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ON THE CUSA-HUYGENS INEQUALITY

YOGESH J. BAGUL, CHRISTOPHE CHESNEAU, AND MARKO KOSTIĆ

ABSTRACT. Sharp bounds of various kinds for the famous unnormalized sinc function $\sin x/x$ are useful in mathematics, physics and engineering. In this paper, we reconsider the Cusa-Huygens inequality by solving the following problem: given real numbers $a, b, c \in \mathbb{R}$ and $T \in (0, \pi/2]$, we find the necessary and sufficient conditions such that the inequalities

$$\frac{\sin x}{x} > a + b \cos^c x, \quad x \in (0, T)$$

and

$$\frac{\sin x}{x} < a + b \cos^c x, \quad x \in (0, T)$$

hold true. We use the elementary methods, only, improving several known results in the existing literature.

1. INTRODUCTION

The following inequality is the main inspiration of this paper:

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad x \in (0, \pi/2). \quad (1.1)$$

In the existing literature, it is known as the Cusa-Huygens inequality [13]; for more details, we refer the reader to [6, 8, 9, 11, 13, 14]. This inequality has been extended and sharpened in many different ways [3, 10, 12, 15–19]. For example, in [9, 16], it is obtained that

$$\left(\frac{2 + \cos x}{3}\right)^\alpha < \frac{\sin x}{x} < \left(\frac{2 + \cos x}{3}\right)^\zeta, \quad x \in (0, \pi/2), \quad (1.2)$$

where $\alpha = \ln(\pi/2)/\ln(3/2) \approx 1.11374$ and $\zeta = 1$ are the best possible constants. A very simple proof of (1.2) is offered in [2]. In [6], it is proved that,

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for every $x \in (0, \pi/2)$, we have

$$\frac{\alpha - 1 + \cos x}{\alpha} < \frac{\sin x}{x} < \frac{\beta - 1 + \cos x}{\beta}, \quad (1.3)$$

$$\frac{2 + \cos x}{3^{\alpha_1}} < \frac{\sin x}{x} < \frac{2 + \cos x}{3^{\beta_1}}, \quad (1.4)$$

$$\frac{2^{\alpha_2} + \cos x}{3} < \frac{\sin x}{x} < \frac{2^{\beta_2} + \cos x}{3}, \quad (1.5)$$

with the best positive constants $\alpha \approx 2.75194$, $\beta = 3$; $\alpha_1 \approx 1.04198$, $\beta_1 = 1$ and $\alpha_2 \approx 0.93345$, $\beta_2 = 1$.

The main aim of this paper is to consider the following problem. Let $a, b, c \in \mathbb{R}$ and $T \in (0, \pi/2]$ be given real numbers; find the necessary and sufficient conditions such that the inequalities

$$\frac{\sin x}{x} > a + b \cos^c x, \quad x \in (0, T) \quad (1.6)$$

and

$$\frac{\sin x}{x} < a + b \cos^c x, \quad x \in (0, T) \quad (1.7)$$

hold true. We completely solve this problem and thus provide generalizations to numerous known special results in the existing literature. We also consider the inequality

$$\frac{2 + \cos^a x}{3} < \frac{\sin x}{x} < \frac{2 + \cos^b x}{3}, \quad x \in (0, T)$$

and propose several auxiliary results of independent interest. For the sake of brevity and better exposition, we will not analyze the hyperbolic analogues of obtained results here.

2. PRELIMINARIES AND LEMMAS

We need to remind ourselves of the statement which is known in the existing literature as l'Hospital's rule of monotonicity; see, e.g., [1]:

Lemma 1. *Let $f(x)$ and $g(x)$ be two real valued functions which are continuous on $[a, b]$ and differentiable on (a, b) , where $-\infty < a < b < \infty$ and $g'(x) \neq 0$, for all $x \in (a, b)$. Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, \quad x \in (a, b)$$

and

$$B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}, \quad x \in (a, b).$$

Then, we have:

- (i) $A(\cdot)$ and $B(\cdot)$ are increasing on (a, b) if $f'(\cdot)/g'(\cdot)$ is increasing on (a, b) .
- (ii) $A(\cdot)$ and $B(\cdot)$ are decreasing on (a, b) if $f'(\cdot)/g'(\cdot)$ is decreasing on (a, b) .

