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This paper is dedicated to Professor Hugo Beirao da Veiga on the occasion of his 70th birthday.

Abstract. In this paper we study the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p$-spaces. This allows us to solve the evolution Stokes problems (1.1) together with the boundary condition (1.3).

1. Introduction

We consider in a bounded cylindrical domain $\Omega\times(0,T)$ the linearised evolution Navier-Stokes problem

$$(1.1)\qquad \begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega\times(0,T), \\ \qquad\qquad\qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega, \end{cases}$$

where the unkowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

To study Problem (1.1) it is necessary to add appropriate boundary conditions. This problem is often studied with Dirichlet boundary conditions, which is not always realistic since it does not reflect necessarily the behavior of the fluid on or near the boundary. In many problems of mathematical physics, Problem (1.1) is studied with other types of boundary conditions called slip boundary conditions.

H. Navier [**17**] has suggested in 1824 a type of boundary conditions based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$(1.2)\qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad 2\,\nu\,[\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = 0 \qquad \text{on } \Gamma\times(0,T),$$

where ν is the viscosity and $\alpha\geq 0$ is the coefficient of friction and $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$.

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The Navier boundary conditions defined above are often used to simulate the flows near rough walls as well as perforated walls. We also mention that such slip boundary conditions are used in the simulation of turbulent flows. Taking use of the vorticity field $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$, and using classical identities, one can observe that in the case of a flat boundary and when $\alpha = 0$ the conditions (1.2) may be replaced by

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \ \text{ on } \qquad \Gamma\times(0,T). \tag{1.3}$$

We call them Navier-type boundary conditions.

Problem (1.1) together with the boundary conditions (1.3) has been studied by several authors and the theory is in recent progress. In a two dimensional, simply connected bounded domain Yudovich [**21**] has established the existence and uniqueness of solution to this problem. These two-dimensional results are based on the fact that the vorticity is scalar and satisfies the maximum principle. However this techniques can not be extended to the three-dimensional case. On the other hand Mitrea and Monniaux [**15**] have employed the Fujita-Kato approach and proved the existence of a local mild solution to Problem (1.1) and (1.3).

In this paper we deal with the Stokes operator with the Navier-type boundary conditions (1.3). Our goal is to obtain a good semi-group theory for the Stokes operator with Navier-type boundary conditions (1.3) on $\boldsymbol{L}^p$-spaces as it is well known for Dirichlet boundary condition (for instance Giga and Sohr [**1, 9, 10, 11, 12**]). Our main result is the following:

THEOREM 1.1. *The Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on* $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

To prove Theorem 1.1 we use a classical approach. We study the resolvent of the Stokes operator. A key observation is that the Stokes operator with Navier-type boundary conditions is equal to the Laplacian operator with Navier-type boundary conditions. For this reason our work is reduced to study the following problem:

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} \;=\; \boldsymbol{f}, \ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. We prove the existence of strong solution to Problem (1.4) satisfying the resolvent estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{1.5}$$

Notice that for $p = 2$ one has estimate (1.5) in a sector $\lambda \in \Sigma_\varepsilon$ for a fixed $\varepsilon \in \,]0,\pi[$. We recall that

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}^*;\ |\arg\lambda| \leq \pi - \varepsilon\}.$$

In the literature, there are several results on the analyticity of the Stokes semi-group with Dirichlet boundary condition in L^p-spaces. In fact, in bounded domains, Giga [**9**] has studied the resolvent of the Stokes operator with Dirichlet boundary condition using the theory of pseudo-differential operators and get the desired result. In exterior domains, Giga and Sohr [**11**] approximate the resolvent of the Stokes operator with Dirichlet boundary condition with the resolvent of the Stokes operator in the entire space to prove this analyticity.

More recently, the analyticity of the Stokes semi-group with Dirichlet boundary condition is studied in spaces of bounded functions by Abe and Giga [**1**]. There approaches here is completely different from the classical approaches. In fact, they have proved a bound for

$$N(\boldsymbol{u},\pi)(x,t) = |\boldsymbol{u}(x,t)| + t^{1/2}\,|\nabla\boldsymbol{u}(x,t)| + t\,|\nabla^2\boldsymbol{u}(x,t)| + t\,|\partial_t\boldsymbol{u}(x,t)| + |\nabla\pi(x,t)|,$$

which is a key to prove the analyticity result. More precisely, they have proved

$$\|N(\boldsymbol{u},\pi)\|_{\boldsymbol{L}^\infty(\Omega\times]0,T_0[)} \leq C\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^\infty(\Omega)}.$$

To establish the last estimate they used a blow-up argument which is often used in the study of nonlinear elliptic and parabolic equations.

Now, concerning the Navier-type boundary conditions, Mitrea and Monniaux [**14**] have studied the resolvent of the Stokes operator with Navier-type boundary conditions in Lipschitz domains and proved estimate (1.5) using the context of differential forms on Lipschitz sub-domains of a smooth compact Riemannian manifold. In addition, when the domain Ω has a sufficiently smooth boundary, estimates of type (1.5) are proved using the fact that the boundary conditions (1.3) are regular elliptic (e.g. [**19**]) and the so called "Agmon trick" (e.g. [**3**]). Moreover, when the domain Ω is of class C^∞, [**16**] shows that the Laplacian with the Navier-type boundary condions (1.3) on $\boldsymbol{L}^p(\Omega)$ leaves the space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ invariant and hence generates a holomorphic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

In this paper we prove estimate (1.5) using (see Lemma 2.6) a formula involving the boundary conditions (1.3) and the following formula: For every $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ one has

$$\begin{aligned}
-\int_\Omega |\boldsymbol{u}|^{p-2}\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x \;=\; &\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega \left|\nabla|\boldsymbol{u}|^{p/2}\right|^2\mathrm{d}\,x\\
&+\;(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\left(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\right)\mathrm{Im}\left(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\right)\mathrm{d}\,x \;-\;\left\langle\frac{\partial\,\boldsymbol{u}}{\partial\boldsymbol{n}},\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\right\rangle_\Gamma,
\end{aligned}$$

where $\langle .\,,\,.\rangle_\Gamma$ is the antiduality between $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,p'}(\Gamma)$.

This paper is organized as follows. In section 2 we give the functional framework and some preliminary results at the basis of our proofs. Next in section 3 we define the Stokes operator with Navier-type boundary conditions, we will see that the Stokes operator with Navier-type boundary conditions (1.3) is equal to the Laplacian operator with conditions (1.3). Section 4 is devoted to our main result and its proof concerning the analyticity of the semi-group generated by the Stokes-operator with Navier-type boundary conditions on $\boldsymbol{L}^p$-spaces. Finally in section 5 we give a new version of the Stokes operator. We give extra assumptions on the Stokes operator that allows us to obtain a bounded and compact inverse as well as an exponential decay of the semi-group generated by the Stokes operator.

