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\sigma, \mathbb{E}, 0_{\mathbb{E}}} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}),$$

that is the diagram

$$(24) \quad \begin{array}{ccc} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) & \xrightarrow{u_1} & \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \\ u_2 \downarrow & \lrcorner & \downarrow \sigma = \rho_* \text{ev}^* s \\ \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \xrightarrow{0_{\mathbb{E}}} & \mathbb{E} = \rho_* \text{ev}^* E \end{array}$$

is cartesian.

The theorem will follow directly from some formal results.

We first note that, since we work in the ∞ -category $\mathfrak{D}\mathfrak{S}t_{\mathbb{k}}$, which as an ∞ -topos is cartesian closed, its the internal hom ∞ -functor $\mathbb{R}Map(-, -)$ is a right-adjoint to taking cartesian product, so by [RV22, Theorem 2.4.2] preserves limits.

Remark 3.2.2. This limit preservation property is also due to the more conceptual reason that, in enriched ∞ -categories, limits and colimits can be defined representably, cf. [Ker21, Proposition 1.1.2.2.8, Example 1.1.2.2.9].

Applying this to our case, we find that $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M})$ is equivalent to the fibre product

$$(25) \quad \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}),$$

with structure morphisms induced by s and the zero section of E . Hence, in order to prove theorem 3.2.1 we only need to identify the two derived stacks over which the fibre products are taken (as well as the two pairs of structure maps), the derived stack of maps to the abelian cone E and the induced cone $\mathbb{E} = \rho_* \text{ev}^* E$.

Proposition 3.2.3. *There is an equivalence of $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ -derived stacks*

$$(26) \quad \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M}) \simeq \mathbb{E}.$$

Proof. Let $a: S \rightarrow \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ be an $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ -stack, with corresponding family $C_a = a^* \mathfrak{C} = S \times_{\mathfrak{M}} \mathfrak{C} \rightarrow S$ (where we implicitly push the structure maps forward along $\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \rightarrow \mathfrak{M}$). Note that, as $\mathcal{P}: \mathfrak{C} \times_{\mathfrak{M}} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \rightarrow \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ is just projection onto the first factor, we have

$$(27) \quad \begin{aligned} \mathcal{P}^{-1}(a) &= S \times_{\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})} (\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\mathfrak{M}} \mathfrak{C}) \\ &= S \times_{\mathfrak{M}} \mathfrak{C} =: C_a, \end{aligned}$$

as seen in the cartesian diagram

$$(28) \quad \begin{array}{ccccc} C_a = S \times_{\mathfrak{M}} \mathfrak{C} & \xrightarrow{\tilde{a}} & \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\mathfrak{M}} \mathfrak{C} & \longrightarrow & \mathfrak{C} \\ \downarrow & \lrcorner & \downarrow \mathcal{P} & \lrcorner & \downarrow \\ S & \xrightarrow{a} & \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \longrightarrow & \mathfrak{M} \end{array}.$$

By lemma 2.1.8, as $\pi: \mathfrak{C} \rightarrow \mathfrak{M}$ was supposed of finite cohomological dimension and morphisms of finite cohomological dimension are stable by base-change, we have

$$(29) \quad \begin{aligned} \mathbb{E}(a) &= \mathbb{V}_{\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\mathcal{P}_* \text{ev}^* \mathcal{E})(a) \\ &= \mathcal{P}_* \mathbb{V}_{\mathfrak{C} \times_{\mathfrak{M}} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\text{ev}^* \mathcal{E})(a) \\ &= \mathbb{V}_{\mathfrak{C} \times_{\mathfrak{M}} \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(\text{ev}^* \mathcal{E})(C_a) \\ &= \text{hom}_{\mathfrak{P} \text{eff}(\mathcal{O}_{C_a})}(\mathcal{O}_{C_a}, \tilde{a}^* \text{ev}^* \mathcal{E}) = \text{hom}_{/X}(C_a, E), \end{aligned}$$

where $\text{ev} \circ \tilde{a}: C_a \rightarrow X$ is the map from a family of curves to X classified by a .

Meanwhile, we have by definition

$$(30) \quad \begin{aligned} &\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a) \\ &= \text{hom}_{\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})}(S, \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \\ &\simeq \text{hom}_{/\mathfrak{M}}(S, \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \times_{\text{hom}_{/\mathfrak{M}}(S, \mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}))} \{a\}. \end{aligned}$$

Indeed⁴, [Lur09, Lemma 5.5.5.12] shows that for any morphism $p: M' \rightarrow M$ in an ∞ -category and any cospan $S \rightarrow M' \leftarrow T$ over M' we have $\text{hom}_{/M'}(S, T) \simeq \text{hom}_{/M}(S, T) \times_{\text{hom}_{/M}(S, M')} \{p\}$; and we can compute

$$(31) \quad \begin{aligned} &\mathbb{R}Map_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a) \\ &\simeq \text{hom}_{/\mathfrak{M}}(S \times_{\mathfrak{M}} \mathfrak{C}, E \times \mathfrak{M}) \times_{\text{hom}_{/\mathfrak{M}}(S \times_{\mathfrak{M}} \mathfrak{C}, X \times \mathfrak{M})} \{a\} \\ &= \text{hom}_{/X \times \mathfrak{M}}(C_a, E \times \mathfrak{M}) = \text{hom}_{/X}(C_a, E). \end{aligned}$$

