

# SET-VALUED ORTHOGONALITY AND NEARNESS

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Dedicated to Professor Antonino Maugeri

**ABSTRACT.** We introduce nearness and orthogonality between set-valued mappings, extending the Campanato nearness and the Birkoff-James orthogonality of single-valued operators.

## 1. INTRODUCTION

In articles ([4, 6]) published at the end of the eighties, S. Campanato introduced and studied the notion of *nearness* between two functions defined on a set  $S$  and taking values in a real normed vector space  $(X, \|\cdot\|)$ . More precisely, a function  $f : S \rightarrow X$  is near function  $g : S \rightarrow X$  in the sense of Campanato, iff there exist two real constants  $\alpha > 0$  and  $\kappa \in [0, 1)$  such that

$$(1) \quad \|(f(s_1) - \alpha g(s_1)) - (f(s_2) - \alpha g(s_2))\| \leq \kappa \|f(s_1) - f(s_2)\|$$

for every  $s_1, s_2 \in S$ .

When  $X$  is a Banach space, S. Campanato proved [4, Theorem 1 & 2] that if  $f$  is bijective and  $f$  is near  $g$  with constants  $\alpha$  and  $\kappa$ , then  $g$  is necessarily a bijection. Moreover, the Lipschitz modulus of the bijective function  $g \circ f^{-1} : X \rightarrow X$  is less than or equal to  $\frac{\alpha}{1-\kappa}$ , namely

$$\|g(f^{-1}(u)) - g(f^{-1}(v))\| \leq \frac{\alpha}{1-\kappa} \|u - v\| \quad \forall u, v \in X.$$

When  $S = X$ ,  $f$  is the identity operator and  $g$  is a linear function, Campanato results boils down to the well-known Neumann's Lemma [8, Lemma 5.1.6], see also [2] for more details and extensions:

**Lemma 1.1 (Neumann's lemma).** *Let  $X$  be a real Banach space, and  $A$  be a bounded linear operator on  $X$ , such that*

$$(2) \quad \exists \alpha > 0, 0 \leq \kappa < 1 : \quad \|Id - \alpha A\| \leq \kappa \|Id\|,$$

where  $\|\cdot\|$  is the operator norm defined by

$$\|A\| := \inf\{c \geq 0 : \|A(x)\| \leq c \|x\|, \forall x \in X\}.$$

Then  $A$  is invertible, and  $\|A^{-1}\| \leq \frac{\alpha}{1-\kappa}$ .

A strongly related notion to the Campanato nearness is the Birkoff-James orthogonality in a locally convex space (see, [9, Definition 1.1]). Given a vector space  $Y$  equipped with a family of semi-norms

$$\mathcal{P} := \{p_\iota : Y \rightarrow \mathbb{R} : \iota \in I\},$$

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we say that the vector  $u \in Y$  is *Birkhoff-James orthogonal* to  $v \in Y$  on  $(Y, \mathcal{P})$  iff

$$(3) \quad p_\iota(u) \leq p_\iota(u - tv) \quad \forall t \in \mathbb{R}, \iota \in I.$$

Of course, when  $Y$  is a normed space,  $\mathcal{P}$  contains only one element, namely the norm of  $Y$ , and we retrieve the original Birkhoff's definition (see [3]; the reader is also referred to the very complete survey [1]).

An abstract notion of the Campanato nearness can be defined as follows: given two vectors  $u$  and  $v$  in a locally convex space  $(Y, \mathcal{P})$ , we say that  $u$  is Campanato near  $v$  iff there exist two real constants  $\alpha > 0$  and  $\kappa \in [0, 1)$  such that

$$(4) \quad p_\iota(u - \alpha v) \leq \kappa p_\iota(u) \quad \forall \iota \in I.$$

The original Campanato nearness is obtained for the particular case of the locally convex space  $(X^S, \mathcal{P})$ , where  $X^S$  denotes the set of all functions from  $S$  to  $X$ , and the set  $\mathcal{P}$  contains all semi-norms of the form

$$(5) \quad p_{s_1, s_2} : X^S \rightarrow \mathbb{R}, \quad p_{s_1, s_2}(f) := \|f(s_1) - f(s_2)\|, \quad \forall f \in X^S,$$

for all the points  $s_1, s_2 \in S$  such that  $s_1 \neq s_2$ . As no confusion risks to occur, we will drop, in the remaining part of this note, the wording “Birkhoff-James” and “Campanato”, and simply speak of orthogonality and nearness.

