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SET-VALUED ORTHOGONALITY AND NEARNESS

ANNAMARIA BARBAGALLO, OCTAVIAN-EMIL ERNST, AND MICHEL THÉRA

Dedicated to Professor Antonino Maugeri

ABSTRACT. We introduce nearness and orthogonality between set-valued mappings, extending the Campanato nearness and the Birkoff-James orthogonality of single-valued operators.

1. INTRODUCTION

In articles ([4, 6]) published at the end of the eighties, S. Campanato introduced and studied the notion of *nearness* between two functions defined on a set S and taking values in a real normed vector space $(X, \|\cdot\|)$. More precisely, a function $f : S \rightarrow X$ is near function $g : S \rightarrow X$ in the sense of Campanato, iff there exist two real constants $\alpha > 0$ and $\kappa \in [0, 1)$ such that

$$(1) \quad \|(f(s_1) - \alpha g(s_1)) - (f(s_2) - \alpha g(s_2))\| \leq \kappa \|f(s_1) - f(s_2)\|$$

for every $s_1, s_2 \in S$.

When X is a Banach space, S. Campanato proved [4, Theorem 1 & 2] that if f is bijective and f is near g with constants α and κ , then g is necessarily a bijection. Moreover, the Lipschitz modulus of the bijective function $g \circ f^{-1} : X \rightarrow X$ is less than or equal to $\frac{\alpha}{1-\kappa}$, namely

$$\|g(f^{-1}(u)) - g(f^{-1}(v))\| \leq \frac{\alpha}{1-\kappa} \|u - v\| \quad \forall u, v \in X.$$

When $S = X$, f is the identity operator and g is a linear function, Campanato results boils down to the well-known Neumann's Lemma [8, Lemma 5.1.6], see also [2] for more details and extensions:

Lemma 1.1 (Neumann's lemma). *Let X be a real Banach space, and A be a bounded linear operator on X , such that*

$$(2) \quad \exists \alpha > 0, 0 \leq \kappa < 1 : \quad \|Id - \alpha A\| \leq \kappa \|Id\|,$$

where $\|\cdot\|$ is the operator norm defined by

$$\|A\| := \inf\{c \geq 0 : \|A(x)\| \leq c \|x\|, \forall x \in X\}.$$

Then A is invertible, and $\|A^{-1}\| \leq \frac{\alpha}{1-\kappa}$.

A strongly related notion to the Campanato nearness is the Birkoff-James orthogonality in a locally convex space (see, [9, Definition 1.1]). Given a vector space Y equipped with a family of semi-norms

$$\mathcal{P} := \{p_\iota : Y \rightarrow \mathbb{R} : \iota \in I\},$$

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we say that the vector $u \in Y$ is *Birkhoff-James orthogonal* to $v \in Y$ on (Y, \mathcal{P}) iff

$$(3) \quad p_\iota(u) \leq p_\iota(u - tv) \quad \forall t \in \mathbb{R}, \iota \in I.$$

Of course, when Y is a normed space, \mathcal{P} contains only one element, namely the norm of Y , and we retrieve the original Birkhoff's definition (see [3]; the reader is also referred to the very complete survey [1]).

An abstract notion of the Campanato nearness can be defined as follows: given two vectors u and v in a locally convex space (Y, \mathcal{P}) , we say that u is Campanato near v iff there exist two real constants $\alpha > 0$ and $\kappa \in [0, 1)$ such that

$$(4) \quad p_\iota(u - \alpha v) \leq \kappa p_\iota(u) \quad \forall \iota \in I.$$

The original Campanato nearness is obtained for the particular case of the locally convex space (X^S, \mathcal{P}) , where X^S denotes the set of all functions from S to X , and the set \mathcal{P} contains all semi-norms of the form

$$(5) \quad p_{s_1, s_2} : X^S \rightarrow \mathbb{R}, \quad p_{s_1, s_2}(f) := \|f(s_1) - f(s_2)\|, \quad \forall f \in X^S,$$

for all the points $s_1, s_2 \in S$ such that $s_1 \neq s_2$. As no confusion risks to occur, we will drop, in the remaining part of this note, the wording “Birkhoff-James” and “Campanato”, and simply speak of orthogonality and nearness.

The main object of this note is to extend the definitions of orthogonality and nearness from the original case of single-valued functions, to the case of set-valued mappings. As many operators of interest in non-smooth optimisation are set-valued - like the subdifferential of a convex function, to pick an example out of many - a correct definition of nearness covering the set-valued case should be a valuable tool in establishing set-valued generalizations of the Neumann and Campanato results.

This note is organized as follows. Section 2 addresses several very simple attempts to define set-valued orthogonality and nearness. Their coherence and possible uses is discussed in the light of several examples.

As a consequence of this analysis, we propose in Section 3 a new definition of orthogonality and nearness in the set-valued setting.