The strictness of the monotonicity of $A(\cdot)$ and $B(\cdot)$ depends on the strictness of monotonicity of $f'(\cdot)/g'(\cdot)$.

We proceed with some original lemmas and observations we will use later on.

Lemma 2. ([4]) *The function*

$$F(x) := \frac{\sin x - x \cos x}{x^2 \sin x}, \quad x \in (0, \pi/2)$$

is positive and strictly increasing on $(0, \pi/2)$.

Remark 1. (i) The proof of [4, Lemma 2.2] contains a small mistake since the function $x \mapsto -x \tan x + 3$, $x \in (0, \pi/2)$ has a unique zero $\zeta \in (0, \pi/2)$ so that an application of l'Hospital's rule of monotonicity shows that the function $F(\cdot)$ is strictly increasing on $(0, \zeta)$, only. But, the result is actually true because in the former step of proof we have $H'_1(x)/H'_2(x) = \sin x/(2 \sin x + x \cos x) = 1/(2 + x \cot x)$, $x \in (0, \pi/2)$, which is strictly increasing on $(0, \pi/2)$.

- (ii) It is clear that, due to Lemma 2, we have that for each number $\sigma \in (-\infty, 2]$ the function

$$F_\sigma(x) = \frac{\sin x - x \cos x}{x^\sigma \sin x}, \quad x \in (0, \pi/2)$$

is positive and strictly increasing on $(0, \pi/2)$. Consider now the case that $\sigma > 2$. Then, for each $x \in (0, \pi/2)$, we have

$$F'_\sigma(x) = (x^\sigma \sin x)^{-2} x^{\sigma-1} \left[x^2 - x \sin x \cos x - \sigma(\sin^2 x - x \sin x \cos x) \right].$$

So, if

$$\sigma \geq \sigma_0 := \sup_{x \in (0, \pi/2)} \frac{x^2 - x \sin x \cos x}{\sin^2 x - x \sin x \cos x} := \sup_{x \in (0, \pi/2)} Q(x) = \frac{\pi^2}{4},$$

then the function $F_\sigma(\cdot)$ is strictly decreasing on $(0, \pi/2)$. In order to prove that $\sigma_0 = \pi^2/4$, we can use the facts that $\lim_{x \rightarrow 0} Q(x) = 2$, established with the help of l'Hospital's rule, and $Q(\cdot)$ is strictly increasing on $(0, \pi/2)$, which follows by applying l'Hospital's rule of monotonicity twice. Strictly speaking, by applying the usual l'Hospital's rule twice,

we get

$$\begin{aligned}\lim_{x \rightarrow 0} Q(x) &= \lim_{x \rightarrow 0} \frac{2x - \sin x \cos x - x \cos 2x}{\sin x \cos x - x \cos 2x} = \lim_{x \rightarrow 0} \frac{1 - \cos 2x + x \sin 2x}{x \sin 2x} \\ &= \lim_{x \rightarrow 0} \left[1 + \frac{1 - \cos 2x}{x \sin 2x} \right] = \lim_{x \rightarrow 0} \left[1 + \frac{\tan x}{x} \right].\end{aligned}$$

For the applications of l'Hospital's rule of monotonicity, we only need to observe yet that

$$\begin{aligned}\sin x \cos x - x \cos 2x &= \cos 2x[(\tan(2x)/2) - x] \\ &= \cos 2x[\tan x(1 - \tan^2 x)^{-1} - x] > \cos 2x[\tan x - x] > 0, \quad x \in (0, \pi/2)\end{aligned}$$

and the function $\tan(\cdot)/\cdot$ is strictly increasing on $(0, \pi/2)$. Finally, the above also implies that for each number $\sigma \in (2, \sigma_0)$ there exists a unique number $x_\sigma \in (0, \pi/2)$ such that the function $F_\sigma(\cdot)$ is strictly decreasing on $(0, x_\sigma)$ and $F_\sigma(\cdot)$ is strictly increasing on $(x_\sigma, \pi/2)$.

