



HAL
open science

Hybrid entanglement between time-bin and wave-like encoding

Elie Gouzien, Floriane Brunel, Sébastien Tanzilli, Virginia d'Auria

► **To cite this version:**

Elie Gouzien, Floriane Brunel, Sébastien Tanzilli, Virginia d'Auria. Hybrid entanglement between time-bin and wave-like encoding. 2020. hal-02474883v1

HAL Id: hal-02474883

<https://hal.science/hal-02474883v1>

Preprint submitted on 11 Feb 2020 (v1), last revised 4 Dec 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hybrid entanglement between time-bin and wave-like encoding

Élie Gouzien,¹ Floriane Brunel,¹ Sébastien Tanzilli,¹ and Virginia D'Auria^{1,*}

¹ *Université Côte d'Azur, CNRS, Institut de Physique de Nice (INPHYNI), Parc Valrose, 06108 Nice Cedex 2, France*

(Dated: February 11, 2020)

We propose a scheme for the generation of hybrid states entangling a single photon time-bin qubit with a coherent state qubit encoded on phases. Compared to other reported solutions, time-bin encoding makes hybrid entanglement particularly well adapted to applications involving long distance propagation in optical fibres. This makes our proposal a promising resource for future out-of-the-laboratory quantum communication. In this perspective, we analyse our scheme by taking into account realistic experimental resources and discuss the impact of their imperfections on the quality of the obtained hybrid state.

I. INTRODUCTION

Over the last years, quantum optics information has traditionally followed two distinct approaches, naturally stemming from light wave-particle complementarity [1]. Discrete variable regime (DV) usually refers to weakly excited optical states, down to single photons, for which information is encoded on discrete spectrum observables such as the polarisation or the number of photons [2]. Conversely, continuous variables regime (CV) relies on multi-photon optical states and to encodings on continuous spectrum observables such as amplitude and phase of a light field [3]. DV encoding is tolerant to losses and allows high-fidelity teleportation, while CV encoding permits deterministic state generation and unambiguous state discrimination [4].

Recently, hybrid states entangling DV and CV encoding have been identified as a key tool to switch from one approach to the other and gather the benefits of both [1, 5–9]. This perspective has motivated an increasing number of theoretical works [10–14] as well as experiments on hybrid state generation [15, 16] or use in proof-of-principle quantum information protocols with single-rail or polarisation encodings for the DV part [17–20].

In view of future applications to quantum networks, we address here the generation of hybrid states with time-bin encoding on their discrete variable part. In the time-bin scheme, information is encoded on two generation/detection times for photons, usually labelled as “early” (e) and “late” (l). Compared to photon-number or polarisation encodings, time-bin allows to comply in a better way with losses, is immune to polarisation dispersion, and is particularly well-adapted to quantum communication over long optical fibre links [21]. As a consequence, the realisation of hybrid entanglement with time-bin encoding is extremely important in view of out-of-laboratory applications. Our scheme permits to generate the desired state in a heralded fashion without any post-selection operation and relies, as enabling resources, on the interference between experimentally achievable

states, *i.e.* an optical Schrödinger cat [22, 23], and a time-bin entangled photon pair [24]. In order to comply with future practical realisations, it has been conceived so as to be fully compatible with existing fiber architectures and with realistic experimental resources, including non-ideal heralding detectors [25, 26] and input states [24].

In the following, we illustrate our scheme in detail. For pedagogical reasons, in section II we analyse it in the case of perfect input states and ideal heralding detectors. In section III, we investigate more realistic situations. We start by considering for the heralding feature a minimum number of ON/OFF single-photon detectors [25, 26], with no photon-number resolving ability and finite efficiency. We then examine the impact of realistic input states such as photon pairs generated by a nonlinear process along with vacuum and multiple pairs [24] on the DV part, and a squeezed vacuum as an approximation of a Schrödinger cat on the CV one [20]. We show that our scheme is resistant to these experimental limitations by discussing the heralded state fidelity with respect to the targeted one.

II. PROPOSED EXPERIMENT

The aim of our work is to generate a DV-CV hybrid state, entangling a DV time-bin qubit with a bright CV qubit encoded on two coherent states with opposite phases. We define such a target state as:

$$|\varphi\rangle = \frac{|1\rangle_{A,e} |+\alpha_f\rangle_B - |1\rangle_{A,l} |-\alpha_f\rangle_B}{\sqrt{2}}, \quad (1)$$

where for the DV part (mode A), e and l stand for the “early” and “late” time bins and the $|1\rangle_{A,i}$ indicates a single photon state in the temporal mode “ i ”. For the CV part (mode B), $|\pm\alpha_f\rangle_B$ is a coherent state of amplitude $\pm\alpha_f$. To simplify the notation, in the Eq. 1 we omitted the vacuum states $|0\rangle_{A,l}$ and $|0\rangle_{A,e}$ that multiply the CV terms $|+\alpha_f\rangle_B$ and $|-\alpha_f\rangle_B$, respectively. The density matrix associated with this state is $|\varphi\rangle\langle\varphi|$.

The experimental scheme at the hearth of our proposal is presented in Fig. 1. Conceptually, it requires pure CV and DV states as initial resources and an interferometric

* virginia.dauria@inphyni.cnrs.fr

$$\begin{aligned}
|\psi^{(1)}\rangle = \frac{1}{2\sqrt{2}\mathcal{N}} & \left[\right. \\
& |+\alpha\rangle_B |1\rangle_{A,e} \left\{ \hat{D}_{C,l} \left(+\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,l} \left(+\frac{r\alpha}{\sqrt{2}} \right) |0\rangle_l \right\} \left[|1\rangle_{C,e} + |1\rangle_{D,e} + |1\rangle_{E,e} + |1\rangle_{F,e} \right] \\
& + |-\alpha\rangle_B |1\rangle_{A,e} |0\rangle_l \left\{ \hat{D}_{C,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,e} \left(+\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,e} \left(+\frac{r\alpha}{\sqrt{2}} \right) \left[|1\rangle_{C,e} + |1\rangle_{D,e} + |1\rangle_{E,e} + |1\rangle_{F,e} \right] \right\} \\
& + |+\alpha\rangle_B |1\rangle_{A,l} |0\rangle_e \left\{ \hat{D}_{C,l} \left(+\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,l} \left(+\frac{r\alpha}{\sqrt{2}} \right) \left[|1\rangle_{C,l} + |1\rangle_{D,l} + |1\rangle_{E,l} + |1\rangle_{F,l} \right] \right\} \\
& \left. + |-\alpha\rangle_B |1\rangle_{A,l} \left\{ \hat{D}_{C,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,e} \left(+\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,e} \left(+\frac{r\alpha}{\sqrt{2}} \right) |0\rangle_e \right\} \left[|1\rangle_{C,l} + |1\rangle_{D,l} + |1\rangle_{E,l} + |1\rangle_{F,l} \right] \right] \quad (6)
\end{aligned}$$

where $|0\rangle_l$ indicates $|0\rangle_{C,l}|0\rangle_{D,l}|0\rangle_{E,l}|0\rangle_{F,l}$ and analogously for $|0\rangle_e$. We note that $\frac{r\alpha}{\sqrt{2}}$ corresponds to the amplitude of the light beam coming from the continuous variable part and reaching the detectors. In the notation $|\psi^{(1)}\rangle$, label 1 indicates that we consider exactly one photon pair on the discrete variable input.