2. THREE ATTEMPTS OF DEFINING SET-VALUED ORTHOGONALITY AND NEARNESS

This section is devoted to the study of three attempts to define orthogonality and nearness for set-valued mappings. The first one is based on the notion of Pompeiu-Hausdorff distance in a metric space X , while the two others are stated in terms of the selections of the two involved set-valued mappings. The limitations of these three attempted definitions are highlighted by some elementary examples.

2.1. Distance-based nearness : first attempt of a definition. Let us consider the Pompeiu-Hausdorff distance, defined between two subsets of X by the following formula:

$$\mathbf{d}(A, B) := \max(\sup_{v \in B} \inf_{u \in A} \|u - v\|, \sup_{u \in A} \inf_{v \in B} \|u - v\|) \quad \forall A, B \subset X.$$

Let us recall that by a set-valued mapping (multifunction, correspondance, point-to-set, in some other terminologies) F , we mean a function between S and the set $\mathcal{P}(X)$ of all the subsets (possibly empty) of X . Throughout the paper, we will use the notation $F : S \rightrightarrows X$.

A simple transposition to the set-valued setting of the original definition of orthogonality and nearness leads us to the following tentative definition.

Definition 1. *Given two set-valued mappings $F, G : S \rightrightarrows X$, we say that F is orthogonal to G iff*

$$d(F(s_1) - tG(s_1), F(s_1) - tG(s_2)) \geq d(F(s_1), F(s_2)) \quad \forall t \in \mathbb{R}, s_1, s_2 \in S,$$

and F is near G iff there are two real constants $\alpha > 0$ and $\kappa \in [0, 1)$ such that

$$d(F(s_1) - \alpha G(s_1), F(s_1) - \alpha G(s_2)) \leq \kappa d(F(s_1), F(s_2)), \quad \forall s_1, s_2 \in S.$$

However, within the setting of this definition, there are set-valued mappings which are not near themselves, as proved by the following result.

Proposition 2.1. *Let S be a set containing at least two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is not near itself, according to Definition 1.*

Proof of Proposition 2.1. Let s_1 and s_2 two different elements from S , and consider the set-valued mapping $F : S \rightrightarrows X$ given by

$$F(s) = \begin{cases} \mathcal{B}_X & \text{if } s = s_1 \\ \{0\} & \text{if } s \neq s_1 \end{cases},$$

where \mathcal{B}_X is the closed unit ball in X .

Let us pick a positive real number t . Since X contains at least a non-null element, the Pompeiu-Hausdorff distance between the sets $t\mathcal{B}_X$ and $\{0\}$ amounts to t . Accordingly, the Pompeiu-Hausdorff distance between the two sets

$$F(s_1) - \alpha F(s_1) = (1 + \alpha) \mathcal{B}_X, \quad F(s_2) - \alpha F(s_2) = \{0\}$$

equals to $1 + \alpha$, and we deduce that

$$d(F(s_1) - \alpha G(s_1), F(s_2) - \alpha G(s_2)) = 1 + \alpha > \kappa = \kappa d(F(s_1), F(s_2))$$

for any two real constants $\alpha > 0$ and $\kappa \in [0, 1)$. \square

Let us remark that the existence of set-valued applications which are not near themselves is still achieved even when the Pompeiu-Hausdorff distance is replaced by one of the numerous distances between sets available in the mathematical literature.

It is easy to verify that, if $\delta : 2^X \times 2^X \rightarrow \mathbb{R}_+$ is any of the distances described in the survey paper ([7]), then the function

$$e : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad e(s) := \delta(\{0\}, s\mathcal{B}_X) \quad \forall s \geq 0$$

is increasing. Accordingly, Proposition 2.1 holds true even if the Pompeiu-Hausdorff distance is replaced in Definition 1 by another distance between sets. It appears thus that no valid definition of the set-valued orthogonality and nearness can be achieved by using distance functions.

2.2. Selections-based nearness: second attempt of a definition. Given $F : S \rightrightarrows X$, we call *selection* of F any function $\sigma_F : S \rightarrow X$ such that $\sigma_F(s) \in F(s)$ for any $s \in S$.

Our second very simple attempt to define a correct set-valued orthogonality and nearness is based on the analysis of all the selections of the two set-valued mappings involved.

Definition 2. *Let $F, G : S \rightrightarrows X$. Then F is orthogonal to (respectively near) G iff any selection of F is orthogonal to (respectively near) any selection of G .*

Once again, this definition leads to the existence of set-valued mappings which are not near themselves.

Proposition 2.2. *Let S be a set containing at least two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is not near itself, according to Definition 2.*

Proof of Proposition 2.2. Let us prove that the constant set-valued mapping $F : S \rightrightarrows X$,

$$F(s) := \mathcal{B}_X \quad \forall s \in S,$$

is not near itself according to Definition 2.