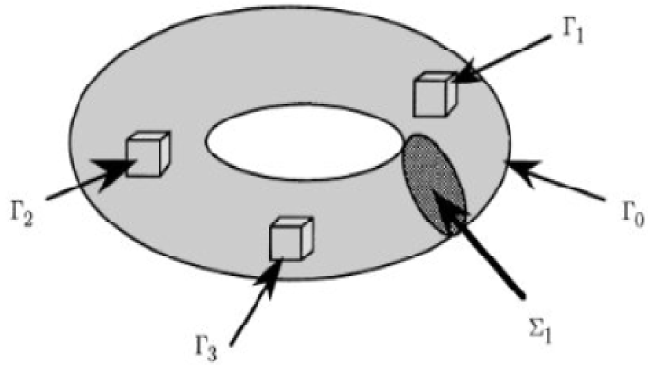
2. Notations and preliminary results

2.1. Functional framework. In this subsection we review some basic notations, definitions and functional framework which are essential in our work.

In what follows, if we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ of class at least $C^{1,1}$ and sometimes of class $C^{2,1}$. Then a unit normal vector to the boundary can be defined almost everywhere it will be denoted by $\boldsymbol{n}$. The generic point in Ω is denoted by $\boldsymbol{x} = (x_1,\, x_2,\, x_3)$.

We do not assume that the boundary Γ is connected and we denote by Γ_i, $0\leq i\leq I$, the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$. We also fix a smooth open set ϑ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in ϑ, and we denote by Ω_i, $0 \leq i \leq I$, the connected component of $\vartheta\backslash\overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\vartheta$ for $i = 0$).

We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ. The intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$ and finally the open set $\Omega^\circ = \Omega\backslash \cup_{j=1}^{J} \Sigma_j$ is simply connected and pseudo-$C^{1,1}$ (see [**4**] for instance).



We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e.* the difference of the traces for $1 \leq j \leq J$. In addition, for any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distribution in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$, it belongs to $\boldsymbol{L}^p(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^p(\Omega)$. In order to distinguish this extension from the gradient of q in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$ we denote it by $\widetilde{\mathbf{grad}}\, q$.

Finally, vector fields, matrix fields and their corresponding spaces defined on Ω will be denoted by bold character. The functions treated here are complex valued functions. We will use also the symbol σ to represent a set of divergence free functions. In other words If $\boldsymbol{E}$ is Banach space, then

$$\boldsymbol{E}_\sigma = \big\{\boldsymbol{v} \in \boldsymbol{E};\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\big\}.$$

Now, we introduce some functional spaces. Let $\boldsymbol{L}^p(\Omega)$ denote the usual vector valued $\boldsymbol{L}^p$-space over Ω. Let us define the spaces:

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\big\},$$

$$\boldsymbol{H}^p(\operatorname{div}, \Omega) = \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\big\},$$

$$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\operatorname{div}, \Omega),$$

equipped with the graph norm. Thanks to [**6**] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ and $\boldsymbol{X}^p(\Omega)$.
We also define the subspaces:

$$\boldsymbol{H}^p_0(\mathbf{curl}, \Omega) = \big\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\},$$

$$\boldsymbol{H}^p_0(\operatorname{div}, \Omega) = \big\{\boldsymbol{v} \in \boldsymbol{H}^p(\operatorname{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\big\},$$

$$\boldsymbol{X}^p_N(\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\},$$

$$\boldsymbol{X}^p_\tau(\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\big\}$$

and

$$\boldsymbol{X}_0^p(\Omega) \,=\, \boldsymbol{X}_N^p(\Omega) \cap \boldsymbol{X}_\tau^p(\Omega).$$

We have denoted by $\boldsymbol{v} \times \boldsymbol{n}$ (respectively by $\boldsymbol{v} \cdot \boldsymbol{n}$) the tangential (respectively normal) boundary value of $\boldsymbol{v}$ defined in $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ (respectively in $W^{-1/p,\,p}(\Gamma)$) as soon as $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively to $\boldsymbol{H}^p(\mathrm{div}, \Omega)$). More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$) has a tangential (respectively normal) trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively $\boldsymbol{v} \cdot \boldsymbol{n}$) in $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ (respectively in $W^{-1/p,\,p}(\Gamma)$) defined by:

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}^{1,\,p'}(\Omega),\ \ \langle \boldsymbol{v} \times \boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma \,=\, \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x \,-\, \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x \tag{2.1}$$

and

$$\forall\, \varphi \in W^{1,\,p'}(\Omega),\ \ \langle \boldsymbol{v} \cdot \boldsymbol{n},\, \varphi\rangle_\Gamma \,=\, \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \overline{\varphi}\,\mathrm{d}\,x \,+\, \int_\Omega \mathrm{div}\, \boldsymbol{v}\, \overline{\varphi}\,\mathrm{d}\,x, \tag{2.2}$$

where $\langle .,. \rangle_\Gamma$ is the anti-duality between $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,\,p'}(\Gamma)$ in (2.1) and between $W^{-1/p,\,p}(\Gamma)$ and $W^{1/p,\,p'}(\Gamma)$ in (2.2). Thanks to **[6]** we know that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)$ and in $\boldsymbol{H}_0^p(\mathrm{div}, \Omega)$.

Finally, we denote by $[\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)]'$ and $[\boldsymbol{H}_0^p(\mathrm{div}, \Omega)]'$ the dual spaces of $\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0^p(\mathrm{div}, \Omega)$ respectively.

Notice that we can characterize these dual spaces as follows: A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)]'$ if and only if there exist functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{curl}\, \boldsymbol{\xi}$. Moreover one has

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^p(\mathbf{curl}, \Omega)]'} \,=\, \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)},\, \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Similarly, a distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^p(\mathrm{div}, \Omega)]'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{grad}\, \chi$ and

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^p(\mathrm{div}, \Omega)]'} \,=\, \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}\,,\, \|\chi\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

2.2. Preliminary results. In this subsection, we review some known results which are essential in our work. First, We recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\,(\mathrm{div}\, \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v}.$$

Next, we have the following lemmas (see **[6]**):

LEMMA 2.1. *The spaces $\boldsymbol{X}_N^p(\Omega)$ and $\boldsymbol{X}_\tau^p(\Omega)$ defined above are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

Consider now the spaces

$$\boldsymbol{X}^{2,p}(\Omega) \,=\, \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\, \boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\big\}$$

and

$$\boldsymbol{Y}^{2,p}(\Omega) \,=\, \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\, \boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)\big\}.$$

LEMMA 2.2. *Assume that Ω is of class $C^{2,1}$, then the spaces $\boldsymbol{X}^{2,p}(\Omega)$ and $\boldsymbol{Y}^{2,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{2,p}(\Omega)$.*

Consider now the space

$$\boldsymbol{E}^{\,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{\,1,p}(\Omega);\ \Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\},$$

which is a Banach space for the norm:

$$\| \boldsymbol{v} \|_{\boldsymbol{E}^{\,p}(\Omega)} = \| \boldsymbol{v} \|_{\boldsymbol{W}^{\,1,p}(\Omega)} + \| \Delta \boldsymbol{v} \|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}\,.$$

Thanks to [**5**, Lemma 4.1] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^{\,p}(\Omega)$. Moreover we have the following Lemma (see [**5**, Corollary 4.2]):

LEMMA 2.3. *The linear mapping $\gamma : \boldsymbol{v} \longrightarrow \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_\tau^{p'}(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ in Ω.

