

Global Sensitivity Analysis: a novel generation of mighty estimators based on rank statistics

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Abstract

Correction to the proof of Lemma B.2 in Global Sensitivity Analysis: a novel generation of mighty estimators based on rank estimators. F. Gamboa, P. Gremaud, T. Klein, and A. Lagnoux. Bernoulli. (2021) Vol. 28, n°4, pp. 2345-2374.

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The purpose of this note is twofold. First, we correct a mistake in the expression of the limiting variance Σ_C that appears in [1, Lemma B.2]. Secondly, we provide simpler expressions for Σ_B in [1, Lemma B.1] and Σ_C . In particular, the expression of Σ_C does no longer depend on the first derivative of the model f . Furthermore, the updated formula of the limiting variance is now consistent with the limiting variance σ_{rank}^2 obtained in [2] by other means.

1 Computation of the asymptotic covariance matrix Σ_B

In [1, Section B.2], we obtained:

$$\Sigma_B^{1,1} = \mathbb{E}[\text{Var}(YY'|X)] + 2\mathbb{E}[\text{Cov}(YY', YY''|X)],$$

where we remind that $Y = f(X, W)$, $Y' = f(X, W')$, and $Y'' = f(X, W'')$ with W' and W'' independent copies of W . Let us introduce the notation

$$\phi(x) = \mathbb{E}[Y|X = x] \quad \text{and} \quad \sigma^2(x) = \text{Var}(Y|X = x).$$

Then, by conditional independence, we get

$$\text{Var}(YY'|X) = \mathbb{E}[(YY')^2|X] - \mathbb{E}[YY'|X]^2 = \mathbb{E}[Y^2|X]^2 - \mathbb{E}[Y|X]^4 = \sigma^4(X) + 2\sigma^2(X)\phi^2(X)$$

so that

$$\mathbb{E}[\text{Var}(YY'|X)] = \mathbb{E}[\sigma^4(X)] + 2\mathbb{E}[\sigma^2(X)\phi^2(X)].$$

Analogously,

$$\mathbb{E}[\text{Cov}(YY', YY''|X)] = \mathbb{E}[\mathbb{E}[Y^2Y'Y''|X]] - \mathbb{E}[\mathbb{E}[YY'|X]\mathbb{E}[YY''|X]] = \mathbb{E}[\sigma^2(X)\phi^2(X)].$$

Finally,

$$\Sigma_B^{1,1} = \mathbb{E}[\sigma^4(X)] + 4\mathbb{E}[\sigma^2(X)\phi^2(X)].$$

Similarly, the other terms are

$$\begin{aligned} \Sigma_B^{2,2} &= \mathbb{E}[\text{Var}(Y|X)] = \mathbb{E}[\sigma^2(X)], \\ \Sigma_B^{3,3} &= \mathbb{E}[\text{Var}(Y^2|X)] = \mathbb{E}[\phi_4(X)] - \mathbb{E}[\phi_2(X)^2], \\ \Sigma_B^{1,2} &= \Sigma_B^{2,1} = 2\mathbb{E}[\text{Cov}(YY', Y|X)] = 2\mathbb{E}[\sigma^2(X)\phi(X)], \\ \Sigma_B^{1,3} &= \Sigma_B^{3,1} = 2\mathbb{E}[\text{Cov}(YY', Y^2|X)] = 2\mathbb{E}[\phi_3(X)\phi(X)] - 2\mathbb{E}[\phi_2(X)\phi^2(X)], \\ \Sigma_B^{2,3} &= \Sigma_B^{3,2} = \mathbb{E}[\text{Cov}(Y, Y^2|X)] = \mathbb{E}[\phi_3(X)] - \mathbb{E}[\phi(X)\phi_2(X)], \end{aligned}$$

where $\phi_i(x) = \mathbb{E}[Y^i|X = x]$ for $i = 2, 3, 4$.

2 Computation of the asymptotic covariance matrix

Σ_C

In [1, Section B.3], we obtained

$$\begin{aligned} \Sigma_C^{1,1} &= \int \psi_{100}(x_1, x'_1, x''_1, \chi_1)\psi_{100}(x_2, x'_2, x''_2, \chi_2)x_1 \wedge x_2 \wedge x'_1 \wedge x'_2 \wedge x''_1 \wedge x''_2 \\ &\quad \times dQ(x_1, x'_1, x''_1, \chi_1)dQ(x_2, x'_2, x''_2, \chi_2) - \left(\int \psi_{100}(x, x', x'', \chi)x \wedge x' \wedge x'' dQ(x, x', x'', \chi) \right)^2 \end{aligned}$$

leading to the corrected expression of $\Sigma_C^{1,1}$;