As the vector space X contains non-null vectors, it follows that there is u , a vector of norm equal to 1. Let s_1 and s_2 two different elements of S . The functions

$$\sigma_1 : S \rightarrow X \quad \sigma_1(s) := \begin{cases} u, & \text{if } s = s_1 \\ -u, & \text{if } s \neq s_1 \end{cases},$$

and

$$\sigma_2 : S \rightarrow X \quad \sigma_2(s) := \begin{cases} u, & \text{if } s \neq s_1 \\ -u, & \text{if } s = s_1 \end{cases},$$

are thus two selections of the set-valued mapping F . But

$$\|(\sigma_1(s_1) - \alpha \sigma_2(s_1)) - (\sigma_1(s_2) - \alpha \sigma_2(s_2))\| = 2(1 + \alpha) > 2\kappa = \kappa \|\sigma_1(s_1) - \sigma_1(s_2)\|$$

for any two real constants $\alpha > 0$ and $\kappa \in [0, 1)$. \square

2.3. Selections-based nearness: third attempt of a definition. The first two attempts to define set-valued nearness were too restrictive. In both cases, set-valued mappings not being near themselves have been identified. Accordingly, in this subsection we address a selection-based definition which is considerably broader than Definitions 1 and 2.

Definition 3. *A set-valued mapping F is orthogonal to (respectively near) G iff any selection of F is orthogonal (respectively near) to at least one selection of G , and for any selection of G , there is at least one selection of F which is orthogonal to (respectively near) it.*

It is now obvious that any set-valued mapping is near itself, in the sense of Definition 3, so the main difficulty plaguing Definitions 1 and 2 is now removed. However, Definition 3 is not satisfactory since, unlike the single-valued case, basic algebraic properties are not inherited from F to G in the case when F is near G , as proved by the following result.

Proposition 2.3. *Let $S := \{s_1, s_2\}$ be a set containing two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is injective, in the sense that $F(s_1) \neq F(s_2)$, and which is near a constant set-valued mapping G , according to Definition 3.*

Proof of Proposition 2.3. Let u be a non-null vector from X , and define

$$F : S \rightrightarrows X, \quad F(s_1) := \{-u, u\}, \quad F(s_2) := \{0, u\}$$

and

$$G : S \rightrightarrows X, \quad G(s_1) = G(s_2) := \{0, u\}.$$

Each of the two set-valued mappings F and G has exactly four selections:

$$\sigma_{F,1}(s_1) := -u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,2}(s_1) := -u, \sigma_{F,2}(s_2) := u$$

$$\sigma_{F,3}(s_1) := u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,4}(s_1) := u, \sigma_{F,2}(s_2) := u$$

are the selections of F , while the selections of G are

$$\sigma_{G,1}(s_1) := 0, \sigma_{G,1}(s_2) := 0, \quad \sigma_{G,2}(s_1) := 0, \sigma_{F,2}(s_2) := u$$

$$\sigma_{G,3}(s_1) := u, \sigma_{F,1}(s_2) := 0, \quad \sigma_{G,4}(s_1) := u, \sigma_{F,2}(s_2) := u.$$

It is straight-forward to prove that $\sigma_{F,1}$ is near $\sigma_{G,2}$, that $\sigma_{F,2}$ is near $\sigma_{G,2}$, that $\sigma_{F,3}$ is near $\sigma_{G,3}$, that $\sigma_{F,4}$ is near $\sigma_{G,1}$, and that $\sigma_{F,4}$ is near $\sigma_{G,4}$. So the set-valued mappings F and G satisfy conditions of Definition 3. Yet, F is injective, while G is constant. \square

3. NEARNESS FOR SET-VALUED FUNCTIONS

In view of the detailed analysis achieved in Section 2, we conclude the correct definition of set-valued orthogonality and nearness should be broader than Definition 2, but more restrictive than Definition 3. In order to attain this objective, let us introduce the following notations. Given $F, G : X \rightrightarrows X$, we say that a binary relation \mathcal{R} on X is (F, G) -compatible if

$$\forall s \in S, \forall u \in F(s), \exists v \in G(s) \quad \text{s.t.} \quad u \mathcal{R} v,$$

and

$$\forall s \in S, \forall v \in G(s), \exists u \in F(s) \quad \text{s.t.} \quad u \mathcal{R} v.$$

Moreover, a selection σ_G of the set-valued mapping G is said \mathcal{R} -compatible with one selection σ_F of F , if $\sigma_F(s) \mathcal{R} \sigma_G(s)$ for any $s \in S$.

We are now ready to state the main notion of this note.

Definition 4. *A set-valued mapping F is said to be orthogonal (respectively near) to G iff there exists a (F, G) -compatible binary relation \mathcal{R} on X such that any selection σ_F of F is orthogonal (respectively near) to any \mathcal{R} -compatible selection σ_G of G .*

It is now easy to see that any set-valued function F is near itself (just take for \mathcal{R} the identity relation, $x \mathcal{R} y \Leftrightarrow x = y$, remark that \mathcal{R} is (F, F) -compatible, and notice that any selection σ_F of F is \mathcal{R} -compatible only with itself).

In a future article, we will investigate the properties of the set-valued mapping F which are inherited by any set-valued mapping G such that F is near G .

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