□

⁴This argument was suggested to the author by Benjamin Hennion

This completes the proof of theorem 3.2.1. \square

In our setting, we will have $\mathfrak{M} = \mathfrak{M}_{g,n}$, $\mathfrak{C} = \mathfrak{C}_{g,n}$, and the morphism of finite cohomological dimension $\pi: \mathfrak{C} \rightarrow \mathfrak{M}$ is $\pi_{g,n}$ the universal curve over the moduli stack of prestable stacky curves of genus g with n marked points.

We will also write $\rho_{g,n} = \rho$, $\text{ev}_{g,n} = \text{ev}$ and $\mathbb{E}_{g,n} = \mathbb{E} = (\rho_{g,n})_* \text{ev}_{g,n}^* \mathbb{E}$.

Corollary 3.2.4 (Geometric quantum Lefschetz principle). *Suppose X satisfies the conditions required for construction 3.1.6 (that is, X is 1-Artin and has a line bundle \mathcal{L}_0 whose stable locus is 1-Deligne–Mumford and quasiprojective with \mathcal{L}_0 as ample polarisation), and fix a class $\beta \in A_1 X$. There is an equivalence of $\mathbb{R}\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$ -derived stacks*

$$(32) \quad \coprod_{i_* \gamma = \beta} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z, \gamma) \simeq \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \times_{\mathbb{E}|\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)} \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta).$$

Proof. Note first that, as Zariski-open immersions are stable by pullbacks, both $\coprod_{i_* \gamma = \beta} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z, \gamma)$ and $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$ are open sub-derived stacks of $\mathbb{R}\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$, so by [STV15, Proposition 2.1] to show that they are equal it is enough to show that their truncations define identical substacks of $t_0 \mathbb{R}\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$.

As explained in remark 3.1.5, the truncation of such a derived mapping stack with (necessarily truncated) source flat over the truncated base is $\text{Map}_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, t_0(Z \times \mathfrak{M}_{g,n}))$, and similarly

$$(33) \quad t_0 \left(\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \right) = \overline{\mathcal{M}}_{g,n}(X, \beta) \times_{t_0 \mathbb{E}} \overline{\mathcal{M}}_{g,n}(X, \beta).$$

In addition, the truncation ∞ -functor commutes with colimits (see [TV08, proof of Lemma 2.2.4.1]) so

$$(34) \quad t_0 \left(\coprod_{i_* \gamma = \beta} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z, \gamma) \right) = \coprod_{i_* \gamma = \beta} \overline{\mathcal{M}}_{g,n}(t_0 Z, \gamma).$$

We now compare the two stacks (which are 1-algebraic, and thus 1-stacks, *i.e.* taking values in 1-groupoids) pointwise. For any $S \rightarrow \mathfrak{M}_{g,n}$ (with corresponding prestable genus- g curve $C_S \rightarrow S$), we have that $(\coprod_{i_* \gamma = \beta} \overline{\mathcal{M}}_{g,n}(t_0 Z, \gamma))(S) = \coprod_{i_* \gamma = \beta} \overline{\mathcal{M}}_{g,n}(t_0 Z, \gamma)(S)$ is tautologically the disjoint union (over $\gamma \in i_*^{-1}(\beta)$) of the groupoids of S -indexed families of stable maps from C_S to $t_0 Z$ of class γ , and

$$(35) \quad \left(\overline{\mathcal{M}}_{g,n}(X, \beta) \times_{t_0 \mathbb{E}} \overline{\mathcal{M}}_{g,n}(X, \beta) \right)(S) \simeq \overline{\mathcal{M}}_{g,n}(X, \beta)(S) \times_{t_0 \mathbb{E}|\overline{\mathcal{M}}_{g,n}(X, \beta)(S)} \overline{\mathcal{M}}_{g,n}(X, \beta)(S)$$

with $t_0 \mathbb{E}|\overline{\mathcal{M}}_{g,n}(X, \beta)(S) = \text{hom}(C_S, t_0 \mathbb{E})$. The latter $(2, 1)$ -fibre product (of groupoids) consists of pairs of stable maps f_1, f_2 from C_S to X equipped with an equivalence between their images $s \circ f_1$ and $0 \circ f_2$ in $t_0 \mathbb{E}$, so that the obvious functor

$$(36) \quad \coprod_{i_* \gamma = \beta} \overline{\mathcal{M}}_{g,n}(t_0 Z, \gamma)(S) \rightarrow \left(\overline{\mathcal{M}}_{g,n}(X, \beta) \times_{t_0 \mathbb{E}} \overline{\mathcal{M}}_{g,n}(X, \beta) \right)(S)$$

sending a stable map $f: C_S \rightarrow Z$ to $(i_{Z \hookrightarrow X} \circ f, i_{Z \hookrightarrow X} \circ f, \mathbb{1}_{C_S})$ is clearly an equivalence. \square

We may now apply proposition 2.2.2 to deduce a proof of theorem B. In the next section 4, we will see how we can recover from this and corollary 2.2.6 the classical (virtual) quantum Lefschetz formula.

