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Extensible Beam Models in Large Deformation Under Distributed Loading: a Numerical Study on Multiplicity of Solutions

Francesco dell'Isola, Alessandro Della Corte, Antonio Battista, and Emilio Barchiesi

Abstract In this paper we present numerical solutions to a geometrically nonlinear version of the extensible Timoshenko beam model under distributed load. The particular cases in which: i) extensional stiffness is infinite (inextensible Timoshenko model), ii) shear stiffness is infinite (extensible Euler model) and iii) extensional and shear stiffnesses are infinite (inextensible Euler model) will be numerically explored. Parametric studies on the axial stiffness in both the Euler and Timoshenko cases will also be shown and discussed.

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2.1 Introduction

In the recent literature the behaviour of a clamped-free nolinear inextensible Euler *Elastica* introduced in Euler (1952); Bernoulli (1843, 1691); see Luongo and Zulli (2013); Eugster (2015); Steigmann and Faulkner (1993) for general reference works, has been mathematically investigated under distributed load (Della Corte et al, 2016). In particular, the set of stable equilibrium configurations has been completely characterized in Della Corte et al (2019). On the other hand, extending such rigorous results to the extensible Euler beam model, or to the Timoshenko beam model is not straightforward. Indeed, the presence of the additional kinematical descriptor accounting for extensibility gives rise to various new mathematical difficulties. The most basic one is that it changes the functional set in which the problem is naturally collocated and in particular prevents it from being a vector space, since a strictly positive local axial deformation has to be prescribed. Of course one can obtain this making suitable assumptions on the energy, but in any case the non-autonomous variational problem that arises in the case of a distributed load will present new difficulties with respect to the inextensible case (Battista et al, 2018).

The problem of existence and stability of equilibrium configurations for extensible Euler and Timoshenko beams in large deformations under distributed load is therefore an open one. Because of this reason, it is interesting to study the behavior of solutions by means of a systematic collection of parametric studies starting from the inextensible *Elastica* and approaching more general beam models. The present work is aimed at performing such kind of investigation. Our main effort will be to show the variety of different (and at times rather exotic) equilibrium configurations that can arise when the value of the load is large enough. Moreover, we want to numerically investigate how fast the number of possible equilibrium solutions increases with the load and how this particular feature is affected by allowing shear deformation.

The study of these exotic configurations is particularly important, nowadays, due to the enhancement of computational methods that make nonlinearity more practically relevant for structural members (Fertis, 2006; Ladevèze, 2012; Antman and Renardy, 1995; Rezaei et al, 2012; Eugster et al, 2014; Steigmann, 2017) as well as in machine mechanics applications (Pepe et al, 2016; Giorgio and Del Vescovo, 2018). Moreover, they are becoming fashionable as an elementary constituent of microstructured objects manufactured with computer-aided techniques (Atai and Steigmann, 1997; dell'Isola et al, 2016b; Ravari and Kadkhodaei, 2015; Ravari et al, 2014; dell'Isola et al, 2016a; Turco et al, 2016, 2017; Milton et al, 2017; Spagnuolo et al, 2017). These objects are potentially advantageous for their mechanical characteristics, as shown in recent literature (Boutin et al, 2017; Giorgio et al, 2017; Scerrato et al, 2016; Eremeyev, 2017; Golaszewski et al, 2019; Turco et al, 2019;

Franciosi et al, 2019) and their theoretical study requires tools from homogenization theory (Boubaker et al, 2007; Ravari et al, 2016; Dos Reis and Ganghoffer, 2012; Reda et al, 2016), discrete mechanics (Turco, 2018; Jawed et al, 2018; Andreaus et al, 2018), theory of generalized continua (Misra et al, 2016; Altenbach et al, 2013; Altenbach and Eremeyev, 2013; Placidi et al, 2017, 2014, 2016) as well as developments in nonlinear elasticity of beams. In this last regard, certainly a full understanding of the onset and the characteristics of multiple solutions for the static problem under distributed load would be an important step forward, and possibly more general beam models will also have to be considered (Diyaroglu et al, 2015; Challamel, 2013; Challamel et al, 2013) because of the exotic properties of microstructured continua (Misra et al, 2018; dell'Isola et al, 2018; Diyaroglu et al, 2017).

The paper is organized as follows: we introduce the general model for a nonlinear version of the Timoshenko beam; we deduce the equilibrium equations by means of the Lagrange multipliers method. Then, imposing stationarity to the energy functional, we deduce an expression for the total energy in stationary points which depends only on the angles formed by the tangent to the deformed shape. Then we show and discuss numerical results on the multiplicity of solutions with large value of the load and on the effect of releasing extensional stiffness in both the Euler and the Timoshenko case. Finally we propose some future research directions.

2.2 The Model

2.2.1 Kinematics and Deformation Energy

Let $\{\boldsymbol{D}_1, \boldsymbol{D}_2\}$ be an orthogonal reference system in which the beam lies in the unstressed configuration along $\boldsymbol{D}_1$. We will denote by s the abscissa along the beam, by the apex $'$ the differentiation with respect to the reference abscissa[1] and by $\boldsymbol{\chi}(s)$ the placement function. The tangent vector to the current configuration of the beam is then:

$$\boldsymbol{\chi}' = \alpha(s)\left[\cos(\theta(s))\boldsymbol{D_1} + \sin(\theta(s))\boldsymbol{D_2}\right] := \alpha(s)\boldsymbol{e}(\theta(s)) \tag{2.1}$$

where $\boldsymbol{e}(\theta(s))$ represents the unit vector parallel to $\boldsymbol{\chi}'$. Therefore $\alpha(s)$ describes the local elongation of the beam:

$$\|\boldsymbol{\chi}'(s)\| = \alpha(s) \tag{2.2}$$

while $\theta(s)$ is the angle between $\boldsymbol{\chi'}(s)$ and $\boldsymbol{D_1}$.

