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ORBITAL STABILITY VS. SCATTERING IN THE CUBIC-QUINTIC SCHRÖDINGER EQUATION

RÉMI CARLES AND CHRISTOF SPARBER

Abstract. We consider the cubic-quintic nonlinear Schrödinger equation in space dimension up to three. The cubic nonlinearity is thereby focusing while the quintic one is defocusing, ensuring global well-posedness of the Cauchy problem in the energy space. The main goal of this paper is paint a more or less complete picture of dispersion and orbital (in-)stability of solitary waves, emanating from nonlinear ground states. In space dimension one, it is already known that solitons are orbitally stable. Here, we establish the analogous result in dimension two. In addition, we show that if the initial data have at most the mass of the ground state for the cubic two-dimensional Schrödinger equation, then the solution is dispersive and asymptotically linear. Finally, in dimension three, relying on some previous results from other authors, we show that solitons may or may not be orbitally stable.

1. Introduction and main results

1.1. **Basic setting.** We consider the nonlinear Schrödinger equation (NLS) with competing cubic-quintic nonlinearities,

$$i\partial_t u + \frac{1}{2}\Delta u = -|u|^2 u + |u|^4 u, \quad x \in \mathbb{R}^d, \tag{1.1}$$

in space dimension $d \leq 3$. The quintic nonlinearity was introduced in several physical situations: typically in optics (see e.g. [27]), or in Bose-Einstein condensation (e.g. [1, 16, 32]). We refer to the review [30] for more precise references. In particular, the incorporation of the defocusing quintic term is motivated by the stabilization of two- and three-dimensional vortex solitons.

Recall some of basic features of this nonlinearity in terms of criticality for the Cauchy problem. Depending on the space dimension, the NLS is seen to be:

- $d = 1$: focusing L^2-subcritical plus defocusing L^2-critical (and H^1-subcritical).
- $d = 2$: focusing L^2-critical plus defocusing L^2-supercritical (and H^1-subcritical).
- $d = 3$: focusing L^2-supercritical plus defocusing H^1-critical.

It is already known from the case of more general, gauge-invariant nonlinearities (see e.g. [6]), that equation (1.1) formally enjoys three basic conservation laws, namely:

- Mass: $M(u) = \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}$,
- Angular momentum: $J(u) = \mathrm{Im} \displaystyle\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx$,
- Energy: $E(u) = \dfrac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} - \dfrac{1}{2}\|u(t,\cdot)\|^4_{L^4(\mathbb{R}^d)} + \dfrac{1}{3}\|u(t,\cdot)\|^6_{L^6(\mathbb{R}^d)}$.

In dimensions 2 and 3, an effect of the quintic term is to *prevent finite time blow-up* which may occur in the purely cubic case (cf. [6]). Indeed, the conservation of the energy, combined with Hölder's inequality,

$$\|u\|^4_{L^4(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}\|u\|^3_{L^6(\mathbb{R}^d)}, \tag{1.2}$$

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shows that the cubic focusing part cannot be an obstruction to global well-posedness, at least in H^1. For $d \leq 2$, global well-posedness then follows from classical results (see e.g. [6]). For $d = 3$, we refer to [45], as the quintic term is energy-critical.

Proposition 1.1 (Global well-posedness)**.** *Let $d \leq 3$. For any $u_0 \in H^1(\mathbb{R}^d)$, (1.1) has a unique global solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ such that $u_{|t=0} = u_0$. The solution obeys the conservation of mass, energy, and momentum.*

Numerically, one observes a kind of oscillatory behavior within the solution u, which is due to the competition of focusing and defocusing effects within (1.1), cf. [38] for more details.

Remark 1.2. Recall that in dimension $d = 2$ or 3, the quintic term is L^2-supercritical, so we cannot hope to solve the Cauchy problem at this regularity level. Moreover, since the cubic-quintic nonlinearity stems from physics, it is sensible to work in H^1, where the energy is well-defined.

Complementing the case of prescribed initial data, we may also want to prescribe asymptotic states (or scattering states) and an asymptotically linear behavior, provided that $d \geq 2$. We thereby recall that in the case $d = 1$, the cubic nonlinearity causes long-range effects, and no non-trivial solution to (1.1) can be asymptotically linear, cf. [2]. However, in dimensions $d = 2, 3$ one can rely on classical techniques (see e.g. [6]) or the results of [45], respectively, to obtain:

Proposition 1.3 (Scattering)**.** *Let $d = 2$ or 3. For any $u_- \in H^1(\mathbb{R}^d)$, (1.1) has a unique global solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ such that*

$$\left\| u(t,\cdot) - e^{i\frac{t}{2}\Delta} u_- \right\|_{H^1(\mathbb{R}^2)} \underset{t\to-\infty}{\longrightarrow} 0.$$

In particular,

$$M(u) = \|u_-\|_{L^2}^2, \quad E(u) = \frac{1}{2}\|\nabla u_-\|_{L^2}^2, \quad \forall t \in \mathbb{R}.$$

As in the case with purely cubic nonlinearity, not every finite-energy solution of (1.1) is necessarily asymptotically linear. Finite time blow-up is of course ruled out in our case, but time-periodic solitary wave solutions also exist.

Definition 1.4. A *standing wave* or *soliton* of (1.1) is a solution of the form $e^{i\omega t}\phi(x)$, with $\omega \in \mathbb{R}$ and ϕ satisfying

$$-\frac{1}{2}\Delta\phi + \omega\phi - |\phi|^2\phi + |\phi|^4\phi = 0, \quad \phi \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{1.3}$$

The associated action is given by

$$S(\phi) = \frac{1}{2}\|\nabla\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - \frac{1}{2}\|\phi\|_{L^4}^4 + \frac{1}{3}\|\phi\|_{L^6}^6.$$

A solution ϕ is a *ground state* if $S(\phi) \leq S(\varphi)$ for any solution φ of (1.3)

As we will see in Section 3, if $d \leq 3$, (1.3) admits a solution if and only if

$$0 < \omega < \tfrac{3}{16}.$$

It turns out that for $d = 1$, explicit solitary wave solutions are available for this range of ω, see below. In the present paper, we are mostly interested in the *stability* of solitary waves, a question which is closely related to dispersive effects in (1.1).

Definition 1.5. Let ϕ be a solution of (1.3). The standing wave $e^{i\omega t}\phi(x)$ is called *orbitally stable* in $H^1(\mathbb{R}^d)$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^d)$ satisfies

$$\|u_0 - \phi\|_{H^1} \leq \delta,$$

then the solution to (1.1) with $u_{|t=0} = u_0$ satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\substack{\theta\in\mathbb{R}\\ y\in\mathbb{R}^d}} \left\| u(t,\cdot) - e^{i\theta}\phi(\cdot - y)\right\|_{H^1(\mathbb{R}^d)} \leq \varepsilon.$$

Otherwise, the standing wave is said to be unstable.

In this paper, we will try to give a more or less complete description of the long time behavior of solutions to (1.1) in terms of scattering versus orbital (in-)stability of solitary waves, depending on the spatial dimension $d \leq 3$.

1.2. **One-dimensional case.** In the case $d = 1$, the overall picture is very neat. Firstly, for $0 < \omega < \frac{3}{16}$, solutions to (1.3) are given by ([11, 34])

$$\phi(x) = 2\sqrt{\frac{\omega}{1 + \sqrt{1 - \frac{16\omega}{3}}\cosh\left(2x\sqrt{2\omega}\right)}}. \tag{1.4}$$

Note that in view of [4], this real-valued solution is unique, up to translation and change of sign. The orbital stability of these nonlinear ground states was established in [33, Theorem 3, case (1)].

Proposition 1.6 (Orbital stability in 1D)**.** *Let $d = 1$, and $0 < \omega < \frac{3}{16}$. The solitary wave $e^{i\omega t}\phi(x)$, where ϕ is given by* (1.4)*, is orbitally stable.*

The proof of this result combines the the well-known *Grillakis-Shatah-Strauss criterion* [18] with the analysis of [20] and an explicit formula for second order ODEs without first order derivatives, a strategy which seems to be restricted to the 1D case and not suited for solutions to (1.3) in $d \geq 2$.

1.3. **Two-dimensional case.** We now turn to the case $d = 2$ and recall that the results of [39] show that for $\|u_0\|_{L^2}$ *sufficiently small*, the solution to (1.1) is asymptotically linear. It turns out that since the cubic term is L^2-critical in 2D, we can in fact be more precise.

