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VOLUME OF REPRESENTATIONS AND MAPPING DEGREE

PIERRE DERBEZ, YI LIU, HONGBIN SUN, AND SHICHENG WANG

ABSTRACT. Given a connected real Lie group and a contractible homogeneous proper G -space X furnished with a G -invariant volume form, a real valued volume can be assigned to any representation $\rho: \pi_1(M) \rightarrow G$ for any oriented closed smooth manifold M of the same dimension as X . Suppose that G contains a closed and cocompact semisimple subgroup, it is shown in this paper that the set of volumes is finite for any given M . From a perspective of model geometries, examples are investigated and applications with mapping degrees are discussed.

1. INTRODUCTION

Let G be a connected real Lie group and H be any maximal compact subgroup of G . After fixing a G -invariant volume form on the homogeneous G -space $X = G/H$, a volume $\text{vol}_G(M, \rho) \in \mathbb{R}$ can be assigned to any oriented closed smooth manifold M of the same dimension as X with respect to a representation $\rho: \pi_1(M) \rightarrow G$, [24]. The value

$$V(M, G) = \sup_{\rho} |\text{vol}_G(M, \rho)|$$

in $[0, +\infty]$ is called the G -representation volume of M . This invariant has been introduced and studied by R. Brooks and W. Goldman [6, 7, 24] as a geometrical analogue of the celebrated simplicial volume of orientable closed manifolds due to M. Gromov [25, 42]. During the past few years, much has been known about the $(\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ -representation volume (the Seifert volume) and the $\text{PSL}(2, \mathbb{C})$ -representation volume (the hyperbolic volume) for 3-manifolds and their finite covers [18, 19, 16, 20, 17]. Those invariants have demonstrated to be useful in studying nonzero degree maps between 3-manifolds, especially when the simplicial volume vanishes. Now it seems a right time to consider higher dimensions, and to give a comprehensive treatment of the theory under reasonably general hypotheses.

In this paper, we are primarily interested in the representation volume for Lie groups that contain closed and cocompact semisimple subgroups. We formulate all the results in the smooth category to avoid technicalities. Our main conclusion is that in this case, the representation volume is a finite and nontrivial invariant for closed orientable smooth manifolds of the applicable dimension:

Theorem 1.1. *Suppose that G is a connected real Lie group which contains a closed cocompact connected semisimple subgroup. Let $X = G/H$ be a homogeneous space*

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furnished with a G -invariant volume form, where H is a maximal compact subgroup of G . Then for any oriented closed smooth manifold M of the same dimension as X , the volume function

$$\text{vol}_G: \mathcal{R}(\pi_1(M), G) \rightarrow \mathbb{R}$$

takes only finitely many values on the space of representations $\mathcal{R}(\pi_1(M), G)$. Moreover, there exists some closed aspherical manifolds M for which vol_G is not constantly zero.

We restrict ourselves to closed manifolds M so as to speak of volume of representations for any connected real Lie group G , (see Section 2). For certain noncompact manifolds and more restricted Lie groups, some authors have established rigidity or flexibility results for generalized volume of representations. For example, when M is a complete hyperbolic n -manifold of finite volume and G is $\text{SO}^+(n, 1)$, the same conclusion as in Theorem 1.1 holds in a general sense for $n \geq 4$, but fails for $n = 2, 3$ in the cusped case, see [10, 36].

On the other hand, the assumptions on G in Theorem 1.1 are quite natural from the perspective of maximal geometries with compact models. According to W. Thurston [42], a *geometry* of dimension n is a pair (X, G) where X is a simply-connected smooth manifold of dimension n and G is a real Lie group that acts transitively and effectively on X by diffeomorphisms with compact isotropy at every point. It is a *maximal geometry* if G cannot be extended to any larger groups, and it is a (compact) *model geometry* if there is a closed n -manifold locally modeled on (X, G) . Two geometries are considered to be equivalent if there is an equivariant diffeomorphism between the spaces with respect to the structure groups. For classifications of geometries of dimension at most 5, we refer the reader to [41, 29, 23]. Furnishing X with a G -invariant volume form ω_X , we can speak of the representation volume of closed orientable smooth n -manifolds with respect to the identity component G° of G .

Corollary 1.2. *Let (X, G') be an n -dimensional geometry where X is contractible and G' is semisimple. For any geometry (X, G) that extends (X, G') and any G -invariant volume form ω_X , the G -representation volume is a homotopy-type invariant of closed orientable smooth n -manifolds, which takes values in $[0, +\infty)$ and does not always vanish.*

The hyperbolic plane $(\mathbb{H}^2, \text{Iso}(\mathbb{H}^2))$ is a well-known nontrivial example in dimension 2. With respect to the hyperbolic metric, it yields the $\text{PSL}(2, \mathbb{R})$ -representation volume which equals $-2\pi\chi(\Sigma)$ for any orientable closed surface Σ of genus at least 2, or 0 otherwise. Higher dimensional hyperbolic spaces $(\mathbb{H}^n, \text{Iso}(\mathbb{H}^n))$ gives rise to the $\text{SO}^+(n, 1)$ -representation volume. It recovers the usual hyperbolic volume for orientable closed hyperbolic n -manifolds, whereas for other n -manifolds the volume may be trivial or not. Other symmetric spaces of noncompact types provide geometric models that yield the representation volume for semisimple Lie groups without compact normal subgroups. In dimension 3, the $(\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ -representation volume arises from the Seifert-space geometry $(\widetilde{\text{SL}}_2(\mathbb{R}), \text{Iso}(\widetilde{\text{SL}}_2(\mathbb{R})))$. It can be greater than the $\widetilde{\text{SL}}_2(\mathbb{R})$ -representation volume in general.

In dimension 5, there is another interesting family which are non-symmetric spaces, namely, the maximal model geometries $(\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\alpha} \widetilde{\text{SL}}_2(\mathbb{R}), \text{Iso}(\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\alpha} \widetilde{\text{SL}}_2(\mathbb{R})))$ for $\alpha \in \mathbb{Q}^*$. The identity component of the structure group is isomorphic to

$$(\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) \times (\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}) / \{(t, \alpha t) \in \mathbb{R} \times \mathbb{R} : t \in \mathbb{R}\},$$

which is 7 dimensional. If one regards the Seifert-space geometry $\widetilde{\text{SL}}_2(\mathbb{R})$ as an affine real line bundle over the hyperbolic plane via the fibration $\text{SO}(2) \rightarrow \widetilde{\text{SL}}_2(\mathbb{R}) \rightarrow \mathbb{H}^2$, the

contractible homogeneous space $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\alpha} \widetilde{\mathrm{SL}}_2(\mathbb{R})$ can be observed as an affine real line bundle over $\mathbb{H}^2 \times \mathbb{H}^2$, where the fibers are modelled on the quotient of the affine real plane $\widetilde{\mathrm{SO}}(2) \times \widetilde{\mathrm{SO}}(2)$ by translations parallel to the $(1, \alpha)$ direction. When $\alpha \in \mathbb{R}$ is irrational, one could also define 5-dimensional geometries with the identity component constructed by the same expression. The corresponding maximal analytic semisimple subgroup is no longer closed. Nevertheless, we do not know if the irrational cases admit compact models anyways, so there is a chance that they only yield vanishing representation volumes.

As a classical application, representation volumes can be used to study mapping degrees. We say that a closed orientable smooth n -manifold N has *Property D* if for all closed orientable smooth manifold M of the same dimension, the set of mapping degrees

$$\mathcal{D}(M, N) = \{|\deg(f)| : f \in C^{\infty}(M, N)\}$$

is always finite. In dimension 2, manifolds with Property D are those of the geometry \mathbb{H}^2 , by considering the simplicial volume. In dimension 3, manifolds with Property D are those which contain non-geometric prime factors, or prime factors of the geometry \mathbb{H}^3 or $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, by [17, Corollary 1.6].

By Corollary 1.2 and the domination inequality of representation volumes (Corollary 3.2), we immediately infer the following:

Theorem 1.3. *If (X, G) is a geometry where X is contractible and G is semisimple, then every orientable closed manifold locally modeled on (X, G) has Property D.*

Note that X is a symmetric space of non-compact type when G has trivial center. In that case, the result has been proved by Besson–Courtois–Gallot [1] in rank one and by Connell–Farb [14] in higher ranks (with a few exceptions), and it also follows from a stronger result that such manifolds have positive simplicial volume by Lafont–Schmidt [38] combined with Savage [40] and Bucher–Karlsson [8]. Both [14] and [38] extend the barycenter method of [1] from rank one to higher ranks. In other cases, Theorem 1.3 is actually talking about a virtual (hyper-)torus bundle over a locally symmetric space, so the simplicial volume has to vanish [25, Section 3.1].

In fact, except for the impossible dimensions 1 and 2 and the unknown dimension 4, there are always n -manifolds with Property D which cannot be detected by the simplicial volume, see [7] in dimension 3 for $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. This result has recently been established by Fauser and Loeh [22, Theorem 1.3]. In our terms, their examples are product manifolds $P \times Q$ such that P and Q are manifolds with Property D, and they show that $P \times Q$ must also have Property D. If we take P to be locally modeled on \mathbb{H}^{n-3} , and Q to be locally modeled on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, then $P \times Q$ is locally modeled on $\mathbb{H}^{n-3} \times \widetilde{\mathrm{SL}}_2(\mathbb{R})$ which satisfies the assumptions of Theorem 1.3, and we obtain an alternate proof of that result.

We make some remarks about Theorem 1.1 before we close the introduction. Although there are some earlier results of special cases, such as Goldman [24] for $\mathrm{Iso}(\mathbb{H}^2)$ and Hartnick–Ott [26] for Hermitian simple Lie groups with finite center, the prototype of Theorem 1.1 is undoubtedly the case of $(\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R})$ -representations, as established by Brooks–Goldman [6, 7]. Their original proof is essentially reduced to the rigidity of certain characteristic class associated with certain $\mathrm{PSL}(2, \mathbb{R})$ -representations, (precisely the Godbillon–Vey class of the associated transversely- $\mathbb{R}P^1$ foliated circle bundles). After their work, generalizations in various directions and alternate approaches are obtained for many semisimple Lie groups G . Generalizations include rigidity of characteristic classes for transversely G -homogeneous foliations, and G -representation volumes for lattices, see [27, 3]. One of the new approaches related to 3-dimensional geometries identifies

representation volumes for $\mathrm{PSL}(2, \mathbb{C})$ and $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ with the Cheeger–Chern–Simons invariants [12, 11], see [37, 35, 16] and their references. This relation is first pointed out by Thurston for discrete and faithful representations in $\mathrm{PSL}(2, \mathbb{C})$ of hyperbolic 3-manifold groups [43]. Another new approach is due to Besson–Courtois–Gallot [2] based on the method they established in [1]. The Lie groups considered there are the structure groups of rank-one symmetric spaces of non-compact type, and a rigidity theorem for $\mathrm{Iso}_+(\mathbb{H}^n)$ is obtained.

Compared to the previous rigidity results the main difficulty of our contribution is that we allow the semisimple Lie groups to have infinite centre (whose rank corresponds to the rank of the homogeneous space X as vector bundle over a symmetric space). In this case the representation variety has no longer finitely many components and the usual argument “local rigidity implies global rigidity” does not hold. In order to solve this problem and to find model geometries which naturally gives rise to representation volume, we have to consider cocompact coextensions of the semisimple Lie groups. This allows us in particular to compare the volume of representations which belong to different components of the original variety.

A significant part of our argument provides more or less a classification of those groups as assumed by Theorem 1.1, (Proposition 8.1). Using that, we can reduce the proof of Theorem 1.1 to a more concrete family, namely, the full central extension of real connected semisimple Lie groups with torsion-free center, (see Section 5). The core argument for the central extension case is contained in the proof of Theorem 7.1. The proof of Theorem 1.1 is summarized in Section 9.

Whitney’s theorem on real affine algebraic varieties and Culler’s lifting criterion are employed, for describing the structure of the associated representation variety (Proposition 5.1). Structure theory for general Lie groups and Borel’s Density Theorem play crucial roles in our characterization of cocompact coextensions of semisimple Lie groups (Proposition 8.1). Based on well-known vanishing theorems, we prove a vanishing lemma for some particular relative cohomology of reductive Lie algebras in some particular dimension (Lemma 7.4).

The organization of this paper is reflected by the table of contents. Efforts have been made to keep the exposition easily accessible to non-experts of Lie groups, and the arguments as self-contained as possible.

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2. VOLUME OF REPRESENTATIONS

In this preliminary section, we revisit this concept and scrutinize the construction in a natural setting.

Let G be any connected real Lie group. Let X be a contractible homogeneous proper G -space, namely, a contractible smooth manifold equipped with a transitive, proper action of G by diffeomorphisms. Suppose that ω_X is a G -invariant volume form on X . Given any such triple (G, X, ω_X) , for any closed oriented smooth manifold M of the same dimension as X , a real-valued function called *volume* can be defined over the G -representation variety $\mathcal{R}(\pi_1(M), G)$ of $\pi_1(M)$.

We confirm in this section that the volume function is continuous with respect to a natural topology of $\mathcal{R}(\pi_1(M), G)$, namely, the algebraic-convergence topology. Moreover, the volume function is essentially determined by G , up to a nonzero scalar factor proportional to the volume form.

In the literature, the G -space X is usually taken to be G/H where H is any maximal compact subgroup of G , so a G -invariant volume form ω_X can be given by any G -invariant Riemannian metric. We first point out the equivalence of that more concrete approach with our abstract setting.

Lemma 2.1. *Let G be any connected real Lie group and let X be a contractible homogeneous proper G -space. Then*

- (1) *the point stabilizers of X are maximal compact subgroups of G . Hence X is determined by G up to a G -equivariant diffeomorphism,*
- (2) *there exists a G -invariant volume form ω_X on X . Moreover, it is unique up to a signed rescaling.*

Proof. Recall that the action of a topological group G on a topological space X is said to be *proper* if the mapping $G \times X \rightarrow X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ is proper, namely, such that the preimage of any compact subset is compact. Since the G -action on X is assumed to be proper, point stabilizers G_x for any $x \in X$ are compact subgroups of G , and the homogeneity assumption implies that X is isomorphic to G/G_x as a G -space. As any maximal compact subgroup is a topologically deformation retract of G , X is homotopy equivalent to a closed manifold H/G_x , for some maximal compact subgroup

H that contains G_x . However, the assumption that X is contractible forces H/G_x to be a point, so G_x is a maximal compact subgroup of the Lie group G . As maximal compact subgroups of G are conjugate to each other, different G -spaces X as assumed are G -equivariantly diffeomorphic to each other. This proves statement (1).

