

Elliptic Problems in Smooth and Non Smooth Domains

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I. Introduction, Motivation and Recall

The purpose of this work is to study the existence and the regularity of solutions of some **elliptic equations**:

$$-\Delta u = f \quad \text{in } \Omega,$$

(or more generally

$$-\operatorname{div}(A \nabla u) + \mathbf{b} \cdot \nabla u + c u = f \quad \text{in } \Omega)$$

with the following boundary conditions

$$u = g \quad \text{or} \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma,$$

where Ω is a **bounded** open set of \mathbb{R}^N , with $N \geq 2$.

We will consider also the **biharmonic problem** with different boundary conditions.

I.1 Traces and Regularity for Ω regular

Theorem 1 (Traces)

- If Ω is of class $C^{0,1}$ (resp. $C^{1,1}$), we recall that

$$\begin{aligned} \gamma_0 : v &\longmapsto v|_{\Gamma}, \\ H^s(\Omega) &\longrightarrow H^{s-1/2}(\Gamma), \end{aligned}$$

is continuous and has a right continuous inverse for

$$\frac{1}{2} < s \leq 1 \quad (\text{resp. } 1 < s \leq 2).$$

- If Ω is of class $C^{1,1}$, we recall that

$$\begin{aligned} (\gamma_0, \gamma_1) : v &\longmapsto (v|_{\Gamma}, \frac{\partial u}{\partial \nu}) \\ H^s(\Omega) &\longrightarrow H^{s-1/2}(\Gamma) \times H^{s-3/2}(\Gamma), \end{aligned}$$

is continuous and has a right continuous inverse for

$$\frac{3}{2} < s \leq 2.$$

Question 1

With the **same hypothesis**, are the **continuity** and the **surjectivity** of the **trace operator** valid for s in a bigger interval?

For example, when Ω is only **Lipchitzian**, for which values of $s > 1$ the following mapping

$$\begin{aligned} \gamma_0 : v &\longmapsto v|_{\Gamma}, \\ H^s(\Omega) &\longrightarrow H^{s-1/2}(\Gamma), \end{aligned}$$

is continuous and has a right continuous inverse?

Consider now the following boundary value problem for the Laplacian:

$$(\mathcal{P}^D) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$

Theorem 2 (Regularity in $W^{m,p}(\Omega)$)

i) Let Ω be of class C^1 , and $1 < p < \infty$. Assume that

$$f \in W^{-1,p}(\Omega), \quad g \in W^{1-1/p,p}(\Gamma),$$

Then, Problem (\mathcal{P}^D) has a unique solution

$$u \in W^{1,p}(\Omega).$$

ii) If moreover Ω is of class $C^{k-1,1}$, with $k \geq 2$ and

$$f \in W^{k-2,p}(\Omega), \quad g \in W^{k-1/p,p}(\Gamma),$$

then,

$$u \in W^{k,p}(\Omega).$$

Remark. We have also corresponding results in fractional Sobolev or Besov spaces $W^{s,p}(\Omega)$, with s real number.

Question 2

Is it possible to obtain the **same regularity** with Ω **less regular** ?

I.2 Regularity $W^{1,p}(\Omega)$, with Ω Lipschitz

In the 80s, Nečas posed the question of solving the problem (\mathcal{P}^D) with the homogeneous boundary condition $g = 0$ on Lipschitz domains, when the RHS $f \in W^{-1,p}(\Omega)$.

The answer to this question, which is a part of Question 2, is given in a paper of Jerison and Kenig (JFA 95):

• **Negative results:**

If $N \geq 3$, then for any $p > 3$ (resp. $p > 4$ if $N = 2$), there is a Lipschitz domain Ω and $f \in C^\infty(\overline{\Omega})$ such that the solution u of Problem (\mathcal{P}^D) with the homogeneous boundary condition $g = 0$ does not belong to $W^{1,p}(\Omega)$.

So, as consequence, we don't have uniqueness in $W^{1,p}(\Omega)$ for $p < 3/2$ if $N \geq 3$ and for $p < 4/3$ if $N = 2$.

Example for $N = 2$. Let us consider the following Lipschitz domain for $\frac{1}{2} < \alpha < 1$:

$$\Omega = \{(r, \theta); 0 < r < 1, 0 < \theta < \frac{\pi}{\alpha}\}$$

We can easily verify that the following function

$$u(r, \theta) = (r^{-\alpha} - r^\alpha)\sin(\alpha\theta)$$

is harmonic in Ω with $u = 0$ on Γ and $u \in W^{1,p}(\Omega)$ for any $p < \frac{2}{\alpha+1}$. Remark that when α is near from $\frac{1}{2}$ then Ω is close to the unit disc and $\frac{2}{\alpha+1}$ is close to $\frac{4}{3}$.

- **Positive results:**

There exist $q > 3$ when $N = 3$ (resp. $q > 4$ when $N = 2$) such that if

$$q' < p < q,$$

then the problem (\mathcal{P}^D) has a unique solution $u \in W^{1,p}(\Omega)$ satisfying the estimate

$$\|u\|_{W^{1,p}(\Omega)} \leq C \|f\|_{W^{-1,p}(\Omega)}.$$

Moreover, if Ω is \mathcal{C}^1 , we can take $q = \infty$.

I.3 Solutions in Fractional Sobolev Spaces

- To obtain existence results in **fractional Sobolev** spaces we will use the **interpolation** theory (real or complex), which leads us to Lizorkin or Besov spaces.

For simplicity, we consider here only the Hilbertian case $p = 2$ and we will use the classical Sobolev spaces $H^s(\Omega)$ which contain the restrictions to Ω of functions in $H^s(\mathbb{R}^N)$.

Recall now the following spaces for $s \in \mathbb{R}^+$:

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s(\Omega)}},$$

$$\tilde{H}^s(\Omega) = \{v|_{\Omega}; v \in H^s(\mathbb{R}^N), \text{ with } \text{supp } v \subset \bar{\Omega}\}.$$

- When $s - \frac{1}{2} \notin \mathbb{N}$

$$H_0^s(\Omega) = \tilde{H}^s(\Omega)$$

and moreover

$$H_0^s(\Omega) = \tilde{H}^s(\Omega) = H^s(\Omega) \quad \text{if } 0 \leq s < \frac{1}{2}.$$

- When $s - \frac{1}{2} = \mu \in \mathbb{N}$,

$$\tilde{H}^{\mu+\frac{1}{2}}(\Omega) = \left\{ u \in H_0^{\mu+\frac{1}{2}}(\Omega); \frac{D^\lambda u}{\delta^{|\lambda|/2}} \in L^2(\Omega), \quad \forall |\lambda| = \mu \right\},$$

where $\delta(\mathbf{x})$ is the distance of the point \mathbf{x} in Ω to the boundary Γ .

This is a strict subspace of $H_0^{\mu+\frac{1}{2}}(\Omega)$ with a strictly finer topology and $\mathcal{D}(\Omega)$ is dense in $\tilde{H}^{\mu+\frac{1}{2}}(\Omega)$ for this finer topology.

Lions-Magenes use the notation

$$H_{00}^{\mu+\frac{1}{2}}(\Omega) \quad \text{instead of } \tilde{H}^{\mu+\frac{1}{2}}(\Omega).$$

- **Definition** : for $s \in \mathbb{R}^+$, we set (like in Lions-Magenes)

$$H^{-s}(\Omega) = [H_0^s(\Omega)]'.$$

When $s - \frac{1}{2} = \mu \in \mathbb{N}$, note that

$$H^{-s}(\Omega) \hookrightarrow [H_{00}^s(\Omega)]'.$$

- The interpolation between two spaces $H_0^s(\Omega)$ is somewhat different from the one between two spaces $H^s(\Omega)$.