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle \mathbf{curl}\boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma.$$

where $\langle .,.\rangle_\Gamma$ denotes the anti-duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle .,.\rangle_\Omega$ denotes the anti-duality between $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)$.

Next we consider the problem:

$$\mathrm{div}\,(\mathbf{grad}\,\pi \,-\, \boldsymbol{f}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi \,-\, \boldsymbol{f}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma. \tag{2.3}$$

We recall the following lemma concerning the weak Neumann problem (see [**18**] for instance).

LEMMA 2.4. *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the Problem (2.3) has a unique solution $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\mathbf{grad}\,\pi\|_{\boldsymbol{L}^p(\Omega)} \,\le\, C_1(\Omega)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

for some constant $C_1(\Omega) > 0$.

The following lemma plays an important role in the proof of the resolvent estimate (1.5):

LEMMA 2.5. *Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then*

$$\begin{aligned} -\int_\Omega |\boldsymbol{u}|^{p-2} \Delta \boldsymbol{u} \cdot \overline{\boldsymbol{u}}\,\mathrm{d}\,x \,&=\, \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2} \int_\Omega \left|\nabla |\boldsymbol{u}|^{p/2}\right|^2 \mathrm{d}\,x \\ &+\, (p-2)\,i \sum_{k=1}^{3} \int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\left(\frac{\partial \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{Im}\left(\frac{\partial \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{d}\,x \,-\, \left\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}},\, |\boldsymbol{u}|^{p-2} \boldsymbol{u} \right\rangle_\Gamma, \end{aligned} \tag{2.4}$$

where $\langle .\,,\,.\rangle_\Gamma$ is the antiduality between $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,p'}(\Gamma)$.

PROOF. Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. We recall that $\boldsymbol{u} = (\ u_1,\ \ u_2,\ \ u_3\ )$ is a vector complex valued function. We recall also that the vectors $\overline{\boldsymbol{u}}$ and $\mathrm{Re}\,\boldsymbol{u}$ given by

$$\overline{\boldsymbol{u}} = (\ \overline{u_1},\ \ \overline{u_2},\ \ \overline{u_3}\ ), \qquad \mathrm{Re}\,\boldsymbol{u} = (\ \mathrm{Re}\,u_1,\ \ \mathrm{Re}\,u_2,\ \ \mathrm{Re}\,u_3\ )$$

are the conjugate and the real part of the vector $\boldsymbol{u}$ respectively. We can easily verify that for any $1 \leq k \leq 3$ one has

$$\frac{\partial\,|\boldsymbol{u}|^2}{\partial x_k} = \sum_{j=1}^{3}\Big[\frac{\partial\,u_j}{\partial x_k}\overline{u_j} + u_j\frac{\partial\,\overline{u_j}}{\partial x_k}\Big] = 2\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big).$$

As a result

$$\frac{\partial\,|\boldsymbol{u}|^{p-2}}{\partial x_k} = (p-2)\,|\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big) \quad\text{and}\quad \Big|\frac{\partial\,|\boldsymbol{u}|^{p/2}}{\partial x_k}\Big|^2 = \frac{p^2}{4}\,|\boldsymbol{u}|^{p-4}\,\Big[\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\Big]^2. \tag{2.5}$$

Now, using (2.5) we have

$$\sum_{k=1}^{3}\frac{\partial\,|\boldsymbol{u}|^{p-2}}{\partial\,x_k}\,\frac{\partial\,\boldsymbol{u}}{\partial\,x_k}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x \;=\; 4\,\frac{p-2}{p^2}\int_\Omega\Big|\nabla|\boldsymbol{u}|^{p/2}\Big|^2\,\mathrm{d}\,x \;+$$
$$(p-2)\,i\sum_{k=1}^{3}\int_\Omega|\boldsymbol{u}|^{p-4}\,\mathrm{Re}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{Im}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x.$$

Finally applying the Green-Formula one gets (2.4). □

Let us now consider any point P on Γ and choose an open neighborhood W of P on Γ small enough to allow the existence of two families of C^2 curves on W. The lengths s_1 and s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves respectively. With these notations we have $\boldsymbol{v}_\tau = \sum_{k=1}^{2} v_k\,\boldsymbol{\tau}_k$, where $v_k = \boldsymbol{v}\cdot\boldsymbol{\tau}_k$. We recall that for all $\boldsymbol{v}$ in $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ the following formula holds:

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} \;=\; \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) \;-\; \Big(\frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}} \;-\; \sum_{j=1}^{2}\Big(\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{v}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j \qquad \text{on}\;\;\Gamma, \tag{2.6}$$

where $\nabla_{\boldsymbol{\tau}}$ is the tangential gradient. More precisely we have the following lemma (see [**5**]):

Lemma 2.6. *Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$. Then $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and satisfies formula (2.6).*

We end this subsection by the definition of a sectorial operator (see [**8**, Chapter 2, page 96]). Let $0 \leq \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta \;=\; \Big\{\lambda\in\mathbb{C}^*;\;\;|\arg\lambda| < \pi-\theta\Big\}.$$

Definition 2.7. Let X be a Banach space. We say that a linear densely defined operator $\mathcal{A}\,:\,D(\mathcal{A})\subseteq X\longmapsto X$ is sectorial if there exists a constant $M>0$ such that

$$\forall\,\lambda\in\Sigma_\theta,\qquad \|R(\lambda,\,\mathcal{A})\|_{\mathcal{L}(X)}\;\leq\;\frac{M}{|\lambda|}, \tag{2.7}$$

where $R(\lambda,\,\mathcal{A}) \;=\; (\lambda\,I\;-\;\mathcal{A})^{-1}$.

This means that the resolvent of a sectorial operator contain a sector Σ_θ for some $0\leq\theta<\pi/2$ and for every $\lambda\in\Sigma_\theta$ one has estimate (2.7).

Moreover thanks to [**8**, Chapter 2, Theorem 4.6, page 101] we have the following theorem:

THEOREM 2.8. *An operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if $\mathcal{A}$ is sectorial.*

Nevertheless, it is not always easy to prove that an operator is sectorial in the sense of Definition 2.7. For this reason in some cases we will use the result of Yosida [**20**] who has proved that it suffices to prove (2.7) in the half plane $\{\lambda \in \mathbb{C}^*;\, \mathrm{Re}\,\lambda \geq w\}$, for some $w \geq 0$. This result is stated in [**7**, Chapter 1, Theorem 3.2, page 30] and proved by K. Yosida .

PROPOSITION 2.9. *Let $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq X \longmapsto X$ be a linear densely defined operator, let $w \geq 0$ and $M > 0$ such that*

$$\forall\, \lambda \in \mathbb{C}^*,\ \ \mathrm{Re}\,\lambda \geq w, \qquad \|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \,\leq\, \frac{M}{|\lambda|}.$$

Then $\mathcal{A}$ is sectorial.