Lemma 3. *For $x \in (0, \pi/2)$, it is true that*

$$\frac{2x^2}{\pi} < \frac{x}{\sin x} - \cos x < \frac{2x^2}{3},$$

which is equivalent to

$$\frac{3}{2x^2 + 3 \cos x} < \frac{\sin x}{x} < \frac{\pi}{2x^2 + \pi \cos x}.$$

Proof. Let us consider the function

$$f(x) = \frac{x^2 \sin x}{x - \sin x \cos x}, \quad x \in (0, \pi/2).$$

We have

$$\begin{aligned}(x - \sin x \cos x)^2 f'(x) &= (x - \sin x \cos x)(x^2 \cos x + 2x \sin x) - 2 \sin^2 x(x^2 \sin x) \\ &= x^3 \cos x + 2x^2 \sin x - x^2 \sin x \cos^2 x - 2x \sin^2 x \cos x - 2x^2 \sin^3 x \\ &= x^3 \cos x + x^2 \sin x \cos x - 2x \sin^2 x \cos x \\ &= x \Delta(x) \cos x,\end{aligned}$$

where $\Delta(x) = x^2 + x \sin x \cos x - 2 \sin^2 x$ ($x \in (0, \pi/2)$). Now, let us notice that

$$\begin{aligned}\Delta'(x) &= 2x + x \cos^2 x - x \sin^2 x + \sin x \cos x - 4 \sin x \cos x \\ &= x + 2x \cos^2 x - 3 \sin x \cos x, \quad x \in (0, \pi/2)\end{aligned}$$

and

$$\begin{aligned}\Delta''(x) &= 1 + 2 \cos^2 x - 4x \sin x \cos x - 3 \cos^2 x + 3 \sin^2 x \\ &= 4 \sin^2 x - 4x \sin x \cos x = 4 \sin x (\sin x - x \cos x), \quad x \in (0, \pi/2).\end{aligned}$$

Since $\sin x - x \cos x > 0$ for $x \in (0, \pi/2)$, we get $\Delta''(x) > 0$, implying that $\Delta'(\cdot)$ is strictly increasing on $(0, \pi/2)$ and, a fortiori, $\Delta(x) > \Delta(0+) = 0$, implying that $f'(x) > 0$. Hence, $f(\cdot)$ is strictly increasing on $(0, \pi/2)$ and $f(0+) < f(x) < f(\pi/2)$, with $f(0+) = 3/2$ by l'Hospital's rule and $f(\pi/2) = \pi/2$. This ends the proof of Lemma 3. \square

In the remainder of paper, it will be always assumed that $T \in (0, \pi/2]$. Observe also that the equation $\sin x/x = 2/3$ has a unique zero ν belonging to the interval $(0, \pi/2)$; furthermore, we have $\nu \approx 1.49578$, as well as $\sin x/x > 2/3$ and $x \in (0, \pi/2)$ iff $x \in (0, \nu)$.

Lemma 4. *For each number $b \in (1/3, F(T))$, there exists a unique number $\sigma_{b,T} \in (0, T)$ such that*

$$\frac{x \cos x - \sin x}{x^2 \sin x} + b = 0$$

holds with $x = \sigma_{b,T}$.

Proof. Follows immediately from Lemma 2 as well as the limit equalities

$$\lim_{x \rightarrow 0+} \frac{x \cos x - \sin x}{x^2 \sin x} = -\frac{1}{3} \quad \text{and} \quad \lim_{x \rightarrow T-} \frac{x \cos x - \sin x}{x^2 \sin x} = -F(T),$$

which can be easily verified. \square

The function

$$W(x) := \cos x \frac{x^2 + x \sin x \cos x - 2 \sin^2 x}{(\sin x - x \cos x)x \sin^2 x}, \quad x \in (0, \pi/2)$$

has an important role in our study, as well. This function is positive (see the proof of Lemma 3, showing that $x^2 + x \sin x \cos x - 2 \sin^2 x > 0$, $x \in (0, \pi/2)$). Moreover, we have $\lim_{x \rightarrow \pi/2-} W(x) = 0$ and $\lim_{x \rightarrow 0+} W(x) = 2/15$ because $(\sin x - x \cos x)x \sin^2 x \sim x^6/3$, $x \rightarrow 0+$ and $x^2 + x \sin x \cos x - 2 \sin^2 x \sim 2x^6/45$, $x \rightarrow 0+$.