To prove statement (2), we may now take X to be G/H as usual, where H is any maximal compact subgroup of G . In this case, it is well-known that there exists a complete G -invariant Riemannian metric on G/H . For example, one may first take an arbitrary inner product $\langle \cdot, \cdot \rangle'_o$ on the tangent space T_oX at the point $o \in X$ which corresponds to the coset H . Then average the inner product over H by defining

$$\langle V, W \rangle_o = \int_H \langle h_* V, h_* W \rangle'_o d\mu_H(h),$$

for all $V, W \in T_oX$, where μ_H denotes the normalized Haar measure of H . After that, a claimed G -invariant Riemannian metric can be obtained by translating the H -invariant inner product at T_oX over X by the action of G . The associated volume form ω_X is hence G -invariant as claimed. To see the uniqueness of the G -invariant volume form up to rescaling, it suffices to observe that any such form is determined by its restriction at o , namely, $\omega_X|_o$, which lies in $\wedge^{\dim X} T_o^*X \cong \mathbb{R}$. As $\omega_X|_o$ must be nonzero, any pair of G -invariant volume forms over X differ only by a multiplication of some nonzero real scalar. This proves statement (2). Note that, however, G -invariant Riemannian metrics on X are not necessarily proportional, unless the isotropy representation of H on T_oX is irreducible. \square

Let (G, X, ω_X) be any triple as assumed. For any oriented (connected) closed smooth manifold M of the same dimension as X , we regard $\pi_1(M)$ as the (discrete) deck transformation group of the universal cover \widetilde{M} of M .

Denote by $\mathcal{R}(\pi_1(M), G)$ the G -representation variety of $\pi_1(M)$. In this paper, for any finitely generated group π , we understand the G -representation variety $\mathcal{R}(\pi, G)$ as the set of all homomorphisms of π into G . This set is inherited with the algebraic-convergence topology, which can be characterized by the property that a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ converges to ρ_∞ in $\mathcal{R}(\pi, G)$ if and only if $\{\rho_n(g)\}_{n \in \mathbb{N}}$ converges to $\rho_\infty(g)$ in G for every $g \in \pi$.

The volume function

$$\text{vol}_{G, X, \omega_X} : \mathcal{R}(\pi_1(M), G) \rightarrow \mathbb{R}$$

can be defined as follows. For any representation

$$\rho : \pi_1(M) \rightarrow G,$$

denote by

$$M \times_\rho X$$

the associated G -flat X -bundle space over M . It is the quotient of the product space $\widetilde{M} \times X$ by the diagonal freely discontinuous action of $\pi_1(X)$, namely, $\sigma \cdot (m, x) = (\sigma \cdot m, \rho(\sigma) \cdot x)$ for all $\sigma \in \pi_1(X)$. The bundle projection $M \times_\rho X \rightarrow M$ is induced by the projection of $\widetilde{M} \times X$ onto the first factor; the flat structure is the induced horizontal foliation on $M \times_\rho X$ with holonomy in G . The pull-back form of ω_X on $\widetilde{M} \times X$, via the projection onto the second factor, is invariant under the diagonal action of $\pi_1(M)$, so there is a naturally induced form ω_X^ρ on $M \times_\rho X$. Take $s : M \rightarrow M \times_\rho X$ to be any C^1 -differentiable section of the bundle.

We define the *volume* of (M, ρ) with respect to the triple (G, X, ω_X) to be

$$(2.1) \quad \text{vol}_{G,X,\omega_X}(\rho) = \int_M s^* \omega_X^\rho.$$

There is a natural way to translate between sections and developing maps. Specifically, every section $s: M \rightarrow M \times_\rho X$ induces a unique lift $\tilde{s}: \widetilde{M} \rightarrow \widetilde{M} \times X$, after choosing a base point lift of M into \widetilde{M} . Then the projection of \tilde{s} to the second factor gives rise to an equivariant map with respect to $\rho: \pi_1(M) \rightarrow G$,

$$D_\rho: \widetilde{M} \rightarrow X,$$

which is called a *developing map*. Conversely, the graph of any developing map D_ρ in $M \times X$ is invariant under the action of $\pi_1(M)$, so $\text{Id}_M \times D_\rho$ induces a section s . In terms of a developing map D_ρ , we may therefore write

$$(2.2) \quad \text{vol}_{G,H,\omega_X}(\rho) = \int_{\mathcal{F}} D_\rho^* \omega_X,$$

where $\mathcal{F} \subset \widetilde{M}$ is any fundamental domain of $\pi_1(M)$.

The dependence of the definition on our auxiliary choices can be summarized by the following:

Lemma 2.2. *With the notations and assumptions above, the following statements are true:*

- (1) *A C^1 -differentiable section s of $M \times_\rho X$ exists and any such section yields one and the same value*

$$\text{vol}_{G,X,\omega_X}(\rho) \in \mathbb{R}.$$

- (2) *For any two G -invariant volume forms ω_X and ω'_X , the associated volume functions agree up to a nonzero scalar factor. Namely, there exists $\lambda \in \mathbb{R}^*$ such that*

$$\text{vol}_{G,X,\omega_X}(\rho) = \lambda \text{vol}_{G,X,\omega'_X}(\rho)$$

for all M and all $\rho \in \mathcal{R}(\pi_1(M), G)$.

Proof. Since X is contractible, the bundle $M \times_\rho X$ admits a continuous section and any two sections $s, s': M \rightarrow M \times_\rho X$ are homotopic. Using standard approximation techniques in differential topology, the sections and homotopy can be approximated by C^r -differentiable ones, for arbitrary $1 \leq r \leq \infty$, (see [30, Chapter 2, Section 2, Exercise 3]). The induced form ω_X^ρ is closed as is ω_X . So the integral over M against the pull-backs of ω_X^ρ via s and s' are equal. This proves statement (1).

Statement (2) follows immediately from Lemma 2.1 (2). \square

The following proposition shows that the volume function is well behaved. The continuity is essentially a consequence of the properness assumption about the G -action on X .

Proposition 2.3. *The function $\text{vol}_{G,X,\omega_X}: \mathcal{R}(\pi_1(M), G) \rightarrow \mathbb{R}$ is conjugation-invariant and continuous.*

Proof. Given a developing map $D_\rho: \widetilde{M} \rightarrow X$ for a representation $\rho \in \mathcal{R}(\pi_1(M), G)$, it is straightforward to check that $g \circ D_\rho$ is a developing map for any conjugation $\tau_g(\rho) \in \mathcal{R}(\pi_1(M), G)$, defined by $\tau_g(\rho)(\sigma) = g\rho(\sigma)g^{-1}$. The volume function is invariant under conjugation of representations by the G -invariance of the volume form ω_X .

It remains to show the continuity. Fix a finite generating set of $\pi_1(M)$ induced by a finite cellular decomposition of M , and denote by \mathcal{F} a fundamental domain which is a

connected union of lifted cells in \widetilde{M} . By the properness of the G -action on X , we can fix a G -invariant Riemannian metric of X , as argued in Lemma 2.1. Fix a Riemannian metric g_M of M .

Suppose that $\{\rho_j\}_{j \in \mathbb{N}}$ is any sequence of representations which converges to a representation ρ_∞ . Take a C^1 -differentiable developing map $D_\infty: \widetilde{M} \rightarrow X$ of ρ_∞ . Then the algebraic convergence allows us to construct a sequence of C^1 -differentiable developing maps $D_n: \widetilde{M} \rightarrow X$ which converge to D_∞ uniformly over \mathcal{F} in the C^1 -norm. For example, a sequence of C^1 -uniformly convergent developing maps $\{D_j\}_{j \in \mathbb{N}}$ can be built first over a neighborhood of the 0-skeleton of \mathcal{F} , and then extended ρ_j -equivariantly over a neighborhood of the 0-skeleton of \widetilde{M} , and inductively over neighborhoods of higher dimensional skeleta of \widetilde{M} . Moreover, as D_∞ is C^1 -differentiable, we can require that in each step of the induction, the maps D_j are constructed to be C^1 -uniformly convergent to D_∞ on the considered neighborhood of the skeleton. Provided with such a sequence of developing maps $\{D_j\}_{j \in \mathbb{N}}$, given any constant $\varepsilon > 0$, and for all sufficiently large j , we have the following estimation by the Stokes Formula:

$$|\text{vol}_{G,X,\omega_X}(\rho_j) - \text{vol}_{G,X,\omega_X}(\rho_\infty)| < C\varepsilon,$$

where C is a constant determined by the Lipschitz constant of D_∞ , and the area of $\partial\mathcal{F}$ with respect to g_M . As $\varepsilon > 0$ is arbitrary, we have

$$\lim_{j \rightarrow \infty} \text{vol}_{G,X,\omega_X}(\rho_j) = \text{vol}_{G,X,\omega_X}(\rho_\infty).$$

This completes the proof. \square

When one wishes to compute the volume for a G -representation, it is important to keep in mind that the exact value depends on a particular normalization of the volume measure. In certain cases, there are canonical ways to specify a normalization. For example, for volume of representations in $\text{PSL}(2, \mathbb{C})$, the normalization is customarily taken as the volume form of hyperbolic geometry \mathbb{H}^3 (of constant curvature -1); for volume of representations in $\widetilde{\text{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$, the normalization used by Brooks and Goldman [6] is specified in terms of the Godbillon–Vey invariant of the naturally associated transversely- $\mathbb{R}P^1$ foliated circle bundle. However, there is no natural way to specify a canonical normalization to any connected Lie group in general.

On the other hand, sometimes one may only be concerned with the ratio between volume values. In other words, only those normalization-free qualities of the volume function are considered to be interesting. The discussion of this paper is of the latter flavor. For this reason, we often adopt the following simpler and more conventional

Notation 2.4. For any connected real Lie group G , we speak of volume of representations with a triple (G, X, ω_X) implicitly assumed as the following. The contractible proper homogeneous G -space X is taken to be G/H where H is a fixed maximal compact subgroup of G . The G -invariant volume form is chosen and fixed. For any oriented closed smooth manifold M , and any representation $\rho: \pi_1(M) \rightarrow G$, we only retain the subscript G and write

$$\text{vol}_G(M, \rho) \in \mathbb{R}$$

for the volume of representation $\text{vol}_{G,X,\omega_X}(\rho)$.

Definition 2.5. Adopting Notation 2.4, we define the G -representation volume of M to be

$$V(M, G) = \sup_{\rho \in \mathcal{R}(\pi_1(M), G)} |\text{vol}_G(M, \rho)|,$$

which is valued in $[0, +\infty]$.

Remark 2.6. Notice that a necessary condition to define the G -representation volume is that the geometry is aspherical, admits a compact quotient M and that $D(M, M)$ is finite.

As a consequence, in dimension ≤ 4 , the isometry groups of the possible geometries always contain a closed cocompact semisimple subgroup.

For dimension 5, the existence of compact quotients seems to be unknown for irrational fiber products of Seifert geometries and the Heisenberg-by-Seifert geometry so that we kept our discussion for that dimension focused only on examples.

3. COMPARISON OF VOLUMES

We consider the behavior of the volume of representations under variation of the manifolds and the Lie groups.

Proposition 3.1. *The following formulas hold for volume of representations.*

- (1) *Let G be a connected real Lie group which acts properly and transitively on a contractible homogeneous G -space X with a G -invariant volume form ω_X . Let M be any closed oriented smooth manifold of the same dimension as X . For any smooth map $f: M' \rightarrow M$ where M' is a closed oriented smooth manifold M' of the same dimension as X , and any representation $\rho: \pi_1(M) \rightarrow G$,*

$$\text{vol}_G(M', f^*(\rho)) = \text{deg}(f) \cdot \text{vol}_G(M, \rho),$$

where $\text{deg}(f) \in \mathbb{Z}$ denotes the signed mapping degree of f .

- (2) *Let (G, X, ω_X) and M be the same as above. For any connected real Lie group G' and any homomorphism $\phi: G' \rightarrow G$ with compact kernel and with closed and cocompact image, the induced action of G' on X is proper and transitive and preserves ω_X . Moreover, for any representation $\rho': \pi_1(M) \rightarrow G'$,*

$$\text{vol}_G(M, \phi_*(\rho')) = \text{vol}_{G'}(M, \rho').$$

- (3) *Let (G, X, ω_X) be the same as above and let I be a finite set. For all $i \in I$, let M_i be a closed oriented smooth manifold of the same dimension as X , and $\rho_i: \pi_1(M_i) \rightarrow G$ be a representation. Then*

$$\text{vol}_G(\#_i M_i, \bigvee_i \rho_i) = \sum_i \text{vol}_G(M_i, \rho_i).$$

- (4) *Let I be a finite set and for all $i \in I$, let (G_i, X_i, ω_{X_i}) be triples similar as above. Let M_i be closed oriented smooth manifolds of the same dimension as X_i , and let $\rho_i: \pi_1(M_i) \rightarrow G_i$ be representations. Then with respect to the triple $(\prod_i G_i, \prod_i X_i, \prod_i \omega_{X_i})$,*

$$\text{vol}_{\prod_i G_i} \left(\prod_i M_i, \prod_i \rho_i \right) = \prod_i \text{vol}_{G_i}(M_i, \rho_i).$$

Proof. To prove (1), we observe that if a developing map $D_\rho: \widetilde{M} \rightarrow X$ defines the section $s: M \rightarrow M \times_\rho X$, then the composed map $D_\rho \circ \tilde{f}: \widetilde{M}' \rightarrow \widetilde{M} \rightarrow X$ is a developing map for $f^*(\rho)$, which defines a section $s': M' \rightarrow M' \times_{\rho \circ f_\#} X$. The form $(s')^* \omega_X^{\rho \circ f_\#}$ on M' is induced by the pull-back of ω_X to \widetilde{M}' via $D_\rho \circ \tilde{f}$, and it coincides with $(s \circ f)^* \omega_X^\rho$ by definition. Turning the expression (2.1) into pairing between homology and cohomology classes, we have

$$\text{vol}_G(M', f^*(\rho)) = \langle [M'], (s')^* [\omega_X^{\rho \circ f_\#}] \rangle = \langle [M'], (s \circ f)^* [\omega_X^\rho] \rangle = \langle f_* [M'], s^* [\omega_X^\rho] \rangle.$$

As $f_*[M'] = \deg(f)[M]$, we have

$$\text{vol}_G(M', f^*(\rho)) = \deg(f) \cdot \langle [M], s^*[\omega_X^\rho] \rangle = \deg(f) \cdot \text{vol}_G(M, \rho).$$

To prove (2), denote by K the kernel of ϕ . Take H to be a maximal compact subgroup of G , for example, the isotropy group of the action of G on X . Since K is assumed to be compact, and $\phi(G')$ is assumed to be closed in G , $H' = \phi^{-1}(H)$ is compact in G . This means that G' acts properly on X via ϕ . Moreover, the maximality of H implies that H' must be a maximal compact subgroup of G . In particular, if we naturally identify X with the coset space G/H , the subspace $\phi(G')H/H$ is an embedded closed contractible subspace of X . Here $\phi(G')H$ denotes the double coset contained in G which consists of elements of the form $\phi(g')h$ for all $g' \in G'$ and $h \in H$. By homotopy equivalences $G/\phi(G') \simeq G/\phi(H') \simeq H/\phi(H')$, we can count the cohomological dimension of the compact spaces H , $\phi(H')$, and $G/\phi(G')$ by our assumption. Then we have $\dim G/\phi(G') = \dim H - \dim \phi(H')$. This means $\phi(G')H/H$ equals X , or in other words, G' acts transitively on X via ϕ . Of course the induced action of G' on X preserves the volume form ω_X .

Observe that for any representation $\rho' : \pi_1(M) \rightarrow G'$, any $\phi_*(\rho')$ -equivariant developing map $\widetilde{M} \rightarrow X$ is by definition a ρ' -equivariant developing map for the induced action of G' , so

$$\text{vol}_G(M, \phi_*(\rho')) = \text{vol}_{G'}(M, \rho').$$

To prove (3), we take a developing map which sends the spheres that the connected sum identifies along to points in X , then the claimed formula follows immediately from the formula 2.2.