We will assume the following energy functional:

[1] Notice that this means that, denoting by θ the angle formed by the tangent to the deformed shape and a reference axis, θ' does not coincide with the geometrical curvature but with the so-called Chebyshev curvature (see Chebyshev, 1878).

$$\mathcal{E}^{\text{def}}(\boldsymbol{\chi},\varphi) := \int_0^L \left\{ \underbrace{\frac{k_e}{2}\left(\|\boldsymbol{\chi}'(s)\|-1\right)^2}_{\text{extensional energy}} + \underbrace{\frac{k_b}{2}(\varphi'(s))^2}_{\text{flexural energy}} + \underbrace{\frac{k_t}{2}(\varphi(s)-\theta(s))^2}_{\text{shear energy}} \right\} ds \tag{2.3}$$

Here φ is the angle between sections of the beam (supposed rigid) and the normal to the neutral axes, while k_b, k_e and k_t are respectively the bending, extensional and shear stiffness. The energy can be rewritten as:

$$\mathcal{E}^{\text{def}} = \int_0^L \left\{ \frac{k_e}{2}\left(\alpha-1\right)^2 + \frac{k_b}{2}(\varphi'(s))^2 + \frac{k_t}{2}(\varphi(s)-\theta(s))^2 \right\} ds. \tag{2.4}$$

We introduce now a uniformly distributed load $\boldsymbol{b}(s)$ and a concentrated load and couple in the endpoint of the beam $s = L$, denoted respectively by $\boldsymbol{R}(L)$ and $\boldsymbol{M}(L)$. The total energy of the system is then:

$$\begin{aligned}\mathcal{E}^{\text{tot}} = &\int_0^L \left\{ \frac{k_e}{2}\left(\alpha-1\right)^2 + \frac{k_b}{2}(\varphi'(s))^2 + \frac{k_t}{2}(\varphi(s)-\theta(s))^2 - \boldsymbol{b}\cdot\boldsymbol{\chi} \right\} ds \\ &- \boldsymbol{R}\cdot\boldsymbol{\chi}(L) - \boldsymbol{M}\varphi(L)\end{aligned} \tag{2.5}$$

The beam model described by the functional (2.5) is a geometrically nonlinear version of the Timoshenko beam model (introduced in Timoshenko, 1921, 1922), which is a particular case of Cosserat continuum introduced in Cosserat and Cosserat (1909); for general references and interesting results see e.g. Altenbach et al (2010); Birsan et al (2012); Forest (2005); Eremeyev and Pietraszkiewicz (2016)). Interesting generalizations of the Timoshenko beam model have been proposed (see e.g. Romano et al, 1992; Serpieri and Rosati, 2014), while a periodic mechanical system whose homogenized limit is the model (2.5) (in the particular case $\alpha \equiv 1$) is shown in Battista et al (2018). It is in fact a microstructured 1D system whose unit cell is an articulated parallelogram and equipped with suitably placed rotational springs, and it can be easily obtained by means of 3D printing. Of course also other (possibly more complex) microstructured systems can have a similar homogenized version (on microstructured continua see e.g. Barchiesi et al, 2019; Engelbrecht and Berezovski, 2015; Engelbrecht et al, 2005; Barchiesi et al, 2018).

2.2.2 Lagrange Multipliers Method

Given the total energy of the system, equilibrium configurations are found as stationary points of the energy functional (2.5). A synthetic formulation of the problem, taking into account together the constrain of Eq. (2.1) and the total energy, is obtained with the introduction of a Lagrange multiplier $\boldsymbol{\Lambda}(s)$. We get the following functional formulation:

$$\mathcal{E}^{\text{tot}} = \int_0^L \left\{ \frac{k_e}{2}(\alpha-1)^2 + \frac{k_b}{2}(\varphi'(s))^2 + \frac{k_t}{2}(\varphi(s)-\theta(s))^2 - \boldsymbol{b}\cdot\boldsymbol{\chi} + \right.$$
$$\left. + \boldsymbol{\Lambda}\cdot(\boldsymbol{\chi}' - \alpha\left[\cos(\theta(s))\boldsymbol{D_1} + \sin(\theta(s))\boldsymbol{D_2}\right]) \right\} ds - \boldsymbol{R}\cdot\boldsymbol{\chi}(L) - \boldsymbol{M}\varphi(L) \tag{2.6}$$

i.e., the energy is a function of the fields $\alpha(s), \varphi(s), \theta(s), \boldsymbol{\chi}(s), \boldsymbol{\Lambda}(s)$. The first variation of the energy with respect to these fields (considered independent) gives the two boundary value problems (BVPs):

$$\begin{cases} -k_b\varphi'' + k_t(\varphi-\theta) = 0 \\ k_b\varphi'(L)\delta\varphi(L) = \boldsymbol{M}\delta\varphi(L) \\ k_b\varphi'(0)\delta\varphi(0) = 0 \end{cases} \tag{2.7}$$

and

$$\begin{cases} -\boldsymbol{\Lambda}' - \boldsymbol{b} = 0 \\ \boldsymbol{\Lambda}(L)\cdot\delta\boldsymbol{\chi}(L) = \boldsymbol{R}\cdot\boldsymbol{\delta}\chi(L) \\ \boldsymbol{\Lambda}(0)\cdot\delta\boldsymbol{\chi}(0) = 0, \end{cases} \tag{2.8}$$

as well as the algebraic relations:

$$k_t(\varphi-\theta) + \alpha\boldsymbol{\Lambda}\cdot\boldsymbol{e}_\perp(\theta) = 0 \tag{2.9}$$

$$k_e(\alpha-1) - \boldsymbol{\Lambda}\cdot\boldsymbol{e}(\theta) = 0 \tag{2.10}$$

and the kinematic constraint Eq. (2.1).

In Eq. (2.9) we introduced:

$$\boldsymbol{e}_\perp(\theta(s)) = -\sin(\theta(s))\boldsymbol{D}_1 + \cos(\theta(s))\boldsymbol{D}_2$$

and the two boundary conditions in 0 are imposed considering a cantilever beam (lying along $\boldsymbol{D}_1$ in the reference configuration and clamped in the extreme $s = 0$).

2.3 Numerical Simulations

2.3.1 Numerical Methods

An increasingly popular approach for the numerical study of nonlinear beams is isogeometric analysis (see e.g. Balobanov and Niiranen, 2018; Niiranen et al, 2017; Cazzani et al, 2016; Greco et al, 2017; Dortdivanlioglu et al, 2017), which is a

suitable variant of the finite element method. This method is very powerful and relatively light from a computational point of view, but just like every energy-related method it is not very suitable to study the multiplicity of arising solutions. For this reason, the numerical technique used here is the same as in Battista et al (2018). Indeed, the boundary value problem for the clamped-free Euler and Timoshenko beams has been solved by means of a shooting technique. We introduce a family of Cauchy problems [2]:

$$\mathcal{P}_k = \begin{cases} \theta'' = -b(1-s)\cos\theta \\ \theta(0) = 0 \\ \theta'(0) = k \end{cases} \tag{2.11}$$

depending on the parameter k. Then we selected the solutions of (2.11) which satisfy (with prescribed accuracy) $\theta'(1) = 0$, so as to obtain a numerical solution of the equilibrium condition $\delta\mathcal{E}^{tot} = 0$. Clearly the solution for the Cauchy problem exists and is unique for every initial datum k.

In Fig. 2.1, we show the plot of $\theta'(1)$ as a function of $k := \theta'(0)$ for an inextensible Euler beam model for $b = 250$. The graph intersects in five different points the horizontal axis, which means that in this case we have five different solutions of the boundary value problem with $\theta(0) = 0$ and $\theta'(1) = 0$. In the next section, the solutions will be shown in the same order as they appear as intersections between $\theta'(1)$ and the horizontal axis. Therefore the absolute minimum of the total energy will be always the rightmost configuration and, of course, the only solution for b small enough.