Light in the four spatial modes C, D, E, F is measured by using single photon detectors; for each of them two temporal modes (l and e) must be considered, thus leading to eight possible heralding modes. The combination of their detection signals heralds the hybrid state on A and B .

We will consider here the ideal case of photon-number resolving detectors with perfect detection efficiency [10, 11] and herald the hybrid state by the simultaneous detection of signals corresponding to the measurement of one photon on detector E in the ‘‘late’’ time bin, one photon on detector F in the ‘‘early’’ time bin and no photon in the six remaining heralding modes. The associated positive operator valued measurement (POVM) [30] reads:

$$\hat{\Pi}^{\text{id}} = |1\rangle\langle 1|_{E,l} |1\rangle\langle 1|_{F,e} \bigotimes_i |0\rangle\langle 0|_i, \quad (7)$$

where the label i indicate the heralding modes (C, e) , (C, l) , (D, e) , (D, l) , (E, e) and (F, l) . In the previous equation, identity is implicit on unmeasured channels A and B . As it can be seen, only the first and the fourth terms of Eq. (6) simultaneously contain light in the (E, l) and the (F, e) modes, and can lead to a detection event as described by $\hat{\Pi}^{\text{id}}$. We note that the detected photons are provided one by the DV and the other by the CV part of the scheme. On the contrary, the second and the third terms contain light only in one of the two time bins and do not contribute to the announced states. Accordingly, as desired, the chosen heralding strategy exactly leads to the target state of Eq. (1):

$$\hat{\rho}^{\text{id}} = \frac{\text{Tr}_{CDEF} \left[\hat{\Pi}^{\text{id}} |\psi^{(1)}\rangle\langle\psi^{(1)}| \right]}{\text{Tr} \left[\hat{\Pi}^{\text{id}} |\psi^{(1)}\rangle\langle\psi^{(1)}| \right]} = |\varphi\rangle\langle\varphi|, \quad (8)$$

where the amplitude of the CV qubit is $\alpha_f = t\alpha$. The

target state is heralded with a probability:

$$P^{\text{id}} = \text{Tr} \left[\hat{\Pi}^{\text{id}} |\psi^{(1)}\rangle\langle\psi^{(1)}| \right] = \frac{1}{16} \frac{|r\alpha|^2 e^{-2|r\alpha|^2}}{1 + e^{-2|\alpha|^2}}. \quad (9)$$

As it can be seen, equality (8) holds true for any choice of the fraction of light, $r\alpha$ that is subtracted from the CV part and mixed with the DV one. However, as shown in Fig. 2 (dashed lines), the heralding probability is strongly affected by $r\alpha$. For a fixed size of the initial cat, *i.e.* for a given α , P^{id} increases with $r\alpha$ up to an optimal value and then decreases for higher $r\alpha$ values. This behaviour is easily justified by the heralding choice. For $r\alpha \rightarrow 0$, the fraction of continuous variable part directed towards detection modes is extremely weak and the probability of obtaining a detection signal from both (E, l) or (F, e) is low. On the contrary, for high $r\alpha$, detection of light on modes $C, D, (E, e)$ and (F, l) have an important probability to be triggered by the continuous variable component of $|\psi^{(1)}\rangle$ and, as for state preparation we impose the absence of photons on these detectors, the overall heralding probability decreases. We also stress that for increasing $r\alpha$, the condition of detecting only one photon on heralding modes (E, l) or (F, e) is no longer respected as multiple photon contributions become non-negligible.

As conclusive remark to this section, it is pertinent to discuss the robustness of the hybrid state with time-bin encoding to transmission losses, occurring when the DV part is transmitted along optical fibre links. This point permits us a direct comparison with hybrid entangled states featuring other kinds of DV encoding. Following a very standard approach, losses can be modelled as a beam-splitter inserted along the path of the DV part of the state (mode A) and exhibiting reflection and transmission coefficients t_{loss} and r_{loss} . The other BS input is in a vacuum state. Due to the action of this fictitious BS, the density matrix of the initial state $|\varphi\rangle\langle\varphi|$ is modified

into (see A):

$$\hat{\rho}_{\text{loss}} = t_{\text{loss}}^2 |\varphi\rangle\langle\varphi| + r_{\text{loss}}^2 |0\rangle\langle 0|_A \frac{|\alpha_f\rangle\langle\alpha_f|_B + |-\alpha_f\rangle\langle-\alpha_f|_B}{2}. \quad (10)$$

where we kept the label A for the DV part after the loss-beam splitter and B for the CV part. We note that an expression similar to $\hat{\rho}_{\text{loss}}$ can be derived starting from $\hat{\rho}^{\text{id}}$. The first term of the previous expression contains the initial state multiplied by t_{loss}^2 and unaffected by decoherence effects. The second terms, proportional to t_{loss}^2 , is in a non-entangled state, with the discrete part, A , is in a vacuum state. We note that in practical protocols, the term proportional to $|0\rangle\langle 0|_A$ can be ultimately traced out by DV detection operation, as it happens in pure DV regime [21]. We observe that in hybrid states where the DV qubit is encoded on the absence, Fock state $|0\rangle$, or presence, $|1\rangle$, of one photon (single-rail encoding), losses change the relative weight of the DV qubit term by reducing the contribution of $|1\rangle$, in favour of $|0\rangle$. This effect cannot be eliminated by post-selection at the detection stage and degrades the hybrid entanglement. A similar discussion holds for polarisation encoding. In this case, losses give, in a first approximation, an expression analogous to $\hat{\rho}_{\text{loss}}$, but polarisation dispersion can convert vertical polarisation (DV state $|1\rangle_V$) into the horizontal one ($|1\rangle_H$) [21], thus leading, as for the photon number encoding, to a deterioration of entanglement.

III. ROBUSTNESS OF THE GENERATION SCHEME AGAINST EXPERIMENTAL LIMITATIONS

III.1. Imperfect detection and simplified heralding strategy

The ideal case discussed in the previous section relies on the possibility of detecting *i.e.* two temporal modes for each of the four spatial modes C, D, E, F . This can be seen from the shape of POVM (7) that explicitly contains projectors over all the eight heralding modes. In experiments, the strategy described in section II would imply that each of the heralding detectors should be able to measure both “early” and “late” time bins. This implies a separation between time bins greater than the detector dead time or, in alternative, the use of extra detectors so as to map each time bin in an additional spatial mode [30]. These solutions have dramatic consequences for the experiment maximum operation rate and required overhead, respectively. At the same time, we note that photon number resolving detectors, although often introduced in the literature on hybrid state [11], are hardly available off-the-shelf and, so far, most of the demonstrations involving some counting ability rely on complex spatial multiplexing strategies based on detector arrays [31, 32].

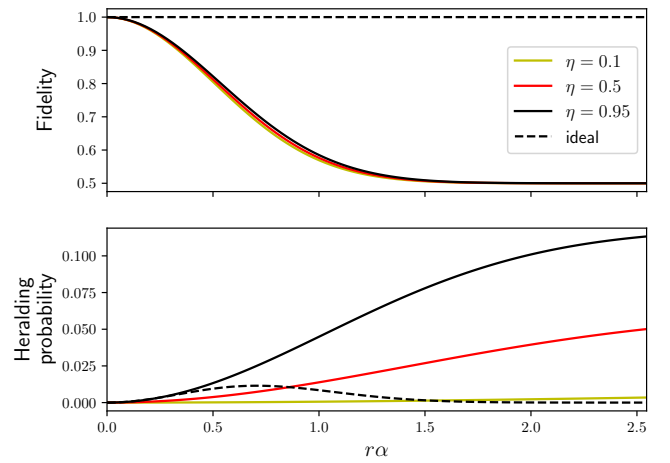


FIG. 2: Fidelity of the heralded hybrid state to the target state $|\varphi\rangle\langle\varphi|$ (top) and heralding probability (bottom) as a function of $r\alpha$, for the two proposed heralding protocols: $\hat{\Pi}^{\text{id}}$ (dashed line) and $\hat{\Pi}$ (plain lines). Different detection efficiencies have been chosen. For the input CV state, we set $\alpha = 2$.

