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Force on a neutron quantized vortex pinned to proton fluxoids in the superfluid core of cold neutron stars

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ABSTRACT

The superfluid and superconducting core of a cold rotating neutron star (NS) is expected to be threaded by a tremendous number of neutron quantized vortices and proton fluxoids. Their interactions are unavoidable and may have important astrophysical implications. In this paper, the various contributions to the force acting on a single vortex to which fluxoids are pinned are clarified. The general expression of the force is derived by applying the variational multifluid formalism developed by Carter and collaborators. Pinning to fluxoids leads to an additional Magnus type force due to proton circulation around the vortex. Pinning in the core of an NS may thus have a dramatic impact on the vortex dynamics, and therefore on the magnetorotational evolution of the star.

Key words: stars: interiors – stars: neutron.

1 INTRODUCTION

Even before their actual discovery, neutron stars (NSs) were expected to be so dense that neutrons and protons in their interior may be in a superfluid state (see e.g. Chamel 2017, and references therein). This theoretical prediction was later confirmed by the very long relaxation times following the first detections of pulsar sudden spin-ups so-called frequency glitches (Haskell & Melatos 2015). Nucleon superfluidity in the core of NSs has recently found additional support from the direct monitoring of the rapid cooling of the young NS in Cassiopeia A (Page et al. 2011; Shternin et al. 2011). However, the interpretation of these observations remains controversial (Posselt & Pavlov 2018; Wijngaarden et al. 2019).

Because NSs are rotating, their interior is threaded by a huge number of neutron quantized vortices, each carrying a quantum $\kappa_{\rm n} = h/(2m_{\rm n}) \simeq 2 \times 10^{-3}$ cm^2 s^{-1} of circulation, where h is the Planck constant and $m_{\rm n}$ is the neutron rest mass. The mean surface density of vortices is proportional to the angular frequency Ω and is given by (Chamel 2017)

$$\mathcal{N}_{\rm n} = 4\, m_{\rm n}\, \Omega/h \simeq 6 \times 10^5/P_{10}\ {\rm cm}^{-2}, \tag{1}$$

where $P_{10} = P/10$ ms is the observed rotation period of the NS. Assuming that protons in NS cores form a type-II superconductor (Baym, Pethick & Pines 1969), the magnetic flux penetrates the stellar interior only via fluxoids, each carrying a quantum magnetic flux $\phi_0 = hc/(2e) \simeq 2 \times 10^{-7}$ G cm^2, where c is the speed of light and e denotes the proton electric charge. For typical NS magnetic fields, the number of fluxoids is considerably larger than that of vortices, their mean surface density being given by (Chamel 2017)

$$\mathcal{N}_{\rm p} = B/\phi_0 \simeq 5 \times 10^{18}\, B_{12}\ {\rm cm}^{-2}, \tag{2}$$

where $B_{12} = B/10^{12}$ G is the stellar internal magnetic field. Interactions between neutron vortices and proton fluxoids are therefore unavoidable, and are pivotal in the magnetorotational evolution of NSs. In particular, vortices may pin to fluxoids (Muslimov & Tsygan 1985; Sauls 1989; Srinivasan et al. 1990; Ruderman, Zhu & Chen 1998, see also Alpar 2017 for a recent review), and this may have important implications for various astrophysical phenomena, such as precession (Sedrakian, Wasserman & Cordes 1999; Link 2006; Glampedakis, Andersson & Jones 2008), r-mode instability (Haskell, Andersson & Passamonti 2009; Haskell, Glampedakis & Andersson 2014), and pulsar glitches (Sedrakian et al. 1995; Sidery & Alpar 2009; Glampedakis & Andersson 2009; Haskell, Pizzochero & Seveso 2013; Haskell & Melatos 2015; Gügercinoğlu 2017; Sourie et al. 2017; Haskell et al. 2018; Graber, Cumming & Andersson 2018). However, the detailed force acting on individual vortices to which fluxoids are pinned remains poorly understood. In particular, the contribution associated with the proton circulation induced by pinned fluxoids has been generally overlooked or treated phenomenologically (see e.g. Glampedakis & Andersson 2011).

Building on the recent study of Gusakov (2019), who determined the force acting on a single fluxoid and clarified the role of degenerate electrons, we derive in this paper the general expression for the force per unit length acting on a neutron vortex to which $N_{\rm p}$ proton fluxoids are pinned. To this end, we follow a general approach originally developed by Carter, Langlois & Prix (2002) in the relativistic framework, and later adapted to the Newtonian context by Carter & Chamel (2005a). The general expression of

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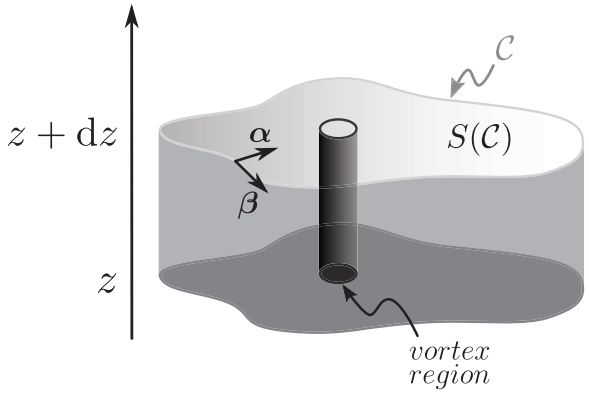

Figure 1. Schematic picture illustrating the fluid element contributing to the force per unit length acting on the vortex directed along the z-axis. The parallel sections $S(\mathcal{C})$ lie in the plane perpendicular to this axis. See the text for details.

the vortex velocity is calculated and the role of pinning on the vortex dynamics is discussed. The implications for pulsar glitches are studied in an accompanying paper (Sourie & Chamel 2020).

2 FORCE ACTING ON A SINGLE VORTEX PINNED TO FLUXOIDS

2.1 General definition

Let us consider a rigid and infinitely long straight neutron superfluid vortex to which $N_{\rm p}$ proton fluxoids are pinned. The medium in which the vortex is embedded is assumed to be asymptotically uniform, stationary and longitudinally invariant, along say the z-axis.

The force density $\boldsymbol{f}$ acting on a matter element is defined by the divergence of the momentum-flux tensor Π^{ij} (i, j denoting space coordinate indices),

$$f_i \equiv \nabla_j \Pi_i^j \,. \tag{3}$$

The force $\mathrm{d}\boldsymbol{F}$ exerted on a vortex segment of length $\mathrm{d}z$ by a matter element whose volume is delimited by a closed contour $\mathcal{C}$ encircling the vortex and the pinned fluxoids, as represented on Fig. 1, is thus given by

$$\begin{aligned}\mathrm{d}F_i &= -\iiint f_i \,\mathrm{d}V = -\iiint \nabla_j \Pi_i^j \,\mathrm{d}V \\ &= \iint_{S(\mathcal{C})} \Pi_i^j(z)\hat{z}_j \,\mathrm{d}S - \iint_{S(\mathcal{C})} \Pi_i^j(z+\mathrm{d}z)\hat{z}_j \,\mathrm{d}S \\ &\quad + \mathrm{d}z \oint_{\mathcal{C}} \Pi_i^j \alpha_j \,\mathrm{d}\ell,\end{aligned} \tag{4}$$

where we have made use of Stokes' theorem, and $\boldsymbol{\alpha}$ is a unit vector perpendicular to both the vortex line and the contour $\mathcal{C}$, and is oriented inside the contour. Longitudinal invariance along the vortex line implies that Π^{ij} is independent of z. The two surface integrals in the second line of equation (4) thus cancel each other. The force per unit length acting on the vortex and the pinned fluxoids can be finally expressed as

$$\mathcal{F}_i \equiv \frac{\mathrm{d}F_i}{\mathrm{d}z} = \oint_{\mathcal{C}} \Pi_i^j \,\alpha_j \,\mathrm{d}\ell \,. \tag{5}$$