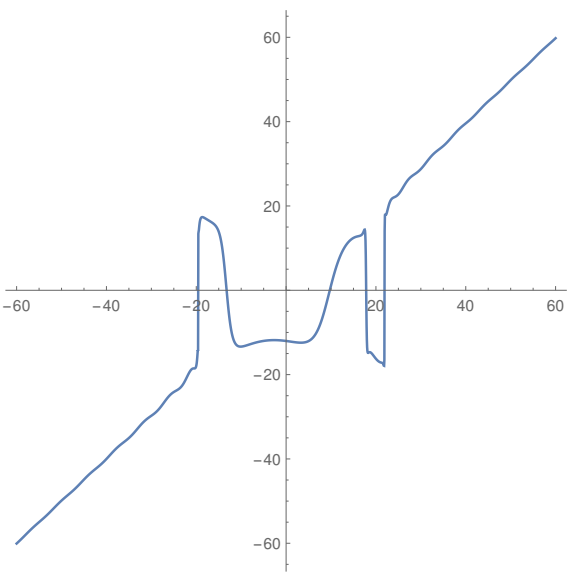

Fig. 2.1 $\theta'(1)$ as a function of $k := \theta'(0)$ for an inextensible Euler beam with a transverse applied load $b = 250$. Solutions to the boundary value problem (2.11) correspond to the intersections of the curve with the horizontal axis.

[2] We consider here the inextensible Euler beam for simplicity, but everything is analogous for the general case.

Remark 2.1. Of course this numerical technique can only capture *regular* equilibrium configurations. It is well-known that the minima (and maxima) of non-autonomous functionals may be not regular enough to satisfy the Euler-Lagrange equations (even in the one-dimensional case as in Ball and Mizel, 1987). This, together with the assessment of the stability of the solutions, is the main reason for which a rigorous study of the problem will be crucial.

2.3.2 The Number of Equilibrium Configurations when the Load Increases

It is generally very difficult to address theoretically the problem of evaluating how fast the number of solutions of a nonlinear parametric dynamical system increases with the parameter (Guckenheimer and Holmes, 1983), which in our case is the external load. It is therefore interesting, as a preliminary step, to address the problem numerically. In Fig. 2.2 the number of solutions for an inextensible (left) and extensible (right) Euler beem model is plotted as a function of the non-dimensional external load. As expected, the behavior is that of a step-function, as it is clear that new branches of solutions arise only when the external load overcomes specific thresholds. The number of equilibrium configurations increases significantly with the load, and again as expected it reaches slightly larger values in case an additional kinematic degree of freedom (i.e. α) is included.

Figure 2.3 is consistent with this. Indeed, in this case inextensible (left) and extensible (right) Timoshenko beam models are considered, which produces an even more rapid increase of the possible equilibrium configurations. It has to be remarked, however, that a straight comparison with the Euler case is difficult because adimensionalizing the load is a different procedure in the two cases.

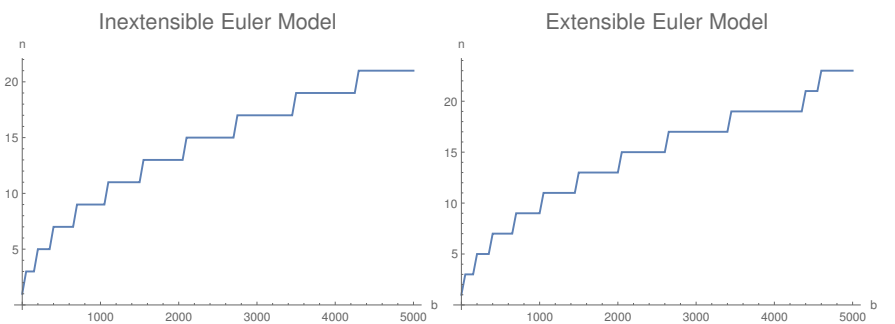

Fig. 2.2 Number of solutions, n, as a function of the transversal applied load, b. Left: inextensible Euler model. Right: extensible Euler model ($k_e = 1.3 \times 10^3$).

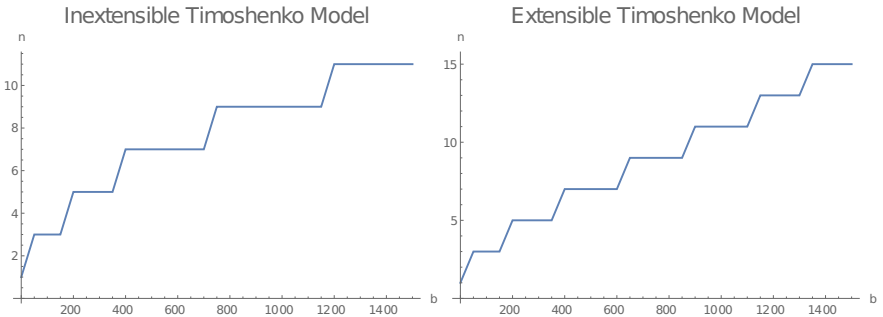

Fig. 2.3 Number of solutions, n, as a function of the transversal applied load, b. Left: inextensible Timoshenko model ($k_t = 1.8 \times 10^4$). Right: extensible Timoshenko model ($k_t = 1.8 \times 10^4$, $k_e = 2 \times 10^3$).

2.3.3 Equilibrium Configurations

In the first gallery of equilibrium solutions we will show the full set of equilibrium configurations when the nondimensional load is $b = 60$, $b = 250$ and $b = 500$. We considered inextensible Euler (Figs. 2.4, 2.8, 2.12), extensible Euler (Figs. 2.5, 2.9, 2.13), inextensible Timoshenko (Figs. 2.6, 2.10, 2.14) and extensible Timoshenko (Figs. 2.7, 2.11, 2.15) beam models.