Lemma 5. *The function $W(\cdot)$ is strictly decreasing on $(0, \pi/2)$.*

Proof. Set

$$W_0(x) := \cos x \frac{x^2 + x \sin x \cos x - 2 \sin^2 x}{x^3 \sin^3 x}, \quad x \in (0, \pi/2).$$

Then, we have $W_0(x) > 0$ for all $x \in (0, \pi/2)$. Therefore, due to Lemma 2 and the decomposition

$$W(x) = \frac{x^2 \sin x}{\sin x - x \cos x} W_0(x), \quad x \in (0, \pi/2),$$

it suffices to show that the function $W_0(\cdot)$ is strictly decreasing on $(0, \pi/2)$. Towards this end, we note that the calculus established with the use of `symlab.com` package shows that, for every $x \in (0, \pi/2)$, we have:

$$\begin{aligned} W'(x) = & -\frac{2x^2 + x \sin 2x - 4 \sin^2 x}{2x^3 \sin^2 x} \\ & + \frac{\cot x \times [-6x^3 \sin 2x - 4x^2 \sin^2 x + 4x^2 \sin^2 \cos 2x + 24 \sin^4 x - 3x^2 \sin^2 x]}{4x^4 \sin^4 x}. \end{aligned}$$

After multiplying the first addend with $2x \sin^2 x$ and grouping the terms, we need to show that, for every $x \in (0, \pi/2)$, we have

$$\begin{aligned} & 4x^2 \cos x \sin x \cos 2x + 24 \cos x \sin^3 x + 8x \sin^4 x - 12x^3 \cos^2 x \\ & < 4x^2 \cos x \sin x + 12x^2 \cos^3 x \sin x + 4x^3 \sin^2 x + 2x^2 \sin 2x \sin^2 x. \end{aligned}$$

Since $1 - \cos 2x = 2 \sin^2 x$ and $\sin^2 x + \cos^2 x = 1$, after collecting similar terms and dividing with 4, the above is equivalent with

$$2x^3 \cos^2 x + x^3 + 3x^2 \sin x \cos x > 6 \cos x \sin^3 x + 2x \sin^4 x, \quad x \in (0, \pi/2). \quad (2.1)$$

Suppose first that $x \in (0, \nu)$. Then, (1.1) implies $\cos x > 3(\sin x/x) - 2 > 0$ so that

$$\begin{aligned} & 2x^3 \cos^2 x + x^3 + 3x^2 \sin x \cos x \\ & > 2x^3 \left(3 \frac{\sin x}{x} - 2\right)^2 + x^3 + 3x^2 \sin x \left(3 \frac{\sin x}{x} - 2\right) \\ & = 27x \sin^2 x + 9x^3 - 18x^2 \sin x. \end{aligned}$$

Therefore, it suffices to show that

$$27x \sin^2 x + 9x^3 > 6 \cos x \sin^3 x + 2x \sin^4 x + 18x^2 \sin x.$$

This follows from the inequalities $9x \sin^2 x + 9x^3 \geq 2\sqrt{9^2 x^4 \sin^2 x} = 18x^2 \sin x$, $18x \sin^2 x < 18x^3$ and $6 \cos x \sin^3 x + 2x \sin^4 x < 6x^3 + 2x^5 < 8x^3$. If $x \in [\nu, \pi/2)$, then the proof is much simpler and follows from the fact that, for such values of variable x , we have $x^3 > 2x \geq 2x \sin^4 x$ and $x^2 > 2 \geq 2 \sin^2 x$, implying that $3x^2 \sin x \cos x > 6 \cos x \sin^3 x$ (see (2.1)). \square

Hence, $\inf\{W(x) : x \in (0, \pi/2)\} = 0$ and $\min\{W(x) : x \in (0, \pi/2)\}$ does not exist as well as $\sup\{W(x) : x \in (0, \pi/2)\} = 2/15$ and $\max\{W(x) : x \in (0, \pi/2)\}$ does not exist.

3. MAIN RESULTS

Now we are in a position to prove the main results of the paper. We start by stating the following

Theorem 1. *Let $a, b \in \mathbb{R}$. Then, the inequality*

$$\frac{\sin x}{x} > a + b \cos x, \quad x \in (0, T) \quad (3.1)$$

holds iff

1. $b \leq 1/3$ and $a \leq (\sin T/T) - b \cos T$, or
2. $b \geq F(T)$ and $a \leq 1 - b$, or
3. $b \in (1/3, F(T))$ and $a \leq \min(1 - b, (\sin T/T) - b \cos T)$.