The force (5) is well-defined provided the contour integral is evaluated at sufficiently large distances from the vortex where the force density vanishes, $f_i = 0$. Indeed, considering two different

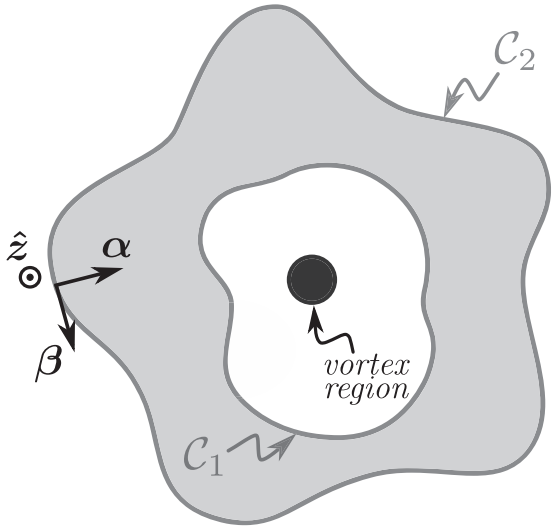

Figure 2. Schematic picture illustrating the surface $\mathcal{S}(\mathcal{C}_2) \setminus \mathcal{S}(\mathcal{C}_1)$ (shaded area) delimited by two different contours $\mathcal{C}_1$ and $\mathcal{C}_2$ around the vortex region.

contours $\mathcal{C}_1$ and $\mathcal{C}_2$, we have

$$\mathcal{F}_i(\mathcal{C}_1) - \mathcal{F}_i(\mathcal{C}_2) = \iint_{\mathcal{S}(\mathcal{C}_2)\setminus\mathcal{S}(\mathcal{C}_1)} f_i \,\mathrm{d}S = 0, \tag{6}$$

where the integration is carried out over the surface area $\mathcal{S}(\mathcal{C}_2) \setminus \mathcal{S}(\mathcal{C}_1)$ delimited by the two contours (see Fig. 2). Therefore, $\mathcal{F}_i(\mathcal{C}_1) = \mathcal{F}_i(\mathcal{C}_2)$.

Considering distances sufficiently far from the vortex for the first-order perturbation theory to hold, the momentum-flux tensor can be decomposed as

$$\Pi_{ij} = \bar{\Pi}_{ij} + \delta\Pi_{ij}, \tag{7}$$

where $\delta\Pi_{ij}$ denotes a small disturbance of the uniform background momentum-flux tensor $\bar{\Pi}_{ij}$. Similarly, any quantity y will be expanded to first order as $y = \bar{y} + \delta y$, where δy denotes a small disturbance of the uniform background quantity $\bar{y}$. Since the force for the unperturbed uniform background flows must evidently vanish by symmetry, the corresponding force in the presence of the vortex (5) will be given to first order by

$$\mathcal{F}_i = \oint_{\mathcal{C}} \delta\Pi_i^j \,\alpha_j \,\mathrm{d}\ell, \tag{8}$$

see also Carter et al. (2002) and Carter & Chamel (2005a).

2.2 Momentum-flux tensor for npe-matter

Let us assume that the vortex and the $N_{\rm p}$ fluxoids pinned to it are evolving in a cold[1] mixture of superfluid neutrons, superconducting protons, and degenerate electrons. Such conditions are expected to be met in the outer core of NSs. The basic model of such a three-component superconducting–superfluid mixture we adopt here has been described by Carter & Langlois (1998). Although developed in the relativistic context, their covariant approach remains formally applicable to the Newtonian space–time since it is based on Cartan's exterior calculus (see also Carter & Chamel

[1]The temperature in mature NSs is expected to be low enough for thermal excitations to be negligible, see e.g. Potekhin, Pons & Page (2015) for a review and Beloin et al. (2018) for recent neutron-star cooling simulations.

2004, 2005a,b for the fully 4D covariant non-relativistic formulation and a discussion of the specificity of the Newtonian space–time). The explicit hydrodynamic equations in the usual 3 + 1 space–time decomposition and based on a similar convective variational action principle have been derived by Prix (2004, 2005).

The momentum-flux tensor can be decomposed as

$$\Pi_{ij} = \Pi_{ij}^{(\mathrm{nuc})} + \Pi_{ij}^{(\mathrm{e})} + \Pi_{ij}^{(\mathrm{em})}, \tag{9}$$

where $\Pi_{ij}^{(\mathrm{nuc})}$, $\Pi_{ij}^{(\mathrm{e})}$, and $\Pi_{ij}^{(\mathrm{em})}$ denote the nucleon, electron, and electromagnetic contributions, respectively. The nucleon part reads (Carter & Langlois 1998)

$$\Pi_i^{(\mathrm{nuc})j} = n_{\mathrm{n}}^j \pi_i^{\mathrm{n}} + n_{\mathrm{p}}^j \left(\pi_i^{\mathrm{p}} - \frac{e}{c} A_i\right) + \Psi\, \delta_i^j, \tag{10}$$

where $n_{\mathrm{n}}^i = n_{\mathrm{n}} v_{\mathrm{n}}^i$ and $n_{\mathrm{p}}^i = n_{\mathrm{p}} v_{\mathrm{p}}^i$ denote the neutron and proton currents respectively, π_i^{n} and π_i^{p} stand for the associated generalized momenta per particle, A_i is the magnetic potential vector, Ψ is the generalized pressure of the nucleon mixture, and δ_i^j is the Kronecker symbol. The generalized momenta π_i^{n} and π_i^{p} are related to the purely nuclear momenta p_i^{n} and p_i^{p} (as obtained in the absence of electromagnetic fields) by the following relations (Carter & Langlois 1998)

$$\pi_i^{\mathrm{n}} = p_i^{\mathrm{n}}, \quad \pi_i^{\mathrm{p}} = p_i^{\mathrm{p}} + \frac{e}{c} A_i\,. \tag{11}$$

As stressed by Carter (1989), the distinction between momenta and currents is crucial. Because neutrons and protons are strongly interacting, they are mutually coupled by non-dissipative entrainment effects of the kind originally discussed in the context of superfluid ^{3}He–^{4}He mixtures by Andreev & Bashkin (1976) such that the nucleon momenta are expressible as (Carter & Langlois 1998)

$$\begin{aligned} p_i^{\mathrm{n}} &= \gamma_{ij}\left(\mathcal{K}^{\mathrm{nn}} n_{\mathrm{n}}^j + \mathcal{K}^{\mathrm{np}} n_{\mathrm{p}}^j\right), \\ p_i^{\mathrm{p}} &= \gamma_{ij}\left(\mathcal{K}^{\mathrm{pn}} n_{\mathrm{n}}^j + \mathcal{K}^{\mathrm{pp}} n_{\mathrm{p}}^j\right), \end{aligned} \tag{12}$$

where γ_{ij} denotes the space metric. The entrainment coefficients $\mathcal{K}^{\mathrm{nn}}$, $\mathcal{K}^{\mathrm{np}} = \mathcal{K}^{\mathrm{pn}}$, and $\mathcal{K}^{\mathrm{pp}}$ are not all independent since Galilean invariance imposes the following relations:

$$\begin{aligned} \mathcal{K}^{\mathrm{nn}} n_{\mathrm{n}} + \mathcal{K}^{\mathrm{np}} n_{\mathrm{p}} &= m_{\mathrm{n}}, \\ \mathcal{K}^{\mathrm{pn}} n_{\mathrm{n}} + \mathcal{K}^{\mathrm{pp}} n_{\mathrm{p}} &= m_{\mathrm{p}}, \end{aligned} \tag{13}$$

where m_{p} is the proton rest mass. The entrainment coefficients depend on the baryon number density and on the composition (see e.g. Gusakov & Haensel 2005; Chamel 2008; Kheto & Bandyopadhyay 2014; Sourie, Oertel & Novak 2016). They may also depend on the relative nucleon currents so that the relations (12) between nucleon momenta and currents are not necessarily linear (Leinson 2017, 2018).