For the chosen values of the other parameters, we have the same number of solutions (when applying the same load) for the four beam models. It has to be remarked that no branch of solutions appear to bifurcate. Instead, when the load increases, new branches appear at some thresholds. It can be seen that $\theta(s)$ is a positive monotonic function for the rightmost solution in all the cases. For $b = 60$ the central equilib-

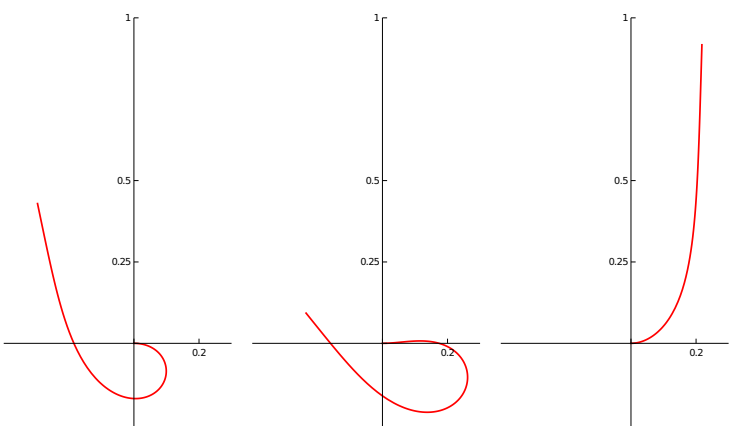

Fig. 2.4 Clamped inextensible Euler beam with a transversal applied load $b = 60$

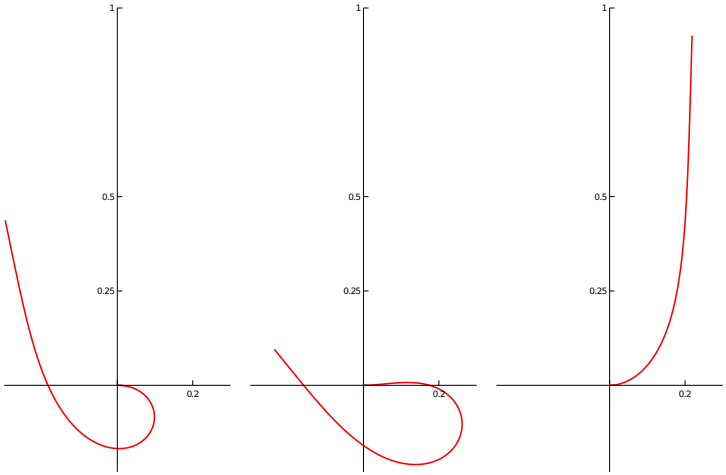

Fig. 2.5 Clamped extensible Euler beam with a transversal applied load $b = 60$ ($k_e = 3000$)

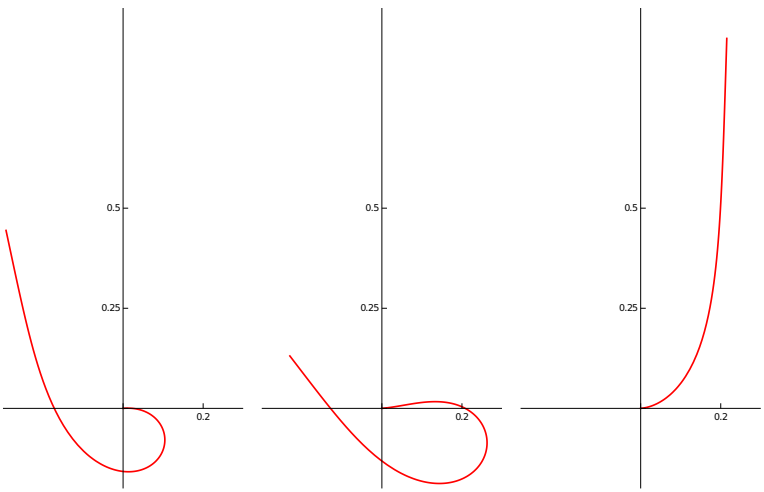

Fig. 2.6 Clamped inextensible Timoshenko beam with a transversal applied load $b = 60$ ($k_t = 7000$)

rium shape is non monotonic, while the leftmost is a negative monotonic function. This is a general trend as the load increases. Indeed, branches that arise with increasingly large values of the load will be made of progressively more numerous monotonic pieces. Of course in the Timoshenko case in general it is $\theta(0) \neq 0$. As the boundary datum for θ in 0 can have more than one solution, we have always chosen the smallest one in absolute value (see also the Appendix). The axial elongation relative to Figs. 2.9 and 2.11 are shown respectively in Figs. 2.16 and 2.18. The shear

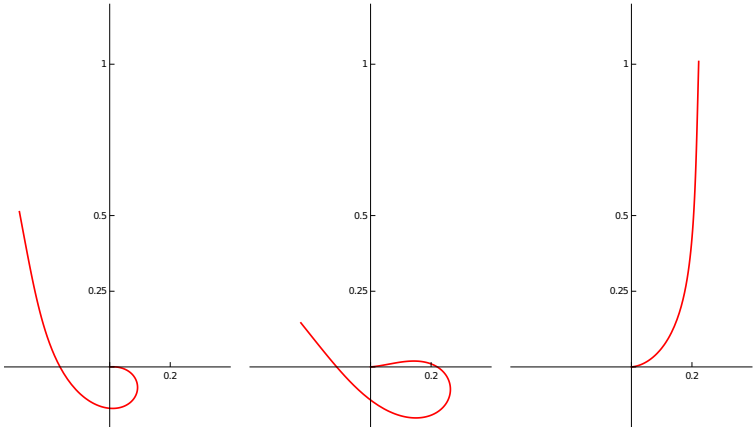

Fig. 2.7 Clamped extensible Timoshenko beam with a transverse applied load $b = 60$ ($k_t = 7000$, $k_e = 3000$)

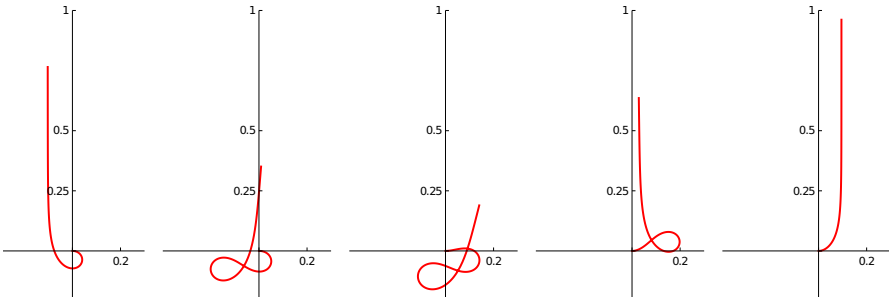

Fig. 2.8 Clamped inextensible Euler beam with a transversal applied load $b = 250$

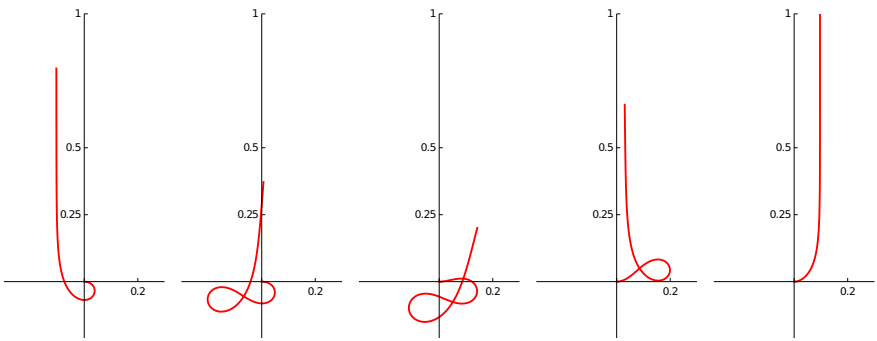

Fig. 2.9 Clamped extensible Euler beam with a transversal applied load $b = 250$ ($k_e = 3000$)

