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REPRESENTATION OF SIGNALS AS SERIES OF ORTHOGONAL FUNCTIONS

E Aristidi¹

Abstract. This paper gives an introduction to the theory of orthogonal projection of functions or signals. Several kinds of decomposition are explored: Fourier, Fourier-Legendre, Fourier-Bessel series for 1D signals, and Spherical Harmonic series for 2D signals. We show how physical conditions and/or geometry can guide the choice of the base of functions for the decomposition. The paper is illustrated with several numerical examples.

1 Introduction

Fourier analysis is one of the most important and widely used techniques of signal processing. Decomposing a signal as sum of sinusoids, probing the energy contained by a given Fourier coefficient at a given frequency, performing operations such as linear filtering in the Fourier plane: all these techniques are in the toolbox of any engineer or scientist dealing with signals.

However the Fourier series is only a particular case of decomposition, and there exist an infinity of Fourier-like expansions over families of functions. The scope of this course is to make an introduction to mathematical concepts underlying these ideas. We shall talk about Hilbert spaces, scalar (or inner) products, orthogonal functions and orthogonal basis. We shall also see that some kind of signals are well-suited to Fourier decomposition, others will be more efficiently represented as series of Legendre polynomials or Bessel functions. Note that this course is not a rigorous mathematical description of the theory of orthogonal decomposition. Readers who wish a more academic presentation may refer to textbooks in the reference list.

The paper is organized as follows. Basics of Fourier series are introduced in Sect. 2. A parallel is made between Fourier series and decomposition of a vector as linear combination of base vectors. In Sect. 3 we generalise the concept to Fourier-Legendre series and present Legendre polynomials as a particular case of the family

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of orthogonal polynomials. Sect. 4 is devoted to the expansion of two-dimensional functions of angular spherical coordinates as series of Spherical Harmonics. Finally Sect. 5 presents another example of decomposition using sets of Bessel functions.

2 Fourier series

First ideas about series expansions arise at the beginning of the 19th century. Pioneering work by Joseph Fourier about the heat propagation (Fourier, 1822) played a fundamental role in the development of mathematical analysis. Heat propagation is described by a second-order partial differential equation (PDE). To integrate this equation, Fourier proposed to represent solutions as trigonometric series (denominated today as “Fourier series”). He laid the foundations of the so-called Fourier analysis, which is now extensively used in a wide range of physical and mathematical topics. A modern and comprehensive presentation of Fourier series can be found in Tolstov (1976).

2.1 Fourier expansion of a periodic signal

The basic idea is that a periodic signal $f(t)$ can be approached by a sum of trigonometric functions. The complex form of the expansion is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi nt}{T}\right) \quad (2.1)$$

where T is the period of the signal. Discussions about the validity of the development can be found in Tolstov (1976). The functions $\phi_n(t) = \exp\left(\frac{2i\pi nt}{T}\right)$ are the *harmonic* components of frequency $\frac{n}{T}$ present in the signal, and the complex coefficient c_n is a weight. This relation suggests that the ensemble of the coefficients $\{c_n\}$ and the period T contain the same information than the signal itself. High frequency harmonics (large n) are generally associated to short-scale variations of the function f (i.e. small details).

Values of the coefficients c_n may be found using the following integral formula (indeed a relation of *orthogonality* of the harmonics, as it will be discussed later)

$$\int_0^T \frac{1}{T} \phi_n(t) \overline{\phi_m(t)} dt = \delta_{mn} \quad (2.2)$$

with δ_{mn} the Kronecker delta. The notation $\bar{\phi}$ means the complex conjugate of ϕ . Combining this formula with Eq. 2.1 gives

$$c_n = \int_0^T \frac{1}{T} f(t) \overline{\phi_n(t)} dt \quad (2.3)$$

2.2 Example

We consider a square wave of period $T = 1$ having value $f(t) = 1$ if $|t| < \frac{1}{4}$ and 0 elsewhere in the interval $[-\frac{1}{2}, \frac{1}{2}]$ (Fig. 1a). The Fourier coefficients c_n calculated from Eq. 2.3 are

$$c_n = \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) \quad (2.4)$$

with $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ the normalized sinc function. The signal $f(t)$ can thus be approached by the sum

$$S_N(t) = \sum_{n=-N}^N \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{2i\pi nt} \quad (2.5)$$

$S_N(t)$ is a partial Fourier series which tends towards $f(t)$ as $N \rightarrow \infty$. Fig 1a and 1b show, for $N = 1$ to 10, the real part of the term $c_n e^{2i\pi nt}$ and the partial sum $S_N(t)$. For $N = 100$ (Fig 1c), the convergence seems fairly good, excepted at the points of discontinuity where it presents oscillations (known as the *Gibbs phenomenon*). These oscillations will vanish as N increases for any point, except the discontinuity itself. Analysis of the Gibbs phenomenon was given by Bôcher (1906).

A plot of Fourier coefficients c_n versus n is displayed in Fig 1d. As it was said, the information contained in this graph and in the signal are the same (c_n is real here, and has no imaginary part). For this example it can be seen that c_n decreases at large n , which is quite common for signals in physics. The decay is low because of discontinuities, but would be faster for smooth signals.

2.3 Fourier series as a scalar product

Representation of functions as sum of harmonics has strong analogies with the decomposition of a vector over an orthogonal basis. An historical introduction of these ideas can be found in Birkhoff & Kreysgig (1984) and references therein. In this section we will pick up some analogies between Fourier series and representations of vectors in the 3-dimensionnal space \mathbb{C}^3 (vectors with complex components).

2.3.1 Scalar product and norm

Vectors Consider two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in \mathbb{C}^3 where u_k and v_k are complex numbers. Their scalar product (indeed *hermitian scalar product* for complex vectors) is the complex number

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^3 u_i \bar{v}_i \quad (2.6)$$

The norm of the vector \mathbf{u} is

$$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} \quad (2.7)$$

Minimum conditions to fulfill for a (hermitian) scalar product is to be

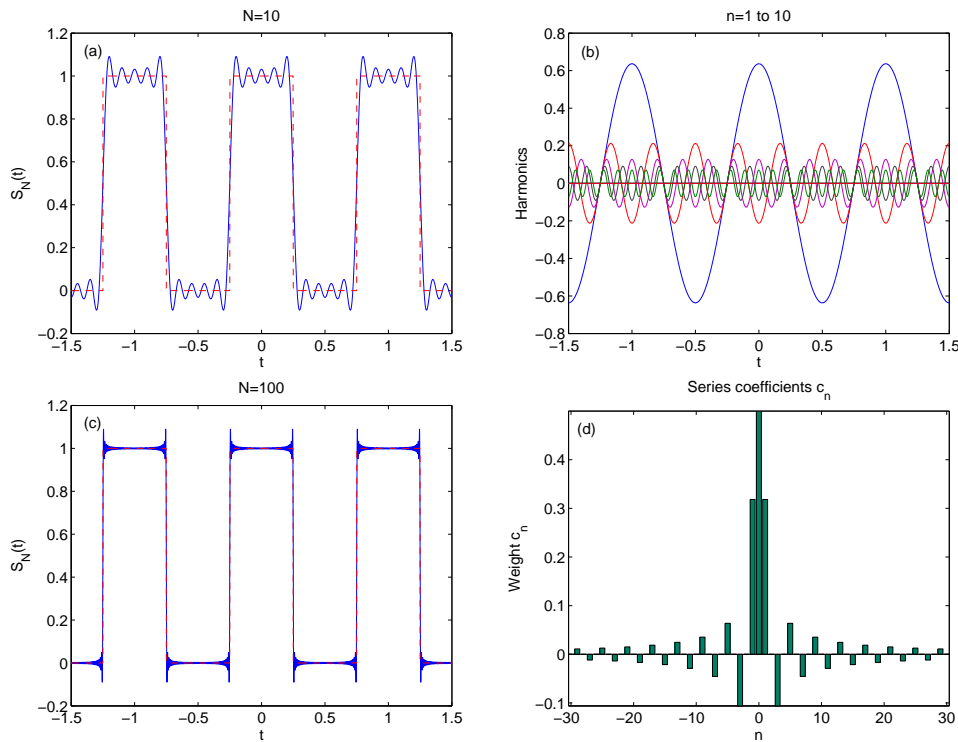


Fig. 1. Fourier series of a square wave of period $T = 1$. (a) Signal (red dashed line) and partial Fourier series with $N = 10$ (blue line). (b) Real part of the terms $c_n \exp^{2i\pi n t}$ (weighted harmonics) for $n = 1$ to 10. (c) Partial Fourier series with $N = 100$. (d) Graph of c_n versus n (all even terms vanish except $n = 0$).