To this end, let Q be the *cubic nonlinear ground state*, i.e., the unique positive radial solution to

$$-\frac{1}{2}\Delta Q + Q - Q^3 = 0, \quad x \in \mathbb{R}^2. \tag{1.5}$$

In view of [42], and noting that we have an extra factor $\frac{1}{2}$ in front of the Laplacian in (1.5) compared to [42], the sharp Gagliardo-Nirenberg inequality reads

$$\|u\|^4_{L^4(\mathbb{R}^2)} \leq \left(\frac{\|u\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}}\right)^2 \|\nabla u\|^2_{L^2(\mathbb{R}^2)}, \quad \forall u \in H^1(\mathbb{R}^2). \tag{1.6}$$

In the focusing cubic case, i.e., without the quintic term, we know from [13] that if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, global existence and scattering hold (see also [26] for the case of radial data u_0). In our first main result below, we shall show that the effect of the additional quintic term is not only to guarantee global well-posedness, but also to extend this dispersive result to the L^2-sphere $\{\|u_0\|_{L^2} = \|Q\|_{L^2}\}$.

Theorem 1.7 (Mass (sub-)critical scattering in 2D)**.** *Let $d = 2$. If $u_0 \in H^1(\mathbb{R}^2)$ is such that*

$$\|u_0\|_{L^2} \leq \|Q\|_{L^2},$$

then the solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^2))$ to (1.1) *such that $u_{|t=0} = u_0$ is asymptotically linear, i.e. there exist $u_\pm \in H^1(\mathbb{R}^2)$ such that*

$$\|u(t,\cdot) - e^{i\frac{t}{2}\Delta}u_\pm\|_{H^1(\mathbb{R}^2)} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

On a heuristic level, we may argue in the same fashion as in [19], and recall that the standard virial computation for (1.1) yields,

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2}|x|^2|u(t,x)|^2dx = 2E(u) + \frac{4}{3}\|u(t)\|^6_{L^6(\mathbb{R}^2)} \geq 2E(u_0),$$

where $E(u) = E(u_0)$ is the conserved energy. In view of the sharp Gagliardo-Nirenberg inequality, we have, under the assumptions of Theorem 1.7, and if in addition $|\cdot|u_0 \in L^2$,

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2}|x|^2|u(t,x)|^2dx \geq \frac{2}{3}\|u_0\|^6_{L^6(\mathbb{R}^2)}.$$

The time-derivative of the virial of u is therefore increasing, a first hint that the solution is dispersive. In order to make this statement rigorous, especially in the limiting case $\|u_0\|_{L^2} = \|Q\|_{L^2}$, we will deploy a profile decomposition technique (see Lemma 2.4 below).

Our second main result concerns the stability of solitary waves:

Theorem 1.8 (Nonlinear ground states in 2D)**.** *Let $d = 2$. Then, for all $\omega \in]0, \frac{3}{16}[$, there exists a solitary wave solution $u(t,x) = e^{i\omega t}\phi(x)$ to (1.1). In addition, we have:*

1. *For any $M > \|Q\|^2_{L^2}$, there exists a ground state such that $\|\phi\|^2_{L^2} = M$.*
2. *The ground state solution is unique, up to translation and multiplication by $e^{i\theta}$, for constant $\theta \in \mathbb{R}$.*
3. *If ϕ is a (real-valued) ground state solution to*

$$-\frac{1}{2}\Delta\phi - \phi^3 + \phi^5 + \omega\phi = 0, \tag{1.7}$$

then the associated solitary wave is orbitally stable.

We emphasize the fact that for any mass *strictly larger* than that of the cubic ground state Q, we can find a soliton of the cubic-quintic NLS, while for a mass less or equal to that of Q, all solutions to (1.1) are asymptotically linear. This is in sharp contrast with the analogous situation in the case of a single pure power nonlinearity, where the critical sphere (in L^2 or other homogeneous Sobolev spaces) *always contains non-dispersive elements*, see e.g. [14, 22, 24, 40].

1.4. **Three-dimensional case.** In $d = 3$, equation (1.1) has already been studied in [25]. However, no statement concerning the (in-)stability of solitary waves is given in there. Here, we shall state the following proposition, the proof of which relies on elements already present in [25]:

Proposition 1.9 (Soliton (in-)stability in 3D)**.** *Let $d = 3$. For all $\omega \in]0, \frac{3}{16}[$, there exists a ground state solution which is unique, up to translation and multiplication by $e^{i\theta}$, for constant $\theta \in \mathbb{R}$. Moreover:*

- *There exists $0 < \omega_0 < \frac{3}{16}$ such that for all $0 < \omega < \omega_0$, the associated soliton is orbitally stable.*
- *There exists $\omega_0 \leq \omega_1 < \frac{3}{16}$ such that for all $\omega_1 < \omega < \frac{3}{16}$, the associated soliton is unstable.*

One expects the equality $\omega_0 = \omega_1$ to hold. More precisely, Conjecture 2.3 from [25], which is supported by numerics, states:

Conjecture ([25])**.** There exists $0 < \omega_* < \frac{3}{16}$ so that $\omega \mapsto M(\phi)$ is strictly decreasing for $\omega < \omega_*$, and strictly increasing for $\omega > \omega_*$.

If this indeed holds true, one can take $\omega_0 = \omega_1 = \omega_*$ in Proposition 1.9, in view of Grillakis-Shatah-Strauss theory [18] (see also [12]).

Somewhat independently from this question, one may wonder about the precise nature of instability. Recall, that in the case of a *single* power nonlinearity, instability is always due to the possibility of finite-time blow-up (see e.g. [6] and references therein). Very recently, Fukuya and Hayashi [15] have established instability results for NLS with a double power nonlinearity, but in their work the *focusing term dominates* the defocusing one (thereby extending the results of [8]). They rely on the possibility of blow-up or invoke the Grillakis-Shatah-Strauss theory, in which case the nature of the instability still remains unclear. For nonlinearly coupled *systems* of NLS, Correia, Oliveira and Silva [10] have shown that instability may correspond to a transfer of mass from one equation to the other. None of these former results, however, apply to our situation. For the latter, one may expect that the stable manifolds analyzed in [28, 35] become open neighborhoods, in the sense that in a full neighborhood of the unstable ground state (not only in a manifold with limited co-dimension), the solution u bifurcates from the solitary wave $e^{i\omega t}\phi(x)$, yielding a behavior of the form

$$u(t,x) = W(t,x) + e^{i\frac{t}{2}\Delta}u_+(x) + o_{L^2}(1) \quad \text{as } t \to \infty,$$

for some $u_+ \in L^2(\mathbb{R}^3)$, and where W is a (possibly different) ground state, modulated by a moving set of parameters (see [28, 35] for details).

The rest of this paper is now organized as follows: In Section 2, we prove Theorem 1.7. In Section 3, we analyze some general results on solitary waves for (1.1), with emphasis on some special properties in the 2D case. Theorem 1.8 is proven in Section 4, and we present the main arguments for Proposition 1.9 in an appendix.

2. Dispersive behavior in 2D

2.1. **Space-time norms.** In this section, our main goal is to prove Theorem 1.7. Recall that for two-dimensional Schrödinger equation, a *Strichartz-pair* (q,r) is *admissible* if

$$\frac{2}{q} + \frac{2}{r} = 1, \quad 2 < q \leq \infty.$$

We denote by

$$\|u\|_{S(I)} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q(I;L^r(\mathbb{R}^2))}.$$

In view of [39, Theorem 1.3], it suffices to prove that for any $u_0 \in H^1(\mathbb{R}^2)$ with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, the global solution u provided by Proposition 1.1 satisfies

$$\|u\|_{S(\mathbb{R})} + \|\nabla u\|_{S(\mathbb{R})} < \infty. \tag{2.1}$$

Remark 2.1. Note that Theorem 1.8 contains the particular information that one can find $u_0 \in H^1(\mathbb{R}^2)$ with $\|u_0\|_{L^2} - \|Q\|_{L^2} > 0$ arbitrarily small, such that $\|u\|_{S(\mathbb{R})} = \infty$.

As a first, basic step, we show that (2.1) can be reduced to the following:

Lemma 2.2 (Reduction step)**.** *Let $d = 2$ and $u_0 \in H^1(\mathbb{R}^2)$. If the global solution provided by Proposition 1.1 satisfies $\|u\|_{S(\mathbb{R})} < \infty$, then we also have*

$$\|\nabla u\|_{S(\mathbb{R})} < \infty,$$

and so u is asymptotically linear,

$$\exists u_\pm \in H^1(\mathbb{R}^2), \quad \left\| u(t) - e^{i\frac{t}{2}\Delta}u_\pm \right\|_{H^1(\mathbb{R}^2)} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

Proof. From [39], we only have to check that $\|u\|_{S(\mathbb{R})} < \infty$ implies $\|\nabla u\|_{S(\mathbb{R})} < \infty$. Let $I = [t_0, t_1]$ be some time interval, with $t_1 \geq t_0 \geq 0$ to simplify notations.