The electromagnetic momentum-flux tensor is given by the usual expression (in Gaussian cgs units)

$$\Pi_i^{(\mathrm{em})j} = -\frac{1}{4\pi}\left(E_i E^j + B_i B^j - \frac{1}{2} E^k E_k \delta_i^j - \frac{1}{2} B^k B_k \delta_i^j\right), \tag{14}$$

where E^i and B^i denote the electric and magnetic fields.

At the vortex scale, electrons do not form a fluid[2] but are in a ballistic regime following classical trajectories. Instead of following the purely hydrodynamic treatment of Carter & Langlois (1998) for the electron momentum-flux tensor $\Pi_{ij}^{(\mathrm{e})}$, we adopt here the expression given by equation (23) of Gusakov (2019).

The total force (8) experienced by the vortex can thus be decomposed as

$$\mathcal{F}_i = \mathcal{F}_i^{(\mathrm{nuc})} + \mathcal{F}_i^{(\mathrm{e})} + \mathcal{F}_i^{(\mathrm{em})}, \tag{15}$$

where the separate contributions

$$\mathcal{F}_i^{(\mathrm{nuc})} = \oint_{\mathcal{C}} \delta\Pi_i^{(\mathrm{nuc})j}\, \alpha_j\, \mathrm{d}\ell, \tag{16}$$

$$\mathcal{F}_i^{(\mathrm{e})} = \oint_{\mathcal{C}} \delta\Pi_i^{(\mathrm{e})j}\, \alpha_j\, \mathrm{d}\ell, \tag{17}$$

and

$$\mathcal{F}_i^{(\mathrm{em})} = \oint_{\mathcal{C}} \delta\Pi_i^{(\mathrm{em})j}\, \alpha_j\, \mathrm{d}\ell, \tag{18}$$

are evaluated in Sections 2.3 - 2.5, respectively.

2.3 Nucleon contribution

The first-order perturbation in the nucleon momentum-flux tensor (10) reads

$$\begin{aligned} \delta\Pi_i^{(\mathrm{nuc})j} = {} & \delta n_{\mathrm{n}}^j\, \bar{\pi}_i^{\mathrm{n}} + \bar{n}_{\mathrm{n}}^j\, \delta\pi_i^{\mathrm{n}} + \delta n_{\mathrm{p}}^j\, \bar{\pi}_i^{\mathrm{p}} + \bar{n}_{\mathrm{p}}^j \left(\delta\pi_i^{\mathrm{p}} - \frac{e}{c}\delta A_i\right) \\ & + \delta\Psi\, \delta_i^j, \end{aligned} \tag{19}$$

using a gauge such that $\bar{A}_i = 0$ (Carter et al. 2002). In the 4D covariant formulations of Carter et al. (2002) and Carter & Chamel (2005a), the first-order perturbation of the nucleon pressure can be expressed as [see equation (A9) of Carter & Chamel (2005b)]

$$\delta\Psi = -\bar{n}_{\mathrm{n}}\, \delta p_0^{\mathrm{n}} - \bar{n}_{\mathrm{p}}\, \delta p_0^{\mathrm{p}} - \bar{n}_{\mathrm{n}}^k\, \delta p_k^{\mathrm{n}} - \bar{n}_{\mathrm{p}}^k\, \delta p_k^{\mathrm{p}}, \tag{20}$$

where p_0^{n} and p_0^{p} correspond to the time components of the neutron and proton four-momenta. These time components can be more explicitly written as [see equation (34) of Prix (2005)]

$$p_0^{\mathrm{x}} = -\mu^{\mathrm{x}} + m_{\mathrm{x}} v_{\mathrm{x}}^2/2 - v_{\mathrm{x}}^k\, p_k^{\mathrm{x}}, \tag{21}$$

with μ^{x} denoting the chemical potential of nucleon species $\mathrm{x} \in \{\mathrm{n},\mathrm{p}\}$. Introducing the time components of the generalized momenta (Carter & Langlois 1998)

$$\pi_0^{\mathrm{n}} = p_0^{\mathrm{n}}, \quad \pi_0^{\mathrm{p}} = p_0^{\mathrm{p}} + eA_0, \tag{22}$$

where $-A_0$ denotes the electric scalar potential, the first-order perturbation in the nucleon momentum-flux tensor (19) can be written as

$$\begin{aligned} \delta\Pi_i^{(\mathrm{nuc})j} = {} & \delta n_{\mathrm{n}}^j\, \bar{\pi}_i^{\mathrm{n}} + \bar{n}_{\mathrm{n}}^j\, \delta\pi_i^{\mathrm{n}} + \delta n_{\mathrm{p}}^j\, \bar{\pi}_i^{\mathrm{p}} + \bar{n}_{\mathrm{p}}^j \left(\delta\pi_i^{\mathrm{p}} - \frac{e}{c}\delta A_i\right) \\ & - \left(\bar{n}_{\mathrm{n}}\delta\pi_0^{\mathrm{n}} + \bar{n}_{\mathrm{p}}\delta\pi_0^{\mathrm{p}} + \bar{n}_{\mathrm{n}}^k\delta\pi_k^{\mathrm{n}} + \bar{n}_{\mathrm{p}}^k\delta\pi_k^{\mathrm{p}}\right)\delta_i^j \\ & + \left(\bar{n}_{\mathrm{p}} e\, \delta A_0 + \frac{e}{c}\bar{n}_{\mathrm{p}}^k \delta A_k\right)\delta_i^j\,. \end{aligned} \tag{23}$$

The nucleon force (16) acting on the vortex can thus be decomposed as

$$\mathcal{F}_i^{(\mathrm{nuc})} = \mathcal{F}_{\mathrm{E}\,i} + \mathcal{F}_{\mathrm{t}\,i} + \mathcal{F}_{\mathrm{Mn}\,i} + \mathcal{F}_{\mathrm{Mp}\,i} + \mathcal{F}_{\Phi\,i}, \tag{24}$$

where the different force terms are given by

$$\mathcal{F}_{\mathrm{E}\,i} = -\oint_{\mathcal{C}} \left(\bar{n}_{\mathrm{n}}\, \delta\pi_0^{\mathrm{n}} + \bar{n}_{\mathrm{p}}\, \delta\pi_0^{\mathrm{p}} - \bar{n}_{\mathrm{p}}\, e\, \delta A_0\right)\alpha_i\, \mathrm{d}\ell, \tag{25}$$

$$\mathcal{F}_{\mathrm{t}\,i} = \oint_{\mathcal{C}} \left(\delta n_{\mathrm{n}}^j\, \bar{\pi}_i^{\mathrm{n}} + \delta n_{\mathrm{p}}^j\, \bar{\pi}_i^{\mathrm{p}}\right)\alpha_j\, \mathrm{d}\ell, \tag{26}$$