Indeed, if

$$s_1 > s_2 \geq 0 \quad \text{such that} \quad s_1, s_2 \notin \left\{ \frac{1}{2} \right\} + \mathbb{N}$$

then we have

$$[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_0^{(1-\theta)s_1 + \theta s_2}(\Omega) \quad \text{if} \quad (1-\theta)s_1 + \theta s_2 \notin \left\{ \frac{1}{2} \right\} + \mathbb{N}$$

and

$$[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_{00}^{(1-\theta)s_1 + \theta s_2}(\Omega) \quad \text{if} \quad (1-\theta)s_1 + \theta s_2 \in \left\{ \frac{1}{2} \right\} + \mathbb{N}$$

- We know that

$$\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

and if moreover Ω is $C^{1,1}$

$$\Delta : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$$

are isomorphisms.

By interpolation, we deduce that the following operators

$$\Delta : H^{s+1} \cap H_0^1(\Omega) \longrightarrow H^{s-1}(\Omega) \quad \text{if } 0 < s < 1 \quad \text{and} \quad s \neq \frac{1}{2}$$

and

$$\Delta : H_0^{3/2}(\Omega) \longrightarrow \left[H_{00}^{1/2}(\Omega) \right]',$$

are isomorphisms, where we note that

$$\left[H^2(\Omega) \cap H_0^1(\Omega), H_0^1(\Omega) \right]_{1/2} = H^{3/2}(\Omega) \cap H_0^1(\Omega) = H_0^{3/2}(\Omega)$$

and

$$\left[L^2(\Omega), H^{-1}(\Omega) \right]_{1/2} = \left(\left[H_0^1(\Omega), L^2(\Omega) \right]_{1/2} \right)' = \left[H_{00}^{1/2}(\Omega) \right]'.$$

- Because

$\mathcal{D}(\Omega)$ is dense in $H^{s+1} \cap H_0^1(\Omega)$

endowed with the topology $H^{s+1}(\Omega)$ when $0 \leq s \leq \frac{1}{2}$, we have

$$H^{s+1}(\Omega) \cap H_0^1(\Omega) = H_0^{s+1}(\Omega) \quad \text{and} \quad H^{s-1}(\Omega) = [H_0^{1-s}(\Omega)]'$$

and then, by **duality**, the following operators

$$\Delta : H_0^{1-s}(\Omega) \longrightarrow H^{-1-s}(\Omega) \quad \text{if} \quad 0 < s < \frac{1}{2}$$

$$\Delta : H_{00}^{1/2}(\Omega) \longrightarrow H^{-3/2}(\Omega),$$

are isomorphisms.

Consequently, assuming Ω of class $\mathcal{C}^{1,1}$,

$$\Delta : H_0^s(\Omega) \longrightarrow H^{s-2}(\Omega) \quad \text{if} \quad \frac{1}{2} < s < \frac{3}{2}$$

is an isomorphism.

- In fact **Jerison** and **Kenig** (95) proved that this result holds when Ω is only **Lipschitz**, but is false for

$$s = \frac{1}{2} \quad \text{or} \quad s = \frac{3}{2}$$

even if Ω is \mathcal{C}^1 .

- However, if we consider the **homogeneous Dirichlet** problem:

$$(\mathcal{P}_H^D) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$

where Ω is supposed **Lipschitz**, the situation is little be different. **Jerison** and **Kenig** proved that for any

$$g \in H^{s-\frac{1}{2}}(\Gamma) \quad \text{with} \quad \frac{1}{2} \leq s \leq \frac{3}{2},$$

the Problem (\mathcal{P}_H^D) has a unique solution

$$u \in H^{s+\frac{1}{2}}(\Omega).$$

We will return to this in Subsection II.1 and II.2.

I.4 Very Weak Solutions. Point of View of Lions-Magenes

- When the data f and g are not regular, but Ω is C^∞ , by using the **transposition method**, **Lions** and **Magenes** proved the existence of **very weak solutions** to Problem (\mathcal{P}^D) belonging to $H^{-s}(\Omega)$, with $s \geq 0$.

Recall that the mapping

$$-\Delta : H^{2+s}(\Omega) \cap H_0^1(\Omega) \longrightarrow H^s(\Omega)$$

is an isomorphism for any $s \geq 0$.

However, the dual of the space $H^s(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$ if and only if $s \leq \frac{1}{2}$. As they are interested in finding solutions $u \in H^{-s}(\Omega)$, then they choose the following subspace

$$X^{2+s}(\Omega) = \{ v \in H^{2+s}(\Omega) \cap H_0^1(\Omega); \Delta v \in H_0^s(\Omega) \}$$

in order to get the following isomorphism:

$$-\Delta : X^{2+s}(\Omega) \longrightarrow H_0^s(\Omega)$$

It implies that for any $\ell \in [X^{2+s}(\Omega)]'$, the linear form

$$\begin{aligned} F &\longmapsto \langle \ell, v \rangle \\ H_0^s(\Omega) &\longrightarrow \mathbb{R}, \end{aligned}$$

where

$$v \in X^{2+s}(\Omega) \quad \text{satisfies} \quad -\Delta v = F \quad \text{in } \Omega,$$

is continuous and then there exists a unique solution

$$u \in H^{-s}(\Omega)$$

such that for any $v \in X^{2+s}(\Omega)$,

$$\langle u, -\Delta v \rangle_{H^{-s}(\Omega) \times H_0^s(\Omega)} = \langle \ell, v \rangle_{[X^{2+s}(\Omega)]' \times X^{2+s}(\Omega)}.$$

- Remark.

In order to interpret the above problem and bring it back to a problem of the form (\mathcal{P}^D) , ℓ should be decomposed as

$$\ell = \ell_1 + \ell_2,$$

so that ℓ_2 leads to the boundary condition $u = g$ in the sense to be specified and where ℓ_1 leads to the equation

$$-\Delta u = f \quad \text{in } \Omega.$$

However, there is an optimal choice for the form ℓ_2 instead of ℓ_1 .

For the choice of the form ℓ_2 , they consider the following mapping

$$v \longmapsto \frac{\partial v}{\partial \mathbf{n}}$$

$$X^{2+s}(\Omega) \longrightarrow H^{1/2+s}(\Gamma),$$

which is continuous and has a right inverse operator which is also continuous.