Proof. It is clear that (3.1) holds iff $a < \inf\{(\sin x/x) - b \cos x : x \in (0, T)\}$ or $a \leq \inf\{(\sin x/x) - b \cos x : x \in (0, T)\}$ if there is no $x \in (0, T)$ such that $(\sin x/x) - b \cos x = a$. Consider the function

$$M(x) := \frac{\sin x}{x} - b \cos x, \quad x \in (0, T).$$

Then, a simple computation shows that

$$M'(x) = \sin x \left[\frac{x \cos x - \sin x}{x^2 \sin x} + b \right], \quad x \in (0, T).$$

Note that $\{(x \cos x - \sin x)/(x^2 \sin x) : x \in (0, T)\} = (-F(T), -1/3)$ (see Lemma 2 and Lemma 4) and $F(0+) = 1 - b$. Hence, if $b \leq 1/3$, then $M'(x) \leq 0$ for all $x \in (0, T)$ and the zeroes of function $M'(\cdot)$ do not form an interval in $(0, T)$ so that the function $M(\cdot)$ is strictly decreasing on $(0, T)$ and therefore (3.1) holds iff $a \leq (\sin T/T) - b \cos T$. Similarly, if $b \geq F(T)$, then the function $M(\cdot)$ is strictly increasing on $(0, T)$ and (3.1) holds iff $a \leq 1 - b$. Finally, if $b \in (1/3, F(T))$, then the function $M(\cdot)$ strictly increases on $(0, \sigma_{b,T})$ and the function $M(\cdot)$ strictly decreases on $(\sigma_{b,T}, T)$ (see Lemma 4) which implies that $\inf\{(\sin x/x) - b \cos x : x \in (0, T)\} = \min(1 - b, (\sin T/T) - b \cos T)$ and (3.1) holds iff $a \leq \min(1 - b, (\sin T/T) - b \cos T)$. \square

Remark 2. Theorem 1 improves the well-known Baricz's inequality

$$\frac{1 + \cos x}{2} \leq \frac{\sin x}{x}, \quad x \in (0, \pi/2);$$

see [5, p. 111].

It is clear that Theorem 1 substantially improves the left hand sides of equations (1.3)-(1.5) with $T = \pi/2$. Strictly speaking, Theorem 1 implies that $\beta = \pi/\pi - 2$ is the smallest positive constant strictly greater than 1 such that the left hand side of (1.3) holds, which can be simply inspected.

We can similarly prove the following extension of the Cusa-Huygens inequality, which taken together with Theorem 1 provides a proper generalization of [6, Theorem 1.6, Theorem 3.9]:

Theorem 2. *Let $a, b \in \mathbb{R}$. Then, the inequality*

$$\frac{\sin x}{x} < a + b \cos x, \quad x \in (0, T)$$

holds iff

1. $b \leq 1/3$ and $a \geq 1 - b$, or
2. $b \geq F(T)$ and $a \geq (\sin T/T) - b \cos T$, or
3. $b \in (1/3, F(T))$ and $a > (\sin \sigma_{b,T}/\sigma_{b,T}) - b \cos \sigma_{b,T}$.

Remark 3. Taken together, Theorem 1 and Theorem 2 improve the well-known results of J. Sándor and R. Oláh-Gal [18, Theorem 1, Theorem 2] and the well-known Oppenheim's double inequality

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (4/\pi) \cos x}{\pi}$$

(see, e.g., [7]).

Suppose now that $a, b, c \in \mathbb{R}$ and $c > 1$. Consider the function

$$M_1(x) := \frac{\sin x}{x} - b \cos^c x, \quad x \in (0, T),$$

whose first derivative is given by

$$M_1'(x) = \cos^{c-1} x \sin x \left[\frac{x \cos x - \sin x}{x^2 \sin x \cos^{c-1} x} + bc \right], \quad x \in (0, T).$$

Using Lemma 2 and the assumption $c > 1$, it follows that the function $x \mapsto (x \cos x - \sin x)/(x^2 \sin x \cos^{c-1} x)$, $x \in (0, T)$ is strictly decreasing; moreover, the range of this function is equal to $(-\infty, -1/3)$, if $T = \pi/2$, resp. $(-F(T) \cos^{1-c} T, -1/3)$, if $T < \pi/2$. Therefore, arguing as above, we may conclude that the following holds:

Theorem 3. *Suppose that $a, b, c \in \mathbb{R}$ and $c > 1$. Then, we have the following:*