3. The Stokes operator with Navier-type boundary conditions

Consider the space

$$\boldsymbol{V}^p_\tau(\Omega) \,=\, \Big\{\boldsymbol{v} \in \boldsymbol{X}^p_\tau(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0\ \text{ in } \Omega\Big\}, \tag{3.1}$$

which is a Banach space for the norm $\boldsymbol{X}^p(\Omega)$. The Stokes operator with Navier-type boundary conditions is defined by

$$\forall\, \boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega),\ \ \forall\, \boldsymbol{v} \in \boldsymbol{V}^{p'}_\tau(\Omega), \qquad \langle A\,\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))' \times \boldsymbol{V}^{p'}_\tau(\Omega)} \,=\, \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

On other words, the Stokes operator with Navier-type boundary conditions is the linear mapping $A : \mathbf{D}_p(A) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where

$$\begin{aligned} \mathbf{D}_p(A) = \big\{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega),\ \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \Omega,& \\ \boldsymbol{u}\cdot\boldsymbol{n} \,=\, 0,\ \ \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} \,=\, \mathbf{0} \text{ on } \Gamma\big\}& \end{aligned} \tag{3.2}$$

and $A\,\boldsymbol{u} \,=\, -P\Delta\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathbf{D}_p(A)$. We recall that $P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is the Helmholtz projection defined by, for all $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $P\boldsymbol{f} \,=\, \boldsymbol{f} \,-\, \mathbf{grad}\,\pi$, where π is the unique solution of Problem (2.3).

PROPOSITION 3.1. *For all $\boldsymbol{u} \in \mathbf{D}_p(A)$, $A\,\boldsymbol{u} = -\Delta\boldsymbol{u}$.*

PROOF. Let $\boldsymbol{u} \in \mathbf{D}_p(A)$, it is clear that $\Delta\boldsymbol{u} \in \boldsymbol{H}^p(\mathrm{div},\Omega)$. Moreover since $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} \,=\, \mathbf{0}$ on Γ then we can easily verify that $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} \,=\, 0$ on Γ. This means that $\Delta\boldsymbol{u}\cdot\boldsymbol{n} \,=\, 0$ on Γ. As a consequence, $\Delta\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $A\boldsymbol{u} \,=\, -P\Delta\boldsymbol{u} \,=\, -\Delta\boldsymbol{u}$. Notice that here the pressure π is a solution of the problem

$$\Delta\pi = 0 \qquad \text{in } \Omega, \qquad \frac{\partial\pi}{\partial\boldsymbol{n}} = \Delta\,\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Thus $\pi = $ Constant and $\mathbf{grad}\,\pi = 0$ in Ω. □

The following two propositions give the density and a regularity property concerning the domain of the Stokes operator.

PROPOSITION 3.2. *The space $\mathbf{D}_p(A)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

PROOF. It is clear that $\boldsymbol{\mathcal{D}}_\sigma(\Omega) \,\subset\, \mathbf{D}_p(A) \,\subset\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Now, since $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then $\mathbf{D}_p(A)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. □

PROPOSITION 3.3. *Suppose that Ω is of class $C^{2,1}$, then*

$$\mathbf{D}_p(A) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\Big\}. \tag{3.3}$$

PROOF. Let $\boldsymbol{u} \in \mathbf{D}_p(A)$ and set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Then $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\mathbf{curl}\,\boldsymbol{z} = -\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{z}\times\boldsymbol{n} = \mathbf{0}$ on Γ. Thus $\boldsymbol{z} \in \boldsymbol{X}^p_N(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Finally observe that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Thanks to Lemma 2.2, we conclude that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, which ends the proof. $\square$

REMARK 3.4. (i) Notice that, thanks to Lemmas 2.1 and 2.2, when Ω is of class $C^{2,1}$ we have

$$\forall\,\boldsymbol{u} \in \mathbf{D}_p(A), \qquad \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

(ii) We recall that, thanks to [**5**, Proposition 4.7], when Ω is of class $C^{2,1}$, for all $\boldsymbol{u} \in \mathbf{D}_p(A)$ such that $\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0$, $1 \le j \le J$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

4. Analyticity results

In this section we will state our main result and its proof. We will prove that the Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$. Since the Hilbertian case is different from the general $\boldsymbol{L}^p$-theory we will treat each case separately.

4.1. The Hilbertian case. Before we state our theorem let us recall the following lemma:

For all $\varepsilon \in\,]0,\pi[$, let Σ_ε be the sector

$$\Sigma_\varepsilon \,=\, \big\{\lambda \in \mathbb{C}^*;\ |\arg\lambda| \le \pi - \varepsilon\big\}.$$

LEMMA 4.1. *Let $\varepsilon \in\,]0,\pi[$ be fixed. There exists a constant $C_\varepsilon > 0$ such that for every positive real numbers a and b one has:*

$$\forall\,\lambda \in \Sigma_\varepsilon, \qquad |\lambda\, a \,+\, b| \ge C_\varepsilon(|\lambda| a \,+\, b). \tag{4.1}$$

Now we want to study the resolvent of the Stokes operator. For that we consider the problem

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f},\ \ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{4.2}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.

REMARK 4.2. Observe that, Problem (4.2) is equivalent to the problem

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f}, & \text{in } \ \Omega,\\ \ \ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \ \Gamma. \end{cases} \tag{4.3}$$

In fact, let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the unique solution of Problem (4.3) and set $\operatorname{div}\boldsymbol{u} = \chi$. It is clear that $\lambda\chi - \Delta\chi = 0$ in Ω. Moreover, since $\boldsymbol{f}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ then $\Delta\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Notice also that the condition $\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ implies that $\mathbf{curl}\,\mathbf{curl}\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Finally since $\Delta\boldsymbol{u} = \mathbf{grad}(\operatorname{div}\boldsymbol{u}) - \mathbf{curl}\,\mathbf{curl}\boldsymbol{u}$ one gets $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$ on Γ. Thus $\chi = 0$ in Ω and the result is proved.

The following theorem gives the solution of the resolvent of the operator A as well as a resolvent estimate.

THEOREM 4.3. *Let* $\varepsilon \in]0, \pi[$ *be fixed,* $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ *and* $\lambda \in \Sigma_\varepsilon$.
(i) *The Problem (4.2) has a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.
(ii) *There exist a constant* $C'_\varepsilon > 0$ *independent of* $\boldsymbol{f}$ *and* λ *such that the solution* $\boldsymbol{u}$ *satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{4.4}$$

and

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.5}$$

($C'_\varepsilon = 1/C_\varepsilon$, where C_ε is the constant in (4.1)).
(iii) *If* Ω *is of class* $C^{2,1}$ *then* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ *and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega,\lambda,\varepsilon)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{4.6}$$

where $C(\Omega,\lambda,\varepsilon) = C(\Omega)(C'_\varepsilon + 1)(|\lambda| + 1)$.

