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THE CORE LOGIC PARADOX

Joseph Vidal-Rosset

Abstract. — This paper provides a proof that Tennant’s logical system entails a paradox that is called “Core logic paradox”, in reference to the new name given by Tennant to his intuitionistic relevant logic [2, 3]. The inconsistency theorem on which this paper ends is preceded by lemmas that prove syntactically and semantically that rule $L\top$ is strongly admissible in the sequent calculus for propositional Core logic. This rule plays a key role in a consistency test on which Tennant’s logical system fails.

1. Two basic claims of Core logic

The idiosyncrasy of Tennant’s logical system lies mainly in this pair of claims:

(1) The First Lewis Paradox that is the sequent $\neg A, A \vdash B$ is *not* provable in Core logic, therefore the formula

$$\neg A, A \not\vdash B \tag{1}$$

is claimed as being valid in Core logic.⁽¹⁾

⁽¹⁾Tennant [5, page 195] wrote $\neg A, A \not\vdash B$, because of the completeness claimed for Core logic, formula (1) is justified.

(2) Every intuitionistic logical *theorem* is provable in Core logic, therefore

$$\vdash \neg A \rightarrow (A \rightarrow B) \quad (2)$$

is provable in Core logic.

The slogan “relevance at the level of the turnstile” [5, p. 15, p. 41, p. 121, p. 263] explains this surprising feature of Tennant’s logical system.

The proofs of lemmas and theorem given in the next section are based on the following rules of sequent calculus for propositional Core logic.

TABLE 1. Rules of sequent calculus for propositional Core logic [5]

$$\begin{array}{c}
 \frac{\Delta \vdash A}{\neg A, \Delta \vdash} L_{\neg} \qquad \frac{}{A \vdash A} Ax. \qquad \frac{A, \Delta \vdash}{\Delta \vdash \neg A} R_{\neg} \\
 \\
 \frac{\Delta \vdash C}{A \wedge B, \Delta \setminus \{A, B\} \vdash C} L_{\wedge} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta, \Gamma \vdash A \wedge B} R_{\wedge} \\
 \text{where } \Delta \cap \{A, B\} \neq \emptyset \\
 \\
 \frac{A, \Delta \vdash C/\perp \quad B, \Gamma \vdash C/\perp}{A \vee B, \Delta, \Gamma \vdash C/\perp} L_{\vee} \qquad \frac{\Delta \vdash A}{\Delta \vdash A \vee B} R_{\vee} \\
 \frac{\Delta \vdash B}{\Delta \vdash A \vee B} R_{\vee} \\
 \\
 \frac{\Delta \vdash A \quad B, \Gamma \vdash C}{A \rightarrow B, \Delta, \Gamma \vdash C} L_{\rightarrow} \qquad \frac{A, \Delta \vdash}{\Delta \vdash A \rightarrow B} R_{\rightarrow(a)} \\
 \frac{\Delta \vdash B}{\Delta \setminus \{A\} \vdash A \rightarrow B} R_{\rightarrow(b)}
 \end{array}$$

2. Inconsistency Proof

Definition 2.1. — $\mathcal{A} = P \cup \{\neg, \vee, \wedge, \rightarrow\} \cup \{\}, \{\}$. \mathcal{A} is the standard alphabet for formulas of propositional Core logic, composed of P as set of propositional variables, of the usual set of connectives that are respectively negation,

disjunction, conjunction, conditional and a set of round brackets. The symbolism of sequent calculus for this logic is also standard: “ Γ, Δ ” stand for sets of formulas, and “ \vdash ” is the usual symbol of syntactic inference.⁽²⁾

Definition 2.2. — \perp is not the absurdity constant but “merely a punctuation device in proofs, used in order to register absurdity” [5, p. 266].

Definition 2.3. — \top here stands for *any theorem of Core propositional logic* and should not be read as the truth constant.

Lemma 2.4. — *The equivalence $\Delta \dashv\vdash \top \wedge \Delta$ is provable “at the level of the turnstile” in Core logic.*

Proof. — Assume $\top = (A \rightarrow A)$, then

$$\frac{\frac{\overline{A \vdash A}^{Ax}}{\vdash A \rightarrow A} \quad \frac{\overline{\Delta \vdash \Delta}^{Ax}}{\Delta \vdash \Delta}^{R\wedge}}{\Delta \vdash (A \rightarrow A) \wedge \Delta} \quad \frac{\overline{\Delta \vdash \Delta}^{Ax}}{(A \rightarrow A) \wedge \Delta \vdash \Delta}^{L\wedge} \quad \text{i.e.} \quad \frac{\overline{\Delta \vdash \Delta}^{Ax}}{A \rightarrow A, \Delta \vdash \Delta}^{Wk.}$$

Therefore, by induction on the structure of these derivations, the equivalence

$$\Delta \dashv\vdash \top \wedge \Delta$$

is proved in Core logic.⁽³⁾ □

⁽²⁾In this paper “ \vdash ” is preferred instead of “ \cdot ” used by Tennant, just because “ $\dashv\vdash$ ” is also used.