- bilinear, in the sense of $(\lambda \mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v})$, and $\mathbf{u} \cdot (\lambda \mathbf{v}) = \bar{\lambda}(\mathbf{u} \cdot \mathbf{v})$ (linear in u , antilinear in v). λ is a constant.
- conjugate symmetric: $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$
- definite positive: the norm of a vector is always positive. If it is zero, then the vector itself is zero.

Functions For two periodic functions f and g with period T , the following integral

$$\langle f, g \rangle = \int_0^T \frac{1}{T} f(t) \overline{g(t)} dt \quad (2.8)$$

is has the 3 properties defined above. It is the generalization for functions of the concept of scalar product, it is generally denoted as the inner product of the

functions f and g . These functions are treated as vectors in a space of functions (Hilbert space).

The associated norm is

$$\langle f, f \rangle = \int_0^T \frac{1}{T} |f(t)|^2 dt \quad (2.9)$$

This definition for the scalar (inner) product is well suited to Fourier series. As we will see later, the definition is likely to change depending on the kind of decomposition, in particular the weighing term $\frac{1}{T}$ inside the integral.

2.3.2 Orthogonality

Two vectors \mathbf{u} and \mathbf{v} are orthogonal if their scalar product $\mathbf{u} \cdot \mathbf{v}$ is zero. Similarly, two functions f and g are said to be orthogonal if their scalar (inner) product $\langle f, g \rangle = 0$.

2.3.3 Orthonormal base

Vectors For 3 dimensional vectors of \mathbb{C}^3 , an orthonormal base is composed of 3 unit vectors $(\hat{x}, \hat{y}, \hat{z})$ verifying

$$\begin{cases} \hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0 \\ \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \end{cases} \quad (2.10)$$

The number of *base vectors* required to construct an complete orthonormal base is the dimension of the vector space, here 3.

Functions This definition can be extended to periodic functions. Eq. 2.2 is indeed the scalar product of the harmonics ϕ_n and ϕ_m and may be rewritten as

$$\langle \phi_n, \phi_m \rangle = \delta_{mn} \quad (2.11)$$

and suggests that the harmonics $\phi_n(t) = \exp\left(\frac{2i\pi nt}{T}\right)$ form an orthonormal base of the space of periodic functions of period T . Like the number of base functions, the dimension of this space is infinite. The number of base functions is infinite, the base is said to be complete if every function of this space can be written as a linear combination on ϕ_n .

2.3.4 Orthogonal decomposition

Vectors Any vector \mathbf{u} of \mathbb{C}^3 can be written as a weighted sum of the base vectors,

$$\mathbf{u} = u_1 \hat{x} + u_2 \hat{y} + u_3 \hat{z} \quad (2.12)$$

where the numbers u_k are given by the scalar products

$$u_1 = \mathbf{u} \cdot \hat{x} \quad u_2 = \mathbf{u} \cdot \hat{y} \quad u_3 = \mathbf{u} \cdot \hat{z} \quad (2.13)$$

i.e. the *projection* of \mathbf{u} on the base vectors.

Functions Analogy to periodic functions is straightforward : the definition of the Fourier series of Eq. 2.1 is indeed a decomposition of the function f on the base vectors ϕ_n :

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \phi_n(t) \quad (2.14)$$

where c_n is given by the integral relation of Eq. 2.3 which is exactly the scalar product between f and the base function ϕ_n .

2.4 Fourier series and differential equations

Fourier series were introduced to solve the differential equation (DE) of heat propagation. It is a PDE whose one dimensionnal form is

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (2.15)$$

The solution $u(x, t)$ is a function of the time t and a space coordinate x . α is a positive real number. A classical technique to solve this kind of equation is to look for solutions of the form $u(x, t) = X(x).T(t)$ where the dependence on x and t is separated. Comprehensive presentations of this method of separation of variables can be found in Mathews and Walker (1970) and in section 2 of Jackson (1998). Substituting u back into equation 2.15 one finds

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha T(t)} \quad (2.16)$$

Since both sides of the equation depend on different variables, they must then be equal to some constant $-\lambda$. The equation for X becomes

$$X'' + \lambda X = 0 \quad (2.17)$$

If λ is positive then Eq. 2.17 is the harmonic equation. Base solutions are $X(x) = e^{\pm ikx}$ where $k = \sqrt{\lambda}$. k is a constant generally determined by boundary conditions. It takes often the form $k = 2\pi \frac{n}{a}$ where n is an integer and a a characteristic length (if one studies the heat propagation inside a bar, then a is the length of the bar). Hence the general solution of Eq. 2.17 is the linear combination of the base solutions for any n :

$$X(x) = \sum_{n=-\infty}^{\infty} c_n e^{2i\pi n \frac{x}{a}} \quad (2.18)$$

which is exactly a Fourier expansion of the function $X(x)$.

3 Legendre polynomials

3.1 Introduction

Legendre polynomials appear for the first time in the work of Legendre (1784) in relation to problems of celestial mechanics. He proposed series expansion of the

Newtonian potential. Consider a punctual mass at position \mathbf{r}' , the potential at position \mathbf{R} is

$$\Phi(\mathbf{R}) \propto \frac{1}{|\mathbf{R} - \mathbf{r}'|} = \frac{1}{R} [1 - 2r \cos \theta + r^2]^{-\frac{1}{2}} \quad (3.1)$$

where θ is the angle between the vectors \mathbf{R} and \mathbf{r}' and $r = \frac{r'}{R}$. A Taylor expansion in r , valid for $r < 1$, is

$$[1 - 2r \cos \theta + r^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} r^n P_n(\cos \theta) \quad (3.2)$$

The coefficients $P_n(\cos \theta)$ are polynoms of degree n , known today as Legendre polynomials. Eq. 3.2 allow for example to write the potential created by any mass (or charge) distribution as a series solution involving integrals of the mass (or charge) density function (multipole expansion). See Jackson, (1998) for a more complete presentation. The function

$$g(x, r) = [1 - 2rx + r^2]^{-\frac{1}{2}} \quad (3.3)$$

is called *generating function* for Legendre polynomials.

3.2 Legendre polynomials

The Taylor expansion as powers of r of the generating function $g(x, r)$ gives the following compact expression for the Legendre polynomials, valid for $|x| \leq 1$, known as Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (3.4)$$

The first polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$. A graph for polynomials P_0 to P_4 is given in Fig. 2. Note that polynomials corresponding to even (resp. odd) degree n are even (resp. odd), and that $P_n(1) = 1 \forall n$. The number of distinct roots in the interval $[-1, 1]$ is also n . At large n polynoms present oscillations whose frequency and amplitude increase as $|x| \rightarrow 1$. Using eq. 8.10.7 of Abramowitz & Stegun (1965), it is possible to obtain the following approximation near the origin $x \simeq 0$

$$P_n(x) \simeq \frac{n!}{\Gamma(n + 1.5)} \sqrt{\frac{2}{\pi}} \cos \left[n \left(x - \frac{\pi}{2} \right) \right] \quad (3.5)$$

Fig. 2 displays the graph of P_{50} and its cosine approximation of Eq. 3.5. A large number of properties and relations between Legendre polynomials are found in Abramowitz & Stegun (1965).

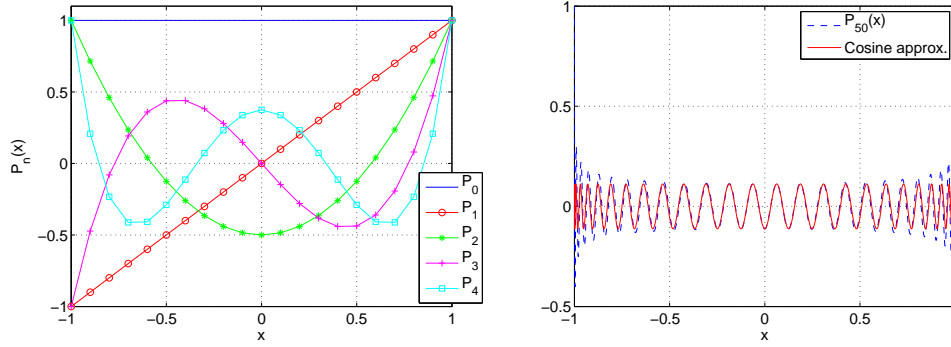


Fig. 2. Left: Legendre polynomials $P_n(x)$ for $n = 0$ to 4. Right: Legendre polynomial for $n = 50$ and its approximation as a cosine function.