Considering the Duhamel's formula associated to (1.1), taking the gradient and applying Strichartz estimates, we find

$$\|\nabla u\|_{S(I)} \lesssim \|\nabla u(t_0)\|_{L^2} + \left\|u^2\nabla u\right\|_{L^{4/3}(I\times\mathbb{R}^2)} + \left\|u^4\nabla u\right\|_{L^{3/2}(I;L^{6/5})},$$

where we have considered the specific admissible pairs $(4,4)$ and $(3,6)$ for the cubic and quintic nonlinearities, respectively. Recall that we already know that $u \in L^\infty(\mathbb{R};H^1(\mathbb{R}^2))$, so the first term on the right hand side is bounded uniformly in time. Write

$$\frac{3}{4} = \frac{2}{4} + \frac{1}{4}, \quad \frac{2}{3} = \frac{4}{12} + \frac{1}{3}, \quad \frac{5}{6} = \frac{4}{6} + \frac{1}{6},$$

in which case, Hölder's inequality yields

$$\begin{aligned}\|\nabla u\|_{S(I)} &\lesssim 1 + \|u\|^2_{L^4(I\times\mathbb{R}^2)}\|\nabla u\|_{L^4(I\times\mathbb{R}^2)} + \|u\|^4_{L^{12}(I;L^6)}\|\nabla u\|_{L^3(I;L^6)}\\ &\lesssim 1 + \|u\|^2_{L^4(I\times\mathbb{R}^2)}\|\nabla u\|_{S(I)} + \|u\|^4_{L^{12}(I;L^6)}\|\nabla u\|_{S(I)}.\end{aligned}$$

Recalling again that $u \in L^\infty(\mathbb{R};H^1(\mathbb{R}^2))$,

$$\|u\|^4_{L^{12}(I;L^6)} \le \|u\|^3_{L^\infty(I;L^6)}\|u\|_{L^3(I;L^6)} \lesssim \|u\|^3_{L^\infty(I;H^1)}\|u\|_{S(I)} \le C\|u\|_{S(I)}.$$

Now since $\|u\|_{S(\mathbb{R})} < \infty$, we can split $\mathbb{R}_t$ into finitely many intervals on which the nonlinear terms in the above estimate can be absorbed by the left hand side, so we conclude $\|\nabla u\|_{S(\mathbb{R})} < \infty$. □

Next, in order to prove $\|u\|_{S(\mathbb{R})} < \infty$ and thus (2.1), we shall in the following distinguish between the case of subcritical mass, i.e. $\|u_0\|_{L^2} < \|Q\|_{L^2}$, and the critical case, where $\|u_0\|_{L^2} = \|Q\|_{L^2}$.

2.2. **Mass subcritical case.** In this subsection, we suppose

$$\|u_0\|^2_{L^2} = (1-\eta)\|Q\|^2_{L^2}, \quad \text{for some } 0<\eta<1. \tag{2.2}$$

Step 1. Consider first the case where not only $u_0 \in H^1(\mathbb{R}^2)$, but we also have finite variance, i.e. $|\cdot|u_0 \in L^2(\mathbb{R}^2)$. Then, we can rely on the pseudo-conformal conservation law (derived initially in [17], see also [6]):

$$\frac{d}{dt}\left(\frac{1}{2}\|(x+it\nabla)u\|^2_{L^2} - \frac{t^2}{2}\|u\|^4_{L^4} + \frac{t^2}{3}\|u\|^6_{L^6}\right) = -\frac{2t}{3}\|u\|^6_{L^6}. \tag{2.3}$$

In view of the standard factorization

$$(x+it\nabla)u = it\, e^{i|x|^2/(2t)}\nabla\left(ue^{-i|x|^2/(2t)}\right), \tag{2.4}$$

we can rewrite

$$\|(x+it\nabla)u\|^2_{L^2} = t^2\|\nabla\left(ue^{-i|x|^2/(2t)}\right)\|^2_{L^2} \equiv t^2\|v\|^2_{L^2}.$$

The sharp Gagliardo–Nirenberg inequality (1.6) applied to v, together with (2.2), then yields

$$\begin{aligned}\|(x+it\nabla)u\|^2_{L^2} - t^2\|u\|^4_{L^4} &\ge \|(x+it\nabla)u\|^2_{L^2} - (1-\eta)\|(x+it\nabla)u\|^2_{L^2}\\ &= \eta\|(x+it\nabla)u\|^2_{L^2}.\end{aligned}$$

Hence, the pseudo-conformal conservation law implies

$$(x+it\nabla)u \in L^\infty(\mathbb{R}_t;L^2(\mathbb{R}^2)).$$

Invoking (2.4) and general Gagliardo–Nirenberg inequalities, for $2 \le r < \infty$,

$$\|u(t)\|_{L^r(\mathbb{R}^2)} \lesssim \|u(t)\|^{1-\theta}_{L^2(\mathbb{R}^2)}\left(\frac{1}{t}\|(x+it\nabla)u\|_{L^2}\right)^\theta, \quad \theta = 1-\frac{2}{r},$$

we infer $u \in L^q(\mathbb{R};L^r(\mathbb{R}^2))$ for all admissible pairs, i.e. $\|u\|_{S(\mathbb{R})} < \infty$.

Step 2. For general $u_0 \in H^1(\mathbb{R}^2)$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, we decompose

$$u_0 = \underbrace{\chi^\varepsilon u_0}_{=:v_0} + \underbrace{(1-\chi^\varepsilon)u_0}_{=:w_0},$$

for $\chi \in C^\infty(\mathbb{R}^2;[0,1])$ some smooth cut-off function equal to 1 on a large ball centered at the origin, so that

$$\|w_0\|_{H^1(\mathbb{R}^2)} \leq \varepsilon.$$

By assumption (2.2), we have

$$\|v_0\|_{L^2}^2 \leq (1-\eta)\|Q\|_{L^2}^2,$$

an estimate which is obviously uniform in ε. Define v as the solution to (1.1), with initial datum v_0. Then, Step 1 above yields, for any $2 \leq r < \infty$,

$$\|v(t)\|_{L^r(\mathbb{R}^2)} \leq \frac{C(r)}{|t|^{1-2/r}}, \quad \forall |t| \geq 1, \tag{2.5}$$

where $C(r)$ is independent of t and ε. Writing $u = v + w$, the remainder w solves an equation of the form

$$i\partial_t w + \frac{1}{2}\Delta w = F(v,w), \quad w_{|t=0} = w_0,$$

where, by using Young's inequality, F satisfies pointwise estimates of the form

$$|F(v,w)| \lesssim |w|^3 + |w|^5 + |v|^2|w| + |v|^4|w|.$$

We shall now briefly recall an argument from [40], which uses the same estimates as the proof of Lemma 2.2: Let $I = [t_0, t_1]$ be some time interval, with $t_1 \geq t_0 \geq 0$. Strichartz estimates and Hölder's inequality yield

$$\begin{aligned}
\|w\|_{S(I)} &\lesssim \|w(t_0)\|_{L^2} + \|w\|_{L^4(I\times\mathbb{R}^2)}^3 + \|w\|_{L^{12}(I;L^6)}^4 \|w\|_{L^3(I;L^6)} \\
&\quad + \|v\|_{L^4(I\times\mathbb{R}^2)}^2 \|w\|_{L^4(I\times\mathbb{R}^2)} + \|v\|_{L^{12}(I;L^6)}^4 \|w\|_{L^3(I;L^6)} \\
&\lesssim \|w(t_0)\|_{L^2} + \|w\|_{S(I)}^3 + \|w\|_{L^{12}(I;L^6)}^4 \|w\|_{S(I)} \\
&\quad + \|v\|_{L^4(I\times\mathbb{R}^2)}^2 \|w\|_{S(I)} + \|v\|_{L^{12}(I;L^6)}^4 \|w\|_{S(I)}.
\end{aligned}$$

Recall that $u, v \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2))$, hence $w \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2))$, and we have

$$\|w\|_{L^{12}(I;L^6)}^4 \leq \|w\|_{L^\infty(I;L^6)}^3 \|w\|_{L^3(I;L^6)} \lesssim \|w\|_{L^\infty(I;H^1)}^3 \|w\|_{S(I)} \leq C\|w\|_{S(I)},$$

for some C independent of I and ε. Thus, we come up with

$$\|w\|_{S(I)} \lesssim \|w(t_0)\|_{L^2} + \|w\|_{S(I)}^3 + \|w\|_{S(I)}^2 + \|v\|_{L^4(I\times\mathbb{R}^2)}^2 \|w\|_{S(I)} + \|v\|_{L^{12}(I;L^6)}^4 \|w\|_{S(I)}.$$

In view of (2.5), $v \in L^4(\mathbb{R}\times\mathbb{R}^2) \cap L^{12}(\mathbb{R}; L^6(\mathbb{R}^2))$ and we can split $\mathbb{R}_t$ into finitely many intervals (this number of intervals being independent of ε) on which the last two terms in the above estimates can be absorbed by the left hand side, so that

$$\|w\|_{S(0,t)} \lesssim \|w(t_0)\|_{L^2} + \|w\|_{S(0,t)}^3 + \|w\|_{S(0,t)}^2 \lesssim \varepsilon + \|w\|_{S(0,t)}^3 + \|w\|_{S(0,t)}^2 \quad \forall t \geq 0.$$

For ε sufficiently small, a bootstrap argument then yields $\|w\|_{S(0,\infty)} \lesssim \varepsilon$, and so

$$\|u\|_{S(0,\infty)} < \infty.$$

The case of negative times is obviously similar, which establishes (2.1) in the case of subcritical mass.

