



**HAL**  
open science

# Orbital stability vs. scattering in the cubic-quintic Schrodinger equation

Rémi Carles, Christof Sparber

► **To cite this version:**

Rémi Carles, Christof Sparber. Orbital stability vs. scattering in the cubic-quintic Schrodinger equation. 2020. hal-02475123v2

**HAL Id: hal-02475123**

**<https://hal.science/hal-02475123v2>**

Preprint submitted on 25 Apr 2020 (v2), last revised 21 May 2020 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ORBITAL STABILITY VS. SCATTERING IN THE CUBIC-QUINTIC SCHRÖDINGER EQUATION

RÉMI CARLES AND CHRISTOF SPARBER

ABSTRACT. We consider the cubic-quintic nonlinear Schrödinger equation in space dimension up to three. The cubic nonlinearity is thereby focusing while the quintic one is defocusing, ensuring global well-posedness of the Cauchy problem in the energy space. The main goal of this paper is paint a more or less complete picture of dispersion and orbital (in-)stability of solitary waves, emanating from nonlinear ground states. In space dimension one, it is already known that solitons are orbitally stable. Here, we establish the analogous result in dimension two. In addition, we show that if the initial data belong to the conformal space and have at most the mass of the ground state for the cubic two-dimensional Schrödinger equation, then the solution is asymptotically linear. Finally, in dimension three, relying on some previous results from other authors, we show that solitons may or may not be orbitally stable.

## 1. INTRODUCTION AND MAIN RESULTS

1.1. **Basic setting.** We consider the nonlinear Schrödinger equation (NLS) with competing cubic-quintic nonlinearities,

$$(1.1) \quad i\partial_t u + \frac{1}{2}\Delta u = -|u|^2 u + |u|^4 u, \quad x \in \mathbb{R}^d,$$

in space dimension  $d \leq 3$ . The quintic nonlinearity was introduced in [38], and adopted in several physical situations: typically in optics (see e.g. [28]), or in Bose-Einstein condensation (e.g. [1, 17, 35]). We refer to the review [32] for more precise references. In particular, the incorporation of the defocusing quintic term is motivated by the stabilization of two- and three-dimensional vortex solitons.

Recall some of basic features of this nonlinearity in terms of criticality for the Cauchy problem. Depending on the space dimension, the NLS is seen to be:

- $d = 1$ : focusing  $L^2$ -subcritical plus defocusing  $L^2$ -critical (and  $H^1$ -subcritical).
- $d = 2$ : focusing  $L^2$ -critical plus defocusing  $L^2$ -supercritical (and  $H^1$ -subcritical).
- $d = 3$ : focusing  $L^2$ -supercritical plus defocusing  $H^1$ -critical.

It is already known from the case of more general, gauge-invariant nonlinearities (see e.g. [6]), that equation (1.1) formally enjoys three basic conservation laws, namely:

- Mass:  $M(u) = \|u(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2$ ,
- Angular momentum:  $J(u) = \text{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) \nabla u(t, x) dx$ ,
- Energy:  $E(u) = \frac{1}{2} \|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{2} \|u(t, \cdot)\|_{L^4(\mathbb{R}^d)}^4 + \frac{1}{3} \|u(t, \cdot)\|_{L^6(\mathbb{R}^d)}^6$ .

In dimensions 2 and 3, an effect of the quintic term is to *prevent finite time blow-up* which may occur in the purely cubic case (cf. [6]). Indeed, the conservation of the energy, combined with Hölder's inequality,

$$(1.2) \quad \|u\|_{L^4(\mathbb{R}^d)}^4 \leq \|u\|_{L^2(\mathbb{R}^d)} \|u\|_{L^6(\mathbb{R}^d)}^3,$$

---

RC is supported by Rennes Métropole through its AIS program. CS acknowledges support by the NSF through grant no. DMS-1348092.

shows that the cubic focusing part cannot be an obstruction to global well-posedness, at least in  $H^1$ . For  $d \leq 2$ , global well-posedness then follows from classical results (see e.g. [6]). For  $d = 3$ , we refer to [50], as the quintic term is energy-critical.

**Proposition 1.1** (Global well-posedness). *Let  $d \leq 3$ . For any  $u_0 \in H^1(\mathbb{R}^d)$ , (1.1) has a unique global solution  $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$  such that  $u|_{t=0} = u_0$ . The solution obeys the conservation of mass, energy, and momentum. If in addition*

$$u_0 \in \Sigma := \{f \in H^1(\mathbb{R}^d), x \mapsto |x|u_0(x) \in L^2(\mathbb{R}^d)\},$$

then  $u \in C(\mathbb{R}; \Sigma)$ .

Numerically, one observes a kind of oscillatory behavior within the solution  $u$ , which is due to the competition of focusing and defocusing effects within (1.1), cf. [43] for more details.

*Remark 1.2.* Recall that in dimension  $d = 2$  or  $3$ , the quintic term is  $L^2$ -supercritical, so we cannot hope to solve the Cauchy problem at this regularity level. Moreover, since the cubic-quintic nonlinearity stems from physics, it is sensible to work in  $H^1$ , where the energy is well-defined.

Complementing the case of prescribed initial data, we may also want to prescribe asymptotic states (or scattering states) and an asymptotically linear behavior, provided that  $d \geq 2$ . We thereby recall that in the case  $d = 1$ , the cubic nonlinearity causes long-range effects, and no non-trivial solution to (1.1) can be asymptotically linear, cf. [3]. However, in dimensions  $d = 2, 3$  one can rely on classical techniques (see e.g. [6]) or the results of [50], respectively, to obtain:

**Proposition 1.3** (Scattering). *Let  $d = 2$  or  $3$ . For any  $u_- \in H^1(\mathbb{R}^d)$ , (1.1) has a unique global solution  $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$  such that*

$$\left\| u(t, \cdot) - e^{i\frac{t}{2}\Delta} u_- \right\|_{H^1(\mathbb{R}^d)} \xrightarrow{t \rightarrow -\infty} 0.$$

In particular,

$$M(u) = \|u_-\|_{L^2}^2, \quad E(u) = \frac{1}{2} \|\nabla u_-\|_{L^2}^2, \quad \forall t \in \mathbb{R}.$$

If in addition  $u_- \in \Sigma$ , then  $u \in C(\mathbb{R}; \Sigma)$  and

$$\left\| e^{-i\frac{t}{2}\Delta} u(t, \cdot) - u_- \right\|_{\Sigma} \xrightarrow{t \rightarrow -\infty} 0.$$

We recall that  $e^{i\frac{t}{2}\Delta}$  is unitary on  $H^1(\mathbb{R}^d)$ , but not on  $\Sigma$  (see e.g. [6]), hence the final formula above.

As in the case with purely cubic nonlinearity, not every finite-energy solution of (1.1) is necessarily asymptotically linear. Finite time blow-up is of course ruled out in our case, but time-periodic solitary wave solutions also exist.

**Definition 1.4.** A *standing wave* or *soliton* of (1.1) is a solution of the form  $e^{i\omega t} \phi(x)$ , with  $\omega \in \mathbb{R}$  and  $\phi$  satisfying

$$(1.3) \quad -\frac{1}{2}\Delta\phi + \omega\phi - |\phi|^2\phi + |\phi|^4\phi = 0, \quad \phi \in H^1(\mathbb{R}^d) \setminus \{0\}.$$

The associated action is given by

$$S(\phi) = \frac{1}{2} \|\nabla\phi\|_{L^2}^2 + \omega \|\phi\|_{L^2}^2 - \frac{1}{2} \|\phi\|_{L^4}^4 + \frac{1}{3} \|\phi\|_{L^6}^6.$$

A solution  $\phi$  is a *ground state* if  $S(\phi) \leq S(\varphi)$  for any solution  $\varphi$  of (1.3).

As we will see in Section 3, if  $d \leq 3$ , (1.3) admits a solution if and only if

$$0 < \omega < \frac{3}{16}.$$

It turns out that for  $d = 1$ , explicit solitary wave solutions are available for this range of  $\omega$ , see below. In the present paper, we are mostly interested in the *stability* of solitary waves, a question which is closely related to dispersive effects in (1.1).

**Definition 1.5.** Let  $\phi$  be a solution of (1.3). The standing wave  $e^{i\omega t}\phi(x)$  is called *orbitally stable* in  $H^1(\mathbb{R}^d)$ , if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $u_0 \in H^1(\mathbb{R}^d)$  satisfies

$$\|u_0 - \phi\|_{H^1} \leq \delta,$$

then the solution to (1.1) with  $u|_{t=0} = u_0$  satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}^d}} \|u(t, \cdot) - e^{i\theta} \phi(\cdot - y)\|_{H^1(\mathbb{R}^d)} \leq \varepsilon.$$

Otherwise, the standing wave is said to be unstable.

In this paper, we will try to give a more or less complete description of the long time behavior of solutions to (1.1) in terms of scattering versus orbital (in-)stability of solitary waves, depending on the spatial dimension  $d \leq 3$ .

**1.2. One-dimensional case.** In the case  $d = 1$ , the overall picture is very neat. Firstly, for  $0 < \omega < \frac{3}{16}$ , solutions to (1.3) are given by ([38], see also [12])

$$(1.4) \quad \phi(x) = 2 \sqrt{\frac{\omega}{1 + \sqrt{1 - \frac{16\omega}{3}} \cosh(2x\sqrt{2\omega})}}.$$

Note that in view of [5], this real-valued solution is unique, up to translation and change of sign. The orbital stability of these nonlinear ground states was established in [36, Theorem 3, case (1)].

**Proposition 1.6** (Orbital stability in 1D). *Let  $d = 1$ , and  $0 < \omega < \frac{3}{16}$ . The solitary wave  $e^{i\omega t}\phi(x)$ , where  $\phi$  is given by (1.4), is orbitally stable.*

The proof of this result combines the the well-known *Grillakis-Shatah-Strauss criterion* [19] with the analysis of [22] and an explicit formula for second order ODEs without first order derivatives, a strategy which seems to be restricted to the 1D case and not suited for solutions to (1.3) in  $d \geq 2$ .