3.3 Legendre differential equation

From the Rodrigues formula of Eq. 3.4, it comes that the polynomials $P_n(x)$ are solution of the following equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \quad (3.6)$$

It is the Legendre DE and it has two independent base solutions: polynoms $P_n(x)$, which are regular on the interval $[-1, 1]$, and functions $Q_n(x)$, infinite at $x = \pm 1$, known as Legendre functions of the second kind. As the equation is linear, any solution is a linear combination of P_n and Q_n . However there is no term in Q_n if the solution is to remain finite at $x = \pm 1$.

3.4 Fourier-Legendre series

As for Fourier series, it is possible to make expansion of a function f as a sum of Legendre polynomials. This concerns only functions with bounded support since $P_n(x)$ is defined for $|x| \leq 1$. This is generally the case in physics or signal processing. Following the ideas on scalar products presented in Section 2.3, one can define a scalar product suited to Legendre polynomials (Kaplan 1992). Let f and g two functions defined on the interval $[-1, 1]$, then

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx \quad (3.7)$$

Note that this definition of the scalar product is different from the one given in Eq. 2.8, valid for Fourier expansion. It can be demonstrated, using multiple integration by parts, that

$$\langle P_n, P_m \rangle = \frac{1}{n + \frac{1}{2}} \delta_{mn} \quad (3.8)$$

The family of polynomials P_n form a complete orthogonal base of the space of functions square-summable on $[-1, 1]$. The base is not orthonormal since the norm $\langle P_n, P_n \rangle$ is not 1. Hence it is possible to expand a function f as the series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad (3.9)$$

with

$$c_n = \left(n + \frac{1}{2}\right) \int_{-1}^1 f(x) P_n(x) dx \quad (3.10)$$

This sum is the Fourier-Legendre expansion of the function f . Its form is similar to a Fourier series (Eqs 2.1 and 2.3). As in the case of Fourier series, polynomials of high degree (large n in the Fourier-Legendre decomposition) are associated to short-scale variations of the function f . Indeed Legendre polynomials for large n approximate to high frequency cosine functions, as displayed in Fig. 2.

3.5 Example of Fourier-Legendre expansion

We consider a Gaussian function $f(x)$ having value

$$f(x) = \exp\left(-\frac{x^2}{2a^2}\right) \quad (3.11)$$

for $|x| \leq 1$ and 0 elsewhere. We took $a = 0.3$ for this example. Fourier-Legendre coefficients c_n are 0 for odd n . Nonzero coefficients were calculated numerically from Eq. 3.10. We then computed partial Fourier-Legendre sums defined as

$$S_N(x) = \sum_{n=0}^N c_n P_n(x) \quad (3.12)$$

Fig. 3a shows the graphs of partial sums $S_N(x)$ for $N = 0$ to 20. The sum converges nicely towards the function f , and the two graphs are coincident for $N \geq 10$ (i.e. a sum of 5 nonzero terms). Fig 3b displays the individual terms of the series $c_n P_n(x)$. Fig 3c is a plot of c_n as a function of n , it is then the Fourier-Legendre spectrum of f and contains the same information.

The convergence of S_N towards f is shown by Fig. 3d. For each value of N , we computed the Euclidian distance between f and S_N , defined as

$$d(N) = \int_{-1}^1 (f(x) - S_N(x))^2 dx \quad (3.13)$$

The convergence is here very fast, and the sum of only 10 nonzero terms ($N = 20$) is enough to obtain a distance of 10^{-14} (the starting point was $d(0) \simeq 0.2$). Every term added to S_N divides the distance by about 50.

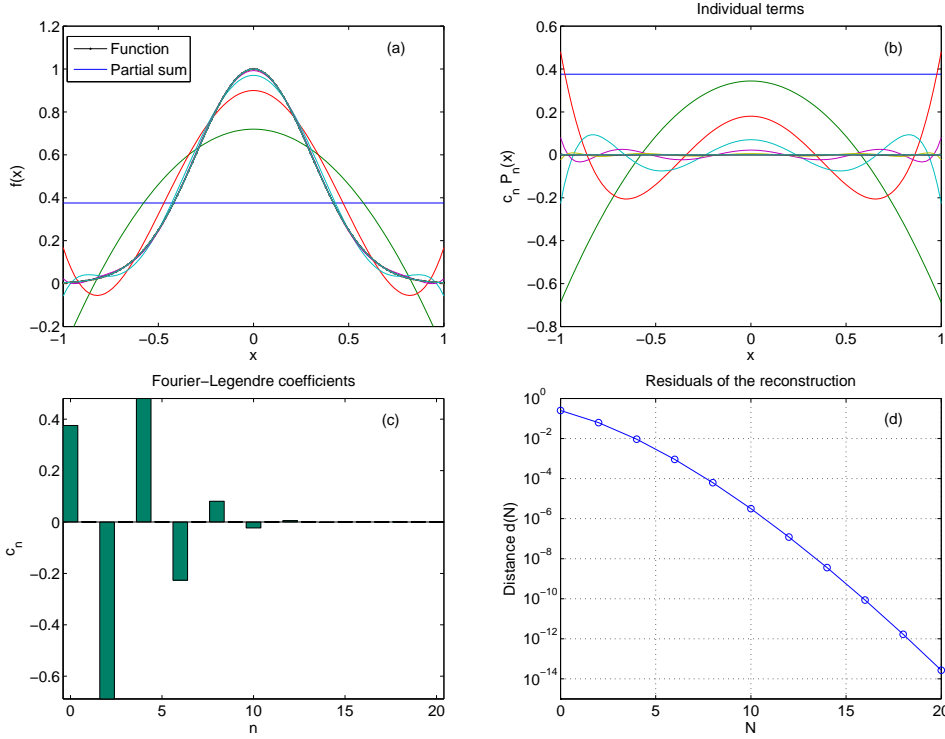


Fig. 3. Fourier-Legendre reconstruction of the Gaussian function of Eq. 3.11 with $a=0.3$. (a) Function and partial sums $S_N(x)$ up to $N=20$. (b) Individual terms of the series $c_n P_n(x)$ for $n=1$ to 20. (c) Graph of c_n versus n (all odd terms vanish). (d) Euclidian distance $d(N)$ between the function and S_N .

3.6 Solution for Laplace equation in spherical coordinates with azimuthal symmetry

As it was said in Sect. 3.1, Legendre polynomials were introduced to express the Newtonian potential of a mass or charge distribution. This connection with physics can be made in a more general way through the DE of the potential. We consider here a charge distribution *presenting the azimuthal symmetry around the z axis*. And we will express the potential in spherical coordinates (r, θ, ϕ) of the Fig. 4 with the condition V independent of ϕ (azimuthal symmetry).

The potential obeys Laplace's equation

$$\Delta V = 0 \quad (3.14)$$

which becomes, in spherical coordinates

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0 \quad (3.15)$$

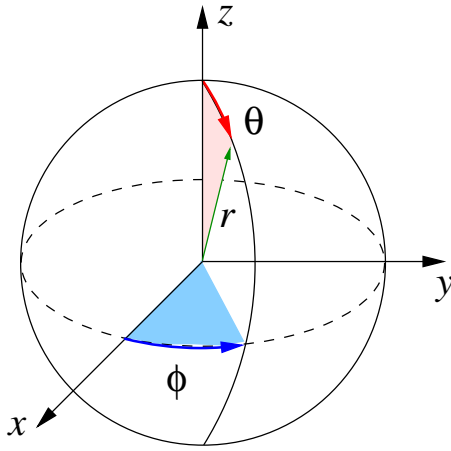


Fig. 4. Spherical coordinate system: r is the distance from the center, θ the colatitude (zero at North pole) and ϕ is the longitude (zero on the x axis).