To prove (4), we take a developing map of the form $\prod_i D_{\rho_i} : \prod_i \widetilde{M} \rightarrow \prod_i X_i$, and again we apply the formula 2.2 to derive the claimed formula. \square

Corollary 3.2. *The following comparisons hold for the representation volume of manifolds:*

- (1) (Domination Inequality). *Given any $f : M' \rightarrow M$ as of Proposition 3.1 (1),*

$$\mathbb{V}(M', G) \geq |\deg(f)| \cdot \mathbb{V}(M, G).$$

- (2) (Induction Inequality). *Given any $\phi : G' \rightarrow G$ as of Proposition 3.1 (2),*

$$\mathbb{V}(M, G') \leq \mathbb{V}(M, G).$$

- (3) (Connected-Sum Inequality). *Given any (G, X, ω_X) and M_i as of Proposition 3.1 (3),*

$$\mathbb{V}(\#_i M_i, G) \leq \sum_i \mathbb{V}(M_i, G).$$

- (4) (Product Inequality). *Given any (G_i, X_i, ω_{X_i}) and M_i as of Proposition 3.1 (4),*

$$\mathbb{V}\left(\prod_i M_i, \prod_i G_i\right) \geq \max_{\sigma \in \mathfrak{S}(I)} \left\{ \prod_i \mathbb{V}(M_i, G_{\sigma(i)}) \right\},$$

where $\mathfrak{S}(I)$ is the permutation group of the finite set I of indices, and the volume $\mathbb{V}(M_i, G_{\sigma(i)})$ is considered to be 0 if the dimension of M_i mismatches that of $X_{\sigma(i)}$.

None of the above inequalities are equalities in general. In fact, the following examples show that each of them can fail to be equal under certain circumstances. One can certainly find tons of such examples, some of which may be trickier to verify. To avoid technicalities,

we only sketch the key points to verify our examples. The reader may safely skip this part for the first reading.

Example 3.3.

- (1) Take G to be either $\mathrm{PSL}(2, \mathbb{C})$ or $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. By [16, Corollary 1.8], there exists a closed oriented 3-manifold M with vanishing $V(M, G)$, whereas $V(M', G) > 0$ holds for some finite cover M' of M . So a strict domination inequality $V(M', G) > |\deg(f)|V(M, G)$ holds for the covering map $f: M' \rightarrow M$.
- (2) Take G to be $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and G' to be $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. Consider an oriented closed 3-manifold M which is a Seifert fibered space with the symbol $(2, 0; 3/2)$. The base 2-orbifold is a closed oriented surface of genus 2 and with a cone point of order 2, so it has Euler characteristic $\chi = -5/2$. The (orbifold) Euler number of the Seifert fibration equals $e = 3/2$. Then $V(M, G') = 4\pi^2 \times (-5/2)^2 / |3/2| = 50\pi^2/3$ by the formula of Seifert volume in this case, (see [16, Section 6.2]). For any representation $\rho: \pi_1(M) \rightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R})$ of nonzero volume, it can be calculated by [16, Proposition 6.3] that ρ must send the regular fiber h to a generator $\mathrm{sh}(\pm 1)$ of the center $\{\mathrm{sh}(n) \in \widetilde{\mathrm{SL}}_2(\mathbb{R}) : n \in \mathbb{Z}\}$. Then $V(M, G) = 4\pi^2 \times 1^2 \times (3/2) = 6\pi^2$. So a strict induction inequality $V(M, G') < V(M, G)$ holds for this case.
- (3) Take G to be $\mathrm{PSL}(2, \mathbb{C})$ and Γ be a torsion-free uniform lattice of G . Denote by $M = \mathbb{H}^3/\Gamma$ be the closed hyperbolic 3-manifold with the induced orientation. By Theorem 1.1, the set of volumes for all representations of $\pi_1(M) \cong \Gamma$ in G consists of finitely many real values $v_1 < v_2 < \dots < v_s$. If we require further that M admits no orientation-reversing self-homeomorphism, it is implied by the volume rigidity that $v_s = \mathrm{Vol}_{\mathbb{H}^3}(M)$ and $|v_1| < v_s$, [2, Theorem 1.4]. Note that the set of volumes of the orientation-reversal $-M$ consists of $-v_s < \dots < -v_2 < -v_1$. Then the set of volumes for the connected sum $M \# (-M)$ consists of all the values $v_i - v_j$, and $V(M \# (-M), G) = |v_s - v_1| < 2v_s$. As $V(M, G) = V(-M, G) = v_s$, we obtain a strict connected-sum inequality $V(M \# (-M), G) < V(M, G) + V(-M, G)$ in this case.
- (4) Take G to be $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times \mathrm{PSL}(2, \mathbb{C})$. Let S be the unit tangent bundle of a closed oriented hyperbolic surface, and H be a closed oriented hyperbolic 3-manifold. Let $M_S = S \times S$ and $M_H = H \times H$.

We argue that $V(M_S, G)$ must be zero. This means that $\mathrm{vol}_G(M_S, \rho) = 0$ holds for every representation $\rho: \pi_1(M_S) \rightarrow G$. Every ρ is of the form (σ, η) where $\sigma: \pi_1(M_S) \rightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R})$, and $\eta: \pi_1(M_S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Via any section $s = (u, v): M_S \rightarrow M_S \times_{(\sigma, \eta)} (\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times \mathbb{H}^3)$, the pull-back of the product volume form can be regarded as a cup product $[u^* \omega_{\widetilde{\mathrm{SL}}_2(\mathbb{R})}^\sigma] \smile [v^* \omega_{\mathbb{H}^3}^\eta]$ in $H^6(M_S; \mathbb{R})$, where $[u^* \omega_{\widetilde{\mathrm{SL}}_2(\mathbb{R})}^\sigma]$ and $[v^* \omega_{\mathbb{H}^3}^\eta]$ are the 3-dimensional factor forms lying in $H^3(M_S; \mathbb{R})$. Note that $H_3(M_S; \mathbb{R}) \cong \bigoplus_{i=0}^3 (H_{3-i}(S; \mathbb{R}) \otimes H_i(S; \mathbb{R}))$. Moreover, the $(3, 0)$ -summand is generated by the 3-cycle $S \times \mathrm{pt}$; the $(2, 1)$ -summand is generated by all the 3-cycles of the form $T_\alpha \times \beta$ where T_α is any vertical torus of S over a non-separating simple closed curve of the base, and β is any loop of S that projects to a non-separating simple closed curve of the base; the $(1, 2)$ -summand and the $(0, 3)$ -summand can be described similarly. The evaluation of $[u^* \omega_{\mathbb{H}^3}^\eta]$ on any of the above cycles is nothing but the volume of the $\mathrm{PSL}(2, \mathbb{C})$ -representations of their fundamental groups induced by η , so it is always trivial by a direct argument. It follows that $[u^* \omega_{\mathbb{H}^3}^\eta] = 0$ in $H^3(M_S; \mathbb{R})$, so

$[u^*\omega_{\widehat{\mathrm{SL}}_2(\mathbb{R})}^\sigma] \smile [v^*\omega_{\mathbb{H}^3}^\eta] = 0$ in $H^6(M_S; \mathbb{R})$. Therefore, $\mathrm{vol}_G(M_S, \rho) = 0$ holds as claimed.

However, it is obvious that $V(M_S \times M_H, G \times G) > 0$. Then we obtain a strict product inequality $V(M_S \times M_H, G \times G) > V(M_S, G) \times V(M_H, G)$ in this case.

We observe the following consequence of the domination inequality:

Corollary 3.4. *For any connected Lie group G , the G -representation volume is a homotopy-type invariant for orientable closed smooth manifolds.*

4. REAL SEMISIMPLE LIE GROUPS

In this section, we recall some standard facts about real semisimple Lie groups. See [32, 28] for general references.

Let G be a connected real Lie group. Denote by \mathfrak{g} the Lie algebra of G , regarded as the tangent space of G at its neutral element. Denote by $\mathrm{GL}(\mathfrak{g})$ the Lie group of invertible \mathbb{R} -linear transformations of \mathfrak{g} . The Lie algebra $\mathrm{gl}(\mathfrak{g})$ of $\mathrm{GL}(\mathfrak{g})$ can be naturally identified with the Lie algebra of \mathbb{R} -endomorphisms $\mathrm{End}(\mathfrak{g})$. The adjoint action of G on \mathfrak{g} , denoted as

$$\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g}),$$

is defined for any $h \in G$ as the derivation (at the neutral element) of the inner automorphism $\sigma(h)(g) = hgh^{-1}$, for all $g \in G$. The derivation of Ad is hence a Lie algebra homomorphism

$$\mathrm{ad}: \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g}).$$

For any $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ can be explicitly given by the Lie bracket, namely, $\mathrm{ad}(X)(Y) = [X, Y]$ for all $Y \in \mathfrak{g}$.

Recall that a Lie algebra \mathfrak{g} is said to be *simple* if it is nonabelian and contains no ideals other than $\{0\}$ and \mathfrak{g} . It is said to be *semisimple* if it is a direct sum of simple Lie algebras. A Lie group G is said to be *semisimple* if its Lie algebra \mathfrak{g} is semisimple.

4.1. Cohomology. Semisimple Lie algebras can be characterized by cohomology through the following [13, Theorem 24.1]. See Section 6 for a review of Lie algebra cohomology.

Theorem 4.1. *Let \mathfrak{g} be a finite dimensional Lie algebra over any field of characteristic 0. Then \mathfrak{g} is semisimple if and only if $H^*(\mathfrak{g}; V) = 0$ holds for all irreducible nontrivial \mathfrak{g} -modules V .*

In particular, $H^*(\mathfrak{g}; \mathfrak{g}^*) = 0$ for real semisimple Lie algebras. Moreover, the semisimplicity of \mathfrak{g} implies that any (finite dimensional) \mathfrak{g} -module W is a direct sum of irreducible factors. It follows from Theorem 4.1 that $H^*(\mathfrak{g}; W)$ is a direct sum of finitely many (possibly zero) copies of $H^*(\mathfrak{g})$. In low dimensions, it can be computed that $H^0(\mathfrak{g}) = \mathbb{R}$, and $H^k(\mathfrak{g}) = 0$ for $k = 1, 2, 4$. However, there is a canonical nonvanishing class of $H^3(\mathfrak{g})$ given by the 3-form $B([\cdot, \cdot], \cdot)$ in $\wedge^3 \mathfrak{g}^*$, where $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ is the Killing form. See Chevalley–Eilenberg [13].

The following relative version of the vanishing result [13, Theorem 28.1] is needed in the rest of this paper.

Theorem 4.2. *Let \mathfrak{g} be a finite dimensional Lie algebra over any field of characteristic 0 and \mathfrak{h} be a Lie subalgebra. If \mathfrak{g} is semisimple, then $H^*(\mathfrak{g}, \mathfrak{h}; V) = 0$ holds for all irreducible nontrivial \mathfrak{g} -modules V .*

4.2. Center and linearity. There are semisimple groups which are not linear, such as $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. However, it is well known that linearity holds after factoring out the center:

Theorem 4.3. *Let $Z(G)$ be the center of a connected real semisimple Lie group G . Then the following statements hold true:*

- (1) *The quotient group $G/Z(G)$ is isomorphic to the identity component of $\mathrm{Aut}(\mathfrak{g})$ as a Lie group.*
- (2) *The quotient homomorphism $G \rightarrow G/Z(G)$ is a covering projection.*
- (3) *The center $Z(G)$ is finitely generated.*

Here $\mathrm{Aut}(\mathfrak{g})$ denotes the group of Lie algebra automorphisms of \mathfrak{g} , which is contained in the real general linear group $\mathrm{GL}(\mathfrak{g})$ as a closed (and Zariski closed) subgroup. In fact, the isomorphism of the first statement can be induced by the adjoint representation $\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})$, which is trivial on $Z(G)$. The kernel of Ad is exactly $Z(G)$ and the image is the entire identity component of $\mathrm{Aut}(\mathfrak{g})$ when G is semisimple, see [28, Chapter II, Corollary 5.2 and Corollary 6.5]. The second statement is an immediate consequence of the semisimplicity of G that the closed normal subgroup $Z(G)$ has to be 0-dimensional, and hence discrete. It follows that $Z(G)$ is a quotient of the fundamental group $\pi_1(G)$, so the finite generation property of the third statement holds by the fact G is homotopy equivalent to any maximal compact subgroup of itself.

Remark 4.4. There is a much stronger result asserting that every connected Lie group G admits a universal linear quotient. When G is semisimple, the result can be stated as follows. Let $\pi: \widetilde{G} \rightarrow G$ denote the universal covering of G and let $\sigma: \widetilde{G} \rightarrow G_{\mathbb{C}}$ be the complexification of \widetilde{G} . Then every linear representation of G factors through $G/\pi(\ker \sigma)$ and there exists a faithful linear representation of $G/\pi(\ker \sigma)$, which is therefore the universal linear quotient of G . See [32, Chapter XVII, Theorem 3.3] and [31].

4.3. Lattices. The existence of torsion-free uniform lattices in any connected semisimple Lie group is a celebrated theorem due to A. Borel [4, Theorem B], which can be paraphrased by the following:

Theorem 4.5. *Let G be a connected real semisimple Lie group and H be a maximal compact subgroup of G . Then G has a discrete subgroup Γ which acts freely discontinuously on the homogeneous G -space G/H with compact quotient.*

In fact, given any torsion-free uniform lattice Γ of G by Borel's theorem, the discreteness of Γ and the compactness of H implies that the action of Γ on G/H is properly discontinuous. Moreover, $\Gamma \cap H$ has to be finite, and therefore trivial as Γ contains no torsion. The cocompactness of the action is equivalent to the uniformity of Γ as H is compact.

5. CENTRAL EXTENSION

Let G be a connected semisimple Lie group. We assume for this section that the center Z of G is torsion-free. Denote by H a maximal compact subgroup of G and X the homogeneous space G/H . In this section, we consider a natural non-splitting central extension $G_{\mathbb{R}}$ of G , which contains a maximal compact subgroup $H_{\mathbb{R}}$ extending H . The associated homogeneous space $G_{\mathbb{R}}/H_{\mathbb{R}}$ can be identified with X , but there are more transformations coming from the enlarged group $G_{\mathbb{R}}$. In our applications, passage from G to $G_{\mathbb{R}}$ allows us more room to deform a representation.

Note that by our assumption and Theorem 4.3, the center Z of G is a finitely generated free abelian group. Denote by $Z_{\mathbb{R}}$ the (additive) torsion-free abelian Lie group $Z \otimes \mathbb{R}$, which is isomorphic to $\mathbb{R}^{\text{rk}(Z)}$ and contains Z naturally as the integral lattice.

