Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay

Fazia Bedouhene, Islam Boussaada, Silviu-Iulian Niculescu

▶ **To cite this version:**

Fazia Bedouhene, Islam Boussaada, Silviu-Iulian Niculescu. Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay. Comptes Rendus. Mathématique, 2020, 358 (9-10), pp.1011-1032. 10.5802/crmath.112 . hal-02476403

HAL Id: hal-02476403

https://hal.science/hal-02476403

Submitted on 12 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay

Fazia Bedouhene

Laboratoire de Mathématiques Pures et Appliquées, Mouloud Mammeri University of Tizi-Ouzou, Algeria

Islam Boussaada

Inria Saclay-Île-de-France, Equipe DISCO & IPSA Ivry sur Seine & L2S, CNRS-CentraleSupélec-Université Paris Sud, Université Paris Saclay, 91192 Gif-sur-Yvette cedex, France

Silviu-Iulian Niculescu

CNRS-CentraleSupélec-Université Paris Sud, Université Paris Saclay, 91192 Gif-sur-Yvette cedex, France

Abstract

This work exploits structural properties of a class of functional Vandermonde matrices to emphasize some qualitative properties of a class of linear autonomous $n-$th order differential equation with forcing term consisting in the delayed dependent-variable. More precisely, it deals with the stabilizing effect of delay parameter coupled with the coexistence of the maximal number of real spectral values. The derived conditions are necessary and sufficient and represent a novelty in the litterature. Under appropriate conditions, such a configuration characterizes the spectral abscissa corresponding to the studied equation. A new stability criterion is proposed. This criterion extends recent results in factorizing quasipolynomial functions. The applicative potential of the proposed method is illustrated through the stabilization of coupled oscillators.

Keywords: Time-delay, asymptotic stability, exponential stability, exponential decay rate, Vandermonde matrix, quasipolynomial factorization, control design.

2010 MSC: 34K20, 39B82, 70Q05, 47N70

Contents

E-mail addresses: fazia.bedouhene@ummto.dz (Fazia Bedouhene)., Islam.Boussaada@l2s.centralesupelec.fr (Islam Boussaada)., Silviu.Niculescu@l2s.centralesupelec.fr (Silviu-Iulian Niculescu).

1. Introduction

Matrices arising from a wide range of problems in mathematics and engineering typically display characteristic structures. In particular, exploiting such a structure in problems from dynamical systems is known to be an engaging aperture for understanding of complex qualitative behaviors and for characterizing system's properties, see, for instance,[1] and references therein. This study is a crossroad between the investigation of the invertibility of a class of such structured matrices which is related to *Multivariate Interpolation Problems* (namely, the well-known *Lagrange Interpolation Problem*) and the localisation of *spectral values* of linear *time-delay systems.* The study of conditions on the time-delay systems parameters that guarantees the exponential stability of solutions is a question of ongoing interest and to the best of the authors' knowledge it remains an open problem. In particular, in frequency-domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic equation, which is an entire function called *characteristic quasipolynomial*), see for instance[2,3,4,5,6,7,8].

The starting point of the present work is a property, discussed in recent studies, called *Multiplicity-Induced-Dominancy*, see for instance[9,10]. As a matter of fact, it is shown that multiple spectral values for time-delay systems can be characterized using a Birkhoff/Vandermonde-based approach; see for instance[11,1,12,13]. More precisely, in previous works, it is emphasized that the admissible multiplicity of the real spectral values is bounded by the generic *Polya and Szegö bound* (denoted PS_B), which is nothing but the *degree* of the corresponding quasipolynomial (i.e the number of the involved polynomials plus their degree minus one), see for instance[14] Problem 206.2, page 144 and page 347. It is worth mentioning that such a bound were recovered using structured matrices in[1] rather than the principle argument as done in[14]. It is important to point out that the multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. To the best of the authors' knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented and discussed in the 50s, see[15]. The dominancy property is further explored and analytically shown in scalar delay equations in[13], then in second-order systems controlled by a delayed proportional controller is proposed in[16,17] where its applicability in damping active vibrations for a piezo-actuated beam is proved. An extension to the delayed proportional-derivative controller case is studied in[18,19] where the dominancy property is parametrically characterized and proven using the argument principle. See also[19,18] which exhibit an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems controlled via a delayed proportional-derivative controller. Recently, in[20] it is shown that under appropriate conditions the coexistence of exactly PS_B distinct negative zeros of quasipolynomial of reduced degree guarantees the exponential stability of the zero solution of the corresponding time-delay system. The dominancy of such real spectral values is shown using an extended factorization technique which generalizes the one provided in[20]. To the best of the authors's knowledge the necessary and sufficient conditions derived in the present paper as well as corresponding control strategy represent a novelty.

The present work investigates the effect of structural properties of a class of functional Vandermonde matrices and its effect on qualitative properties of a corresponding *linear autonomous time-delay system of retarded type*. More precisely, the aim of this work is two-fold: firstly, it emphasizes the link between the invertibility of a class of structured functional Vandermonde matrices and the coexistence of distinct real spectral values of linear time-delay systems, which allows to recover the maximal number of distinct real spectral values that may coexist for a given time-delay system. Furthermore, if the number of coexistent real spectral values reaches the PS_B, then a necessary and sufficient condition for the asymptotic stability is provided (which is equivalent to the exponential stability[21] p79), see also[22] for an estimate of the exponential decay rate for stable linear delay systems. Notice also that the constructive approach we propose, which consists in providing an appropriate factorization of a given quasipolynomial function and then to focus on the location of zeros of one of its factors, gives further insights on such a qualitative property. Namely, it furnishes the exact exponential decay rate rather than just counting the number of the quasipolynomial roots on the left-half plane as may be done by using the principle argument, see for instance[5].

The class of dynamical systems we consider is an $n-$th order linear autonomous ordinary differential equations with a forcing term consisting in the delayed dependent variable. This class of systems has an applicative interest particularly in control design problems. As a matter of fact, the forcing term may be seen as a delayed-input able to stabilize the system's solutions. The idea of exploiting the delay effect in controllers design was first introduced in[23] where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus can replace the proportional-derivative controller, see also[24]. Furthermore, it was stressed in[25] that time-delay has a stabilizing effect in the control design.

Indeed, the closed-loop stability is guaranteed precisely by the existence of the delay. Also in [26] it is shown that a chain of n integrators can be stabilized using n distinct delay blocks, where a delay block is described by two parameters: a "gain" and a"delay". The interest of considering control laws of the form $\sum_{k=1}^{m} \gamma_k \, y(t-\tau_k)$ lies in the simplicity of the controller as well as in its easy practical implementation.

From a control theory point of view, the problem we consider and the approach we propose give rise to an *exponential decay assignment* method using two parameters a "gain" and a"delay". Notice that the idea of using roots assignment for controller-design for time-delay system is not new. For instance, in [27] a feedback law yields a finite spectrum of the closed-loop system, located at an arbitrarily preassigned set of points in the complex plane. In the case of systems with delays in control only, a necessary and sufficient condition for finite spectrum assignment is obtained. Notice that the resulting feedback law involves integrals over the past control. In case of delays in state variables it is shown that a technique based on the finite Laplace transform leads to a constructive design procedure. The resulting feedback consists of proportional and (finite interval) integral terms over present and past values of state variables. In [28], a similar finite pole placement for time-delay systems with commensurate delays is proposed. Feedback laws defined in terms of Volterra equations are obtained thanks to the properties of the Bezout ring of operators including derivatives, localized and distributed delays. Other analytical/numerical placement methods for retarded time-delay systems are proposed in [29,30], see also [31] for further insights on pole-placement methods for retarded time-delays systems with proportional-integral-derivative controller-design.

The remaining paper is organized as follows. In Section 2, the problem formulation is presented and some technical lemmas are derived. Section 3 is devoted to the main results of the paper. Section 4 gives an illustrative example showing the applicative perspectives of the derived results. Some concluding remarks end the paper. Finally, the reader finds proofs of the technical lemmas in the Appendix.

2. Problem settings and prerequisites

In this paper, we are interested in studying the stabilizing effect of the coexistence of the maximal number of real spectral values for the generic n-order ordinary differential equation perturbed by a forcing term depending in the delayed dependent variable

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \alpha y(t-\tau) = 0, \quad t \in \mathbb{R}_+, \tag{1}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau, 0], \mathbb{R})$ which is an infinite-dimensional differential equation with a single constant delay $\tau > 0$.

From a control theory point of view, the aim is to establish a delayed-output-feedback controller $u(t) = -\alpha \, y(t-\tau)$ able to stabilize solutions of the following control system:

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = u(t). \tag{2}$$

The particular cases of first and second order equations are considered in [20], where a stabilizing effect of the coexistence of respectively 2 and 3 negative real roots is shown. By this paper, one generalizes such a result for arbitrary order n.

In the Laplace domain, the corresponding quasipolynomial characteristic function defined by $\Delta_n : \mathbb{C}\times\mathbb{R}_+^* \longrightarrow \mathbb{C}$ writes

$$\Delta_n(s,\tau) := s^n + \sum_{k=0}^{n-1} a_k s^k + \alpha e^{-\tau s}. \tag{3}$$

One can prove that the quasipolynomial function (3) admits an infinite number of zeros, see for instance the references [2,32,33]. The study of zeros of an entire function [33] of the form (3) plays a crucial role in the analysis of asymptotic stability of the zero solution of Equation (1). Indeed, the zero solution is asymptotically stable if, and only if, all the zeros of (3) are in the open left-half complex plane [7].