Remark 3.2.5. Further evidence for this geometric form of the quantum Lefschetz principle can also be found by comparing the tangent complexes. Let us write temporarily $M(X)$ and $M(Z)$ for the moduli stacks of stable maps $\mathbb{R}\overline{\mathcal{M}}_{g,b}(X, \beta)$ and $\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathcal{M}}_{g,n}(Z, \gamma)$, and $M'(Z)$ for the zero locus $M(X) \times_{\mathbb{E}} M(X)$. The universal property of the latter stack induces a canonical morphism denoted $\Upsilon: M(Z) \rightarrow M'(Z)$ such that $u'_i \circ \Upsilon = u_i$ for $i = 1, 2$ where $u_i: M(Z) \hookrightarrow M(X)$ and $u'_i: M'(Z) \hookrightarrow M(X)$ are the canonical arrows (as in eq. (24)).

We know from proposition 3.1.2 that $\mathbb{T}_{M(X)/\mathfrak{M}_{g,n}} \simeq \mathcal{P}_{g,n,*} \text{ev}_{g,n}^* \mathbb{T}_X|_{M(X)}$. There is a fibre sequence $i_1^* \mathbb{L}_X \rightarrow \mathbb{L}_Z \rightarrow \mathbb{L}_{i_1}: Z/X$, and as Z sit by definition in a cartesian square we have that $\mathbb{L}_{i_1} = i_2^* \mathbb{L}_{X/\mathbb{E}} = \mathcal{E}^\vee[1]|_Z$ (where once again we have written $i_{1,2}: Z \hookrightarrow X$ the two canonical inclusions). As both pushforward and pullback preserve fibre sequences, we obtain finally that $\mathbb{T}_{M(Z)/\mathfrak{M}_{g,n}}$ is the fibre of the morphism $\mathcal{P}_{g,n,*} \text{ev}_{g,n}^* \mathbb{T}_X|_{M(Z)} \rightarrow \mathcal{P}_{g,n,*} \text{ev}_{g,n}^* \mathcal{E}|_{M(Z)}$.

Following the same logic, writing $M'(Z)$ for the zero locus, we see that $\mathbb{T}_{M'(Z)}$ is the fibre of $\mathbb{T}_{M(X)} = \mathcal{P}_{g,n,*} \text{ev}_{g,n}^* \mathbb{T}_X|_{M(X)} \rightarrow \mathcal{P}_{g,n,*} \text{ev}_{g,n}^* \mathcal{E}|_{M'(Z)}$. But we have seen that $\Upsilon^* \circ u'_i{}^* = u_i^*$ so it is clear that $\Upsilon^* \mathbb{T}_{M'(Z)} \simeq \mathbb{T}_{M(Z)}$.

As it is sufficient and necessary for a morphism of derived stacks to be an equivalence that it induce an isomorphism on the truncation and that its (co)tangent complex vanish, this is another way of proving theorem 3.2.1.

Example 3.2.6. Let $(X, f: X \rightarrow \mathbb{A}^1)$ be a Landau–Ginzburg model, from which we deduce the perfect cone $T^\vee X = \mathbb{V}_X(\mathbb{L}_X)$ and section $\text{d}_{\text{dR}} f$, whose zero locus is by definition the critical locus $\mathbb{R}\text{Crit}(f)$ (which is the intersection of two Lagrangians in a 0-shifted symplectic derived stack and thus carries a canonical (-1) -shifted symplectic form). Then the derived moduli stack of stable maps to $\mathbb{R}\text{Crit}(f)$ is the zero locus of the induced section of

$$(37) \quad \mathcal{P}_* \text{ev}^* T^\vee X = \mathbb{V}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)}(\mathcal{P}_*(\text{ev}^* \mathbb{L}_X))$$

But notice that

$$(38) \quad T^\vee \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta) \simeq \mathbb{V}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)}((\mathcal{P}_* \text{ev}^* \mathbb{T}_X)^\vee) \simeq \mathbb{V}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)}(\mathcal{P}_! \text{ev}^* \mathbb{L}_X)$$

where $\mathcal{P}_!: \mathcal{F} \mapsto \mathcal{P}_*(\mathcal{F}^\vee)^\vee \simeq \mathcal{P}_*(\mathcal{F} \otimes \omega_{\mathcal{P}})$ is the left adjoint to \mathcal{P}^* (by [Lur19, Proposition 6.4.5.3]), so $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{R}\text{Crit}(f), \beta)$ is not a critical locus and so cannot in general be expected to carry a (-1) -shifted symplectic structure if (g, n) differs from $(0, 1)$ or $(1, 0)$.