From this result, the optimal choice for ℓ_2 is given by:

$$\langle \ell_2, v \rangle = \left\langle g, \frac{\partial v}{\partial \mathbf{n}} \right\rangle_{H^{-s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma)},$$

for any $v \in X^{2+s}(\Omega)$,

Concerning the choice of the form ℓ_1 , they consider a Hilbert space $K^{2+s}(\Omega)$ satisfying

$$X^{2+s}(\Omega) \hookrightarrow K^{2+s}(\Omega),$$

and such that

$$\mathcal{D}(\Omega) \text{ is dense in } K^{2+s}(\Omega).$$

The dual space $K^{-2-s}(\Omega)$ of $K^{2+s}(\Omega)$ could be identified with a subspace of $\mathcal{D}'(\Omega)$ and if f belongs to $K^{-2-s}(\Omega)$, then the form

$$v \in X^{2+s}(\Omega), \quad \langle \ell_1, v \rangle = \langle f, v \rangle_{K^{-2-s}(\Omega) \times K^{2+s}(\Omega)},$$

also defines a continuous linear form on $X^{2+s}(\Omega)$. Therefore, the linear form

$$\begin{aligned} \langle \ell, v \rangle &= \langle \ell_1, v \rangle + \langle \ell_2, v \rangle \\ &= \langle f, v \rangle_{K^{-2-s}(\Omega) \times K^{2+s}(\Omega)} + \left\langle g, \frac{\partial v}{\partial \mathbf{n}} \right\rangle_{H^{-s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma)} \end{aligned}$$

is continuous on $X^{2+s}(\Omega)$.

In order to consider all kinds of boundary conditions, Lions-Magenes choose to take the space

$$K^{2+s}(\Omega) = \Xi^{2+s}(\Omega)$$

where for $m \in \mathbb{N}$,

$$\Xi^m(\Omega) = \left\{ v \in L^2(\Omega); \delta^{|\alpha|} D^\alpha v \in L^2(\Omega), \quad |\alpha| \leq m \right\}$$

and for $s = m + \theta$ with $0 < \theta < 1$,

$$\Xi^s(\Omega) = [\Xi^{m+1}(\Omega), \Xi^m(\Omega)]_{1-\theta},$$

where δ is a regular positive function in Ω and vanishes on the boundary Γ , of the same order as the distance d to the boundary Γ .

- We have the following properties:

i)

$$\forall s \geq 0, \quad \mathcal{D}(\Omega) \text{ is dense in } \Xi^s(\Omega).$$

ii) The dual space $\Xi^{-s}(\Omega)$ of $\Xi^s(\Omega)$ is then a subspace of distributions in Ω .

iii) The space $\mathcal{D}(\bar{\Omega})$ is dense in

$$D_{\Delta}^{-s}(\Omega) = \{v \in H^{-s}(\Omega); \Delta v \in \Xi^{-2-s}(\Omega)\}.$$

and the linear mapping

$$u \longmapsto u|_{\Gamma}$$

defined on $\mathcal{D}(\bar{\Omega})$ can be continuously extended to a **continuous** linear mapping from

$$D_{\Delta}^{-s}(\Omega) \quad \text{to} \quad H^{-s-1/2}(\Gamma)$$

and we have the following Green formula:

$$\forall u \in D_{\Delta}^{-s}(\Omega), \quad \forall \varphi \in X^{2+s}(\Omega),$$

$$\begin{aligned} \langle \Delta u, \varphi \rangle_{\Xi^{-2-s}(\Omega) \times \Xi^{2+s}(\Omega)} &= \langle u, \Delta \varphi \rangle_{H^{-s}(\Omega) \times H_0^s(\Omega)} + \\ &+ \left\langle u, \frac{\partial \varphi}{\partial n} \right\rangle_{H^{-s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma)}, \end{aligned}$$

where we recall that

$$X^{2+s}(\Omega) = \left\{ v \in H^{2+s}(\Omega) \cap H_0^1(\Omega); \quad \Delta v \in H_0^s(\Omega) \right\}$$

which is included in $\Xi^{2+s}(\Omega)$.

Definition. For

$$f \in \Xi^{-2-s}(\Omega) \quad \text{and} \quad g \in H^{-s-1/2}(\Gamma)$$

we say that

$u \in H^{-s}(\Omega)$ is a **very weak solution** to Problem (\mathcal{P}^D)

if for any $\varphi \in X^{2+s}(\Omega)$,

$$\begin{aligned} \langle u, \Delta\varphi \rangle_{H^{-s}(\Omega) \times H_0^s(\Omega)} &= \langle f, \varphi \rangle_{\Xi^{-2-s}(\Omega) \times \Xi^{2+s}(\Omega)} + \\ &+ \left\langle g, \frac{\partial\varphi}{\partial n} \right\rangle_{H^{-s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma)}. \end{aligned}$$

The next theorem, proved by **Lions-Magenes**, gives a result concerning the existence and uniqueness of very weak solutions to Problem (\mathcal{P}^D) .

Theorem 3 (Very Weak Solutions to (\mathcal{P}^D) in $H^{-s}(\Omega)$)

Assume Ω is bounded and of class C^∞ . Let $s \geq 0$ and $f \in \Xi^{-2-s}(\Omega)$ and $g \in H^{-s-1/2}(\Gamma)$. Then there exists a unique solution

$$u \in H^{-s}(\Omega) \quad \text{of } (\mathcal{P}^D).$$

satisfying the estimate as follows

$$\|u\|_{H^{-s}(\Omega)} \leq C \left(\|f\|_{\Xi^{-2-s}(\Omega)} + \|g\|_{H^{-s-1/2}(\Gamma)} \right).$$

Remark In particular if

$$f \in H^{-2}(\Omega) \quad \text{with } \text{supp } f \text{ compact in } \Omega \quad \text{and} \quad g \in H^{-1/2}(\Gamma)$$

then there exists a unique solution

$$u \in L^2(\Omega) \quad \text{of } (\mathcal{P}^D).$$

Remark.

If we consider now the following Neumann problem for the Laplacian:

$$(\mathcal{P}^N) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \Gamma. \end{cases}$$

for any

$$f \in \Xi^{-2-s}(\Omega) \quad \text{and} \quad h \in H^{-s-3/2}(\Gamma),$$

with

$$\langle f, 1 \rangle_{\Xi^{-2-s}(\Omega) \times \Xi^{2+s}(\Omega)} = \langle h, 1 \rangle_{H^{-s-3/2}(\Gamma) \times H^{s+3/2}(\Gamma)}$$

there exists a unique solution

$$u \in H^{-s}(\Omega) \quad \text{of } (\mathcal{P}^N),$$

up an additive constant.

II. The Dirichlet and Neumann problems for the Laplacian in Lipschitz Domains

Assume Ω is a bounded Lipschitz open.

II.1 Links between (\mathcal{P}^D) and (\mathcal{P}^N) Rellich equality

First, we will recall some known results of Jerison-Kenig (JFA 95).

Theorem 4 (Jerison-Kenig)

Let $0 \leq \alpha \leq 1$. Suppose that u is a harmonic function in Ω . Let $k \in \mathbb{N}$. Then the following properties are equivalent:

- i) $u \in H^{k+\alpha}(\Omega)$,
- ii) $u \in L^2(\Omega)$, $\nabla^k u \in L^2(\Omega)$, $\delta^{1-\alpha} \nabla^{k+1} u \in L^2(\Omega)$ where δ is the distance to the boundary Γ .

Remark. 1) To prove that i) \Rightarrow ii), it suffices to use the mean-value formula for the harmonic functions.