**1.3. Two-dimensional case.** We now turn to the case  $d = 2$  and recall that the results of [44] show that for  $\|u_0\|_{L^2}$  *sufficiently small*, the solution to (1.1) is asymptotically linear. It turns out that since the cubic term is  $L^2$ -critical in 2D, we can in fact be more precise.

To this end, let  $Q$  be the *cubic nonlinear ground state*, i.e., the unique positive radial solution to

$$(1.5) \quad -\frac{1}{2}\Delta Q + Q - Q^3 = 0, \quad x \in \mathbb{R}^2.$$

In view of [47], and noting that we have an extra factor  $\frac{1}{2}$  in front of the Laplacian in (1.5) compared to [47], the sharp Gagliardo-Nirenberg inequality reads

$$(1.6) \quad \|u\|_{L^4(\mathbb{R}^2)}^4 \leq \left( \frac{\|u\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}} \right)^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2, \quad \forall u \in H^1(\mathbb{R}^2).$$

In the focusing cubic case, i.e., without the quintic term, we know from [14] that if  $u_0 \in L^2$  and  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , global existence and scattering hold (see also [27] for the case of radial data  $u_0$ ). In the presence of (1.1), it was proved in [8] that for  $u_0 \in H^1$  with  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , scattering holds as well, relying on the cubic

case from [14]. In our first main result below, we shall show that the effect of the additional quintic term is not only to guarantee global well-posedness, but also to extend this dispersive result to the  $L^2$ -sphere  $\{\|u_0\|_{L^2} = \|Q\|_{L^2}\}$ . We emphasize that we assume  $u_0 \in \Sigma$ , not simply  $u_0 \in H^1(\mathbb{R}^2)$ , see Section 2 for a more precise discussion of this aspect.

**Theorem 1.7** (Mass (sub-)critical scattering in 2D). *Let  $d = 2$ . If  $u_0 \in \Sigma$  with*

$$\|u_0\|_{L^2} \leq \|Q\|_{L^2},$$

*then the solution  $u \in C(\mathbb{R}; \Sigma)$  to (1.1) such that  $u|_{t=0} = u_0$  is asymptotically linear, i.e. there exist  $u_{\pm} \in \Sigma$  such that*

$$\|e^{-i\frac{t}{2}\Delta}u(t, \cdot) - u_{\pm}\|_{\Sigma} \xrightarrow{t \rightarrow \pm\infty} 0.$$

On a heuristic level, we may argue in the same fashion as in [21], and recall that the standard virial computation for (1.1) yields,

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx = 2E(u) + \frac{4}{3} \|u(t)\|_{L^6(\mathbb{R}^2)}^6 \geq 2E(u_0),$$

where  $E(u) = E(u_0)$  is the conserved energy. In view of the sharp Gagliardo-Nirenberg inequality, we have, under the assumptions of Theorem 1.7,

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx \geq \frac{2}{3} \|u_0\|_{L^6(\mathbb{R}^2)}^6.$$

The time-derivative of the virial of  $u$  is therefore increasing, a first hint that the solution is dispersive. In order to make this statement rigorous, especially in the limiting case  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , we rely on a conformal transform, and rigidity results regarding the concentration phenomenon in nonlinear Schrödinger equations.

Our second main result concerns the stability of solitary waves:

**Theorem 1.8** (Nonlinear ground states in 2D). *Let  $d = 2$ . Then, for all  $\omega \in ]0, \frac{3}{16}[$ , there exists a solitary wave solution  $u(t, x) = e^{i\omega t} \phi(x)$  to (1.1). In addition, we have:*

- (1) *For any  $M > \|Q\|_{L^2}^2$ , there exists a ground state such that  $\|\phi\|_{L^2}^2 = M$ .*
- (2) *The ground state solution is unique, up to translation and multiplication by  $e^{i\theta}$ , for constant  $\theta \in \mathbb{R}$ .*
- (3) *If  $\phi$  is a (real-valued) ground state solution to*

$$(1.7) \quad -\frac{1}{2} \Delta \phi - \phi^3 + \phi^5 + \omega \phi = 0,$$

*then the associated solitary wave is orbitally stable.*

We emphasize the fact that for any mass *strictly larger* than that of the cubic ground state  $Q$ , we can find a soliton of the cubic-quintic NLS, while for a mass less or equal to that of  $Q$ , all solutions to (1.1) are asymptotically linear. This is in sharp contrast with the analogous situation in the case of a single pure power nonlinearity, where the critical sphere (in  $L^2$  or other homogeneous Sobolev spaces) *always contains non-dispersive elements*, see e.g. [15, 24, 25, 45].

**1.4. Three-dimensional case.** In  $d = 3$ , equation (1.1) has already been studied in [26]. However, no statement concerning the (in-)stability of solitary waves is given in there. Here, we shall state the following proposition, the proof of which relies on elements already present in [26]:

**Proposition 1.9** (Soliton (in-)stability in 3D). *Let  $d = 3$ . For all  $\omega \in ]0, \frac{3}{16}[$ , there exists a ground state solution which is unique, up to translation and multiplication by  $e^{i\theta}$ , for constant  $\theta \in \mathbb{R}$ . Moreover:*

- There exists  $0 < \omega_0 < \frac{3}{16}$  such that for all  $0 < \omega < \omega_0$ , the associated soliton is orbitally stable.
- There exists  $\omega_0 \leq \omega_1 < \frac{3}{16}$  such that for all  $\omega_1 < \omega < \frac{3}{16}$ , the associated soliton is unstable.

One expects the equality  $\omega_0 = \omega_1$  to hold. More precisely, Conjecture 2.3 from [26], which is supported by numerics, states:

**Conjecture** ([26]). There exists  $0 < \omega_* < \frac{3}{16}$  so that  $\omega \mapsto M(\phi)$  is strictly decreasing for  $\omega < \omega_*$ , and strictly increasing for  $\omega > \omega_*$ .

If this indeed holds true, one can take  $\omega_0 = \omega_1 = \omega_*$  in Proposition 1.9, in view of Grillakis-Shatah-Strauss theory [19] (see also [13]).

Somewhat independently from this question, one may wonder about the precise nature of instability. Recall, that in the case of a *single* power nonlinearity, instability is always due to the possibility of finite-time blow-up (see e.g. [6] and references therein). Very recently, Fukuya and Hayashi [16] have established instability results for NLS with a double power nonlinearity, but in their work the *focusing term dominates* the defocusing one (thereby extending the results of [9]). They rely on the possibility of blow-up or invoke the Grillakis-Shatah-Strauss theory, in which case the nature of the instability still remains unclear. For nonlinearly coupled *systems* of NLS, Correia, Oliveira and Silva [11] have shown that instability may correspond to a transfer of mass from one equation to the other. None of these former results, however, apply to our situation. For the latter, one may expect that the stable manifolds analyzed in [29, 40] become open neighborhoods, in the sense that in a full neighborhood of the unstable ground state (not only in a manifold with limited co-dimension), the solution  $u$  bifurcates from the solitary wave  $e^{i\omega t}\phi(x)$ , yielding a behavior of the form

$$u(t, x) = W(t, x) + e^{i\frac{t}{2}\Delta}u_+(x) + o_{L^2}(1) \quad \text{as } t \rightarrow \infty,$$

for some  $u_+ \in L^2(\mathbb{R}^3)$ , and where  $W$  is a (possibly different) ground state, modulated by a moving set of parameters (see [29, 40] for details).

The rest of this paper is now organized as follows: In Section 2, we prove Theorem 1.7. In Section 3, we analyze some general results on solitary waves for (1.1), with emphasis on some special properties in the 2D case. Theorem 1.8 is proven in Section 4, and we present the main arguments for Proposition 1.9 in an appendix.

## 2. DISPERSIVE BEHAVIOR IN 2D

**2.1. Space-time norms.** In this section, our main goal is to prove Theorem 1.7. Recall that for two-dimensional Schrödinger equation, a *Strichartz-pair*  $(q, r)$  is *admissible* if

$$\frac{2}{q} + \frac{2}{r} = 1, \quad 2 \leq r < \infty.$$

We denote by

$$\|u\|_{S(I)} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q(I; L^r(\mathbb{R}^2))}.$$

In view of [44, Theorem 1.3], to prove Theorem 1.7 with  $\Sigma$  replaced by the larger space  $H^1(\mathbb{R}^2)$ , it suffices to prove that for any  $u_0 \in H^1(\mathbb{R}^2)$  with  $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ , the global solution  $u$  provided by Proposition 1.1 satisfies

$$(2.1) \quad \|u\|_{S(\mathbb{R})} + \|\nabla u\|_{S(\mathbb{R})} < \infty.$$

*Remark 2.1.* Note that Theorem 1.8 contains the particular information that one can find  $u_0 \in H^1(\mathbb{R}^2)$  with  $\|u_0\|_{L^2} - \|Q\|_{L^2} > 0$  arbitrarily small, such that  $\|u\|_{S(\mathbb{R})} = \infty$ .