- (i) *The inequality (1.6) holds iff:*
 1. $bc \leq 1/3$ and $a \leq 2/\pi$, or $bc > 1/3$ and $a \leq \min(1 - b, 2/\pi)$, provided that $T = \pi/2$.
 2. $bc \leq 1/3$ and $a \leq (\sin T/T) - b \cos^c T$, or $bc \geq F(T) \cos^{1-c} T$ and $a \leq 1 - b$, or $bc \in (1/3, F(T))$ and $a \leq \min(1 - b, (\sin T/T) - b \cos^c T)$, provided that $T < \pi/2$.
- (ii) *The inequality (1.7) holds iff:*

1. $bc \leq 1/3$ and $a \geq 1 - b$, or $bc > 1/3$ and $a > (\sin \sigma_{b,c}/\sigma_{b,c}) - b \cos^c \sigma_{b,c}$, where $\sigma_{b,c}$ denotes the unique solution of equation

$$\frac{x \cos x - \sin x}{x^2 \sin x \cos^{c-1} x} + bc = 0$$

on $(0, \pi/2)$, provided that $T = \pi/2$.

2. $bc \leq 1/3$ and $a \geq 1 - b$, or $bc \geq F(T) \cos^{1-c} T$ and $a \geq (\sin T/T) - b \cos^c T$, or $bc \in (1/3, F(T))$ and $a > (\sin \zeta_{b,c,T}/\zeta_{b,c,T}) - b \cos^c \zeta_{b,c,T}$, where $\zeta_{b,c,T}$ denotes the unique solution of equation

$$\frac{x \cos x - \sin x}{x^2 \sin x \cos^{c-1} x} + bc = 0, \quad x \in (0, T),$$

provided that $T < \pi/2$.

Therefore, it remains to consider the case in which $a, b, c \in \mathbb{R}$ and $c < 1$. In this case, we have

$$\begin{aligned} \left(\frac{x \cos x - \sin x}{x^2 \sin x \cos^{c-1} x} \right)' &= \cos^{-c} x \frac{x \cos x - \sin x}{x^2} \\ &\times \left[\cos x \frac{x^5 \sin^2 x - x^2 (\sin x - x \cos x) (2x \sin x + x^2 \cos x)}{(\sin x - x \cos x) x^4 \sin^2 x} + c - 1 \right] \\ &= \cos^{-c} x \frac{x \cos x - \sin x}{x^2} \left[\cos x \frac{x^2 + x \sin x \cos x - 2 \sin^2 x}{(\sin x - x \cos x) x \sin^2 x} + c - 1 \right] \\ &= \cos^{-c} x \frac{x \cos x - \sin x}{x^2} [W(x) + c - 1], \quad x \in (0, T). \end{aligned}$$

Based on this computation, Lemma 5 and the analysis preceding it, we can clarify the following extension of Lemma 2:

Proposition 1. *Let $d \in \mathbb{R}$. Then, the function*

$$Q_d(x) := \frac{x \cos x - \sin x}{x^2 \sin x \cos^{d-1} x}, \quad x \in (0, T)$$

has the following properties:

- (i) $Q_d(\cdot)$ is strictly increasing on $(0, T)$ iff $d \geq 1 - W(T)$.
- (ii) $Q_d(\cdot)$ is strictly decreasing on $(0, T)$ iff $d \leq 13/15$.
- (iii) If $d \in (13/15, 1 - W(T))$, then there exists a unique number $\theta_{d,T} \in (0, T)$ such that the function $Q_d(\cdot)$ is strictly increasing on $(0, \theta_{d,T})$ and $Q_d(\cdot)$ is strictly decreasing on $(\theta_{d,T}, T)$.

We continue with the analysis of case $a, b, c \in \mathbb{R}$ and $c < 1$. In actual fact, the following theorem holds true:

Theorem 4. (i) *Let $c \leq 13/15$. Then, (1.6), resp. (1.7), holds iff:*

1. $bc \geq 1/3$ and $a \leq 1 - b$, resp. $bc \geq 1/3$ and $a \geq 2/\pi$ if $c > 0$ and $T = \pi/2$ [$bc \geq 1/3$ and $a \geq M_1(T)$ if $T < \pi/2$], or
2. $bc \leq F(T) \cos^{1-c} T$ and $a \leq (\sin T/T) - b \cos^c T$ if $T < \pi/2$ or $T = \pi/2$ and $c \geq 0$, or $a \in \mathbb{R}$, $b < 0$, $c < 0$, $T = \pi/2$ or $bc \leq 0$, $a \leq 2/\pi$, $b = 0$, $c < 0$, $T = \pi/2$, resp. $bc \leq F(T) \cos^{1-c} T$ and $a \geq 1 - b$, or
3. $bc \in (F(T) \cos^{1-c} T, 1/3)$. Then, there exists a unique number $\sigma_{b;c;T} \in (0, T)$ such that the equation

$$\frac{\sin x - x \cos x}{x^2 \sin x} \cos^{1-c} x = bc$$

holds with $x = \sigma_{b;c;T}$; in this case, (1.6), resp. (1.7), holds iff $a < (\sin \sigma_{b;c;T}/\sigma_{b;c;T}) - b \cos^c \sigma_{b;c;T}$, resp. $c > 0$ and $a \geq \max(1 - b, M_1(T-))$.

- (ii) Let $c \geq 1 - W(T)$ and $T < \pi/2$. Then, (1.6), resp. (1.7), holds iff:
 1. $bc \leq F(T) \cos^{1-c} T$ and $a \leq \sin T/T$, resp. $a \geq 1 - b$, or
 2. $bc \geq 1/3$ and $a \leq 1 - b$, resp. $bc \geq 1/3$ and $a \geq M_1(T)$, or
 3. $bc \in (F(T) \cos^{1-c} T, 1/3)$ and $a < M_1(\eta_{b;c;T})$, where $\eta_{b;c;T}$ denotes the unique solution of equation $Q_c(x) + bc = 0$ on $(0, T)$, resp. $bc \in (F(T) \cos^{1-c} T, 1/3)$ and $a \geq \max(1 - b, M_1(T))$.
- (iii) Let $c \in (13/15, 1 - W(T))$. Then, (1.6), resp. (1.7), holds iff:
 1. $bc \leq -Q_c(\theta_{c;T})$ and $a \leq M_1(T-)$, resp. $bc \leq -Q_c(\theta_{c;T})$ and $a \geq 1 - b$, or
 2. $bc \geq -\min(-1/3, Q_c(T))$ and $a \leq 1 - b$, resp. $bc \geq -\min(-1/3, Q_c(T))$ and $a \geq M_1(T-)$, or
 3. $-bc \in (\min(-1/3, Q_c(T)), Q_c(\theta_{c;T}))$ and the equation $Q_c(x) + bc = 0$ has exactly one zero $\zeta_{b,c;T}$ in $(0, T)$. Then, (1.6), resp. (1.7), holds iff $a < M_1(\zeta_{b,c;T})$, resp. $a \geq \max(1 - b, M_1(T-))$.
 4. $-bc \in (\min(-1/3, Q_c(T)), Q_c(\theta_{c;T}))$ and the equation $Q_c(x) + bc = 0$ has exactly two zeroes $\zeta_{b,c;T}^1$ and $\zeta_{b,c;T}^2$ in $(0, T)$ such that, say, $\zeta_{b,c;T}^1 < \zeta_{b,c;T}^2$. Then, (1.6), resp. (1.7), holds iff $a \leq M_1(T-) < M_1(\zeta_{b,c;T}^1)$ or $a < M_1(\zeta_{b,c;T}^1) \leq M_1(T-)$, resp. $a \geq 1 - b > M_1(\zeta_{b,c;T}^2)$ or $a > M_1(\zeta_{b,c;T}^2) \geq 1 - b$.

Proof. The proof of theorem is very similar to those of Theorem 1 and Theorem 2, and we will only outline the main details for the statement (iii). If $c \in (13/15, 1 - W(t))$ and $bc \leq -Q_c(\theta_{c;T})$, we have that the function $M_1(\cdot)$ is strictly decreasing on $(0, T)$ since its first derivative is non-negative, which simply implies that the statement 1. holds true. Similarly, if we have $bc \geq -\min(-1/3, Q_c(T))$, then the function $M_1(\cdot)$ is strictly increasing on $(0, T)$ and the statement 2. follows as above. If $-bc \in (\min(-1/3, Q_c(T)), Q_c(\theta_{c;T}))$, the equation can have exactly one or exactly two zeroes belonging to the

interval $(0, T)$. In the first case, the function $M_1(\cdot)$ is strictly decreasing on $(0, \zeta_{b,c,T})$ and the function $M_1(\cdot)$ is strictly decreasing on $(\zeta_{b,c,T}, T)$, while in the second case, the function $M_1(\cdot)$ is strictly decreasing on the intervals $(0, \zeta_{b,c,T}^1)$ and $(\zeta_{b,c,T}^2, T)$ and the function $M_1(\cdot)$ is strictly increasing on $(\zeta_{b,c,T}^1, \zeta_{b,c,T}^2)$. \square