The main object of this note is to extend the definitions of orthogonality and nearness from the original case of single-valued functions, to the case of set-valued mappings. As many operators of interest in non-smooth optimisation are set-valued - like the subdifferential of a convex function, to pick an example out of many - a correct definition of nearness covering the set-valued case should be a valuable tool in establishing set-valued generalizations of the Neumann and Campanato results.

This note is organized as follows. Section 2 addresses several very simple attempts to define set-valued orthogonality and nearness. Their coherence and possible uses is discussed in the light of several examples.

As a consequence of this analysis, we propose in Section 3 a new definition of orthogonality and nearness in the set-valued setting.

## 2. THREE ATTEMPTS OF DEFINING SET-VALUED ORTHOGONALITY AND NEARNESS

This section is devoted to the study of three attempts to define orthogonality and nearness for set-valued mappings. The first one is based on the notion of Pompeiu-Hausdorff distance in a metric space  $X$ , while the two others are stated in terms of the selections of the two involved set-valued mappings. The limitations of these three attempted definitions are highlighted by some elementary examples.

**2.1. Distance-based nearness : first attempt of a definition.** Let us consider the Pompeiu-Hausdorff distance, defined between two subsets of  $X$  by the following formula:

$$\mathbf{d}(A, B) := \max(\sup_{v \in B} \inf_{u \in A} \|u - v\|, \sup_{u \in A} \inf_{v \in B} \|u - v\|) \quad \forall A, B \subset X.$$

Let us recall that by a set-valued mapping (multifunction, correspondance, point-to-set, in some other terminologies)  $F$ , we mean a function between  $S$  and the set  $\mathcal{P}(X)$  of all the subsets (possibly empty) of  $X$ . Throughout the paper, we will use the notation  $F : S \rightrightarrows X$ .

A simple transposition to the set-valued setting of the original definition of orthogonality and nearness leads us to the following tentative definition.

**Definition 1.** *Given two set-valued mappings  $F, G : S \rightrightarrows X$ , we say that  $F$  is orthogonal to  $G$  iff*

$$d(F(s_1) - tG(s_1), F(s_1) - tG(s_2)) \geq d(F(s_1), F(s_2)) \quad \forall t \in \mathbb{R}, s_1, s_2 \in S,$$

and  $F$  is near  $G$  iff there are two real constants  $\alpha > 0$  and  $\kappa \in [0, 1)$  such that

$$d(F(s_1) - \alpha G(s_1), F(s_1) - \alpha G(s_2)) \leq \kappa d(F(s_1), F(s_2)), \quad \forall s_1, s_2 \in S.$$

However, within the setting of this definition, there are set-valued mappings which are not near themselves, as proved by the following result.

**Proposition 2.1.** *Let  $S$  be a set containing at least two points, and  $X$  be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping  $F : S \rightrightarrows X$  which is not near itself, according to Definition 1.*

*Proof of Proposition 2.1.* Let  $s_1$  and  $s_2$  two different elements from  $S$ , and consider the set-valued mapping  $F : S \rightrightarrows X$  given by

$$F(s) = \begin{cases} \mathcal{B}_X & \text{if } s = s_1 \\ \{0\} & \text{if } s \neq s_1 \end{cases},$$

where  $\mathcal{B}_X$  is the closed unit ball in  $X$ .