Define the *full central extension* of G to be

$$G_{\mathbb{R}} = G \times_Z Z_{\mathbb{R}},$$

namely, the quotient of $G \times Z_{\mathbb{R}}$ by the Z -action

$$z \cdot (g, x) \mapsto (gz, x - z \otimes 1)$$

for all $z \in Z$. It is clear that $G_{\mathbb{R}}$ naturally contains G as a closed subgroup, which intersects the center $Z_{\mathbb{R}}$ in the naturally embedded Z . Denote by \overline{G} the quotient group G/Z . We have the following commutative diagram of Lie group homomorphisms:

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & \overline{G} \longrightarrow \{1\} \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ \{0\} & \longrightarrow & Z_{\mathbb{R}} & \longrightarrow & G_{\mathbb{R}} & \longrightarrow & \overline{G} \longrightarrow \{1\} \end{array}$$

where the rows are exact sequences.

5.1. Representation varieties and lift obstruction. For any finitely generated group π , there are maps between the representation varieties

$$\mathcal{R}(\pi, G) \longrightarrow \mathcal{R}(\pi, G_{\mathbb{R}}) \longrightarrow \mathcal{R}(\pi, \overline{G})$$

which are naturally induced by the group homomorphisms

$$G \longrightarrow G_{\mathbb{R}} \longrightarrow \overline{G}.$$

Proposition 5.1. *Suppose that G is a connected real semisimple Lie group with torsion-free center Z . There is a characteristic class for \overline{G} -representations of finitely generated groups, namely, a natural assignment*

$$e_Z: \mathcal{R}(\pi, \overline{G}) \rightarrow H^2(\pi; Z)$$

for any finitely generated group π . Moreover, the following statements are true:

- (1) *The space of representations $\mathcal{R}(\pi, \overline{G})$ is a finite union of path-connected components of an affine real algebraic variety. The characteristic class e_Z is constant over each path-connected component of $\mathcal{R}(\pi, \overline{G})$.*
- (2) *The space of representations $\mathcal{R}(\pi, G)$ is an $H^1(\pi; Z)$ -principal bundle over the union of the path-connected components of $\mathcal{R}(\pi, \overline{G})$ on which e_Z is trivial.*
- (3) *The space of representations $\mathcal{R}(\pi, G_{\mathbb{R}})$ is an $H^1(\pi; Z_{\mathbb{R}})$ -principal bundle over the union of the path-connected components of $\mathcal{R}(\pi, \overline{G})$ on which e_Z is torsion.*

Proof. For any finitely generated group π and any \overline{G} -representation η of π , the characteristic class $e_Z(\eta) \in H^2(\pi; Z)$ can be constructed concretely as follows:

Take any CW complex K which realizes the Eilenberg–MacLane space $K(\pi, 1)$. Denote by P_{η} the associated \overline{G} -principal bundle $K \times_{\eta} \overline{G}$ over K . Let $\{U_{\alpha}\}_{\alpha \in A}$ be a locally finite open cover of K over which P_{η} is locally trivialized by sections $s_{\alpha}: U_{\alpha} \rightarrow P_{\eta}|_{U_{\alpha}}$. The transition functions $g_{\alpha\beta} \in \overline{G}$ are given by $s_{\beta} = s_{\alpha}g_{\alpha\beta}$ over any $U_{\alpha} \cap U_{\beta}$, which satisfy the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ over any $U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$. Take a lift $\tilde{g}_{\alpha\beta} \in G$ for each

transition function in such a way that $\tilde{g}_{\alpha\beta} = \tilde{g}_{\beta\alpha}^{-1}$. It follows that over each $U_\alpha \cap U_\beta \cap U_\gamma$, the lift determines an element

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \in Z$$

It is easy to check that this determines a Z -valued Čech 2-cocycle which represents a cohomology class

$$e_Z(\eta) \in \check{H}^2(K; Z) \cong H^2(\pi; Z)$$

which depends only on the isomorphism class of the bundle, which means that $e_Z(\eta)$ depends only on η and it is functorial with respect to homomorphisms between finitely generated groups.

To prove statement (1), we first observe that e_Z is by definition constant as the representation η varies continuously, so it is constant on each path-connected component of $\mathcal{R}(\pi, \overline{G})$. This follows from the fact that when η varies continuously the corresponding bundles are isomorphic so e_Z is constant. To bound the number of the path-connected components, we apply Theorem 4.3 about semisimple Lie groups. Denote by \mathfrak{g} the Lie algebra of G . The adjoint representation $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$ induces an embedding of \overline{G} as the identity component of $\text{Aut}(\mathfrak{g})$. Note that for any continuous path of representations $\rho_t: \pi \rightarrow \text{Aut}(\mathfrak{g})$, the image of every element of π must remain in some component of $\text{Aut}(\mathfrak{g})$ all the time. It follows that $\mathcal{R}(\pi, \overline{G})$ can be identified with those path-connected components of $\mathcal{R}(\pi, \text{Aut}(\mathfrak{g}))$ which consist of all the representations into the identity component of $\text{Aut}(\mathfrak{g})$. Since $\text{Aut}(\mathfrak{g})$ is a Zariski closed subset of $\text{GL}(\mathfrak{g})$, (thus an affine real algebraic group,) $\mathcal{R}(\pi, \text{Aut}(\mathfrak{g}))$ is an affine real algebraic variety (namely, a Zariski closed subset of \mathbb{R}^N for some large N). By H. Whitney [44, Theorem 3], $\mathcal{R}(\pi, \text{Aut}(\mathfrak{g}))$ has at most finitely many path-connected components, so the same holds for $\mathcal{R}(\pi, \overline{G})$. Note that \overline{G} is an analytic linear group but not an algebraic group in general.

To prove statement (2), regard any element $\alpha \in H^1(\pi; Z)$ as a homomorphism $\alpha: \pi \rightarrow Z$. For any representation $\rho: \pi \rightarrow G$, we can twist ρ by α , namely, setting $\alpha \cdot \rho: \pi \rightarrow G$ by $(\alpha \cdot \rho)(g) = \alpha(g)\rho(g)$. This induces an action of $H^1(\pi; Z)$ on $\mathcal{R}(\pi, G)$ which is clearly free. Note that $Z(G)$ is discrete by Theorem 4.3, so the action is discontinuous with respect to the algebraic-convergence topology, as can be easily checked on a finite set of generators. This action endows $\mathcal{R}(\pi, G)$ with a $H^1(\pi; Z)$ -principal bundle structure. Every fiber of $\mathcal{R}(\pi, G)$ is clearly projected to a single point of $\mathcal{R}(\pi, \overline{G})$. As G/Z is connected and covered by G , by M. Culler [15, Theorem 4.1], any continuous path of representations in $\mathcal{R}(\pi, G/Z)$ can be lifted to $\mathcal{R}(\pi, G)$ provided that some representation of the path can be lifted. Therefore, $\mathcal{R}(\pi, G)$ are projected onto a sub-union of path-connected components of $\mathcal{R}(\pi, G/Z)$. By the construction of e_Z , a \overline{G} -representation η can be lifted to a G -representation exactly when $e_Z(\eta) \in H^2(\pi; Z)$ vanishes.

Statement (3) can be proved in a similar way. In fact, since e_Z is constant on the components of $\mathcal{R}(\pi, \overline{G})$, which are at most finitely many, we may take D to be the least common multiple of the orders of those e_Z -values which are torsion. Using a finite central extension $G_{D^{-1}\mathbb{Z}} = G \times_Z (Z \otimes D^{-1}\mathbb{Z})$ instead of G , we see that all the e_Z -torsion components of $\mathcal{R}(\pi, \overline{G})$ are projected onto by $\mathcal{R}(\pi, G_{D^{-1}\mathbb{Z}})$, and hence by $\mathcal{R}(\pi, G_{\mathbb{R}})$. Representations of the other components of $\mathcal{R}(\pi, \overline{G})$ cannot be lifted to $\mathcal{R}(\pi, G_{\mathbb{R}})$ since e_Z survives in $H^2(\pi; Z_{\mathbb{R}})$. The $H^1(\pi; Z_{\mathbb{R}})$ -principal bundle structure of $\mathcal{R}(\pi, G_{\mathbb{R}})$ follows from the same argument as in (2). This completes the proof. \square

5.2. Associated homogeneous space. From the proper action of G on its associated contractible homogeneous space $X = G/H$, one can extract a concrete model of the central extension $G_{\mathbb{R}}$ as follows.

Proposition 5.2. *Let G be a connected semisimple Lie group. Assume that the center Z of G is torsion-free. Let H be a maximal compact subgroup of G . There exists a connected closed torsion-free abelian subgroup R of G with the following properties:*

- *The subgroup of G generated by H and R is a direct product HR .*
- *The intersection of R and Z is a lattice of R and a finite-index subgroup of Z .*
- *The center Z of G is contained in HR .*
- *The quotient HR/Z is embedded as a maximal compact subgroup of \overline{G} as induced by the projection.*

Therefore, the center Z of G is embedded as a lattice of HR/H via the quotient projection, and there is a uniquely induced isomorphism

$$G_{\mathbb{R}} \cong G \times_Z (HR/H).$$

Proof. Let H be a maximal compact subgroup of G . Let \overline{K} be a maximal compact subgroup of \overline{G} which contains the projected image of H . Denote by K the preimage of \overline{K} in G . Since maximal compact subgroups are topologically deformation retracts of any virtually connected Lie group, we have homotopy equivalences $H \simeq G$ and $\overline{K} \simeq \overline{G}$ via the inclusions. Therefore, we have $H \simeq K \simeq G$ via the inclusions because K is the covering space of \overline{K} that corresponds the subgroup $\pi_1(G)$ of $\pi_1(\overline{G})$. Moreover, \overline{K} and K are both connected as \overline{K} is homotopy equivalent to the connected group \overline{G} .

By the structure theory of compact Lie groups, the universal covering group \tilde{K} of the connected compact Lie group \overline{K} is isomorphic to a direct product $\tilde{K}_1 \times \cdots \times \tilde{K}_m \times \mathbb{R}^n$ where \tilde{K}_i are simply-connected simple compact Lie groups, [33, Theorem 6.6]. The center $Z(\tilde{K})$ of \tilde{K} is the direct product of the finite centers $Z(\tilde{K}_i)$ and the abelian factor \mathbb{R}^n . The kernel of the covering projection $\tilde{K} \rightarrow \overline{K}$ can be identified with $Z(\tilde{G})$ by the homotopy equivalence $\overline{K} \simeq \overline{G}$. Denote by L_K the kernel of $\tilde{K} \rightarrow K$. There is an induced short exact sequence of discrete abelian groups

$$\{0\} \longrightarrow L_K \longrightarrow Z(\tilde{G}) \longrightarrow Z \longrightarrow \{0\},$$

which splits as Z is assumed to be torsion-free. It follows that the intersection of L_K with \mathbb{R}^n is a finite index subgroup of L_K which is contained in a unique \mathbb{R} -subspace V_K of \mathbb{R}^n . Take any lift \tilde{Z} of Z into $Z(\tilde{G})$. The intersection between \tilde{Z} and the factor subgroup \mathbb{R}^n is a finite index subgroup of \tilde{Z} , and it is contained in a unique \mathbb{R} -subspace \tilde{R} of \mathbb{R}^n . However, note that $Z(\tilde{G})$ may not be a subgroup of the connected component \mathbb{R}^n of $Z(\tilde{K})$, even if it is torsion-free, so one might not be able to make \tilde{Z} contained in \mathbb{R}^n . The fact that \overline{K} is compact implies the relation

$$\dim V_K + \dim \tilde{R} = \dim Z(\tilde{K}),$$

or more concretely, there is a direct-sum decomposition $V_K + \tilde{R} = \mathbb{R}^n$. We take the claimed abelian subgroup R of G to be the projected image of \tilde{R} .

Note that the preimage of H in \tilde{K} is exactly $\tilde{H} = \tilde{K}_1 \times \cdots \times \tilde{K}_m \times V_K$, which contains the kernel L_K of $\tilde{K} \rightarrow K$. Since \tilde{R} meets \tilde{H} trivially and centralizes \tilde{H} , the double coset HR forms a subgroup of G which is the direct product of H and R . The intersection of Z and R is a finite-index subgroup of Z and a lattice of R because the same holds for \tilde{Z} and \tilde{R} . The fact that Z is contained in HR follows from that \tilde{Z} is contained in $Z(\tilde{G}) \subset Z(\tilde{H})\tilde{R}$. The quotient HR/Z is a maximal compact subgroup of \overline{G} because it is exactly $\tilde{H}\tilde{R}/Z(\tilde{G}) = \tilde{K}/Z(\tilde{G}) = \overline{K}$. Therefore, we have verified the claimed properties about R .

To complete the proof, we see that Z is embedded into HR/H as a lattice because it is torsion-free and it meets R in a finite index subgroup. The induced isomorphism

$$G_{\mathbb{R}} \cong G \times_Z (HR/H)$$

is immediately implied by the unique isomorphism

$$Z_{\mathbb{R}} \cong HR/H$$

which respects the inclusions of Z . \square

In the following, we keep the notation HR/H even though it is canonically isomorphic to R . We denote the elements of HR/H as cosets rH where $r \in R$. In this way, we remind ourselves that Z is not automatically contained in R as subgroups of G .

The homogeneous space $X = G/H$ is equipped with a proper transitive action

$$G \times_Z (HR/H) \rightarrow \text{Diff}(X)$$

defined by

$$(g, rH) \cdot (xH) = (gxr^{-1})H.$$

In this concrete model, there is a distinguished point

$$O_X \in X,$$

namely, the coset $H \in X$. The stabilizer of O_X , or the isotropy group of the action, is the image of the diagonal embedding

$$\Delta: HR/Z \rightarrow G \times_Z (HR/H)$$

defined by $\Delta(hrZ) = (hr, rH) \bmod Z$. We denote the isotropy group as

$$H_{\mathbb{R}} = \Delta(HR/Z),$$

while we identify $G_{\mathbb{R}}$ with $G \times_Z (HR/H)$. Therefore, we have an identification between homogeneous spaces

$$X \cong G_{\mathbb{R}}/H_{\mathbb{R}}$$

via the induced equivariant diffeomorphism.

5.3. Representations and reduction of bundles. Let M be a connected closed smooth manifold. Denote by \widetilde{M} the universal cover of M . For any representation $\rho: \pi_1(M) \rightarrow G_{\mathbb{R}}$, after choosing a ρ -equivariant smooth developing map $D_{\rho}: \widetilde{M} \rightarrow X$, there is an induced principal $H_{\mathbb{R}}$ -bundle $H_{\mathbb{R}}(\widetilde{M}, D_{\rho})$ over \widetilde{M} , namely, the pull-back of the canonical principal $H_{\mathbb{R}}$ -bundle $G_{\mathbb{R}} \rightarrow X$ via D_{ρ} . The bundle space is the fiber product

$$H_{\mathbb{R}}(\widetilde{M}, D_{\rho}) = \widetilde{M} \times_{D_{\rho}} G_{\mathbb{R}},$$

which consists of those points $(\tilde{m}, g) \in \widetilde{M} \times G_{\mathbb{R}}$ such that $D_{\rho}(\tilde{m}) = g \cdot O_X$. The bundle projection takes any point (\tilde{m}, g) to \tilde{m} . Moreover, $H_{\mathbb{R}}(\widetilde{M}, D_{\rho})$ is equipped with a natural action of $\pi_1(M)$ given by $\sigma \cdot (\tilde{m}, g) \rightarrow (\sigma \cdot \tilde{m}, \rho(\sigma) \cdot g)$, which commutes with the bundle projection, so the quotient by the action yields a principal $H_{\mathbb{R}}$ -bundle $H_{\mathbb{R}}(M, D_{\rho})$ over M . This is a model of the $H_{\mathbb{R}}$ -reduction of the flat principal $G_{\mathbb{R}}$ -bundle $M \times_{\rho} G_{\mathbb{R}}$ over

M , induced by the section s of $M \times_\rho X \rightarrow M$ that corresponds to D_ρ , in the sense that the following diagram of maps commutes:

$$\begin{array}{ccc} H_{\mathbb{R}}(M, D_\rho) & \xrightarrow{\text{incl.}} & M \times_\rho G_{\mathbb{R}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{s} & M \times_\rho X \end{array}$$

The isomorphism type of $H_{\mathbb{R}}$ -reductions of flat principal $G_{\mathbb{R}}$ -bundles $M \times_\rho G_{\mathbb{R}}$ over M depends only on the path-connected component of $\rho \in \mathcal{R}(\pi_1(M), G_{\mathbb{R}})$. However, the models are different as ρ varies. In the following, we exhibit a construction that turns a smooth path of representations into a smoothly varying family of $H_{\mathbb{R}}$ -reductions tied to a fixed model. The procedure should be routine and easy in principle, but we need to properly formulate and examine a few details.