In view of future experimental realisations, in this section, we discuss our scheme in the case of a more realistic detection scenario. A practical simplification of our protocol consists in heralding the target state only upon the detection signals from the “late” time bin in mode E and from the “early” time bin in mode F , while disregarding all other six heralding modes. Such an approach reduces the number of detected modes from eight to two. By doing so, only two gated detectors are required and, as each of them must detect only one temporal mode, dead times are no longer a limiting factor to the operation rate [25]. At the same time, we consider, at the heralding modes, ON/OFF single-photon detectors, with non unit detection efficiency η and no photon number resolving ability [26]. These systems are able to provide only the two generic responses: ON, *i.e.* “at least one photon has been detected”, and OFF, *i.e.* “no photon has been detected” and they represent the vast majority of available single-photon counters. Their action is described by the positive operators [25, 30, 31]:

$$\hat{\Pi}_i^{\text{OFF}} = \sum_{k=0}^{+\infty} (1 - \eta)^k |k\rangle\langle k|_i, \quad (11a)$$

$$\hat{\Pi}_i^{\text{ON}} = \mathbf{1} - \hat{\Pi}_i^{\text{OFF}}, \quad (11b)$$

where i indicates the heralding mode under investigation and, only for the previous equations, $|k\rangle_i$ indicate a Fock state of mode i containing k photons.

The measurement positive valued operator associated with the simplified heralding strategy and with ON/OFF gated detectors reads:

$$\hat{\Pi} = \hat{\Pi}_{E,l}^{\text{ON}} \otimes \hat{\Pi}_{F,e}^{\text{ON}}, \quad (12)$$

identity being implicit on all unmentioned modes as they are not measured. We note, in particular, that with this measurement strategy, the scheme no longer relies on projection on vacuum states. Analogously to what is done in Eqs. (8) and (9), the heralded state can be computed from $\hat{\Pi}$:

$$\hat{\rho}^{(1)} = \frac{1}{2} \left\{ |\alpha_f\rangle\langle\alpha_f|_B |1\rangle\langle 1|_{A,e} - \frac{\eta \frac{|r\alpha|^2}{2} e^{-2|r\alpha|^2}}{1 - e^{-\eta \frac{|r\alpha|^2}{2}}} \left[|\alpha_f\rangle\langle -\alpha_f|_B |1\rangle\langle 0|_{A,e} |0\rangle\langle 1|_{A,l} + |-\alpha_f\rangle\langle +\alpha_f|_B |0\rangle\langle 1|_{A,l} |1\rangle\langle 0|_{A,e} \right] + |-\alpha_f\rangle\langle -\alpha_f|_B |1\rangle\langle 1|_{A,l} \right\}. \quad (13)$$

Its associated heralding probability reads:

$$P^{(1)} = \frac{\eta}{8} \frac{1 - e^{-\eta \frac{|r\alpha|^2}{2}}}{1 + e^{-2|\alpha|^2}}, \quad (14)$$

where, as before, we used the notation $t\alpha = \alpha_f$.

The state given by (13) belongs to the qubit subspace generated by $|+\alpha_f\rangle_B |1\rangle_{A,e}$ and $|-\alpha_f\rangle_B |1\rangle_{A,l}$, as the density matrix $|\varphi\rangle\langle\varphi|$, and correctly tends to it when $r\alpha \rightarrow 0$. This can be seen from the fidelity of the heralded state to the target state:

$$\mathcal{F}^{(1)} = \langle\varphi|\hat{\rho}^{(1)}|\varphi\rangle = \frac{1}{2} \left[1 + \frac{\eta \frac{|r\alpha|^2}{2} e^{-2|r\alpha|^2}}{1 - e^{-\eta \frac{|r\alpha|^2}{2}}} \right]. \quad (15)$$

The heralding probability, P_1 , and fidelity $\mathcal{F}^{(1)}$ are plotted in Fig. 2 as functions of $r\alpha$ for different detector efficiencies η . In this regard we underline that the fidelity is a commonly-adopted and pertinent criterium to check the validity of any scheme aiming at the generation of a given target state [10, 11]. The efficiency η has a clear impact on the heralding probability, P_1 , on the order of η^2 in the limit of $r\alpha \rightarrow 0$. Compared to the ideal case describe in the previous section (dashed lines), the heralding probability is no longer decreasing for high $r\alpha$, as the heralding strategy is no longer sensitive to spurious events firing one of the disregarded modes C , D and (E, e) , (F, l) . Nevertheless, the bad effect of multiphoton contributions coming from the CV part can be clearly seen on the state fidelity, $\mathcal{F}^{(1)}$, that decreases for increasing $r\alpha$. Detection efficiency η has little influence on the fidelity. This weak dependency arises from the shape of the ON operator in Eq. (11) [25].

In experiments, for a fixed input state $|\text{cat}+\rangle_3$, *i.e.* for a given α , and for given detector quantum efficiency η , the reflection coefficient r can be chosen so as to reach a desired value for the fidelity. As an example, we consider $\alpha = 2$ as in Refs. [15, 16, 22, 23] and realistic single-photon detectors based on superconducting nanowire technology exhibiting $\eta = 0.95$ [33, 34] working in a gated operation mode. With these parameters, a

high target fidelity $\mathcal{F}^{(1)} = 0.99$ imposes $\frac{r\alpha}{\sqrt{2}} = 0.075$, thus giving $r \approx 0.053$. Correspondingly, the heralding probability $P^{(1)} \approx 6.4 \cdot 10^{-4}$. Working with a 1 GHz repetition rate laser [35], and fast single photon detectors such as those reported in Ref. [34], leads to a heralding rate of 0.64 MHz. Note that, as the fraction of light subtracted from the CV input is very small, the size of the CV part of the final hybrid state is $\alpha_f \approx 2$.

To conclude, we note that, beside studying the fidelity, in some applied situations, it can be useful to quantify the produced states in terms of specific touchstones [36, 37], that can be chosen according to the specific quantum information protocol. The non-classicality of the hybrid states as resources for quantum communication is often formalised by their ability to produce a non-classical continuous variable state when performing a suitable projecting measurement on the discrete variable subsystem. This operational notion has been theoretically investigated and experimentally validated [18, 37]. In Fig. 3 left, we plot the negativity of the Wigner function of the CV state obtained after having projected the DV part of $\hat{\rho}^{(1)}$ on the state $\frac{|1\rangle_{A,e} + |1\rangle_{A,l}}{\sqrt{2}}$. Analogously to what is observed for the fidelity, while being weakly dependent on the detection efficiency η , the negativity of the CV part clearly decreases with $r\alpha$; at the same time, the Wigner function correctly shows negative values for low $r\alpha$ that correspond to the best fidelity with the hybrid target state $|\varphi\rangle\langle\varphi|$. In order to focus more on the hybrid entanglement quantification, as discussed in [38], a valuable tool is the negativity of the partial transpose (NPT). NPT is proportional to the sum of the negative eigenvalues of a partially-transposed state density matrix and verify $0 \leq \text{NPT} \leq 1$; separable and Bell states are respectively valued at 0 and 1. The NPT of the state $\hat{\rho}^{(1)}$ is shown by Fig. 3 right. The state is maximally entangled, NPT is close to 1, when $r\alpha$ tends to zero, and becomes factorisable when $r\alpha$ increases (NPT is then close to 0).