Let us pick a positive real number  $t$ . Since  $X$  contains at least a non-null element, the Pompeiu-Hausdorff distance between the sets  $t\mathcal{B}_X$  and  $\{0\}$  amounts to  $t$ . Accordingly, the Pompeiu-Hausdorff distance between the two sets

$$F(s_1) - \alpha F(s_1) = (1 + \alpha) \mathcal{B}_X, \quad F(s_2) - \alpha F(s_2) = \{0\}$$

equals to  $1 + \alpha$ , and we deduce that

$$d(F(s_1) - \alpha G(s_1), F(s_2) - \alpha G(s_2)) = 1 + \alpha > \kappa = \kappa d(F(s_1), F(s_2))$$

for any two real constants  $\alpha > 0$  and  $\kappa \in [0, 1)$ .  $\square$

Let us remark that the existence of set-valued applications which are not near themselves is still achieved even when the Pompeiu-Hausdorff distance is replaced by one of the numerous distances between sets available in the mathematical literature.

It is easy to verify that, if  $\delta : 2^X \times 2^X \rightarrow \mathbb{R}_+$  is any of the distances described in the survey paper ([7]), then the function

$$e : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad e(s) := \delta(\{0\}, s\mathcal{B}_X) \quad \forall s \geq 0$$

is increasing. Accordingly, Proposition 2.1 holds true even if the Pompeiu-Hausdorff distance is replaced in Definition 1 by another distance between sets. It appears thus that no valid definition of the set-valued orthogonality and nearness can be achieved by using distance functions.

**2.2. Selections-based nearness: second attempt of a definition.** Given  $F : S \rightrightarrows X$ , we call *selection* of  $F$  any function  $\sigma_F : S \rightarrow X$  such that  $\sigma_F(s) \in F(s)$  for any  $s \in S$ .

Our second very simple attempt to define a correct set-valued orthogonality and nearness is based on the analysis of all the selections of the two set-valued mappings involved.

**Definition 2.** *Let  $F, G : S \rightrightarrows X$ . Then  $F$  is orthogonal to (respectively near)  $G$  iff any selection of  $F$  is orthogonal to (respectively near) any selection of  $G$ .*

Once again, this definition leads to the existence of set-valued mappings which are not near themselves.

**Proposition 2.2.** *Let  $S$  be a set containing at least two points, and  $X$  be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping  $F : S \rightrightarrows X$  which is not near itself, according to Definition 2.*

*Proof of Proposition 2.2.* Let us prove that the constant set-valued mapping  $F : S \rightrightarrows X$ ,

$$F(s) := \mathcal{B}_X \quad \forall s \in S,$$

is not near itself according to Definition 2.

As the vector space  $X$  contains non-null vectors, it follows that there is  $u$ , a vector of norm equal to 1. Let  $s_1$  and  $s_2$  two different elements of  $S$ . The functions

$$\sigma_1 : S \rightarrow X \quad \sigma_1(s) := \begin{cases} u, & \text{if } s = s_1 \\ -u, & \text{if } s \neq s_1 \end{cases},$$

and

$$\sigma_2 : S \rightarrow X \quad \sigma_2(s) := \begin{cases} u, & \text{if } s \neq s_1 \\ -u, & \text{if } s = s_1 \end{cases},$$

are thus two selections of the set-valued mapping  $F$ . But

$$\|(\sigma_1(s_1) - \alpha \sigma_2(s_1)) - (\sigma_1(s_2) - \alpha \sigma_2(s_2))\| = 2(1 + \alpha) > 2\kappa = \kappa \|\sigma_1(s_1) - \sigma_1(s_2)\|$$

for any two real constants  $\alpha > 0$  and  $\kappa \in [0, 1)$ .  $\square$

**2.3. Selections-based nearness: third attempt of a definition.** The first two attempts to define set-valued nearness were too restrictive. In both cases, set-valued mappings not being near themselves have been identified. Accordingly, in this subsection we address a selection-based definition which is considerably broader than Definitions 1 and 2.

**Definition 3.** *A set-valued mapping  $F$  is orthogonal to (respectively near)  $G$  iff any selection of  $F$  is orthogonal (respectively near) to at least one selection of  $G$ , and for any selection of  $G$ , there is at least one selection of  $F$  which is orthogonal to (respectively near) it.*

It is now obvious that any set-valued mapping is near itself, in the sense of Definition 3, so the main difficulty plaguing Definitions 1 and 2 is now removed. However, Definition 3 is not satisfactory since, unlike the single-valued case, basic algebraic properties are not inherited from  $F$  to  $G$  in the case when  $F$  is near  $G$ , as proved by the following result.