As a first, basic step, we show that (2.1) can be reduced to the following:

**Lemma 2.2** (Reduction step). *Let  $d = 2$  and  $u_0 \in H^1(\mathbb{R}^2)$ . If the global solution provided by Proposition 1.1 satisfies  $\|u\|_{S(\mathbb{R})} < \infty$ , then we also have*

$$\|\nabla u\|_{S(\mathbb{R})} < \infty,$$

and so  $u$  is asymptotically linear,

$$\exists u_{\pm} \in H^1(\mathbb{R}^2), \quad \left\| u(t) - e^{i\frac{t}{2}\Delta} u_{\pm} \right\|_{H^1(\mathbb{R}^2)} \xrightarrow{t \rightarrow \pm\infty} 0.$$

If in addition  $u_0 \in \Sigma$ , then  $u_{\pm} \in \Sigma$  and

$$\left\| e^{-i\frac{t}{2}\Delta} u(t) - u_{\pm} \right\|_{H^1(\mathbb{R}^2)} \xrightarrow{t \rightarrow \pm\infty} 0.$$

*Proof.* From [44], we only have to check that  $\|u\|_{S(\mathbb{R})} < \infty$  implies  $\|\nabla u\|_{S(\mathbb{R})} < \infty$ . Let  $I = [t_0, t_1]$  be some time interval, with  $t_1 \geq t_0 \geq 0$  to simplify notations. Considering the Duhamel's formula associated to (1.1), taking the gradient and applying Strichartz estimates, we find

$$\|\nabla u\|_{S(I)} \lesssim \|\nabla u(t_0)\|_{L^2} + \|u^2 \nabla u\|_{L^{4/3}(I \times \mathbb{R}^2)} + \|u^4 \nabla u\|_{L^{3/2}(I; L^{6/5})},$$

where we have considered the specific admissible pairs (4, 4) and (3, 6) for the cubic and quintic nonlinearities, respectively. Recall that we already know that  $u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2))$ , so the first term on the right hand side is bounded uniformly in time. Write

$$\frac{3}{4} = \frac{2}{4} + \frac{1}{4}, \quad \frac{2}{3} = \frac{4}{12} + \frac{1}{3}, \quad \frac{5}{6} = \frac{4}{6} + \frac{1}{6},$$

in which case, Hölder's inequality yields

$$\begin{aligned} \|\nabla u\|_{S(I)} &\lesssim 1 + \|u\|_{L^4(I \times \mathbb{R}^2)}^2 \|\nabla u\|_{L^4(I \times \mathbb{R}^2)} + \|u\|_{L^{12}(I; L^6)}^4 \|\nabla u\|_{L^3(I; L^6)} \\ &\lesssim 1 + \|u\|_{L^4(I \times \mathbb{R}^2)}^2 \|\nabla u\|_{S(I)} + \|u\|_{L^{12}(I; L^6)}^4 \|\nabla u\|_{S(I)}. \end{aligned}$$

Recalling again that  $u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2))$ ,

$$\|u\|_{L^{12}(I; L^6)}^4 \leq \|u\|_{L^\infty(I; L^6)}^3 \|u\|_{L^3(I; L^6)} \lesssim \|u\|_{L^\infty(I; H^1)}^3 \|u\|_{S(I)} \leq C \|u\|_{S(I)}.$$

Now since  $\|u\|_{S(\mathbb{R})} < \infty$ , we can split  $\mathbb{R}_t$  into finitely many intervals on which the nonlinear terms in the above estimate can be absorbed by the left hand side, so we conclude  $\|\nabla u\|_{S(\mathbb{R})} < \infty$ .

Now if  $u_0 \in \Sigma$ , we introduce the Galilean operator

$$J(t) = x + it\nabla,$$

which commutes with the free Schrödinger operator, i.e.

$$[J, i\partial_t + \frac{1}{2}\Delta] = 0.$$

Moreover,

$$(2.2) \quad J(t)u = it e^{i|x|^2/(2t)} \nabla \left( u e^{-i|x|^2/(2t)} \right),$$

which implies that  $J(t)u$  can be estimated like  $\nabla u$  above. Thus  $\|Ju\|_{S(\mathbb{R})} < \infty$  and scattering in  $\Sigma$  follows along the same lines as scattering in  $H^1$  (see e.g. [6]).  $\square$

Next, in order to prove  $\|u\|_{S(\mathbb{R})} < \infty$  and thus (2.1), we shall in the following distinguish between the case of subcritical mass, i.e.  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , and the critical case, where  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ .

**2.2. Mass subcritical case.** In this subsection, we suppose

$$(2.3) \quad \|u_0\|_{L^2}^2 = (1 - \eta)\|Q\|_{L^2}^2, \quad \text{for some } 0 < \eta < 1.$$

Since  $u_0 \in \Sigma$ , we can rely on the pseudo-conformal conservation law (derived initially in [18], see also [6]):

$$(2.4) \quad \frac{d}{dt} \left( \frac{1}{2} \|J(t)u\|_{L^2}^2 - \frac{t^2}{2} \|u\|_{L^4}^4 + \frac{t^2}{3} \|u\|_{L^6}^6 \right) = -\frac{2t}{3} \|u\|_{L^6}^6.$$

In view of (2.2), we can rewrite

$$\|J(t)u\|_{L^2}^2 = t^2 \left\| \nabla \left( u e^{-i|x|^2/(2t)} \right) \right\|_{L^2}^2.$$

The sharp Gagliardo–Nirenberg inequality (1.6) when applied to  $u(t, x) e^{-i|x|^2/(2t)}$ , then yields, together with (2.3), that

$$\begin{aligned} \|J(t)u\|_{L^2}^2 - t^2 \|u\|_{L^4}^4 &\geq \|J(t)u\|_{L^2}^2 - (1 - \eta) \|J(t)u\|_{L^2}^2 \\ &= \eta \|J(t)u\|_{L^2}^2. \end{aligned}$$

Hence, the pseudo-conformal conservation law implies

$$J(t)u \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2)).$$

Invoking (2.2) and general Gagliardo–Nirenberg inequalities, for  $2 \leq r < \infty$ ,

$$(2.5) \quad \|u(t)\|_{L^r(\mathbb{R}^2)} \lesssim \|u(t)\|_{L^2(\mathbb{R}^2)}^{1-\theta} \left( \frac{1}{t} \|J(t)u\|_{L^2} \right)^\theta, \quad \theta = 1 - \frac{2}{r},$$

we infer  $u \in L^q(\mathbb{R}; L^r(\mathbb{R}^2))$  for all admissible pairs, i.e.  $\|u\|_{S(\mathbb{R})} < \infty$ .

It was proved very recently in [8], that the assumption  $u_0 \in \Sigma$  can be relaxed to  $u_0 \in H^1(\mathbb{R}^2)$ , by combining the Kenig–Merle roadmap ([24], see also [39]) with a profile decomposition in  $L^2$ . Such a decomposition arises naturally, even though the nonlinearity is not homogeneous. Indeed, if the solution  $u$  is dispersive, with fixed  $L^2(\mathbb{R}^2)$  norm, we expect it to behave like

$$|u(t, x)| \underset{t \rightarrow \infty}{\sim} \frac{1}{\lambda(t)} U \left( \frac{x}{\lambda(t)} \right),$$

with some fixed profile  $U$  and some scaling factor  $\lambda(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . In the case of scattering (i.e., asymptotically linear behavior), we have  $\lambda(t) \approx t$ . Equivalence, however, is difficult to establish. In general, for  $\lambda(t) \rightarrow \infty$ , we can only infer that the quintic nonlinearity for large times  $t$  becomes negligible when compared to the cubic one. The large time stability estimate in [8] relies precisely on the solution to

$$(2.6) \quad i\partial_t v + \frac{1}{2}\Delta v = -|v|^2 v,$$

considered as a reference solution. Since this equation is  $L^2$ -critical, we can follow [34] and expect a profile decomposition to arise at the  $L^2$ -level (as opposed to  $\dot{H}^s$ -level for  $s > 0$ , see, e.g. [15, 21, 24]). The solution to (2.6) is guaranteed to be global in time and asymptotically linear if  $\|v\|_{L^2} < \|Q\|_{L^2}$ . We emphasize that this property, conjectured for a long time, is actually hard to prove (see [14], and [37] for the historical perspective), and, hence, moving from the assumption  $u_0 \in \Sigma$  to  $u_0 \in H^1(\mathbb{R}^2)$  requires lots of technicalities.

In the case  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , the solution to the focusing cubic Schrödinger equation may develop singularities, so the approach of [8] seems doomed. Supposing that we know that  $u$  is dispersive, a critical aspect in our analysis below is to prove that  $\lambda(t) \approx t$ .



**2.3. Mass critical case.** Let now  $u_0 \in \Sigma$  with  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ . The pseudo-conformal conservation law yields

$$(2.7) \quad \underbrace{\frac{1}{2}\|J(t)u\|_{L^2}^2 - \frac{t^2}{2}\|u\|_{L^4}^4 + \frac{t^2}{3}\|u\|_{L^6}^6}_{\geq 0 \text{ from (1.6) and (2.2)}} = \frac{1}{2}\|xu_0\|_{L^2}^2 - \int_0^t \frac{2s}{3}\|u(s, \cdot)\|_{L^6}^6 ds,$$

hence

$$(2.8) \quad \|u(t, \cdot)\|_{L^6}^6 \lesssim \frac{1}{1+t^2} \text{ and } \int_0^\infty t\|u(t, \cdot)\|_{L^6}^6 dt < \infty.$$

This does not rule out a behavior of the form

$$\|u(t, \cdot)\|_{L^6}^6 \approx \frac{1}{t^2(\log t)^2} \text{ as } t \rightarrow \infty,$$

in which case  $u \notin L_t^3 L_x^6$  (recall that (3, 6) is an admissible pair). In other words, a direct use of the pseudo-conformal conservation law seems hopeless in the mass critical case, since we cannot access a convenient bound on  $\|J(t)u\|_{L^2}$ . In fact, we do not even have a moderate growth of this quantity, like  $\mathcal{O}(t^\gamma)$  for some  $\gamma < 1/2$ , as was exploited in [46].

*Remark 2.3.* Note that if the defocusing nonlinearity was weaker, for instance quartic,

$$i\partial_t u + \frac{1}{2}\Delta u = -|u|^2 u + |u|^3 u, \quad x \in \mathbb{R}^2,$$

then the same approach as above would yield

$$\|u(t, \cdot)\|_{L^5}^5 \lesssim \frac{1}{1+t^2},$$

and so  $u \in L_t^{10/3} L_x^5$ , an admissible pair. We could then proceed as in the subcritical mass case.