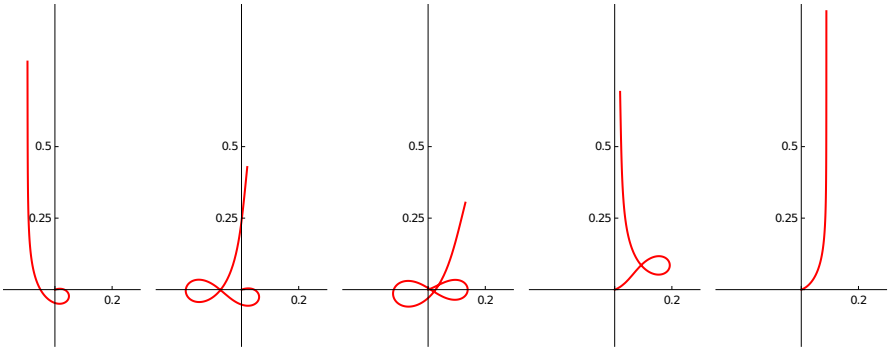

Fig. 2.10 Clamped inextensible Timoshenko beam with a transversal applied load $b = 250$ ($k_t = 7000$)

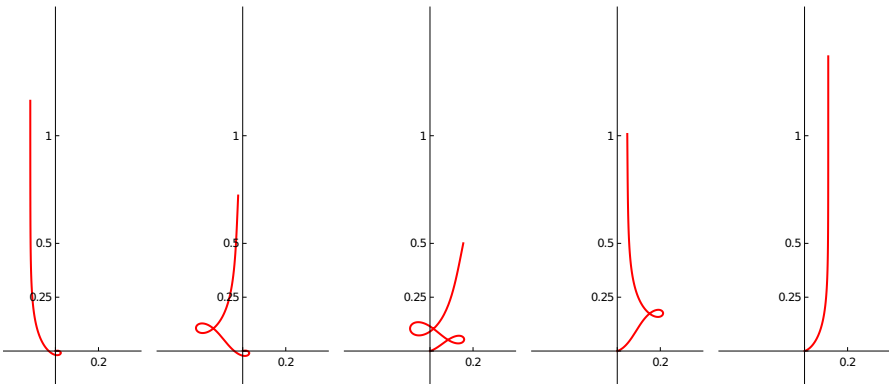

Fig. 2.11 Clamped extensible Timoshenko beam with a transverse applied load $b = 250$ ($k_t = 7000$, $k_e = 3000$)

deformation relative to Figs. 2.10 and 2.11 are shown respectively in Figs. 2.17 and 2.19.

It is not clear whether the solutions shown herein for the extensible Euler and Timoshenko models can be stable—while in Della Corte et al (2019) it has been proved that for the inextensible Euler case only the left and right configurations of Figs. 2.4 and 2.8 can be stable.

2.3.4 Parametric Study on the Extensional Stiffness

When the parameter k_e diverges, the beam model tends to inextensibility. The effect of decreasing k_e is evaluated for the Euler and Timoshenko models respectively in Figs. 2.24 and 2.26 for $b = 120$. In Figs. 2.25 and 2.27 the local elongation $\alpha(s)$ is

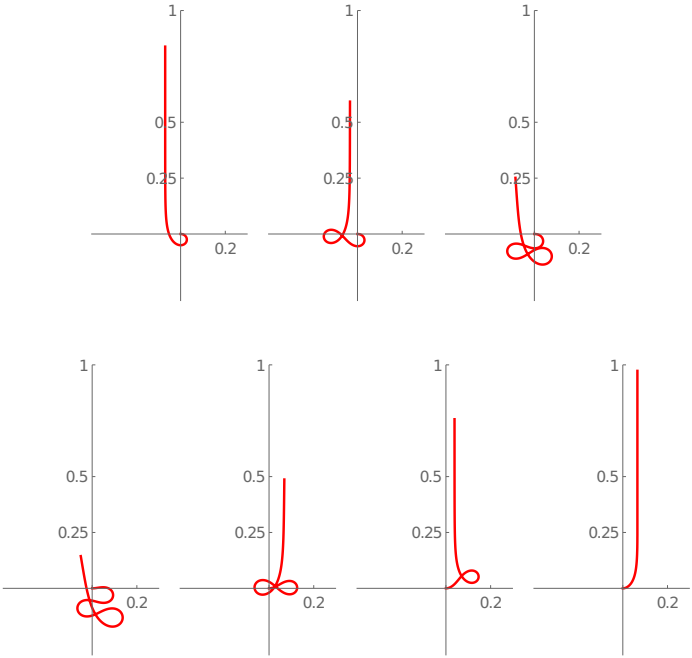

Fig. 2.12 Clamped inextensible Euler beam with a transverse applied load $b = 500$.

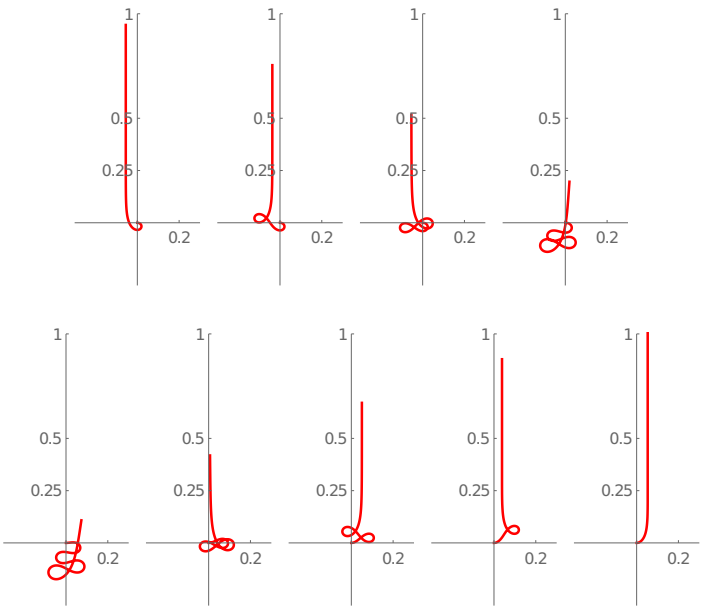

Fig. 2.13 Clamped extensible Euler beam with a transverse applied load $b = 500$ and $k_e = 5000$.