[2]We recall here that we are dealing with scales large compared to the size of the vortex-fluxoid configuration, but small with respect to the typical intervortex distance $d_{\mathrm{n}} \simeq 1/\sqrt{\mathcal{N}_{\mathrm{n}}} \sim 10^{-3}$ cm. The electron mean-free path is generally much larger than d_{n}, see e.g. Shternin & Yakovlev (2008) and Glampedakis, Andersson & Samuelsson (2011).

$$\mathcal{F}_{\mathrm{Mn}\,i} = \oint_{\mathcal{C}} \left(\bar{n}^j_{\mathrm{n}}\, \delta\pi^{\mathrm{n}}_i - \delta^j_i\, \bar{n}^k_{\mathrm{n}}\, \delta\pi^{\mathrm{n}}_k \right) \alpha_j \, \mathrm{d}\ell, \tag{27}$$

$$\mathcal{F}_{\mathrm{Mp}\,i} = \oint_{\mathcal{C}} \left(\bar{n}^j_{\mathrm{p}}\, \delta\pi^{\mathrm{p}}_i - \delta^j_i\, \bar{n}^k_{\mathrm{p}}\, \delta\pi^{\mathrm{p}}_k \right) \alpha_j \, \mathrm{d}\ell, \tag{28}$$

and

$$\mathcal{F}_{\Phi\,i} = -\frac{e}{c} \oint_{\mathcal{C}} \left(\bar{n}^j_{\mathrm{p}}\, \delta A_i - \delta^j_i\, \bar{n}^k_{\mathrm{p}}\, \delta A_k \right) \alpha_j \, \mathrm{d}\ell \,. \tag{29}$$

Let us first focus on $\mathcal{F}_{\mathrm{E}\,i}$. The equations of motion for the neutron superfluid and for the proton superconductor, as expressed as the vanishing of a suitably generalized vorticity tensor, thus take a very simple form in the 4D covariant approach [see equations (15), (17), and (18) of Carter & Langlois (1998), or equations (161) and (171) of Carter & Chamel (2004)]. In the usual space–time decomposition, the stationary limit of these equations reduces to $\nabla_i\, \pi^{\mathrm{n}}_0 = \nabla_i\, \pi^{\mathrm{p}}_0 = 0$. As shown in Appendix A, this result can also be obtained from equations (26)–(29) of Prix (2005) or from equations (B4) and (B5) of Gusakov (2019) in the particular case in which mutual neutron–proton entrainment effects are neglected. The above equations imply that both π^{n}_0 and π^{p}_0 are uniform, i.e. $\pi^{\mathrm{n}}_0 = \bar{\pi}^{\mathrm{n}}_0$ and $\pi^{\mathrm{p}}_0 = \bar{\pi}^{\mathrm{p}}_0$, or equivalently, $\delta\pi^{\mathrm{p}}_0 = \delta\pi^{\mathrm{n}}_0 = 0$. Therefore, we get

$$\mathcal{F}_{\mathrm{E}\,i} = \bar{n}_{\mathrm{p}}\, e \oint_{\mathcal{C}} A_0\, \alpha_i \, \mathrm{d}\ell, \tag{30}$$

using the fact that $\delta A_0 = A_0 - \bar{A}_0 = A_0$ since the uniform background value vanishes (in an appropriate gauge). Equation (30) can thus be interpreted as the (opposite of the) force acting on charged protons due to the electric field.

Introducing the coefficients $\mathcal{D}_{\mathrm{n}}$ and $\mathcal{D}_{\mathrm{p}}$ as

$$\mathcal{D}_{\mathrm{n}} = \oint_{\mathcal{C}} n^j_{\mathrm{n}}\, \alpha_j \, \mathrm{d}\ell \quad \text{and} \quad \mathcal{D}_{\mathrm{p}} = \oint_{\mathcal{C}} n^j_{\mathrm{p}}\, \alpha_j \, \mathrm{d}\ell, \tag{31}$$

and recalling that $\bar{p}^x_i = \bar{\pi}^x_i$, the force term $\mathcal{F}_{\mathrm{t}\,i}$ can be rewritten as

$$\mathcal{F}_{\mathrm{t}\,i} = \delta\mathcal{D}_{\mathrm{n}}\, \bar{p}^{\mathrm{n}}_i + \delta\mathcal{D}_{\mathrm{p}}\, \bar{p}^{\mathrm{p}}_i, \tag{32}$$

where $\delta\mathcal{D}_{\mathrm{n}} = \mathcal{D}_{\mathrm{n}} - \bar{\mathcal{D}}_{\mathrm{n}}$ and $\delta\mathcal{D}_{\mathrm{p}} = \mathcal{D}_{\mathrm{p}} - \bar{\mathcal{D}}_{\mathrm{p}}$. Using Stokes' theorem, $\mathcal{D}_{\mathrm{n}}$ and $\mathcal{D}_{\mathrm{p}}$ can be equivalently expressed as

$$\mathcal{D}_{\mathrm{n}} = -\iint_{\mathcal{S}(\mathcal{C})} \nabla_k n^k_{\mathrm{n}}\, \mathrm{d}S \quad \text{and} \quad \mathcal{D}_{\mathrm{p}} = -\iint_{\mathcal{S}(\mathcal{C})} \nabla_k n^k_{\mathrm{p}}\, \mathrm{d}S, \tag{33}$$

where the integrals are over the surface $\mathcal{S}(\mathcal{C})$ delimited by the contour $\mathcal{C}$ and $\mathrm{d}S$ is the corresponding surface element. Therefore, the background values vanish and $\delta\mathcal{D}_{\mathrm{n}} = \mathcal{D}_{\mathrm{n}}$ and $\delta\mathcal{D}_{\mathrm{p}} = \mathcal{D}_{\mathrm{p}}$. The force $\mathcal{F}_{\mathrm{t}\,i}$ is thus associated with transfusive processes, whereby particles of different species are converted into each other by nuclear reactions (Carter & Chamel 2005b). If each species is separately conserved, $\nabla_k n^k_{\mathrm{n}} = 0$ and $\nabla_k n^k_{\mathrm{p}} = 0$ must hold everywhere throughout the fluids. In such a case, we deduce that $\mathcal{D}_{\mathrm{n}} = \mathcal{D}_{\mathrm{p}} = 0$, which leads to $\mathcal{F}_{\mathrm{t}\,i} = 0$.