We apply the technique of separation of variables (see Sect. 2.4) and seek solutions of the form

$$V(r, \theta) = f(r) \cdot g(\theta) \tag{3.16}$$

Eq. 3.15 can be rewritten as

$$\frac{1}{f} \frac{d}{dr} (r^2 f') = -\frac{1}{g \sin \theta} \frac{d}{d\theta} (\sin \theta g') \tag{3.17}$$

the two sides of this equation must be constant since they depend on different variables, so that

$$\left[\begin{array}{l} \frac{1}{f} \frac{d}{dr} (r^2 f') = \text{Cte} = \alpha(\alpha + 1) \\ \frac{1}{g \sin \theta} \frac{d}{d\theta} (\sin \theta g') = -\alpha(\alpha + 1) \end{array} \right. \tag{3.18}$$

where the constant was written as $\alpha(\alpha + 1)$ for convenience. The solution for f is of the form

$$f(r) = A_1 r^\alpha + A_2 r^{-1-\alpha} \tag{3.19}$$

where A_1 and A_2 are constants. For g , the substitution $x = \cos \theta$ gives

$$(1 - x^2)g'' - 2xg' + \alpha(\alpha + 1)g = 0 \tag{3.20}$$

which has regular solutions at $x = 1$ if $\alpha = n$ with n a positive integer. This equation is then exactly the Legendre's DE of Eq. 3.6. Hence g is the linear

combination of the base solutions corresponding to any n

$$g(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos \theta) \quad (3.21)$$

and the complete solution for the potential is the Fourier-Legendre series

$$V(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-1-n}) P_n(\cos \theta) \quad (3.22)$$

Coefficients A_n and B_n are determined by boundary conditions. This example shows how a Fourier-Legendre series appear as the natural expansion of V in this coordinate system with the condition of azimuthal symmetry.

3.7 Other orthogonal polynomials

The Sturm-Liouville problem Fourier and Fourier-Legendre series are indeed particular cases of a more general problem explored by Sturm and Liouville in the early 1800 (Liouville, 1836). See the book by Brown & Churchill, (2011) for a modern presentation. The Sturm-Liouville (SL) theory explores solutions of second-order DE of the form

$$\frac{d}{dx} [p(x)y'] + (\lambda\rho(x) + q(x))y = 0 \quad (3.23)$$

where $y(x)$ is the solution, defined on an interval $x \in [a, b]$ with fixed boundary conditions on y and y' at $x = a$ and $x = b$. The key result of the SL theory is that the problem has a solution for particular values λ_n of λ . And that solutions y_n corresponding to each value of n are orthogonal with respect to the weight function $\rho(x)$. Series expansion on Fourier harmonics or Legendre polynomials can be considered as a special case of the SL boundary value problem. As an example, the Legendre equation of Eq. 3.6 is equivalent to the following form

$$\frac{d}{dx} [(1-x^2)y'] + n(n+1)y = 0 \quad (3.24)$$

i.e. a SL problem with $p(x) = (1-x^2)$, $q(x) = 0$, $\rho(x) = 1$ and $\lambda_n = n(n+1)$.

SL problems occur frequently in physics since a wide class of phenomena are described by a second order PDE. Depending on the coordinate system or circumstances, one may find solutions as series expansion of different kind of functions. Fourier series are likely to occur in cartesian coordinates while Fourier-Legendre series are often met in spherical coordinates for problems having azimuthal symmetry (Jackson, 1998).

Hereafter we give two examples of orthogonal polynom families, related to particular Sturm-Liouville problems. Hermite and Laguerre polynoms have applications in quantum mechanics (Griffiths, 2005). And in Sect. 5 we present another example with series of Bessel functions.

Laguerre polynomials $L_n(x)$

Definition domain	$\mathcal{D} = [0, +\infty[$
Weight of the scalar product	$\rho(x) = e^{-x}$
Orthogonality	$\langle f, g \rangle = \int_0^\infty e^{-x} L_n(x) L_m(x) dx = \delta_{mn}$
Coefficient determination for the expansion $f(x) = \sum_{n=0}^\infty c_n L_n(x)$	$c_n = \langle f, L_n \rangle = \int_0^\infty e^{-x} f(x) L_n(x) dx$
Generating function (for $ t < 1$) :	$\frac{1}{(1-t)} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^\infty L_n(x) t^n$
Rodrigues formula	$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})$
DE	$xy'' + (1-x)y' + ny = 0$

Hermite polynomials $H_n(x)$

Definition domain	$\mathcal{D} =]-\infty, +\infty[$
Weight of the scalar product	$\rho(x) = e^{-x^2}$
Orthogonality	$\langle f, g \rangle = \int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{mn}$
Coefficient determination for the expansion $f(x) = \sum_{n=0}^\infty c_n H_n(x)$	$c_n = \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^\infty e^{-x^2} f(x) H_n(x) dx$
Generating function :	$\exp(2xt - t^2) = \sum_{n=0}^\infty H_n(x) \frac{t^n}{n!}$
Rodrigues formula	$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$
DE	$y'' - 2xy' + 2ny = 0$

4 Spherical harmonics

4.1 Associated Legendre functions

The associated Legendre functions are defined as

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad (4.1)$$

with m a positive integer and $m \leq l$. The relation

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \quad (4.2)$$

makes it possible to define P_l^m for $-l \leq m \leq l$. For $m = 0$ we have

$$P_l^0(x) = P_l(x) \quad (4.3)$$

Legendre associated functions are not polynomials if m is odd. They can be considered as a generalization of Legendre polynomials. Orthogonality relations exist for the P_l^m , for fixed l or m (Abramowitz & Stegun, 1965).

As for Legendre polynomials, the P_l^m functions obey a Sturm-Liouville DE:

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0 \quad (4.4)$$

This equation is the associated Legendre DE. It has regular solutions at $x = 1$ if $-l \leq m \leq l$.

4.2 Laplace equation and spherical harmonics

In Sect. 3.6 we showed that the general solution of the Laplace equation in spherical coordinates is a Fourier-Legendre series in case of azimuthal symmetry. In the general case where the potential V is a function of the three coordinates (r, θ, ϕ) , the Laplace equation expresses as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 V}{\partial \phi^2} = 0 \quad (4.5)$$

Seeking solutions of the form $V(r, \theta, \phi) = f(r) \cdot g(\theta) \cdot h(\phi)$ gives a system of 3 DE similar to Eq. 3.18. The equation for f is unchanged. The equation for $h(\phi)$ is a harmonic equation whose solution takes the form

$$h(\phi) = \alpha_1 e^{im\phi} + \alpha_2 e^{-im\phi} \quad (4.6)$$

with m an integer (since the potential must have 2π periodicity in ϕ), and α_1 and α_2 complex coefficients. The DE for g is slightly different from the azimuthal symmetry problem (second equation of the system 3.18)

$$\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta g') + \left(\alpha(\alpha + 1) - \frac{m^2}{\sin^2 \theta} \right) g = 0 \quad (4.7)$$

which takes the form of the associated Legendre DE (Eq. 4.4) with the substitution $x = \cos \theta$. Thus, g is an associated Legendre function $P_l^m(\cos \theta)$. And the final solution of the Laplace equation in spherical coordinates takes the form of the following series

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (A_{lm} r^l + B_{lm} r^{-1-l}) Y_l^m(\theta, \phi) \quad (4.8)$$

where

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi} \quad (4.9)$$

is the *spherical harmonic* (SH) function, first introduced by Laplace (1782), though the denomination ‘‘spherical harmonic’’ was introduced later (see MacRobert & Sneddon, 1967 for more details about the history of the SH). The number l is the *degree* of the SH, and the number m is its *order* (we recall that $-l \leq m \leq l$)

Sometimes a multiplicative factor $(-1)^m$ is prepended to the definition of the SH. This factor is the Condon-Shortley phase and may rather be included in the definition of the associated Legendre functions (Eq. 4.1), as it is the case here.

We shall see in Sect. 4.4 that the SH functions are orthogonal. They are indeed a basis for representing 2D functions $f(\theta, \phi)$ defined for fixed r over the sphere. They are the spherical analogue of the 1D Fourier series and play a very important role in physics.

4.3 Some properties and symmetries

The first SH are

$$\left. \begin{aligned} Y_0^0 &= \frac{1}{\sqrt{4\pi}} \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_1^1 &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} = -\overline{Y_1^{-1}} \end{aligned} \right| \begin{aligned} Y_2^0 &= \sqrt{\frac{5}{4\pi}} P_2(\cos \theta) \\ Y_2^1 &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} = -\overline{Y_2^{-1}} \\ Y_2^2 &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi} = \overline{Y_2^{-2}} \end{aligned} \quad (4.10)$$

SH are functions of the two angles (θ, ϕ) of the spherical coordinates. Various graphic representation are found in the literature. Fig. 5 displays the real part of the functions Y_l^m for $l = 1$ and $m = -1, 0, +1$. Plots (d), (e) and (f) of Fig. 5 are spherical representations: the surface of a sphere is painted in false colours according to the variation of the SH with spherical coordinates angles (θ, ϕ) .