Remark 4. Theorem 4 improves the well-known inequality

$$\sqrt[3]{\cos x} < \frac{\sin x}{x}, \quad x \in (0, \pi/2)$$

due to by D. D. Adamović and D. S. Mitrinović (see, e.g., [13, p. 238]).

Theorem 4 is an abstract result and the direct use of l'Hospital's rule of monotonicity is sometimes a much better choice for examining the best possible constants for which certain concrete inequalities hold true. The theorem below presents the sharp bounds of the form $(2 + \cos^u x)/3$ for $\sin x/x$. In order to formulate this theorem, let us recall that $\nu \approx 1.49578$ denotes the unique solution of equation $\sin x/x = 2/3$ on $(0, \pi/2)$.

Theorem 5. *Let $\lambda \in (0, \alpha)$ with $\alpha \approx 1.49578$. Then, the best possible constants a and b such that*

$$\frac{2 + \cos^a x}{3} < \frac{\sin x}{x} < \frac{2 + \cos^b x}{3}, \quad x \in (0, \lambda)$$

are $\log(3 \sin \lambda/\lambda - 2)/\log(\cos \lambda)$ and 1, respectively.

Proof. Let us consider the function

$$f(x) = \frac{\log(3 \sin x/x - 2)}{\log(\cos x)} = \frac{f_1(x)}{f_2(x)}, \quad x \in (0, \lambda).$$

For such values of x , we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{3(\sin x - x \cos x)}{x^2 \sin x} \frac{x \cos x}{3 \sin x - 2x} = f_3(x)f_4(x),$$

where $f_3(x) = 3(\sin x - x \cos x)/(x^2 \sin x)$ and $f_4(x) = x \cos x/(3 \sin x - 2x)$. It follows by Lemma 2 that $f_3(\cdot)$ is positive and strictly increasing. Now, remark that

$$(3 \sin x - 2x)^2 f_4'(x) = -3x + 3 \sin x \cos x + 2x^2 \sin x,$$

which is positive by Lemma 3. Hence, $f_4(\cdot)$ is also positive and strictly increasing. Therefore, $f_1'(\cdot)/f_2'(\cdot)$ is strictly increasing and, by l'Hospital's rule of monotonicity (see Lemma 1), $f(\cdot)$ is strictly increasing on $(0, \lambda)$ with $f(0+) < f(x) < f(\lambda)$. We end the proof of Theorem 5 by noticing that $f(0+) = 1$ by l'Hospital's rule and $f(\lambda) = \log(3 \sin \lambda/\lambda - 2)/\log(\cos \lambda)$. \square

Figure 1 displays the functions involved in the inequalities of Theorem 5.

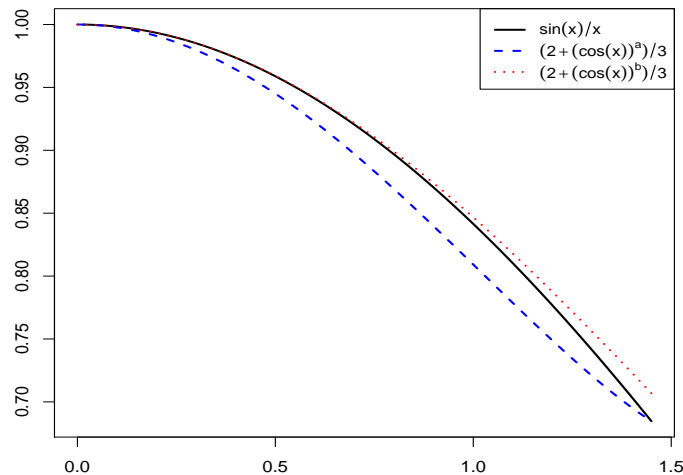


FIGURE 1. Illustration of the inequalities in Theorem 5.

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