2.1. Counting quasipolynomial roots in horizontal strips

The following result was first introduced and claimed in the problems collection published in 1925 by G. Pólya and G. Szegö. In the fourth edition of their book [14] Problem 206.2, page 144 and page 347, G. Pólya and G. Szegö emphasize that the proof was obtained by N. Obreschkoff in 1928 using the principle argument, see [34]. Such a result gives a bound for the number of quasipolynomial's roots in any horizontal strip. As a consequence, a bound for the number of quasipolynomial's real roots can be easily deduced.

Theorem 1 ([14]). *Let $\tau_1, \ldots, \tau_N$ denote real numbers such that $\tau_1 < \tau_2 < \ldots < \tau_N$ and $d_1, \ldots, d_N$ positive integers such that $d_1 + d_2 + \ldots + d_N = D$. Let $f_{i,j}(s)$ stand for the function $f_{i,j}(s) = s^{i-1}e^{(\tau_j s)}$, for $1 \leq i \leq d_j$ and $1 \leq j \leq N$. Let $\sharp$ be the number of zeros of the function*

$$f(s) = \sum_{\substack{1\leq j\leq N \\ 1\leq i\leq d_j}} c_{i,j} f_{i,j}(s) \tag{4}$$

that are contained in the horizontal strip $\alpha \leq \mathrm{Im}(z) \leq \beta$. Assuming that

$$\sum_{1\leq k\leq d_1} |c_{k,1}| > 0 \quad \textit{and} \quad \sum_{1\leq k\leq d_N} |c_{k,N}| > 0$$

then

$$\frac{(\tau_N - \tau_1)(\beta-\alpha)}{2\pi} - D + 1 \leq \sharp \leq \frac{(\tau_N - \tau_1)(\beta-\alpha)}{2\pi} + D + N - 1. \tag{5}$$

Setting $\alpha = \beta = 0$, the above theorem yields $\sharp_{PS} \leq D + N - 1$ where D stands for the sum of the degrees of the polynomials involved in the quasipolynomial function f and N designates the associated number of polynomials. This gives a sharp bound for the number of f's real roots. Notice that $D + N - 1$ corresponds to the degree of the quasipolynomial f.[1]

[1] The quasipolynomial degree is defined as the sum of degrees of the involved polynomials plus the corresponding number of delays

Let's investigate the coexistence of $n+1$ real (negative) roots for the quasipolynomial $\Delta_n(.,\tau)$. Due to the linearity of Δ_n with respect to its coefficients $(a_k)_{0\le k\le n-1}$ and α, one reduces the system $\Delta_n(s_1,\tau) = \ldots = \Delta_n(s_{n+1},\tau) = 0$ to the linear system $V_n(X^{n+1},\tau).\mathcal{V} = b$ where $\mathcal{V} = (a_{n-1},\ldots,a_0,\alpha)^T$, $b = -(s_1^n,\ldots,s_{n+1}^n)^T$ and $X^{n+1} \triangleq (s_1, s_2, \cdots, s_{n+1})$:

$$V_n\left(X^{n+1},\tau\right) = \begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & 1 & e^{-\tau s_1} \\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & 1 & e^{-\tau s_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & 1 & e^{-\tau s_n} \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}} \end{bmatrix}. \tag{6}$$

In the sequel, such a matrix is called *structured functional Vandermonde type matrix* due to its form and its structural properties.

2.2. Structured matrices appearing in the control of dynamical systems

Initially, Birkhoff and Vandermonde matrices are derived from the problem of polynomial interpolation of some unknown function g, this can be presented in a general way by describing the interpolation conditions in terms of *incidence matrices*, see for instance[35]. For given integers $n \ge 1$ and $r \ge 0$, the matrix

$$\mathcal{E} = \begin{pmatrix} e_{1,0} & \ldots & e_{1,r} \\ \vdots & & \vdots \\ e_{n,0} & \ldots & e_{n,r} \end{pmatrix},$$

is called an incidence matrix if $e_{i,j} \in \{0,1\}$ for every i and j. Such a matrix contains the data providing the known information about a sufficiently smooth function $g : \mathbb{R} \mapsto \mathbb{R}$. Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ such that $x_1 < \ldots < x_n$, the problem of determining a polynomial $\hat{P} \in \mathbb{R}[x]$ with degree less or equal to ι ($\iota + 1 = \sum_{1\le i\le n,\ 1\le j\le r} e_{i,j}$) that interpolates g at $(x,\mathcal{E})$, i.e. which satisfies the conditions:

$$\hat{P}^{(j)}(x_i) = g^{(j)}(x_i),$$

is known as the *Birkhoff interpolation problem.* Recall that $e_{i,j} = 1$ when $g^{(j)}(x_i)$ is known, otherwise $e_{i,j} = 0$. Furthermore, an incidence matrix $\mathcal{E}$ is said to be *poised* if such a polynomial $\hat{P}$ is unique. This amounts to saying that, if, $n = \sum_{i=1}^n \sum_{j=1}^r e_{i,j}$ then the coefficients of the interpolating polynomial $\hat{P}$ are solutions of a linear square system with associated square matrix Υ that we call *Birkhoff matrix* in the sequel. This matrix is parametrized in x and is shaped by $\mathcal{E}$. It turns out that the incidence matrix $\mathcal{E}$ is poised if, and only if, the Birkhoff matrix Υ is non singular for all x such that $x_1 < \ldots < x_n$. The characterization of poised incidence matrices is solved for interpolation problems of low degrees. As a matter of fact, the problem is still unsolved for any degree greater than six, see for instance[36,37].

Remark 2.1. *Unlike Hermite interpolation problem, for which the knowledge of the value of a given order derivative of the interpolating polynomial at a given interpolating point impose the values of all the lower orders derivatives of the interpolating polynomial at that point, the Birkhoff interpolation problem release such a restriction. Thereby justifying the qualification of "lacunary" to describe the Birkhoff interpolation problem.*

In the spirit of the definition of functional confluent Vandermonde matrices introduced in [38], the following functional Birkhoff matrices were introduced in [1]

Definition 2.1. *The square* functional Birkhoff matrix Υ *is associated to a sufficiently regular function* ϖ *and an incidence matrix* $\mathcal{E}$ *(or equivalently an incidence vector* $\mathcal{V}$*) and is defined by:*

$$\Upsilon = [\Upsilon^1 \, \Upsilon^2 \, \ldots \, \Upsilon^M] \in \mathcal{M}_\delta(\mathbb{R}), \tag{7}$$

where

$$\Upsilon^i = [\kappa^{(k_{i_1})}(x_i) \, \kappa^{(k_{i_2})}(x_i) \, \ldots \, \kappa^{(k_{i_{d_i}})}(x_i)], \tag{8}$$

such that $k_{i_l} \geq 0$ *for all* $(i,l) \in \{1,\ldots,M\} \times \{1,\ldots,d_i\}$ *and* $\sum_{i=1}^{M} d_i = \delta$ *where*

$$\kappa(x_i) = \varpi(x_i)[1 \, \ldots \, x_i^{\delta-1}]^T, \quad \textit{for} \quad 1 \leq i \leq M. \tag{9}$$

Analogously to the Birkhoff interpolation problem, in [1] the non degeneracy of such functional Birkhoff matrices represent a fondamental assumption for investigating the codimension of the zero spectral values for time-delay systems.

To the best of the author's knowledge, the first time the Vandermonde matrix appears in a control problem is reported in [39] p. 121, where the controllability of a finite dimensional dynamical system is guaranteed by the invertibility of such a matrix, see also [38,40]. Next, in the context of time-delay systems, the use of the standard Vandermonde matrix properties was proposed by [26,7] when controlling one chain of integrators by delay blocks. Here we further exploit the algebraic properties of such structured matrices into a different context.

2.3. The determinant of a structured functional Vandermonde type matrix

The following auxiliary result gives explicitly the determinant of the structured functional Vandermonde type matrix (6). Its proof is presented in the Appendix. In the following we adopt the notation $[x,y]_t$ to designate the $t-$convex combination of the real (or complex) numbers x and y, that is: $[x,y]_t = tx + (1-t)\,y$ for $t \in [0,1]$.

Theorem 2. *For any distinct real numbers* $s_{n+1} < \cdots < s_2 < s_1$*, and* $\tau > 0$*, the structured functional Vandermonde type matrix* $V_n\left(X^{n+1}, \tau\right)$ *is invertible. Moreover, its determinant is*

$$Q_n(X^{n+1}, \tau) = \det\left(V_n\left(X^{n+1}, \tau\right)\right) = \tau^n \prod_{\substack{i<j \\ i,j=1}}^{n+1} (s_i - s_j)\, F_{\tau,n}\left(X^{n+1}\right), \tag{10}$$

which is always positive and where $F_{\tau,n}:\mathbb{R}^{n+1}\to\mathbb{R}_+^*$ is defined as follows:

$$F_{\tau,n}\left(X^{n+1}\right)=\underbrace{\int_0^1\cdots\int_0^1}_{n\ times}\prod_{k=1}^{n-1}(1-t_k)^{n-k}\,.e^{-\tau\left[s_1,\left[s_2,\cdots\left[s_n,s_{n+1}\right]_{t_n}\cdots\right]_{t_2}\right]_{t_1}}dt_n\cdots dt_1$$

Remark 2.2. *It is worth mentioning that the product in the expression of Q_n given by* (10) *corresponds to the determinant of the standard Vandermonde matrix, see for instance*[41].

2.4. Symmetry property

The multivariate function $F_{\tau,n}$ admits an invariance property that will be emphasized in the following Lemma which will be used in the proof of the main results. Its proof is presented in the appendix.