2) For the Implication ii) \Rightarrow i), it is not necessary to suppose u harmonic. In fact, we can prove the following result:

Theorem 5 (A-Moussaoui-Nguyen)

Let $0 \leq \alpha \leq 1$, $k \in \mathbb{N}$ and $u \in \mathcal{D}'(\Omega)$ such that

$$\delta^{1-\alpha} \nabla^{k+1} u \in L^2(\Omega).$$

Then

$$u \in H^{k+\alpha}(\Omega).$$

Proof. i) The case $\alpha = 1$ is proved by **A-Girault** (1994): if u is any distribution satisfying

$$\nabla^{k+1}u \in L^2(\Omega),$$

then

$$u \in H^{k+1}(\Omega).$$

ii) Note that $u \in H_{\text{loc}}^{k+1}(\Omega)$. It suffices to consider the case $k = 0$ and the more difficult case $\alpha = \frac{1}{2}$, which means that

$$\delta^{1/2}\nabla u \in L^2(\Omega).$$

For any $\varphi \in \mathcal{D}(\Omega)$,

$$\left| \int_{\Omega} \frac{\partial u}{\partial x_j} \varphi \right| = \left| \int_{\Omega} \sqrt{\delta} \frac{\partial u}{\partial x_j} \frac{\varphi}{\sqrt{\delta}} \right| \leq \left\| \sqrt{\delta} \frac{\partial u}{\partial x_j} \right\|_{L^2(\Omega)} \|\varphi\|_{H_{00}^{1/2}(\Omega)}.$$

Using the density of $\mathcal{D}(\Omega)$ in $H_{00}^{1/2}(\Omega)$, we have

$$\nabla u \in [H_{00}^{1/2}(\Omega)]'.$$

But, we know that (see [A-Girault 1994](#))

$$v \in \mathcal{D}'(\Omega), \quad \nabla v \in H^{-1}(\Omega) \quad \Longrightarrow \quad v \in L^2(\Omega)$$

and

$$v \in \mathcal{D}'(\Omega), \quad \nabla v \in L^2(\Omega) \quad \Longrightarrow \quad v \in H^1(\Omega).$$

So by interpolation, we have

$$v \in \mathcal{D}'(\Omega), \quad \nabla v \in [H_{00}^{1/2}(\Omega)]' \quad \Longrightarrow \quad v \in H^{1/2}(\Omega)$$

We conclude that $u \in H^{1/2}(\Omega)$.

(Observe that the converse is true:

$$u \in H^{1/2}(\Omega) \Longrightarrow \nabla u \in [H_{00}^{1/2}(\Omega)]' \text{ but not in } [H^{1/2}(\Omega)]')$$

Theorem 6 (Necas Property)

Let

$$u \in H^1(\Omega) \quad \text{with} \quad \Delta u \in L^2(\Omega).$$

Then we have the Nečas property:

$$u|_{\Gamma} \in H^1(\Gamma) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial \nu} \in L^2(\Gamma) \quad \Longleftrightarrow \quad \nabla u \in L^2(\Gamma).$$

Moreover, we have the following estimates :

$$\inf_{k \in \mathbb{R}} \|u + k\|_{H^1(\Gamma)} \leq C(\|\Delta u\|_{L^2(\Omega)} + \|\frac{\partial u}{\partial \nu}\|_{L^2(\Gamma)})$$

and

$$\|\frac{\partial u}{\partial \nu}\|_{L^2(\Gamma)} \leq C(\|\Delta u\|_{L^2(\Omega)} + \inf_{k \in \mathbb{R}} \|u + k\|_{H^1(\Gamma)})$$

Sketch of the proof: To prove **Necas property**, we recall first that there exist $\alpha > 0$ and $\mathbf{h} \in C^\infty(\overline{\Omega})^N$ satisfying (see the book of Grisvard):

$$\mathbf{h} \cdot \boldsymbol{\nu} \geq \alpha > 0 \quad \text{a.e. on } \Gamma.$$

We use then a simplified variant of **Rellich identity**, which holds for general elliptic operator like $-\operatorname{div}(A\nabla)$:

$$\begin{aligned} \int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |\nabla u|^2 &= 2 \int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |\nabla_{\boldsymbol{\tau}} u|^2 - 2 \int_{\Gamma} \mathbf{h}_{\boldsymbol{\tau}} \cdot \nabla u \frac{\partial u}{\partial \boldsymbol{\nu}} - \int_{\Omega} (\operatorname{div} \mathbf{h}) |\nabla u|^2 \\ &+ 2 \sum_{k=1}^N \int_{\Omega} \frac{\partial \mathbf{h}}{\partial x_k} \cdot \nabla u \frac{\partial u}{\partial x_k} + 2 \int_{\Omega} \mathbf{h} \cdot \nabla u \Delta u \end{aligned}$$

where on the boundary,

$$\begin{aligned} \nabla u &= \nabla_{\boldsymbol{\tau}} u + \frac{\partial u}{\partial \boldsymbol{\nu}} \boldsymbol{\nu} \\ |\nabla u|^2 &= |\nabla_{\boldsymbol{\tau}} u|^2 + \left| \frac{\partial u}{\partial \boldsymbol{\nu}} \right|^2 \end{aligned}$$

When u is **harmonic**, we deduce by **Necas Property** that the **Steklov-Poincaré** operator

$$S : u \mapsto \frac{\partial u}{\partial \nu}$$

is continuous from

$$H^1(\Gamma) \text{ into } L^2(\Gamma).$$

and

$$S : H^1(\Gamma)/\mathbb{R} \longrightarrow L^2(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

Because $S^* = S$, we deduce then by duality that

$$S : L^2(\Gamma) \rightarrow H^{-1}(\Gamma)$$

is continuous and

$$S : L^2(\Gamma)/\mathbb{R} \rightarrow H^{-1}(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

Then by interpolation for all $0 \leq s \leq 1$,

$$S : H^s(\Gamma) \rightarrow H^{s-1}(\Gamma)$$

is continuous and

$$S : H^s(\Gamma)/\mathbb{R} \rightarrow H^{s-1}(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

II.2 Homogeneous Dirichlet Problem

Let us consider now the **homogeneous Dirichlet** problem:

$$(\mathcal{P}_H^D) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma \end{cases}$$

and recall the following result:

Theorem 7 (Solutions in $H^s(\Omega)$, Jerison-Kenig)

Let $0 \leq s \leq 1$ and $g \in H^s(\Gamma)$. Then there exists a unique solution $u \in H^{s+\frac{1}{2}}(\Omega)$ of (\mathcal{P}_H^D) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \leq C \|g\|_{H^s(\Gamma)}.$$

Moreover, with $\delta = d(x, \Gamma)$, we have

$$\sqrt{\delta} \nabla u \in L^2(\Omega) \quad \text{if } s = 0 \quad \text{and} \quad \sqrt{\delta} \nabla^2 u \in L^2(\Omega) \quad \text{if } s = 1.$$

II.3 Homogeneous Neumann Problem

Consider now the **homogeneous Neumann** problem:

$$(\mathcal{P}_H^N) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \Gamma, \end{cases}$$

and recall the following result:

Theorem 8 (Solutions in $H^s(\Omega)$, Jerison-Kenig)

Let $0 \leq s \leq 1$ and $h \in H^{s-1}(\Gamma)$ satisfying the compatibility condition $\langle h, 1 \rangle = 0$. Then there exists a unique solution, up to an additive constant, $u \in H^{s+\frac{1}{2}}(\Omega)$ of (\mathcal{P}_H^N) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \leq C \|h\|_{H^{s-1}(\Gamma)}. \quad (1)$$

- To prove the previous theorem, it suffices to use Theorem 7 for Dirichlet problem and the invertibility of the Steklov-Poincaré operator from $H^s(\Gamma)$ and $H^{s-1}(\Gamma) \perp \mathbb{R}$.

Remark.