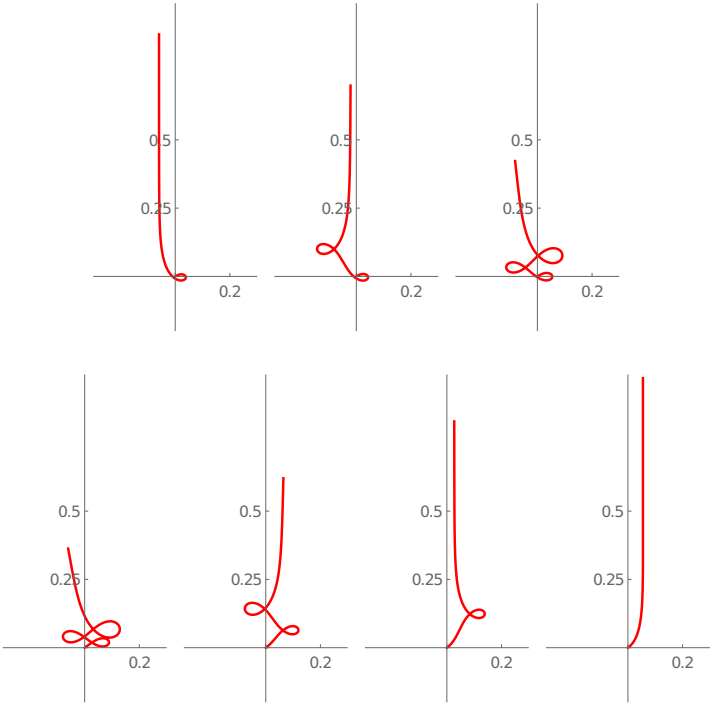

Fig. 2.14 Clamped inextensible Timoshenko beam with a transverse applied load $b = 500$ and $k_t = 7000$.

shown for the two previous cases respectively. The elongation reaches its maximum (minimum) value where the beam lies parallel to the load with the same (opposite) direction. Instead, it is close to 1 where the beam lies orthogonal to the load. While in the Euler case the change in k_e causes a minimal change in the deformed shape, in the Timoshenko case there is a much more relevant influence of k_e on the configuration. In particular, decreasing k_e allows a much larger maximum value of the local geometrical curvature $\gamma := \theta'/\alpha$ of the beam. This maximum is attained when $\theta(s) = -\pi/2$; we will define s_0 the point at which this occurs. It has to be noted, however, that the Chebyshev curvature $\theta'(s_0)$ takes similar values in the two cases. For instance, in the rightmost simulation of Fig. 2.24 we have $\theta'(s_0) \approx -16.4$ and $\gamma(s_0) \approx -17.9$, while in the rightmost simulation of Fig. 2.26 we have $\theta'(s_0) \approx -16.3$ and $\gamma(s_0) \approx -111.9$ (we point out that $s_0 \approx 0.303$ in the right panel of Fig. 2.24 and $s_0 \approx 0.288$ in the right panel of Fig. 2.26).

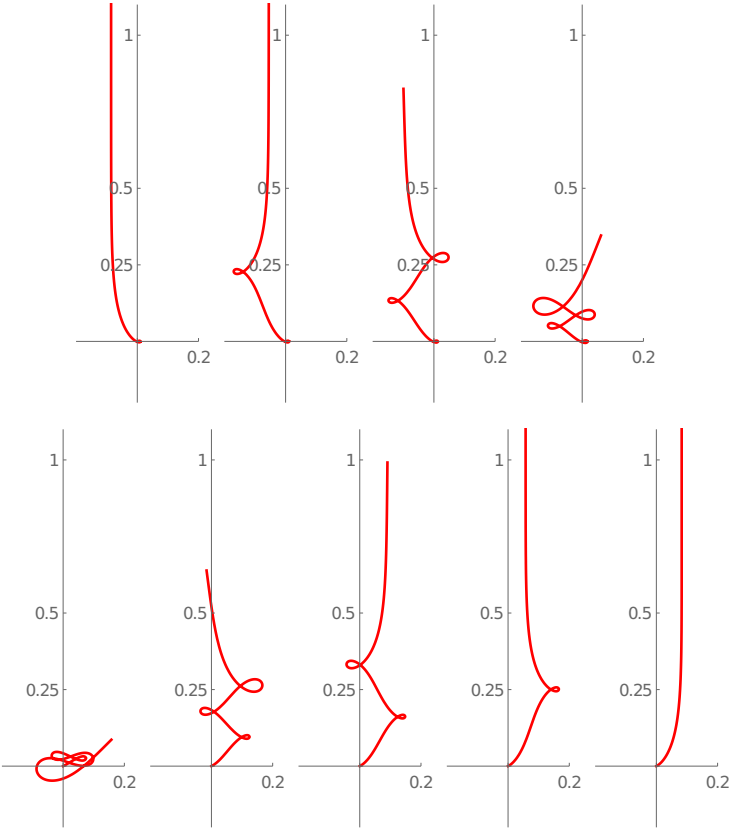

Fig. 2.15 Clamped extensible Timoshenko beam with a transverse applied load $b = 500$ and $k_t = 7000$ and $k_e = 5000$.

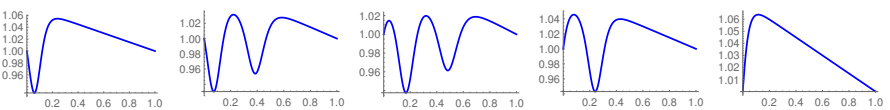

Fig. 2.16 Plot of α for the five configurations in Fig. 2.9.

2.4 Conclusions

In this paper we numerically studied clamped-free Euler and Timoshenko beams in large deformation under distributed load. Extensibility has been taken into account and results on the static behavior of the beam under different values of the load and of the axial stiffness has been shown. The main interest of the results consists in the multiplicity of solutions that arise as the load increases, not as a bifurcation of ex-

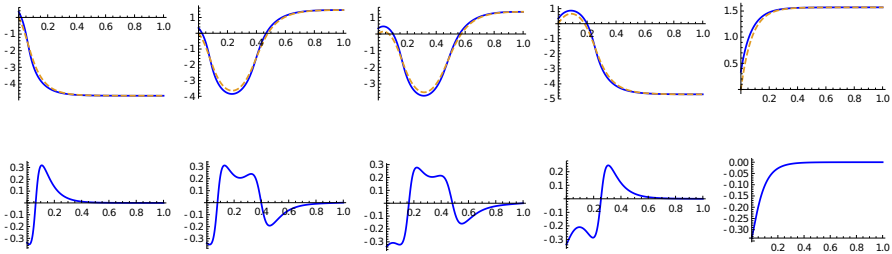

Fig. 2.17 Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig. 2.10. Below: the corresponding plot of $\phi - \theta$.