Lemma 2.1. *For any positive delay τ the functional $F_{\tau,n}$ is invariant for any permutation of the finite sequence $(s_1,s_2,\cdots,s_{n+1})$, namely, for any permutation σ of X^{n+1}, we have*

$$F_{\tau,n}\left(X^{n+1}\right)=F_{\tau,n}\left(\sigma\left(X^{n+1}\right)\right).$$

For instance, for $n=2$, Lemma 2.1 allows to say that for all $(x,y,z)\in\mathbb{R}^3$,

$$F_{\tau,2}(x,y,z)=\int_0^1\int_0^1(1-t_1)\,e^{-\tau(t_1x+(1-t_1)\,(t_2\,y+(1-t_2)\,z))}dt_1\,dt_2$$

and

$$F_{\tau,2}(x,y,z)=F_{\tau,2}(x,z,y)=F_{\tau,2}(y,x,z)=F_{\tau,2}(y,z,x)=F_{\tau,2}(z,x,y)=F_{\tau,2}(z,y,x)\,.$$

Remark 2.3. *The symmetry property emphasized in the above Lemma 2.1 is justified by the convexity property on the argument of the exponential kernel. Its proof which can be found in the appendix relies on simple change of coordinates.*

2.5. Shifting properties

The following Lemmas exhibit some shifting properties which will be used in the proof of the main results. Their proofs are presented in the appendix.

Lemma 2.2. *Let $(s_i)_{i=1}^{n+1}$ be a sequence of distinct real numbers. Let $(s_{i_k})_{k=1}^{k=m+1}\subset\mathbb{R}$ be any subsequence from $(s_i)_{i=1}^{n+1}\subset\mathbb{R}$. For $1\le M\le n-1$, let*

$$\mathrm{I}_{m,M}=\left\{(i_1,i_2,\cdots,i_m)\in\mathbb{N}^m,\ \sum_{j=1}^m i_j=M\right\}.$$

Then

$$\sum_{(i_1,i_2,\cdots,i_m)\in\mathrm{I}_{m,M}}\prod_{k=1}^m s_{j_k}^{i_k}-\sum_{(i_1,i_2,\cdots,i_m)\in\mathrm{I}_{m,M}}\prod_{k=1}^m s_{j_{k+1}}^{i_k}=\left(s_{j_1}-s_{j_{m+1}}\right)\sum_{(i_1,i_2,\cdots,,i_{m+1})\in\mathrm{I}_{m+1,M-1}}\prod_{k=1}^{m+1}s_{j_k}^{i_k}$$

Lemma 2.3. *Let $\tau > 0$ and $n \geq 1$. Let $(s_i)_{i=1}^{n+1}$ be a sequence of distinct real numbers. For any subsequence $(s_{i_k})_{k=1}^{k=m+1}$ from $(s_i)_{i=1}^{n+1}$, the function $F_{\tau,m}$ satisfies*

$$F_{\tau,m-1}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}\right) - F_{\tau,m-1}\left(s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right) = -\tau\left(s_{i_1} - s_{i_{m+1}}\right) F_{\tau,m}\left(s_{i_1}, s_{i_2}, \cdots, s_{i_m}, s_{i_{m+1}}\right).$$

Remark 2.4.

- *Lemma 2.2 and Lemma 2.3 remain valid even if the elements of the sequence $(s_i)_{1\leq i\leq n+1}$ are distinct and complex.*
- *Under the conditions of Lemma 2.3 it is obvious that $F_{\tau,n} > 0$ for any $\tau > 0$.*

2.6. Factorization property

The following Lemma provides a way to factorize a given quasipolynomial function (3) having at least n distinct real roots. This will be used in the proof of the main results.

Lemma 2.4. *Assume that the quasipolynomial* (3) *admits n distinct real roots $s_n < \ldots < s_1$ then it can be written under the following factorized form:*

$$\Delta_n(s,\tau) = \prod_{i=1}^{n}(s - s_i)\left[1 + (-\tau)^n \alpha F_{\tau,n}\left(s, s_1, \cdots, s_n\right)\right]. \tag{11}$$

3. Main results

In this section, we provide mainly two theorems exploiting the structural properties of the considered class of functional Vandermonde matrices to give some qualitative properties of the solutions of (1). Namely, the first theorem gives conditions on the coexistence of real roots of the quasipolynomial Δ_n. The next theorem emphasizes the effect of the coexistence of such real roots on the remaining roots of Δ_n. Finally, the combination of those results allows to give some important insights on the exponential stability of the solutions of (1).

3.1. Coexistence of $n+1$ real roots of Δ_n

The following Theorem 3 allows to recover PS_B as a bound of the admissible number of coexisting real roots for the quasipolynomial (3), see for instance [14]. This provides an alternative constructive analytical proof based on factorization technique. Furthermore, explicit conditions on the parameters guaranteeing the coexistence of such a number of real roots is provided allowing to Vieta's-like formulas for quasipolynomials.

Theorem 3.

i) The maximal number of coexisting real roots of the quasipolynomial (3) *is $n+1$.*

ii) *For a fixed* $\tau > 0$*, Equation* (3) *admits* $n+1$ *distinct real spectral values* s_{n+1}*,* s_n*,* $\cdots$*,* s_2 *and* s_1 *with* $s_{n+1} < \cdots < s_2 < s_1$ *if, and only if, the coefficients* $(a_k)_{0 \le k \le n-1}$ *and* α *are respectively given by the following functions in* τ *and* $X^{n+1} = (s_1, \ldots, s_{n+1})$

$$a_0\left(X^{n+1}, \tau\right) = \frac{1}{Q_n\left(X^{n+1}, \tau\right)} \det \begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & -s_1^n & e^{-\tau s_1} \\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & -s_2^n & e^{-\tau s_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & -s_n^n & e^{-\tau s_n} \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & -s_{n+1}^n & e^{-\tau s_{n+1}} \end{bmatrix}, \tag{12}$$

and for $1 \le k \le n-1$ *one has:*

$$a_k\left(X^{n+1}, \tau\right) = \frac{1}{Q_n\left(X^{n+1}, \tau\right)} \det \begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1^{k+1} & -s_1^n & s_1^{k-1} & \cdots & s_1 & 1 & e^{-\tau s_1} \\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2^{k+1} & -s_2^n & s_2^{k-1} & \cdots & s_2 & 1 & e^{-\tau s_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n^{k+1} & -s_n^n & s_n^{k-1} & \cdots & s_n & 1 & e^{-\tau s_n} \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1}^{k+1} & -s_{n+1}^n & s_{n+1}^{k+1} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}} \end{bmatrix}, \tag{13}$$

and

$$\alpha\left(X^{n+1}, \tau\right) = \frac{1}{Q_n\left(X^{n+1}, \tau\right)} \det \begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & s_1 & 1 & -s_1^n \\ s_2^{n-1} & s_2^{n-2} & \cdots & s_2 & 1 & -s_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_n^{n-1} & s_n^{n-2} & \cdots & s_n & 1 & -s_n^n \\ s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & -s_{n+1}^n \end{bmatrix}. \tag{14}$$

Remark 3.1.

- *From a control theory point of view, let recall the design problem presented in* (2)*, which consists in tuning the controller gain* α *and the delay parameter* τ *such that the closed-loop system's solution is asymptotically stable. In such a problem the sign of the controller gain is important with respect to the system structure. Here one has to emphasize that in the design induced from the result we propose, the coefficient* α *is of alternate sign with respect to the parity of the derivative order* n*.*
- *One can observe that the asymptotic expansion of the coefficients* a_k *allows to recover the well-know Vieta's formulas. This comes from the fact that when* $\tau \to \infty$ *the quasipolynomial* Δ_n *reduces to a polynomial of degree* n*. So here the important fact to emphasize is the disappearance of the* $(n+1)$*-th real root of the quasipolynomial* Δ_n*.*

Proof of Theorem 3. Let us start by the proof of *ii)* and we conclude by *i)*.

ii) Assume that (3) admits $n+1$ real spectral values $s_1 > s_2 > \cdots > s_{n+1}$. This means that the coefficients

$(a_k)_{0\leq k\leq n-1}$ and α satisfy the linear system

$$\Delta_n(s_i,\tau)=s_i^n+\sum_{k=0}^{n-1}a_k s_i^k+\alpha e^{-\tau s_i}=0,\quad \text{for all } i=1,\cdots,n+1. \tag{15}$$

Thanks to the invertibility of structured functional Vandermonde type matrix $V_n\left(X^{n+1},\tau\right)$ as asserted in Theorem 2, one deals with a Cramer system with respect to the coefficients $(a_k)_{0\leq k\leq n-1}$ and α. So that, one easily computes these coefficients with the standard formulas allowing to (12), (13) and (14) respectively. In particular, the expression of $\alpha\left(X^{n+1},\tau\right)$ is reduced to

$$\alpha\left(X^{n+1},\tau\right)=\frac{(-1)^{n+1}\prod\limits_{\substack{i<j\\ i,j=1}}^{n+1}(s_i-s_j)}{\det V_n\left(X^{n+1},\tau\right)}=(-1)^{n+1}\left[\tau^n F_{\tau,n}\left(X^{n+1}\right)\right]^{-1}. \tag{16}$$

showing the alternating sign of α.

i) Let proceed by contradiction in assuming the coexistence of $n+2$ real roots of (3). We shall use the factorization of (3) derived in Lemma 2.4, that is:

$$\Delta_n(s,\tau)=\prod_{i=1}^{n}(s-s_i)\left[1+(-\tau)^n\alpha F_{\tau,n}\left(s,s_1,\cdots,s_n\right)\right].$$

Since we assumed that s_{n+1} and s_{n+2} are two distinct real roots of Δ_n then one has

$$\begin{cases}\Delta_n(s_{n+1},\tau)=\prod\limits_{i=1}^{n}(s_{n+1}-s_i)\left[1+(-\tau)^n\alpha F_{\tau,n}\left(s_{n+1},s_1,\cdots,s_n\right)\right]=0,\\ \Delta_n(s_{n+2},\tau)=\prod\limits_{i=1}^{n}(s_{n+2}-s_i)\left[1+(-\tau)^n\alpha F_{\tau,n}\left(s_{n+2},s_1,\cdots,s_n\right)\right]=0.\end{cases} \tag{17}$$

Hence, $F_{\tau,n}\left(s_{n+2},s_1,\cdots,s_n\right)-F_{\tau,n}\left(s_{n+2},s_1,\cdots,s_n\right)=0$, which, using the shift property given in Lemma 2.3, proves the inconsistency in assuming the coexistence of $n+2$ distinct real roots.