It is also possible to go the other way, that is to obtain a Landau–Ginzburg model from our general setting. If $\omega: E^\vee \rightarrow X$ is the dual of the perfect cone with section s , then the section $\omega^* s$ of $\omega^* E$ can be paired with the tautological section t of $\omega^* E^\vee$, defining a function $w_s = \langle s, t \rangle$ on the total space E^\vee . By [Isi12, Corollary 3.8], if X is smooth, there is an equivalence $\mathcal{Coh}^b(Z) \simeq \text{Sing}(\mathbb{R}\text{Zero}(w_s)/\mathbb{G}_m)$ with the \mathbb{G}_m -equivariant dg-category of singularities of $\mathbb{R}\text{Zero}(w_s)$ (where \mathbb{G}_m acts by rescaling on the fibres of E^\vee). However we only have $Z = \mathbb{R}\text{Crit}(w_s)$ if Z is smooth (see [CJW19, Lemma 2.2.2] in the regular and underived case).

4. FUNCTORIALITY IN INTERSECTION THEORY BY THE CATEGORIFICATION OF VIRTUAL PULLBACKS

We have obtained (as theorem B) a categorified form of the quantum Lefschetz principle, which in the cases where $\mathbb{E} = \rho_* \text{ev}^* E$ is a vector bundle we can by corollary 2.2.6 decategorify by passing to the G_0 -theory groups (or, more generally, the G -theory spectra) of the derived moduli stacks. To show that our statement is indeed a categorification of the quantum Lefschetz principle, it remains to compare it with the virtual statement, in the G -theory of the truncated moduli stacks. As explained in the introduction, this will be obtained through an appropriate construction of virtual pullbacks. These were defined in [Man12] (and in [Qu18] for G_0 -theory) from perfect obstruction theories. Following the understanding of virtual classes and the constructions of [MR18], we will give an alternative construction from derived thickenings. To ensure consistency, we show in subsection 4.2 that our construction coincides with that of [Qu18] when both are defined, and we use it in subsection 4.3 to get back the virtual form of the quantum Lefschetz formula.

Remark 4.0.1. The derived origin of virtual pullbacks was already considered in [Sch11, Section 7], where it is shown that any morphism of DM stacks which is the classical truncation of a morphism of derived DM stacks, with the induced obstruction theory, carries the compatibility necessary for the construction of a virtual pullback. However, the origin of the virtual classes and their precise relation to derived thickenings was still considered mysterious, and no direct construction of the virtual pullbacks from derived algebraic geometry was given.

4.1. Definition from derived geometry. Let $f: X \rightarrow Y$ be a quasi-smooth morphism of derived stacks, that is its cotangent complex $\mathbb{L}_f: X/Y$ is of perfect Tor-amplitude in $[-1, 0]$.

Remark 4.1.1. By [GR17a, Chapter 4, Lemma 3.1.3], as the quasi-smooth morphism f is of finite Tor-amplitude, the pullback of quasicohherent sheaves f^* maps $\mathcal{Coh}^b(Y)$ to $\mathcal{Coh}^b(X)$. As we work in G -theory, which is the K -theory of the stable ∞ -category of bounded coherent sheaves, the notation f^* will be understood in this section to mean the restriction of the pullback operation to coherent sheaves.

Recall that, due to the theorem of the heart (cf. [Bar15, Theorem 6.1]) and [Lur19, Corollary 2.5.9.2 with $n = 0$], the closed embedding $j_X: \mathcal{t}_0 X \hookrightarrow X$ induces an equivalence $j_{X,*}: G(\mathcal{t}_0 X) \xrightarrow{\sim} G(X)$ in G -theory, whose inverse at the level of G_0 -groups is given by $(\pi_0(j_{X,*}))^{-1}: G_0(X) \ni \mathcal{G} \mapsto \sum_{i \geq 0} (-1)^i [\pi_i(\mathcal{G})] \in G_0(\mathcal{t}_0 X)$.

It is therefore natural to define the virtual pullback along $\mathcal{t}_0 f$ to be given by the actual pullback along f , intertwined with these isomorphisms.

However we wish to consider the virtual pullback as a bivariate class, that is defined as a collection of maps $G(Y') \rightarrow G(X \times_Y Y') = G(X \times_Y^{\mathcal{t}} Y')$ indexed by all $\mathcal{t}_0 Y$ -schemes $Y' \rightarrow \mathcal{t}_0 Y$, or more generally by all derived Y -schemes $Y' \rightarrow Y$. Then the virtual pullback we defined should be the map corresponding to the $\mathcal{t}_0 Y$ -scheme $\mathbb{1}_{\mathcal{t}_0 Y}: \mathcal{t}_0 Y = \mathcal{t}_0 Y$.

We recall that we use the notation $\times^{\mathcal{t}}$ (a fibre product decorated by \mathcal{t}) to differentiate the “truncated” (1-2-categorical, for our moduli 1-stacks) fibre products of classical stacks from the implicitly ∞ -categorical fibre products of derived stacks.

Definition 4.1.2. The **bivariant virtual pullback** along f is the collection, indexed by all Y -schemes $\alpha: Y' \rightarrow Y$, of maps $(\epsilon_0 f)_{\text{DAG}}^{\alpha}: G(\epsilon_0 Y') \rightarrow G(\epsilon_0(Y' \times_Y X))$ defined as follows.