2.3. **Mass critical case.** Let now $u_0 \in H^1(\mathbb{R}^2)$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. If we suppose, like in the previous subsection, that additionally $|\cdot|u_0 \in L^2(\mathbb{R}^2)$, then the pseudo-conformal conservation law yields only

$$\|u(t)\|_{L^6}^6 \lesssim \frac{1}{1+t^2}, \quad \int_0^\infty t\|u(t)\|_{L^6}^6 dt < \infty,$$

which does not rule out a behavior of the form

$$\|u(t)\|_{L^6}^6 \approx \frac{1}{t^2(\log t)^2} \quad \text{as } t \to \infty,$$

in which case $u \notin L_t^3 L_x^6$ (recall that $(3,6)$ is an admissible pair). In other words, a direct use of the pseudo-conformal conservation law seems hopeless in the mass critical case, since we cannot access a convenient bound on $\|(x+it\nabla)u\|_{L^2}$ (we do not even have a moderate growth of this quantity, like $\mathcal{O}(t^\gamma)$ for some $\gamma < 1$, as was exploited in [41]). Moreover, if we try to proceed like in Step 2, we lose the uniformity in ε for v, which will cause the bootstrap argument to break down.

Remark 2.3. Note that if the defocusing nonlinearity was weaker, for instance quartic,

$$i\partial_t u + \frac{1}{2}\Delta u = -|u|^2 u + |u|^3 u, \quad x \in \mathbb{R}^2,$$

then the same approach as above would yield

$$\|u(t)\|_{L^5}^5 \lesssim \frac{1}{1+t^2},$$

and so $u \in L_t^{10/3} L_x^5$, an admissible pair. We could then proceed as in the subcritical mass case.

To overcome these issues, we use a strategy based on profile decompositions, which has by now become a classical tool for critical problems. More precisely, we shall adapt and partially repeat parts of the strategy from [14, 19], treating the 3D cubic nonlinear Schrödinger equation (which is $\dot{H}^{1/2}$-critical). To begin with, we have a profile decomposition without scales, as in [14] (scales are not relevant in our context, since (1.1) is not scale-invariant).

Lemma 2.4 (Profile decomposition)**.** *Let $(\phi_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$ be uniformly bounded. For each $M \in \mathbb{N}$, there exists a subsequence, also denoted ϕ_n, and*

(1) *for each $1 \le j \le M$, there exists a fixed profile $\psi^j(x)$ in $H^1(\mathbb{R}^2)$,*
(2) *for each $1 \le j \le M$, there exists a sequence (in n) of time shifts $t_n^j \in \mathbb{R}$,*
(3) *for each $1 \le j \le M$, there exists a sequence (in n) of space shifts $x_n^j \in \mathbb{R}^2$,*
(4) *there exists a sequence (in n) of remainders $W_n^M \in H^1(\mathbb{R}^2)$, such that*

$$\phi_n(x) = \sum_{j=1}^M e^{-i\frac{t_n^j}{2}\Delta}\psi^j(x - x_n^j) + W_n^M(x).$$

The time and space shifts have a pairwise divergence property: for $1 \le j \ne k \le M$,

$$\lim_{n\to\infty} \left(|t_n^j - t_n^k| + |x_n^j - x_n^k|\right) = \infty. \tag{2.6}$$

The remainder sequence has the following asymptotic smallness property,

$$\lim_{M\to\infty}\left(\lim_{n\to\infty}\left\|W_n^M\right\|_{L^2(\mathbb{R}^2)}\right) = 0.$$

For fixed M, we have the asymptotic Pythagorean expansion

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^M \|\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1), \quad 0 \le s \le 1.$$

The only difference with the statement of [14, Lemma 2.1] is that the asymptotic smallness of the free evolution of W_n^M is stated for $\dot H^{1/2}$-admissible pairs there, while we consider L^2-admissible pairs. In turn, we simply replace $L_t^\infty L_x^3$-norms in the proof of [14, Lemma 2.1] (corresponding to an $\dot H^{1/2}$-admissible pair) with $L_t^\infty L_x^2$-norms, which is obviously L^2-admissible, in a similar fashion as in the linear profile decomposition of [5] (up to the fact that the present problem is not scale invariant, as pointed out above).

We then argue as in the proof of [19, Proposition 5.5]: Let $t_n \to \infty$, and set $\phi_n(x) \equiv u(t_n, x)$, solution to (1.1). This sequence is bounded in $H^1(\mathbb{R}^2)$, in view of Proposition 1.1, so we may apply Lemma 2.4. Now, consider ϕ_n to be a sequence of initial data to equation (1.1). We then want to show that for $n \in \mathbb{N}$ sufficiently large

$$\|u\|_{S(t_n,\infty)} < \infty.$$

To this end, we distinguish two cases: more than one profile ψ^j is non-zero, or at most one is, since in the case where all profiles are zero, we have nothing left to prove.

First case: more than one profile is non-zero. In this case, the asymptotic Pythagorean expansion with $s = 0$ shows that each ψ^j has a mass strictly smaller than that of Q. Up to passing to a subsequence in n, we have three cases:

$$t_n^j \to -\infty, \quad t_n^j \to +\infty, \quad \text{or } t_n^j \to T \text{ finite.}$$

In each case, invoking Propositions 1.1 and 1.3 (where we can of course replace the limit $t \to -\infty$ with $t \to +\infty$), the subcritical mass case presented in the previous subsection implies that we have a profile $\tilde\psi^j$ in H^1 such that

$$\left\| \mathrm{NLS}(-t_n^j)\tilde\psi^j - e^{-i\frac{t_n^j}{2}\Delta}\psi^j \right\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0,$$

where $\mathrm{NLS}(t)f$ stands for the solution at time t to (1.1) with initial data f. We infer

$$\phi_n(x) = \sum_{j=1}^M \mathrm{NLS}(-t_n^j)\tilde\psi^j(x - x_n^j) + \tilde W_n^M,$$

where

$$\lim_{M\to\infty}\left(\lim_{n\to\infty}\left\|\tilde W_n^M\right\|_{L^2(\mathbb{R}^2)}\right) = 0.$$

Now, let

$$v^j(t) = \mathrm{NLS}(t)\tilde\psi^j, \quad u_n(t) = \mathrm{NLS}(t)\phi_n, \quad \tilde u_n(t) = \sum_{j=1}^M v^j\left(t - t_n^j, x - x_n^j\right).$$

The subcritical mass case implies that for each j,

$$\|v^j\|_{S(\mathbb{R})} < \infty.$$

By construction, $\tilde u_n$ solves (1.1) up to an asymptotically small source term,

$$i\partial_t \tilde u_n + \frac{1}{2}\Delta\tilde u_n = -|\tilde u_n|^2\tilde u_n + |\tilde u_n|^4\tilde u_n + e_n + f_n,$$

where

$$e_n(t,x) = |\tilde u_n(t,x)|^2\tilde u_n(t,x) - \sum_{j=1}^M \left|v^j\left(t - t_n^j, x - x_n^j\right)\right|^2 v^j\left(t - t_n^j, x - x_n^j\right)$$

$$f_n(t,x) = -|\tilde u_n(t,x)|^4\tilde u_n(t,x) + \sum_{j=1}^M \left|v^j\left(t - t_n^j, x - x_n^j\right)\right|^4 v^j\left(t - t_n^j, x - x_n^j\right).$$

Let $\varepsilon > 0$. For $M_1(\varepsilon)$ sufficiently large, $M \leq M_1$, and $n_1 = n_1(M)$ sufficiently large, we have,

$$\left\| e^{i\frac{t}{2}\Delta}\left(u_n(0,\cdot) - \tilde{u}_n(0,\cdot)\right)\right\|_{S(\mathbb{R})} = \left\| e^{i\frac{t}{2}\Delta} W_n^M \right\|_{S(\mathbb{R})} \leq \varepsilon, \quad \forall n \geq n_1.$$

Using orthogonality properties based on (2.6) (as established initially in [23, 31]), we find that given M and $\varepsilon > 0$, for $n \geq n_2$ sufficiently large,

$$\|e_n\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^2)} + \|f_n\|_{L^{3/2}(\mathbb{R};L^{6/5})} \leq \varepsilon,$$

where we have used the same (dual) admissible pairs as in the proof of Lemma 2.2. Recall that from the mass-subcritical case, we know that

$$\|\tilde{u}_n\|_{S(\mathbb{R})} < \infty.$$

Then using a long-time perturbation argument, as in the second step of the mass-subcritical case (based on bootstrap), we infer

$$\|u_n\|_{S(\mathbb{R})} < \infty.$$

Second case: only one profile is non-zero, say $\psi^1 \neq 0$ and $\psi^j = 0$ for $j \geq 2$, and so

$$\phi_n(x) = e^{-i\frac{t_n^1}{2}\Delta}\psi^j(x - x_n^1) + W_n(x), \quad \lim_{n\to\infty}\left\| e^{i\frac{t}{2}\Delta} W_n \right\|_{S(\mathbb{R})} = 0. \tag{2.7}$$

Like above, up to passing to a subsequence in n, we have three cases: $t_n^1 \to -\infty$, $t_n^1 \to \infty$, or $t_n^1 \to T$ finite. It turns out that only the first case is possible.