To keep concrete, we say that a path of representations ρ_t in $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$ is *smooth* if at every element $\sigma \in \pi_1(M)$, the path $\rho_t(\sigma)$ is smooth in $G_{\mathbb{R}}$. According to the description of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$ from Proposition 5.1, any pair of representations on a path-connected component of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$ can be connected by a piecewise smooth path of representations

$$\rho_t: \pi_1(M) \rightarrow G_{\mathbb{R}},$$

parametrized by $t \in [0, 1]$.

Lemma 5.3. *Given a smooth path of representations $\rho_t: \pi_1(M) \rightarrow G_{\mathbb{R}}$, parametrized by $t \in [0, 1]$, there exists a path of smooth developing maps*

$$D_t: \widetilde{M} \rightarrow X$$

which is ρ_t -equivariant for each t and which varies smoothly with respect to t .

Proof. Suppose that ρ_t is a smooth path of representations. Write $M_{[0,1]}$ for the product $[0, 1] \times M$, and M_t for any slice $\{t\} \times M$. We take the induced smooth bundle over $M_{[0,1]}$ with fibers diffeomorphic to X , denoted as $M_{[0,1]} \times_\rho X$. The total space is the quotient of $[0, 1] \times \widetilde{M} \times X$ by the induced action of $\pi_1(M)$, namely, $\sigma \cdot (t, \tilde{m}, x) = (t, \sigma \cdot \tilde{m}, \rho_t(\sigma) \cdot x)$. The bundle projection is induced by the projection of the universal cover onto the first two factors. Since the fiber X is contractible, the bundle admits a continuous section, which can be perturbed to be a smooth section $s: M_{[0,1]} \rightarrow M_{[0,1]} \times_\rho X$ by standard techniques of approximations [30, Chapter 2, see Section 2, Exercise 3]. Any elevation $\tilde{s}: [0, 1] \times \widetilde{M} \rightarrow [0, 1] \times \widetilde{M} \times X$ gives rise to a path of developing maps $D_t: \widetilde{M} \rightarrow X$ such that $D_t(\tilde{m})$ is the X -component of $\tilde{s}(t, \tilde{m})$. It follows that $D_t(\sigma \cdot \tilde{m}) = \rho_t(\sigma) \cdot D_t(\tilde{m})$ for all deck transformations $\sigma \in \pi_1(M)$ and D_t varies smoothly for $t \in [0, 1]$. Therefore, D_t is a smooth path of ρ_t -equivariant smooth developing maps, as claimed. \square

Proposition 5.4. *For every path-connected component C of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$, there is a unique principal $H_{\mathbb{R}}$ -bundle*

$$H_{\mathbb{R}}(M)_C \rightarrow M$$

whose pull back $H_{\mathbb{R}}(\widetilde{M})_C \rightarrow \widetilde{M}$ over \widetilde{M} satisfies the following property:

Given any smooth path of ρ_t -equivariant developing maps $D_t: \widetilde{M} \rightarrow X$ with respect to a smooth path of representations ρ_t on C , parametrized by $t \in [0, 1]$, there exists a path of $\pi_1(M)$ -equivariant isomorphisms of principal $H_{\mathbb{R}}$ -bundles

$$\phi_t: H_{\mathbb{R}}(\widetilde{M})_C \rightarrow H_{\mathbb{R}}(\widetilde{M}, D_t)$$

which is smooth in the sense that coordinate functions of the following composed maps

$$H_{\mathbb{R}}(\widetilde{M})_C \xrightarrow{\phi_t} H_{\mathbb{R}}(\widetilde{M}, D_t) \xrightarrow{\text{incl.}} \widetilde{M} \times G_{\mathbb{R}}.$$

vary smoothly with respect to t in any local coordinates in $\widetilde{M} \times G_{\mathbb{R}}$.

Proof. Given any smooth path of ρ_t -equivariant developing maps $D_t: \widetilde{M} \rightarrow X$ with respect to a smooth path of representations ρ_t , parametrized by $t \in [0, 1]$, the principal $H_{\mathbb{R}}$ -bundles $H_{\mathbb{R}}(\widetilde{M}, D_t)$ can be put together as a smooth principal $H_{\mathbb{R}}$ -bundle

$$H_{\mathbb{R}}(\widetilde{M}_{[0,1]}, D_{[0,1]}) \rightarrow \widetilde{M}_{[0,1]},$$

where $\widetilde{M}_{[0,1]}$ is the product $[0, 1] \times \widetilde{M}$. The bundle space $H_{\mathbb{R}}(\widetilde{M}_{[0,1]}, D_{[0,1]})$ consists of the points (t, \tilde{m}, g) of $[0, 1] \times \widetilde{M} \times G_{\mathbb{R}}$ with the property that $D_t(\tilde{m}) = g \cdot O_X$, and the bundle projection takes (t, \tilde{m}, g) to (t, \tilde{m}) . As $\pi_1(M)$ acts freely on $H_{\mathbb{R}}(\widetilde{M}_{[0,1]}, D_{[0,1]})$ by $\sigma \cdot (t, \tilde{m}, g) = (t, \sigma \cdot \tilde{m}, \rho_t(\sigma)g)$, which commutes with the bundle projection, there is an induced smooth principal $H_{\mathbb{R}}$ -bundle

$$H_{\mathbb{R}}(M_{[0,1]}, D_{[0,1]}) \rightarrow M_{[0,1]},$$

where $M_{[0,1]}$ is the product $[0, 1] \times M$. There exists a smooth isomorphism of principal $H_{\mathbb{R}}$ -bundles

$$\begin{array}{ccc} [0, 1] \times H_{\mathbb{R}}(M, D_0) & \xrightarrow{\Psi} & H_{\mathbb{R}}(M_{[0,1]}, D_{[0,1]}) \\ \downarrow & & \downarrow \\ [0, 1] \times M & \xrightarrow{\text{id}} & M_{[0,1]} \end{array}$$

which can be constructed as follows.

Observe that both $[0, 1] \times H_{\mathbb{R}}(M, D_0)$ and $H_{\mathbb{R}}(M_{[0,1]}, D_{[0,1]})$ restricts to be $H_{\mathbb{R}}(M, D_0)$ over the 0-slice M_0 of the base space. As M_0 is a deformation retract of $M_{[0,1]}$, there is a continuous isomorphism Ψ of principal $H_{\mathbb{R}}$ -bundles which fits into the claimed commutative diagram. We can modify Ψ to be smooth by the following argument: Cover M by compact disks U_1, \dots, U_N , such that each U_i is contained in an open regular neighborhood U'_i . The product of U_i and U'_i with $[0, 1]$ are denoted as $U_{i,[0,1]}$ and $U'_{i,[0,1]}$ accordingly. Then the smooth principal $H_{\mathbb{R}}$ -bundles $H_{\mathbb{R}}(\widetilde{M}_{[0,1]}, D_{[0,1]})$ and $[0, 1] \times H_{\mathbb{R}}(\widetilde{M}, D_0)$ can be trivialized over $U'_{i,[0,1]}$ by smooth local sections $s_{i,[0,1]}$ and $\text{id}_{[0,1]} \times s_{i,0}$ accordingly, such that the $H_{\mathbb{R}}$ -valued transition functions on overlaps are all smooth for both bundles. With the local trivializations given by the local sections, the continuous bundle-isomorphism Ψ is determined by its local functions $h_i: U_{i,[0,1]} \rightarrow H_{\mathbb{R}}$. Those functions are defined by the relation

$$\Psi((\text{id}_{[0,1]} \times s_{i,0})(t, \tilde{m})) = s_{i,[0,1]}(t, \tilde{m}) \cdot h_i(t, \tilde{m}),$$

where the right action of $h_i(t, \tilde{m})$ applies only to the $G_{\mathbb{R}}$ -component by right multiplication. Using the standard approximation techniques [30, Chapter 2], we can slightly perturb h_1 over $U'_{1,[0,1]}$ to make it smooth in an open neighborhood of U_1 . Modify Ψ accordingly over $U'_{1,[0,1]}$. Proceeding inductively, we can modify the next h_i over $U'_{i,[0,1]}$ without affecting the already modified Ψ on $U_1 \cup \dots \cup U_{i-1}$. After finitely many steps, we eventually obtain a modified bundle-isomorphism Ψ which is smooth.

In particular, we see that the restriction of Ψ to the t -slice $\{t\} \times H_{\mathbb{R}}(M, D_0)$ induces a smooth path of smooth isomorphisms of principal $H_{\mathbb{R}}$ -bundles over M :

$$\Psi_t: H_{\mathbb{R}}(M, D_0) \xrightarrow{\cong} H_{\mathbb{R}}(M, D_t).$$

After passing to covering spaces, it gives rise to a smooth path of smooth $\pi_1(M)$ -equivariant isomorphisms of principal $H_{\mathbb{R}}$ -bundles over \widetilde{M} :

$$\psi_t: H_{\mathbb{R}}(\widetilde{M}, D_0) \cong H_{\mathbb{R}}(\widetilde{M}, D_t).$$

To conclude the proof, it suffices to show that given any path-connected component C of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$, any representation ρ_C and a smooth ρ_C -equivariant developing map $D_C: \widetilde{M} \rightarrow X$, the isomorphism class of the principal $H_{\mathbb{R}}$ -bundle $H_{\mathbb{R}}(M, D_C)$ depends only on C . Taking this fact for granted for the moment, we can fix any such model as the claimed $H_{\mathbb{R}}(M)_C$. In fact, given any smooth path of ρ_t -equivariant developing maps $D_t: \widetilde{M} \rightarrow X$ with respect to a smooth path of representations ρ_t on C , parametrized by $t \in [0, 1]$, the claimed smooth path of $\pi_1(M)$ -equivariant isomorphisms ϕ_t can be taken as the following path of composed isomorphisms

$$H_{\mathbb{R}}(\widetilde{M})_C \xrightarrow{\cong} H_{\mathbb{R}}(\widetilde{M}, D_0) \xrightarrow{\psi_t} H_{\mathbb{R}}(\widetilde{M}, D_t),$$

where the first isomorphism does not depend on t .

To see that the isomorphism class of $H_{\mathbb{R}}(M, D_C)$ depends only on C , we argue as follows. For any fixed ρ_C of C , different choices of smooth ρ_C -equivariant developing maps can be joined by a smooth path of ρ_C -equivariant developing maps, for example, using the argument of Lemma 2.2 and the developing map interpretation of sections. So the isomorphism type of $H_{\mathbb{R}}(M, D_C)$ depends only on ρ_C . For any other choice ρ'_C of C , and any D'_C accordingly, there is a piecewise smooth path connecting ρ_C and ρ'_C . So by Lemma 5.3 applied to each smooth piece, and the induced isomorphism ψ_1 constructed above, we obtain an isomorphism between $H_{\mathbb{R}}(M, D_C)$ and $H_{\mathbb{R}}(M, D'_C)$, through a finite sequence of intermediate isomorphisms. Therefore, the isomorphism type of $H_{\mathbb{R}}(M, D_C)$ depends only on C . This completes the proof. \square

6. COHOMOLOGY OF LIE ALGEBRAS

Cohomology of Lie algebra supplies a useful device for computation of volume for representations. From the level of forms it can be derived through several standard operations associated with the exterior algebra. The constructions that we recall in this section apparently work for arbitrary ground fields, but we specialize our discussion to real Lie algebras.

6.1. Operations on forms. Let \mathfrak{g} be a real Lie algebra. The exterior algebra

$$A^*(\mathfrak{g}) = \bigoplus_k \wedge^k \mathfrak{g}^*$$

is a graded algebra over \mathbb{R} . We regard any exterior k -form over \mathfrak{g} as an antisymmetric \mathbb{R} -multilinear function of k variables in \mathfrak{g} and valued in \mathbb{R} . More generally, given any \mathfrak{g} -module V , namely, a \mathbb{R} -vector space with a Lie algebra homomorphism $\mathfrak{g} \rightarrow \text{End}(V)$, we can consider V -valued exterior k -forms over \mathfrak{g} . Such twisted forms constitute

$$A^*(\mathfrak{g}; V) = \bigoplus_k \text{Hom}(\wedge^k \mathfrak{g}, V),$$

which is a graded left $A^*(\mathfrak{g})$ -module with respect to the wedge product. Elements of $\text{Hom}(\wedge^k \mathfrak{g}, V)$ are regarded as antisymmetric \mathbb{R} -multilinear maps $f: \mathfrak{g} \times \cdots \times \mathfrak{g} \rightarrow V$.

On any $A^*(\mathfrak{g}; V)$ there are three natural operations, namely, the differential, Lie derivatives, and the interior product. The differential $\delta: A^k(\mathfrak{g}; V) \rightarrow A^{k+1}(\mathfrak{g}; V)$ is defined

by

$$\begin{aligned} (\delta f)(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i \cdot f(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) + \\ &\quad \sum_{i < j} (-1)^{i+j} f([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

The differential of $A^*(\mathfrak{g})$ is denoted particularly as d , and there the first summation is gone. Given any $X \in \mathfrak{g}$, the Lie derivative $L_X: A^k(\mathfrak{g}; V) \rightarrow A^k(\mathfrak{g}; V)$ is defined by

$$(L_X f)(X_1, \dots, X_k) = X \cdot f(X_1, \dots, X_k) + \sum_i f(X_1, \dots, [X_i, X], \dots, X_k),$$

and the interior product $i_X: A^k(\mathfrak{g}; V) \rightarrow A^{k-1}(\mathfrak{g}; V)$ is defined by

$$(i_X f)(X_1, \dots, X_{k-1}) = f(X, X_1, \dots, X_{k-1}).$$

Recall that L_X, i_X and δ are related by

$$(6.1) \quad L_X = \delta \circ i_X + i_X \circ \delta.$$

If we take a basis e_1, \dots, e_n of \mathfrak{g} and let e_1^*, \dots, e_n^* denote the dual basis of \mathfrak{g}^* , then the usual differential can be calculated by

$$(6.2) \quad 2d\omega = \sum_i e_i^* \wedge L_{e_i} \omega,$$

for $\omega \in A^*(\mathfrak{g})$, [5, Ch. I, Sec. 1.1, Formula (7)]. In particular, if ω is a 1-form, then

$$(6.3) \quad d\omega = \sum_{i < j} \omega([e_j, e_i]) e_i^* \wedge e_j^*.$$

Here and throughout this paper, we adopt the wedge product notation

$$e_{i_1}^* \wedge \dots \wedge e_{i_k}^* = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) e_{i_{\sigma(1)}}^* \otimes \dots \otimes e_{i_{\sigma(k)}}^*.$$

6.2. Relative cohomology with module coefficients. With the notations above, it can be checked that $(A^*(\mathfrak{g}; V), \delta)$ becomes a cochain complex over \mathbb{R} , and the induced cohomology is denoted by

$$H^*(\mathfrak{g}; V) = H^*(A^*(\mathfrak{g}; V)).$$

When V is trivially \mathbb{R} , the cohomology is usually called the cohomology of the Lie algebra \mathfrak{g} , denoted simply by $H^*(\mathfrak{g})$.