III.2. Vacuum and multiple pairs in the DV input

In this section, we further modify our model so as to take into account possible limitations due a more realistic model for the discrete variable input. The case of a non-ideal CV input will be discussed in the last subsection.

So far, we have considered a perfect time-bin entangled state, $|\xi\rangle_{1,2}$, at the input modes 1 and 2. In usual experiments, the generation of time-bin entangled photons typically relies on a parametric down conversion (SPDC) nonlinear process, where a pump photon is converted in a pair of photons (signal and idler) [21]. However, this kind of process suffers from unwanted generation of vacuum and multiple pair components [39]. In order to discuss their effect, we explicitly include these contributions to the DV input state. Accordingly we replace the initial

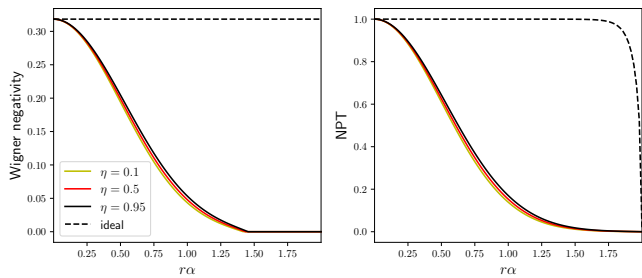


FIG. 3: Negativity of the Wigner function obtained after having projected the DV part of $\hat{\rho}^{(1)}$ onto the state $\frac{|1\rangle_{A,e} + |1\rangle_{A,l}}{\sqrt{2}}$ (left). In the ideal case discussed in the section II, the negativity of the Wigner function of the conditional state is constant with $r\alpha$. The optimal value of $\frac{2}{\pi} \approx 0.64$ corresponds to the Wigner function definition $W(x, p) = \frac{1}{\pi} \int \langle x + \frac{u}{2} | \hat{\rho}^{(1)} | x - \frac{u}{2} \rangle e^{-2ipu} du$. Negativity of the partial transpose (NPT) as a function of $r\alpha$ for the state $\hat{\rho}^{(1)}$ (right). Different detection efficiencies have been chosen.

with $|\psi^{(1)}\rangle$ as given by (6),

$$|\psi^{(0)}\rangle = \frac{1}{\mathcal{N}} \left[|t\alpha\rangle_B |0\rangle_A \hat{D}_{C,l} \left(\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,l} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,l} \left(\frac{r\alpha}{\sqrt{2}} \right) |0\rangle_{C,D,E,F} \right. \\ \left. + |-t\alpha\rangle_B |0\rangle_A \hat{D}_{C,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{D,e} \left(\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{E,e} \left(-\frac{r\alpha}{\sqrt{2}} \right) \hat{D}_{F,e} \left(\frac{r\alpha}{\sqrt{2}} \right) |0\rangle_{C,D,E,F} \right], \quad (17b)$$

and $|\psi^{(\varepsilon)}\rangle$ a normed state including the contribution due to multiple pairs coming from the discrete part input, $|\xi\rangle_{1,2}$.

As previously, the state is heralded on the simultaneous detection signals (E, l) and (F, e) . As can be seen from (17b), for each of the terms of $|\psi^{(0)}\rangle$, only one of the two temporal modes (e, l) is populated. Accordingly, $|\psi_0\rangle$ has a zero probability to give the heralding signal and it will not contribute to the final heralded state. Conceptually, the density matrix of the heralded state has the following form:

$$\hat{\rho}' = \frac{\text{Tr}_{CDEF} [\hat{\Pi} |\psi'\rangle\langle\psi'|]}{\text{Tr} [\hat{\Pi} |\psi'\rangle\langle\psi'|]} \\ = \frac{1}{1 + \frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}}} \left[\hat{\rho}^{(1)} + \sqrt{\frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}}} \hat{\rho}^{(1,\varepsilon)} + \frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}} \hat{\rho}^{(\varepsilon)} \right]. \quad (18)$$

with a corresponding heralding probability:

$$P' = \text{Tr} [\hat{\Pi} |\psi'\rangle\langle\psi'|] = p_1 P^{(1)} + p_\varepsilon P^{(\varepsilon)}. \quad (19)$$

In the previous expressions, $P^{(1)}$ is the same as given in Eq. (14) and $P^{(\varepsilon)}$ is the probability for $|\psi^{(\varepsilon)}\rangle$ to give

state of Eq. (5), with the state:

$$|\xi'\rangle_{1,2} = \sqrt{p_0} |0\rangle_{1,2} + \sqrt{p_1} \frac{|1\rangle_{1,e} |1\rangle_{2,e} + |1\rangle_{1,l} |1\rangle_{2,l}}{\sqrt{2}} + \sqrt{p_\varepsilon} |\varepsilon\rangle_{1,2}, \quad (16)$$

with $p_0 + p_1 + p_\varepsilon = 1$, and where $|\varepsilon\rangle_{1,2}$ represents all multi-pair contributions. A more detailed discussion on the shape of $|\xi'\rangle_{1,2}$ is given in B. In experimental situations, $p_0 \gg p_1 \gg p_\varepsilon$, thus making the vacuum the most important contribution to $|\xi'\rangle_{1,2}$.

As in Section II, we first write the state right before the detection. This reads:

$$|\psi'\rangle = \sqrt{p_0} |\psi^{(0)}\rangle + \sqrt{p_1} |\psi^{(1)}\rangle + \sqrt{p_\varepsilon} |\psi^{(\varepsilon)}\rangle, \quad (17a)$$

a heralding signal. The explicit expression of $P^{(\varepsilon)}$ can be analytically computed for a given multi-pair contribution, represented by $|\varepsilon\rangle_{1,2}$. Similarly, $\hat{\rho}^{(1)}$ the density matrix already given in Eq. (13), $\hat{\rho}^{(\varepsilon)}$ a density operator containing multiple photons on mode A, and $\hat{\rho}^{(1,\varepsilon)}$ a traceless operator whose coefficients are on the order of 1, as required to preserve the positivity of $\hat{\rho}'$. For the sake of simplicity, we do not provide the explicit expressions of $\hat{\rho}^{(\varepsilon)}$ and $\hat{\rho}^{(1,\varepsilon)}$ but only discuss, their relative weight compared to $\hat{\rho}^{(1)}$. We note that, as expected, the vacuum component of the discrete variable input is entirely rejected by the heralding process and does not enter the expression of the announced state $\hat{\rho}'$.