We emphasize that in view of (2.5), it suffices to show  $Ju \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2))$ , which is actually a stronger property than  $\|u\|_{S(\mathbb{R})} < \infty$ . Suppose on the contrary that

$$(2.9) \quad \|J(t_n)u\|_{L^2(\mathbb{R}^2)} \xrightarrow{n \rightarrow \infty} \infty \text{ for some } t_n \rightarrow \infty.$$

Consider  $\psi$  given by

$$(2.10) \quad \psi(t, x) = \frac{1}{t} u\left(\frac{-1}{t}, \frac{x}{t}\right) e^{i|x|^2/(2t)}, \quad t \neq 0.$$

As is well-known, this transform exchanges the large time and finite time régimes. In the case of the  $L^2$ -critical nonlinear Schrödinger equation,  $u$  and  $\psi$  solve the same equation. In the present case,  $\psi$  solves the non-autonomous equation

$$i\partial_t \psi + \frac{1}{2}\Delta \psi = -|\psi|^2 \psi + t^2 |\psi|^4 \psi.$$

In view of (2.2) and (2.9),

$$(2.11) \quad \|\nabla \psi(\tau_n, \cdot)\|_{L^2(\mathbb{R}^2)} = \left\| J\left(\frac{-1}{\tau_n}\right) u \right\|_{L^2(\mathbb{R}^2)} \xrightarrow{n \rightarrow \infty} \infty, \quad \tau_n := \frac{-1}{\tau_n} \xrightarrow{n \rightarrow \infty} 0^-.$$

We obviously have  $\|\psi(t)\|_{L^2} = \|u_0\|_{L^2} = \|Q\|_{L^2}$ . We use this property to rely on fine rigidity results associated to the  $L^2$ -critical nonlinear Schrödinger equation (with critical mass), as first proved in [33], and revisited in [20]. More precisely, as will be proven below, (2.9) implies

$$\rho_n e^{i\theta_n} \psi(\tau_n, \rho_n x + x_n) \xrightarrow{n \rightarrow \infty} Q(x) \text{ in } H^1(\mathbb{R}^2),$$

for some  $\rho_n, \theta_n \in \mathbb{R}$  and  $x_n \in \mathbb{R}^2$ . The crucial aspect is that this convergence is strong in  $H^1(\mathbb{R}^2)$ . We then show that  $|\rho_n| \lesssim |\tau_n|$ . This behavior rules out the first part of (2.8), and hence (2.9) cannot hold.

**Step 1. A-priori estimates.** The idea of this proof by contradiction is to rely on fine properties established for mass-critical, blowing-up solutions in the case of  $L^2$ -critical nonlinear Schrödinger equations. We start with the fact that, in view of (2.7),

$$0 \leq \|J(t)u\|_{L^2}^2 - t^2 \|u(t)\|_{L^4}^4 \leq \|xu_0\|_{L^2}^2.$$

Hence, in terms of  $\psi$ , defined via (2.10), we have

$$(2.12) \quad 0 \leq \|\nabla\psi(t)\|_{L^2}^2 - \|\psi(t)\|_{L^4}^4 \leq \|xu_0\|_{L^2}^2.$$

Next, we recall standard estimates on the virial quantity: The identity

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx = 2 \operatorname{Im} \int_{\mathbb{R}^2} \bar{u}(t, x) x \cdot \nabla u(t, x) dx,$$

together with Cauchy-Schwarz and the boundedness  $u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2))$  yields

$$\int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx \lesssim 1 + |t|, \quad \forall t \in \mathbb{R}.$$

In turn, (2.10) implies

$$(2.13) \quad \int_{\mathbb{R}^2} |x|^2 |\psi(t, x)|^2 dx \lesssim 1, \quad \forall t \in [-1, 0[.$$

Recall that the (conserved) energy associated to the focusing cubic Schrödinger equation

$$i\partial_t v + \frac{1}{2}\Delta v = -|v|^2 v,$$

is

$$E_{\text{cub}}(v) = \|\nabla v\|_{L^2}^2 - \|v\|_{L^4}^4.$$

This will be used to prove a-priori estimates on truncated versions of the virial quantity. Following [20], we introduce a nonnegative radial  $\chi \in C_0^\infty(\mathbb{R}^2)$ , such that

$$\chi(x) = |x|^2 \quad \text{if } |x| < 1, \quad |\nabla\chi(x)|^2 \lesssim \chi(x),$$

and for  $\underline{x} \in \mathbb{R}^2$  and every  $p \in \mathbb{N}^*$ , we define

$$\chi_{p, \underline{x}}(x) = p^2 \chi\left(\frac{x - \underline{x}}{p}\right), \quad g_{p, \underline{x}}(t) = \int_{\mathbb{R}^2} \chi_{p, \underline{x}}(x) |\psi(t, x)|^2 dx.$$

The function  $g_{p, \underline{x}}$  is a truncated virial, centered at point  $\underline{x} \in \mathbb{R}^2$ . A computation shows that

$$(2.14) \quad \dot{g}_{p, \underline{x}}(t) = 2 \operatorname{Im} \int_{\mathbb{R}^2} \bar{\psi}(t, x) \nabla \chi_{p, \underline{x}}(x) \cdot \nabla \psi(t, x) dx.$$

We now invoke an argument introduced by Valeria Banica in [2]: The fact that  $\|\psi\|_{L^2} = \|Q\|_{L^2}$  and (1.6) imply that for every  $t \in [-1, 0[$ ,

$$E_{\text{cub}}(e^{is\chi_{p, \underline{x}}}\psi(t)) \geq 0, \quad \forall s \in \mathbb{R}.$$

The above quantity is a polynomial of degree two in  $s$ ,

$$\begin{aligned} E_{\text{cub}}(e^{is\chi_{p, \underline{x}}}\psi(t)) &= E_{\text{cub}}(\psi(t)) - 2s \operatorname{Im} \int_{\mathbb{R}^2} \bar{\psi}(t, x) \nabla \chi_{p, \underline{x}}(x) \cdot \nabla \psi(t, x) dx \\ &\quad + s^2 \int_{\mathbb{R}^2} |\nabla \chi_{p, \underline{x}}(x)|^2 |\psi(t, x)|^2 dx, \end{aligned}$$

so its discriminant is non-positive, i.e.

$$\begin{aligned} \left| 2 \operatorname{Im} \int_{\mathbb{R}^2} \bar{\psi}(t, x) \nabla \chi_{p, \underline{x}}(x) \cdot \nabla \psi(t, x) dx \right|^2 &\leq 4 E_{\text{cub}}(\psi(t)) \int_{\mathbb{R}^2} |\nabla \chi_{p, \underline{x}}(x)|^2 |\psi(t, x)|^2 dx \\ &\lesssim \|x u_0\|_{L^2}^2 g_{p, \underline{x}}(t). \end{aligned}$$

Here we have used (2.12) and the definition of  $\chi$ . Together with (2.14) this yields

$$|\dot{g}_{p, \underline{x}}(t)| \lesssim \sqrt{g_{p, \underline{x}}(t)},$$

and, hence, by integration,

$$(2.15) \quad \left| \sqrt{g_{p, \underline{x}}(t)} - \sqrt{g_{p, \underline{x}}(\tau)} \right| \lesssim |t - \tau|, \quad -1 \leq t \leq \tau < 0.$$

**Step 2. Blow-up profile along the sequence  $\tau_n$ .** Following [20], we set

$$(2.16) \quad \rho_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla \psi(\tau_n, \cdot)\|_{L^2}}, \quad v_n(x) := \rho_n \psi(\tau_n, \rho_n x).$$

We have  $\|v_n\|_{L^2} = \|Q\|_{L^2}$ ,  $\|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}$  for all  $n \in \mathbb{N}$ . In view of (2.11),

$$E_{\text{cub}}(v_n) = \rho_n^2 E_{\text{cub}}(\psi(\tau_n)) \xrightarrow{n \rightarrow \infty} 0,$$

since  $E_{\text{cub}}(\psi(t))$  is bounded from (2.12). We have all the ingredients necessary to invoke Theorem 2.3 from [20], *along the sequence  $\tau_n$* : there exist  $x_n \in \mathbb{R}^2$  and  $\theta_n \in \mathbb{R}$  such that

$$(2.17) \quad \rho_n e^{i\theta_n} \psi(\tau_n, \rho_n x + x_n) \xrightarrow{n \rightarrow \infty} Q, \quad \text{strongly in } H^1(\mathbb{R}^2).$$

**Step 3. Lower bound on  $\|\nabla \psi(\tau_n, \cdot)\|_{L^2}$ .** The convergence (2.17) implies that, in the sense of measures,

$$|\psi(\tau_n, x)|^2 dx - Q(x) \delta_{x=x_n} \rightarrow 0.$$

This and the bound (2.13) imply that the sequence  $(x_n)_{n \in \mathbb{N}}$  is bounded, hence we can extract a subsequence  $(x_{n'})_{n' \in \mathbb{N}}$  such that

$$x_{n'} \rightarrow \underline{x}, \quad \underline{x} \in \mathbb{R}^2.$$

Using  $\chi_{p, \underline{x}}$  as a test function, the above convergence implies that  $g_{p, \underline{x}}(\tau_{n'}) \rightarrow 0$ , as  $\chi_p(0) = 0$ , and thus, (2.15) yields

$$\left| \sqrt{g_{p, \underline{x}}(t)} \right| \lesssim |t|, \quad -1 \leq t < 0.$$

Now letting  $p \rightarrow \infty$ , Fatou's lemma implies

$$\int_{\mathbb{R}^2} |x - \underline{x}|^2 |\psi(t, x)|^2 dx \lesssim t^2.$$

Using of the classical uncertainty principle, i.e.

$$\|\psi(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 \leq \left( \int_{\mathbb{R}^2} |x - \underline{x}|^2 |\psi(t, x)|^2 dx \right)^{1/2} \|\nabla \psi(t, \cdot)\|_{L^2(\mathbb{R}^2)},$$

allows us to conclude

$$\|\nabla \psi(t, \cdot)\|_{L^2(\mathbb{R}^2)} \gtrsim \frac{1}{t}.$$

In particular, along the sequence  $\tau_n$ , we find

$$0 < \rho_n \lesssim \tau_n.$$

**Step 4. Conclusion.** In view of (2.17), and since  $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$ ,

$$\|\psi(\tau_n, \cdot)\|_{L^6}^6 \underset{n \rightarrow \infty}{\sim} \frac{1}{\rho_n^4} \|Q\|_{L^6}^6 \gtrsim \frac{1}{\tau_n^4}.$$

Recalling that  $\psi$  and  $u$  are linked via (2.10), we find

$$\|u(t_n, \cdot)\|_{L^6}^6 = \frac{1}{t_n^4} \left\| \psi \left( \frac{-1}{t_n}, \cdot \right) \right\|_{L^6}^6 \gtrsim 1,$$

which is incompatible with the first part of (2.8). Therefore, (2.9) cannot hold, i.e.  $Ju \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2))$ , and hence Theorem 1.7 follows from Lemma 2.2.