• Suppose $t_n^1 \to -\infty$, and fix $\varepsilon > 0$, (q,r) an admissible pair with $r > 2$ (or equivalently $q < \infty$). For n sufficiently large.

$$\|e^{i\frac{t}{2}\Delta}\phi_n\|_{L^q(0,\infty;L^r)} \leq \|e^{i\frac{t}{2}\Delta}\psi^1\|_{L^q(-t_n^1,\infty;L^r)} + \left\| e^{i\frac{t}{2}\Delta} W_n \right\|_{S(\mathbb{R})} \leq \varepsilon,$$

where the first term on the right hand side goes to zero as $n \to \infty$ since $e^{i\frac{t}{2}\Delta}\psi^1 \in L^q(\mathbb{R};L^r)$ thanks to Strichartz estimates. On the other hand, we know that $u \in L^\infty H^1$, so using the same estimates as in the proof of Lemma 2.2, we find, for n sufficiently large and $t \geq t_n$,

$$\|u\|_{\dot{X}^0(t_n,t)} := \|u\|_{L^4((t_n,t)\times\mathbb{R}^2)} + \|u\|_{L^3(t_n,t;L^6)} \leq \varepsilon + C\|u\|^2_{\dot{X}^0(t_n,t)} + C\|u\|^3_{\dot{X}^0(t_n,t)},$$

where C depends only on the $L^\infty H^1$ norm of u. Here, and in the following we resume the same notation as in [40]

$$\dot{X}^0(I) = L^4(I\times\mathbb{R}^2) \cap L^3(I;L^6(\mathbb{R}^2)).$$

A bootstrap argument then implies than for ε sufficiently small (n sufficiently large), $\|u\|_{\dot{X}^0(t_n,\infty)} < \infty$, hence $\|u\|_{S(t_n,\infty)} < \infty$ by using Strichartz inequalities again.

• Suppose $t_n^1 \to +\infty$. We then use the same idea as in the first case, but going backwards in time. For $t > 0$, Duhamel's formula for u reads, in view of (2.7),

$$\begin{aligned} u(t_n - t, x) = &\underbrace{e^{-i\frac{t+t_n^1}{2}\Delta}\psi^1(x - x_n^1) + e^{-i\frac{t}{2}\Delta}W_n(x)}_{=:u_n^{\mathrm{lin}}(t,x)} \\ &- i\int_0^{-t} e^{-i\frac{t+s}{2}\Delta}\left(-|u|^2u + |u|^4u\right)(s)ds. \end{aligned}$$

We have

$$\|u_n^{\mathrm{lin}}\|_{\dot{X}^0(0,\infty)} \underset{n\to\infty}{\longrightarrow} 0,$$

so by the same bootstrap argument as in the first case,

$$\|u\|_{\dot{X}^0(-\infty,t_n)} \underset{n\to\infty}{\longrightarrow} 0,$$

which is in contradiction with the fact that we consider a non-zero initial data $u_0 \in H^1$.

- Suppose $t_n^1 \to T \in \mathbb{R}$. Then by the same estimates as above,

$$u(t_n, x) = e^{-i\frac{T}{2}\Delta}\psi^1(x - x_n^1) + o_n(1) \text{ in } L^2(\mathbb{R}^2),$$

hence, since $u \in L^\infty(\mathbb{R}; H^1)$,

$$u(t_n, x) = e^{-i\frac{T}{2}\Delta}\psi^1(x - x_n^1) + o_n(1) \text{ in } H^s(\mathbb{R}^2), \quad \forall\, 0 \le s < 1.$$

In particular, unless $\psi^1 \equiv 0$ (a case which has been ruled out at the beginning of the discussion),

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^6(\mathbb{R}^2)} > 0. \tag{2.8}$$

Recall that in the case $\|u_0\|_{L^2} \le \|Q\|_{L^2}$, where additionally $|\cdot|u_0 \in L^2(\mathbb{R}^2)$, the pseudo-conformal conservation law implies

$$\|u(t)\|_{L^6}^6 \lesssim \frac{1}{t^2}, \quad \forall t \ge 1,$$

uniformly in $\|Q\|_{L^2} - \|u_0\|_{L^2} \ge 0$, hence a contradiction to (2.8) in the case where u_0 has finite variance. For general $u_0 \in H^1(\mathbb{R}^2)$ and $\varepsilon > 0$, let v be as in the Step 2 of the subcritical mass case: we have

$$\|v_0\|_{L^2} \le \|Q\|_{L^2}, \quad \|u_0 - v_0\|_{H^1} \le \varepsilon, \quad \|v(t)\|_{L^6}^6 \le \frac{C}{t^2}, \quad \forall t \ge 1,$$

where C does not depend on ε. Strichartz estimates yield

$$\begin{aligned}\|u - v\|_{\dot{X}^0(0,t)} \lesssim \varepsilon + \|u - v\|_{\dot{X}^0(0,t)}^3 + \left(\|u\|_{L^{12}(0,t;L^6)}^4 + \|v\|_{L^{12}(0,t;L^6)}^4\right)\|u - v\|_{\dot{X}^0(0,t)} \\ + \left(\|u\|_{L^4((0,t)\times\mathbb{R}^2)}^2 + \|v\|_{L^4((0,t)\times\mathbb{R}^2)}^2\right)\|u - v\|_{\dot{X}^0(0,t)}.\end{aligned}$$

Using the property $u, v \in L_t^\infty H_x^1$ (with bounds independent of ε), and Sobolev embedding, yields

$$\|u - v\|_{\dot{X}^0(0,t)} \lesssim \varepsilon + \max\left(t^{1/2}, t^{1/3}\right)\|u - v\|_{\dot{X}^0(0,t)}.$$

Repeating this argument on different time intervals, we have the uniform bound

$$\|u - v\|_{\dot{X}^0(0,t)} \le C\varepsilon e^{ct^\alpha},$$

for some $\alpha > 0$ whose optimal value is irrelevant. The important aspect is that the constants α, c and C are independent of ε, thus showing, up to a suitable choice of ε_n in terms of t_n,

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^6(\mathbb{R}^2)} = 0,$$

hence a contradiction to (2.8).

3. Existence of solitons and first properties

3.1. **A priori estimates.** Suppose we have a solution $u(t,x) = e^{i\omega t}\phi(x)$, with ϕ sufficiently smooth and localized. Then (1.1) becomes

$$-\frac{1}{2}\Delta\phi - |\phi|^2\phi + |\phi|^4\phi + \omega\phi = 0. \tag{3.1}$$

Proposition 3.1 (A priori estimates for solitary waves)**.** *Let $1 \le d \le 3$. If $\phi \in H^1(\mathbb{R}^d)$ solves* (3.1)*, then we have:*

(1) *Pohozaev identities:*

$$\frac{1}{2}\int_{\mathbb{R}^d}|\nabla\phi|^2\,dx-\int_{\mathbb{R}^d}|\phi|^4\,dx+\int_{\mathbb{R}^d}|\phi|^6\,dx+\omega\int_{\mathbb{R}^d}|\phi|^2\,dx=0, \tag{3.2}$$

$$\frac{d-2}{2}\int_{\mathbb{R}^d}|\nabla\phi|^2\,dx-\frac{d}{2}\int_{\mathbb{R}^d}|\phi|^4\,dx+\frac{d}{3}\int_{\mathbb{R}^d}|\phi|^6\,dx+\omega d\int_{\mathbb{R}^d}|\phi|^2\,dx=0. \tag{3.3}$$

(2) *If $\phi\not\equiv 0$, then $0<\omega<\frac{3}{16}$.*

(3) *If $d=2$ and $\phi\neq 0$, then $\|\phi\|_{L^2}>\|Q\|_{L^2}$, where Q is cubic ground state solution to* (1.5).