For a morphism of schemes $\alpha: Y' \rightarrow Y$, we have the diagram

$$(39) \quad \begin{array}{ccccc} \epsilon_0(Y' \times_Y X) \simeq \epsilon_0 Y' \times_{\epsilon_0 Y} \epsilon_0 X & \xrightarrow{j_{Y' \times_Y X}} & Y' \times_Y X & \xrightarrow{\tilde{f}} & Y' \\ & \searrow & \downarrow & \lrcorner & \downarrow \alpha \\ & & X & \xrightarrow{f} & Y \end{array}$$

Then we set $(\epsilon_0 f)_{\text{DAG}}^{\alpha} := (j_{Y' \times_Y X, *})^{-1} \circ \tilde{f}^* \circ j_{Y', *}$.

Lemma 4.1.3. *The virtual pullback only depends on $\epsilon_0 \alpha: \epsilon_0 Y' \rightarrow \epsilon_0 Y$. That is, for any $\alpha_1, \alpha_2: Y'_1, Y'_2 \rightarrow Y$ with $\epsilon_0 \alpha_1 = \epsilon_0 \alpha_2$, the virtual pullbacks $f_{\text{DAG}}^{\alpha_1}$ and $f_{\text{DAG}}^{\alpha_2}$ induced by α_1 and α_2 are equivalent.*

Proof. For any $\alpha: Y' \rightarrow Y$, we compare the virtual pullbacks induced by α and $\epsilon_0 Y' \xrightarrow{\epsilon_0 \alpha} \epsilon_0 Y \xrightarrow{j_Y} Y$.

$$(40) \quad \begin{array}{ccccc} \epsilon_0 Y' \times_{\epsilon_0 Y} \epsilon_0 X & \xrightarrow{\quad} & Y' \times_Y X & \xrightarrow{\tilde{f}} & Y' \\ & \searrow & \downarrow & \lrcorner & \downarrow \alpha \\ & & \epsilon_0(Y') \times_Y X & \xrightarrow{\tilde{f}} & \epsilon_0 Y' \\ & & \downarrow & \searrow j_Y \circ \epsilon_0 \alpha & \\ & & X & \xrightarrow{f} & Y \end{array}$$

The back square (the one exhibiting $\tilde{f} \circ i \simeq j_{Y'} \circ \hat{f}$) is cartesian and its side $j_{Y'}$ is a closed immersion and thus proper, so the base-change formula gives $\tilde{f}^* \circ j_{Y', *} = i_* \hat{f}^*$. Commutativity of the leftmost triangle implies that $i_* j_{\epsilon_0 Y' \times_Y X, *} = j_{Y' \times_Y X, *}$, and as both closed immersions involved induce isomorphisms in G -theory, we have $(j_{Y' \times_Y X, *})^{-1} i_* = (j_{\epsilon_0 Y' \times_Y X, *})^{-1}$. Putting the ingredients together, we finally obtain that

$$(41) \quad f_{\text{DAG}}^{\alpha} := (j_{Y' \times_Y X, *})^{-1} \tilde{f}^* j_{Y', *} = (j_{Y' \times_Y X, *})^{-1} i_* \hat{f}^* = (j_{\epsilon_0 Y' \times_Y X, *})^{-1} \hat{f}^* = f_{\text{DAG}}^{j_Y \circ \epsilon_0 \alpha}.$$

□

Remark 4.1.4 (Functoriality). The virtual pullbacks satisfy obvious functoriality properties. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable arrows, and let $\alpha: Z' \rightarrow Z$ be a Z -scheme. We have the commutative diagram

$$(42) \quad \begin{array}{ccccccc} \epsilon_0(Z' \times_Z X) & & \epsilon_0(Z' \times_Z X) & & & & \\ & \searrow j_{Z' \times_Z X} & & \searrow j_{Z' \times_Z Y} & & & \\ & & Z' \times_Z X & \xrightarrow{\tilde{f}} & Z' \times_Z Y & \xrightarrow{\tilde{g}} & Z' \\ & & \downarrow & & \downarrow b & & \downarrow \alpha \\ & & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

It follows by associativity of fibre products that

$$\begin{aligned}
 (43) \quad (\mathcal{t}_0 f)_{\text{DAG}}^{\dagger, b} \circ (\mathcal{t}_0 g)_{\text{DAG}}^{\dagger, a} &= (j_{(Z' \times_Z Y) \times_Y X, *})^{-1} \circ \tilde{f}^* \circ (j_{Z' \times_Z Y, *}) \circ (j_{Z' \times_Z Y, *})^{-1} \circ \tilde{g}^* \circ j_{Z', *} \\
 &= (j_{Z' \times_Z X, *})^{-1} \circ \tilde{g}^* \circ j_{Z', *} = (\mathcal{t}_0(gf))_{\text{DAG}}^{\dagger, a}.
 \end{aligned}$$