More generally, suppose that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . We denote by

$$A^*(\mathfrak{g}, \mathfrak{h}; V) \subset A^*(\mathfrak{g}; V)$$

the forms that are annihilated by both L_X and i_X for every $X \in \mathfrak{h}$. Note that $A^*(\mathfrak{g}, \mathfrak{h}; V)$ is closed under δ by the formula (6.1), so δ restricts to be a differential on $A^*(\mathfrak{g}, \mathfrak{h}; V)$. This allows us to define the relative cohomology:

$$H^*(\mathfrak{g}, \mathfrak{h}; V) = H^*(A^*(\mathfrak{g}, \mathfrak{h}; V)).$$

If $(\mathfrak{g}, \mathfrak{h})$ are the Lie algebras of a real Lie group G and a connected closed subgroup H , the relative cohomology $H^*(\mathfrak{g}, \mathfrak{h})$ can be identified with the cohomology of G -invariant differential forms of the homogeneous space $X = G/H$. Namely, denote by $A^*(X)^G$

the graded algebra of G -invariant differential forms of X , and by $H^*(A^*(X)^G)$ its corresponding cohomology. The evaluation at the origin yields an isomorphism $A^*(X)^G \rightarrow A^*(\mathfrak{g}, \mathfrak{h})$ that commutes with the differential, so it induces an identification

$$H^*(A^*(X)^G) \cong H^*(\mathfrak{g}, \mathfrak{h}),$$

see [5, Chapter 1, Section 1.6].

6.3. The coadjoint module. The dual vector space \mathfrak{g}^* can be endowed with a coadjoint representation

$$-\text{ad}^*: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$$

to become a \mathfrak{g} -module. The action $\mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is therefore defined by

$$(6.4) \quad (X, \omega) \mapsto L_X \omega = \omega([\cdot, X]).$$

Remark 6.1. Some authors adopt a different convention, using the coaction ad^* of \mathfrak{g} on \mathfrak{g}^* . The difference is that $-\text{ad}^*: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$ is a Lie algebra homomorphism which preserves the Lie bracket. In fact, the Jacobi identity implies that

$$(6.5) \quad \text{ad}^*([X, Y]) = \text{ad}^*(Y) \circ \text{ad}^*(X) - \text{ad}^*(X) \circ \text{ad}^*(Y).$$

For this reason, our formula of the differential δ is not the same as [13, (23.1), page 115], but the convention here agrees with [5, Chapter 1].

There exists a natural homomorphism of graded \mathbb{R} -modules of degree -1 :

$$J: A^*(\mathfrak{g}) \rightarrow A^{*-1}(\mathfrak{g}; \mathfrak{g}^*)$$

defined by the formula:

$$(6.6) \quad (J\omega)(X_1, \dots, X_{k-1})(X) = i_X(\omega(\cdot, X_1, \dots, X_{k-1})) = \omega(X, X_1, \dots, X_{k-1}),$$

for any $\omega \in A^k(\mathfrak{g})$.

Lemma 6.2. *The homomorphism J satisfies the following commutative relations:*

$$(6.7) \quad \delta \circ J = -J \circ d$$

$$(6.8) \quad i_X \circ J = -J \circ i_X$$

$$(6.9) \quad L_X \circ J = J \circ L_X$$

Proof. Assuming the first two relations, we can derive the third relation by:

$$L_X \circ J = (\delta \circ i_X + i_X \circ \delta) \circ J = J \circ (d \circ i_X + i_X \circ d) = J \circ L_X.$$

For any $\omega \in A^k(\mathfrak{g})$, the first two relations can be verified directly as follows. For any $X_1, \dots, X_k, Y \in \mathfrak{g}$, we compute:

$$\begin{aligned}
& ((\delta J\omega)(X_1, \dots, X_k))(Y) \\
&= \sum_i (-1)^{i+1} \left(-\text{ad}^*(X_i) \cdot ((J\omega)(X_1, \dots, \widehat{X}_i, \dots, X_k)) \right) (Y) + \\
&\quad \sum_{i < j} (-1)^{i+j} \left((J\omega)([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \right) (Y) \\
&= \sum_i (-1)^{i+1} \omega([Y, X_i], X_1, \dots, \widehat{X}_i, \dots, X_k) + \\
&\quad \sum_{i < j} (-1)^{i+j} \omega(Y, [X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \\
&= (-1) \times \sum_i (-1)^{1+(i+1)} \omega([Y, X_i], X_1, \dots, \widehat{X}_i, \dots, X_k) + \\
&\quad (-1) \times \sum_{i < j} (-1)^{(i+1)+(j+1)} \omega([X_i, X_j], Y, X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \\
&= -(\text{d}\omega)(Y, X_1, \dots, X_k) \\
&= -((J\text{d}\omega)(X_1, \dots, X_k))(Y).
\end{aligned}$$

This shows $\delta \circ J = -J \circ \text{d}$. For any $X, X_1, \dots, X_{k-1}, Y \in \mathfrak{g}$, we compute:

$$\begin{aligned}
((i_X J\omega)(X_1, \dots, X_{k-1}))(Y) &= ((J\omega)(X, X_1, \dots, X_{k-1}))(Y) \\
&= \omega(Y, X, X_1, \dots, X_{k-1}),
\end{aligned}$$

and

$$\begin{aligned}
((J i_X \omega)(X_1, \dots, X_{k-1}))(Y) &= (i_X \omega)(Y, X_1, \dots, X_{k-1}) \\
&= \omega(X, Y, X_1, \dots, X_{k-1}).
\end{aligned}$$

This shows $i_X \circ J = -J \circ i_X$. \square

As an immediate consequence, for any Lie subalgebra \mathfrak{h} of \mathfrak{g} , there is an induced homomorphism of differential graded \mathbb{R} -modules of degree -1 :

$$J: A^*(\mathfrak{g}, \mathfrak{h}) \rightarrow A^{*-1}(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}^*),$$

and an induced homomorphism of graded \mathbb{R} -modules of degree -1 :

$$J': H^*(\mathfrak{g}, \mathfrak{h}) \rightarrow H^{*-1}(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}^*).$$

7. VOLUME RIGIDITY OF REPRESENTATIONS

In this section, we prove that for the full central extension of a connected real semisimple Lie group with torsion-free center, the volume of representations for any given manifold is locally constant.

Theorem 7.1. *Let G be a connected real semisimple Lie group. Suppose that the center Z of G is torsion-free, and denote by $G_{\mathbb{R}}$ the full central extension of G . For any closed oriented smooth manifold M , the volume of representations*

$$\text{vol}_{G_{\mathbb{R}}}: \mathcal{R}(\pi_1(M), G_{\mathbb{R}}) \rightarrow \mathbb{R}$$

is constant on every path-connected component of the representation space $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$.

The rest of this section is devoted to the proof of Theorem 7.1. Fix a maximal compact subgroup H of G and denote by $X = G/H$ the associated contractible homogeneous G -space. We identify the proper action of $G_{\mathbb{R}}$ on X , following Proposition 5.2, so that the extended isotropy group is a maximal compact subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Fix a $G_{\mathbb{R}}$ -invariant volume form ω_X of X . Therefore, the volume of representations of this section are considered with respect to the triple

$$(G_{\mathbb{R}}, X, \omega_X)$$

in accordance with Notation 2.4. Suppose that M is a closed oriented manifold of the same dimension as X , as in other dimensions the volume is constantly zero by our convention.

Let C be any path-connected component of the representation space $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$. According to the description of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$ from Proposition 5.1, any pair of representations on C can be connected by a piecewise smooth path of representations. Therefore, it suffices to show that $\text{vol}_{G_{\mathbb{R}}}$ is constant restricted to any smooth path of representations on C .

7.1. Derivative of developing paths. Suppose that

$$\rho_t: \pi_1(M) \rightarrow G_{\mathbb{R}}$$

is a smooth path of representations on C , parametrized by $t \in [0, 1]$. By Lemma 5.3, there exists a smooth path of smooth ρ_t -equivariant developing maps

$$D_t: \widetilde{M} \rightarrow X.$$

It follows that there is an induced path of homomorphisms between differential graded \mathbb{R} -algebras:

$$D_t^{\sharp}: A^*(X)^{G_{\mathbb{R}}} \rightarrow A^*(\widetilde{M})^{\pi_1(M)},$$

namely, \mathbb{R} -linear homomorphisms which preserve the wedge product, the exterior differential, and the grading. The $G_{\mathbb{R}}$ -invariant differential forms on X are naturally identified with exterior forms on the tangent space $T_H X$ at the coset H of the homogeneous space $X = G/H$, where the isotropy group is $H_{\mathbb{R}}$. The $\pi_1(M)$ -invariant differential forms on \widetilde{M} are naturally identified with the pull-backs of the differential forms on M via the covering. Therefore, the induced path of homomorphisms between differential graded \mathbb{R} -algebras can be identified as

$$D_t^{\sharp}: A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \rightarrow A^*(M)$$

This path is smooth with respect to t in the sense that the coordinate functions of the image of D_t^{\sharp} are smooth in t for any local coordinates in M . The derivative of D_t^{\sharp} with respect to t gives rise to a smooth path of homomorphisms between differential graded \mathbb{R} -modules

$$\dot{D}_t^{\sharp}: A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \rightarrow A^*(M).$$

On any open subset U of M with local coordinates (u_1, \dots, u_m) ,

$$D_t^{\sharp}(\omega) = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k}^{\omega}(t, u_1, \dots, u_m) du_{i_1} \wedge \dots \wedge du_{i_k},$$

where the coefficient functions are linear in $\omega \in A^k(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$ and smooth in (t, u_1, \dots, u_m) , and

$$\dot{D}_t^{\sharp}(\omega) = \sum_{i_1 < \dots < i_k} \left. \frac{\partial f_{i_1, \dots, i_k}^{\omega}}{\partial t} \right|_{(t, u_1, \dots, u_m)} du_{i_1} \wedge \dots \wedge du_{i_k}.$$

It follows that there are induced homomorphisms of graded \mathbb{R} -algebras

$$D_t^*: H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \rightarrow H^*(M; \mathbb{R}),$$

and homomorphisms of graded \mathbb{R} -modules

$$\dot{D}_t^*: H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \rightarrow H^*(M; \mathbb{R}).$$

Lemma 7.2. *Given a class $[\omega] \in H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$, the smooth path of classes $D_t^*[\omega]$ is constant in $H^*(M; \mathbb{R})$ if and only if the derivative classes $\dot{D}_t^*[\omega]$ vanishes for all t .*

Proof. It is obvious from the local expression that derivation and integration with respect to t commute with the differentials of the differential graded \mathbb{R} -modules $A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$ and $A^*(M)$. As it can be checked completely locally, the conclusion is a simple fact of multi-variable calculus. \square

7.2. Factorization of derivatives.

Lemma 7.3. *Given any smooth path of representations $\rho_t \in \mathcal{R}(\pi_1(M), G_{\mathbb{R}})$, parametrized by $t \in [0, 1]$, and any smooth path of smooth ρ_t -equivariant developing maps $D_t: \widetilde{M} \rightarrow X$, there exists a factorization of \dot{D}_t^{\sharp} as a composition of homomorphisms, of degree -1 and $+1$, between differential graded \mathbb{R} -modules*

$$A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \xrightarrow{J} A^{*-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) \xrightarrow{\dot{F}_t} A^*(M).$$

Therefore, there exists a factorization of \dot{D}_t^* accordingly

$$H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \xrightarrow{J'} H^{*-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) \xrightarrow{\dot{F}'_t} H^*(M).$$

Proof. Denote by C the path-connected component of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$ which contains the path ρ_t . Denote by $H_{\mathbb{R}}(M, D_t) \rightarrow M$ the principal $H_{\mathbb{R}}$ -bundles which are the $H_{\mathbb{R}}$ -reductions induced by D_t , of the flat $G_{\mathbb{R}}$ -bundles $M \times_{\rho_t} X \rightarrow M$ with fiber X , (Subsection 5.3).

Fix a model of the $H_{\mathbb{R}}$ -reduction associated with C , namely, a principal $H_{\mathbb{R}}$ -bundle $H_{\mathbb{R}}(M)_C \rightarrow M$ as asserted by Proposition 5.4. With the pull-backs to \widetilde{M} denoted by $H_{\mathbb{R}}(\widetilde{M})_C$ and $H_{\mathbb{R}}(M, D_t)$ accordingly, there are smooth isomorphisms of $\pi_1(M)$ -equivariant principal $H_{\mathbb{R}}$ -bundles ϕ_t , as guaranteed by Proposition 5.4, which fit into the following sequence of smooth morphisms of $\pi_1(M)$ -equivariant principal $H_{\mathbb{R}}$ -bundles:

$$\begin{array}{ccccccc} H_{\mathbb{R}}(\widetilde{M})_C & \xrightarrow{\phi_t} & H_{\mathbb{R}}(\widetilde{M}, D_t) & \xrightarrow{\text{incl.}} & \widetilde{M} \times G_{\mathbb{R}} & \xrightarrow{\text{proj.}} & G_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{M} & \xrightarrow{\text{id}} & \widetilde{M} & \xrightarrow{\text{id} \times D_t} & \widetilde{M} \times X & \xrightarrow{\text{proj.}} & X \end{array}$$

(The action of $\pi_1(M)$ on any $G_{\mathbb{R}}$ or X factors are understood as by ρ_t .) The composition along the lower row of the commutative diagram is nothing but the developing maps D_t . We denote by

$$\Delta_t: H_{\mathbb{R}}(\widetilde{M})_C \rightarrow G_{\mathbb{R}}$$

the maps obtained by composition along the upper row, which are therefore ρ_t -equivariant, and vary smoothly as parametrized by $t \in [0, 1]$. The induced maps of invariant differential forms

$$\Delta_t^{\sharp}: A^*(G_{\mathbb{R}})^{G_{\mathbb{R}}} \rightarrow A^*(H_{\mathbb{R}}(\widetilde{M})_C)^{\pi_1(M)}$$

can be identified with the homomorphisms between differential graded \mathbb{R} -algebras

$$\Delta_t^{\sharp}: A^*(\mathfrak{g}_{\mathbb{R}}) \rightarrow A^*(H_{\mathbb{R}}(M)_C).$$

The derivative of Δ_t^\sharp with respect to t are homomorphisms between differential graded \mathbb{R} -modules

$$\dot{\Delta}_t^\sharp: A^*(\mathfrak{g}_\mathbb{R}) \rightarrow A^*(H_\mathbb{R}(M)_C).$$

Via the bundle projection of the principal $H_\mathbb{R}$ -bundles, the derivatives \dot{D}_t^\sharp can be identified with the restriction of $\dot{\Delta}_t^\sharp$, as seen by the commutative diagram of homomorphisms between differential graded \mathbb{R} -modules

$$\begin{array}{ccc} A^*(\mathfrak{g}_\mathbb{R}) & \xrightarrow{\dot{\Delta}_t^\sharp} & A^*(H_\mathbb{R}(M)_C) \\ \uparrow & & \uparrow \\ A^*(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}) & \xrightarrow{\dot{D}_t^\sharp} & A^*(M) \end{array}$$

where the vertical homomorphisms are inclusions, and forms of $A^*(M)$ are identified with forms of $A^*(H_\mathbb{R}(M)_C)$ that are $H_\mathbb{R}$ -horizontal and $H_\mathbb{R}$ -invariant.