REMARK 4.4. We note that for $\lambda > 0$ the constant C'_ε is equal to 1 and we recover the m-accretiveness property of the Stokes operator on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

PROOF. **(i) Existence and uniqueness**: Consider the space $\boldsymbol{V}^2_\tau(\Omega)$ given by (3.1) (for $p = 2$). It is clear that $\boldsymbol{V}^2_\tau(\Omega)$ is a closed subspace of $\boldsymbol{X}^2_\tau(\Omega)$ and it is an Hilbert space for the inner product of $\boldsymbol{X}^2(\Omega)$. We also recall that on $\boldsymbol{V}^2_\tau(\Omega)$ the norm of $\boldsymbol{X}^2_\tau(\Omega)$ is equivalent to the norm of $\boldsymbol{H}^1(\Omega)$.

Now, consider the variational problem: find $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x, \tag{4.7}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x.$$

We can easily verify that a is a continuous sesqui-linear form on $\boldsymbol{V}^2_\tau(\Omega)$. For the coercivity, observe that since $\lambda \in \Sigma_\varepsilon$, thanks to Lemma 4.1 there exists a constant C_ε such that

$$\begin{aligned} |\,a(\boldsymbol{v},\boldsymbol{v})| &= \left|\lambda\,\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right| \\ &\geq C_\varepsilon\left(|\lambda|\,\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\ &\geq C_\varepsilon\,\min(|\lambda|\,,1)\|\boldsymbol{v}\|^2_{\boldsymbol{X}^2_\tau(\Omega)}. \end{aligned}$$

Then for all $\lambda \in \Sigma_\varepsilon$ a is a sesqui-linear continuous coercive form on $\boldsymbol{V}^2_\tau(\Omega)$. Due to Lax-Milgram Lemma, Problem (4.7) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{V}^2_\tau(\Omega))'$.

Now, using the same argument as in the proof of [**5**, Proposition 4.3] we prove that the two problems (4.2) and (4.7) are equivalent. Thus we obtain the existence and the uniqueness of solution to Problem (4.2).

(ii) Estimates: Multiplying the first equation of System (4.2) by $\overline{\boldsymbol{u}}$ and integrating both sides one gets

$$\lambda\int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}\,x + \int_\Omega |\mathbf{curl}\,\boldsymbol{u}|^2\,\mathrm{d}\,x = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x.$$

Now as described above, since $\lambda\in\Sigma_\varepsilon$, there exists a constant $C'_\varepsilon = 1/C_\varepsilon$ such that

$$\begin{aligned} |\lambda|\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} &\le C'_\varepsilon\big|\lambda\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\big| \\ &= C'_\varepsilon\Big|\int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x\Big| \\ &\le C'_\varepsilon\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

As a result

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

which is estimate (4.4). In addition, it is clear that

$$\begin{aligned} \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} &\le C'_\varepsilon\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \frac{C'^2_\varepsilon}{|\lambda|}\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

which is estimate (4.5). We recall that C_ε is the constant in (4.1).

(iii) Regularity: The regularity of the solution is a direct application of Proposition 3.3. Let us prove estimate (4.6). Thanks to (4.4) it is clear that

$$\|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{f}-\lambda\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le (C'_\varepsilon+1)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.8}$$

Now, since $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ one has estimate (4.6). □

Remark 4.5. Consider the sesqui-linear form (see [**4**]):

$$\forall\,\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{V}^2_\tau(\Omega),\qquad a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{4.9}$$

If Ω is simply connected, we know that for all $\boldsymbol{v}\in\boldsymbol{V}^2_\tau(\Omega)$ one has

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2(\Omega)} \le C\,\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.10}$$

As a result, the sesqui-linear form a is coercive and we can apply Lax-Milgram Lemma to find solution to the problem: find $\boldsymbol{u}\in\boldsymbol{V}^2_\tau(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x,$$

where $\boldsymbol{f}\in\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$. This means that the operator $A:\mathbf{D}_2(A)\subset\boldsymbol{L}^2_{\sigma,\tau}(\Omega)\longmapsto\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is bijective.

Now, if Ω is multiply-connected, the inequality (4.10) is false. Indeed we introduce the Kernel $\boldsymbol{K}^2_\tau(\Omega)$:

$$\boldsymbol{K}^2_\tau(\Omega) = \big\{\boldsymbol{v}\in\boldsymbol{X}^2_\tau(\Omega);\ \operatorname{div}\boldsymbol{v}=0,\ \mathbf{curl}\,\boldsymbol{v}=\mathbf{0}\ \text{in}\ \Omega\big\}. \tag{4.11}$$

Thanks to [**4**, Proposition 3.14] we know that this kernel is not trivial, it is of finite dimension and it is spanned by the functions $\widetilde{\mathbf{grad}}\,q^\tau_j$, $1\le j\le J$, where q^τ_j is the unique solution up to an additive constant of the problem:

$$\begin{cases} -\Delta q^\tau_j = 0 & \text{in }\ \Omega^\circ, \\ \partial_n q^\tau_j = 0 & \text{on }\ \Gamma, \\ \big[q^\tau_j\big]_k = \text{constant}, & 1\le k\le J, \\ \big[\partial_n q^\tau_j\big]_k = 0; & 1\le k\le J, \\ \langle\partial_n q^\tau_j\,,\,1\rangle_{\Sigma_k} = \delta_{jk}, & 1\le k\le J. \end{cases} \tag{4.12}$$

Moreover, thanks to [**4**, Corollary 3.16], for all $\boldsymbol{v} \in \boldsymbol{X}^2_\tau(\Omega)$ we have the following Poincaré-type inequality:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2_\tau(\Omega)} \leq C_2(\Omega)(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}|). \tag{4.13}$$

The following theorem gives us the analyticity of the semi-group generated by the Stokes operator on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

Theorem 4.6. *The operator $-A$ generates a bounded analytic semi-group on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.*

Proof. Thanks to Theorem 2.8 it suffices to prove that $-A$ is sectorial which is a direct application of Theorem 4.3. We recall that, with the Navier-type boundary conditions (1.3) the Stokes operator coincides with the $-\Delta$ operator. □

Remark 4.7. We recall that the restriction of an analytic semi-group to the non negative real axis is C_0 semi-group. Thanks to Remark 4.4 the restriction of our analytic semi-group to the real axis gives a C_0 semi-group of contraction.

4.2. $\boldsymbol{L}^p$-theory. We have seen that the Hilbert case can be obtained easily using Lax-Milgram Lemma. However the general case $p \neq 2$ is not as easy as the particular case $p = 2$ and demand extra work. In this section we extend Theorem 4.3 to every $1 < p < \infty$. We start by the existence theorem:

Theorem 4.8. *Let $\lambda \in \mathbb{C} \in \Sigma_\varepsilon$ and let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. The Problem (4.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Moreover, if Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.*

Proof. As in [**5**, Proposition 4.3], we can easily verify that Problem (4.2) is equivalent to the variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^{p'}_\tau(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\;\mathrm{d}x \;=\; \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x,$$

where $\boldsymbol{V}^p_\tau(\Omega)$ is given by (3.1). The proof is done in three steps:
(i) Case $2 \leq p \leq 6$. Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the unique solution of Problem (4.2). We write Problem (4.2) in the form:

$$\begin{cases} -\Delta\boldsymbol{u} = \boldsymbol{F}, \qquad \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{4.14}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda\,\boldsymbol{u}$. Thans to the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ one has $\boldsymbol{F} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

It remains to verify (see [**5**, Proposition 4.3]) that $\boldsymbol{F}$ satisfies the compatibility condition

$$\forall\,\boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega), \qquad \int_\Omega \boldsymbol{F}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0, \tag{4.15}$$

where

$$\boldsymbol{K}^{p'}_\tau(\Omega) \;=\; \big\{\boldsymbol{v} \in \boldsymbol{X}^{p'}_\tau(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega\big\}.$$

To this end let $\boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega)$, thanks to Lemma 2.3 one has:

$$\int_\Omega \boldsymbol{F}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; -\int_\Omega \Delta\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x - \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\,,\,\boldsymbol{v}\rangle_\Gamma \;=\; 0.$$

Now applying [**5**, Proposition 4.3], our solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$.