(4) *If in addition $\phi\in L^\infty\cap C^2$ is real-valued, then*

$$\|\phi\|_{L^\infty(\mathbb{R}^d)}\leq\sqrt{\frac{1+\sqrt{1-4\omega}}{2}}. \tag{3.4}$$

Proof. For item (1), we quickly recall the method to derive Pohozaev identities formally, and refer to [4] for a rigorous justification via density type arguments. Firstly, multiplying (3.1) by $\bar\phi$ and integrating yields (3.2). In particular, we infer $\omega\in\mathbb{R}$. Secondly, by multiplying (3.1) with $x\cdot\nabla\bar\phi$ and integrating by parts we obtain (3.3). For $d=2$, subtracting (3.3) from (3.2) yields

$$\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2\,dx+\frac{1}{3}\int_{\mathbb{R}^2}|\phi|^6\,dx=\omega\int_{\mathbb{R}^2}|\phi|^2\,dx,$$

hence $\omega>0$ unless $\phi\equiv 0$. In the case $d=3$, we obtain similarly

$$\frac{1}{2}\int_{\mathbb{R}^3}|\phi|^4\,dx=2\omega\int_{\mathbb{R}^3}|\phi|^2\,dx,$$

and thus we arrive at the same conclusion.

(2) From now on, we shall denote

$$F(s)=\frac{1}{4}s^4-\frac{1}{6}s^6$$

and set

$$\omega^*=\sup\left\{\omega>0;\quad \frac{\omega}{2}s^2-F(s)<0 \text{ for some } s>0\right\}.$$

A computation then shows $\omega^*=\frac{3}{16}$. In particular, if $\omega\geq\frac{3}{16}$, we have the pointwise relation

$$-\frac{1}{2}|\phi|^4+\frac{1}{3}|\phi|^6+\omega|\phi|^2\geq 0,\quad \forall x\in\mathbb{R}^d,$$

which, in view (3.3), implies $\phi\equiv 0$ for $d\geq 2$. In the case $d=1$, the conclusion follows from ODE arguments, and more precisely [4, Theorem 5].

(3) We suppose $d=2$ and introduce

$$\gamma=\frac{\|\phi\|_{L^4}^4}{\|\nabla\phi\|_{L^2}^2},$$

which allows us to rewrite (3.2) as

$$\left(\frac{1}{2}-\gamma\right)\int_{\mathbb{R}^2}|\nabla\phi|^2+\int_{\mathbb{R}^2}\phi^6+\omega\int_{\mathbb{R}^2}\phi^2=0,$$

Similarly, we can rewrite (3.3) for $d=2$, by using γ, in the following form

$$\frac{\gamma}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2-\frac{1}{3}\int_{\mathbb{R}^2}|\phi|^6-\omega\int_{\mathbb{R}^2}|\phi|^2=0.$$

Combining these identities, we infer

$$\int_{\mathbb{R}^2}|\phi|^6=\frac{3(\gamma-1)}{4}\int_{\mathbb{R}^2}|\nabla\phi|^2,$$

and in particular $\gamma > 1$, i.e. $\|\phi\|_{L^4}^4 > \|\nabla\phi\|_{L^2}^2$. In view of the sharp Gagliardo-Nirenberg inequality (1.6), this consequently implies that the mass of the cubic-quintic ground states satisfies $\|\phi\|_{L^2} > \|Q\|_{L^2}$.

(4) Let $\phi \in C^2$ be a real-valued bounded solution to (3.1). Suppose that ϕ reaches its maximum at $x_0 \in \mathbb{R}^d$. Then $\Delta\phi(x_0) \leq 0$, and hence

$$(-\phi^3 + \phi^5 + \omega\phi)_{|x=x_0} \leq 0.$$

Writing

$$\omega\phi - \phi^3 + \phi^5 = \phi\left(\phi^2 - \frac{1-\sqrt{1-4\omega}}{2}\right)\left(\phi^2 - \frac{1+\sqrt{1-4\omega}}{2}\right),$$

we see that

$$\phi(x_0) \leq \sqrt{\frac{1+\sqrt{1-4\omega}}{2}}.$$

Reasoning similarly for a minimum of ϕ, we infer (3.4). □

3.2. **Existence and uniqueness.** Denote $f(s) = s^3 - s^5$ and

$$F(s) = \int_0^s f(\tau)\,d\tau = \frac{1}{4}s^4 - \frac{1}{6}s^6,$$

as before. We have already seen that

$$\sup\left\{\omega > 0; \quad \frac{\omega}{2}s^2 - F(s) < 0 \text{ for some } s > 0\right\} = \frac{3}{16}.$$

Then according to [4] (treating the case $d = 1$ or $d = 3$) and [3] (treating the case $d = 2$), for all $\omega \in]0, \omega^*[$, there exists a solution ϕ_ω of (3.1). Uniqueness of ϕ_ω in $d = 1$ is proven in [4], while in $d = 3$ uniqueness follows from [36], as pointed out in [25]. Finally, for $d = 2$, we infer uniqueness from the results of [21], where we emphasize that the assumptions made there correspond more closely to those made to prove existence.

We summarize all of these results in the proposition below, we recall that the action, defined in the introduction, is given by

$$S(\phi) = \frac{1}{2}\|\nabla\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - 2V(\phi), \quad \text{where} \quad V(\phi) = \int_{\mathbb{R}^d} F(\phi(x))\,dx,$$

and satisfies

$$S(\phi) = E(\phi) + \omega\|\phi\|_{L^2}^2. \tag{3.5}$$

Proposition 3.2 (Uniqueness of radial ground states)**.** *Let $1 \leq d \leq 3$. Suppose that*

$$0 < \omega < \frac{3}{16}.$$

Then (3.1) *has a unique solution ϕ_ω such that*

(1) *$\phi_\omega > 0$ on $\mathbb{R}^d$.*
(2) *ϕ_ω is radially symmetric, $\phi_\omega(x) = \phi(r)$, where $r = |x|$, and ϕ is a non-increasing function of r.*
(3) *$\phi_\omega \in C^2(\mathbb{R}^d)$.*
(4) *The derivatives of order at most two of ϕ_ω decay exponentially:*

$$\exists\delta > 0, \quad |\partial^\alpha\phi_\omega(x)| \lesssim e^{-\delta|x|}, \quad |\alpha| \leq 2.$$

(5) *For every solution φ to* (3.1)*,*

$$0 < S(\phi_\omega) \leq S(\varphi).$$

3.3. **Further properties.** We collect here some further, asymptotic properties of cubic-quintic ground states. These have been established in [25, Theorem 2.2] in the case $d = 3$, but can easily be adapted to the case $d = 2$. We therefore assume $d = 2$ or 3 and note that the map $\omega \mapsto \phi_\omega$ given by Proposition 3.2 is real analytic.

First, we want to better understand the limit $\omega \to 0$ in (3.1) which, unfortunately, is singular. To turn it into a regular limit, we change the unknown function ϕ_ω into

$$\psi_\omega(x) = \frac{1}{\sqrt{\omega}}\phi_\omega\left(\frac{x}{\sqrt{\omega}}\right).$$

Then (3.1) is equivalent to

$$-\frac{1}{2}\Delta\psi_\omega + \psi_\omega - \psi_\omega^3 + \omega\psi_\omega^5 = 0,$$

and it can be shown that $\psi_\omega \to Q$ in $H^1(\mathbb{R}^d)$ as $\omega \to 0$, where Q is the unique non-negative, radially symmetric ground state solution to

$$-\frac{1}{2}\Delta Q + Q - Q^3 = 0, \quad x \in \mathbb{R}^d.$$

In particular, in the case $d = 2$, we infer

$$\|\phi_\omega\|_{L^2(\mathbb{R}^2)} = \|\psi_\omega\|_{L^2(\mathbb{R}^2)} \underset{\omega\to 0}{\longrightarrow} \|Q\|_{L^2(\mathbb{R}^2)},$$

thus showing that ground states for the cubic-quintic NLS in 2D have mass strictly larger but arbitrarily close to that of the cubic ground state Q.

Second, one may wonder about the limit as $\omega \to \frac{3}{16}$. In this case, we shall see that the mass of ϕ_ω grows to infinity. For $d = 3$, this is established in Theorem 2.2, (v), in [25], where the authors also prove that

$$\left(\tfrac{3}{16} - \omega\right)^{-3} \lesssim M(\phi_\omega) \lesssim \left(\tfrac{3}{16} - \omega\right)^{-3},$$

Here, we propose a simpler argument in the case $d = 2$, which however only shows divergence of the mass in the limit (but no asymptotic behavior).