⁽³⁾The rule of Weakening on the left is *admissible* in Core logic, but, to prevent the derivation of the negated-conclusion version of the First Lewis Paradox i.e. $\neg A, A \vdash \neg B$, Weakening on the left is banned in this case:

$$\frac{\neg A, A \vdash \perp}{B, \neg A, A \vdash \perp} Wk.$$

Lemma 2.5. — *The following couple of rules $L\top$ is strongly admissible⁽⁴⁾ in Core logic:*

$$\frac{\Delta \vdash C}{\top, \Delta \vdash C} L\top \qquad \frac{\Delta \not\vdash C}{\top, \Delta \not\vdash C} L\top$$

Proof. — $\{\top, \Delta\}$ and $\{\Delta\}$ are equivalent (see the previous proof). \square

First proof by contradiction. — 1. Assume $\Delta \not\vdash C$, therefore $\not\vdash \Delta \rightarrow C$ and therefore $\neg(\Delta \rightarrow C)$ is satisfiable.

2. Assume $\top, \Delta \vdash B$, therefore $\top \wedge \Delta \vdash B$, therefore $\vdash (\top \wedge \Delta) \rightarrow C$ and therefore $(\top \wedge \Delta) \rightarrow C$ is satisfiable.

3. $\overline{\vdash \top}^{Ax}$ is trivially admissible in any sequent calculus for propositional logic.

4. The following derivation proves in propositional Core logic that the conjunction of assumptions 1 and 2 is *inconsistent*, in other words *unsatisfiable*:

$$\frac{\frac{\frac{\overline{\Delta \vdash \Delta}^{Ax} \quad \overline{\vdash \top}^{Ax}}{\Delta \vdash \top \wedge \Delta} R\wedge \quad \overline{C \vdash C}^{Ax}}{\Delta, (\top \wedge \Delta) \rightarrow C \vdash C} L\rightarrow \quad \overline{(\top \wedge \Delta) \rightarrow C \vdash \Delta \rightarrow C} R\rightarrow}{\neg(\Delta \rightarrow C), (\top \wedge \Delta) \rightarrow C \vdash} L\neg$$

\square

Second proof by contradiction. — 1. Assume $\Delta \vdash C$, therefore $\vdash \Delta \rightarrow C$ and therefore $\Delta \rightarrow C$ is satisfiable.

⁽⁴⁾ The rule

$$\frac{S}{S'}$$

is strongly admissible if and only if for every derivation of height n of an instance of S there is a derivation of height $\leq n$ of the corresponding instance of S' [1, p. 1501]. Rules $L\top$ are strongly admissible according to this definition given by Dyckhoff and Negri [1, p. 1501], and invertible, hence the double inference line.

2. Assume $\top, \Delta \not\vdash C$, therefore $\top \wedge \Delta \not\vdash C$, therefore $\not\vdash (\top \wedge \Delta) \rightarrow C$ and therefore $\neg((\top \wedge \Delta) \rightarrow C)$ is satisfiable.

3. The following derivation proves in propositional Core logic that the conjunction of assumptions 1 and 2 is *inconsistent*, in other words *unsatisfiable*:

$$\frac{\frac{\frac{\overline{\Delta \vdash \Delta}^{Ax}}{\top \wedge \Delta \vdash \Delta}^{L\wedge} \quad \frac{\overline{C \vdash C}^{Ax}}{\top \wedge \Delta, \Delta \rightarrow C \vdash C}^{L\rightarrow}}{\Delta \rightarrow C \vdash (\top \wedge \Delta) \rightarrow C}^{R\rightarrow}}{\neg((\top \wedge \Delta) \rightarrow C), \Delta \rightarrow C \vdash}^{L\neg}$$

□

Lemma 2.6. — *Rules $L\top$ in Core logic are sound:*

$$\frac{\overline{\Delta \vdash C}}{\top, \Delta \vdash C}^{L\top} \quad \text{therefore } \top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$$

$$\frac{\overline{\Delta \not\vdash C}}{\top, \Delta \not\vdash C}^{L\top} \quad \text{therefore } \top, \Delta \not\vdash C \Leftrightarrow \Delta \not\vdash C$$

Equivalence proof: $\top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$. — In Core logic semantics, like in minimal logic's, there are only three possible cases [4, pp. 758-762] that lead all to the same conclusion:

1. Assume that C is a contradiction i.e. $C = \perp$, then $\top, \Delta \vDash C$ is valid if and only if \perp can be deduced from Δ , therefore $\top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$.