■

3.2. On qualitative properties of s_1 as a root of Δ_n

To study the stability of solutions of Equation (3), one needs to study the negativity as well as the dominancy of the root s_1 by using an adequate factorization of the quasipolynomial $\Delta_n(s,\tau)$ in (3).

Theorem 4. *The following assertions hold:*

i) **(Negativity)** *The spectral value s_1 is negative if, and only if, there exists $\tau^*>0$ such that*

$$a_{n-1}(\tau^*)+\sum_{k=2}^{n}s_k=0. \tag{18}$$

ii) **(Dominancy)** *The spectral value s_1 is the spectral abscissa of Equation* (1).

Proof of Theorem 4.

i) Let assume that $s_1 < 0$. Since the parameter a_{n-1} given by (13) is a continuous function with respect to the delay τ and thanks to the l'Hospital's rule one asserts that its asymptotic behavior is described by: $\lim_{\tau\to 0} a_{n-1}(\tau) = -\infty$ and $\lim_{\tau\to\infty} a_{n-1}(\tau) = -\sum_{k=1}^{n} s_k > 0$, which proves the existence of $\tau^* > 0$ such that $a_{n-1}(\tau^*) + \sum_{k=2}^{n} s_k = 0$. Conversely, to show the negativity of s_1, one exploites determinant expressions provided in Theorem 2, allowing to write for any $\tau > 0$ one has:

$$a_{n-1}(\tau) = -\sum_{k=1}^{n} s_k - \frac{1}{\tau}\,\frac{F_{n-1,\tau}(s_1,\dots,s_n)}{F_{n,\tau}(s_1,\dots,s_{n+1})}.$$

In particular

$$a_{n-1}(\tau^*) + \sum_{k=2}^{n} s_k = -s_1 - \frac{1}{\tau^*}\,\frac{F_{n-1,\tau^*}(s_1,\dots,s_n)}{F_{n,\tau^*}(s_1,\dots,s_{n+1})}.$$

Using (18) and the positivity of τ^* as well as the positivity of both F_{n,τ^*} and F_{n-1,τ^*} one concludes

$$s_1 = -\frac{1}{\tau^*}\,\frac{F_{n-1,\tau^*}(s_1,\dots,s_n)}{F_{n,\tau^*}(s_1,\dots,s_{n+1})} < 0.$$

ii) The proof is based on the quasipolynomial factorization established in the proof of Theorem 3, more precisely, see formula (11).

To prove dominancy property for s_1, let us assume that there exists some $s_0 = \zeta + j\eta$ a root of $\Delta_n(s,\tau) = 0$ such that $\zeta > s_1$. This means that $P(s_0,\tau) = 0$. Hence

$$\begin{aligned} 1 &= (-1)^{n+1}\tau^n \alpha F_{\tau,n}(s_0, s_1, \cdots, s_n) = (-1)^{n+1}\tau^n \alpha \mathrm{Re}\left(F_{\tau,n}(s_0, s_1, \cdots, s_n)\right) \\ &= \tau^n |\alpha| \mathrm{Re}\left(F_{\tau,n}(s_0, s_1, \cdots, s_n)\right) \le \tau^n |\alpha| F_{\tau,n}(\zeta, s_1, \cdots, s_n). \end{aligned} \tag{19}$$

Denote by $x_{2,n}$the quantity $\left[s_2, \cdots [s_{n-1}, s_n]_{t_n} \cdots\right]_{t_3}$. Rewriting the term $\left[\zeta, [x_{2,n}, s_1]_{t_2}\right]_{t_1}$ as follows

$$\begin{aligned} \left[\zeta, [x_{2,n}, s_1]_{t_2}\right]_{t_1} &= t_1(\zeta - s_1) + s_1 + t_2(1-t_1)(x_{2,n} - s_1) \\ &= t_1(\zeta - s_1) + [x_{2,n}, s_1]_{t_2(1-t_1)} \\ &= t_1(\zeta - s_1) + \left[s_1, [x_{2,n}, s_1]_{t_2}\right]_{t_1}. \end{aligned}$$

Then, using the following estimates

$$\left[s_1, [x_{2,n}, s_1]_{t_2}\right]_{t_1} > \left[s_1, [x_{2,n}, s_{n+1}]_{t_2}\right]_{t_1} \quad \text{and} \quad e^{-\tau t_1(\zeta - s_1)} < 1, \ \forall t_1 \in]0,1[$$

we get from (19) and Lemma 2.1

$$\begin{aligned} 1 &\le \tau^n |\alpha| \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k} e^{-\tau t_1(\zeta - s_1)} e^{-\tau\left[\zeta,[x_{2,n},s_1]_{t_2}\right]_{t_1}} dt_n \cdots dt_1 \\ &< \tau^n |\alpha| \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k} e^{-\tau\left[s_1,[x_{2,n},s_{n+1}]_{t_2}\right]_{t_1}} dt_n \cdots dt_1 \end{aligned}$$

$$= \quad \tau^n \, |\alpha| \, F_{\tau,n} \left(s_1, s_2, \cdots, s_{n+1}\right) = 1 \ \ \text{(thanks to (16))},$$

which is inconsistent. This proves the dominancy of s_1. The proof of Theorem 4 is achieved.

■

Remark 3.2. *Note that the factorization* (11) *of* $\Delta_n(.,\tau)$ *allows to retrieve the explicit expression of the coefficient* α *defined in* (14)*, since* s_{n+1} *is a root of quasipolynomial* $\Delta_n(.,\tau)$*. Just replace* s *by* s_{n+1} *in* (11)*.*

3.3. Exponential stability

Note that for a linear retarded functional differential equations the exponential stability is equivalent to the uniform asymptotic stability,[21] p79. Further, for the linear autonomous retarded functional differential equations, asymptotic stability implies uniform asymptotic stability and, hence, exponential stability. Recall that Theorem 3 gives necessary and sufficient conditions for the coexistence of $n+1$ real roots of (3). Theorem 4 gives a necessary and sufficient conditions for the negativity of all such real roots and asserts that the roots of (3) have necessarily $\Re(s) < s_1$. So the following result which is a direct consequence of Theorems 3-4 allows to the exponential stability.

Corollary 3.1. *If equation* (3) *admits* $(n+1)$ *distinct real spectral values* $s_{n+1} < \ldots < s_1$ *and* (18) *is satisfied then the trivial solution of* (1) *is exponentially stable with* s_1 *as a decay rate.*

4. Stabilizing coupled oscillators using delayed output feedback

To show the applicative potential of the obtained results, let consider as an illustrative example a system consisting in two coupled oscillators. Coupled oscillations occur when two or more oscillating systems are connected in such a way the motion energy is transferred between them. The dynamics of coupled oscillators plays an important role in a variety of systems in nature and technology, see for instance[42] and references therein. Their ability to display complex self-organized dynamical phenomena makes them an important tool to explain fundamental mechanism of emergent dynamics in coupled systems. It is known that when the coupling is small then each oscillator operates at its natural frequency and the system is then said to be incoherent. However, when the coupling exceeds a certain threshold then the system spontaneously synchronizes. Here we consider the mechanical system of two coupled oscillators as depicted in Figure 1 and we aim to design a stabilizing delayed controller, which corresponds to *oscillation quenching*. Using the fundamental principle of dynamics and the standard assumption about the linearity of the damping lead to the following differential equations governing the motion of the system:

$$\begin{cases} m_1 \ddot{x}_1(t) = -b_1 \dot{x}_1(t) - k_1 x_1(t) + k_2 \left(x_2(t) - x_1(t)\right) + f(t), \\ m_2 \ddot{x}_2(t) = -k_2 \left(x_2(t) - x_1(t)\right). \end{cases} \tag{20}$$

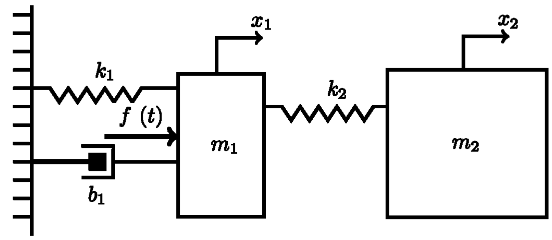

Figure 1: Coupled damped oscillators.