- Let $f \in \left[H^{\frac{3}{2}-s}(\Omega) \right]'$ and $h \in H^{s-1}(\Gamma)$, with $0 < s < 1$ and satisfying the compatibility condition:

$$\langle f, 1 \rangle_{\Omega} + \langle h, 1 \rangle_{\Gamma} = 0.$$

Then the following **Neumann*** problem

Find $u \in H^{s+\frac{1}{2}}(\Omega)$ such that for any $\varphi \in H^{\frac{3}{2}-s}(\Omega)$,

$$\langle \nabla u, \nabla \varphi \rangle_{H^{s-\frac{1}{2}}(\Omega) \times H^{-s+\frac{1}{2}}(\Omega)} = \langle f, \varphi \rangle_{\Omega} + \langle h, \varphi \rangle_{\Gamma},$$

has a unique solution, up to an additive constant.

Such a solution satisfies:

$$-\Delta u = f|_{\Omega} \quad \text{in } \Omega$$

but not

$$\frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma.$$

III. The Dirichlet and Neumann problems for the Laplacian in $C^{1,1}$ Domains

Assume Ω is a bounded $C^{1,1}$ open. Using again the Rellich identity, but with more technical difficulties, we can prove the following extension of Necas property:

Theorem 9 (Extension of Necas Property, A-Moussaoui-Nguyen)

Let

$$u \in H^2(\Omega) \quad \text{with} \quad \Delta u \in H^1(\Omega).$$

Then we have:

$$u|_{\Gamma} \in H^2(\Gamma) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial \nu} \in H^1(\Gamma) \quad \Longleftrightarrow \quad \nabla^2 u \in L^2(\Gamma).$$

Moreover, we have the following estimates:

$$\inf_{k \in \mathbb{R}} \|u + k\|_{H^2(\Gamma)} \leq C(\|\Delta u\|_{H^1(\Omega)} + \|\frac{\partial u}{\partial \nu}\|_{H^1(\Gamma)})$$

and

$$\|\frac{\partial u}{\partial \nu}\|_{H^1(\Gamma)} \leq C(\|\Delta u\|_{H^1(\Omega)} + \inf_{k \in \mathbb{R}} \|u + k\|_{H^2(\Gamma)})$$

When u is **harmonic**, we deduce by **Necas Property** that the **Steklov-Poincaré** operator

$$S : u \mapsto \frac{\partial u}{\partial \nu}$$

is continuous from

$$H^2(\Gamma) \text{ into } H^1(\Gamma).$$

and

$$S : H^2(\Gamma)/\mathbb{R} \longrightarrow H^1(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

As above we deduce then by duality that

$$S : H^{-1}(\Gamma) \rightarrow H^{-2}(\Gamma)$$

is continuous and

$$S : H^{-1}(\Gamma)/\mathbb{R} \rightarrow H^{-2}(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

Then by interpolation for all $0 \leq s \leq 1$,

$$S : H^{s+1}(\Gamma) \rightarrow H^s(\Gamma)$$

is continuous and

$$S : H^{s+1}(\Gamma)/\mathbb{R} \rightarrow H^s(\Gamma) \perp \mathbb{R}$$

is one to one and onto.

Using then the result of Jerison and Kenig, we can prove the following theorem:

Theorem 10 (Solutions in $H^s(\Omega)$, A-Moussaoui-Nguyen)

Let $0 < s \leq 1$ and $g \in H^{s+1}(\Gamma)$. Then there exists a unique solution $u \in H^{s+\frac{3}{2}}(\Omega)$ of (\mathcal{P}_H^D) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{3}{2}}(\Omega)} \leq C \|g\|_{H^{s+1}(\Gamma)}. \quad (2)$$

Moreover, we have

$$\sqrt{\delta} \nabla^3 u \in L^2(\Omega) \quad \text{if} \quad s = 1.$$

For the **homogeneous Neumann** problem, we have similarly

Theorem 11 (Solutions in $H^s(\Omega)$, A-Moussaoui-Nguyen)

Let $0 < s \leq 1$ and $h \in H^s(\Gamma)$ satisfying the compatibility condition $\int_{\Gamma} h = 0$. Then there exists a unique solution, up an additive constant, $u \in H^{s+\frac{3}{2}}(\Omega)$ of (\mathcal{P}_H^N) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{3}{2}}(\Omega)} \leq C \|h\|_{H^s(\Gamma)}. \quad (3)$$

IV. Dirichlet and Neumann Problem for the Bilaplacian

We suppose here that Ω is Lipschitz or more regular. Consider now the following **homogeneous Dirichlet and "Neumann" Biharmonic** (used for **plates elasticity** theory) problems:

$$(\mathcal{B}_D^H) \quad \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = g_0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g_1 & \text{on } \Gamma. \end{cases}$$

and

$$(\mathcal{B}_N^H) \quad \begin{cases} \Delta^2 v = 0 & \text{in } \Omega, \\ \mu \Delta v + (1 - \mu) \frac{\partial^2 v}{\partial \nu^2} = \Lambda_0 & \text{on } \Gamma, \\ \frac{\partial \Delta v}{\partial \nu} + \frac{1 - \mu}{2} \frac{\partial}{\partial \tau_{ij}} \left(\frac{\partial^2 v}{\partial \nu \partial \tau_{ij}} \right) = \Lambda_1 & \text{on } \Gamma, \end{cases}$$

where μ is a constant known as the Poisson ratio,

$$\forall 1 \leq j < k \leq N, \quad \tau_{ij} = (0, \dots, 0, -\nu_k, 0, \dots, 0, \nu_j, 0, \dots, 0)$$

and $\frac{\partial}{\partial \tau_{ij}} = \tau_{ij} \cdot \nabla$ are the tangential derivatives.

Theorem 12 (Solutions in $H^s(\Omega)$, Dahlberg-Kenig-Verchota 86)

Assume that Ω is a **Lipschitz** bounded open set. For any

$$g_0 \in H^1(\Gamma) \quad \text{and} \quad g_1 \in L^2(\Gamma),$$

the problem (\mathcal{B}_D^H) has a unique solution $u \in H^{3/2}(\Omega)$ satisfying $\sqrt{\rho} \nabla^2 u \in L^2(\Omega)$. Moreover

$$\|u\|_{H^{3/2}(\Omega)} + \|\sqrt{\rho} \nabla^2 u\|_{L^2(\Omega)} \leq C (\|g_0\|_{H^1(\Gamma)} + \|g_1\|_{L^2(\Gamma)}). \quad (4)$$

Theorem 13 (Variant of the Rellich Identity, A-Moussaoui-Nguyen)

Suppose that Ω is $\mathcal{C}^{1,1}$. Let

$$u \in H^3(\Omega) \quad \text{with} \quad \Delta^2 u \in L^2(\Omega).$$

Then we have the following Rellich equality:

$$\begin{aligned} \int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |\nabla^2 u|^2 &= I + 2 \int_{\Gamma} \nabla(\mathbf{h} \cdot \nabla u) \cdot \frac{\partial \nabla u}{\partial \boldsymbol{\nu}} + \\ &+ 2 \int_{\Omega} (\mathbf{h} \cdot \nabla u) \Delta^2 u - 2 \langle \frac{\partial \Delta u}{\partial \boldsymbol{\nu}}, \mathbf{h} \cdot \nabla u \rangle_{\Gamma} \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ is the duality bracket $H^{-1}(\Gamma) \times H^1(\Gamma)$ and

$$I = \int_{\Omega} (\operatorname{div} \mathbf{h}) |\nabla^2 u|^2 - 2 \int_{\Omega} \left[\frac{\partial^2 \mathbf{h}}{\partial x_j \partial x_k} \cdot \nabla u + 2 \frac{\partial \mathbf{h}}{\partial x_k} \cdot \nabla \frac{\partial u}{\partial x_j} \right] \frac{\partial^2 u}{\partial x_j \partial x_k}$$

Introduce now the following differential operator

$$L_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} + \theta \delta_{ij} \Delta,$$

with θ satisfying the relation

$$\mu = \frac{2\theta + N\theta^2}{1 + 2\theta + N\theta^2}.$$

We can prove that if $\theta \geq -1/N$, or equivalently $\frac{1}{1-N} \leq \mu < 1$, then

$$L_{ij}(u)L_{ij}(u) \geq |\nabla^2 u|^2 - \frac{1}{N} |\Delta u|^2,$$

with the implicit summation on the repeated indexes.