### 3. EXISTENCE OF SOLITONS AND FIRST PROPERTIES

**3.1. A priori estimates.** Suppose we have a solution  $u(t, x) = e^{i\omega t}\phi(x)$ , with  $\phi$  sufficiently smooth and localized. Then (1.1) becomes

$$(3.1) \quad -\frac{1}{2}\Delta\phi - |\phi|^2\phi + |\phi|^4\phi + \omega\phi = 0.$$

**Proposition 3.1** (A priori estimates for solitary waves). *Let  $1 \leq d \leq 3$ . If  $\phi \in H^1(\mathbb{R}^d)$  solves (3.1), then we have:*

(1) *Pohozaev identities:*

$$(3.2) \quad \frac{1}{2} \int_{\mathbb{R}^d} |\nabla\phi|^2 dx - \int_{\mathbb{R}^d} |\phi|^4 dx + \int_{\mathbb{R}^d} |\phi|^6 dx + \omega \int_{\mathbb{R}^d} |\phi|^2 dx = 0,$$

$$(3.3) \quad \frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla\phi|^2 dx - \frac{d}{2} \int_{\mathbb{R}^d} |\phi|^4 dx + \frac{d}{3} \int_{\mathbb{R}^d} |\phi|^6 dx + \omega d \int_{\mathbb{R}^d} |\phi|^2 dx = 0.$$

(2) *If  $\phi \neq 0$ , then  $0 < \omega < \frac{3}{16}$ .*

(3) *If  $d = 2$  and  $\phi \neq 0$ , then  $\|\phi\|_{L^2} > \|Q\|_{L^2}$ , where  $Q$  is cubic ground state solution to (1.5).*

(4) *If in addition  $\phi \in L^\infty \cap C^2$  is real-valued, then*

$$(3.4) \quad \|\phi\|_{L^\infty(\mathbb{R}^d)} \leq \sqrt{\frac{1 + \sqrt{1 - 4\omega}}{2}}.$$

*Proof.* For item (1), we quickly recall the method to derive Pohozaev identities formally, and refer to [5] for a rigorous justification via density type arguments. Firstly, multiplying (3.1) by  $\bar{\phi}$  and integrating yields (3.2). In particular, we infer  $\omega \in \mathbb{R}$ . Secondly, by multiplying (3.1) with  $x \cdot \nabla\bar{\phi}$  and integrating by parts we obtain (3.3). For  $d = 2$ , subtracting (3.3) from (3.2) yields

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla\phi|^2 dx + \frac{1}{3} \int_{\mathbb{R}^2} |\phi|^6 dx = \omega \int_{\mathbb{R}^2} |\phi|^2 dx,$$

hence  $\omega > 0$  unless  $\phi \equiv 0$ . In the case  $d = 3$ , we obtain similarly

$$\frac{1}{2} \int_{\mathbb{R}^3} |\phi|^4 dx = 2\omega \int_{\mathbb{R}^3} |\phi|^2 dx,$$

and thus we arrive at the same conclusion.

(2) From now on, we shall denote

$$F(s) = \frac{1}{4}s^4 - \frac{1}{6}s^6$$

and set

$$\omega^* = \sup \left\{ \omega > 0; \frac{\omega}{2}s^2 - F(s) < 0 \text{ for some } s > 0 \right\}.$$

A computation then shows  $\omega^* = \frac{3}{16}$ . In particular, if  $\omega \geq \frac{3}{16}$ , we have the pointwise relation

$$-\frac{1}{2}|\phi|^4 + \frac{1}{3}|\phi|^6 + \omega|\phi|^2 \geq 0, \quad \forall x \in \mathbb{R}^d,$$

which, in view (3.3), implies  $\phi \equiv 0$  for  $d \geq 2$ . In the case  $d = 1$ , the conclusion follows from ODE arguments, and more precisely [5, Theorem 5].

(3) We suppose  $d = 2$  and introduce

$$\gamma = \frac{\|\phi\|_{L^4}^4}{\|\nabla\phi\|_{L^2}^2},$$

which allows us to rewrite (3.2) as

$$\left(\frac{1}{2} - \gamma\right) \int_{\mathbb{R}^2} |\nabla\phi|^2 + \int_{\mathbb{R}^2} \phi^6 + \omega \int_{\mathbb{R}^2} \phi^2 = 0,$$

Similarly, we can rewrite (3.3) for  $d = 2$ , by using  $\gamma$ , in the following form

$$\frac{\gamma}{2} \int_{\mathbb{R}^2} |\nabla\phi|^2 - \frac{1}{3} \int_{\mathbb{R}^2} |\phi|^6 - \omega \int_{\mathbb{R}^2} |\phi|^2 = 0.$$

Combining these identities, we infer

$$\int_{\mathbb{R}^2} |\phi|^6 = \frac{3(\gamma - 1)}{4} \int_{\mathbb{R}^2} |\nabla\phi|^2,$$

and in particular  $\gamma > 1$ , i.e.  $\|\phi\|_{L^4}^4 > \|\nabla\phi\|_{L^2}^2$ . In view of the sharp Gagliardo-Nirenberg inequality (1.6), this consequently implies that the mass of the cubic-quintic ground states satisfies  $\|\phi\|_{L^2} > \|Q\|_{L^2}$ .

(4) Let  $\phi \in C^2$  be a real-valued bounded solution to (3.1). Suppose that  $\phi$  reaches its maximum at  $x_0 \in \mathbb{R}^d$ . Then  $\Delta\phi(x_0) \leq 0$ , and hence

$$(-\phi^3 + \phi^5 + \omega\phi)|_{x=x_0} \leq 0.$$

Writing

$$\omega\phi - \phi^3 + \phi^5 = \phi \left( \phi^2 - \frac{1 - \sqrt{1 - 4\omega}}{2} \right) \left( \phi^2 - \frac{1 + \sqrt{1 - 4\omega}}{2} \right),$$

we see that

$$\phi(x_0) \leq \sqrt{\frac{1 + \sqrt{1 - 4\omega}}{2}}.$$

Reasoning similarly for a minimum of  $\phi$ , we infer (3.4).  $\square$

**3.2. Existence and uniqueness.** Denote  $f(s) = s^3 - s^5$  and

$$F(s) = \int_0^s f(\tau) d\tau = \frac{1}{4}s^4 - \frac{1}{6}s^6,$$

as before. We have already seen that

$$\sup \left\{ \omega > 0; \frac{\omega}{2}s^2 - F(s) < 0 \text{ for some } s > 0 \right\} = \frac{3}{16}.$$

Then according to [5] (treating the case  $d = 1$  or  $d = 3$ ) and [4] (treating the case  $d = 2$ ), for all  $\omega \in ]0, \omega^*[$ , there exists a solution  $\phi_\omega$  of (3.1). Uniqueness of  $\phi_\omega$  in  $d = 1$  is proven in [5], while in  $d = 3$  uniqueness follows from [41], as pointed out in [26]. Finally, for  $d = 2$ , we infer uniqueness from the results of [23], where we emphasize that the assumptions made there correspond more closely to those made to prove existence.

We summarize all of these results in the proposition below, we recall that the action, defined in the introduction, is given by

$$S(\phi) = \frac{1}{2} \|\nabla\phi\|_{L^2}^2 + \omega \|\phi\|_{L^2}^2 - 2V(\phi), \quad \text{where} \quad V(\phi) = \int_{\mathbb{R}^d} F(\phi(x)) dx,$$

and satisfies

$$(3.5) \quad S(\phi) = E(\phi) + \omega \|\phi\|_{L^2}^2.$$

**Proposition 3.2** (Uniqueness of radial ground states). *Let  $1 \leq d \leq 3$ . Suppose that*

$$0 < \omega < \frac{3}{16}.$$

Then (3.1) has a unique solution  $\phi_\omega$  such that

- (1)  $\phi_\omega > 0$  on  $\mathbb{R}^d$ .
- (2)  $\phi_\omega$  is radially symmetric,  $\phi_\omega(x) = \phi(r)$ , where  $r = |x|$ , and  $\phi$  is a non-increasing function of  $r$ .
- (3)  $\phi_\omega \in C^2(\mathbb{R}^d)$ .
- (4) The derivatives of order at most two of  $\phi_\omega$  decay exponentially:

$$\exists \delta > 0, \quad |\partial^\alpha \phi_\omega(x)| \lesssim e^{-\delta|x|}, \quad |\alpha| \leq 2.$$

- (5) For every solution  $\varphi$  to (3.1),

$$0 < S(\phi_\omega) \leq S(\varphi).$$

**3.3. Further properties.** We collect here some further, asymptotic properties of cubic-quintic ground states. These have been established in [26, Theorem 2.2] in the case  $d = 3$ , but can easily be adapted to the case  $d = 2$ . We therefore assume  $d = 2$  or  $3$  and note that the map  $\omega \mapsto \phi_\omega$  given by Proposition 3.2 is real analytic.