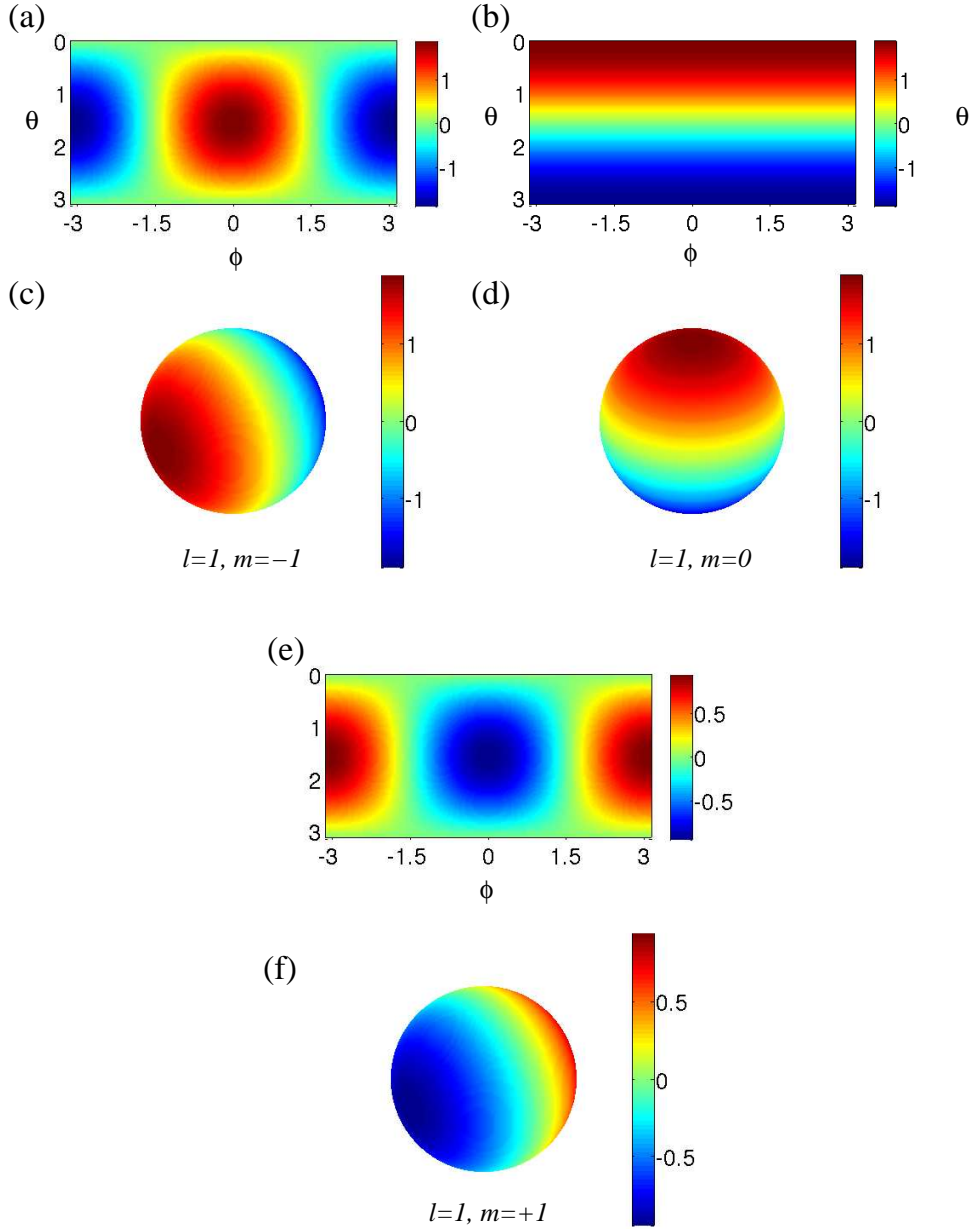


Fig. 5. Representations of the real part of the SH $Y_l^m(\theta, \phi)$ for $l = 1$ and the 3 possible values of m . (a) and (c): $m = -1$, (b) and (d): $m = 0$, (e) and (f): $m = 1$. Graphs (a), (b) and (e) presents false colour plots of the SH in the plane (θ, ϕ) . Graphs (c), (d) and (f) displays the same information on the surface of a sphere, as if the sphere was painted with false colours according to the variation of the SH. Angles θ and ϕ are the colatitude and longitude as defined in Fig. 4.

The case $m = 0$ corresponds to azimuthal symmetry and the corresponding SH $Y_l^0(\theta)$ is proportionnal to the Legendre polynomial $P_l(\cos\theta)$. The series solution of the Laplace equation in the general case (Eq. 4.8) is identical to the Fourier-Legendre expansion valid for azimuthal symmetry (Eq. 3.22).

As for Legendre functions, a number of relations exist between SH (Abramowitz & Stegun, 1965). For negative m one can use the expression

$$Y_l^{-m}(\theta, \phi) = (-1)^m \overline{Y_l^m(\theta, \phi)} \quad (4.11)$$

Parity in $\cos\theta$ is the following

- if $l + m$ is even, the SH is even in $\cos\theta$, i.e. the equator ($z = 0$) is a plane of symmetry
- if $l + m$ is odd, the SH is odd in $\cos\theta$ and the equator is a plane of anti-symmetry

Examples of both cases are displayed in Fig. 6. In the SH expansion of the solution to Laplace equation (Eq. 4.8), a charge distribution presenting a symmetry plane at the equator will create a potential having only even SH in its expansion.

The period of the SH $Y_l^m(\theta, \phi)$ in the direction ϕ is $\frac{2\pi}{m}$ (if $m \neq 0$). The real part of this function has $2|m|$ zeros in the interval $\phi \in [0, 2\pi[$. In the direction θ , the SH has $l - |m|$ zeros in the interval $\theta \in]0, \pi[$ (excluding the poles). In the spherical representation, the lines corresponding to $\Re[Y_l^m(\theta, \phi)] = 0$ are denoted as *nodal lines*. In the direction ϕ they are meridian circles passing through the poles. In the direction θ they are latitude circles parallel to the equator. Fig. 7 shows nodal lines corresponding to the cases $(l, m) = (10, 1)$ and $(10, 4)$.

SH corresponding to large $|m|$ (resp. large $l - |m|$) have a high angular frequency along the axis ϕ (resp. θ). In the SH expansion of a function $f(\theta, \phi)$ they play the same role than high frequency harmonics in a Fourier series: they are associated to short scale variation (small details) of the function f .

4.4 Orthogonality and series expansion

We consider a function of two angular variables $f(\theta, \phi)$, defined on a sphere in the 3D space, for $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. At it was said before, it is possible to make an expansion of f as a sum of SH. One needs to define the following scalar (inner) product between two complex-valued functions f and g :

$$\langle f, g \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\theta, \phi) \overline{g(\theta, \phi)} \sin\theta \, d\theta \, d\phi \quad (4.12)$$

It is easy to show that

$$\langle Y_l^m, Y_{l'}^{m'} \rangle = \delta_{ll'} \delta_{mm'} \quad (4.13)$$

i.e. that SH forms an orthonormal basis of the space of functions defined on the sphere. The series expansion in SH is then

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi) \quad (4.14)$$

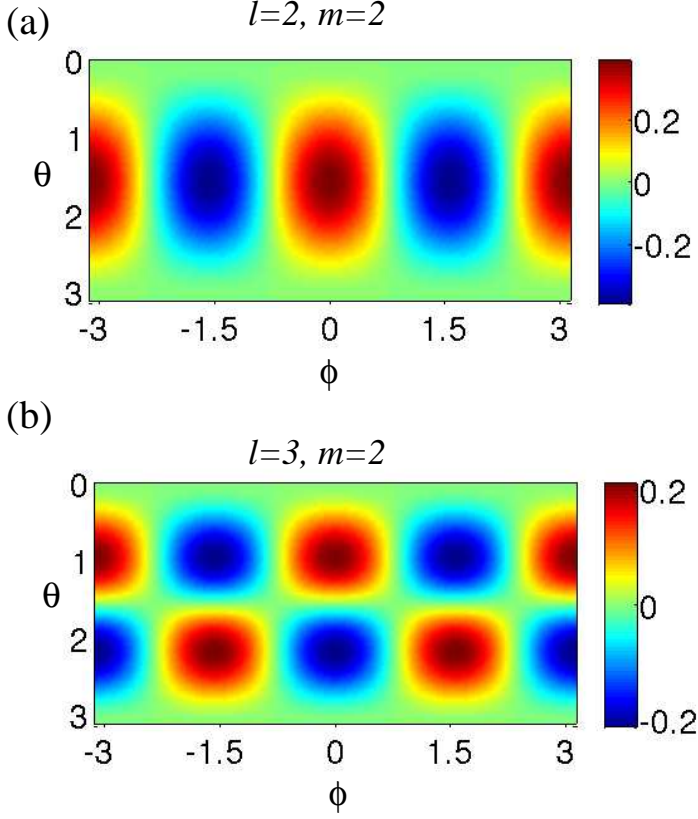


Fig. 6. False color plots of the real part of the SH $Y_l^m(\theta, \phi)$ for (a): $l = 2, m = 2$ ($l + m$ even, parity in $\cos \theta$, symmetry plane at the equator) and (b): $l = 3, m = 2$ ($l + m$ odd, anti-symmetry plane at the equator).