Theorem 14 (Second Variant of the Rellich Identity, Verchota)

Suppose that Ω is $C^{1,1}$. Let

$$u \in H^3(\Omega) \quad \text{with} \quad \Delta^2 u \in L^2(\Omega).$$

Then we have the following Rellich equality:

$$\begin{aligned} \frac{1-\mu}{2} \int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |L_{ij}(u)|^2 &= J - \langle K_{\mu}(u), \mathbf{h} \cdot \nabla u \rangle_{\Gamma} + \\ &+ \int_{\Gamma} \frac{\partial}{\partial \boldsymbol{\nu}} (\mathbf{h} \cdot \nabla u) M_{\mu}(u) \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ is the duality bracket $H^{-1}(\Gamma) \times H^1(\Gamma)$ and

$$J = \int_{\Omega} E_{ij}(\mathbf{h}, u, \mu) L_{ij}(u) + \mu \int_{\Omega} \mathbf{h} \cdot \nabla u \Delta^2 u$$

$$M_{\mu}(u) = \mu \Delta u + (1-\mu) \frac{\partial^2 u}{\partial \boldsymbol{\nu}^2}, \quad K_{\mu}(u) = \frac{\partial \Delta u}{\partial \boldsymbol{\nu}} + \frac{1-\mu}{2} \frac{\partial}{\partial \tau_{ij}} \left(\frac{\partial^2 u}{\partial \boldsymbol{\nu} \partial \tau_{ij}} \right)$$

Lemma 15 (Coercive estimate on the boundary, Verchota)

Assume that Ω is $C^{1,1}$, $\frac{1}{1-N} \leq \mu < 1$ and

$$u \in H^2(\Omega) \quad \text{with} \quad \Delta^2 u = 0 \quad \text{and} \quad \langle \Delta u, 1 \rangle_{\Gamma} = 0.$$

Then we have the following inequality

$$\int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |\nabla^2 u|^2 \leq C_1 \int_{\Gamma} \mathbf{h} \cdot \boldsymbol{\nu} |L_{ij}(u)|^2 + C_2 \|K_{\mu}(u)\|_{H^{-1}(\Gamma)}^2$$

where C_2 may be taken to be zero when $0 \leq \mu < 1$, i.e $\theta \geq 0$.

Lemma 16 (Weak Necas Property, A-Moussaoui-Nguyen)

Let Ω be a Lipschitz bounded open subset of \mathbb{R}^N . Let $u \in L^2(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then we have the following estimates:

$$\begin{aligned} \left\| \frac{\partial u}{\partial n} \right\|_{H^{-1}(\Gamma)} &\leq C(\inf_{k \in \mathbb{R}} \|u + k\|_{L^2(\Gamma)} + \|\Delta u\|_{L^2(\Omega)}) \\ \inf_{k \in \mathbb{R}} \|u + k\|_{L^2(\Gamma)} &\leq C\left(\left\| \frac{\partial u}{\partial n} \right\|_{H^{-1}(\Gamma)} + \|\Delta u\|_{L^2(\Omega)}\right) \end{aligned} \tag{5}$$

Theorem 17 (Necas Property for Bilaplacian, A-Moussaoui-Nguyen)

Let Ω be a bounded open subset of \mathbb{R}^N of class of $C^{1,1}$. Let $u \in H^2(\Omega)$ such that

$$\Delta^2 u = f \in L^2(\Omega).$$

i) If $u \in H^2(\Gamma)$ and $\frac{\partial u}{\partial \mathbf{n}} \in H^1(\Gamma)$, then

$$\nabla^2 u \in L^2(\Gamma) \quad (6)$$

with the following estimate

$$\|\nabla^2 u\|_{L^2(\Gamma)} \leq C \left(\|f\|_{L^2(\Omega)} + \inf_{q \in \mathcal{P}_1} (\|u + q\|_{H^2(\Gamma)}) + \left\| \frac{\partial(u+q)}{\partial \mathbf{n}} \right\|_{H^1(\Gamma)} \right) \quad (7)$$

ii) If $\frac{\partial \nabla u}{\partial \mathbf{n}} \in L^2(\Gamma)$ and $\frac{\partial \Delta u}{\partial \mathbf{n}} \in H^{-1}(\Gamma)$, then we have (6) with the following estimate

$$\|\nabla u\|_{H^1(\Gamma)} + \|\nabla^2 u\|_{L^2(\Gamma)} \leq C \left(\|f\|_{L^2(\Omega)} + \left\| \frac{\partial \nabla u}{\partial \mathbf{n}} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial \Delta u}{\partial \mathbf{n}} \right\|_{H^{-1}(\Gamma)} \right) \quad (8)$$

Theorem 18 (Solutions in $H^s(\Omega)$, A-Moussaoui-Nguyen)

Assume that Ω is $C^{1,1}$. For any

$$g_0 \in H^2(\Gamma) \quad \text{and} \quad g_1 \in H^1(\Gamma),$$

the problem (\mathcal{P}_H^B) has a unique solution $u \in H^{5/2}(\Omega)$ satisfying the following estimate

$$\|u\|_{H^{5/2}(\Omega)} \leq C (\|g_0\|_{H^2(\Gamma)} + \|g_1\|_{H^1(\Gamma)}). \quad (9)$$

Consequently for any $0 < s < 1$ and

$$g_0 \in H^{1+s}(\Gamma) \quad \text{and} \quad g_1 \in H^s(\Gamma),$$

the problem (\mathcal{P}_H^B) has a unique solution $u \in H^{3/2+s}(\Omega)$

Sketch of Proof. We use

- the previous Rellich identity,
- the existence result of Dahlberg-Jerison and Kenig for Δ^2 ,
- the results obtained for the Laplacian
- the following result:

if Ω is (only) $C^{0,1}$, for any distribution f such that

$$\sqrt{\delta}f \in L^2(\Omega),$$

the solution

$$u \in H_0^1(\Omega) \quad \text{satisfying} \quad \Delta u = f \quad \text{in } \Omega$$

verifies

$$\sqrt{\delta}\nabla^2 u \in L^2(\Omega).$$

We introduce now the following **Steklov-Poincaré** operator for the **Bilaplacian**: for biharmonic function u ,

$$\begin{aligned} S_B : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) &\longrightarrow H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \\ (u, \frac{\partial u}{\partial \nu}) &\longmapsto (K_\mu(u), M_\mu(u)) \end{aligned}$$

which is continuous as well as its inverse. Using Lemma 15 and Theorem 17,

$$\begin{aligned} S_B : H^2(\Gamma) \times H^1(\Gamma) &\longrightarrow H^{-1}(\Gamma) \times L^2(\Gamma) \\ (u, \frac{\partial u}{\partial \nu}) &\longmapsto (K_\mu(u), M_\mu(u)) \end{aligned}$$

is also continuous as well as its inverse.