First, we want to better understand the limit  $\omega \rightarrow 0$  in (3.1) which, unfortunately, is singular. To turn it into a regular limit, we change the unknown function  $\phi_\omega$  into

$$\psi_\omega(x) = \frac{1}{\sqrt{\omega}} \phi_\omega\left(\frac{x}{\sqrt{\omega}}\right).$$

Then (3.1) is equivalent to

$$-\frac{1}{2}\Delta\psi_\omega + \psi_\omega - \psi_\omega^3 + \omega\psi_\omega^5 = 0,$$

and it can be shown that  $\psi_\omega \rightarrow Q$  in  $H^1(\mathbb{R}^d)$  as  $\omega \rightarrow 0$ , where  $Q$  is the unique non-negative, radially symmetric ground state solution to

$$-\frac{1}{2}\Delta Q + Q - Q^3 = 0, \quad x \in \mathbb{R}^d.$$

In particular, in the case  $d = 2$ , we infer

$$\|\phi_\omega\|_{L^2(\mathbb{R}^2)} = \|\psi_\omega\|_{L^2(\mathbb{R}^2)} \xrightarrow{\omega \rightarrow 0} \|Q\|_{L^2(\mathbb{R}^2)},$$

thus showing that ground states for the cubic-quintic NLS in 2D have mass strictly larger but arbitrarily close to that of the cubic ground state  $Q$ .

Second, one may wonder about the limit as  $\omega \rightarrow \frac{3}{16}$ . In this case, we shall see that the mass of  $\phi_\omega$  grows to infinity. For  $d = 3$ , this is established in Theorem 2.2, (v), in [26], where the authors also prove that

$$\left(\frac{3}{16} - \omega\right)^{-3} \lesssim M(\phi_\omega) \lesssim \left(\frac{3}{16} - \omega\right)^{-3},$$

Here, we propose a simpler argument in the case  $d = 2$ , which however only shows divergence of the mass in the limit (but no asymptotic behavior).

Let  $d = 2$ , and suppose that  $\|\phi_\omega\|_{L^2}$  is bounded as  $\omega \rightarrow \frac{3}{16}$ . Then, using Hölder's inequality (1.2) in the Pohozaev identity (3.3) we infer

$$\int_{\mathbb{R}^2} |\phi_\omega|^6 \leq \left(3 - \frac{2\omega}{3}\right) \|\phi_\omega\|_{L^2}^2.$$

This implies that also  $\|\phi_\omega\|_{L^6}$  remains bounded as  $\omega \rightarrow \frac{3}{16}$ . Using Hölder's inequality once more, we conclude that  $\|\phi_\omega\|_{L^4}$  remains bounded as well, which, together with (3.2), yields the boundedness of  $\phi_\omega$  in  $H^1(\mathbb{R}^2)$ . Since  $\phi_\omega$  is radial, the sequence

$(\phi_\omega)_\omega$  is compact in  $L^2 \cap L^6(\mathbb{R}^2)$ , in view of Strauss' lemma [42]. Therefore, up to the choice of a suitable subsequence,

$$\phi_\omega \xrightarrow{\omega \rightarrow 3/16} \Phi \quad \text{strongly in } L^2 \cap L^6(\mathbb{R}^2).$$

Passing to the limit in (3.3) (recall that  $d = 2$  here), we get

$$\int_{\mathbb{R}^2} \left( -\frac{1}{4}\Phi^4 + \frac{1}{6}\Phi^6 + \frac{\omega}{2}\Phi^2 \right) \Big|_{\omega=3/16} = 0.$$

But  $\omega = \frac{3}{16}$  was obtained as

$$\frac{3}{16} = \sup \left\{ \omega > 0; \quad \frac{\omega}{2}s^2 - \frac{s^4}{4} + \frac{s^6}{6} < 0 \text{ for some } s > 0 \right\},$$

so we infer  $\Phi \equiv 0$ , which contradicts the property

$$\|\phi_\omega\|_{L^2} > \|Q\|_{L^2}, \quad \forall \omega \in ]0, \frac{3}{16}[.$$

Therefore, possibly along some subsequence,  $\|\phi_\omega\|_{L^2} \rightarrow \infty$  as  $\omega \rightarrow \frac{3}{16}$ , so the range of the map  $\omega \mapsto \|\phi_\omega\|_{L^2}$  is exactly  $] \|Q\|_{L^2}, \infty[$ , as  $\omega$  varies in  $]0, \frac{3}{16}[$ .

*Remark 3.3.* It turns out that this map is non-decreasing, as a consequence of the orbital stability established in the next section, and the Grillakis-Shatah-Strauss theory (the spectral assumptions needed to apply this theory can be obtained by adapting the argument of [26] from the 3D to the 2D case, see the appendix).

#### 4. ORBITAL STABILITY OF GROUND STATES IN 2D

To prove orbital stability in the context of nonlinear Schrödinger equations, there are two main strategies: the first one, historically, is due to Cazenave and Lions [7] and based on concentration-compactness arguments. The second one, known as Grillakis-Shatah-Strauss theory, was introduced in [19] (see also [13]), and generalized the ideas developed by M. Weinstein in [48, 49]. Here, we shall follow the former strategy.

**Proposition 4.1** (Energy minimization). *Let  $d = 2$  and  $\omega \in ]0, \frac{3}{16}[$ . For  $\rho > \|Q\|_{L^2(\mathbb{R}^2)}^2$ , consider*

$$\Gamma = \left\{ u \in H^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} |u|^2 = \rho \right\},$$

and denote

$$-\nu = \inf \{ E(u); u \in H^1(\mathbb{R}^2), M(u) = \rho \}.$$

Then the following properties hold:

(1) *The minimization problem*

$$(4.1) \quad u \in \Gamma, \quad E(u) = \min \{ E(v), v \in \Gamma \}$$

*has a solution.*

(2) *If  $(u_n)_{n \in \mathbb{N}} \subset \Gamma$  satisfies  $M(u_n) \rightarrow \rho$  and  $E(u_n) \rightarrow -\nu$ , then there exist a subsequence, still denoted by  $u_n$ , and a sequence  $y_n \in \mathbb{R}^2$  such that  $u_n(\cdot - y_n)$  has a strong limit  $u$  in  $H^1(\mathbb{R}^2)$ . In particular,  $u$  satisfies (4.1).*

*Proof. First step.* We show that

$$\inf \{ E(u); u \in H^1(\mathbb{R}^2), M(u) = \rho \} = -\nu,$$

for some finite  $\nu > 0$ . To prove that the infimum is finite, we use Hölder's inequality (1.2), to infer

$$E(u) \geq \frac{1}{2} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 - \frac{\sqrt{\rho}}{2} \|u(t)\|_{L^6(\mathbb{R}^d)}^3 + \frac{1}{3} \|u(t)\|_{L^6(\mathbb{R}^d)}^6,$$

and thus  $E(u)$  is bounded from below. To see that the infimum is negative, consider the  $L^2$ -invariant scaling, for  $\lambda > 0$ ,

$$u_\lambda(x) = \lambda^{d/2}u(\lambda x),$$

which, for  $d = 2$ , implies

$$E(u_\lambda) = \frac{\lambda^2}{2} \left( \|\nabla u\|_{L^2}^2 - \|u\|_{L^4}^4 + \frac{2}{3}\lambda^2\|u\|_{L^6}^6 \right).$$

In view of the sharp Gagliardo-Nirenberg inequality, and since  $\|u\|_{L^2}^2 > \|Q\|_{L^2}^2$ , we may choose a profile  $u \in H^1$  so that the terms independent of  $\lambda$  inside the parentheses become negative, e.g., take

$$u = \left( \frac{\rho}{M(Q)} \right)^{1/2} Q, \text{ with } \lambda > 0 \text{ sufficiently small.}$$

**Second step.** Any minimizing sequence is bounded away from zero in  $L^4$ . Let  $(u_n)_{n \geq 0}$  be a minimizing sequence: for  $n$  sufficiently large,  $E(u_n) \leq -\nu/2$ , hence

$$\|u_n\|_{L^4}^4 \geq \nu > 0.$$

**Third step.** In view of [31] (see also [6, Proposition 1.7.6]), we have the standard trichotomy of concentration compactness. From the second step, vanishing is ruled out, so we have to rule out dichotomy to infer compactness. Arguing by contradiction, suppose that, after extraction of suitable subsequences, there exist  $(v_k)_{k \geq 0}$ ,  $(w_k)_{k \geq 0}$  in  $H^1(\mathbb{R}^2)$ , such that

$$\text{supp } v_k \cap \text{supp } w_k = \emptyset, \quad |v_k| + |w_k| \leq |u_{n_k}|, \quad \|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C\|u_{n_k}\|_{H^1},$$

satisfying

$$\|v_k\|_{L^2}^2 \xrightarrow{k \rightarrow \infty} \theta\rho, \quad \|w_k\|_{L^2}^2 \xrightarrow{k \rightarrow \infty} (1 - \theta)\rho, \quad \text{for some } \theta \in ]0, 1[,$$

and

$$\begin{aligned} \liminf_{k \rightarrow \infty} \left( \int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \right) &\geq 0, \\ \left| \int |u_{n_k}|^p - \int |v_k|^p - \int |w_k|^p \right| &\xrightarrow{k \rightarrow \infty} 0, \end{aligned}$$

for all  $2 \leq p < \infty$ . We infer

$$\liminf_{k \rightarrow \infty} (E(u_{n_k}) - E(v_k) - E(w_k)) \geq 0,$$

hence

$$(4.2) \quad \limsup_{k \rightarrow \infty} (E(v_k) + E(w_k)) \leq -\nu.$$

Following an idea from [10], we then use a scaling argument rather than a multiplicative one as in [7]. Let

$$\begin{aligned} \tilde{v}_k(x) &= v_k \left( \lambda_k^{-1/2} x \right), \quad \lambda_k = \frac{\rho}{\|v_k\|_{L^2}^2} \\ \tilde{w}_k(x) &= w_k \left( \mu_k^{-1/2} x \right), \quad \mu_k = \frac{\rho}{\|w_k\|_{L^2}^2}. \end{aligned}$$