where the parameters values are chosen accordingly to an experimental settings: $b_1 = \frac{1}{2}$, $k_1 = 0.836$, $k_2 = 1$, $m_1 = 0.15$, $m_2 = 3$. If forcing term f acts on the system as an input and takes a *proportional-minus-delay* structure as suggested in [23]; that is $f(t) = -\alpha_0\, x_2(t) - \alpha_1\, x_2(t-\tau)$ and by setting $\xi(t) = (x_1(t), \dot{x}_1(t), x_2(t), \dot{x}_2(t))$ the above system writes

$$\dot{\xi}(t) = A_0 \xi(t) + A_1 \xi(t-\tau), \tag{21}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_2+k_1}{m_1} & -\frac{b_1}{m_1} & \frac{k_2}{m_1} - \frac{\alpha_0}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_1}{m_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding characteristic quasipolynomial function has the form (3) and writes explicitely as follows:

$$\Delta_4(s,\tau) = s^4 + \frac{b_1}{m_1} s^3 + \frac{(k_2 m_2 + m_1 k_2 + k_1 m_2)}{m_1 m_2} s^2 + \frac{b_1 k_2}{m_1 m_2} s + \frac{k_1 k_2 + k_2 \alpha_0}{m_1 m_2} + \frac{k_2 \alpha_1 \mathrm{e}^{-s\tau}}{m_1 m_2}. \tag{22}$$

The aim is to establish values for controller's gains α_0 and α_1 as well as the value of the delay parameter τ enabling us to assign $PS_B = 5$ real roots of the quasipolynomial (22) guaranteeing the exponential stability of the trivial solution of the closed-loop system as asserted in Theorem 4. To simplify the design task, we consider the case of equidistributed negative spectral values where the distance between two consecutive roots is $d = \frac{1}{2}$. By setting a targeted decay rate or equivalently the rightmost root, for instance at $s_1 = -1$; that is $s_k = s_1 - \frac{k-1}{2}$ for $k = 2, \ldots, 5$ one then applies Theorem 4 and a simple parameter identification to recover the gains values $\alpha_0 \approx 5.29, \alpha_1 \approx -4.54$ and the delay value $\tau \approx 0.81$, the spectrum distribution illustrated in Figure 2.

5. Concluding remarks

By this paper, we investigated conditions on the coefficients of the $n-th$ order linear ordinary differential equations with delayed-state forcing term guaranteeing the coexistence of the maximal number of real spectral

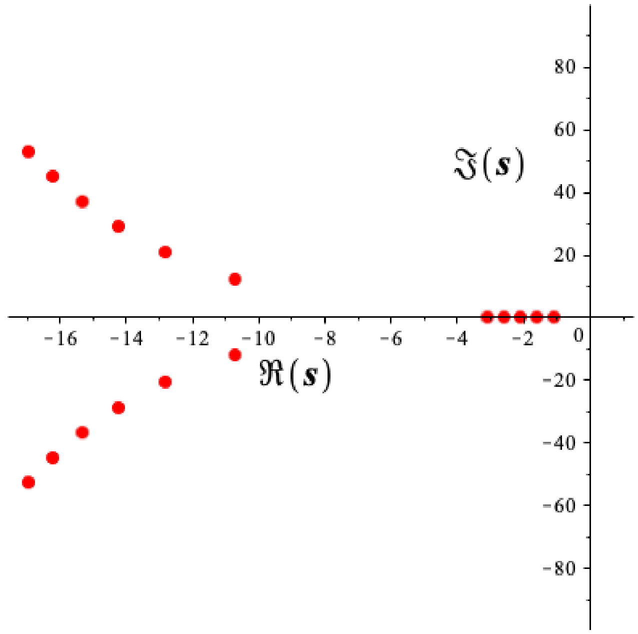

Figure 2: Spectrum distribution of the closed-loop system (21) using a proportional-minus-delay controller. The parameters values are given in Section 4.

values, which itself corresponds to the well-known Polya and Szegö bound for quasipolynomial's real roots. Such a bound was recovered using an analytical constructive approach. Furthermore, an easy to check criterion was provided, which allows to characterize the stabilizing effect of the coexistence of such spectral values. It is worth noting that such a configuration guarantees the exponential stability and explicitly describes the corresponding exponential decay rate. The applicative potential of the presented results is illustrated through the problem of stabilizing controller design for the system of coupled oscillators.

Acknowledgements

IB & SIN are partially financially supported by a public grant overseen by the French National Research Agency (ANR) as part of the "Investissement d'Avenir" program, through the "iCODE Institute project" funded by the IDEX Paris-Saclay and a grant from Hubert Curien (PHC) BALATON, project number 40502NM. The authors warmly thank Jean-Jacques Loiseau (LS2N Nantes, France) for discussion and insights on pole-placement for delay systems.

Appendix: Proof of the technical lemmas

Proof of Lemma 2.1. Without loss of generality, it suffices to consider the following permutation

$$\sigma_i\left(s_1, s_2, \cdots, s_i, s_{i+1}, \cdots, s_{n+1}\right) = \left(s_1, s_2, \cdots, s_{i+1}, s_i, \cdots, s_{n+1}\right),$$

where $1 \leq i \leq n$. Any other permutation of sets of indices is none other than the composition of such permutations. For example, if $\sigma_{i,j}$, with $j - i > 1$, is such that

$$\sigma_{i,j}\left(s_1, s_2, \cdots, s_i, \cdots, s_j, \cdots, s_{n+1}\right) = \left(s_1, s_2, \cdots, s_j, \cdots, s_i, \cdots, s_{n+1}\right),$$

then

$$\sigma_{i,j} = \sigma_i \circ \sigma_{i+1} \circ \cdots \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \cdots \circ \sigma_{i+1} \circ \sigma_i.$$

Write $\left[s_1, \left[s_2, \cdots \left[s_n, s_{n+1}\right]_{t_n} \cdots\right]_{t_2}\right]_{t_1}$ as

$$t_1 s_1 + (1 - t_1) t_2 s_2 + \cdots + \prod_{k=1}^{i-1} (1 - t_k) t_i s_i + \prod_{k=1}^{i} (1 - t_k) t_{i+1} s_{i+1} + \cdots + \prod_{k=1}^{n-1} (1 - t_k) t_n s_n + \prod_{k=1}^{n} (1 - t_k) s_{n+1}.$$

It is then necessary to introduce a suitable change of variable, that switches the coefficient of s_i with the coefficient of s_{i+1}, without affecting the other coefficients. Let

$$\left\{\begin{array}{l} \left\{\begin{array}{lcl} u_k & = & t_k, \quad k \neq i \wedge i+1, \\ u_i & = & (1 - t_i) t_{i+1} \\ u_{i+1} & = & \dfrac{t_i}{1 - t_{i+1} + t_i t_{i+1}} \end{array}\right. \quad \text{if} \quad 1 \leq i \leq n-1 \\ \\ \text{and} \\ \\ \left\{\begin{array}{lcl} u_k & = & t_k, \ 1 \leq k \leq n-1 \\ u_n & = & 1 - t_n \end{array}\right. \quad \text{if} \quad i = n \end{array}\right. \tag{23}$$

Clearly, $u_i \in]0,1[$ for all $1 \leq i \leq n-1$. Moreover, from $(1 - t_i)(1 - t_{i+1}) > 0$, we have $1 - t_{i+1} + t_i t_{i+1} > t_i > 0$, hence $u_{i+1} \in]0,1[$. The jacobian matrix $J = \frac{D(u_1, u_2, \cdots, u_n)}{D(t_1, t_2, \cdots, t_n)}$ is such that

$$\det J = \frac{t_i - 1}{t_i t_{i+1} - t_{i+1} + 1} \neq 0.$$

So, (23) defines a C^1-diffeomorphism from $]0,1[^n$ into $]0,1[^n$, for all $1 \leq i \leq n-1$, and the following properties

$$\begin{gathered} t_i \prod_{k=1}^{i-1} (1 - t_k) = u_{i+1} \prod_{k=1}^{i} (1 - u_k), \\ t_{i+1} \prod_{k=1}^{i} (1 - t_k) = u_i \prod_{k=1}^{i-1} (1 - u_k), \\ t_m \prod_{k=1}^{m-1} (1 - t_k) = u_m \prod_{k=1}^{m-1} (1 - u_k), \quad \forall m \in \{2, \cdots, n\}, \ m \neq i \wedge i+1 \end{gathered}$$

are satisfied.