The impact of multi-pair contributions to the heralded state can be quantified in terms of the fidelity of $\hat{\rho}'$ with the target density matrix $|\varphi\rangle\langle\varphi|$. This reads as:

$$\mathcal{F}' = \frac{1}{1 + \frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}}} \mathcal{F}^{(1)}, \quad (20)$$

with $\mathcal{F}^{(1)}$ as expressed in Eq. (15). For both the announced state, $\hat{\rho}'$, and the fidelity, \mathcal{F}' , multi-photon contributions increase with $\frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}}$. In order to evaluate the general form of this ratio, we note that in most of the situations, multi-pair terms in Eq. (16) are dominated

by double-pair contributions [35] and, as a consequence, $P^{(\varepsilon)} \approx P^{(2)}$, the probability of a double-pair to give the heralding signal. In the limit of a small fraction of light coming from the CV part, it is reasonable to expect $P^{(2)} \underset{r\alpha \rightarrow 0}{=} O(\eta^2)$ [35], while, based on Eq. (14), for small α , $P^{(1)} \underset{r\alpha \rightarrow 0}{=} \eta^2 |r\alpha|^2 / 32$. Correspondingly, we obtain:

$$\frac{p_\varepsilon P^{(\varepsilon)}}{p_1 P^{(1)}} \underset{r\alpha \rightarrow 0}{\approx} \frac{p_\varepsilon}{p_1} \frac{1}{O(|r\alpha|^2)}. \quad (21)$$

In this limit, the fidelity \mathcal{F}' can be written as:

$$\mathcal{F}' \underset{r\alpha \rightarrow 0}{\approx} \frac{1}{1 + \frac{p_\varepsilon}{p_1} \frac{1}{O(|r\alpha|^2)}} \left[1 - \left(1 - \frac{\eta}{8}\right) |r\alpha|^2 \right], \quad (22)$$

where the numerator is given by (15) for $r\alpha \rightarrow 0$.

Eq. (21) shows that unwanted multi-pair contributions to the final heralded state can be avoided provided $p_\varepsilon \ll p_1 |r\alpha|^2$. In particular, for sources based on second order non-linear effects, $p_\varepsilon \sim p_1^2$, and the criterion reads $p_1 \ll |r\alpha|^2$. By adequately choosing the single pair generation rate p_1 and product $r\alpha$ it is thus possible to neglect the components containing multiple photons on the discrete part of the output state.

To conclude with an example, we consider the case of sources based on spontaneous parametric down conversion. In this case, the explicit expression for $|\xi'\rangle_{1,2}$ can be obtained from the general expression of the output of the SPDC process [25, 30], as a function of an excitation parameter λ^2 proportional to the pump intensity and to the square of the non-linear coefficient of the source. The time-bin state generation can be seen as the result of an SPDC process on mode e and an SPDC process on mode l . By combining the coefficients of the two individual processes, we obtain for the overall generation the weights (see B):

$$\begin{aligned} p_0 &= (1 - \lambda^2)^2 \\ p_1 &= 2(1 - \lambda^2)^2 \lambda^2 \\ p_2 &= 3(1 - \lambda^2)^2 \lambda^4 \end{aligned} \quad (23)$$

The explicit expression for the $P^{(2)}$ is:

$$P^{(2)} = \frac{\eta}{48} \frac{12 - \eta - (12 - 2\eta + \frac{\eta^2}{2} |r\alpha|^2) e^{-\frac{\eta}{2} |r\alpha|^2} + \eta e^{-2|r\alpha|^2}}{1 + e^{-2|r\alpha|^2}}. \quad (24)$$

In the limit of $r\alpha \rightarrow 0$, this expression leads to $P^{(2)} \underset{r\alpha \rightarrow 0}{=} \eta^2 / 48$ and, as expected, correctly behaves as η^2 . More explicitly, based on Eqs. (23) and (24), we find for the SPDC:

$$\frac{p_2 P^{(2)}}{p_1 P^{(1)}} \underset{r\alpha \rightarrow 0}{\approx} \frac{\lambda^2}{|r\alpha|^2} \quad (25)$$

The Fig. (4, dashed line) shows the fidelity \mathcal{F}' as a function of the parameter λ^2 . We considered as for the previous section, $\eta = 0.95$, $\frac{r\alpha}{\sqrt{2}} = 0.075$, with $\alpha = 2$ and

$r = 0.05$. As seen, for these values, the fidelity in absence of multi-pairs contributions is $\mathcal{F} \approx 0.99$. The fidelity \mathcal{F}' approaches \mathcal{F} when $\lambda^2 \ll |r\alpha|^2$ (see Eq. (25)), *i.e.* $\lambda^2 \ll 0.01$, to decrease when the contribution of multi-pair increases.

III.3. Squeezed states at the CV input

In this last section, we briefly discuss the case of non-ideal states on the CV input. So far, we have considered at the CV input a perfect Schrödinger cat as described by Eq. (2). Nevertheless, these states are difficult to generate experimentally and they are often replaced by Schrödinger kittens, of small size α , generated in a heralded fashion [22, 23]. At the same time, the use of such heralded states as starting resource for the generation of hybrid-entanglement implies a further heralding signal to be combined with the ones considered so far (*i.e.* (E, l) and (F, e)). This would imply that the overall protocol success would rely on a 3-fold coincidence signal with a dramatic effect for the generation rate.

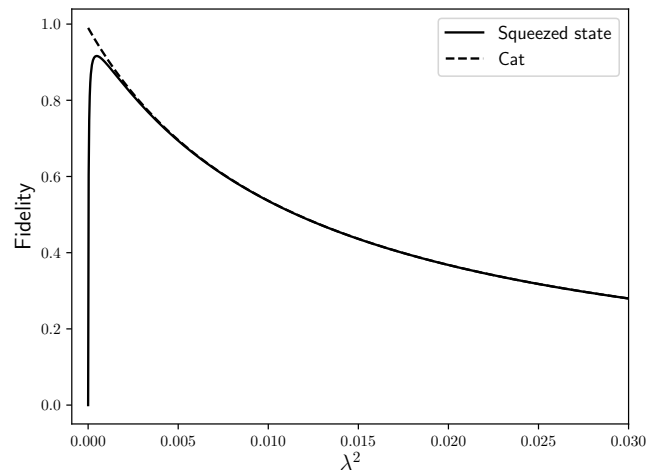


FIG. 4: Fidelities with respect to $|\varphi\rangle$ with multiple pairs on the discrete variable input as functions of the probability to have one photon on the discrete variable part, p_1 , for a realistic source based on parametric down conversion. Cat and squeezed states were considered for the continuous variable input. Detection efficiency $\eta = 0.95$. We chose $\frac{r\alpha}{\sqrt{2}} = 0.075$, with a value of $\alpha = 0.25$.

In view of realistic realisations, a convenient approximation to the state $|\text{cat}+\rangle$ with $\alpha < 1$ consists in using at the input 3 a squeezed vacuum state [20], $\hat{S}(\zeta)|0\rangle_3$, where $\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$ is the single mode squeezing operator and ζ the squeezing parameter. These states are deterministically available at the output of many nonlinear optical systems with a huge simplification of required experimental resources. The main drawback of this approach is that the interference of light reflected from BS1

with the coherent input $|\alpha\rangle$ at BS2 is no-longer perfect, thus modifying the shape of the state of Eq. (4) and, in turns of $|\psi^{(1)}\rangle$. Under these conditions, the heralding signals (E, l) , (F, e) could in principle be triggered from the sole CV part, and, occurring even in the absence of photons in the discrete variable. This would lead to an announced state containing vacuum contributions in its DV part.