Since  $\tilde{v}_k$  and  $\tilde{w}_k$  have mass  $\rho$ ,

$$E(\tilde{v}_k), E(\tilde{w}_k) \geq -\nu.$$

On the other hand, we compute

$$E(\tilde{v}_k) = \lambda_k \left( \frac{1}{2\lambda_k} \int |\nabla v_k|^2 - \frac{1}{2} \int |v_k|^4 + \frac{1}{3} \int |v_k|^6 \right),$$



and so

$$E(v_k) = \frac{1}{\lambda_k} E(\tilde{v}_k) + \frac{1 - \lambda_k^{-1}}{2} \int |\nabla v_k|^2 \geq \frac{-\nu}{\lambda_k} + \frac{1 - \lambda_k^{-1}}{2} \int |\nabla v_k|^2.$$

Doing the same for  $E(w_k)$ , we find

$$\begin{aligned} E(v_k) + E(w_k) &\geq -\nu \left( \frac{1}{\lambda_k} + \frac{1}{\mu_k} \right) + \frac{1 - \lambda_k^{-1}}{2} \int |\nabla v_k|^2 + \frac{1 - \mu_k^{-1}}{2} \int |\nabla w_k|^2 \\ &\geq -\nu \left( \frac{1}{\lambda_k} + \frac{1}{\mu_k} \right) + \frac{1 - \lambda_k^{-1}}{2 \|v_k\|_{L^2}^2} \|v_k\|_{L^4}^4 + \frac{1 - \mu_k^{-1}}{2 \|w_k\|_{L^2}^2} \|w_k\|_{L^4}^4, \end{aligned}$$

where in the second step, we have used the Gagliardo-Nirenberg inequality. Passing to the limit, yields

$$\liminf_{k \rightarrow \infty} (E(v_k) + E(w_k)) \geq -\nu + \frac{1}{2} \min \left( \frac{1 - \theta}{\theta \rho}, \frac{\theta}{(1 - \theta) \rho} \right) \liminf_{k \rightarrow \infty} \|u_{n_k}\|_{L^4}^4,$$

and hence a contradiction to (4.2), in view of the second step and  $\theta \in ]0, 1[$ .  $\square$

Recall the formula  $S(\phi) = E(\phi) + \omega \|\phi\|_{L^2}^2$ . Hence, for fixed mass, minimizing the action is equivalent to minimizing the energy. We then have the analogue of [6, Corollary 8.3.8], to which we also refer for the details of the proof:

**Lemma 4.2** (Equivalence of minimizers). *Let  $d = 2$  and  $0 < \omega < \frac{3}{16}$ . Then  $\phi \in H^1(\mathbb{R}^2)$  is a ground state of (3.1) if and only if  $\phi$  solves the minimization problem*

$$(4.3) \quad \phi \in \Gamma, \quad S(\phi) = \min\{S(\varphi), \varphi \in \Gamma\}.$$

*In addition, the problems (4.1) and (4.3) are equivalent.*

**Conclusion.** At this stage, we have all the arguments to conclude in the classical way. Assume, by contradiction, that there exist a sequence  $(u_{0,n})_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^2)$ , such that

$$(4.4) \quad \|u_{0,n} - \phi\|_{H^1} \xrightarrow{n \rightarrow \infty} 0,$$

and a sequence  $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ , such that the sequence of solutions  $u_n$  to (1.1) associated to the initial data  $u_{0,n}$  satisfies

$$(4.5) \quad \inf_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}^2}} \|u_n(t_n, \cdot) - e^{i\theta} \phi(\cdot - y)\|_{H^1(\mathbb{R}^2)} > \varepsilon,$$

for some  $\varepsilon > 0$ . Introducing  $v_n = u_n(t_n, \cdot)$ , the above inequality also reads

$$\inf_{\varphi \in G} \|v_n - \varphi\|_{H^1(\mathbb{R}^2)} > \varepsilon,$$

where  $G$  is the set of all possible ground states, as given in Definition 1.4. In view of (4.4) and Lemma 4.2,

$$\int_{\mathbb{R}^2} |u_{0,n}|^2 \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^2} |\phi|^2, \quad S(u_{0,n}) \xrightarrow{n \rightarrow \infty} S(\phi) = \inf\{S(v), v \in \Gamma\}.$$

The conservation laws for mass and energy imply

$$\int_{\mathbb{R}^2} |v_n|^2 \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^2} |\phi|^2, \quad S(v_n) \xrightarrow{n \rightarrow \infty} S(\phi),$$

so  $(v_n)_n$  is a minimizing sequence for the problem (4.3), and hence also for the problem (4.1). From Proposition 4.1, there exists  $y_n$  in  $\mathbb{R}^2$  and a solution  $u$  to (4.1) such that

$$\|v_n - u(\cdot - y_n)\|_{H^1} \xrightarrow{n \rightarrow \infty} 0.$$

However, in view of Lemma 4.2,  $u$  is a ground state and so is  $u(\cdot - y_n)$ , hence a contradiction.

## APPENDIX A. STABILITY AND INSTABILITY OF THREE-DIMENSIONAL GROUND STATES

**A.1. Grillakis-Shatah-Strauss theory.** The proof of Proposition 1.9 relies on the application of the theory developed in [19], and all the ingredients necessary to do so are already present in [26]. In view of the existence results given in Proposition 1.1 and Proposition 3.2, we only have to check the spectral Assumption 3 imposed in [19] and analyze the monotonicity of the map  $\omega \mapsto M(\phi_\omega)$ . To state the spectral assumption, we write the second order derivative of the action as

$$\langle S''(\phi_\omega)w, w \rangle = \frac{1}{2} \langle L_1 u, u \rangle + \frac{1}{2} \langle L_2 v, v \rangle,$$

where  $w = u + iv$ . In our case, we have

$$\begin{aligned} L_1 &= -\frac{1}{2}\Delta + \omega - 3\phi_\omega^2 + 5\phi_\omega^4, \\ L_2 &= -\frac{1}{2}\Delta + \omega - \phi_\omega^2 + \phi_\omega^4. \end{aligned}$$

We then need to check:

**Assumption A.1.** For each  $\omega \in ]0, \frac{3}{16}[$ , the Hessian  $S''(\phi_\omega)$  has exactly one negative eigenvalue; its kernel is spanned by  $i\phi_\omega$  and  $\nabla\phi_\omega$ , and the rest of its spectrum is positive and bounded away from zero.

If this holds true, then:

- (a) If  $\frac{d}{d\omega}M(\phi_\omega) > 0$ , then the standing wave  $e^{i\omega t}\phi_\omega(x)$  is orbitally stable.
- (b) If  $\frac{d}{d\omega}M(\phi_\omega) < 0$ , then the standing wave  $e^{i\omega t}\phi_\omega(x)$  is unstable.

Indeed, the authors of [26] proved that Assumption A.1. holds true for the cubic-quintic NLS:

**Proposition A.2** (Proposition 2.4 from [26]). *Fix  $\ell = 0, 1, 2, \dots$ , and consider the restriction of  $L_1$  to functions of the form  $f(|x|)Y(x/|x|)$ , where  $Y$  is a spherical harmonic of degree  $\ell$ .*

- (1) *When  $\ell = 0$ , the operator has exactly one negative eigenvalue; it is simple.*
- (2) *When  $\ell = 1$ , there are no negative eigenvalues. Zero is an eigenvalue and its eigenspace is spanned by the three components of  $\nabla\phi_\omega$ .*
- (3) *When  $\ell \geq 2$ , the operator is positive definite.*

The proof of this result relies on Sturm Oscillation Theorem, since the analysis boils down to second order ODEs for the radial function  $f$ . Note that the proof from [26] can be readily adapted to the 2D case, by replacing spherical harmonics with functions of the form  $e^{i\ell\theta}$  in radial coordinates. The above proposition is complemented by the following one:

**Proposition A.3** (Proposition 2.5 from [26]). *Let  $\delta = \delta(r)$  be the solution to*

$$-\frac{1}{2}\delta'' - \frac{1}{r}\delta' + (5\phi_\omega^4 - 3\phi_\omega^2 + \omega)\delta = 0$$

*obeying  $\delta(0) = 1$ . Then  $\delta(r) \rightarrow -\infty$  as  $r \rightarrow \infty$ . Correspondingly, zero is not an eigenvalue of  $L_1$  restricted to radial functions.*

We emphasize that the above result is proved in a more general setting (including the two-dimensional case) in [30] (in particular, Lemma 3 there). With this in hand, we can now proceed to prove (in-)stability of ground states, depending on the frequency  $\omega \in ]0, \frac{3}{16}[$ .

A.2. **Instability for small  $\omega$ .** Citing [26, Theorem 2.2], we have

$$\frac{d}{d\omega}M(\phi_\omega) < \frac{3\beta(\omega) - 1}{2\omega}M(\phi_\omega),$$

where

$$\beta(\omega) = \frac{\|\phi_\omega\|_{L^6}^6}{\|\nabla\phi_\omega\|_{L^2}^2}.$$

Therefore, we are in the unstable case (b) provided  $\beta < \frac{1}{3}$ . In particular, this is guaranteed for  $\omega \in ]0, \omega_0[$  where  $\omega_0 \ll 1$ , since  $\beta(\omega) = \mathcal{O}(\omega)$  as  $\omega \rightarrow 0$ , see [26, Theorem 2.2, (iv)].