Neutron and proton superflows far from the vortex must obey the irrotationality condition [see equations (15), (17), and (18) of Carter & Langlois (1998) or equations (161) and (171) of Carter & Chamel (2004)]

$$\varepsilon^{ijk} \nabla_j \pi^{\mathrm{n}}_k = 0 \quad \text{and} \quad \varepsilon^{ijk} \nabla_j \pi^{\mathrm{p}}_k = 0, \tag{34}$$

where ε_{ijk} is the Levi–Civita symbol. The longitudinal invariance along the vortex, in association with the previous irrotationality conditions, lead to $\hat{z}^j \delta\pi^{\mathrm{n}}_j = 0$ and $\hat{z}^j \delta\pi^{\mathrm{p}}_j = 0$ (Carter & Chamel 2005a). Defining $\perp^i_j$ as the operator of projection orthogonal to the vortex, i.e. $\perp^i_j = \delta^i_j - \hat{z}^i \hat{z}_j$, we thus have

$$\perp^j_i \, \delta\pi^{\mathrm{n}}_j = \delta\pi^{\mathrm{n}}_i, \qquad \perp^j_i \, \delta\pi^{\mathrm{p}}_j = \delta\pi^{\mathrm{p}}_i \,. \tag{35}$$

The force term $\mathcal{F}_{\mathrm{Mn}\,i}$ (27) can therefore be recast as

$$\mathcal{F}_{\mathrm{Mn}\,i} = \oint_{\mathcal{C}} \bar{n}^j_{\mathrm{n}} \delta\pi^{\mathrm{n}}_k \left[\perp^k_i \alpha_j - \perp^k_j \alpha_i \right] \mathrm{d}\ell \,. \tag{36}$$

Introducing the unit vector β^i along the contour such that $\alpha_i = -\varepsilon_{ijk} \beta^j \hat{z}^k$ as illustrated in Fig. 2, and making use of the identity

$$\perp^k_i \, \varepsilon_{jlm} - \perp^k_j \, \varepsilon_{ilm} = \perp^k_l \, \varepsilon_{jim}, \tag{37}$$

the force $\mathcal{F}_{\mathrm{Mn}\,i}$ can be equivalently written as

$$\mathcal{F}_{\mathrm{Mn}\,i} = -\varepsilon_{ijk} \hat{z}^j \bar{n}^k_{\mathrm{n}} \delta\mathcal{C}^{\mathrm{n}}, \tag{38}$$

where the neutron momentum integral $\mathcal{C}^{\mathrm{n}}$ is given by

$$\mathcal{C}^{\mathrm{n}} = \oint_{\mathcal{C}} \pi^{\mathrm{n}}_k \mathrm{d}\ell^k, \tag{39}$$

with $\mathrm{d}\ell^k = \mathrm{d}\ell\, \beta^k$. Since the background value $\bar{\mathcal{C}}^{\mathrm{n}}$ vanishes and using the fact that the neutron momentum circulation integral is quantized, we have

$$\delta\mathcal{C}^{\mathrm{n}} = \mathcal{C}^{\mathrm{n}} = m_{\mathrm{n}}\, \kappa_{\mathrm{n}} \tag{40}$$

in the presence of a single vortex line. The force term $\mathcal{F}_{\mathrm{Mn}\,i}$ finally reads

$$\mathcal{F}_{\mathrm{Mn}\,i} = -\bar{\rho}_{\mathrm{n}}\, \kappa_{\mathrm{n}}\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{n}}, \tag{41}$$

where $\rho_{\mathrm{n}} = m_{\mathrm{n}}\, n_{\mathrm{n}}$. This force can thus be recognized as the Magnus force induced by the (quantized) momentum circulation of the neutron superfluid around the vortex. Similar arguments also apply to the proton superconductor, with the only difference that the proton momentum circulation integral reads

$$\mathcal{C}^{\mathrm{p}} = \oint_{\mathcal{C}} \pi^{\mathrm{p}}_k \mathrm{d}\ell^k = m_{\mathrm{p}}\, N_{\mathrm{p}}\, \kappa_{\mathrm{p}} \tag{42}$$

with $\kappa_{\mathrm{p}} = h/(2m_{\mathrm{p}})$, if N_{p} fluxoids are enclosed inside the contour. The force term $\mathcal{F}_{\mathrm{Mp}\,i}$ thus reads

$$\mathcal{F}_{\mathrm{Mp}\,i} = -\bar{\rho}_{\mathrm{p}}\, N_{\mathrm{p}}\, \kappa_{\mathrm{p}}\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{p}} \,, \tag{43}$$

where $\rho_{\mathrm{p}} = m_{\mathrm{p}}\, n_{\mathrm{p}}$.

Finally, since the magnetic field $B^i = \varepsilon^{ijk} \nabla_j A_k$ carried by the vortex-fluxoid configuration is directed along $\hat{z}^i$, the z-component of the vector potential A_i must vanish, i.e. $A_z = 0$ [see e.g. equations (7) and (9) of Gusakov (2019)]. As a consequence, we have $\perp^j_i \, \delta A_j = \delta A_i$. Following a procedure similar to the one used to derive $\mathcal{F}_{\mathrm{Mn}\,i}$ and $\mathcal{F}_{\mathrm{Mp}\,i}$, we obtain

$$\mathcal{F}_{\Phi\,i} = \bar{n}_{\mathrm{p}} \frac{e}{c} \Phi\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{p}}, \tag{44}$$

where

$$\Phi = \oint_{\mathcal{C}} A_k \mathrm{d}\ell^k \tag{45}$$

is the magnetic flux through the surface $S(\mathcal{C})$ delimited by the contour $\mathcal{C}$, and using the fact that the uniform background flux $\bar{\Phi}$ necessarily vanishes. Equation (44) can thus be interpreted as the (opposite of the) force acting on charged protons due to the magnetic field.

Collecting terms in equation (24), the nucleon force contribution is finally expressible as

$$\mathcal{F}^{(\mathrm{nuc})}_i = -\bar{\rho}_{\mathrm{n}}\, \kappa_{\mathrm{n}}\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{n}} - \bar{\rho}_{\mathrm{p}}\, N_{\mathrm{p}}\, \kappa_{\mathrm{p}}\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{p}} + \bar{n}_{\mathrm{p}}\, e \oint_{\mathcal{C}} A_0\, \alpha_i \, \mathrm{d}\ell + \bar{n}_{\mathrm{p}} \frac{e}{c} \Phi\, \varepsilon_{ijk} \hat{z}^j \bar{v}^k_{\mathrm{p}} \,. \tag{46}$$

Let us recall that this expression only holds in the absence of transfusive processes, i.e. assuming that each species is separately conserved.

2.4 Electron contribution

Since electrons are in a ballistic regime at the scale of interest here, the calculation of $\mathcal{F}_i^{(\mathrm{e})}$ needs a specific treatment, see e.g. Gusakov (2019). Although Gusakov (2019) mainly focused on a single proton fluxoid, several conclusions of this work are actually very general and can be readily transposed to the vortex-fluxoid configuration under consideration. In particular, the electron force term (17) can be decomposed into two parts, i.e.

$$\mathcal{F}_i^{(\mathrm{e})} = \mathcal{F}_i^{(\mathrm{e,sc})} + \mathcal{F}_i^{(\mathrm{e,ind})}, \tag{47}$$

see equation (B27) of Gusakov (2019), where

$$\mathcal{F}_i^{(\mathrm{e,sc})} = \oint_{\mathcal{C}} \delta\Pi_i^{(\mathrm{e,sc})j} \alpha_j \, \mathrm{d}\ell \tag{48}$$

is associated with the scattering of electrons off the vortex-fluxoid system and

$$\mathcal{F}_i^{(\mathrm{e,ind})} = \oint_{\mathcal{C}} \delta\Pi_i^{(\mathrm{e,ind})j} \alpha_j \, \mathrm{d}\ell \tag{49}$$

is an 'induced' contribution related to the fact that the scattered electrons carry a charge and thus generate a weak electric field far from the vortex. Using equation (B15) of Gusakov (2019), this induced contribution reads[3]

$$\mathcal{F}_i^{(\mathrm{e,ind})} = -\bar{n}_{\mathrm{e}} \, e \oint_{\mathcal{C}} A_0 \, \alpha_i \, \mathrm{d}\ell \, . \tag{50}$$