where the coefficients a_{lm} are calculated by the scalar product

$$a_{lm} = \langle f, Y_l^m \rangle = \int_0^\pi \int_0^{2\pi} f(\theta, \phi) \overline{Y_l^m(\theta, \phi)} \sin \theta d\theta d\phi \quad (4.15)$$

The norm of the function f is

$$\langle f, f \rangle = \sum_{l=0}^{\infty} \left[\sum_{m=-l}^l |a_{lm}|^2 \right] \quad (4.16)$$

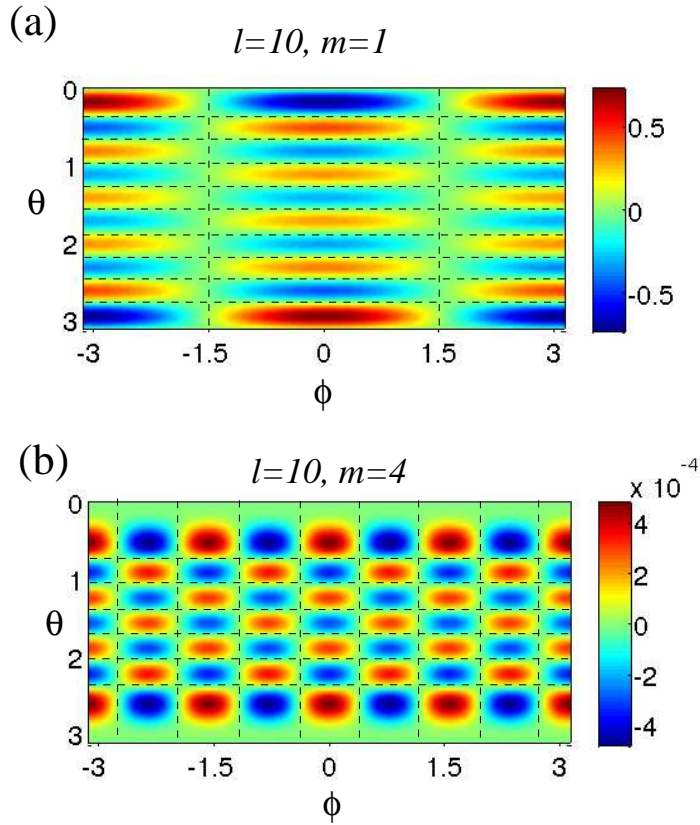


Fig. 7. False color plots of the real part of the SH $Y_l^m(\theta, \phi)$ for (a): ($l = 10, m = 1$) and (b): ($l = 10, m = 4$). Dashed lines are nodal lines corresponding to $\Re[Y_l^m(\theta, \phi)] = 0$. The number of nodal lines is $2|m|$ in the ϕ direction and $l - |m|$ along the θ axis.

The term between brackets is the usual definition of the *angular power spectrum* of f .

4.5 Example

As an example of SH decomposition, we consider the function

$$f(\theta, \phi) = \exp\left(-\frac{(\theta - \theta_0)^2 + \phi^2}{\delta^2}\right) \quad (4.17)$$

with $\theta_0 = 0.5$ rad and $\delta = 0.5$ rad. The function is drawn in the (θ, ϕ) plane in Fig. 8a and over a sphere in Fig. 8b. It may represent a bright spot at the surface of a star. Coefficients of the SH decomposition of f were numerically computed using Eq. 4.15. We then computed the truncated SH decomposition defined as

$$S_L(\theta, \phi) = \sum_{l=0}^L \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi) \quad (4.18)$$

The results are shown in Fig. 8c, d, e, and f for $L = 1, 2, 5$ and 12 . For a given L this corresponds to $(L + 1)^2$ terms in the sum S_L . We can see that the function appears to be well reconstructed for $L = 12$. Fig. 8g (top graph) shows the modulus of coefficients a_{lm} in the (l, m) plane, for positive m (negative m verify $a_{l,-m} = (-1)^m a_{lm}$ since f is even in ϕ). Largest coefficients are obtained for small m and l , just as it is for 1D Fourier decomposition of a Gaussian signal. The angular power spectrum $\sum_{m=-l}^l |a_{lm}|^2$ is plotted in the bottom graph and confirms this trend. The convergence of the series is shown in Fig. 8h. It is a plot of the least-square distance between f and S_L as a function of L . The distance was normalized to 1 for $L = 0$.

SH decomposition is widely used in domains of physics described by a second order PDE such as the Laplace, Schrödinger or the wave equation. For example stellar oscillations are often described in terms of standing waves whose angular part in spherical coordinates is a SH (see chapter 8 of Collins, 1989 for a review about stellar pulsations). Another application is to use SH decomposition for decomposition of functions defined over the sphere. It is indeed an image processing technique analog to 2D Fourier decomposition of an image defined on a rectangle. For example in cosmology, the brightness distribution over the whole sky of the Cosmic Microwave Background (CMB) is analyzed in terms of SH series. The angular power spectrum provides informations about the statistical properties of the CMB. Hinshaw et al., (2007) present an extensive analysis of data from the WMAP spacecraft.

5 Bessel functions

Bessel functions were introduced after the work by Bessel (1824) about the motion of the planets around the Sun. Bessel expressed the position $r(t)$ of the planet as a temporal Fourier series whose coefficients were defined by integrals. In his memoir, Bessel made systematic investigation of the properties of these integrals who now bear his name. An extensive treatise about Bessel functions is that of Watson (1966) and includes an historical introduction (one can also refer to Dukta, 1995).