$$\begin{aligned} \Sigma_C^{1,1} &= \mathbb{E}[(Y_1 + Y_1'')(Y_2 + Y_2'')f_x(X_1, W_1')f_x(X_2, W_2')(X_1 \wedge X_2)] - \mathbb{E}[(Y + Y'')f_x(X, W')X]^2 \\ &= 4\mathbb{E}[Y_1Y_2f_x(X_1, W_1')f_x(X_2, W_2')(X_1 \wedge X_2)] - 4\mathbb{E}[Yf_x(X, W')X]^2, \end{aligned}$$

where we remind that $Y = f(X, W)$, $Y' = f(X, W')$, and $Y'' = f(X, W'')$ with W' and W'' independent copies of W (and analogously for Y_1 and Y_2). Hence, we remark that $\Sigma_C^{1,1} = \Sigma_C^{3,3}$. Considering the second term in $\Sigma_C^{1,1}$, we get

$$\begin{aligned}\mathbb{E}[Y f_x(X, W')X] &= \mathbb{E}[f(X, W)f_x(X, W')X] = \mathbb{E}[\mathbb{E}[f(X, W)f_x(X, W')X|X]] \\ &= \mathbb{E}[X\mathbb{E}[f(X, W)|X]\mathbb{E}[f_x(X, W')|X]] \quad \text{by conditional independence} \\ &= \frac{1}{2}\mathbb{E}[X\phi(X)\phi'(X)] = \frac{1}{2}(\phi^2(1) - \mathbb{E}[\phi^2(X)]) \quad \text{by integration by parts.}\end{aligned}$$

Similarly,

$$4\mathbb{E}[Y_1 f_x(X_1, W'_1)Y_2 f_x(X_2, W'_2)(X_1 \wedge X_2)] = \phi^4(1) - 2\phi^2(1)\mathbb{E}[\phi^2(X)] + \mathbb{E}[\phi^4(X)]$$

that yields

$$\Sigma_C^{1,1} = \phi^4(1) - 2\phi^2(1)\mathbb{E}[\phi^2(X)] + \mathbb{E}[\phi^4(X)] - (\phi^2(1) - \mathbb{E}[\phi^2(X)])^2 = \text{Var}(\phi^2(X)).$$

Finally, the remaining terms of Σ_C (after correction) are:

$$\begin{aligned}\Sigma_C^{2,2} &= \mathbb{E}[f_x(X_1, W_1)f_x(X_2, W_2)(X_1 \wedge X_2)] - \mathbb{E}[f_x(X, W)X]^2 = \text{Var}(\phi(X)) \\ \Sigma_C^{1,2} &= \mathbb{E}[(Y_1 + Y_1'')f_x(X_1, W'_1)f_x(X_2, W_2)(X_1 \wedge X_2)] - \mathbb{E}[(Y + Y'')f_x(X, W')X]\mathbb{E}[f_x(X, W)X] \\ &= 2\mathbb{E}[Y_1 f_x(X_1, W'_1)f_x(X_2, W_2)(X_1 \wedge X_2)] - 2\mathbb{E}[Y f_x(X, W')X]\mathbb{E}[f_x(X, W)X] \\ &= \text{Cov}(\phi(X), \phi^2(X)) \\ \Sigma_C^{1,3} &= 2\mathbb{E}[(Y_1 + Y_1'')f_x(X_1, W'_1)Y_2 f_x(X_2, W'_2)(X_1 \wedge X_2)] - 2\mathbb{E}[(Y + Y'')f_x(X, W')X]\mathbb{E}[Y f_x(X, W')X] \\ &= 4\mathbb{E}[Y_1 f_x(X_1, W'_1)Y_2 f_x(X_2, W'_2)(X_1 \wedge X_2)] - 4\mathbb{E}[Y f_x(X, W')X]^2 = \Sigma_C^{1,1} \\ \Sigma_C^{2,3} &= 2\mathbb{E}[f_x(X_1, W_1)Y_2 f_x(X_2, W'_2)(X_1 \wedge X_2)] - 2\mathbb{E}[f_x(X, W)X]\mathbb{E}[Y f_x(X, W')X] = \Sigma_C^{1,2}.\end{aligned}$$

Evidently, $\Sigma_C^{2,1} = \Sigma_C^{1,2}$, $\Sigma_C^{3,1} = \Sigma_C^{1,3}$, and $\Sigma_C^{3,2} = \Sigma_C^{2,3}$.

References

- [1] F. Gamboa, P. Gremaud, T. Klein, and A. Lagnoux. Global sensitivity analysis: A novel generation of mighty estimators based on rank statistics. *Bernoulli*, 28(4):2345–2374, 2022.
- [2] T. Klein and P. Rochet. Efficiency of the averaged rank-based estimator for first order Sobol index inference. working paper or preprint, June 2023.