The described events are linked to vacuum contributions in the DV input $|\xi'\rangle_{1,2}$. Accordingly, in analogy to the formalism of the previous paragraph, we will call $P^{(0)}$ their corresponding probability and write the overall heralding probability as $P'_s = p_0 P^{(0)} + p_1 P^{(1)} + p_\epsilon P^{(\epsilon)}$, with $P^{(1)}$ and $P^{(\epsilon)}$ already discussed in the previous sections. We note that in realistic situations, to comply with multi-pair events, the non-linear process providing the DV input $|\xi'\rangle_{1,2}$ is weakly pumped. This condition comes at the price of a high vacuum contribution represented by a p_0 close to 1. By taking into account both the effects of vacuum and multi-pairs due to non-ideal DV and CV inputs, we can express the fidelity as:

$$\mathcal{F}'_s = \frac{1}{1 + \frac{p_0 P^{(0)}}{p_1 P^{(1)}} + \frac{p_\epsilon P^{(\epsilon)}}{p_1 P^{(1)}}} \mathcal{F}^{(1)}, \quad (26)$$

where we have neglected minor changes on $P^{(1)}$ and $\mathcal{F}^{(1)}$. The possibility of having a heralding signal with vacuum on the DV input further reduces the fidelity with respect to the situation described in Sec. III.2. To avoid the effect of vacuum contribution, $\frac{p_0 P^{(0)}}{p_1 P^{(1)}} \ll 1$. As for the Sec. III.2, it is pertinent to analyse this ratio in the limit $r\alpha \rightarrow 0$ where $\mathcal{F}^{(1)}$ is close to 1. The analytical expression of $P^{(0)}$ is reported in the C. It depends on the input squeezing level, ζ , on the product $|r\alpha|$ and on the detection efficiency η . Its asymptotic behaviour gives $P^{(0)} \underset{r\alpha \rightarrow 0}{=} O\left(\eta^2 \left(\frac{|r\alpha|^2}{r}\right)^4\right)$. Accordingly, by considering $p_0 = O(1)$ and by taking the limit of Eq. (14) for $P^{(1)}$, we obtain:

$$\frac{p_0 P^{(0)}}{p_1 P^{(1)}} \underset{r\alpha \rightarrow 0}{\propto} \frac{|r\alpha|^6}{p_1 r^4}. \quad (27)$$

By putting together Eqs. (21) and (27), we obtain that, with realistic input states and detectors, optimal fidelity \mathcal{F}'_s is obtained when:

$$\frac{|r\alpha|^6}{r^4} \ll p_1 \ll |r\alpha|^2. \quad (28)$$

Compared to the case of an ideal input cat state $|\text{cat}+\rangle$, where the fidelity is maximised by taking an arbitrary low value of p_1 below $|r\alpha|^2$, when considering at the CV input a squeezed state, an optimal value of p_1 must be chosen so as to comply with both conditions of the Eq. (28). Better approximation of Schrödinger cat than squeezed state [22, 23] would even reduce the lower bound of the previous inequality, thus allowing to reach a better fidelity with respect to $|\varphi\rangle$.

To conclude, in Fig. 4 we illustrate the behaviour of the fidelity \mathcal{F}'_s in the case of a DV input state $|\xi'\rangle_{1,2}$ produced by SPDC as given by Eqs. (23). The fidelity is reported as a function of the SPDC excitation parameter, λ^2 and it is compared with the \mathcal{F}' corresponding to an ideal CV input state, $|\text{cat}+\rangle_3$ (dashed line). In the numerical computation, we have considered, as for the previous paragraph, $\eta = 0.95$ and $\frac{r\alpha}{\sqrt{2}} = 0.075$. In particular, we have imposed for the CV input state the squeezing level ζ minimising $P^{(0)}$ under the condition $\frac{r\alpha}{\sqrt{2}} = 0.075$ and by choosing $\alpha = 0.25$. In these regards, we note that, in experiments, low α values allow to satisfy the approximation of a Schrödinger kitten with a squeezed state [16]. With this set of values, the minimum $P^{(0)} = 1.3 \times 10^{-8}$ is obtained for a squeezing parameter $\zeta = -0.061$ and leads to the fidelity given in Fig. (4). The optimal value, $\mathcal{F}'_s = 0.92$, corresponds to $\lambda^2 = 9.4 \times 10^{-4}$ and to a heralding probability of $P'_s \approx 3 \times 10^{-7}$. We note that this value is consistent with similar estimated [10, 11] and measured [17] values in schemes where hybrid state generation is conditioned upon a double detection signal. By operating the experiment at 1 GHz repetition rate [35], the heralding rate reaches 300 Hz. If needed, a higher heralding probability can be reached, at the cost of lower values of the fidelity [16, 20].

IV. CONCLUSION

In this article, we have presented an experimental scheme able to generate time-bin encoded hybrid entangled state of the form $|\varphi\rangle = \frac{|1\rangle_{A,\epsilon}|+\alpha_f\rangle_B - |1\rangle_{A,l}|-\alpha_f\rangle_B}{\sqrt{2}}$. Our protocol is fully compatible with experimental realisations with off-the-shelf fibre components. The required input resources are a coherent state, an optical Schrödinger cat $|\text{cat}+\rangle$ and a time-bin entangled photon pair $|\xi\rangle_{1,2}$. We have shown that with ideal perfect photon number resolving detectors, the process exactly generates the desired state $|\varphi\rangle$. In the second part of the paper, we studied the case of non-ideal detectors and/or input states. In particular, we have shown that even with realistic detectors, *i.e.* available from today's commercial technology, an arbitrary close approximation of $|\varphi\rangle$ can be obtained by playing on the scheme parameters. A major advantage of the presented scheme lies in its ability to tolerate both predominance of vacuum on its discrete variable input, as well as multiple pairs, when operated with well-chosen parameters. In this context, we have studied the case of a realistic DV input as well as the one of a CV one in a squeezed vacuum state instead of the Schrödinger cat. Our study shows that vacuum and multi-pairs effect can be neglected as long as the experiment is carried out with a p_1 in the DV input respecting the condition $\frac{|r\alpha|^6}{r^4} \ll p_1 \ll |r\alpha|^2$.

Experimental generation of time-bin coded hybrid states, compatible with standard telecommunication systems, will permit pushing the applications of hybrid-

states of light out-of-the-laboratory with a high impact in the context of future development of fibre quantum network systems.

ACKNOWLEDGMENTS

All the authors would like to thank Jean Etesse and Anthony Martin for stimulating discussions. This work has been conducted within the framework of the project HyLight (ANR-17-CE30-0006-01) granted by the Agence Nationale de la Recherche (ANR). The authors also acknowledge financial support from the European Union and the Région PACA by means of the Fond Européen de Développement Regional (FEDER) through the project OPTIMAL, and the French government through its program “Investments for the Future” under the Université Côte d’Azur UCA–JEDI project (under the label Quantum@UCA) managed by the ANR (grant agreement ANR-15-IDEX-01).