On the other hand, from

$$du_1 du_2 \cdots du_n = \left|\det \frac{D(u_1, u_2, \cdots, u_n)}{D(t_1, t_2, \cdots, t_n)}\right| dt_1 dt_2 \cdots dt_n = \frac{1 - t_i}{1 - u_i} dt_1 dt_2 \cdots dt_n,$$

we get

$$\prod_{k=1}^{n-1}(1-t_k)^{n-k}\,dt_1dt_2\cdots dt_n=\prod_{k=1}^{n-1}(1-u_k)^{n-k}\,du_1du_2\cdots du_n.$$

The case $i=n$ is simpler so omitted. The symmetry property is well proven. ■

Proof of Lemma 2.2. Let us first observe that the preceding sums (or the homogeneous forms of degree M and $M-1$ respectively) are invariant under any permutation between the s_{i_k}, for $k\in\{1,\cdots,m+1\}$. Thus, using the well-known factorization

$$s_{j_1}^{i_{m-1}}-s_{j_{m+1}}^{i_{m-1}}=\left(s_{j_1}-s_{j_{m+1}}\right)\sum_{i_m=0}^{i_{m-1}-1}s_{j_1}^{i_m}s_{j_{m+1}}^{i_{m-1}-i_m-1}$$

we have

$$\begin{aligned}
&\sum_{(i_1,i_2,\cdots,i_m)\in\mathrm{I}_{m,M}}\prod_{k=1}^{m}s_{j_k}^{i_k}-\sum_{(i_1,i_2,\cdots,i_m)\in\mathrm{I}_{m,M}}\prod_{k=1}^{m}s_{j_{k+1}}^{i_k}\\
&=\sum_{i_1=0}^{M}\cdots\sum_{i_k=0}^{i_{k-1}}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}s_{j_1}^{i_{m-1}}s_{j_2}^{i_{m-2}-i_{m-1}}\cdots s_{j_k}^{i_{m-k}-i_{m-k+1}}\cdots s_{j_{m-1}}^{i_1-i_2}s_{j_m}^{M-i_1}\\
&\quad-\sum_{i_1=0}^{M}\cdots\sum_{i_k=0}^{i_{k-1}}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}s_{j_{m+1}}^{i_{m-1}}s_{j_2}^{i_{m-2}-i_{m-1}}\cdots s_{j_k}^{i_{m-k}-i_{m-k+1}}\cdots s_{j_m}^{M-i_1}\\
&=\sum_{i_1=0}^{M}\cdots\sum_{i_k=0}^{i_{k-1}}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}\left(s_{j_1}^{i_{m-1}}-s_{j_{m+1}}^{i_{m-1}}\right)\left(s_{j_2}^{i_{m-2}-i_{m-1}}\cdots s_{j_k}^{i_{m-k}-i_{m-k+1}}\cdots s_{j_m}^{M-i_1}\right)\\
&=\left(s_{j_1}-s_{j_{m+1}}\right)\sum_{i_1=0}^{M}\cdots\sum_{i_k=0}^{i_{k-1}}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}\sum_{i_m=0}^{i_{m-1}-1}\left(s_{j_1}^{i_m}s_{j_{m+1}}^{i_{m-1}-i_m-1}\right)\left(s_{j_2}^{i_{m-2}-i_{m-1}}\cdots s_{j_k}^{i_{m-k}-i_{m-k+1}}\cdots s_{j_m}^{M-i_1}\right)\\
&=\left(s_{j_1}-s_{j_{m+1}}\right)\sum_{(i_1,i_2,\cdots,,i_{m+1})\in\mathrm{I}_{m+1,M-1}}\prod_{k=1}^{m+1}s_{j_k}^{i_k}
\end{aligned}$$

■

Proof of Lemma 2.3. In view of Lemma 2.1, we have $F_{\tau,m-1}\left(s_{i_1},s_{i_2},\cdots,s_{i_m}\right)=F_{\tau,m-1}\left(s_{i_2},\cdots,s_{i_m},s_{i_1}\right)$, hence

$$\begin{aligned}
&F_{\tau,m-1}\left(s_{i_1},s_{i_2},\cdots,s_{i_m}\right)-F_{\tau,m-1}\left(s_{i_2},\cdots,s_{i_m},s_{i_{m+1}}\right)\\
&=\underbrace{\int_0^1\cdots\int_0^1}_{(m-1)\ \text{times}}\prod_{k=1}^{m-2}(1-t_k)^{m-1-k}\cdot\left(e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m},s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}}-e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m},s_{i_{m+1}}\right]_{t_{m-1}}\cdots\right]_{t_1}}\right)dt_{m-1}\cdots dt_1
\end{aligned}$$

Using the judicious form of the exponential in brackets,

$$e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m},s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}}=e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2},t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}}e^{-\tau\prod_{k=1}^{m-1}(1-t_k)s_{i_1}}$$

which allows to isolate the last term of the convex combination, we obtain by virtue of the mean value theorem applied to $x \longmapsto e^{-\tau \prod\limits_{k=1}^{m-1}(1-t_k)x}$:

$$\begin{aligned}
& e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m},s_{i_1}\right]_{t_{m-1}}\cdots\right]_{t_1}} - e^{-\tau\left[s_{i_2},\cdots\left[s_{i_m},s_{i_{m+1}}\right]_{t_{m-1}}\cdots\right]_{t_1}} \\
= \; & e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2},t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}} \left(-\tau \prod_{k=1}^{m-1}\left(1-t_k\right)\left(s_{i_1}-s_{i_{m+1}}\right) \int_0^1 e^{-\tau\left(t_m s_{i_1}+(1-t_m)s_{i_{m+1}}\right)} dt_m\right) \\
= \; & -\tau \prod_{k=1}^{m-1}\left(1-t_k\right)\left(s_{i_1}-s_{i_{m+1}}\right) e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2},t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1}} \int_0^1 e^{-\tau \prod\limits_{k=1}^{m-1}(1-t_k)\left(t_m s_{i_1}+(1-t_m)s_{i_{m+1}}\right)} dt_m
\end{aligned}$$

Hence, thanks to the two properties

$$\prod_{k=1}^{m-1}\left(1-t_k\right) . \prod_{k=1}^{m-2}\left(1-t_k\right)^{m-1-k} = \prod_{k=1}^{m-1}\left(1-t_k\right)^{m-k}$$

and

$$\begin{aligned}
& \left[s_{i_2},\cdots\left[s_{m-2,}t_{m-1}s_{i_m}\right]_{t_{m-2}}\cdots\right]_{t_1} + \prod_{k=1}^{m-1}\left(1-t_k\right)\left(t_m s_{i_1}+\left(1-t_m\right)s_{i_{m+1}}\right) \\
& \qquad = \left[s_{i_2},\cdots\left[s_{m-2,}\left[s_{i_m},\left[s_{i_{m+1}},s_{i_1}\right]_{t_m}\right]_{t_{m-1}}\right]_{t_{m-2}}\cdots\right]_{t_1}
\end{aligned}$$

and Lemma 2.1, we get

$$\begin{aligned}
& F_{\tau,m-1}\left(s_{i_1},s_{i_2},\cdots,s_{i_m}\right) - F_{\tau,m-1}\left(s_{i_2},\cdots,s_{i_m},s_{i_{m+1}}\right) \\
& = -\tau\left(s_{i_1}-s_{i_{m+1}}\right) \underbrace{\int_0^1\cdots\int_0^1}_{m \text{ times}} \prod_{k=1}^{m-1}\left(1-t_k\right)^{m-k} . e^{-\tau\left[s_{i_2},\cdots\left[s_{m-2,}\left[s_{i_m},\left[s_{i_{m+1}},s_{i_1}\right]_{t_m}\right]_{t_{m-1}}\right]_{t_{m-2}}\cdots\right]_{t_1}} dt_m dt_{m-1}\cdots dt_1 \\
& \qquad = -\tau\left(s_{i_1}-s_{i_{m+1}}\right) F_{\tau,m}.
\end{aligned}$$

This achieves the proof of the lemma. ■

Proof of Theorem 2. The calculation is done in n steps. The idea is to have at each step k in the penultimate column only "1". Then, a linear combination of the lines makes it possible to reduce the size of the determinant of a unit, as well as to recover the factors $(s_i - s_j)$, with $i - j = k$, using Lemmas 2.3 and 2.2. To do so, denoting by L_i the i−th line of $V_n(\tau) := V_n(s_1, s_2, \cdots, s_{n+1}, \tau)$. Replacing L_i by $L_i - L_{i+1}$, for $i = 1, \cdots, n$, in $\det V_n(\tau)$, we get

$$\det V_n(\tau) = \det \begin{bmatrix}
s_1^{n-1}-s_2^{n-1} & s_1^{n-2}-s_2^{n-2} & \cdots & s_1-s_2 & 0 & e^{-\tau s_1}-e^{-\tau s_2} \\
s_2^{n-1}-s_3^{n-1} & s_2^{n-2}-s_3^{n-2} & \cdots & s_2-s_3 & 0 & e^{-\tau s_2}-e^{-\tau s_3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
s_n^{n-1}-s_{n+1}^{n-1} & s_n^{n-2}-s_{n+1}^{n-2} & \cdots & s_n-s_{n+1} & 0 & e^{-\tau s_n}-e^{-\tau s_{n+1}} \\
s_{n+1}^{n-1} & s_{n+1}^{n-2} & \cdots & s_{n+1} & 1 & e^{-\tau s_{n+1}}
\end{bmatrix}.$$

Using the mean value theorem as follows,

$$e^{-\tau s_i} - e^{-\tau s_{i+1}} = -\tau \left(s_i - s_{i+1}\right) \int_0^1 e^{-\tau(ts_i+(1-t)s_{i+1})} dt, \quad i = 1, \cdots, n,$$

we obtain

$$\det V_n(\tau) = \tau \prod_{k=1}^{n} (s_k - s_{k+1}) \times$$
$$\det \begin{bmatrix} \sum\limits_{i=0}^{n-2} s_1^i s_2^{n-2-i} & \sum\limits_{i=0}^{n-3} s_1^i s_2^{n-3-i} & \cdots & s_1 + s_2 & 1 & \int_0^1 e^{-\tau[s_1,s_2]_{t_1}} dt_1 \\ \sum\limits_{i=0}^{n-2} s_2^i s_3^{n-2-i} & \sum\limits_{i=0}^{n-3} s_2^i s_3^{n-3-i} & \cdots & s_2 + s_3 & 1 & \int_0^1 e^{-\tau[s_2,s_3]_{t_1}} dt_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum\limits_{i=0}^{n-2} s_n^i s_{n+1}^{n-2-i} & \sum\limits_{i=0}^{n-3} s_n^i s_{n+1}^{n-3-i} & \cdots & s_n + s_{n+1} & 1 & \int_0^1 e^{-\tau[s_n,s_{n+1}]_{t_1}} dt_1 \end{bmatrix}$$