Let $d = 2$, and suppose that $\|\phi_\omega\|_{L^2}$ is bounded as $\omega \to \frac{3}{16}$. Then, using Hölder's inequality (1.2) in the Pohozaev identity (3.3) we infer

$$\int_{\mathbb{R}^2} |\phi_\omega|^6 \le \left(3 - \frac{2\omega}{3}\right)\|\phi_\omega\|_{L^2}^2.$$

This implies that also $\|\phi_\omega\|_{L^6}$ remains bounded as $\omega \to \frac{3}{16}$. Using Hölder's inequality once more, we conclude that $\|\phi_\omega\|_{L^4}$ remains bounded as well, which, together with (3.2), yields the boundedness of ϕ_ω in $H^1(\mathbb{R}^2)$. Since ϕ_ω is radial, the sequence $(\phi_\omega)_\omega$ is compact in $L^2 \cap L^6(\mathbb{R}^2)$, in view of Strauss' lemma [37]. Therefore, up to the choice of a suitable subsequence,

$$\phi_\omega \underset{\omega\to 3/16}{\longrightarrow} \Phi \quad \text{strongly in } L^2 \cap L^6(\mathbb{R}^2).$$

Passing to the limit in (3.3) (recall that $d = 2$ here), we get

$$\int_{\mathbb{R}^2}\left(-\frac{1}{4}\Phi^4 + \frac{1}{6}\Phi^6 + \frac{\omega}{2}\Phi^2\right)\Big|_{\omega=3/16} = 0.$$

But $\omega = \frac{3}{16}$ was obtained as

$$\frac{3}{16} = \sup\left\{\omega > 0; \quad \frac{\omega}{2}s^2 - \frac{s^4}{4} + \frac{s^6}{6} < 0 \text{ for some } s > 0\right\},$$

so we infer $\Phi \equiv 0$, which contradicts the property

$$\|\phi_\omega\|_{L^2} > \|Q\|_{L^2}, \quad \forall \omega \in \left]0, \tfrac{3}{16}\right[.$$

Therefore, possibly along some subsequence, $\|\phi_\omega\|_{L^2} \to \infty$ as $\omega \to \frac{3}{16}$, so the range of the map $\omega \mapsto \|\phi_\omega\|_{L^2}$ is exactly $]\|Q\|_{L^2}, \infty[$, as ω varies in $]0, \frac{3}{16}[$.

Remark 3.3. It turns out that this map is non-decreasing, as a consequence of the orbital stability established in the next section, and the Grillakis-Shatah-Strauss theory (the spectral assumptions needed to apply this theory can be obtained by adapting the argument of [25] from the 3D to the 2D case, see the appendix).

4. Orbital stability of ground states in 2D

To prove orbital stability in the context of nonlinear Schrödinger equations, there are two main strategies: the first one, historically, is due to Cazenave and Lions [7] and based on concentration-compactness arguments. The second one, known as Grillakis-Shatah-Strauss theory, was introduced in [18] (see also [12]), and generalized the ideas developed by M. Weinstein in [43, 44]. Here, we shall follow the former strategy.

Proposition 4.1 (Energy minimization)**.** *Let $d = 2$ and $\omega \in]0, \frac{3}{16}[$. For $\rho > \|Q\|^2_{L^2(\mathbb{R}^2)}$, consider*

$$\Gamma = \left\{ u \in H^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} |u|^2 = \rho \right\},$$

and denote

$$-\nu = \inf \left\{ E(u)\,;\, u \in H^1(\mathbb{R}^2),\, M(u) = \rho \right\}.$$

Then the following properties hold:

(1) *The minimization problem*

$$u \in \Gamma, \quad E(u) = \min\{E(v),\ v \in \Gamma\} \tag{4.1}$$

has a solution.

(2) *If $(u_n)_{n\in\mathbb{N}} \subset \Gamma$ satisfies $M(u_n) \to \rho$ and $E(u_n) \to -\nu$, then there exist a subsequence, still denoted by u_n, and a sequence $y_n \in \mathbb{R}^2$ such that $u_n(\cdot - y_n)$ has a strong limit u in $H^1(\mathbb{R}^2)$. In particular, u satisfies* (4.1)*.*

Proof. **First step.** We show that

$$\inf \left\{ E(u)\,;\, u \in H^1(\mathbb{R}^2),\, M(u) = \rho \right\} = -\nu,$$

for some finite $\nu > 0$. To prove that the infimum is finite, we use Hölder's inequality (1.2), to infer

$$E(u) \geq \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{\sqrt{\rho}}{2}\|u(t)\|^3_{L^6(\mathbb{R}^d)} + \frac{1}{3}\|u(t)\|^6_{L^6(\mathbb{R}^d)},$$

and thus $E(u)$ is bounded from below. To see that the infimum is negative, consider the L^2-invariant scaling, for $\lambda > 0$,

$$u_\lambda(x) = \lambda^{d/2} u(\lambda x),$$

which, for $d = 2$, implies

$$E(u_\lambda) = \frac{\lambda^2}{2}\left(\|\nabla u\|^2_{L^2} - \|u\|^4_{L^4} + \frac{2}{3}\lambda^2 \|u\|^6_{L^6} \right).$$

In view of the sharp Gagliardo-Nirenberg inequality, and since $\|u\|^2_{L^2} > \|Q\|^2_{L^2}$, we may choose a profile $u \in H^1$ so that the terms independent of λ inside the parentheses become negative, e.g., take

$$u = \left(\frac{\rho}{M(Q)} \right)^{1/2} Q, \text{ with } \lambda > 0 \text{ sufficiently small.}$$

Second step. Any minimizing sequence is bounded away from zero in L^4. Let $(u_n)_{n\geq 0}$ be a minimizing sequence: for n sufficiently large, $E(u_n) \leq -\nu/2$, hence

$$\|u_n\|_{L^4}^4 \geq \nu > 0.$$

Third step. In view of [29] (see also [6, Proposition 1.7.6]), we have the standard trichotomy of concentration compactness. From the second step, vanishing is ruled out, so we have to rule out dichotomy to infer compactness. Arguing by contradiction, suppose that, after extraction of suitable subsequences, there exist $(v_k)_{k\geq 0}$, $(w_k)_{k\geq 0}$ in $H^1(\mathbb{R}^2)$, such that

$$\operatorname{supp} v_k \cap \operatorname{supp} w_k = \emptyset,\ |v_k| + |w_k| \leq |u_{n_k}|,\ \|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C\|u_{n_k}\|_{H^1},$$

satisfying

$$\|v_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} \theta\rho, \quad \|w_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} (1-\theta)\rho, \quad \text{for some } \theta \in]0,1[,$$

and

$$\begin{aligned}
&\liminf_{k\to\infty} \left(\int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \right) \geq 0,\\
&\left| \int |u_{n_k}|^p - \int |v_k|^p - \int |w_k|^p \right| \underset{k\to\infty}{\longrightarrow} 0,
\end{aligned}$$

for all $2 \leq p < \infty$. We infer

$$\liminf_{k\to\infty} \left(E\left(u_{n_k}\right) - E(v_k) - E(w_k) \right) \geq 0,$$

hence

$$\limsup_{k\to\infty} \left(E(v_k) + E(w_k) \right) \leq -\nu. \tag{4.2}$$

Following an idea from [9], we then use a scaling argument rather than a multiplicative one as in [7]. Let

$$\begin{aligned}
\tilde{v}_k(x) &= v_k\left(\lambda_k^{-1/2} x\right), \quad \lambda_k = \frac{\rho}{\|v_k\|_{L^2}^2}\\
\tilde{w}_k(x) &= w_k\left(\mu_k^{-1/2} x\right), \quad \mu_k = \frac{\rho}{\|w_k\|_{L^2}^2}.
\end{aligned}$$

Since $\tilde{v}_k$ and $\tilde{w}_k$ have mass ρ,

$$E(\tilde{v}_k),\ E(\tilde{w}_k) \geq -\nu.$$

On the other hand, we compute

$$E(\tilde{v}_k) = \lambda_k \left(\frac{1}{2\lambda_k} \int |\nabla v_k|^2 - \frac{1}{2} \int |v_k|^4 + \frac{1}{3} \int |v_k|^6 \right),$$

and so

$$E(v_k) = \frac{1}{\lambda_k} E(\tilde{v}_k) + \frac{1-\lambda_k^{-1}}{2} \int |\nabla v_k|^2 \geq \frac{-\nu}{\lambda_k} + \frac{1-\lambda_k^{-1}}{2} \int |\nabla v_k|^2.$$