Because $S_B^* = S_B$, we deduce then by duality that

$$\begin{aligned} S_B : H^1(\Gamma) \times L^2(\Gamma) &\longrightarrow H^{-2}(\Gamma) \times H^{-1}(\Gamma) \\ (u, \frac{\partial u}{\partial \nu}) &\longmapsto (K_\mu(u), M_\mu(u)) \end{aligned}$$

is also continuous.

Then by interpolation for all $0 \leq s \leq 1$,

$$\begin{aligned} S_B : H^{1+s}(\Gamma) \times H^s(\Gamma) &\longrightarrow H^{s-2}(\Gamma) \times H^{s-1}(\Gamma) \\ (u, \frac{\partial u}{\partial \nu}) &\longmapsto (K_\mu(u), M_\mu(u)) \end{aligned}$$

is also continuous.

V. New Results for the Traces

Extending the previous results for Δ^2 , we prove the following result:

Theorem 19 (Traces), A-Moussaoui-Nguyen

- If Ω is of class $\mathcal{C}^{0,1}$ (resp. $\mathcal{C}^{1,1}$), we have

$$\begin{aligned} \gamma_0 : v &\longmapsto v|_{\Gamma}, \\ H^s(\Omega) &\longrightarrow H^{s-1/2}(\Gamma), \end{aligned}$$

is continuous for

$$\frac{1}{2} < s < \frac{3}{2} \quad (\text{resp.} \quad \frac{3}{2} \leq s < \frac{5}{2})$$

and has a right continuous inverse for

$$\frac{1}{2} \leq s \leq \frac{3}{2} \quad (\text{resp.} \quad \frac{3}{2} < s \leq \frac{5}{2})$$

- If Ω is of class $\mathcal{C}^{1,1}$, we have

$$(\gamma_0, \gamma_1) : v \longmapsto (v|_{\Gamma}, \frac{\partial v}{\partial \nu}) \\ H^s(\Omega) \longrightarrow H^{s-1/2}(\Gamma) \times H^{s-3/2}(\Gamma),$$

is continuous for

$$\frac{3}{2} < s < \frac{5}{2}.$$

and has a right continuous inverse for

$$\frac{3}{2} < s \leq \frac{5}{2}.$$

Remark.

- i) This theorem is well known for the case Ω Lipschitz.
- ii) For $s = \frac{3}{2}$, we need only that Ω be Lipschitz to ensure the surjectivity of (γ_0, γ_1) .

VI. Inhomogeneous Dirichlet Problem

Theorem 20 (Solutions in $H^s(\Omega)$, Jerison-Kenig)

Assume Ω is Lipschitz. Let $0 < s < 1$ and $f \in H^{s-\frac{3}{2}}(\Omega)$. Then there exists a unique solution $u \in H_0^{s+\frac{1}{2}}(\Omega)$ of (\mathcal{P}_0^D) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \leq C \|f\|_{H^{s-\frac{3}{2}}(\Omega)}. \quad (10)$$

To prove this theorem, it suffices firstly to extend f in $\tilde{f} \in H^{s-\frac{3}{2}}(\mathbb{R}^N)$, secondly to solve a Poisson equation in the whole space, thirdly to use the continuity properties of the trace operator and finally to solve a corresponding homogeneous Dirichlet problem by using the previous theorem.

With the same ideas, we have:

Theorem 21 (Solutions in $H^s(\Omega)$, A-Moussaoui-Nguyen)

Assume Ω is $\mathcal{C}^{1,1}$. Let $0 \leq s < 1$ and $f \in H^{s-\frac{1}{2}}(\Omega)$. Then there exists a unique solution $u \in H^{s+\frac{3}{2}}(\Omega)$ of (\mathcal{P}_0^D) satisfying the estimate as follows

$$\|u\|_{H^{s+\frac{3}{2}}(\Omega)} \leq C \|f\|_{H^{s-\frac{1}{2}}(\Omega)}. \quad (11)$$

- Remark

- The previous result is well known for $0 \leq s \leq \frac{1}{2}$.
- We can prove more general regularity results when Ω is $\mathcal{C}^{k-1,1}$, with $k \geq 3$.

VII. Revisitation of the Very Weak Solutions of Lions-Magenes

For $m \in \mathbb{N}$, with $m \geq 2$, we introduce the following space

$$\mathfrak{T}^m(\Omega) = \left\{ v \in H_0^{3/2}(\Omega); \delta^{|\lambda| - \frac{3}{2}} D^\lambda v \in L^2(\Omega), 2 \leq |\lambda| \leq m \right\}$$

For a real s such that $s = m + \theta$ with $m \in \mathbb{N}, m \geq 2$ and $0 < \theta < 1$:

$$\mathfrak{T}^s(\Omega) = [\mathfrak{T}^{m+1}(\Omega), \mathfrak{T}^m(\Omega)]_{1-\theta}.$$

From this definition, we have the following embeddings

$$H^s(\Omega) \cap H_0^{3/2}(\Omega) \hookrightarrow \mathfrak{T}^s(\Omega) \hookrightarrow \mathfrak{T}^t(\Omega) \hookrightarrow H_0^{3/2}(\Omega),$$

if $s > t \geq 2$.

Theorem 22 (Density Result, A-Moussaoui-Nguyen)

$\mathcal{D}(\Omega)$ is dense in $\mathfrak{T}^s(\Omega)$ for any real $s \geq 2$.

- Remark

i) Therefore, the space $\mathfrak{T}^s(\Omega)$ is a normal space of distributions on Ω and its dual $\mathfrak{T}^{-s}(\Omega)$ may be identified to a subspace of distributions on Ω .

ii) It suffices to prove the density of $\mathcal{D}(\Omega)$ in $\mathfrak{T}^m(\Omega)$, for any integer $m \geq 2$, but this density is not easy instead of the density of $\mathcal{D}(\Omega)$ in

$$\left\{ v \in H_{00}^{3/2}(\Omega); \delta^{|\lambda|-\frac{3}{2}} D^\lambda v \in L^2(\Omega), 2 \leq |\lambda| \leq m \right\}$$

• Let $s \geq 0$, k integer such that $s < k - \frac{3}{2}$. Let Ω be of class $\mathcal{C}^{k-1,1}$. As in the case of Lions-Magenes, we have the following properties:

i) The space $\mathcal{D}(\overline{\Omega})$ is dense in

$$F_{\Delta}^{-s}(\Omega) = \{v \in H^{-s}(\Omega); \Delta v \in \mathfrak{T}^{-2-s}(\Omega)\}.$$

and the linear mapping $v \mapsto v|_{\Gamma}$ defined on $\mathcal{D}(\overline{\Omega})$ can be continuously extended to a **continuous** linear mapping from $F_{\Delta}^{-s}(\Omega)$ to $H^{-s-1/2}(\Gamma)$

ii) and we have the following **Green formula**: For all $u \in F_{\Delta}^{-s}(\Omega)$ and $\varphi \in X^{2+s}(\Omega)$,

$$\langle \Delta u, \varphi \rangle_{\Omega} = \langle u, \Delta \varphi \rangle_{H^{-s}(\Omega) \times H_0^s(\Omega)} + \left\langle u, \frac{\partial \varphi}{\partial n} \right\rangle_{H^{-s-1/2}(\Gamma) \times H^{s+1/2}(\Gamma)},$$

where $\langle \cdot, \cdot \rangle_{\Omega}$ denotes the duality bracket between $\mathfrak{T}^{-2-s}(\Omega)$ and $\mathfrak{T}^{2+s}(\Omega)$ and where we recall that

$$X^{2+s}(\Omega) = \{v \in H^{2+s}(\Omega) \cap H_0^1(\Omega); \Delta v \in H_0^s(\Omega)\} \hookrightarrow \mathfrak{T}^{2+s}(\Omega)$$

Using again the transposition method we get the following result.