we get using the same linear combination as above

$$\det V_n(\tau) = \tau^2 \prod_{k=1}^{n} (s_k - s_{k+1}) (s_1 - s_3) (s_2 - s_4) \cdots (s_{n-1} - s_{n+1}) \times$$
$$\det \begin{bmatrix} \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k\geq 0}} s_1^i s_2^j s_3^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k\geq 0}} s_1^i s_2^j s_3^k & \cdots & \sum\limits_{l=1}^{3} s_l & 1 & \int_0^1\int_0^1 (1-t)\, e^{-\tau\left[s_1,[s_2,s_3]_\theta\right]_t} d\theta dt \\ \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k\geq 0}} s_2^i s_3^j s_4^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k\geq 0}} s_2^i s_3^j s_4^k & \cdots & \sum\limits_{l=2}^{4} s_l & 1 & \int_0^1\int_0^1 (1-t)\, e^{-\tau\left[s_2,[s_3,s_4]_\theta\right]_t} d\theta dt \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum\limits_{\substack{i+j+k=n-3 \\ i,j,k\geq 0}} s_{n-1}^i s_n^j s_{n+1}^k & \sum\limits_{\substack{i+j+k=n-4 \\ i,j,k\geq 0}} s_{n-1}^i s_n^j s_{n+1}^k & \cdots & \sum\limits_{l=n-1}^{n+1} s_l & 1 & \int_0^1\int_0^1 (1-t)\, e^{-\tau\left[s_{n-1},[s_n,s_{n+1}]_\theta\right]_t} d\theta dt \end{bmatrix}$$

Repeating the same process as above. In the last step, only the term $(s_1 - s_{n+1})$ remains to be recovered. The determinant is reduced to the following:

$$\det V_n(\tau) = \tau^{n-1} \prod_{0<i-j\leq n} (s_i - s_j) \det \begin{bmatrix} 1 & F_{\tau,n}(s_1, s_2, \cdots, s_n) \\ 1 & F_{\tau,n}(s_2, \cdots, s_n, s_{n+1}) \end{bmatrix}.$$

Thanks to Lemma 2.3, we get

$$\det V_n(\tau) = \tau^n \prod_{\substack{i<j \\ i,j=1}}^{n+1} (s_i - s_j)\, F_{\tau,n}(s_1, s_2, \cdots, s_{n+1}),$$

which is always positive since $F_{\tau,n}$ is positive and $s_i > s_j$. ■

Proof of Lemma 2.4. We start first by carrying out the following factorization of Δ_n by writing $\Delta_n(s,\tau) = \prod_{i=1}^{n} (s - s_i) . P_n(s,\tau)$ with

$$P_n(s,\tau) = \left[\prod_{i=1}^{n} (s - s_i)\right]^{-1} \left(s^n + \sum_{k=1}^{n} a_{n-k} s^{n-k} + \alpha \exp(-\tau s)\right). \tag{24}$$

Let introduce the following coefficients which come from the standard Vieta's formulas

$$\widetilde{a}_{n-k} = (-1)^k \left(\sum_{i_1 < \cdots < i_k}^{n} \prod_{j=1}^{k} s_{i_j} \right), \qquad \text{for } k = 1, \cdots, n.$$

Then by performing an Euclidean division in (24) one gets:

$$P_n(s,\tau) = 1 + \frac{\sum\limits_{k=1}^{n} \left(a_{n-k} - \widetilde{a}_{n-k}\right) s^{n-k}}{\prod\limits_{i=1}^{n} (s - s_i)} + \frac{\alpha \exp(-\tau s)}{\prod\limits_{i=1}^{n} (s - s_i)}.$$

Let B_n be defined as follows: $B_n(s) := \sum\limits_{k=1}^{n} \left(a_{n-k} - \widetilde{a}_{n-k}\right) s^{n-k}$, which satisfies

$$B_n(s_k) = s_k^n + a_{n-1} s_k^{n-1} + \cdots + a_1 s_k + a_0, \quad \forall k = 1, \cdots, n.$$

Then one performs the partial fractions corresponding to $\dfrac{B_n(s)}{\prod\limits_{i=1}^{n} (s - s_i)}$ and $\dfrac{1}{\prod\limits_{i=1}^{n} (s - s_i)}$:

$$\frac{B_n(s)}{\prod\limits_{i=1}^{n} (s - s_i)} = \sum_{k=1}^{n} \frac{c_k}{s - s_k} \quad \text{and} \quad \frac{1}{\prod\limits_{i=1}^{n} (s - s_i)} = \sum_{k=1}^{n} \frac{d_k}{s - s_k} \tag{25}$$

where

$$\begin{aligned} c_k &= \frac{B_n(s_k)}{\prod\limits_{i=1, i \neq k}^{n} (s_k - s_i)} = \frac{s_k^n + \sum\limits_{i=1}^{n} a_{n-i} s_k^{n-i}}{\prod\limits_{i=1, i \neq k}^{n} (s_k - s_i)}, \qquad k = 1, \cdots, n. \\ d_k &= \left[\prod_{i=1, i \neq k}^{n} (s_k - s_i) \right]^{-1}, \qquad k = 1, \cdots, n. \end{aligned}$$

Thus

$$P_n(s,\tau) = 1 + \sum_{k=1}^{n} \left[\frac{B_n(s_k)}{(s - s_k) \prod\limits_{i=1, i \neq k}^{n} (s_k - s_i)} + \frac{\alpha e^{-\tau s}}{(s - s_k) \prod\limits_{i=1, i \neq k}^{n} (s_k - s_i)} \right]$$

Next, one substitutes $B_n(s_k)$ as $-\alpha\, e^{-\tau s_k}$ for $1 \leq k \leq n$ which allows to:

$$P_n(s,\tau) = 1 + \alpha \sum_{k=1}^{n} \left(\frac{e^{-\tau s} - e^{-\tau s_k}}{(s - s_k) \prod\limits_{i=1, i \neq k}^{n} (s_k - s_i)} \right).$$

At this step, we use the following integral representation

$$e^{-\tau s} - e^{-\tau s_k} = -\tau \,(s - s_k) \int_0^1 e^{-\tau [s, s_k]_{t_1}} \, dt_1 \tag{26}$$

to get

$$P_n(s,\tau) = 1 - \tau\alpha \sum_{k=1}^{n} \left(\left[\prod_{i=1, i\neq k}^{n} (s_k - s_i) \right]^{-1} \int_0^1 e^{-\tau[s,s_k]_{t_1}} dt_1 \right) \tag{27}$$

From the property $\sum_{k=1}^{n} \left[\prod_{i=1, i\neq k}^{n} (s_k - s_i) \right]^{-1} = 0$ which can be easily shown using the decomposition in partial fractions as illustrated in the second equality in (25). Hence, one can extract the first term (corresponding to $k = 1$) of (27) in terms of the remaining $n-1$-terms, and namely one gets:

$$\left[\prod_{i=2}^{n} (s_1 - s_i) \right]^{-1} \int_0^1 e^{-\tau[s,s_1]_{t_1}} dt_1 = -\sum_{k=2}^{n} \left[\prod_{i=1, i\neq k}^{n} (s_k - s_i) \right]^{-1} \int_0^1 e^{-\tau[s,s_1]_{t_1}} dt_1.$$

The expression of P_n given by (27) becomes:

$$P_n(s,\tau) = 1 - \tau\alpha \sum_{k=2}^{n} \left(\left[\prod_{i=1, i\neq k}^{n} (s_k - s_i) \right]^{-1} \int_0^1 \left(e^{-\tau[s,s_k]_{t_1}} - e^{-\tau[s,s_1]_{t_1}} \right) dt_1 \right).$$

From Lemma 2.3

$$\int_0^1 \left(e^{-\tau[s,s_k]_{t_1}} - e^{-\tau[s,s_1]_{t_1}} \right) dt_1 = -\tau\left(s_k - s_1\right) \int_0^1 \int_0^1 (1-t_1)\, e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}} dt_2 dt_1, \quad \text{for } k = 2, \cdots, n$$

we get

$$P_n(s,\tau) = 1 + (-\tau)^2 \alpha \sum_{k=2}^{n} \left(\left[\prod_{i=2, i\neq k}^{n} (s_k - s_i) \right]^{-1} \int_0^1 \int_0^1 (1-t_1)\, e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}} dt_2 dt_1 \right).$$

Observe that the coefficients $\left(\left[\prod_{i=2, i\neq k}^{n} (s_k - s_i) \right]^{-1} \right)_{2\leq k\leq n}$ satisfy

$$\sum_{k=2}^{n} \left[\prod_{i=2, i\neq k}^{n} (s_k - s_i) \right]^{-1} = 0,$$

(and are independent of s_1), hence repeating the previous step, by rewriting the first term in the last sum as follows:

$$\left[\prod_{i=3}^{n} (s_2 - s_i) \right]^{-1} = -\sum_{k=3}^{n} \left[\prod_{i=2, i\neq k}^{n} (s_k - s_i) \right]^{-1}$$

one obtains

$$P_n(s,\tau) \quad = \quad 1 + (-\tau)^2 \alpha \sum_{k=3}^{n} \left(\frac{\int_0^1 \int_0^1 (1-t_1) \left(e^{-\tau\left[s,[s_1,s_k]_{t_2}\right]_{t_1}} - e^{-\tau\left[s,[s_1,s_2]_{t_2}\right]_{t_1}} \right) dt_2 dt_1}{\prod_{i=2, i\neq k}^{n} (s_k - s_i)} \right)$$

$$= \quad 1 + (-\tau)^3 \alpha \sum_{k=3}^{n} \left(\frac{\int_0^1 \int_0^1 \int_0^1 (1-t_1)^2 (1-t_2)\, e^{-\tau \left[s, \left[s_1, \left[s_2, s_k\right]_{t_3}\right]_{t_2}\right]_{t_1}} dt_3 dt_2 dt_1}{\prod_{i=3, i \neq k}^{n} (s_k - s_i)} \right).$$