5.1 Bessel differential equation

Bessel functions are often introduced via the Bessel DE

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad (5.1)$$

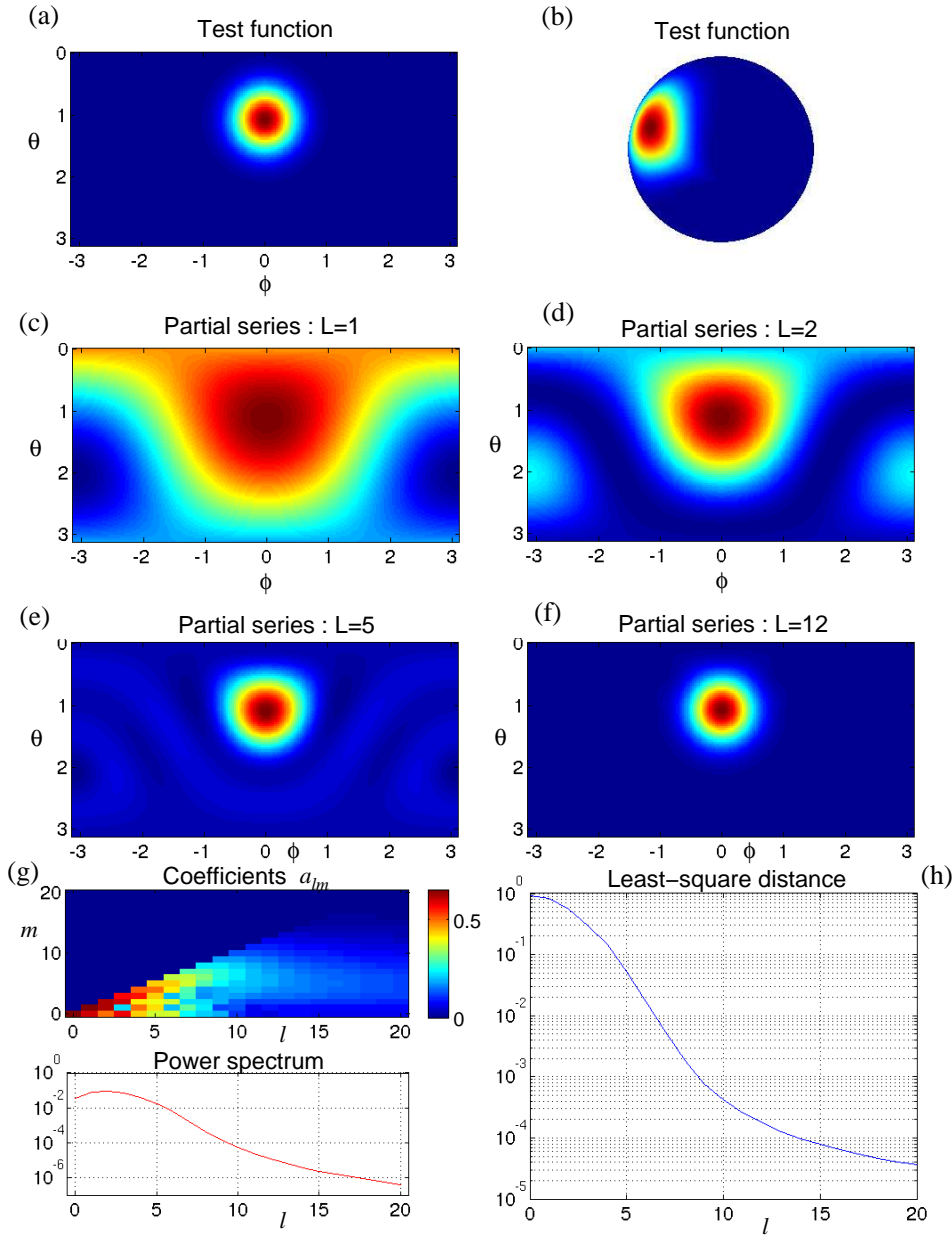


Fig. 8. Example of SH decomposition of a 2D Gaussian test function (Eq. 4.17). (a) false color plot of the test function in the plane (θ, ϕ) . (b): spherical representation. (c) to (f): plot of the partial SH sum (Eq. 4.18) up to $L = 1, 2, 5$ and 12 . (g): the top graph is a false color plot of the amplitude of the coefficients a_{lm} for $m \geq 0$ in the plane (l, m) . The bottom graph is the angular power spectrum as a function of l . (h) is the least-square distance between the test function and the partial SH sum.

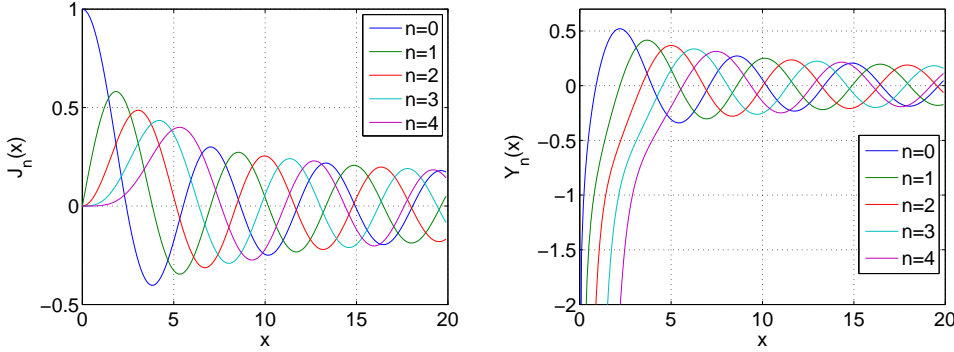


Fig. 9. Plots of the first Bessel functions for positive argument x . Left: $J_n(x)$ for $n = 0$ to 4. Right: $Y_n(x)$ for $n = 0$ to 4.

where n is an arbitrary complex number, but in the present paper we shall restrict to integer n . The Bessel equation is indeed a Sturm-Liouville problem (see Sect. 3.7) with $p(x) = x$, $q(x) = x$, $\rho(x) = -\frac{1}{x}$ and $\lambda = n^2$. Series solutions of the DE can be obtained by the Frobenius method (Mathews & Walker, 1970). This equation has two independent base solutions: first kind and second kind Bessel functions $J_n(x)$ and $Y_n(x)$. The function J_n is regular at $x = 0$ for all n , while Y_n diverges at $x = 0$ and is complex-valued for $x < 0$. Plots of the first J_n and Y_n are shown in Fig. 9.

The zoology includes also modified Bessel functions I_n and K_n , Hankel functions H_n and cylindrical Bessel functions j_n and y_n . These functions are of common use in various domains (for example the spherical Bessel functions have applications in quantum mechanics, see chapter 4 of Griffiths 2005). In this paper we shall focus our presentation to the J_n family.

5.2 Bessel functions of the first kind $J_n(x)$

In Sect. 3.1 we introduced the Legendre polynomials by means of a Taylor expansion of a generating function. The same can be made for J_n , the generating function $g(x, t)$ being (Watson, 1966):

$$g(x, t) = \exp \left[\frac{x}{2} \left(t - \frac{1}{t} \right) \right] = \sum_{n=-\infty}^{\infty} t^n J_n(x) \quad (5.2)$$

Or the equivalent trigonometric form (variable change $t = e^{i\phi}$)

$$e^{ix \sin \phi} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\phi} \quad (5.3)$$

which is the Fourier series of the function $e^{ix \sin \phi}$, 2π -periodic in ϕ , and $J_n(x)$ is the Fourier coefficient calculated by Eq. 2.3

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \phi} e^{-in\phi} d\phi \quad (5.4)$$

This important formula is an integral representation of J_n which has many applications, for example in optics to calculate diffraction patterns of screens having rotational invariance (Born & Wolf, 2000). It is also very close to the coefficients introduced in the memoir of Bessel (1824).

Basic properties of the functions J_n include

- Parity: same as n
- Negative order: $J_{-n}(x) = (-1)^n J_n(x)$
- Value at the origin: $J_n(0) = 0$ for $n > 0$ and $J_0(0) = 1$
- Behaviour for small x : $J_n(x) \simeq \frac{1}{n!} \left(\frac{x}{2}\right)^n$
- Asymptotic limit ($x \rightarrow \infty$): $J_n(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$

All J_n functions behave like damped sinusoids with an infinite number of roots on the real axis. The position of these zeros is not periodic. They play a role in Fourier-Bessel expansion, as it will be further discussed in Sect. 5.3.

Various expansions of functions in series of Bessel functions exist, see Watson (1966) for a review. Eqs. 5.2 and 5.3 are two examples of Neumann series of the type

$$f(x) = \sum_{n=-\infty}^{\infty} a_n J_n(x) \quad (5.5)$$

In next section, we focus on a special case of orthogonal expansion involving Bessel functions, which has many connections with physics: the Fourier-Bessel series.

5.3 Fourier-Bessel series

5.3.1 Vibrations of a drum membrane

In Sect. 2.4, 3.6 and 4.2 we showed that a DE representing a particular problem would lead to series solutions depending on the geometry: Fourier expansion for rectangular coordinates, Fourier-Legendre or SH series for spherical coordinates. Here we explore solutions of the equation describing oscillations of a drum circular membrane of radius R in polar coordinates (ρ, ϕ) (origin at the center of the membrane). The function $s(\rho, \phi, t)$ describing the displacement of the membrane obeys the wave equation

$$\Delta s - \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} = 0 \quad (5.6)$$

where c is the speed of propagation. We apply the technique of separation of variable (as in Sect.2.4 and 4.2) and look for solutions of the form $s(\rho, \phi, t) = f(\rho).g(\phi).h(t)$. The wave equation is then equivalent to the following system

$$\begin{cases} h'' + k^2 c^2 h = 0 \\ g'' + n^2 g = 0 \\ \rho^2 f'' + \rho f' + (k^2 \rho^2 - n^2) f = 0 \end{cases} \quad (5.7)$$

where k and n are constant (k is positive to have oscillatory solutions for $h(t)$, n is integer to ensure 2π periodicity in ϕ). A more detailed step-by-step calculation can be found in Asmar (2004). It can be noticed that the equation for f is indeed a Bessel DE (Eq. 5.1) whose base solution is

$$f(\rho) = J_n(k\rho) \quad (5.8)$$

An important condition is the boundary value, i.e. the membrane must be motionless at its extremities ($\rho = R$), so that $f(R) = 0$ whatever n . Hence k must have the form $k = \frac{\alpha_{nm}}{R}$ with α_{nm} the m -th positive root of the function J_n . The general solution is the linear combination of base solutions (i.e. *modes*) for every possible m and n :

$$s(\rho, \phi, t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{mn} J_n\left(\frac{\alpha_{nm}\rho}{R}\right) e^{in\phi} e^{i\omega_{mn}t} \quad (5.9)$$

with A_{nm} a constant and $\omega_{mn} = kc = \frac{\alpha_{nm}c}{R}$ the temporal pulsation of the mode. This double sum is indeed a Fourier series for the variable ϕ and a Fourier-Bessel series for the variable ρ .