Appendix A: Action of propagation losses on the DV part

We consider losses on DV part of state $|\varphi\rangle$, defined by Eq. (1). Similar expression can be found for the states obtained in non ideal conditions and discussed in different sections of the papers. Losses can be modelled by an unbalanced beam-splitter of transmittivity t_{loss} and reflectivity r_{loss} , that couple the DV channel A with a vacuum populated channel v . We will label as A' the BS transmitted output and as “loss” the reflected one, that also corresponds to the lost part. Hence, the state after the beam-splitter is:

$$|\varphi'\rangle = \frac{1}{\sqrt{2}} \left[t_{\text{loss}} |1\rangle_{A',e} |+\alpha_f\rangle_B + r_{\text{loss}} |1\rangle_{\text{loss},e} |+\alpha_f\rangle_B - t_{\text{loss}} |1\rangle_{A',l} |-\alpha_f\rangle_B - r_{\text{loss}} |1\rangle_{\text{loss},l} |-\alpha_f\rangle_B \right]. \quad (\text{A1})$$

By tracing out the modes (loss, e) and (loss, l) , we obtain the state after losses, described by the density matrix:

$$\begin{aligned} \hat{\rho}_{\text{loss}} &= \text{Tr}_{\text{loss}} [|\varphi'\rangle\langle\varphi'|] \\ &= t_{\text{loss}}^2 |\varphi\rangle\langle\varphi| \\ &\quad + r_{\text{loss}}^2 |0\rangle\langle 0|_{A'} \frac{|+\alpha_f\rangle\langle+\alpha_f|_B + |-\alpha_f\rangle\langle-\alpha_f|_B}{2}. \end{aligned} \quad (\text{A2})$$

It is clear that for any projection operation (or measurement) on A' that is insensitive to vacuum, the result is the same as the one obtained from $|\varphi\rangle\langle\varphi|$, with a success probability multiplied by t_{loss}^2 . As an example, we can consider a remote preparation experiment with the measurement described by the projector on $\frac{|1\rangle_{A',e} + |1\rangle_{A',l}}{\sqrt{2}}$:

$$\hat{\Pi}_{A'} = \frac{1}{2} \left(|1\rangle_{A',e} + |1\rangle_{A',l} \right) \left(\langle 1|_{A',e} + \langle 1|_{A',l} \right). \quad (\text{A3})$$

Unnormalised state after such a conditioning reads:

$$\begin{aligned} \text{Tr}_{A'} [\hat{\Pi}_{A'} \hat{\rho}_{\text{losses}}] &= \frac{t_{\text{loss}}^2}{4} [|+\alpha_f\rangle_B - |-\alpha_f\rangle_B] [\langle +\alpha_f|_B - \langle -\alpha_f|_B] \\ &\propto |\text{cat-}\rangle\langle\text{cat-}|_B. \end{aligned} \quad (\text{A4})$$

The obtained state is exactly the same as it would be without losses.

Appendix B: Time-bin entangled photons source

Generic form Typically, the time-bin entangled photon pair $|\xi'\rangle_{1,2}$ (see Eq. (16)) can be seen as the result of two identical entangled photon pair generation processes, one for “early” (e) and other for the “late” (l) mode. In the Fock basis, whose elements are indicated here as $\{|k\rangle\}$, the generated state can be written as:

$$\begin{aligned} |\xi'\rangle_{1,2} &= \left(\sqrt{p_0^m} |0\rangle_{1,e} |0\rangle_{2,e} + \sqrt{p_1^m} |1\rangle_{1,e} |1\rangle_{2,e} + \sqrt{p_2^m} |2\rangle_{1,e} |2\rangle_{2,e} + \dots \right) \\ &\quad \otimes \left(\sqrt{p_0^m} |0\rangle_{1,l} |0\rangle_{2,l} + \sqrt{p_1^m} |1\rangle_{1,l} |1\rangle_{2,l} + \sqrt{p_2^m} |2\rangle_{1,l} |2\rangle_{2,l} + \dots \right). \end{aligned} \quad (\text{B1})$$

By explicitly taking into account only terms containing at most two photons per spatial mode, we can write:

$$|\xi'\rangle_{1,2} \approx p_0^m |0\rangle + \sqrt{2p_0^m p_1^m} \frac{|1\rangle_{1,e} |1\rangle_{2,e} + |1\rangle_{1,l} |1\rangle_{2,l}}{\sqrt{2}} + \sqrt{2p_0^m p_2^m} \frac{|2\rangle_{1,e} |2\rangle_{2,e} + |2\rangle_{1,l} |2\rangle_{2,l}}{\sqrt{2}} + p_1^m |1\rangle_{1,e} |1\rangle_{1,l} |1\rangle_{2,e} |1\rangle_{2,l}, \quad (\text{B2})$$

where, as before, $|1\rangle$ and $|2\rangle$ are single and two photon Fock states, respectively.

By comparing the previous expression with the general form of $|\xi'\rangle_{1,2}$ cut at the second order ($p_e \approx p_l$), we

obtain the values of coefficients appearing in Eq. (16):

$$\begin{cases} p_0 = (p_0^m)^2 \\ p_1 = 2p_0^m p_1^m \\ p_2 = 2p_0^m p_2^m + (p_1^m)^2 \end{cases} \quad (\text{B3})$$

with the second order term being:

$$|\epsilon\rangle_{1,2} = \sqrt{\frac{2p_0^m p_2^m}{2p_0^m p_2^m + (p_1^m)^2}} \frac{|2\rangle_{1,e} |2\rangle_{2,e} + |2\rangle_{1,l} |2\rangle_{2,l}}{\sqrt{2}} + \frac{p_1^m}{\sqrt{2p_0^m p_2^m + (p_1^m)^2}} |1\rangle_{1,e} |1\rangle_{1,l} |1\rangle_{2,e} |1\rangle_{2,l} \quad (\text{B4})$$

SPDC case The explicit expression of coefficients p_k^m ($k=0,1,2,\dots$) in the previous equations depends on the specific generation process. In the special case of SPDC, $p_k^m = (1-\lambda^2)(\lambda^2)^k$ [25, 30], with λ the SPDC excitation parameter as described in the text. By injecting these expressions in Eq. (B3), we obtain the results of Eq. (23):

$$\begin{cases} p_0 = (1-\lambda^2)^2 \\ p_1 = 2(1-\lambda^2)^2 \lambda^2 \\ p_2 = 3(1-\lambda^2)^2 \lambda^4. \end{cases} \quad (\text{B5})$$

$$P_0 = \frac{e^{-|r\alpha|^2}}{\cosh(\zeta)} \left[\sum_{y=0}^{+\infty} \sum_{z=0}^y \sum_{q=0}^z \left[1 - \left(1 - \frac{\eta}{4}\right)^{z-q} \right] \left[1 - \left(1 - \frac{\eta}{4}\right)^q \right] \left| \sum_{\substack{0 \leq k_4 \leq z \\ k_4 \equiv y[2]}} t^{y-z} r^{z-k_4} \sqrt{\frac{(y-k_4)!}{(z-k_4)!}} \frac{(r\alpha)^{k_4}}{\sqrt{k_4!}} \sqrt{\frac{(y-k_4)!}{2^{y-k_4}}} \left[\frac{-\tanh \zeta}{2} \right]^{\frac{y-k_4}{2}} \frac{C_{k_4, z-k_4, q}}{\sqrt{2}^z} \right|^2 \right] \quad (\text{C1})$$

In the previous expression:

$$C_{k,l,x} = \begin{cases} \sqrt{\binom{k+l-x}{k} \binom{l}{x}} {}_2F_1(-k, -x, l-x+1, -1) & \text{if } 0 \leq x \leq l \\ (-1)^{x-l} \sqrt{\binom{x}{l} \binom{k}{x-l}} {}_2F_1(-l, -(k+l-x), x-l+1, -1) & \text{if } l \leq x \leq k+l, \end{cases} \quad (\text{C2})$$

with ${}_2F_1(p, q, t, w)$ the hypergeometric function.

