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NO VICIOUS CIRCULARITY WITHOUT CLASSICAL LOGIC

Joseph Vidal-Rosset

Abstract. — This paper proves that the paradoxes called "vicious circles" like Russell's, require classical logic, since the double negation elimination plays a key role in their derivation.

It seems that, at least confusedly, Quine (1986) recognized the existence of a relationship between the paradoxes of set theory and the use of classical logic when he wrote:

> Moreover, the paradoxes of set theory put an added premium on constructivism; for what we accomplish within its restraints is pretty clearly immune to the threat of contradiction that lingers outside.(Quine, 1986, p. 88)

The rejection of the excluded middle being an expression of constructivism, if the adoption of constructivism excludes the paradoxes of set theory, it is of course possible to suppose that this principle of classical logic can be one of the culprits of these paradoxes. But, of course, this suspicion does not explain the role of classical logic in the expression of these paradoxes.

To unravel this question, let us recall, first of all, that Russell's paradox starts from the definition of a set whose all elements are sets that do not belong themselves, a definition that can be translated in turn by this first order formula:

$$\exists y \forall x (x \notin x \leftrightarrow x \in y) \tag{1}$$

and, by instantiation, an absurdity follows:

$$\forall x (x \notin x \leftrightarrow x \in a) \tag{2}$$

$$a \notin a \leftrightarrow a \in a \tag{3}$$

The trouble is that it is provable in intuitionistic logic that (1) implies a contradiction, and it probably explains why Tennant claimed that "excluded

middle plays no essential role in the derivation of the paradoxes" (Tennant, 2017, p. 285) in disagreement with Field (2008) and Priest (1983) on this question. The analysis made by Tennant is all the more disturbing because, in spite of his philosophical commitment in favor of intuitionism, he disagrees with Brouwer by exempting the law of excluded middle of any responsibility in the derivation of paradoxes of set theory. Therefore, Tennant's analysis constrains to clarify this question: to what extent are the logical laws specific to classical logic involved in the derivation of these paradoxes?[(1)]

The distinction between use and mention could be useful here. The intuitionist logician who seeks to understand Russell's paradox translates the excluded middle as meaning that, for any set, it is *decidable* that it *is not* an element of itself or that it *is* an element of itself, to point out that the excluded middle does not provide any method of construction to build the set of all sets that are not elements of themselves. On the other hand, for classical logicians like Russell and many others, this definition is robust, because it is based on the law of the excluded third, hence the fact that Russell mentioned this law twice, explaining the form of this paradox. :

> If x be a predicate, x may or may not be predicable of itself. Let us assume that "not-predicable of oneself " is a predicate. Then to suppose either that this predicate is, or that it is not, predicable of itself, is self-contradictory. The conclusion, in this case, seems obvious: "not-predicable of oneself " is not a predicate. (Russell, 1903, p. 102)

That is this last expression of Russell's paradox that has been translated by the famous Grelling–Nelson paradox published by Grelling and Nelson (1907). Here is this paradox. A word is "autological" if and only if it describes itself. For example "short" is a short word, "short" is short. A word that is not autological is heterological. Now, let us see how the law of excluded middle and the rule of double negation elimination are *used* in the derivation of Grelling-Nelson paradox.

Definition 0.1 **(Rule *aut*)**. — For any term t, the assumption that t is *autological* entails the following disquotation: "t" is t.

Definition 0.2 **(Rule *het*)**. — For any term t, the assumption that t is *heterological* entails the following negation and disquotation: "t" is not t.

[(1)]The famous distinction made by Ramsey (Ramsey, 1990, pp. 183–184) between logical paradoxes and epistemological paradoxes is not taken into account in this paper, where I wonder only on the logic that is necessary to the derivation of paradoxes described by Russell (1903).

Proposition 0.3. — *Any term that is not heterological is autological (from the* definiens *of "heterological" and the double negation elimination).*

Proposition 0.4. — *Any term is either heterological or not heterological (excluded middle).*

Proposition 0.5. — *With Definitions 0.1 and 0.2, Propositions 0.3 and 0.4 entail a contradiction.*

Proof. — A consequence of proposition 0.4 is that the term "heterological" is either heterological or not heterological. This instance of the excluded middle offers a choice of two incompatible assumptions. The trouble is that it is provable that each assumption entails its negation, hence the paradox.

– Assumption 1: "heterological" is heterological. Therefore, *via* rule *het*

$$\text{“heterological” is not heterological} \tag{4}$$

But (4) is the negation of Assumption 1.