Note that $H_\mathbb{R}(M)_C$ is a principal $H_\mathbb{R}$ -bundle. For any $X \in \mathfrak{h}_\mathbb{R}$, the flow on $H_\mathbb{R}(M)_C$ given by the right action of the 1-parameter subgroup $\exp(sX)$ gives rise to a vector field of $H_\mathbb{R}(M)_C$, still denoted as X for simplicity. Then the interior product i_X and Lie derivative L_X are operators on $A^*(H_\mathbb{R}(M)_C)$. Therefore, a form of $A^*(H_\mathbb{R}(M)_C)$ is $H_\mathbb{R}$ -horizontal if and only if it is annihilated by i_X for all $X \in \mathfrak{h}_\mathbb{R}$, and $H_\mathbb{R}$ -invariant if and only if it is annihilated by L_X .

There is a natural factorization of $\dot{\Delta}_t^\sharp$ into homomorphisms between differential graded \mathbb{R} -modules

$$A^*(\mathfrak{g}_\mathbb{R}) \xrightarrow{J} A^{*-1}(\mathfrak{g}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*) \xrightarrow{F_t} A^*(H_\mathbb{R}(M)_C),$$

which is just the Leibniz rule of derivatives. More precisely, the homomorphism J is the canonical homomorphism introduced in Subsection 6.3, (Formula 6.6). Identifying $A^{k-1}(\mathfrak{g}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*)$ with $(\wedge^{k-1} \mathfrak{g}_\mathbb{R}^*) \otimes \mathfrak{g}_\mathbb{R}^*$, the homomorphism F_t is defined by specifying

$$F_t(\xi \otimes \omega) = \dot{\Delta}_t^\sharp(\omega) \wedge \Delta_t^\sharp(\xi).$$

The factorization is readily checked by

$$\begin{aligned} \dot{\Delta}_t^\sharp(\omega_1 \wedge \cdots \wedge \omega_k) &= \sum_{i=1}^k \Delta_t^\sharp(\omega_1) \wedge \cdots \wedge \dot{\Delta}_t^\sharp(\omega_i) \wedge \cdots \wedge \Delta_t^\sharp(\omega_k) \\ &= \sum_{i=1}^k (-1)^{i+1} \dot{\Delta}_t^\sharp(\omega_i) \wedge \Delta_t^\sharp(\omega_1 \wedge \cdots \wedge \widehat{\omega}_i \wedge \cdots \wedge \omega_k) \\ &= F_t(J(\omega_1 \wedge \cdots \wedge \omega_k)). \end{aligned}$$

Furthermore, it is easy to verify that

$$d \circ F_t = -F_t \circ \delta,$$

and that for any $X \in \mathfrak{h}$,

$$i_X \circ F_t = -F_t \circ i_X,$$

and

$$L_X \circ F_t = F_t \circ L_X.$$

This means that F_t is indeed a homomorphism of differential graded \mathbb{R} -modules, and its restriction induces a homomorphism of differential graded \mathbb{R} -modules

$$F_t: A^{*-1}(\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}; \mathfrak{g}_\mathbb{R}^*) \rightarrow A^*(M).$$

The restriction of J is a homomorphism of differential graded \mathbb{R} -modules

$$J: A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \rightarrow A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*),$$

(Subsection 6.3). Passing to the restrictions, therefore, we obtain a factorization of the derivative \dot{D}_t^{\sharp} into homomorphism of differential graded \mathbb{R} -modules

$$A^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}) \xrightarrow{J} A^{*-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) \xrightarrow{F_t} A^*(M),$$

as claimed. The cohomological factorization is an immediate consequence of the factorization on the form level. This completes the proof. \square

7.3. A vanishing lemma.

Lemma 7.4.

$$H^{\dim(X)-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) = 0.$$

Proof. We compute the cohomology $H^1(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}})$ directly and derive the claimed vanishing by Poincaré duality. Here the coefficient $\mathfrak{g}_{\mathbb{R}}$ is considered as the $\mathfrak{g}_{\mathbb{R}}$ -module with the adjoint action.

First observe that

$$H^1(\mathfrak{g}, \mathfrak{h}) = 0.$$

In fact, for any $\omega \in A^1(\mathfrak{g}, \mathfrak{h})$ we have $(d\omega)(X, Y) = -\omega([X, Y])$. If $d\omega = 0$, the semisimplicity of \mathfrak{g} implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so $\omega = 0$. Using the decomposition of $\mathfrak{g}_{\mathbb{R}}$ into the center and the simple factors,

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{z}(\mathfrak{g}_{\mathbb{R}}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

we obtain an induced decomposition of $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_{\mathbb{R}})$ into the direct sum of $H^1(\mathfrak{g}, \mathfrak{h}) \otimes_{\mathbb{R}} \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$ and $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_i)$, which are all trivial by the above observation and Theorem 4.2. Therefore,

$$H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_{\mathbb{R}}) = 0.$$

The cohomology classes of $H^1(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}})$ are by definition represented by \mathbb{R} -linear maps

$$f: \mathfrak{g}_{\mathbb{R}} \rightarrow \mathfrak{g}_{\mathbb{R}}$$

with the property that

$$i_X(f) = f(X) = 0$$

for all $X \in \mathfrak{h}_{\mathbb{R}}$ and

$$(\delta f)(X, Y) = [X, f(Y)] - [Y, f(X)] - f([X, Y]) = 0$$

for all $X, Y \in \mathfrak{g}_{\mathbb{R}}$ and

$$L_X(f)(Y) = [X, f(Y)] - f([X, Y]) = 0$$

for all $X \in \mathfrak{h}_{\mathbb{R}}$ and $Y \in \mathfrak{g}_{\mathbb{R}}$. For any such f , the restriction of f to \mathfrak{g} is a 1-cocycle of $A^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_{\mathbb{R}})$.

When \mathfrak{h} is nontrivial, the restriction of f to \mathfrak{g} is 0 because $A^0(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_{\mathbb{R}})$ is trivial and $H^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}_{\mathbb{R}}) = 0$. Then f is trivial on $\mathfrak{g}_{\mathbb{R}}$ since it is also trivial on $\mathfrak{h}_{\mathbb{R}}$.

When \mathfrak{h} is trivial, the vanishing cohomology $H^1(\mathfrak{g}; \mathfrak{g}_{\mathbb{R}}) = 0$ implies that there exists some 0-cochain $U \in \mathfrak{g}_{\mathbb{R}}$, and

$$f(X) = [X, U]$$

for all $X \in \mathfrak{g}$. We argue that $f(X) = 0$ if X lies in the center $\mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$. In fact, the formula of δf implies that $[Y, f(X)] = 0$ for all $Y \in \mathfrak{g}_{\mathbb{R}}$ if $X \in \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$. So $f(X) \in \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$. On the other hand, recall that $\mathfrak{h}_{\mathbb{R}}$ is the diagonal of the subalgebra $\mathfrak{r} \oplus \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$, where \mathfrak{r} is the

Lie algebra of the subgroup R of G , according to the description of $H_{\mathbb{R}}$ from Proposition 5.2. It follows that for any $X \in \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$, there exists some $X' \in \mathfrak{r}$ such that $X + X' \in \mathfrak{h}_{\mathbb{R}}$. So we have $f(X) = -f(X') = -[X', U]$, which lies in \mathfrak{g} . It follows that $f(X)$ lies in $\mathfrak{z}(\mathfrak{g}_{\mathbb{R}}) \cap \mathfrak{g} = \{0\}$ whenever X lies in $\mathfrak{z}(\mathfrak{g}_{\mathbb{R}})$. This shows

$$f(X) = [X, U] = (\delta U)(X)$$

for all $X \in \mathfrak{g}_{\mathbb{R}}$. In other words, any 1-cocycle f is a coboundary δU of some 0-cochain U .

Therefore, $H^1(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}})$ always vanishes. By the Poincaré duality [5, Chapter I, Proposition 1.5],

$$H^{\dim(X)-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) \cong H^1(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}})^* = 0,$$

which completes the proof. \square

7.4. Proof of Theorem 7.1. We summarize the discussion so far and finish the proof of Theorem 7.1. Let G be a connected semisimple Lie group with torsion-free center Z . Adopt the notations $(G_{\mathbb{R}}, X, \omega_X)$ as before. For any closed oriented smooth manifold M of the dimension the same as X , and for any path-connected component C of $\mathcal{R}(\pi_1(M), G_{\mathbb{R}})$, it suffices to show that $\text{vol}_{G_{\mathbb{R}}}(M, \rho_t)$ is constant for every smooth path of representations on C , as we have argued at the beginning of this section.

Take a smooth path of ρ_t -equivariant developing maps $D_t: \widetilde{M} \rightarrow X$ as in Subsection 7.1. Adopting the notations there, we have

$$\text{vol}_{G_{\mathbb{R}}}(M, \rho_t) = \int_{\mathcal{F}} D_t^* \omega_X = \langle D_t^*[\omega_X], [M] \rangle.$$

Here the integral is over a fixed fundamental domain \mathcal{F} of \widetilde{M} , and the pairing is the canonical pairing between the top-dimensional cohomology and homology of M . The derivative homomorphism

$$\dot{D}_t^*: H^{\dim(X)}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathbb{R}) \rightarrow H^{\dim(X)}(M; \mathbb{R})$$

is constantly zero as it factors through $H^{\dim(X)-1}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}; \mathfrak{g}_{\mathbb{R}}^*) = 0$, by Lemmas 7.3 and 7.4. It follows that the classes $D_t^*[\omega_X]$ in $H^{\dim(X)}(M; \mathbb{R})$ are constant independent of t , (Lemma 7.2). Therefore, the volume $\text{vol}_{G_{\mathbb{R}}}(M, \rho_t)$ is constant as t varies.

This completes the proof of Theorem 7.1.

8. COCOMPACT COEXTENSION

In this section, we study the structure of connected real Lie groups that contain closed cocompact connected semisimple Lie subgroups.

Proposition 8.1. *Let G be a connected real Lie group. Suppose that G contains a closed cocompact connected semisimple Lie subgroup, then there exists an exact sequence of homomorphisms of Lie groups*

$$\{0\} \longrightarrow Z(G)_{\text{tor}} \longrightarrow G \longrightarrow \widehat{G}_{\mathbb{R}} \longrightarrow T \longrightarrow \{0\}$$

where $Z(G)_{\text{tor}}$ is the maximal compact central subgroup of G , T is a connected compact abelian Lie group, and $\widehat{G}_{\mathbb{R}}$ is the full central extension of a connected semisimple Lie group \widehat{G} with torsion-free center.

In fact, the Lie groups $\widehat{G}_{\mathbb{R}}$, and T are determined by G up to isomorphism, as the proof indicates. Note also that the maximal compact central subgroup $Z(G)_{\text{tor}}$ is a possibly disconnected closed normal subgroup of G . Before we prove Proposition 8.1, we point out a partial converse as the following:

Lemma 8.2. *Suppose that G is a real connected Lie group that fits into an exact sequence as of Proposition 8.1. If $Z(G)_{\text{tor}}$ is finite, then G contains a closed cocompact connected semisimple normal subgroup.*

Proof. Since \widehat{G} is a closed cocompact connected semisimple normal subgroup of $\widehat{G}_{\mathbb{R}}$, which necessarily lies in the image of G by the exact sequence, the identity component of the preimage of \widehat{G} in G yields a closed cocompact connected semisimple normal subgroup of G as claimed. \square

The rest of this section is devoted to the proof of Proposition 8.1. To this end, suppose that G_1 is a closed connected semisimple Lie subgroup of a real Lie group G , such that the coset space G/G_1 is compact.

We first show that G has a reductive Lie algebra, and that the commutator subgroup $[G, G]$ is a connected closed cocompact semisimple Lie subgroup of G . Then we derive the exact sequence by studying the universal covering $\widetilde{G} \rightarrow G$.

By the structure theory of Lie groups, the universal covering group \widetilde{G} of G is a Lie-group semidirect product of its maximal connected solvable subgroup with a maximal simply-connected semisimple subgroup, as induced by the Lie-algebra decomposition as a semidirect product, (see [32, Chapter XII, Theorem 1.2, and Chapter XI]). The kernel of the covering is a discrete central subgroup of \widetilde{G} [32, Chapter I, Exercise 1]. Therefore, the maximal normal solvable subgroup G_{sol} of G is closed, and possibly disconnected, and the quotient group \overline{G}_{ss} is a connected semisimple real Lie group with trivial center.

Denote by \overline{G}_1 the image of G_1 in \overline{G}_{ss} . The preimage of \overline{G}_1 in G is obviously the double-coset $G_{\text{sol}}G_1$, which forms a subgroup of G that contains G_{sol} . We have a commutative diagram of Lie-group homomorphisms

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & G_{\text{sol}} & \longrightarrow & G & \longrightarrow & \overline{G}_{\text{ss}} \longrightarrow \{1\} \\ & & \uparrow \text{id} & & \uparrow & & \uparrow \\ \{1\} & \longrightarrow & G_{\text{sol}} & \longrightarrow & G_{\text{sol}}G_1 & \longrightarrow & \overline{G}_1 \longrightarrow \{1\} \end{array}$$

where the horizontal arrows have closed image and the rows are short exact sequences. The following lemma implies that vertical arrows are all closed maps as well.

Lemma 8.3. *The subgroup \overline{G}_1 of \overline{G}_{ss} is closed and cocompact.*

Proof. The intersection of $G_1 \cap G_{\text{sol}}$ is a closed normal solvable subgroup of G_1 . It is necessarily discrete and central in G_1 , because the Lie algebra of that subgroup has to be the trivial ideal by the semisimplicity of G_1 . It follows that \overline{G}_1 is a connected (analytic) semisimple subgroup of \overline{G}_{ss} . Since \overline{G}_{ss} is semisimple with trivial center, it is a closed real algebraic linear group (Theorem 4.3). In this case, it is known that \overline{G}_1 must be a closed subgroup of \overline{G}_{ss} , [28, Chapter II, Exercise and Further Results D.4]. \square

Lemma 8.4. *The maximal solvable normal subgroup G_{sol} of G is abelian. In other words, the Lie algebra of G is reductive.*

Proof. By Lemma 8.3, the subgroup $G_{\text{sol}}G_1$, which equals the preimage of \overline{G}_1 in G , is a closed subgroup of G . By the assumption, G_1 is a closed subgroup of G , so the projection of G onto the coset space G/G_1 is an analytic open mapping, [28, Chapter II Section 4]. It follows that the image $(G_{\text{sol}}G_1)/G_1$ is closed in G/G_1 . By the assumption, G/G_1 is compact, so $(G_{\text{sol}}G_1)/G_1$ is a compact (possibly disconnected) manifold without boundary. The component of $(G_{\text{sol}}G_1)/G_1$ that contains the identity coset G_1 can be identified with the coset space $G_{\text{sol}}^\circ/(G_{\text{sol}}^\circ \cap G_1)$, where G_{sol}° the identity component of G_{sol} .