(ii) Case $p \geq 6$. Since $\boldsymbol{f} \in \boldsymbol{L}^6(\Omega)$, Problem (4.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Now proceeding in the same way as above one gets that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.

(iii) Case $p \leq 2$. As described above, for $p \geq 2$ the operator $\lambda I + A$ is an isomorphism from $\boldsymbol{V}^p_\tau(\Omega)$ to $(\boldsymbol{V}^{p'}_\tau(\Omega))'$. Then the adjoint operator which is equal to $\lambda I + A$ is an isomorphism from $\boldsymbol{V}^{p'}_\tau(\Omega)$ to $(\boldsymbol{V}^p_\tau(\Omega))'$ for $p' \leq 2$. This means that, the operator $\lambda I + A$ is an isomorphism for $p \leq 2$, which ends the proof. Notice that the operator $\lambda I + A \in \mathcal{L}(\boldsymbol{V}^p_\tau(\Omega), (\boldsymbol{V}^{p'}_\tau(\Omega))')$ is defined by: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^p_\tau(\Omega)$, for all $\boldsymbol{\xi} \in \boldsymbol{V}^{p'}_\tau(\Omega)$

$$\langle(\lambda I + A)\boldsymbol{\varphi}, \boldsymbol{\xi}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))' \times \boldsymbol{V}^{p'}_\tau(\Omega)} = \lambda \int_\Omega \boldsymbol{\varphi} \cdot \overline{\boldsymbol{\xi}} \, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \mathbf{curl}\, \overline{\boldsymbol{\xi}} \, \mathrm{d}x.$$

□

Now, we want to prove a resolvent estimate similar to the estimate (4.4) for all $1 < p < \infty$. But this case is not as obvious as the case $p = 2$ and the proof will be done in several steps.

PROPOSITION 4.9. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $|\lambda| \geq \lambda_0$, where $\lambda_0 = \lambda_0(\Omega, p)$ is defined in (4.25). Moreover, let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where $1 < p < \infty$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (4.2). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_1(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.16}$$

where the constant $\kappa_1(\Omega, p)$ is independent of λ and $\boldsymbol{f}$. Moreover, for $\frac{4}{3} \leq p \leq 4$ the constant κ_1 is independent of Ω and p.

PROOF. Suppose that $p \geq 2$, multiplying the first equation of Problem (4.2) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and integrating both sides one gets thanks to Lemma 2.5

$$\begin{aligned}\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &+ (p-2)\,i \sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\operatorname{Re}\Big(\frac{\partial\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\operatorname{Im}\Big(\frac{\partial\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\ &= \int_\Gamma |\boldsymbol{u}|^{p-2}\Big(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma + \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x.\end{aligned} \tag{4.17}$$

Notice that the integral on Γ is well defined. In fact, thanks to Lemma 2.6 and to the boundary conditions satisfied by $\boldsymbol{u}$ we have $\left(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}} = -\sum_{j=1}^{2}\left(\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{s}_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\right)\boldsymbol{\tau}_j$. Moreover, since Ω is of class $C^{1,1}$ then $\boldsymbol{n} \in \boldsymbol{W}^{1,\infty}(\Gamma)$ and since $\boldsymbol{u}_{\boldsymbol{\tau}}$ belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{L}^p(\Gamma)$. As a result $\left(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\right)_{\boldsymbol{\tau}}$ belongs to $\boldsymbol{L}^p(\Gamma)$. In addition, it is clear that $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}} \in \boldsymbol{W}^{1,p'}(\Omega)$ and then its trace belongs to $\boldsymbol{W}^{1-1/p',p'}(\Gamma) \hookrightarrow \boldsymbol{L}^{p'}(\Gamma)$. Which justify the integral on Γ.

Now observe that

$$\Big(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}_{\boldsymbol{\tau}} = -\sum_{j=1}^{2}\Big(\frac{\partial\boldsymbol{n}}{\partial s_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j\cdot\sum_{k=1}^{2}\overline{u}_k\,\boldsymbol{\tau}_k$$

$$= \boldsymbol{n}\cdot\sum_{j,k=1}^{2}\overline{u}_j\,u_k\,\frac{\partial\boldsymbol{\tau}_k}{\partial s_j}. \tag{4.18}$$

Next we put together the two formulas (4.17) and (4.18), we study separately the real and the imaginary parts of formula (4.17) and using the fact that Ω is of class $C^{1,1}$ one gets

$$\mathrm{Re}\,\lambda\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \tag{4.19}$$
$$\leq C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}$$

and

$$|\mathrm{Im}\,\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \leq \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \tag{4.20}$$
$$+ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)},$$

for some constant $C_1(\Omega)>0$. Now putting together (4.19) and (4.20) one has

$$|\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \tag{4.21}$$
$$\leq \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, thanks to [**13**, Chapter 1, Theorem 1.5.1.10, page 41] we know that:

$$\int_\Gamma|w|^2\,\mathrm{d}\,\sigma \leq \varepsilon\int_\Omega|\nabla w|^2\,\mathrm{d}\,x + C_\varepsilon\int_\Omega|w|^2\,\mathrm{d}\,x, \tag{4.22}$$

for all $w\in H^1(\Omega)$ and for all $\varepsilon\in\,]0,1[$. Applying formula (4.22) to $w=|\boldsymbol{u}|^{p/2}$ and substituting in (4.21) one gets

$$|\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \tag{4.23}$$
$$\leq \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,C_1(\Omega)\Big[\varepsilon\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x + C_\varepsilon\int_\Omega|\boldsymbol{u}|^p\,\mathrm{d}\,x\Big]$$
$$+ 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

We chose $\varepsilon>0$ such that $\varepsilon\,C_1(\Omega)=\frac{p-2}{p^2}$. As a result the constant C_ε in (4.23) depends on p and Ω. Then by setting $C_\varepsilon=C_2(\Omega,p)$ one has

$$|\lambda|\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq C_3(\Omega,p)\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)},$$

where

$$C_3(\Omega,p) = 2\,C_1(\Omega)C_2(\Omega,p). \tag{4.24}$$

We define

$$\lambda_0 = 2\, C_3(\Omega, p). \tag{4.25}$$

Now, for $|\lambda| \geq \lambda_0$ one has

$$\frac{|\lambda|}{2}\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\, \frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x$$
$$\leq\, \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

In fact we have two different cases.