A.3. **Stability for large  $\omega$ .** We know from [26, Theorem 2.2, (v)] that there are constants  $c_1, c_2 > 0$  such that

$$c_1 \left(\frac{3}{16} - \omega\right)^{-3} \leq M(\phi_\omega) \leq c_2 \left(\frac{3}{16} - \omega\right)^{-3},$$

On the other hand, for  $\omega$  sufficiently close to  $\frac{3}{16}$ , we can quote [26, Theorem 4.1, (iv)], which ensures that for  $M(\phi_\omega) = m$  sufficiently large

$$\inf \{E(u); u \in H^1(\mathbb{R}^3), M(u) = m\} = [E_{\min}(m), \infty[,$$

with  $E_{\min}(m) < 0$ . It is then possible to resume the argument presented in Section 4, and conclude to orbital stability via the Cazenave-Lions argument.

The analysis of [26] yields more precise information regarding the set of  $\omega$ 's for which orbital stability holds, comforting Conjecture 2.3 from [26], whose statement was recalled in the introduction.

## REFERENCES

- [1] F. K. ABDULLAEV, A. GAMMAL, L. TOMIO, AND T. FREDERICO, *Stability of trapped Bose-Einstein condensates*, Phys. Rev. A, 63 (2001), p. 043604.
- [2] V. BANICA, *Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 3 (2004), pp. 139–170.
- [3] J. E. BARAB, *Nonexistence of asymptotically free solutions for nonlinear Schrödinger equation*, J. Math. Phys., 25 (1984), pp. 3270–3273.
- [4] H. BERESTYCKI, T. GALLOUËT, AND O. KAVIAN, *Équations de champs scalaires euclidiens non linéaires dans le plan*, C. R. Acad. Sci. Paris Sér. I Math., 297 (1983), pp. 307–310.
- [5] H. BERESTYCKI AND P.-L. LIONS, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., 82 (1983), pp. 313–345.
- [6] T. CAZENAVE, *Semilinear Schrödinger equations*, vol. 10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [7] T. CAZENAVE AND P.-L. LIONS, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., 85 (1982), pp. 549–561.
- [8] X. CHENG, *Scattering for the mass super-critical perturbations of the mass critical nonlinear Schrödinger equations*. Archived at <https://arxiv.org/abs/1904.11826>. To appear in Illinois J. Math.
- [9] X. CHENG, C. MIAO, AND L. ZHAO, *Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities in the radial case*, J. Differential Equ., 261 (2016), pp. 2881–2934.
- [10] M. COLIN, L. JEANJEAN, AND M. SQUASSINA, *Stability and instability results for standing waves of quasi-linear Schrödinger equations*, Nonlinearity, 23 (2010), pp. 1353–1385.
- [11] S. A. CORREIA, F. OLIVEIRA, AND J. D. SILVA, *Mass-transfer instability of ground-states for Hamiltonian Schrödinger systems*. preprint, archived at <http://www.arxiv.org/abs/1912.09691>, 2019.
- [12] S. COWAN, R. H. ENNS, S. S. RANGNEKAR, AND S. S. SANGHERA, *Quasi-soliton and other behaviour of the nonlinear cubic-quintic Schrödinger equation*, Revue canadienne phys., 64 (1986), pp. 311–315.
- [13] S. DE BIÈVRE, F. GENOUD, AND S. ROTA NODARI, *Orbital stability: analysis meets geometry*, in Nonlinear optical and atomic systems, vol. 2146 of Lecture Notes in Math., Springer, Cham, 2015, pp. 147–273.

- [14] B. DODSON, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*, Adv. Math., 285 (2015), pp. 1589–1618.
- [15] T. DUYNCKAERTS, J. HOLMER, AND S. ROUDENKO, *Scattering for the non-radial 3D cubic nonlinear Schrödinger equation*, Math. Res. Lett., 15 (2008), pp. 1233–1250.
- [16] N. FUKUYA AND M. HAYASHI, *Instability of algebraic standing waves for nonlinear Schrödinger equations with double power nonlinearities*. preprint, archived at <http://www.arxiv.org/abs/2001.08488>, 2020.
- [17] A. GAMMAL, T. FREDERICO, L. TOMIO, AND P. CHOMAZ, *Atomic Bose-Einstein condensation with three-body interactions and collective excitations*, J. Phys. B, 33 (2000), pp. 4053–4067.
- [18] J. GINIBRE AND G. VELO, *On a class of nonlinear Schrödinger equations. II Scattering theory, general case*, J. Funct. Anal., 32 (1979), pp. 33–71.
- [19] M. GRILLAKIS, J. SHATAH, AND W. STRAUSS, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal., 74 (1987), pp. 160–197.
- [20] T. HMIDI AND S. KERAANI, *Blowup theory for the critical nonlinear Schrödinger equations revisited*, Int. Math. Res. Not., (2005), pp. 2815–2828.
- [21] J. HOLMER AND S. ROUDENKO, *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys., 282 (2008), pp. 435–467.
- [22] I. D. ILIEV AND K. P. KIRCHEV, *Stability and instability of solitary waves for one-dimensional singular Schrödinger equations*, Differential Integral Equ., 6 (1993), pp. 685–703.
- [23] J. JANG, *Uniqueness of positive radial solutions of  $\Delta u + f(u) = 0$  in  $\mathbb{R}^N$ ,  $N \geq 2$* , Nonlinear Anal., 73 (2010), pp. 2189–2198.
- [24] C. E. KENIG AND F. MERLE, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math., 166 (2006), pp. 645–675.
- [25] S. KERAANI, *On the blow up phenomenon of the critical nonlinear Schrödinger equation*, J. Funct. Anal., 235 (2006), pp. 171–192.
- [26] R. KILLIP, T. OH, O. POCOVNICU, AND M. VIŞAN, *Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on  $\mathbb{R}^3$* , Arch. Ration. Mech. Anal., 225 (2017), pp. 469–548.
- [27] R. KILLIP, T. TAO, AND M. VIŞAN, *The cubic nonlinear Schrödinger equation in two dimensions with radial data*, J. Eur. Math. Soc. (JEMS), 11 (2009), pp. 1203–1258.
- [28] Y. S. KIVSHAR AND G. P. AGRAWAL, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, 2003.
- [29] J. KRIEGER AND W. SCHLAG, *Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension*, J. Amer. Math. Soc., 19 (2006), pp. 815–920.
- [30] M. LEWIN AND S. ROTA NODARI, *Uniqueness and non-degeneracy for a nuclear nonlinear Schrödinger equation*, NoDEA Nonlinear Differential Equations Appl., 22 (2015), pp. 673–698.
- [31] P.-L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–145.
- [32] B. MALOMED, *Vortex solitons: Old results and new perspectives*, Physica D, 399 (2019), pp. 108–137.
- [33] F. MERLE, *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power*, Duke Math. J., 69 (1993), pp. 427–454.
- [34] F. MERLE AND L. VEGA, *Compactness at blow-up time for  $L^2$  solutions of the critical nonlinear Schrödinger equation in 2D*, Internat. Math. Res. Notices, (1998), pp. 399–425.
- [35] H. MICHINEL, J. CAMPO-TÁBOAS, R. GARCÍA-FERNÁNDEZ, J. R. SALGUEIRO, AND M. L. QUIROGA-TEIXEIRO, *Liquid light condensates*, Phys. Rev. E, 65 (2002), p. 066604.
- [36] M. OHTA, *Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity*, Kodai Math. J., 18 (1995), pp. 68–74.
- [37] F. PLANCHON, *Existence globale et scattering pour les solutions de masse finie de l'équation de Schrödinger cubique en dimension deux [d'après Benjamin Dodson, Rowan Killip, Terence Tao, Monica Vişan et Xiaoyi Zhang]*, no. 348, 2012, pp. Exp. No. 1042, x, 425–447. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042.
- [38] K. I. PUSHKAROV, D. I. PUSHKAROV, AND I. V. TOMOV, *Self-action of light beams in nonlinear media: soliton solutions*, Optical Quantum Electronics, 11 (1979), pp. 471–478.
- [39] P. RAPHAËL, *Concentration compacité à la Kenig-Merle*, no. 352, 2013, pp. Exp. No. 1046, vii, 121–146. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.
- [40] W. SCHLAG, *Stable manifolds for an orbitally unstable nonlinear Schrödinger equation*, Ann. of Math. (2), 169 (2009), pp. 139–227.
- [41] J. SERRIN AND M. TANG, *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J., 49 (2000), pp. 897–923.

- [42] W. A. STRAUSS, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., 55 (1977), pp. 149–162.
- [43] C. SULEM AND P.-L. SULEM, *The nonlinear Schrödinger equation*, vol. 139 of Applied Mathematical Sciences, Springer-Verlag, New York, 1999. Self-focusing and wave collapse.
- [44] T. TAO, M. VISAN, AND X. ZHANG, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Diff. Eq., 32 (2007), pp. 1281–1343.
- [45] T. TAO, M. VISAN, AND X. ZHANG, *Minimal-mass blowup solutions of the mass-critical NLS*, Forum Math., 20 (2008), pp. 881–919.
- [46] Y. TSUTSUMI AND K. YAJIMA, *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Amer. Math. Soc. (N.S.), 11 (1984), pp. 186–188.
- [47] M. I. WEINSTEIN, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys., 87 (1982/83), pp. 567–576.
- [48] ———, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal., 16 (1985), pp. 472–491.
- [49] ———, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., 39 (1986), pp. 51–67.
- [50] X. ZHANG, *On the Cauchy problem of 3-D energy-critical Schrödinger equations with sub-critical perturbations*, J. Differential Equ., 230 (2006), pp. 422–445.

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE  
E-mail address: Remi.Carles@math.cnrs.fr

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, M/C 249, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN STREET CHICAGO, IL 60607, USA  
E-mail address: sparber@uic.edu