Observe that $G_{\text{sol}}^\circ \cap G_1$ is virtually central in G_{sol}° , or in other words, a finite-index subgroup of $G_{\text{sol}}^\circ \cap G_1$ is central in G_{sol}° . In fact, the adjoint action induces a linear representation $\alpha: G_1 \rightarrow \text{Aut}(\mathfrak{g}_{\text{sol}})$, and any element of $G_{\text{sol}}^\circ \cap G_1$ is central in G_{sol}° if and only if it lies in the kernel of α . However, as G_1 is semisimple, the kernel of α meets the center of G_1 in a finite-index subgroup of that center, by [32, Chapter XVII Theorem 3.3 and Theorem 2.1]. Since $G_{\text{sol}}^\circ \cap G_1$ is central in G_1 , some finite-index subgroup of $G_{\text{sol}}^\circ \cap G_1$ is central in G_{sol}° .

Therefore, $G_{\text{sol}}^\circ \cap G_1$ is a cocompact discrete virtually central subgroup of G_{sol}° , and the coset space $G_{\text{sol}}^\circ/(G_{\text{sol}}^\circ \cap G_1)$ is a quotient Lie group. The universal cover G_{sol}° is a simply-connected solvable Lie group, which is in particular contractible. It follows from the structure of compact Lie groups that $G_{\text{sol}}^\circ/(G_{\text{sol}}^\circ \cap G_1)$ is abelian. In particular, the Lie algebra of the maximal solvable normal subgroup G_{sol} of G is abelian, so the Lie algebra of G is reductive. \square

By Lemma 8.4, the commutator subgroup of G is the semisimple part of G , namely, it is the maximal semisimple analytic subgroup of G . We denote

$$G_{\text{ss}} = [G, G].$$

Note that G_{ss} covers \overline{G}_{ss} under the projection of G . Denote by K the maximal compact connected normal subgroup of G_{ss} , and by \overline{K} its projected image in \overline{G}_{ss} . We also observe that G_1 is contained in G_{ss} , since the corresponding Lie algebras satisfy

$$\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1] \leq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_{\text{ss}}.$$

The double coset KG_1 forms a subgroup of G as K is normal.

Lemma 8.5. *The commutator subgroup G_{ss} of G equals KG_1 . Therefore, G_{ss} is closed and cocompact in G .*

Proof. We observe that the subgroup KG_1 is closed and cocompact in G . In fact, since G_1 is closed and K is compact, $(KG_1)/G_1$ is compact in the coset space G/G_1 , so KG_1 is closed in G . Since G_1 is cocompact in G , the subgroup KG_1 is cocompact as well.

Therefore, it suffices to show that the Lie algebra of KG_1 equals the Lie algebra of G_{ss} , or equivalently, that $\overline{KG_1}$ equals \overline{G}_{ss} .

Because \overline{G}_1 is closed in \overline{G}_{ss} (Lemma 8.3), and \overline{K} is compact in \overline{G}_{ss} , the coset space $(\overline{KG_1})/\overline{G}_1$ is compact in $\overline{G}_{\text{ss}}/\overline{G}_1$. It follows that $\overline{KG_1}$ is a closed and cocompact subgroup of \overline{G}_{ss} . Since \overline{K} is normal and compact, it suffices to show that the quotient Lie group $(\overline{KG_1})/\overline{K}$ equals $\overline{G}_{\text{ss}}/\overline{K}$.

In other words, possibly after passing to the quotient, we may assume that the semisimple Lie group G_{ss} contains no compact normal subgroups. We must show under this assumption that any connected cocompact closed semisimple subgroup \overline{G}_1 equals \overline{G}_{ss} . This is a consequence of Borel's Density Theorem. In fact, consider the adjoint representation $\text{Ad}: \overline{G}_{\text{ss}} \rightarrow \text{Aut}(\mathfrak{g}_{\text{ss}})$. Since the subspace \mathfrak{g}_1 is invariant under the action of \overline{G}_1 , Borel's

Density Theorem [39, Theorem 5.28] implies that \mathfrak{g}_1 is invariant under the action of \overline{G}_{ss} . This means that \mathfrak{g}_1 is an ideal of \mathfrak{g}_{ss} , or equivalently, \overline{G}_1 is a normal subgroup of \overline{G}_{ss} . The quotient Lie group $\overline{G}_{\text{ss}}/\overline{G}_1$ has to be trivial, since otherwise it would be a semisimple Lie group of noncompact type, which is impossible provided that \overline{G}_1 is cocompact in \overline{G}_{ss} . This shows that \overline{G}_1 equals \overline{G}_{ss} , which completes the proof. \square

Therefore, we see that G has a reductive Lie algebra, and its commutator subgroup is closed and cocompact.

Lemma 8.6. *The quotient Lie group $G/Z(G)_{\text{tor}}$ has a reductive Lie algebra, and its commutator subgroup is closed and compact. Furthermore, the center of $G/Z(G)_{\text{tor}}$ is torsion-free.*

Proof. The center of $G/Z(G)_{\text{tor}}$ is obviously torsion-free, otherwise $Z(G)_{\text{tor}}$ would not be the maximal compact central subgroup. The Lie algebra of $G/Z(G)_{\text{tor}}$ is reductive as the quotient only factors out some abelian ideal of the Lie algebra of G . The preimage of the commutator subgroup of $G/Z(G)_{\text{tor}}$ in G is the double coset $Z(G)_{\text{tor}}G_{\text{ss}}$. Note that $Z(G)_{\text{tor}}G_{\text{ss}}$ forms a subgroup of G since $Z(G)_{\text{tor}}$ is normal. Moreover, it is the preimage of the image of $Z(G)_{\text{tor}}$ in the coset space G/G_{ss} , so the cocompactness of G_{ss} implies that $Z(G)_{\text{tor}}G_{\text{ss}}$ is closed in G . It is cocompact in G since G_{ss} is already cocompact. Therefore, the image of $(Z(G)_{\text{tor}}G_{\text{ss}})/Z(G)_{\text{tor}}$ in the quotient group $G/Z(G)_{\text{tor}}$ is a closed and cocompact subgroup as asserted. \square

With Lemma 8.6, we may argue for $G/Z(G)_{\text{tor}}$ instead of G , so the claimed exact sequence can be obtained by the following lemma.

Lemma 8.7. *Suppose in addition that the center of G is torsion-free. Then there exists a homomorphism of Lie groups*

$$G \rightarrow \widehat{G}_{\mathbb{R}}$$

where $\widehat{G}_{\mathbb{R}}$ is the full central extension of a connected real semisimple Lie group \widehat{G} with torsion-free center. Moreover, the homomorphism is injective and the image is a closed and cocompact normal subgroup of $\widehat{G}_{\mathbb{R}}$ that contains \widehat{G} .

Proof. Denote by \widetilde{G} the universal covering Lie group of G , and by Λ the kernel of the covering projection, which is a discrete, central subgroup of \widetilde{G} . Denote by G' the commutator subgroup of G , which is closed and cocompact in G . Since the Lie algebra of G is reductive, \widetilde{G} is the Lie-group direct product of the identity component of the center $Z^\circ(\widetilde{G})$ and the universal covering Lie group \widetilde{G}' of G' . Note that \widetilde{G}' can be identified with the commutator subgroup of \widetilde{G} , and it is the maximal connected semisimple subgroup of \widetilde{G} , which is a direct product of simply-connected simple Lie subgroups.

By the assumption that G has no central torsion, the identity component of the center $Z^\circ(\widetilde{G})$ is isomorphic to $\mathbb{R}^{\dim(Z^\circ(\widetilde{G}))}$. Moreover, Λ contains the torsion subgroup $Z(\widetilde{G}')_{\text{tor}}$ of the center $Z(\widetilde{G}')$ of \widetilde{G}' . Note that $Z(\widetilde{G}')$ is a finitely generated abelian group. We take

$$\widetilde{G}'_{\text{free}} = \widetilde{G}'/Z(\widetilde{G}')_{\text{tor}},$$

namely, the characteristic free abelian covering group of G' . Note that the center $Z(\widetilde{G}'_{\text{free}})$ is discrete and torsion-free. It follows from the above description that the quotient of \widetilde{G} by $Z(\widetilde{G}')_{\text{tor}}$ is a direct product $\widetilde{G}'_{\text{free}} \times Z^\circ(\widetilde{G})$.

The kernel of the covering projection $\widetilde{G}'_{\text{free}} \times Z^\circ(\widetilde{G}) \rightarrow G$ is the free central subgroup,

$$\Lambda_{\text{free}} \subset Z(\widetilde{G}'_{\text{free}}) \times Z^\circ(\widetilde{G}),$$

namely, the quotient group $\Lambda/Z(\tilde{G}')_{\text{tor}}$. Again by the assumption that G has no central torsion, the quotient of $Z(\tilde{G}'_{\text{free}}) \times Z^\circ(\tilde{G})$ by Λ_{free} is torsion-free abelian. This implies that the projection of Λ_{free} to $Z(\tilde{G}'_{\text{free}})$ is a direct summand of $Z(\tilde{G}'_{\text{free}})$. The assumption that G' is closed and cocompact in G implies that the projection of Λ_{free} to $Z^\circ(\tilde{G})$ is discrete and cocompact in $Z^\circ(\tilde{G})$, or in other words, the projection is a lattice of $Z^\circ(\tilde{G})$. This implies a direct-sum decomposition

$$Z(\tilde{G}'_{\text{free}}) = A \oplus B \oplus C$$

such that $A \oplus B$ is isomorphic with Λ_{free} via the projection $\Lambda_{\text{free}} \rightarrow Z(\tilde{G}'_{\text{free}})$ and that B is identified with the kernel of the projection $\Lambda_{\text{free}} \rightarrow Z^\circ(\tilde{G})$. Note that B is contained in Λ_{free} , and the rank of A equals $\dim(Z^\circ(\tilde{G}))$. We take a connected semisimple Lie group

$$\hat{G} = \tilde{G}'_{\text{free}}/B,$$

which covers G' and has torsion-free center isomorphically projected to $A \oplus C$. The above description shows that there exists an isomorphism of Lie groups

$$G \cong \hat{G} \times_A (A \otimes \mathbb{R}).$$

The right-hand side can be canonically embedded into $\hat{G} \times_{A \oplus C} ((A \oplus C) \otimes \mathbb{R})$, which is the full central extension $\hat{G}_{\mathbb{R}}$ of \hat{G} . It is clear that this embedding has closed and cocompact image. It is normal because it already contains \hat{G} . Therefore, the Lie group \hat{G} is as desired. \square

By Lemmas 8.6 and 8.7, we obtain an exact sequence

$$\{1\} \longrightarrow G/Z(G)_{\text{tor}} \longrightarrow \hat{G}_{\mathbb{R}} \longrightarrow T \longrightarrow \{0\},$$

where T is the connected compact abelian Lie group which is the cokernel of the homomorphism in the middle. Combining with the canonical short exact sequence

$$\{0\} \longrightarrow Z(G)_{\text{tor}} \longrightarrow G \longrightarrow G/Z(G)_{\text{tor}} \longrightarrow \{1\},$$

we obtain the exact sequence as claimed. This completes the proof of Proposition 8.1.

9. VOLUME FINITENESS AND NONTRIVIALITY

In this section, we prove Theorem 1.1, namely, the finiteness and nontriviality of the volume function for real connected Lie groups that contain a closed cocompact semisimple subgroup.

Proof of Theorem 1.1. Since G contains a closed and cocompact semisimple Lie subgroup G_1 we can take a torsion-free uniform lattice Γ of G_1 , by Borel's theorem on the existence of uniform lattices (Theorem 4.5). Then Γ is also uniform in G . Take M to be the orbit space of X quotiented by Γ . Note that M is aspherical as it is covered by a contractible space X . The inclusion $\rho_0: \Gamma \rightarrow G$ is a discrete faithful representation of $\pi_1(M) \cong \Gamma$, and

$$\text{vol}_G(M, \rho_0) = \int_M \omega_X > 0.$$

Therefore, the volume function vol_G is nontrivial for some closed oriented smooth manifold M .

It remains to show that vol_G takes only finitely many values for any M . By the formula for the volume of induced representations (Proposition 3.1 (2)) and the characterization of G (Proposition 8.1), it suffices to show that $\text{vol}_{G_{\mathbb{R}}}$ takes only finitely many values for the

full central extension $G_{\mathbb{R}}$ of any real connected semisimple Lie group G with torsion-free center.

In this case, by Proposition 5.1, there are at most finitely many path-connected components of the space of representations $\mathcal{R}(\pi_1(M); G_{\mathbb{R}})$. By Theorem 7.1, the volume function is constant on every path-connected component. Therefore, the volume function $\text{vol}_{G_{\mathbb{R}}}$ takes only finitely many values. This completes the proof. \square

10. CONCLUSIONS

In conclusion, for any connected Lie group that contains a closed cocompact semisimple subgroup, the representation volume of that group is a finite nontrivial homotopy-type invariant for closed orientable smooth manifolds. In this case, the representation volume is virtually essentially the representation volume associated to the semisimple subgroup of the considered Lie group. For semisimple Lie groups with infinite center, the associated representation volumes are useful invariants to detect Property D for certain manifolds with vanishing simplicial volume.

In the following we propose a few further questions.

Question 1. In every dimension $n \geq 4$, is there an orientable closed aspherical smooth n -manifold M with Property D, whereas the representation volume $V(M', G)$ vanishes for every possible Lie group G and every finite cover M' of M ? Here we require that G defines a finite representation volume $V(-, G)$ in the dimension n .

For dimension 3, any orientable closed 3-manifold with property D has a finite cover with nonvanishing Seifert volume, see [17]. In dimension 4, the similar question for the simplicial volume is open [22]. On the other hand, surface bundles over surfaces are known to have the simplicial volume $\|X\|$ at least $6\chi(X) = 6\chi(F)\chi(B) > 0$, provided that $\chi(F) < 0$ and $\chi(B) < 0$ for the fiber F and the base B , [34] and [9]. However, very little is known about their virtual representations.

Question 2. Are there simply-connected closed manifolds of any dimension that have Property D?

Highly connected manifolds tend to fail Property D. For example, it is known that any $(n-1)$ -connected $2n$ -manifold M admits self-maps of nonzero degree, [21]. In particular, admissible self-mapping degrees of any simply-connected 4-manifold are all the perfect squares $0, 1, 4, 9, 16, \dots$. For simply-connected manifolds, representation volumes are by definition zero so they provide no obstruction.

Question 3. What is the largest class of connected Lie groups that define nontrivial representation volumes?

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