– Assumption 2: "heterological" is not heterological. Therefore

$$\text{“heterological” is autological} \tag{5}$$

by proposition 0.3 (double negation elimination). Therefore

$$\text{“heterological” is heterological} \tag{6}$$

via rule *aut*. But (6) is the negation Assumption 2.

□

Kleene (Kleene, 1952, p. 37) and Schwartz (Schwartz, 1991, p. 28) pointed out that Russell's paradox lies in the fact that each member of the alternation $(a \notin a) \vee (a \in a)$ is proved, while clearly these two formulas contradict each other. Grelling-Nelson paradox has the same structure. Is classical logic necessary to express *exactly* this form of paradox? *Pace* Tennant, it is provable that classical logic is indeed *necessary* to translate the famous "vicious circle" on which Russell (1903) insisted.

Proposition 0.6. — *No vicious circularity* as such *can be proved without help of classical logic, that is without the rule of double negation elimination.*

Proof. — The vicious circularity of paradoxes related to Russell's can be described as a complete reversal of an instance of excluded middle:

$$A \vee \neg A \tag{7}$$

becomes

$$\neg A \wedge A \tag{8}$$

because from assumption A, $\neg A$ is proved, *and*, from assumption $\neg A$, A is proved in turn. The biconditional

$$A \leftrightarrow \neg A \tag{9}$$

is exactly the form of the conclusion of Russell's paradox as well as Grelling-Nelson paradox: (9) is a *deducible* vicious circle, that is a conjunction of two conditional formulas

$$(A \to \neg A) \wedge (\neg A \to A) \tag{10}$$

that are both provable. But if the first member of this conjunction is reducible to $\neg A$ in minimal logic, the second member is not reducible to A without the use of classical logic. Indeed, if A is deducible from $\neg A$, it is because $\neg\neg A$ is provable, but A is deducible from $\neg\neg A$ *only in classical logic*, because of the rule of double negation elimination. Therefore, no vicious circularity *as such* can be proved without help of classical logic. □

Remark 0.7. — It is clear that classical logic is *not* required to deduce $\perp$ from (9) as open assumption, because the sequent

$$A \leftrightarrow \neg A \vdash \perp \tag{11}$$

is provable in minimal logic, [(2)] but there is a crucial difference between assuming a formula and deriving it (von Plato, 2013, p. 83), and it is unquestionable that the derivation of (9) is not possible without the double negation elimination rule of classical logic.

It must therefore be admitted that the disturbing beauty of vicious circles can not be unveiled without the rule of double negation elimination. Quine might have concluded that this was ultimately one of the signs of the superiority of classical logic over intuitionistic logic; nevertheless, he wrote:

> Intuitionism antedates Godel's proof that there can be no complete proof procedure for number theory. This great result of Godel's adds force, however, to the intuitionist protest. The excess of admitted questions over possible answers seems especially regrettable when the questions are mathematical and the answers mathematically impossible. (Quine, 1986, p. 87)

But Quine never considered that this "excess of admitted questions over possible answers" was caused by the classical interpretation of the excluded middle.

David et al. (2004)

[(2)] To get the proof of (11) in normal form see von Plato (2000) and (Tennant, 2017, pp. 289–293).

References

David, R., Nour, K., and Raffalli, C. (2004). *Introduction à la logique: théorie de la démonstration*. Dunod, Paris.

Field, H. (2008). *Saving Truth From Paradox*. Oxford University Press, New York.

Grelling, K. and Nelson, L. (1907). Bemerkungen zu den Paradoxien von Russell und Burali-Forti. *Abhandlungen Der Fries'schen Schule (Neue Serie)*, 2:300–334.

Kleene, S. C. (2010, First published 1952). *Introduction to Metamathematics*. Ishi Press International, New York, U.S.A.

Priest, G. (1983). The Logical Paradoxes and the Law of Excluded Middle. *The Philosophical Quarterly*, 33(131):160–165.

Quine, W. V. (1986). *Philosophy of Logic*. Harvard University Press, Cambridge, Mass., 2 edition.

Ramsey, F. P. (1990). *Philosophical Papers*. Cambridge University Press.

Russell, B. (2010 - First published 1903). *The Principles of Mathematics*. Routledge Classics. Routledge.

Schwartz, L. (1991). *Analyse, 1. Théorie des ensemble et topologie*. Collection Enseignement des sciences 42. Hermann.

Tennant, N. (2017). *Core Logic*. Oxford University Press, 1 edition.

von Plato, J. (2000). A Problem of Normal Form in Natural Deduction. *Mathematical Logic Quarterly*, 46(1):121–124.

von Plato, J. (2013). *Elements of Logical Reasoning*. Cambridge University Press.

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