**Proposition 2.3.** *Let  $S := \{s_1, s_2\}$  be a set containing two points, and  $X$  be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping  $F : S \rightrightarrows X$  which is injective, in the sense that  $F(s_1) \neq F(s_2)$ , and which is near a constant set-valued mapping  $G$ , according to Definition 3.*

*Proof of Proposition 2.3.* Let  $u$  be a non-null vector from  $X$ , and define

$$F : S \rightrightarrows X, \quad F(s_1) := \{-u, u\}, \quad F(s_2) := \{0, u\}$$

and

$$G : S \rightrightarrows X, \quad G(s_1) = G(s_2) := \{0, u\}.$$

Each of the two set-valued mappings  $F$  and  $G$  has exactly four selections:

$$\sigma_{F,1}(s_1) := -u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,2}(s_1) := -u, \sigma_{F,2}(s_2) := u$$

$$\sigma_{F,3}(s_1) := u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,4}(s_1) := u, \sigma_{F,2}(s_2) := u$$

are the selections of  $F$ , while the selections of  $G$  are

$$\sigma_{G,1}(s_1) := 0, \sigma_{G,1}(s_2) := 0, \quad \sigma_{G,2}(s_1) := 0, \sigma_{F,2}(s_2) := u$$

$$\sigma_{G,3}(s_1) := u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{G,4}(s_1) := u, \sigma_{F,2}(s_2) := u.$$

It is straight-forward to prove that  $\sigma_{F,1}$  is near  $\sigma_{G,2}$ , that  $\sigma_{F,2}$  is near  $\sigma_{G,2}$ , that  $\sigma_{F,3}$  is near  $\sigma_{G,3}$ , that  $\sigma_{F,4}$  is near  $\sigma_{G,1}$ , and that  $\sigma_{F,4}$  is near  $\sigma_{G,4}$ . So the set-valued mappings  $F$  and  $G$  satisfy conditions of Definition 3. Yet,  $F$  is injective, while  $G$  is constant.  $\square$

### 3. NEARNESS FOR SET-VALUED FUNCTIONS

In view of the detailed analysis achieved in Section 2, we conclude the correct definition of set-valued orthogonality and nearness should be broader than Definition 2, but more restrictive than Definition 3. In order to attain this objective, let us introduce the following notations. Given  $F, G : X \rightrightarrows X$ , we say that a binary relation  $\mathcal{R}$  on  $X$  is  $(F, G)$ -compatible if

$$\forall s \in S, \forall u \in F(s), \exists v \in G(s) \quad \text{s.t.} \quad u \mathcal{R} v,$$

and

$$\forall s \in S, \forall v \in G(s), \exists u \in F(s) \quad \text{s.t.} \quad u \mathcal{R} v.$$

Moreover, a selection  $\sigma_G$  of the set-valued mapping  $G$  is said  $\mathcal{R}$ -compatible with one selection  $\sigma_F$  of  $F$ , if  $\sigma_F(s) \mathcal{R} \sigma_G(s)$  for any  $s \in S$ .

We are now ready to state the main notion of this note.

**Definition 4.** *A set-valued mapping  $F$  is said to be orthogonal (respectively near) to  $G$  iff there exists a  $(F, G)$ -compatible binary relation  $\mathcal{R}$  on  $X$  such that any selection  $\sigma_F$  of  $F$  is orthogonal (respectively near) to any  $\mathcal{R}$ -compatible selection  $\sigma_G$  of  $G$ .*

It is now easy to see that any set-valued function  $F$  is near itself (just take for  $\mathcal{R}$  the identity relation,  $x \mathcal{R} y \Leftrightarrow x = y$ , remark that  $\mathcal{R}$  is  $(F, F)$ -compatible, and notice that any selection  $\sigma_F$  of  $F$  is  $\mathcal{R}$ -compatible only with itself).

In a future article, we will investigate the properties of the set-valued mapping  $F$  which are inherited by any set-valued mapping  $G$  such that  $F$  is near  $G$ .

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