Besides, the force term $\mathcal{F}_i^{(\mathrm{e,sc})}$ is found to be expressible as

$$\mathcal{F}_i^{(\mathrm{e,sc})} = D_{\mathrm{e}} \, \bar{v}_{\mathrm{e}\,i} + D'_{\mathrm{e}} \, \varepsilon_{ijk} \hat{z}^j \bar{v}_{\mathrm{e}}^k, \tag{51}$$

see equations (29) and (18) of Gusakov (2019), where $\bar{v}_{\mathrm{e}}^i$ denotes the asymptotically uniform electron velocity (in the frame where the vortex is at rest). Furthermore, Gusakov (2019) derived the expressions for the coefficients D_{e} and D'_{e} in the particular context where $\mathcal{F}_i^{(\mathrm{e,sc})}$ is governed by the scattering of electrons off the magnetic field carried by a single fluxoid. His expression for D'_{e} [see equation (61) of Gusakov (2019)], i.e.

$$D'_{\mathrm{e}} = -\frac{e}{c} \bar{n}_{\mathrm{e}} \Phi, \tag{52}$$

where Φ is given by equation (45), happens to be very general in the sense that it does not depend on the detailed structure of the quantized lines carrying the magnetic flux (as long as electrons follow classical trajectories). Therefore, equation (52) remains valid in the present case where electrons are scattered off the magnetic field carried by the vortex and the pinned fluxoids.[4] On the other hand, the expression (60) of Gusakov (2019) for the drag coefficient D_{e} is not applicable here since it depends strongly on the configuration of the quantized line(s) carrying the flux Φ. The determination of D_{e} would thus require (i) to know the exact geometry and structure of the vortex and the N_{p} fluxoids pinned to it, and (ii) to generalize the microscopic scattering calculations carried out by Gusakov (2019) to the present case.

[3]The quantities e_{e}, ϕ and $n_{\mathrm{e}0}$ appearing in equation (B15) of Gusakov (2019) correspond here to $-e$, $-A_0$, and $\bar{n}_{\mathrm{e}}$, respectively.

[4]Each proton fluxoid pinned to the vortex carries a quantum of magnetic flux ϕ_0. Besides, the neutron vortex itself is magnetized due to neutron–proton entrainment effects, and thus carries a fractional quantum of magnetic flux $\phi_{\mathrm{n}} \simeq -\varepsilon_{\mathrm{p}} \phi_0$, where ε_{p} characterizes the importance of entrainment effects (see e.g. Sedrakian & Shakhabasian 1980; Alpar, Langer & Sauls 1984).

From the previous considerations, the electron force (17) finally reads

$$\mathcal{F}_i^{(\mathrm{e})} = -\bar{n}_{\mathrm{p}} \, e \oint_{\mathcal{C}} A_0 \, \alpha_i \, \mathrm{d}\ell + D_{\mathrm{e}} \, \bar{v}_{\mathrm{p}\,i} - \frac{e}{c} \bar{n}_{\mathrm{p}} \Phi \, \varepsilon_{ijk} \hat{z}^j \bar{v}_{\mathrm{p}}^k, \tag{53}$$

where we have made use of the electric charge neutrality condition $\bar{n}_{\mathrm{e}} = \bar{n}_{\mathrm{p}}$ and the so-called screening condition $\bar{v}_{\mathrm{e}}^i = \bar{v}_{\mathrm{p}}^i$, see e.g. Glampedakis et al. (2011) and Gusakov & Dommes (2016).

2.5 Electromagnetic contribution

As can be seen from equation (14), the first-order perturbation in the electromagnetic stress tensor $\delta\Pi_i^{(\mathrm{em})j}$ only involves terms of the kind $\delta E^j \bar{E}_i$ and $\delta B^j \bar{B}_i$. However, the asymptotically uniform magnetic and electric fields must vanish in view of the Meissner effect, i.e. $\bar{B}_i = 0$ and $\bar{E}_i = 0$ (Carter et al. 2002; Gusakov 2019). From $\delta\Pi_i^{(\mathrm{em})j} = 0$, we conclude that

$$\mathcal{F}_i^{(\mathrm{em})} = 0 \, . \tag{54}$$

2.6 Final expression and comparison with previous studies

Combining equations (46), (53), and (54), the total force per unit length (15) acting on a neutron vortex to which N_{p} proton fluxoids are pinned is finally expressible as

$$\mathcal{F}_i = -\bar{\rho}_{\mathrm{n}} \kappa_{\mathrm{n}} \varepsilon_{ijk} \hat{z}^j \bar{v}_{\mathrm{n}}^k - \bar{\rho}_{\mathrm{p}} N_{\mathrm{p}} \kappa_{\mathrm{p}} \varepsilon_{ijk} \hat{z}^j \bar{v}_{\mathrm{p}}^k + D_{\mathrm{e}} \, \bar{v}_{\mathrm{p}\,i} \, . \tag{55}$$

Note that the electron force proportional to D'_{e} derived by Gusakov (2019), i.e. the last term in equation (53), is exactly cancelled by the proton force $\mathcal{F}_{\Phi\,i}$. Let us remark that equation (55) is also applicable to determine the force acting on a neutron–proton vortex cluster of the kind proposed by Sedrakian & Sedrakian (1995).

In the absence of pinning ($N_{\mathrm{p}} = 0$), a situation considered by Alpar et al. (1984), a neutron vortex still experiences a drag force due to the scattering of electrons off the magnetic field induced by the circulation of entrained protons. By setting $N_{\mathrm{p}} = 1$ and $\kappa_{\mathrm{n}} = 0$, our expression (55) reduces to that obtained by Gusakov (2019) for the force acting on a single fluxoid. For NSs with a rotation period $P_{10} \sim 1$ and a typical magnetic field $B_{12} \sim 1$, the number of pinned fluxoids may potentially be as large as $N_{\mathrm{p}} \sim \mathcal{N}_{\mathrm{p}}/\mathcal{N}_{\mathrm{n}} \sim 10^{13}$ so that the second term in equation (55) may have a very strong impact on the vortex dynamics. To illustrate the relative importance of the different force terms on the vortex motion, we give the expression of the vortex velocity in the next section.

2.7 Vortex motion

Once expressed in a frame where the neutron vortex moves at the velocity v_{L}^i, the vortex motion can be obtained from the force balance equation $\mathcal{F}_i = 0$ (neglecting the masses of the different quantized lines as in previous studies). Following the classical approach of Hall & Vinen (1956) and considering velocities orthogonal to $\hat{z}^i$, the vortex velocity is found to be given by

$$\begin{aligned} v_{\mathrm{L}}^i &= \bar{v}_{\mathrm{p}}^i + \mathcal{B} \, \varepsilon^{ijk} \hat{z}_j \bar{w}_{\mathrm{pn}\,k} + \left(1 - \mathcal{B}'\right) \varepsilon^{ijk} \hat{z}_j \varepsilon_{klm} \hat{z}^l \bar{w}_{\mathrm{pn}}^m \\ &= \bar{v}_{\mathrm{n}}^i + \mathcal{B} \, \varepsilon^{ijk} \hat{z}_j \bar{w}_{\mathrm{pn}\,k} - \mathcal{B}' \varepsilon^{ijk} \hat{z}_j \varepsilon_{klm} \hat{z}^l \bar{w}_{\mathrm{pn}}^m, \end{aligned} \tag{56}$$

where $\bar{w}_{\mathrm{pn}}^i = \bar{v}_{\mathrm{p}}^i - \bar{v}_{\mathrm{n}}^i$ denotes the relative velocity far from the quantized lines, see Appendix B for details (the coefficients usually denoted by $\mathcal{B}$ and $\mathcal{B}'$ in the neutron-star literature were indicated by α and α' in the standard textbook of Donnelly 2005). In this

expression, the coefficients $\mathcal{B}$ and $\mathcal{B}'$ are expressible as[5]

$$\mathcal{B} = \frac{\xi}{\xi^2 + (1+X)^2} \quad \text{and} \quad 1 - \mathcal{B}' = \frac{1+X}{\xi^2 + (1+X)^2}, \tag{57}$$

where the drag-to-lift ratio ξ and the momentum circulation ratio X read

$$\xi = \frac{D_\mathrm{e}}{\bar{\rho}_\mathrm{n}\kappa_\mathrm{n}} \quad \text{and} \quad X = \frac{\bar{n}_\mathrm{p}}{\bar{n}_\mathrm{n}} N_\mathrm{p}\,. \tag{58}$$