4.2. Comparison with the construction from obstruction theories.

Construction 4.2.1 (Virtual pullbacks from perfect obstruction theories). Let $g: V \rightarrow W$ be a morphism of Artin stacks of Deligne–Mumford type (*i.e.* relatively DM) endowed with a perfect obstruction theory $\varphi: E \rightarrow \mathbb{L}_g: V/W$, inducing the closed immersion $\varphi^\vee: \mathfrak{C}_g: V/W \hookrightarrow \mathfrak{E}$, where $\mathfrak{E} = \mathcal{t}_0(\mathbb{V}_V(E[1]^\vee))$ is the vector bundle (Picard) stack associated with E and \mathfrak{C}_g is the intrinsic normal cone of g (constructed in [BF97]). As in [MR18] we define a derived thickening $\mathbb{R}^\varphi V$ of V as the derived intersection

$$(44) \quad \begin{array}{ccc} \mathbb{R}^\varphi V & \xrightarrow{q} & \mathfrak{C}_g \\ p \downarrow & \lrcorner & \downarrow \varphi^\vee \\ V & \xrightarrow{0_{\mathfrak{E}}} & \mathfrak{E} \end{array}$$

Note that the arrow p is a retract of j_V , and provides a splitting of the induced perfect obstruction theory $j_V^* \mathbb{L}_{\mathbb{R}^\varphi V} \rightarrow \mathbb{L}_V$. We may use it to define a map of derived stacks $\mathbb{R}^\varphi g: \mathbb{R}^\varphi V \xrightarrow{p} V \xrightarrow{f} W$ which is a derived thickening of g .

We also recall the construction of the virtual pullback $g_\varphi^!$, or $g_{\text{POT}}^!$, from the perfect obstruction φ , defined in [Man12] for Chow homology then [Qu18] for G_0 -theory.

Let $\alpha: W' \rightarrow W$ and write $g': V' \rightarrow W'$ the base-change of g . Recall that one may define a deformation space (constructed in [KR19, Theorem 4.1.13] for quasi-smooth closed immersions of derived stacks, and extended by [Hek21, Proposition 7.6.2] to arbitrary closed immersions) $\mathfrak{D}_{V'} W'$ over $\mathbb{P}_{\mathbb{A}^1}^1$, with general fibre W' giving the open immersion $j: W' \times \mathbb{A}_{\mathbb{A}^1}^1 \hookrightarrow \mathfrak{D}_{V'} W'$, and special fibre $\mathfrak{C}_{g'}$ giving the complementary closed immersion $i: \mathfrak{C}_{g'} \times \{\infty\} \hookrightarrow \mathfrak{D}_{V'} W'$. It follows that there is an exact sequence of abelian groups $G_0(\mathfrak{C}_{g'}) \rightarrow G_0(\mathfrak{D}_{V'} W') \rightarrow G_0(W' \times \mathbb{A}^1) \rightarrow 0$ (coming from the fibred sequence of G -theory spectra). Furthermore, as (by excess intersection) $i^* i_*$ is equivalent to tensoring by the symmetric algebra on the conormal bundle of $\mathfrak{C}_{g'}$ in $\mathfrak{D}_{V'} W'$ and as the latter is trivial, we have $i^* i_* = 0$, inducing a map $G_0(W' \times \mathbb{A}^1) \rightarrow G_0(\mathfrak{C}_{g'})$: concretely, any section j^{*-1} of j^* gives the same map when post-composed with i^* so we do have a well-defined map $i^* j^{*-1}$. The specialisation map $\text{sp}: G_0(W') \rightarrow G_0(\mathfrak{C}_{g'})$ is then defined by precomposing it by $\text{pr}^*: G_0(W') \rightarrow G_0(W' \times \mathbb{A}^1)$. Finally, the cartesian square defining V' induces by [Man12, Proposition 2.26] a closed immersion $c: \mathfrak{C}_{g'} \hookrightarrow \alpha^* \mathfrak{C}_g = V' \times_V \mathfrak{C}_g$, and the virtual pullback $g_\varphi^!$ along g is constructed as the composite

$$(45) \quad g_\varphi^!: G_0(W') \xrightarrow{\text{sp}} G_0(\mathfrak{C}_{g'}) \xrightarrow{c_*} G_0(\alpha^* \mathfrak{C}_g) \xrightarrow{(\alpha^* \varphi^\vee)_*} G_0(\alpha^* \mathfrak{E} = V' \times_V \mathfrak{E}) \xrightarrow{0_{\alpha^* \mathfrak{E}}^*} G_0(V').$$

Lemma 4.2.2. *The virtual pullback $(\mathcal{t}_0 \mathbb{R}^\varphi g)_{\text{DAG}}^!$ as defined above for the map $\mathbb{R}^\varphi g$ coincides with the virtual pullback $g_\varphi^!$ of [Man12; Qu18]: for any $\alpha: W' \rightarrow W$, we have $(\mathcal{t}_0 \mathbb{R}^\varphi g)_{\text{DAG}}^{\dagger, a} = g_\varphi^{\dagger, a}: G_0(W') \rightarrow G_0(V')$.*

Note that this is essentially also proved as [Kha22, Proposition 6.8].

Proof. We adapt the results of [Jos10, Proposition 3.5] to the more general case of a morphism that need not be a regular embedding.