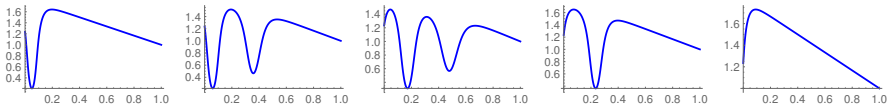

Fig. 2.18 Plot of α for the five configurations in Fig. 2.11.

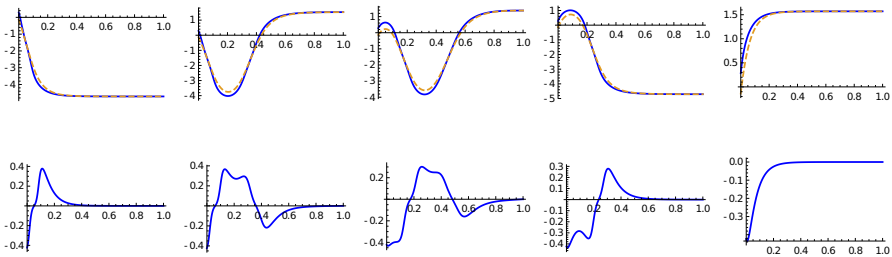

Fig. 2.19 Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig. 2.11. Below: the corresponding plot of $\phi - \theta$.

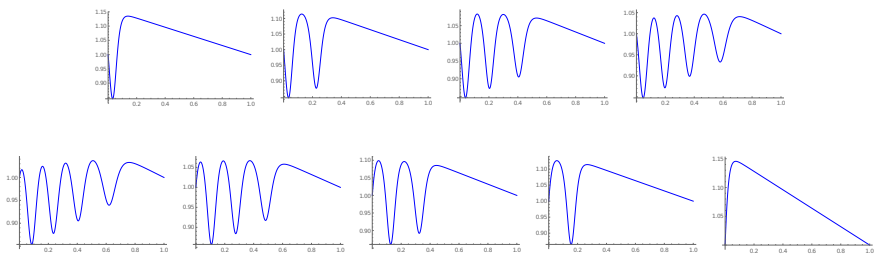

Fig. 2.20 Plot of α for the configurations in Fig. 2.13.

isting branches of solutions but as new branches that arise when the load overcomes a series of progressively larger threshold-values. Future investigations are required to establish whether these multiple solutions can be stable. In this regard, an analysis of the small oscillations of the beam around candidate stable equilibria would be useful to assess numerically the question. Moreover, addressing theoretically the

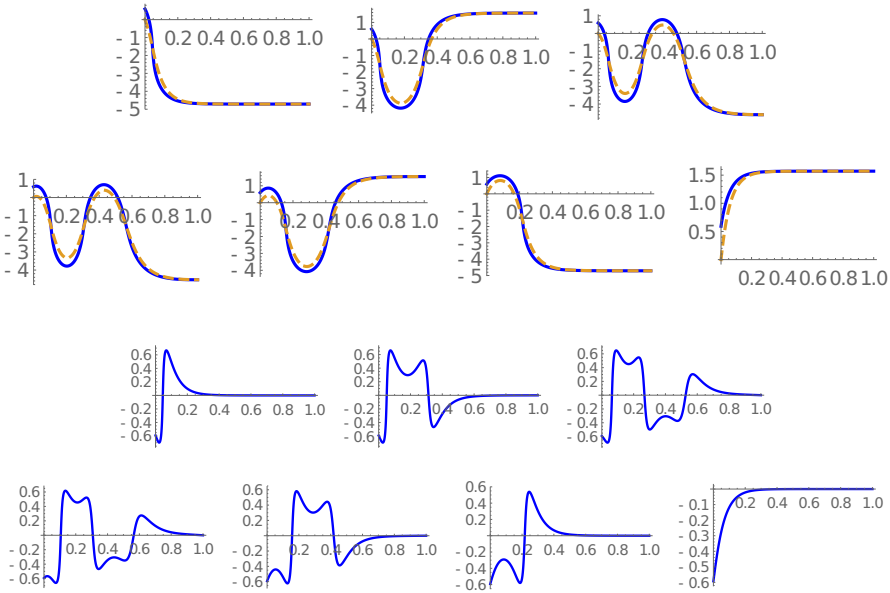

Fig. 2.21 Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig. 2.14. Below: the corresponding plot of $\phi - \theta$.

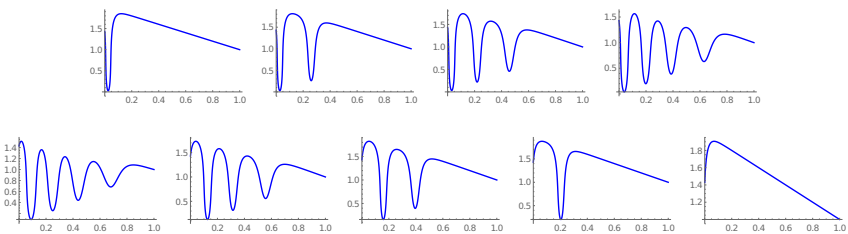

Fig. 2.22 Plot of α for the configurations in Fig. 2.15.

dynamical behavior of the nonlinear version of the Timoshenko beam model proposed here is a challenging task. The results developed in Berezovski et al (2018); Luongo and D'Annibale (2013); Piccardo et al (2015b); Taig et al (2015); Piccardo et al (2015a); Chróścielewski et al (2019) may prove useful in this direction.

Apart from the models introduced here, an independent and novel approach to understand the large deformation of beams to be investigated in the future is to regard a beam as the boundary curve of a two-dimensional manifold in a three-dimensional space. In doing so, not only is the curve endowed with its own energy similar to that in the context of lower-dimensional energetics (Javili et al, 2013b) but also in a geometrically nonlinear framework (Javili et al, 2014) and in accordance with higher gradient elasticity accounting for boundary energetics elaborated in Javili et al (2013a). The advantage of this approach, particularly from a computational

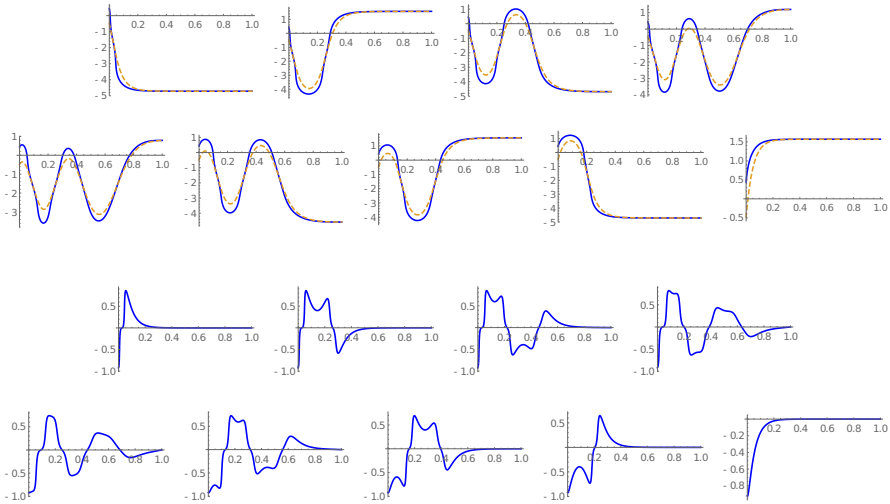

Fig. 2.23 Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig. 2.15. Below: the corresponding plot of $\phi - \theta$.