In reiterating the same process, one observes that the order of denominator decreases by one at each step, and one gets the general formula for the intermediate step l:

$$P_n(s,\tau) = 1 + (-\tau)^l \alpha \sum_{k=l}^{n} \frac{F_{\tau,l}(s, s_1, \dots, s_{l-1}, s_k)}{\prod_{i=l, i \neq k}^{n} (s_k - s_i)}$$

which, by induction, allows to obtain at the step $n-1$

$$P_n(s,\tau) = 1 + (-\tau)^{n-1} \alpha \frac{F_{\tau,n-1}(s, s_1, \dots, s_{n-2}, s_{n-1}) - F_{\tau,n-1}(s, s_1, \dots, s_{n-2}, s_n)}{s_{n-1} - s_n}$$

Finally, using the shift formula given in Lemma 2.3, one gets

$$P_n(s,\tau) = 1 + (-\tau)^n \alpha \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} \prod_{k=1}^{n-1} (1-t_k)^{n-k} e^{-\tau \left[s, \left[s_1, \cdots \left[s_{n-1}, s_n\right]_{t_n} \cdots \right]_{t_2}\right]_{t_1}} dt_n \cdots dt_1$$

which allows to the following factorization of the quasipolynomial $\Delta_n(.,\tau)$

$$\Delta_n(s,\tau) = \prod_{i=1}^{n} (s - s_i) \left[1 + (-\tau)^n \alpha F_{\tau,n}(s, s_1, \cdots, s_n)\right].$$

■

References

1. I. Boussaada, S.-I. Niculescu, Characterizing the codimension of zero singularities for time-delay systems, Acta Applicandae Mathematicae 145 (1) (2016) 47–88.

2. R. Bellman, K. Cooke, Differential-difference equations, Academic Press, New York, 1963.

3. K.-L. Cooke, P. van den Driessche, On zeroes of some transcendental equations, Funkcial. Ekvac. 29 (1) (1986) 77–90.

4. K. Walton, J. E. Marshall, Direct method for tds stability analysis, IEE Proceedings D - Control Theory and Applications 134 (2) (1987) 101–107.

5. G. Stépán, Retarded Dynamical Systems: Stability and Characteristic Functions, Pitman research notes in mathematics series, Longman Sci. and Tech., 1989.

6. J. K. Hale, S. M. V. Lunel, Introduction to functional differential equations, Vol. 99 of Applied Mathematics Sciences, Springer Verlag, New York, 1993.

7. W. Michiels, S.-I. Niculescu, Stability and stabilization of time-delay systems, Vol. 12 of Advances in Design and Control, SIAM, 2007.

8. R. Sipahi, S. i. Niculescu, C. T. Abdallah, W. Michiels, K. Gu, Stability and stabilization of systems with time delay, IEEE Control Systems 31 (1) (2011) 38–65.

9. I. Boussaada, S.-I. Niculescu, A. E. Ati, R. Perez-Ramos, K. Trabelsi, Multiplicity-induced-dominancy in parametric second-order delay differential equations: Analysis and application in control design, (submitted) (2019) 32pp.

10. F. Bedouhene, I. Boussaada, S.-I. Niculescu, On pole placement and spectral abscissa characterization for time-delay systems, in: Proceeding of the 15th IFAC Workshop on Time Delay Systems TDS 2019.

11. I. Boussaada, S. I. Niculescu, Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials: A Vandermonde-based approach, IEEE Transactions on Automatic Control 61 (2016) 1601–1606.

12. I. Boussaada, S.-I. Niculescu, Computing the codimension of the singularity at the origin for delay systems: The missing link with Birkhoff incidence matrices, 21st International Symposium on Mathematical Theory of Networks and Systems (2014) 1 – 8.

13. I. Boussaada, H. Unal, S.-I. Niculescu, Multiplicity and stable varieties of time-delay systems: A missing link, in: Proceeding of the 22nd International Symposium on Mathematical Theory of Networks and Systems, 2016, pp. 1–6.

14. G. Pólya, G. Szegő, Problems and Theorems in Analysis, Vol. I: Series, Integral Calculus, Theory of Functions, Springer-Verlag, New York, Heidelberg, and Berlin, 1972.

15. N. D. Hayes, Roots of the transcendental equation associated with a certain difference-differential equation, Journal of the London Mathematical Society s1-25 (3) (1950) 226–232.

16. I. Boussaada, S. Tliba, S.-I. Niculescu, H. Unal, T. Vyhlídal, Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. application to the control of a mechanical system, Linear Algebra and its Applications 542 (2018) 589 – 604, proceedings of the 20th ILAS Conference, Leuven, Belgium 2016.

17. I. Boussaada, S.-I. Niculescu, S. Tliba, T. Vyhlídal, On the coalescence of spectral values and its effect on the stability of time-delay systems: Application to active vibration control, Procedia IUTAM 22 (Supplement C) (2017) 75 – 82.

18. I. Boussaada, S.-I. Niculescu, On the dominancy of multiple spectral values for time-delay systems with applications, IFAC-PapersOnLine 51 (14) (2018) 55 – 60, 14th IFAC Workshop on Time Delay Systems TDS 2018.

19. I. Boussaada, S.-I. Niculescu, K. Trabelsi, Toward a decay rate assignment based design for time-delay systems with multiple spectral values, in: Proceeding of the 23rd International Symposium on Mathematical Theory of Networks and Systems, 2018, pp. 864–871.

20. S. Amrane, F. Bedouhene, I. Boussaada, S.-I. Niculescu, On qualitative properties of low-degree quasipolynomials: Further remarks on the spectral abscissa and rightmost-roots assignment, Bull. Math. Soc. Sci. Math. Roumanie Tome 61. No. 4 (109) (2018) 361–381.

21. V. Kolmanovskii, V. Nosov, Stability of Functional Differential Equations, Vol. 180 of Mathematics in Science and Engineering, Elsevier, 1986.

22. T. Mori, N. Fukuma, M. Kuwahara, On an estimate of the decay rate for stable linear delay systems, International Journal of Control 36 (1) (1982) 95–97.

23. I. Suh, Z. Bien, Proportional minus delay controller, IEEE Trans. on Aut. Cont. 24 (1979) 370–372.

24. K. Pyragas, Continuous control of chaos by self-controlling feedback, Phys. Lett. A 170 (1992) 421–428.

25. S.-I. Niculescu, W. Michiels, K. Gu, C. Abdallah, Delay Effects on Output Feedback Control of Dynamical Systems, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010, pp. 63–84.

26. S.-I. Niculescu, W. Michiels, Stabilizing a chain of integrators using multiple delays, IEEE Trans. on Aut. Cont. 49 (5) (2004) 802–807.

27. A. Manitius, A. Olbrot, Finite spectrum assignment problem for systems with delays, IEEE Transactions on Automatic Control 24 (4) (1979) 541–552. doi:10.1109/TAC.1979.1102124.

28. J. Loiseau, Invariant factors assignment for a class of time-delay systems, Kybernetika 37 (3) (2001) 265 – 275.

29. W. Michiels, K. Engelborghs, P. Vansevenant, D. Roose, Continuous pole placement for delay equations, Automatica 38 (5) (2002) 747 – 761.

30. S. Mondie, J. Loiseau, Finite spectrum assignment for input delay systems, IFAC Proceedings Volumes 34 (23) (2001) 201 – 206.

31. P. Zitek, J. Fiser, T. Vyhlidal, Dimensional analysis approach to dominant three-pole placement in delayed pid control loops, Journal of Process Control 23 (8) (2013) 1063 – 1074.

32. L. V. Ahlfors, Complex Analysis, McGraw-Hill, Inc., 1979.

33. B. Levin, Distribution of zeros of entire functions, Translations of Mathematical Monographs, AMS, Providence, Rhode Island, 1964.

34. N. Obreschkoff, Nullstellen linearer kombinationen von exponentialfunktionen, Jber. der Deutsch. Math. Verein. 37 (1928) 81–84.

35. G. Lorentz, K. Zeller, Birkhoff interpolation, SIAM Journal on Numerical Analysis 8 (1) (1971) pp. 43–48.

36. L. Gonzalez-Vega, Applying quantifier elimination to the Birkhoff interpolation problem, J. Symbolic. Comp. 22 (1) (1996) 83–104.

37. F. Rouillier, M. Din, E. Schost, Solving the Birkhoff interpolation problem via the critical point method: An experimental study, in: J. Richter-Gebert, D. Wang (Eds.), Automated Deduction in Geom., Vol. 2061 of LNCS, Springer Berlin Heidelberg, 2001, pp. 26–40.

38. T. Ha, J. Gibson, A note on the determinant of a functional confluent Vandermonde matrix and controllability, Linear Algebra and its Applications 30 (0) (1980) 69 – 75.

39. T. Kailath, Linear Systems, Prentice-Hall information and system sciences series, Prentice Hall International, 1998.

40. J. S. Respondek, On the confluent Vandermonde matrix calculation algorithm, App. Math. Letters 24 (2) (2011) 103 – 106.

41. P. Olver, On multivariate interpolation, Stud. Appl. Math. 116 (2006) 201–240.

42. A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge Nonlinear Science Series, 2003.