Let again $\alpha: W' \rightarrow W$ and write $g': V' \rightarrow W'$ the base-change of g . We now review our construction of the virtual pullback from derived thickenings from the point of view of the perfect obstruction theory. The map $g_{\text{DAG}}^! \alpha$ of definition 4.1.2 is computed in the following way: we define a derived thickening $\mathbb{R}^\varphi V'$ of $V' = V \times_W^\ell W'$ as $V' \times_{\mathfrak{e}} \mathfrak{C}_g$; note that we have $\mathbb{R}^\varphi V' = V' \times_V \mathbb{R}^\varphi V$ and writing $p': \mathbb{R}^\varphi V' \rightarrow V'$ we obtain a derived thickening $\mathbb{R}^\varphi g' = g' \circ p': \mathbb{R}^\varphi V' \rightarrow W'$ of g' . Then $g_{\text{DAG}}^! \alpha$ is the pullback along $\mathbb{R}^\varphi g'$ followed by the inverse of $j_{\mathbb{R}^\varphi V', *}$.

We also note that the fibred product $V' \times_{\alpha^* \mathfrak{e}} (\alpha^* \mathfrak{C}_g)$ is the base-change of $V \times_{\mathfrak{e}} \mathfrak{C}_g$ along $\alpha': V' \rightarrow V$, so the square

$$(46) \quad \begin{array}{ccc} \mathbb{R}^\varphi V' & \xrightarrow{q'} & \alpha^* \mathfrak{C}_g \\ p' \downarrow & \lrcorner & \downarrow \alpha^* \varphi^\vee \\ V' & \xrightarrow{0_{\alpha^* \mathfrak{e}}} & \alpha^* \mathfrak{C} \end{array}$$

is cartesian. As p' is proper, we have $0_{\alpha^* \mathfrak{e}}^* (\alpha^* \varphi^\vee)_* = p'_* q'^*$; concomitantly, as p' is a retract of $j_{\mathbb{R}^\varphi V', *}$ we have in G-theory $(j_{\mathbb{R}^\varphi V', *})^{-1} = p'_*$. We conclude that the virtual pullback of [Qu18] coincides with $(j_{\mathbb{R}^\varphi V', *})^{-1} \circ q'^* \circ c_* \circ \text{sp}$, and thus it only remains to check that the latter part specialises to $(\mathbb{R}^\varphi g')^* = p'^* \circ g'^*$. But the deformation space $\mathfrak{D}_{V', W'}$ provides exactly an interpolation between $g': V' \rightarrow W'$ and $V' \hookrightarrow \mathfrak{C}_{g'}$, so by transporting this comparison along the \mathbb{A}^1 -invariance of G-theory the lemma is proved. \square

Recall that for any quasi-smooth morphism $f: X \rightarrow Y$ of derived Artin stacks, by [STV15, Proposition 1.2] the canonical map $\varphi: j_X^* \mathbb{L}_f \rightarrow \mathbb{L}_{\mathcal{t}_0 f}$ is a perfect obstruction theory.

Proposition 4.2.3. *Let $f: X \rightarrow Y$ be a quasi-smooth relatively DM map of derived Artin stacks. The virtual pullback $(\mathcal{t}_0 f)_{\text{DAG}}^!$ defined with derived geometry is equal to $(\mathcal{t}_0 \mathbb{R}^\varphi \mathcal{t}_0 f)_{\text{DAG}}^!$, and thus to the virtual pullback $(\mathcal{t}_0 f)_\varphi^!$ of [Man12; Qu18], induced by the obstruction theory $\varphi: j_X^* \mathbb{L}_f \rightarrow \mathbb{L}_{\mathcal{t}_0 f}$.*

Proof. The proof is similar to the one sketched in [MR18, Proposition 4.3.2] for the comparison of the virtual classes defined from perfect obstruction theories and derived geometry, which mainly followed [LS12]: one constructs a deformation to the normal bundle of the closed immersion $j_X: \mathcal{t}_0 X \hookrightarrow X$, and finally uses that G-theory is \mathbb{A}^1 -invariant. Note that the main ingredient which was missing to make the proof of [MR18] precise, deformation to the normal cone for derived stacks, has now been constructed by [Hek21]. \square

We shall henceforth simply write $(\mathcal{t}_0 f)^!$ for the virtual pullback along f .

Example 4.2.4 (Virtual classes). Suppose $Y = \text{Spec}(\mathbb{k})$ so $f: X \rightarrow \text{Spec}(\mathbb{k})$ is the structure morphism. The virtual structure sheaf of $\mathcal{t}_0 X$ is $[\mathcal{O}_{\mathcal{t}_0 X}^{\text{vir}}] = f^!, \mathbb{L}_X([\mathcal{O}_{\text{Spec}(\mathbb{k})}]) = (j_{X,*})^{-1}([\mathcal{O}_X])$.

Example 4.2.5. Suppose that the classical map g is already a quasi-smooth immersion, so that $\mathbb{1}_{\mathbb{L}_g}$ is a perfect obstruction theory. Then the virtual pullback is given by the Gysin pullback $g^!$, studied in details for example in [Jos10].