We also observe that for SPDC, the two photon component reads:

$$|\epsilon\rangle_{1,2} = \sqrt{\frac{2}{3}} \frac{|2\rangle_{1,e} |2\rangle_{2,e} + |2\rangle_{1,l} |2\rangle_{2,l}}{\sqrt{2}} + \sqrt{\frac{1}{3}} |1\rangle_{1,e} |1\rangle_{1,l} |1\rangle_{2,e} |1\rangle_{2,l} = \frac{|2\rangle_{1,e} |2\rangle_{2,e} + |2\rangle_{1,l} |2\rangle_{2,l} + |1\rangle_{1,e} |1\rangle_{1,l} |1\rangle_{2,e} |1\rangle_{2,l}}{\sqrt{3}} \quad (\text{B6})$$

Appendix C: Heralding probability with an CV input squeezed vacuum state and with no-photon from the DV input

We consider at the CV a squeezed vacuum state, $\hat{S}(\zeta)|0\rangle_3$, with $\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$ the single mode squeezing operator and ζ the squeezing parameter. In this case, the probability of having an announced signal with no-photon from the DV part is not zero and it is a function of ζ , of the detection efficiency, η , and of the amplitude $r\alpha$ of the coherent state entering the system via the input labeled as 4. As for the previous cases, r and t are the amplitude reflection and transmission coefficients of BS1.

-
- [1] U. L. Andersen, J. S. Neergaard-Nielsen, P. van Loock, and A. Furusawa, *Nature Physics* **11**, 713 (2015), 1409.3719.
[2] T. E. Northup and R. Blatt, *Nature Photonics* **8**, 356 (2014), 1708.00424.
[3] G. Adesso, S. Ragy, and A. R. Lee, *Open Systems & Information Dynamics* **21**, 1440001 (2014), 1401.4679.
[4] S. H. Lie and H. Jeong, *Photonics Research* **7**, A7 (2019).
[5] J. B. Brask, I. Rigas, E. S. Polzik, U. L. Andersen, and

- A. S. Sørensen, *Physical Review Letters* **105**, 160501 (2010), 1004.0083.
[6] S.-W. Lee and H. Jeong, *Physical Review A* **87**, 022326 (2013), 1112.0825.
[7] R. C. Parker, J. Joo, M. Razavi, and T. P. Spiller, *Journal of Optics* **19**, 104004 (2017), 1706.08492.
[8] Y. Lim, J. Joo, T. P. Spiller, and H. Jeong, *Physical Review A* **94**, 062337 (2016), 1608.04882.
[9] P. van Loock, W. J. Munro, K. Nemoto, T. P. Spiller,

- T. D. Ladd, S. L. Braunstein, and G. J. Milburn, *Physical Review A* **78**, 022303 (2008), [quant-ph/0701057](#).
- [10] H. Kwon and H. Jeong, *Physical Review A* **91**, 012340 (2015), [1410.6823](#).
- [11] S. Li, H. Yan, Y. He, and H. Wang, *Physical Review A* **98**, 022334 (2018).
- [12] M. Bergmann and P. van Loock, *Physical Review A* **99**, 032349 (2019), [1708.09322](#).
- [13] S. Omkar, Y. S. Teo, and H. Jeong, (2019), [1906.04864](#).
- [14] K. Park, S.-W. Lee, and H. Jeong, *Physical Review A* **86**, 062301 (2012), [1209.3095](#).
- [15] H. Jeong, A. Zavatta, M. Kang, S.-W. Lee, L. S. Costanzo, S. Grandi, T. C. Ralph, and M. Bellini, *Nature Photonics* **8**, 564 (2014), [1309.6192](#).
- [16] O. Morin, K. Huang, J. Liu, H. Le Jeannic, C. Fabre, and J. Laurat, *Nature Photonics* **8**, 570 (2014), [1309.6191](#).
- [17] A. E. Ulanov, D. Sychev, A. A. Pushkina, I. A. Fedorov, and A. I. Lvovsky, *Physical Review Letters* **118**, 160501 (2017), [1605.08649](#).
- [18] A. Cavallès, H. Le Jeannic, J. Raskop, G. Guccione, D. Markham, E. Diamanti, M. D. Shaw, V. B. Verma, S. W. Nam, and J. Laurat, *Physical Review Letters* **121**, 170403 (2018), [1811.00036](#).
- [19] H. Le Jeannic, A. Cavallès, J. Raskop, K. Huang, and J. Laurat, *Optica* **5**, 1012 (2018), [1809.10700](#).
- [20] D. V. Sychev, A. E. Ulanov, E. Tiunov, A. A. Pushkina, A. Kuzhamuratov, V. Novikov, and A. I. Lvovsky, *Nature Communications* **9**, 3672 (2018), [1712.10206](#).
- [21] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Reviews of Modern Physics* **74**, 145 (2002), [quant-ph/0101098](#).
- [22] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouiri, and P. Grangier, *Nature* **448**, 784 (2007).
- [23] D. V. Sychev, A. E. Ulanov, A. A. Pushkina, M. W. Richards, I. A. Fedorov, and A. I. Lvovsky, *Nature Photonics* **11**, 379 (2017), [1609.08425](#).
- [24] O. Alibart, V. D'Auria, M. De Micheli, F. Doutré, F. Kaiser, L. Labonté, T. Lunghi, E. Picholle, and S. Tanzilli, *Journal of Optics* **18**, 104001 (2016), [1608.01100](#).
- [25] É. Gouzien, B. Fedrici, A. Zavatta, S. Tanzilli, and V. D'Auria, *Physical Review A* **98**, 013833 (2018), [1807.00722](#).
- [26] R. H. Hadfield, *Nature Photonics* **3**, 696 (2009).
- [27] R. J. Glauber, *Physical Review* **131**, 2766 (1963).
- [28] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, *Physical Review A* **65**, 032323 (2002), [quant-ph/0106136](#).
- [29] M. G. A. Paris, *Physics Letters A* **217**, 78 (1996).
- [30] V. D'Auria, N. Lee, T. Amri, C. Fabre, and J. Laurat, *Physical Review Letters* **107**, 050504 (2011), [1105.4090](#).
- [31] A. F. Feito Boirac, J. S. Lundeen, H. B. Coldenstrodtr-Ronge, J. Eisert, M. B. Plenio, and I. A. Walmsley, *New Journal of Physics* **11**, 093038 (2009), [0906.3440](#).
- [32] E. Schmidt, E. Reutter, M. Schwartz, H. Vural, K. Ilin, M. Jetter, P. Michler, and M. Siegel, *IEEE Transactions on Applied Superconductivity* , 1 (2019), [1810.11256](#).
- [33] T. Yamashita, S. Miki, H. Terai, and Z. Wang, *Optics Express* **21**, 27177 (2013), [1305.2672](#).
- [34] F. Marsili, V. B. Verma, J. A. Stern, S. Harrington, A. E. Lita, T. Gerrits, I. Vayshenker, B. Baek, M. D. Shaw, R. P. Mirin, and S. W. Nam, *Nature Photonics* **7**, 210 (2013), [1209.5774](#).
- [35] L. A. Ngah, O. Alibart, L. Labonté, V. D'Auria, and S. Tanzilli, *Laser & Photonics Reviews* **9**, L1 (2015), [1412.5427](#).
- [36] K. Kreis and P. van Loock, *Physical Review A* **85**, 032307 (2012), [1111.0478](#).
- [37] E. Agudelo, J. Sperling, L. S. Costanzo, M. Bellini, A. Zavatta, and W. Vogel, *Physical Review Letters* **119**, 120403 (2017), [1702.04257](#).
- [38] L. S. Costanzo, A. Zavatta, S. Grandi, M. Bellini, H. Jeong, M. Kang, S.-W. Lee, and T. Ralph, *Physica Scripta* **90**, 074045 (2015).
- [39] A. Christ and C. Silberhorn, *Physical Review A* **85**, 023829 (2012), [1111.4095](#).