(i) Case $2 \leq p \leq 4$. One has

$$\frac{|\lambda|}{2}\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{4-p}{2} \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\, \frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x \leq$$
$$2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, \frac{4}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.26}$$

which is the required estimate.

(ii) Case $p > 4$. We write Problem (4.2) in the form (4.14). Thanks to [**5**, Proposition 4.3] we have

$$\big\|\boldsymbol{u} \,-\, \sum_{j=1}^{J} \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau \big\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_4(\Omega)\, \|\boldsymbol{f} \,-\, \lambda\, \boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq \big\| \sum_{j=1}^{J} \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau \big\|_{\boldsymbol{W}^{1,4}(\Omega)} \,+\, C_4(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} + \tag{4.27}$$
$$+\, C_4(\Omega)\, |\lambda|\, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

On the other hand, thanks to [**6**, Lemma 3.2] and (4.26) we have

$$|\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j}| \,\leq\, C_5(\Omega)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \,\leq\, \frac{C_5(\Omega)}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \leq \frac{C_5(\Omega)}{\lambda_0}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

As a result, using (4.26) with $p = 4$ and substituting in (4.27) one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \,\leq\, C_7(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)},$$

where $C_7(\Omega) = C_6(\Omega)\, \frac{C_5(\Omega)}{\lambda_0} \,+\, 5\, C_4(\Omega)$ and $\|\widetilde{\mathbf{grad}}\, q_j^\tau\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_6(\Omega)$.

Now since $\boldsymbol{W}^{1,4}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, then

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} &\leq C_8(\Omega)\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \,\leq\, C_8(\Omega)\, C_7(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}\\ &\leq C_8(\Omega)\, C_7(\Omega) (\mathrm{mes}\,\Omega)^{(p-4)/4p} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

Consequently

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C_9(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.28}$$

where

$$C_9(\Omega) = C_8(\Omega)C_7(\Omega)(\mathrm{mes}\,\Omega)^{1/4}. \tag{4.29}$$

Notice that

$$\begin{aligned} \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &= \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \\ &\leq C_9(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{4.30}$$

Thus proceeding exactly as above and putting together (4.19), (4.22) and (4.30) one has

$$\begin{aligned} \mathrm{Re}\,\lambda\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ \leq (C_3(\Omega,p)C_9(\Omega)+1)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

As a result one has

$$\mathrm{Re}\,\lambda\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{10}(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.31}$$

$$\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \leq C_{10}(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{4.32}$$

and

$$2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \leq C_{10}(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{4.33}$$

where

$$C_{10}(\Omega,p) = 1 + C_3(\Omega,p)C_9(\Omega). \tag{4.34}$$

In addition, using (4.20), (4.32) and (4.33) one has

$$|\mathrm{Im}\,\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{11}(\Omega,p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.35}$$

Thus putting together (4.31) and (4.35) one gets for $p > 4$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{12}(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.36}$$

which ends the case $p > 4$.

Finally putting together (4.26) and (4.36) we conclude that for $p \geq 2$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_1(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.37}$$

with

$$\kappa_1(\Omega,p) = \max(4, C_{12}(\Omega,p)). \tag{4.38}$$

By duality we obtain estimate (4.37) for all $1 < p < \infty$. □

PROPOSITION 4.10. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and $0 < |\lambda| \leq \lambda_0$, with λ_0 as in Proposition (4.9). Moreover, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem 4.2. Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_2(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.39}$$

For some constant $\kappa_2(\Omega,p)$ independent of λ and $\boldsymbol{f}$.

PROOF. Thanks to (4.4) with $\varepsilon = \frac{\pi}{2}$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{C_{13}}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

and

$$\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \le \frac{C_{13}^2}{|\lambda|}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Moreover we know that

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\le C_{14}(\Omega)(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)})\\ &\le C_{14}(\Omega)C_{13}^2\frac{1+|\lambda|}{|\lambda|^2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Now because $|\lambda| \le \lambda_0$ we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le \frac{C_{15}(\Omega)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

where

$$C_{15}(\Omega) = C_{13}\sqrt{C_{14}(\Omega)(1+\lambda_0)}. \tag{4.40}$$

In fact we have two different cases.

(i) Case $2 \le p \le 6$. Because $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ we have

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\le C_{16}(\Omega,p)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\\ &\le \frac{C_{16}(\Omega,p)C_{15}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{L^2(\Omega)} \le \frac{C_{17}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{L^p(\Omega)},\end{aligned} \tag{4.41}$$

where

$$C_{17}(\Omega,p) = (\mathrm{mes}\,\Omega)^{(p-2)/2p}\,C_{15}(\Omega,p)C_{16}(\Omega). \tag{4.42}$$

(ii) Case $p \ge 6$. Proceeding in a similar way as in Proposition 4.9 (case $p > 4$), we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C_{18}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.43}$$

Finally putting together (4.41) and (4.43), we deduce the estimate (4.39) with

$$\kappa_2(\Omega,p) = \max(C_{17}(\Omega,p)\,,C_{18}(\Omega,p)). \tag{4.44}$$

□

As a conclusion of Propositions 4.9 and 4.10 we have the following theorem:

THEOREM 4.11. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \ge 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (4.2). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{\kappa_3(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.45}$$

where $\kappa_3(\Omega,p) = \max\,(\kappa_1(\Omega,p),\kappa_2(\Omega,p))$.

In addition, if Ω is of class $C^{2,1}$ we have the following estimate

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{\kappa_4(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.46}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \kappa_5(\Omega, p)\,\frac{1+|\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.47}$$

PROOF. The proof of estimate (4.45) is a conclusion of Propositions 4.9 and 4.10. Let us prove estimate (4.46). The proof is done in two steps.
(i) Case $\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$. Thanks to [**5**, Proposition 4.7] we know that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. Now, using the Gagliardo-Nirenberg inequality (see [**2**, Chapter IV, Theorem 4.14, Theorem 4.17] for instance) we have

$$\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega,p)\,\|\Delta\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\
&= C(\Omega,p)\,\|\boldsymbol{f}-\lambda\,\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\
&\leq \frac{C(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

(ii) General case. Let $\boldsymbol{u} \in \mathbf{D}_p(A)$ be the unique solution of Problem (4.2) and set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \sum_{j=1}^{J}\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^\tau.$$

As a result, thanks to the previous case we have

$$\|\mathbf{curl}\,\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\Delta\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{curl}\,\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\Delta\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} = \|\Delta\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, thanks to [**6**, Lemma 3.2] we know that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence we deduce estimate (4.46).