**Theorem 23 (Very Weak Solutions in $H^{-s}(\Omega)$,
A-Moussaoui-Nguyen)**

Let $s \geq 0$, $k \geq 2$ integer such that $s < k - \frac{3}{2}$. Let Ω be of class $\mathcal{C}^{k-1,1}$. Assume $f \in \mathfrak{T}^{-2-s}(\Omega)$ and $g \in H^{-s-1/2}(\Gamma)$. Then there exists a unique solution $u \in H^{-s}(\Omega)$ of Problem (\mathcal{P}^D) satisfying the estimate as follows

$$\|u\|_{H^{-s}(\Omega)} \leq C \left(\|f\|_{\mathfrak{T}^{-2-s}(\Omega)} + \|g\|_{H^{-s-1/2}(\Gamma)} \right).$$

Observe that with this theorem, we improve the existence and regularity result of Lions-Magenes since

$$\mathfrak{T}^{2+s}(\Omega) \hookrightarrow \Xi^{2+s}(\Omega) \quad \text{with density}$$

and then

$$\Xi^{-2-s}(\Omega) \hookrightarrow \mathfrak{T}^{-2-s}(\Omega)$$

VIII. Solvability in Weighted Sobolev Spaces

For simplicity, we consider only one particular case. The following result is well known when Ω is of class $C^{1,1}$.

Theorem 24 (A-Moussaoui-Nguyen)

Let Ω be an bounded Lipschitz open subset of \mathbb{R}^N . The following mapping

$$\Delta : H_{\delta}^2(\Omega) \longrightarrow L_{\delta}^2(\Omega)$$

is an isomorphism, where $\delta(x)$ is the distance of x at the origine

$$H_{\delta}^2(\Omega) = \left\{ u \in H_0^1(\Omega); \delta D^{\lambda} u \in L^2(\Omega), \forall |\lambda| = 2 \right\}$$

and

$$L_{\delta}^2(\Omega) = \left\{ u \text{ measurable; } \int_{\Omega} \delta^2 |u|^2 < \infty \right\}.$$

- **Remark.** Recall that thanks to **Hardy Inequality**, we have

$$L^2_\delta(\Omega) \hookrightarrow H^{-1}(\Omega).$$

As $\mathcal{D}(\Omega)$ is dense in $H^2_\delta(\Omega)$ and in $L^2_\delta(\Omega)$, we deduce by duality that

$$\Delta : L^2_{\frac{1}{\delta}}(\Omega) \longrightarrow [H^2_\delta(\Omega)]'$$

is also an isomorphism.

- Now, we know that if moreover Ω is of class $C^{1,1}$

$$\Delta : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$$

is an isomorphism. Because

$$\Delta : H_\delta^2(\Omega) \longrightarrow L_\delta^2(\Omega)$$

is also an isomorphism and

$$[H^2(\Omega) \cap H_0^1(\Omega), H_\delta^2(\Omega)]_{1-\theta} = \left\{ v \in H^{1+\theta}(\Omega) \cap H_0^1(\Omega); \delta^{1-\theta} D^2 v \in L^2(\Omega) \right\}$$

we deduce by interpolation the following corollary:

Corollary 25 (A-Moussaoui-Nguyen)

Let Ω be a bounded open subset of \mathbb{R}^N of class $C^{1,1}$. Then for any $0 < \theta < 1$, the following mapping

$$\Delta : \left\{ v \in H^{2-\theta}(\Omega) \cap H_0^1(\Omega); \quad \delta^\theta D^2 v \in L^2(\Omega) \right\} \longrightarrow L_{\delta^\theta}^2(\Omega)$$

is an isomorphism. In particular, if $\theta = \frac{1}{2}$, then

$$\Delta : \mathfrak{T}^2(\Omega) \longrightarrow L_{\sqrt{\delta}}^2(\Omega)$$

is an isomorphism. Consequently,

$$\Delta : L_{\frac{1}{\sqrt{\delta}}}^2(\Omega) \longrightarrow \mathfrak{T}^{-2}(\Omega)$$

is also an isomorphism.

- i) We have already seen that for any $f \in \mathfrak{T}^{-2}(\Omega)$, there exists a unique $u \in L^2(\Omega)$ satisfying

$$\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma$$

Corollary 25 proves in fact that u is better because $u \in L^2_{\frac{1}{\sqrt{\delta}}}(\Omega)$.

Recall also that the result of Lions-Magenes state that

$$f \in \Xi^{-2}(\Omega) \leftrightarrow \mathfrak{T}^{-2}(\Omega) \implies u \in L^2(\Omega) \leftrightarrow L^2_{\frac{1}{\sqrt{\delta}}}(\Omega).$$

More strangely, we deduce the following property:

$$\text{if } u \in L^2_{\frac{1}{\sqrt{\delta}}}(\Omega), \quad \text{then } u|_{\Gamma} \in H^{-\frac{1}{2}}(\Gamma)$$

without any other assumption. Moreover, because $\mathcal{D}(\Omega)$ is dense in $L^2_{\frac{1}{\sqrt{\delta}}}(\Omega)$, then

$$u = 0 \quad \text{on } \Gamma \quad \text{in the sense } H^{-\frac{1}{2}}(\Gamma).$$

A fortiori the functions of $H^{\frac{1}{2}}_{00}(\Omega)$, instead of $H^{\frac{1}{2}}(\Omega)$, have a trace (in $L^2(\Gamma)$?) which is **equal to zero**.

- ii) Because

$$H_{00}^{1/2}(\Omega) \hookrightarrow L_{\frac{1}{\sqrt{\delta}}}^2(\Omega) \quad \text{and} \quad H^{-3/2}(\Omega) \hookrightarrow \mathfrak{T}^{-2}(\Omega)$$

the following isomorphism







$$\Delta : H_{00}^{1/2}(\Omega) \longrightarrow H^{-3/2}(\Omega),$$

can be considered as a regularity result of the following isomorphism.

$$\Delta : L_{\frac{1}{\sqrt{\delta}}}^2(\Omega) \longrightarrow \mathfrak{T}^{-2}(\Omega)$$

Perspectives

- Case of L^p -Theory
- Extension to the $\operatorname{div}(A\nabla)$ operator (in progress with L. Santos)
- Extension to the **biharmonic** problem (in progress)
- Extension to systems: **Stokes** (in progress), **elasticity**, ...
- Extension to different boundary conditions: **Fourier-Robin**, **Navier boundary condition**, ... (in progress)
- Evolution case : **heat equation**, **waves equation**, **Stokes equations**, ...

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