5.3.2 Fourier-Bessel expansion

As for Fourier or Fourier-Legendre series, it is possible to make orthogonal expansions of functions as series of Bessel functions. This concerns continuous functions with bounded support $[0, 1]$ who verify $f(1) = 0$. One needs to define the following scalar product:

$$\langle f, g \rangle = \int_0^1 x f(x) g(x) dx \quad (5.10)$$

Note that this scalar product, suited for Fourier-Bessel expansion, is different from those defined in Eqs 2.8 and 3.7. It can be shown (Watson, 1966) that

$$\langle J_n(\alpha_{nm}x), J_n(\alpha_{np}x) \rangle = \frac{J_{n+1}^2(\alpha_{nm})}{2} \delta_{mp} \quad (5.11)$$

so that the functions $j_m(x) = J_n(\alpha_{nm}x)$ form a complete set of orthogonal functions. As an example Fig. 10 shows the graph of the base functions $J_0(\alpha_{0m}x)$

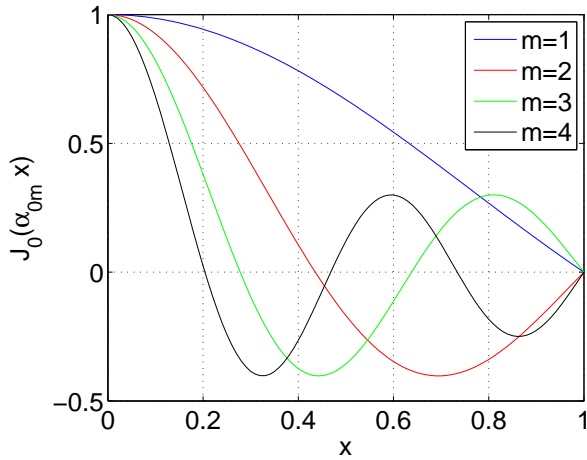


Fig. 10. Graph of the functions $J_0(\alpha_{0m}x)$ for $m = 1$ to 4. Corresponding roots of J_0 are $\alpha_{01} = 2.4$, $\alpha_{02} = 5.5$, $\alpha_{03} = 8.7$, $\alpha_{04} = 11.8$. These functions are orthogonal with respect to the scalar product defined by Eq. 5.10. All functions vanish at $x = 1$.

for $m = 1$ to 4. All of them are stretched version of J_0 with a stretching factor depending on the root α_{0m} . Any function $f(x)$ continuous on $[0, 1]$ with $f(1) = 0$ can be expanded as a series of j_m

$$f(x) = \sum_{m=1}^{\infty} c_m J_n(\alpha_{nm}x) \tag{5.12}$$

with the coefficient

$$c_m = \frac{2}{J_{n+1}^2(\alpha_{nm})} \langle f, j_m \rangle = \frac{2}{J_{n+1}^2(\alpha_{nm})} \int_0^1 x f(x) J_n(\alpha_{nm}x) dx \tag{5.13}$$

This expansion is denoted as Fourier-Bessel expansion of f . Note that the choice of n , the order of the Bessel function, is arbitrary (or suggested by physics): an infinity of Fourier-Bessel expansions exist for a given function.

5.3.3 Exemple of Fourier-Bessel expansion

We consider the following function for $x \in [0, 1]$

$$f(x) = e^{-3x} \cos\left(\frac{3\pi x}{2}\right) \tag{5.14}$$

This function vanishes at $x = 1$, as required to make a Fourier-Bessel expansion. Also since $f(0) = 1$ we choose to expand it on the basis of $J_0(\alpha_{0m}x)$. Coefficients

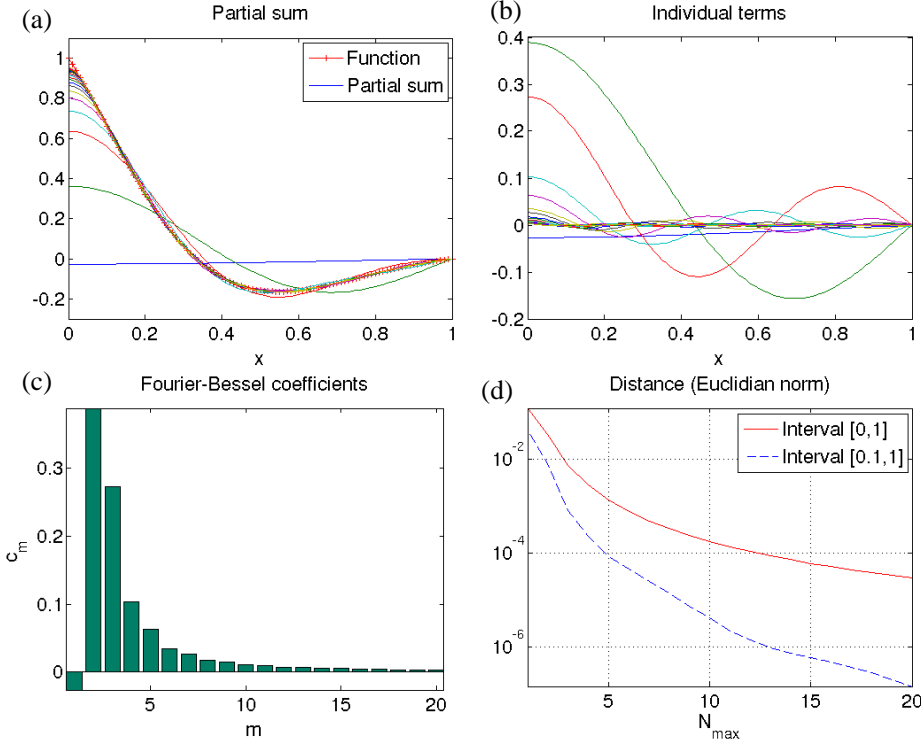


Fig. 11. Fourier-Bessel reconstruction of the function defined by Eq. 5.14, on the base of orthogonal function $J_0(\alpha_{0m}x)$. (a) function and partial sums $S_N(x)$ for $N = 1$ to 20. (b) individual terms of the series. (c) graph of c_m versus m . (d) Least square distance d_N between the function and the sum for two intervals (Eq. 5.16).

of the series were calculated numerically from Eq. 5.13. We computed the partial series

$$S_N(x) = \sum_{m=1}^N c_m J_n(\alpha_{nm}x) \quad (5.15)$$

up to $N = 20$. Results are shown in Fig. 11. The convergence is fairly good for $N = 20$, excepted near $x = 0$ where more terms are required to fit the function. This is due to the fact that J_0 derivative is zero at $x = 0$ while it is not for the function. In Fig. 11d, we display the least-square distance d_N between the function and the partial sum as a function of N :

$$d_N = \int_a^1 (f(x) - S_N(x))^2 dx \quad (5.16)$$

Two curves are shown: one for $a = 0$ (whole interval), the other one for $a = 0.1$.

As we see, the second curve is two decades below the first for $N = 20$, it means that the interval $[0, 0.1]$ contains 99% of the reconstruction residuals.

Fourier-Bessel expansion can be made using higher order Bessel functions J_n , but for this particular example the convergence is very slow near $x = 0$ and hundreds of terms are required in the series.

6 Conclusion

This presentation aimed at introducing the mathematical concept of representing a function or a signal as a series of orthogonal functions. We focused on four particular cases: the Fourier series, the Fourier-Legendre series, the Spherical Harmonic decomposition and the Fourier-Bessel expansion. A basic presentation of each set of orthogonal functions was given, together with a list of references for further reading. The central concept of inner/scalar product of functions was introduced by analogy with “classical” vectors in the three-dimensional space. All series expansions were illustrated by a numerical example.

The connexion with physics was enlightened each time, and we showed that the link is often made via differential equations of the physical phenomenon. The geometry often gives a “natural” basis for series expansion. Hence Fourier series are often used in rectangular coordinates, SH are the natural base in spherical coordinates and Fourier-Bessel expansions are met in cylindrical or polar coordinates.

A wide variety of series expansions on orthogonal set of functions exist in the field of functional analysis, and this presentation is far from exhaustive. For example we did not explore the decomposition of a two-dimensional function in Zernike modes, widely used in adaptive optics (Noll, 1976).

Also, it is not the sake of this paper to present FFT-like fast algorithms to compute various transforms such Fourier-Legendre or Spherical Harmonics. There is indeed an abundant literature on this topic and the reader may refer to O’Neil et al. (2010) and references therein.

This paper could hopefully be used as a starting point or a reminder, in particular for students. The reference list gives a number of classical textbooks with much details and exercises.

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