Finally, when Ω is of class $C^{2,1}$, on $\mathbf{D}_p(A)$ the norm of $\boldsymbol{W}^{2,p}(\Omega)$ is equivalent to the graph norm of the Stokes operator with Navier-type boundary conditions (1.3). As a result when has estimate (4.47). □

As in the Hilbertian case, Proposition 3.2 and Theorems 4.8 allow us to deduce the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

THEOREM 4.12. *The operator $-A$ generates a bounded analytic semigroup on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.*

PROOF. The proof is a direct application of Proposition 2.9 with $w = 0$. In fact, thanks to Proposition 3.2 and Theorems 4.8 and 4.11 the operator $-A$ satisfies the assumptions of Proposition 2.9. This justify the analyticity of the semi-group generated by the operator $-A$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p \leq \infty$. □

REMARK 4.13. Notice that, unlike the Hilbertian case, we can not use the result of [**8**, Chapter II, Theorem 4.6, page 101] to prove the analyticity of the semi-group generated by the Stokes operator in the $\boldsymbol{L}^p$-space where we have supposed that $\operatorname{Re}\lambda \geq 0$.

REMARK 4.14. Consider the two problems:

$$\left\{\begin{array}{lll}\lambda\boldsymbol{u}-\Delta\boldsymbol{u}=\boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u}=0 & \text{in} & \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\mathbf{0} & \text{on} & \Gamma\end{array}\right. \tag{4.48}$$

and

$$\left\{\begin{array}{llll}\lambda\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 & \text{in} & \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=0, & [\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}=0 & \text{on} & \Gamma,\end{array}\right. \tag{4.49}$$

where $\lambda\in\mathbb{C}^*$ is such that $\operatorname{Re}\lambda\geq 0$ and $\boldsymbol{f}\in\boldsymbol{L}^p_\sigma(\Omega)$ (respectively $\boldsymbol{f}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$).

In a forthcoming two papers we will study the two Problems (4.48) and (4.49). In fact, proceeding in a similar way as in Theorem 4.8 and Propositions 4.9 and 4.10 we prove that these two Problems have a unique solution $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ (respectively $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$) that satisfy the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\leq\frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Moreover when Ω is of class $C^{2,1}$, we have $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$. This means that the Laplacian operator with normal boundary conditions and the Stokes operator with Navier boundary conditions generate a bounded analytic semi-group on $\boldsymbol{L}^p_\sigma(\Omega)$ and $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ respectively .

This analyticity allows us to solve the evolutionary Stokes Problem with normal boundary condition and pressure boundary condition:

$$\left\{\begin{array}{llll}\frac{\partial\boldsymbol{u}}{\partial t}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 & \text{in} & \Omega\times(0,T),\\ \boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}, & \pi=0 & \text{on} & \Gamma\times(0,T),\\ & \boldsymbol{u}(0)=\boldsymbol{u}_0 & \text{in} & \Omega,\end{array}\right. \tag{4.50}$$

as well as the evolutionary Stokes Problem (1.1) with Navier-boundary condition (1.2) for a given $\boldsymbol{f}\in L^q(0,T;\,\boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0\in\boldsymbol{L}^p_\sigma(\Omega)$ (respectively $\boldsymbol{u}_0\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$).

5. Stokes operator with flux boundary conditions

In this section we will also consider the Stokes operator associated to Problem (4.2) but with adding an extra boundary condition which is the flux through the cuts Σ_j, $1\leq j\leq J$. This last condition enables the Stokes operator to be invertible with bounded and compact inverse.

Consider the space

$$\boldsymbol{X}_p\;=\;\big\{\boldsymbol{f}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega);\;\int_\Omega\boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x=0,\;\forall\;\boldsymbol{v}\in\boldsymbol{K}^{p'}_\tau(\Omega)\big\} \tag{5.1}$$

(do not confuse between this space and the space $\boldsymbol{X}^p(\Omega)$ defined in the subsection 2.1).

Next, we define the operator A' : $\mathbf{D}_p(A')\subset\boldsymbol{X}_p\longmapsto\boldsymbol{X}_p$ by:

$$\mathbf{D}_p(A')=\big\{\boldsymbol{u}\in\mathbf{D}_p(A);\;\langle\boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}=0,\;1\leq j\leq J\big\}$$

and $A'\boldsymbol{u}=A\boldsymbol{u}$, for all $\boldsymbol{u}\in\mathbf{D}_p(A')$. On other words, the operator A' is the restriction of the Stokes operator to the space $\boldsymbol{X}_p$. It is clear that when Ω is simply connected the Stokes operator A coincides with the operator A'.

REMARK 5.1. Let $\boldsymbol{u}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, it is important to know that (see [**6**, Lemma 3.2, Corollary 3.4]) the condition $\int_\Omega\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x=0$ for all $\boldsymbol{v}\in\boldsymbol{K}^{p'}_\tau(\Omega)$ is equivalent to the condition $\langle\boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}=0,\;1\leq j\leq J$.

PROPOSITION 5.2. *The operator A' is a well defined operator of dense domain.*

PROOF. Thanks to Remark 5.1 it is clear that $\mathbf{D}_p(A') \subset \boldsymbol{X}_p$. Moreover, using Lemma 2.3 we can easily verify that for all $\boldsymbol{v} \in \boldsymbol{K}^{p'}_{\tau}(\Omega)$, $\int_\Omega \Delta \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x = 0$. As a result $A'\boldsymbol{u} \in \boldsymbol{X}_p$ and A' is a well defined operator.

Now, for the density, let $\boldsymbol{w} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ such that $\langle \boldsymbol{w} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$ for all $1 \leqslant j \leqslant J$. We know that there exists a sequence $(\boldsymbol{w}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{w}_k \longrightarrow \boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. As a consequence for all $1 \leqslant j \leqslant J$, $\langle \boldsymbol{w}_k \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \longrightarrow \langle \boldsymbol{w} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$, as $k \to +\infty$.
Now for all $k \in \mathbb{N}$, setting $\widetilde{\boldsymbol{w}}_k = \boldsymbol{w}_k - \sum_{j=1}^J \langle \boldsymbol{w}_k \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q^\tau_j$. We can easily verify that $(\widetilde{\boldsymbol{w}}_k)_k$ is in $\mathbf{D}_p(A')$ and converges to $\boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. ☐

Now we will study the resolvent of the operator A'. For this reason we consider the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0, & 1 \leq j \leq J, \end{cases} \tag{5.2}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{X}_p$. We skip the proof of the following theorem because it is similar to the proof of [**5**, Proposition 4.3], Theorem 4.8 and 4.11.

THEOREM 5.3. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{X}_p$. The Problem (5.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ that satisfies the estimates (4.45)-(4.46). In addition, when Ω is of class $C^{2,1}$ the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{5.3}$$

where $C(\Omega, p)$ is independent of λ and $\boldsymbol{f}$.

As a result we have the following theorem

THEOREM 5.4. *The operator $-A'$ generates a bounded analytic semi-group on $\boldsymbol{X}_p$ for all $1 < p < \infty$.*

REMARK 5.5. Let $(S(t))_{t\geqslant 0}$ be the semi-group generated by $-A'$ on $\boldsymbol{X}_p$. We notice that $S(t) = T(t)_{|\boldsymbol{X}_p}$ where $(T(t))_{t\geq 0}$ is the analytic semi-group generated by the operator $-A$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

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