2. Assume that C is a theorem i.e. $C = \top$, by definition, $\top, \Delta \vDash C$ is reducible to $\vDash C$, therefore $\top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$,

3. Assume that C is neither a contradiction nor a tautology, but only a satisfiable formula. In this case, $\top, \Delta \vDash C$ is valid if and only if C is deducible

from Δ , therefore $\top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$.

Therefore, in Core logic, $\top, \Delta \vDash C \Leftrightarrow \Delta \vDash C$. \square

Equivalence proof: $\top, \Delta \not\vDash C \Leftrightarrow \Delta \not\vDash C$. — Again, there are only three possible cases that lead all to the same conclusion:

1. Assume that C is a contradiction i.e. $C = \perp$. $\top \not\vDash \perp$ is trivial: provided the consistency of any deductive system S , the negation of a theorem in S is always unprovable in S , therefore it is provable that $\top, \Delta \not\vDash \perp$ if and only if no contradiction is deducible from Δ , therefore $\top, \Delta \not\vDash C \Leftrightarrow \Delta \not\vDash C$.

2. Assume that C is a theorem i.e. $C = \top$, then by definition the formula $\not\vDash C$ i.e. $\not\vDash \top$ is always false and therefore $\top, \Delta \not\vDash C \Leftrightarrow \Delta \not\vDash C$.

3. Assume that C is neither a contradiction nor a tautology, but only a satisfiable formula. In this case, $\top, \Delta \not\vDash C$ is provable if and only if it is provable that C is *not* deducible from Δ because, again, it is trivial that $\top \not\vDash C$ when the truth value of C can be \perp , therefore $\top, \Delta \not\vDash C \Leftrightarrow \Delta \not\vDash C$.

Therefore, in Core logic, $\top, \Delta \not\vDash C \Leftrightarrow \Delta \not\vDash C$. \square

Lemma 2.7. — *In Core proofs, rules $L\top$ must be written as follows*

$$\frac{\overline{\top \top}^{Ax} \quad \Delta \vdash C}{\top, \Delta \vdash C} L\top$$

$$\frac{\overline{\top \top}^{Ax} \quad \Delta \not\vDash C}{\top, \Delta \not\vDash C} L\top$$

Reason. — Axiom $\overline{\top \top}^{Ax}$ is trivially admissible in any sequent calculus for propositional logic. \top and $\overline{\top \top}^{Ax}$ must be replaced respectively by the mention of a theorem and its proof, symbol \top being not used in Core logic. Last, the double inference line is useless, the use and the reading of rule $L\top$ being top-down. \square

Theorem 2.8. — *Tennant's propositional Core logic is inconsistent.*

Proof. — Here is a consistency test *via* rule $L\top$ that leads to the addition of theorem (2) on the left of sequent (1). Indeed, the following formula

$$\neg A \rightarrow (A \rightarrow B), \neg A, A \not\vdash B \quad (3)$$

is provable in Core logic, *via* this derivation \mathcal{D}_1 where the last rule is $L\top$:

$$\frac{\frac{\frac{\overline{A \vdash A}^{Ax}}{\neg A, A \vdash} L\neg}{\neg A \vdash A \rightarrow B} R \rightarrow (a)}{\vdash \neg A \rightarrow (A \rightarrow B)} R \rightarrow (b) \quad \neg A, A \not\vdash B}{\neg A \rightarrow (A \rightarrow B), \neg A, A \not\vdash B} L\top$$

But the trouble is that the sequent

$$\neg A \rightarrow (A \rightarrow B), \neg A, A \vdash B \quad (4)$$

is also provable in Core logic, *via* this derivation \mathcal{D}_2 :

$$\frac{\frac{\frac{\overline{A \vdash A}^{Ax}}{\neg A, A \vdash} L\neg}{\neg A \vdash \neg A} R\neg \quad \frac{\frac{\overline{A \vdash A}^{Ax} \quad \overline{B \vdash B}^{Ax}}{A \rightarrow B, A \vdash B} L \rightarrow}{\neg A \rightarrow (A \rightarrow B), \neg A, A \vdash B} L \rightarrow}{\neg A \rightarrow (A \rightarrow B), \neg A, A \vdash B} L \rightarrow$$

Therefore, a contradiction is derivable from propositional Core logic:

$$\frac{\begin{array}{c} \mathcal{D}_1 \\ \vdots \\ \neg A \rightarrow (A \rightarrow B), \neg A, A \not\vdash B \end{array} \quad \begin{array}{c} \mathcal{D}_2 \\ \vdots \\ \neg A \rightarrow (A \rightarrow B), \neg A, A \vdash B \end{array}}{\perp}$$

Consequently, according to the well-known definition of an inconsistent theory,⁽⁵⁾ Tennant's propositional Core logic is inconsistent. \square

⁽⁵⁾A theory Σ is inconsistent if and only if a contradiction is deducible from Σ .

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