Doing the same for $E(w_k)$, we find

$$\begin{aligned}
E(v_k) + E(w_k) &\geq -\nu \left(\frac{1}{\lambda_k} + \frac{1}{\mu_k} \right) + \frac{1-\lambda_k^{-1}}{2} \int |\nabla v_k|^2 + \frac{1-\mu_k^{-1}}{2} \int |\nabla w_k|^2\\
&\geq -\nu \left(\frac{1}{\lambda_k} + \frac{1}{\mu_k} \right) + \frac{1-\lambda_k^{-1}}{2\|v_k\|_{L^2}^2} \|v_k\|_{L^4}^4 + \frac{1-\mu_k^{-1}}{2\|w_k\|_{L^2}^2} \|w_k\|_{L^4}^4,
\end{aligned}$$

where in the second step, we have used the Gagliardo-Nirenberg inequality. Passing to the limit, yields

$$\lim_{k\to\infty} \inf \left(E(v_k) + E(w_k) \right) \geq -\nu + \frac{1}{2} \min \left(\frac{1-\theta}{\theta\rho}, \frac{\theta}{(1-\theta)\rho} \right) \liminf_{k\to\infty} \|u_{n_k}\|_{L^4}^4,$$

and hence a contradiction to (4.2), in view of the second step and $\theta \in]0,1[$. □

Recall the formula $S(\phi) = E(\phi) + \omega\|\phi\|_{L^2}^2$. Hence, for fixed mass, minimizing the action is equivalent to minimizing the energy. We then have the analogue of [6, Corollary 8.3.8], to which we also refer for the details of the proof:

Lemma 4.2 (Equivalence of minimizers)**.** *Let $d = 2$ and $0 < \omega < \frac{3}{16}$. Then $\phi \in H^1(\mathbb{R}^2)$ is a ground state of* (3.1) *if and only if ϕ solves the minimization problem*

$$\phi \in \Gamma, \quad S(\phi) = \min\{S(\varphi),\ \varphi \in \Gamma\}. \tag{4.3}$$

In addition, the problems (4.1) *and* (4.3) *are equivalent.*

Conclusion. At this stage, we have all the arguments to conclude in the classical way. Assume, by contradiction, that there exist a sequence $(u_{0,n})_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$, such that

$$\|u_{0,n} - \phi\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.4}$$

and a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, such that the sequence of solutions u_n to (1.1) associated to the initial data $u_{0,n}$ satisfies

$$\inf_{\substack{\theta\in\mathbb{R}\\ y\in\mathbb{R}^2}} \left\|u_n(t_n,\cdot) - e^{i\theta}\phi(\cdot - y)\right\|_{H^1(\mathbb{R}^2)} > \varepsilon, \tag{4.5}$$

for some $\varepsilon > 0$. Introducing $v_n = u_n(t_n,\cdot)$, the above inequality also reads

$$\inf_{\varphi\in G} \|v_n - \varphi\|_{H^1(\mathbb{R}^2)} > \varepsilon,$$

where G is the set of all possible ground states, as given in Definition 1.4. In view of (4.4) and Lemma 4.2,

$$\int_{\mathbb{R}^2} |u_{0,n}|^2 \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^2} |\phi|^2, \quad S\left(u_{0,n}\right) \underset{n\to\infty}{\longrightarrow} S(\phi) = \inf\{S(v),\ v \in \Gamma\}.$$

The conservation laws for mass and energy imply

$$\int_{\mathbb{R}^2} |v_n|^2 \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^2} |\phi|^2, \quad S\left(v_n\right) \underset{n\to\infty}{\longrightarrow} S(\phi),$$

so $(v_n)_n$ is a minimizing sequence for the problem (4.3), and hence also for the problem (4.1). From Proposition 4.1, there exists y_n in $\mathbb{R}^2$ and a solution u to (4.1) such that

$$\|v_n - u(\cdot - y_n)\|_{H^1} \underset{n\to\infty}{\longrightarrow} 0.$$

However, in view of Lemma 4.2, u is a ground state and so is $u(\cdot - y_n)$, hence a contradiction.

Appendix A. Stability and instability of three-dimensional ground states

A.1. **Grillakis-Shatah-Strauss theory.** The proof of Proposition 1.9 relies on the application of the theory developed in [18], and all the ingredients necessary to do so are already present in [25]. In view of the existence results given in Proposition 1.1 and Proposition 3.2, we only have to check the spectral Assumption 3 imposed in [18] and analyze the monotonicity of the map $\omega \mapsto M(\phi_\omega)$. To state the spectral assumption, we write the second order derivative of the action as

$$\langle S''(\phi_\omega)w, w\rangle = \frac{1}{2}\langle L_1 u, u\rangle + \frac{1}{2}\langle L_2 v, v\rangle,$$

where $w = u + iv$. In our case, we have

$$L_1 = -\frac{1}{2}\Delta + \omega - 3\phi_\omega^2 + 5\phi_\omega^4,$$
$$L_2 = -\frac{1}{2}\Delta + \omega - \phi_\omega^2 + \phi_\omega^4.$$

We then need to check:

Assumption A.1. For each $\omega \in]0, \frac{3}{16}[$, the Hessian $S''(\phi_\omega)$ has exactly one negative eigenvalue; its kernel is spanned by $i\phi_\omega$ and $\nabla\phi_\omega$, and the rest of its spectrum is positive and bounded away from zero.

If this holds true, then:

(a) If $\frac{d}{d\omega}M(\phi_\omega) > 0$, then the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable.
(b) If $\frac{d}{d\omega}M(\phi_\omega) < 0$, then the standing wave $e^{i\omega t}\phi_\omega(x)$ is unstable.

Indeed, the authors of [25] proved that Assumption A.1. holds true for the cubic-quintic NLS:

Proposition A.2 (Proposition 2.4 from [25])**.** *Fix $\ell = 0, 1, 2, \dots,$ and consider the restriction of L_1 to functions of the form $f(|x|)Y(x/|x|)$, where Y is a spherical harmonic of degree ℓ.*

(1) *When $\ell = 0$, the operator has exactly one negative eigenvalue; it is simple.*
(2) *When $\ell = 1$, there are no negative eigenvalues. Zero is an eigenvalue and its eigenspace is spanned by the three components of $\nabla\phi_\omega$.*
(3) *When $\ell \geq 2$, the operator is positive definite.*

The proof of this result relies on Sturm Oscillation Theorem, since the analysis boils down to second order ODEs for the radial function f. Note that the proof from [25] can be readily adapted to the 2D case, by replacing spherical harmonics with functions of the form $e^{i\ell\theta}$ in radial coordinates. The above proposition is complemented by the following one:

Proposition A.3 (Proposition 2.5 from [25])**.** *Let $\delta = \delta(r)$ be the solution to*

$$-\frac{1}{2}\delta'' - \frac{1}{r}\delta' + \left(5\phi_\omega^4 - 3\phi_\omega^2 + \omega\right)\delta = 0$$

obeying $\delta(0) = 1$. Then $\delta(r) \to -\infty$ as $r \to \infty$. Correspondingly, zero is not an eigenvalue of L_1 restricted to radial functions.

With this in hand, we can now proceed to prove (in-)stability of ground states, depending on the frequency $\omega \in]0, \frac{3}{16}[$.

A.2. **Instability for small ω.** Citing [25, Theorem 2.2], we have

$$\frac{d}{d\omega}M(\phi_\omega) < \frac{3\beta(\omega) - 1}{2\omega}M(\phi_\omega),$$

where

$$\beta(\omega) = \frac{\|\phi_\omega\|_{L^6}^6}{\|\nabla\phi_\omega\|_{L^2}^2}.$$

Therefore, we are in the unstable case (b) provided $\beta < \frac{1}{3}$. In particular, this is guaranteed for $\omega \in]0, \omega_0[$ where $\omega_0 \ll 1$, since $\beta(\omega) = \mathcal{O}(\omega)$ as $\omega \to 0$, see [25, Theorem 2.2, (iv)].

A.3. **Stability for large ω.** We know from [25, Theorem 2.2, (v)] that there are constants $c_1, c_2 > 0$ such that

$$c_1 \left(\tfrac{3}{16} - \omega\right)^{-3} \leq M(\phi_\omega) \leq c_2 \left(\tfrac{3}{16} - \omega\right)^{-3},$$

On the other hand, for ω sufficiently close to $\frac{3}{16}$, we can quote [25, Theorem 4.1, (iv)], which ensures that for $M(\phi_\omega) = m$ sufficiently large

$$\inf\left\{E(u);\ u \in H^1(\mathbb{R}^3),\ M(u) = m\right\} = [E_{\min}(m), \infty[,$$

with $E_{\min}(m) < 0$. It is then possible to resume the argument presented in Section 4, and conclude to orbital stability via the Cazenave-Lions argument.

The analysis of [25] yields more precise information regarding the set of ω's for which orbital stability holds, comforting Conjecture 2.3 from [25], whose statement was recalled in the introduction.

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