Remark 4.2.6 (Virtual pullbacks in generalised motivic homology theories). Our construction of virtual pullbacks only relies on the fact that G-theory is insensitive to the non-reduced structure, and the identification with the classical definition requires simply the specialisation morphism and, more generally, the \mathbb{A}^1 -invariance. These ingredients are present in motivic homotopy theory (by construction for the \mathbb{A}^1 -invariance, and by [Kha19a, Corollary 3.2.9] for the insensitivity to derived structures), so the virtual pullbacks in motivic cohomology theories also admit the derived geometric interpretation.

In fact such virtual pullbacks were constructed for motivic Borel–Moore homology with coefficients in any étale motivic spectrum in [Kha19b, Construction 3.4] from the virtual pullbacks canonically associated with a quasi-smooth derived enhancement (through its derived deformation space).

4.3. Recovering the quantum Lefschetz formula.

Proposition 4.3.1. *With the notations of subsection 2.2, if \mathcal{F} is a vector bundle then $(\mathcal{t}_0 u)_* [\mathcal{O}_T^{\text{vir}}] = [\mathcal{O}_M^{\text{vir}}] \otimes \lambda_{-1}(\pi_0 \mathcal{F}^\vee)$ in $G(M)$.*

Proof. By naturality of the transformation j , we have $(\mathcal{t}_0 u)_* = (j_{M,*})^{-1} u_* j_{T,*}$ so that $(\mathcal{t}_0 u)_* (\mathcal{t}_0 u)^! = (j_{M,*})^{-1} u_* u^* j_{M,*} = (j_{M,*})^{-1} (j_{M,*}(-) \otimes \lambda_{-1}(\mathcal{F}^\vee))$ by corollary 2.2.6. Hence $(\mathcal{t}_0 u)_* [\mathcal{O}_T^{\text{vir}}] = (\mathcal{t}_0 u)_* (\mathcal{t}_0 u)^! [\mathcal{O}_M^{\text{vir}}] = (j_{M,*})^{-1} (\lambda_{-1}(\mathcal{F}^\vee))$.

By [Lur19, Corollary 25.2.3.3], as \mathcal{F}^\vee is flat over \mathcal{O}_M so are its exterior powers $\bigwedge^n(\mathcal{F}^\vee)$. In particular, by [TV08, Proposition 2.2.2.5. (4)] they are strong \mathcal{O}_M -modules, meaning that $\pi_i(\bigwedge^n \mathcal{F}^\vee) \simeq \pi_i(\mathcal{O}_M) \otimes_{\pi_0(\mathcal{O}_M)} \pi_0(\bigwedge^n \mathcal{F}^\vee)$ for all natural integers i and n , and we conclude that

$$\begin{aligned}
 (\mathcal{t}_0 u)_* [\mathcal{O}_T^{\text{vir}}] &= \sum_{i \geq 0} (-1)^i \sum_{n \geq 0} (-1)^n \left[\pi_i \left(\bigwedge^n \mathcal{F}^\vee \right) \right] \\
 (47) \qquad &= \sum_{i \geq 0} (-1)^i [\pi_i(\mathcal{O}_M)] \otimes \sum_{n \geq 0} (-1)^n \left[\bigwedge^n \pi_0(\mathcal{F}^\vee) \right]
 \end{aligned}$$

as required. \square

Remark 4.3.2. In the setting of the quantum Lefschetz principle, the only cases in which $\mathbb{E}_{g,n}$ is a vector bundle are when E is convex, that is $\mathbb{R}^1 p_* f^* \mathcal{E} = 0$ for any stable map $(p: C \rightarrow S, f: C \rightarrow X)$ from a rational curve C , and thus the genus is $g = 0$, which is the setting in which the quantum Lefschetz principle is already known. We conclude that it is not possible to relax the hypotheses for the quantum Lefschetz principle in G-theory, and that the more general version is thus only valid in its categorified form.

One may also notice that as the cotangent complex of u is $\mathcal{L}_* \text{ev}^* \mathcal{E}^\vee[1]$, which has Tor-amplitude in $[-2, 0]$ (in fact $[-2, -1]$) unless the above conditions are satisfied, so that u is not quasi-smooth and the virtual pullback along it cannot be defined.

Corollary 4.3.3. *If $\mathbb{E}_{0,n} = \rho_{0,n;*} \mathrm{ev}_{0,n}^* E$ is a vector bundle (that is if E is convex), the G-theoretic quantum Lefschetz formula of theorem A holds:*

$$(48) \quad (\epsilon_0 u)_* \sum_{i_* \gamma = \beta} \left[\mathcal{O}_{\mathcal{M}_{0,n}(Z, \gamma)}^{\mathrm{vir}} \right] = \left[\mathcal{O}_{\mathcal{M}_{0,n}(X, \beta)}^{\mathrm{vir}} \right] \otimes \lambda_{-1}(\pi_0 \rho_{0,n;*} \mathrm{ev}_{0,n}^* \mathcal{E}^\vee).$$

□

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