Note that, regardless of the actual values of ξ and X, the following inequalities hold

$$\mathcal{B} \leq 1/2 \quad \text{and} \quad \mathcal{B}' \leq 1\,. \tag{59}$$

In the absence of pinning ($N_\mathrm{p} = 0$), equation (57) reduces to well-known expressions (see e.g. Carter 2001). In particular, the vortex velocity v_L^i coincides with $\bar{v}_\mathrm{n}^i$ and $\bar{v}_\mathrm{p}^i$ in the weak ($\xi \ll 1$) and strong ($\xi \gg 1$) drag regimes, respectively. The motion of a vortex is more complicated if proton fluxoids are pinned to it. In particular, the vortex will move with velocity $\bar{v}_\mathrm{p}^i$ even in the weak drag limit if the number of pinned fluxoids N_p is large enough such that $X \gg 1$ and $\gg \xi$. However, it should be stressed that the drag-to-lift ratio ξ itself depends on N_p and may thus be also very large (Ding, Cheng & Chau 1993; Sedrakian & Sedrakian 1995). Because pinning may lead to a dramatic reduction of the coefficient $\mathcal{B}$, it may also have important implications for the onset of superfluid turbulence, which is thought to be governed by the parameter $q = \mathcal{B}/\left(1 - \mathcal{B}'\right)$ (Finne et al. 2003).

3 CONCLUSIONS

Following an approach originally proposed by Carter and collaborators (Carter et al. 2002; Carter & Chamel 2005a), we have derived the expression for the force per unit length acting on a quantized neutron vortex to which N_p proton fluxoids are attached, see equation (55). Our expression is very general and can be applied to describe various situations. In particular, equation (55) extends the expression recently obtained by Gusakov (2019) for the force per unit length acting on a single fluxoid.

By clarifying the different contributions to the force, we have shown that the proton-momentum circulation around the vortex induced by the presence of pinned fluxoids gives rise to a Magnus type force. Due to mutual entrainment effects, the distinction between momenta (usually improperly introduced in terms of 'superfluid velocities') and currents is crucial to obtain the correct expression of the force. Unlike the drag force, this Magnus force does not depend on the microscopic arrangement of pinned fluxoids, as a consequence of the quantization of the proton circulation.

Although the vortex velocity takes a similar form as in the absence of pinning, see equation (56), the friction coefficients $\mathcal{B}$ and $\mathcal{B}'$ are found to depend on the dimensionless ratio $N_\mathrm{p} \times \bar{n}_\mathrm{p}/\bar{n}_\mathrm{n}$ in addition to the drag-to-lift ratio ξ. Because N_p may be potentially as large as $\sim 10^{13}$, pinning may have a dramatic impact on the vortex motion and the onset of superfluid turbulence. A major complication comes from the fact that N_p, thereby the coefficients $\mathcal{B}$ and $\mathcal{B}'$, may vary along the vortex trajectory: N_p may increase as the vortex moves and encounters more and more fluxoids, but N_p may also decrease as fluxoids get unpinned. The evolution of N_p will generally depend on the spatial distribution of fluxoids in the outer core of an NS, which in turn reflects the geometry of the internal magnetic field. Pinning of neutron vortices to proton fluxoids should thus be taken into account in the modelling of the magnetorotational evolution of NSs.

[5] As shown in the accompanying letter (Sourie & Chamel 2020), the coefficient $\mathcal{B}$ ($\mathcal{B}'$) is associated with the dissipative (conservative) contribution in the smooth-averaged mutual-friction force arising at scales large compared to the intervortex spacing.

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APPENDIX A: STATIONARY EQUATIONS OF MOTION FOR THE NEUTRON SUPERFLUID AND FOR THE PROTON SUPERCONDUCTOR

In this appendix, we show how the stationary equations of motion for the neutron superfluid and for the proton superconductor, i.e.

$$\nabla_i \pi_0^{\rm n} = 0 \ \text{ and } \ \nabla_i \pi_0^{\rm p} = 0, \tag{A1}$$

can be derived from equations (26)–(29) of Prix (2005) and from equations (B4) and (B5) of Gusakov (2019), although these two studies rely on different approaches.

A1 Derivation from Prix (2005)

Using equations (26)–(29) of Prix (2005), the canonical force densities acting on the neutron superfluid and the proton superconductor are respectively given by

$$f_i^{\rm n} = n_{\rm n}\left(\partial_t p_i^{\rm n} - \nabla_i p_0^{\rm n}\right) - \varepsilon_{ijk} n_{\rm n} v_{\rm n}^j \varepsilon^{klm} \nabla_l p_m^{\rm n} + n_{\rm n} m_{\rm n} \nabla_i \varphi + \Gamma_{\rm n} \pi_i^{\rm n}, \tag{A2}$$

$$f_i^{\rm p} = n_{\rm p}\left(\partial_t p_i^{\rm p} - \nabla_i p_0^{\rm p}\right) - \varepsilon_{ijk} n_{\rm p} v_{\rm p}^j \varepsilon^{klm} \nabla_l p_m^{\rm p} + n_{\rm p} m_{\rm p} \nabla_i \varphi + \Gamma_{\rm p} \pi_i^{\rm p} - n_{\rm p}\, e \left(E_i + \frac{1}{c}\varepsilon_{ijk} v_{\rm p}^j B^k\right), \tag{A3}$$

where φ denotes the gravitational gauge field and we have used the short-hand notations

$$\Gamma_{\rm n} = \partial_t n_{\rm n} + \nabla_i\left(n_{\rm n} v_{\rm n}^i\right) \ \text{ and } \ \Gamma_{\rm p} = \partial_t n_{\rm p} + \nabla_i\left(n_{\rm p} v_{\rm p}^i\right). \tag{A4}$$

In the absence of external forces, as characterized by $f_i^{\rm n} = f_i^{\rm p} = 0$, and ignoring any transfusive processes, i.e. $\Gamma_{\rm n} = \Gamma_{\rm p} = 0$, the stationary equations of motion reduce to

$$0 = -\nabla_i p_0^{\rm n} - \varepsilon_{ijk} v_{\rm n}^j \varepsilon^{klm} \nabla_l p_m^{\rm n}, \tag{A5}$$

$$0 = -\nabla_i p_0^{\rm p} - \varepsilon_{ijk} v_{\rm p}^j \varepsilon^{klm} \nabla_l p_m^{\rm p} - e\left(E_i + \frac{1}{c}\varepsilon_{ijk} v_{\rm p}^j B^k\right), \tag{A6}$$

where we have neglected the small variations of the gravitational field φ on the scales of interest. Making use of the irrotationality conditions (34), rewritten as

$$\varepsilon^{ijk}\nabla_j p_k^{\rm n} = 0 \ \text{ and } \ \varepsilon^{ijk}\nabla_j p_k^{\rm p} + \frac{e}{c} B^i = 0, \tag{A7}$$

in combination with the definition (22) for $\pi_0^{\rm n}$ and $\pi_0^{\rm p}$, equations (A5) and (A6) reduce to equation (A1), as expected.