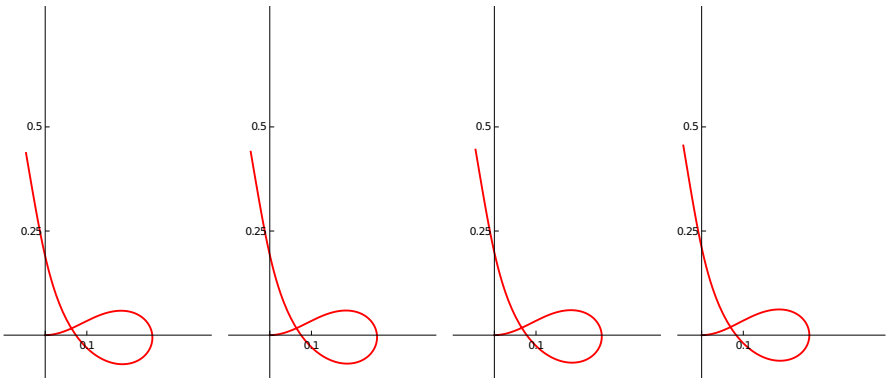

Fig. 2.24 Parametric study on a clamped extensible Euler beam with a transversal applied load $b = 120$ with $k_e = 2500$, $k_e = 2000$, $k_e = 1500$, $k_e = 1000$.

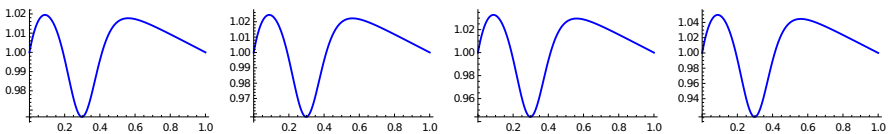

Fig. 2.25 The local elongation $\alpha(s)$ relative to the equilibrium shapes shown in Fig. 2.24.

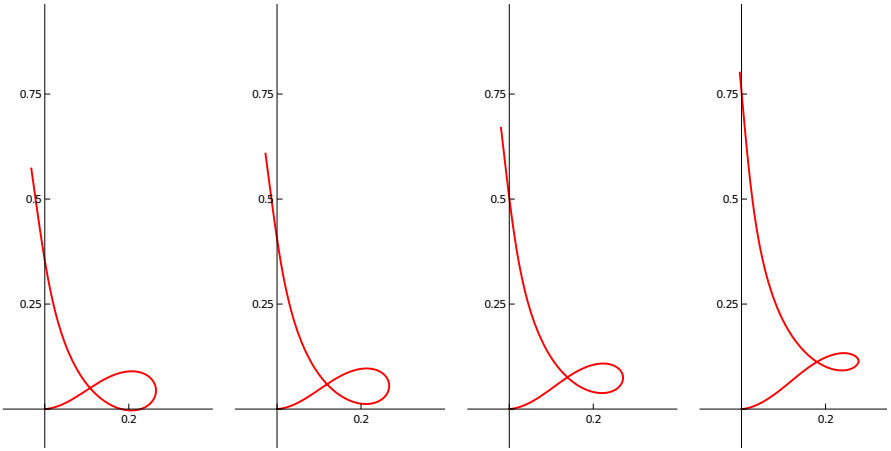

Fig. 2.26 Parametric study on a clamped extensible Timoshenko beam with a transversal applied load $b = 120$ with $k_e = 2500$, $k_e = 2000$, $k_e = 1500$, $k_e = 1000$ ($k_t = 18000$).

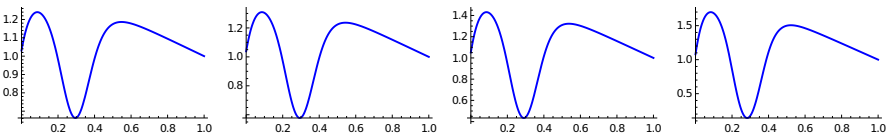

Fig. 2.27 The local elongation $\alpha(s)$ relative to the equilibrium shapes shown in Fig. 2.26.

viewpoint, is that the bulk material acts to regularize the behavior of the beam especially important to analyze the instabilities associated with thin beams similar to the instabilities of thin films on an elastic foundation (Javili et al, 2015). Obviously, in the limiting case of the vanishing bulk one would recover exactly the beam theory.

Appendix

We show here the Euler–Lagrange boundary value problem associated with the functional (2.5). We recall that the solutions $\theta(s)$ of this BVP are the scalar fields of angles formed by the tangent to the deformed configuration and a reference axis at the equilibrium for a clamped-free, extensible Timoshenko beam, in large deformation regime, under distributed load.

$$\theta'' = \left[k_b\left(bk_e(1-s)\sin(\theta) - (b(1-s))^2(\cos(2\theta)) + k_e k_t\right)\right]^{-1} \\ \left[bk_b(1-s)\theta^2\cos(\theta)(4b(1-s)\sin(\theta)+k_e) - 2bk_b\theta(2b(1-s)\cos(2\theta)+ \right. \\ \left. -k_e\sin(\theta)) + \cos(\theta)\left(2b^2k_b\sin(\theta) - bk_t(1-s)(b(1-s)\sin(\theta)+k_e)\right)\right] \tag{2.12}$$

with boundary conditions:

$$\begin{aligned} &\left(\theta - \frac{b^2(1-s)^2\sin(\theta)\cos(\theta)}{k_e k_e} + \frac{b(1-s)\cos(\theta)}{k_e}\right)\Big|_{s=0} = 0 \\ &\left(\theta - \frac{b^2(1-s)^2\sin(\theta)\cos(\theta)}{k_e k_e} + \frac{b(1-s)\cos(\theta)}{k_e}\right)'\Big|_{s=1} = 0 \end{aligned} \tag{2.13}$$

Remark 2.2. The boundary conditions in the previous problem have in general more than one solution. In the numerical simulations, we always considered the value $\theta(0)$ which was smaller in absolute value.

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