A2 Derivation from Gusakov (2019)

In the alternative approach followed by Gusakov (2019), the conservation equations for the neutron and proton momenta read

$$\partial_t G_{{\rm n}\,i} = -\nabla_k\left(m_{\rm n} n_{\rm n} v_{\rm n}^k v_{{\rm n}\,i}\right) - n_{\rm n}\nabla_i \mu^{\rm n}, \tag{A8}$$

$$\partial_t G_{{\rm p}\,i} = -\nabla_k\left(m_{\rm p} n_{\rm p} v_{\rm p}^k v_{{\rm p}\,i}\right) - n_{\rm p}\nabla_i \mu^{\rm p} + n_{\rm p}\, e\left(E_i + \frac{1}{c}\varepsilon_{ijk} v_{\rm p}^j B^k\right), \tag{A9}$$

where the variations of the gravitational field have been neglected, see equations (B5) and (B4) of Gusakov (2019). The quantities $G_{{\rm n}\,i} = n_{\rm n}\, p_i^{\rm n}$ and $G_{{\rm p}\,i} = n_{\rm p}\, p_i^{\rm p}$ denote the neutron and proton momentum densities, respectively, with $p_i^{\rm n} = m_{\rm n} v_{{\rm n}\,i}$ and $p_i^{\rm p} = m_{\rm p} v_{{\rm p}\,i}$ in the absence of mutual neutron–proton entrainment effects [see equations (12) and (13) with $\mathcal{K}^{\rm np} = 0$], as considered in Gusakov (2019). Ignoring transfusive processes and focusing on stationary situations only, equations (A8) and (A9) reduce to

$$0 = -m_{\rm n} v_{\rm n}^k \nabla_k v_{{\rm n}\,i} - \nabla_i \mu^{\rm n}, \tag{A10}$$

$$0 = -m_{\rm p} v_{\rm p}^k \nabla_k v_{{\rm p}\,i} - \nabla_i \mu^{\rm p} + e\left(E_i + \frac{1}{c}\varepsilon_{ijk} v_{\rm p}^j B^k\right). \tag{A11}$$

Making use of the relation

$$A^j \nabla_j A_i = \frac{1}{2}\nabla_i\left(A^j A_j\right) - \varepsilon_{ijk} A^j \varepsilon^{klm} \nabla_l A_m, \tag{A12}$$

valid for any vector field A^i, these equations can then be rewritten as

$$0 = -\nabla_i\left(\frac{1}{2} v_{\rm n}^k p_k^{\rm n} + \mu^{\rm n}\right) + \varepsilon_{ijk} v_{\rm n}^j \varepsilon^{klm} \nabla_l p_m^{\rm n}, \tag{A13}$$

$$0 = -\nabla_i\left(\frac{1}{2} v_{\rm p}^k p_k^{\rm p} + \mu^{\rm p}\right) + \varepsilon_{ijk} v_{\rm p}^j \varepsilon^{klm} \nabla_l p_m^{\rm p} + e\left(E_i + \frac{1}{c}\varepsilon_{ijk} v_{\rm p}^j B^k\right). \tag{A14}$$

Using equation (21), which can be recast as

$$p_0^x = -\mu^x - m_x v_x^2/2 = -\mu^x - v_x^k p_k^x/2, \tag{A15}$$

in the absence of entrainment effects, equations (A13) and (A14) are found to be equivalent to (the opposite of) equations (A5) and (A6). The irrotationality conditions (34) therefore lead to equation (A1), as in Appendix A1.

APPENDIX B: VORTEX VELOCITY

In a frame where the neutron vortex and the $N_{\rm p}$ fluxoids pinned to it move at the velocity $v_{\rm L}^i$, the total force per unit length (55) acting on the quantized lines reads

$$\mathcal{F}_i = -\bar{\rho}_{\rm n}\kappa_{\rm n}\varepsilon_{ijk}\hat{z}^j\left(\bar{v}_{\rm n}^k - v_{\rm L}^k\right) - \bar{\rho}_{\rm p} N_{\rm p}\kappa_{\rm p}\varepsilon_{ijk}\hat{z}^j\left(\bar{v}_{\rm p}^k - v_{\rm L}^k\right) + D_{\rm e}\left(\bar{v}_{{\rm p}\,i} - v_{{\rm L}\,i}\right). \tag{B1}$$

Neglecting the masses of the quantized lines (as in previous studies), the force balance equation $\mathcal{F}_i = 0$ leads to

$$v_{\rm L}^i = \bar{v}_{\rm p}^i - \frac{1}{\xi}\varepsilon^{ijk}\hat{z}_j\left(\bar{v}_{{\rm n}\,k} - v_{{\rm L}\,k}\right) - \frac{X}{\xi}\varepsilon^{ijk}\hat{z}_j\left(\bar{v}_{{\rm p}\,k} - v_{{\rm L}\,k}\right), \tag{B2}$$

where ξ and X are given by equation (58). Projecting this latter equation along $\varepsilon_{ijk}\hat{z}^j$ yields

$$\varepsilon_{ijk}\hat{z}^j v_{\rm L}^k = \varepsilon_{ijk}\hat{z}^j \bar{v}_{\rm p}^k - \frac{1}{\xi}\varepsilon_{ijk}\hat{z}^j\varepsilon^{klm}\hat{z}_l\left(\bar{v}_{{\rm n}\,m} - v_{{\rm L}\,m}\right) - \frac{X}{\xi}\varepsilon_{ijk}\hat{z}^j\varepsilon^{klm}\hat{z}_l\left(\bar{v}_{{\rm p}\,m} - v_{{\rm L}\,m}\right) \tag{B3}$$

$$= \varepsilon_{ijk}\hat{z}^j\bar{v}_{\rm p}^k + \frac{1}{\xi}\left(\bar{v}_{{\rm n}\,i} - v_{{\rm L}\,i}\right) + \frac{X}{\xi}\left(\bar{v}_{{\rm p}\,i} - v_{{\rm L}\,i}\right), \tag{B4}$$

where we have used the fact that $\varepsilon_{ijk}\varepsilon^{klm} = \delta_i^l\delta_j^m - \delta_i^m\delta_j^l$, δ_j^i being the Kronecker delta, and we have only considered velocities orthogonal to $\hat{z}^i$. Using equation (B4) in the right-hand side of equation (B2) now leads to

$$v_{\rm L}^i\left[1 + \frac{(1+X)^2}{\xi^2}\right] = \bar{v}_{\rm p}^i\left[1 + \frac{X\,(1+X)}{\xi^2}\right] + \bar{v}_{\rm n}^i\frac{1+X}{\xi^2} + \frac{1}{\xi}\varepsilon^{ijk}\hat{z}_j\left(\bar{v}_{{\rm p}\,k} - \bar{v}_{{\rm n}\,k}\right), \tag{B5}$$

which can be eventually recast in the equivalent forms (56).

This paper has been typeset from a TEX/LATEX file prepared by the author.