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SET-VALUED ORTHOGONALITY AND NEARNESS

ANNAMARIA BARBAGALLO, OCTAVIAN-EMIL ERNST, AND MICHEL THÉRA

Dedicated to Professor Antonino Maugeri

Abstract. We introduce nearness and orthogonality between set-valued mappings, extending the Campanato nearness and the Birkoff-James orthogonality of single-valued operators.

1. Introduction

In articles ([4, 6]) published at the end of the eighties, S. Campanato introduced and studied the notion of nearness between two functions defined on a set $S$ and taking values in a real normed vector space $(X, \|\cdot\|)$. More precisely, a function $f : S \to X$ is near function $g : S \to X$ in the sense of Campanato, iff there exist two real constants $\alpha > 0$ and $\kappa \in [0, 1)$ such that

$$\| (f(s_1) - \alpha g(s_1)) - (f(s_2) - \alpha g(s_2)) \| \leq \kappa \| f(s_1) - f(s_2) \|$$

for every $s_1, s_2 \in S$.

When $X$ is a Banach space, S. Campanato proved [4, Theorem 1 & 2] that if $f$ is bijective and $f$ is near $g$ with constants $\alpha$ and $\kappa$, then $g$ is necessarily a bijection. Moreover, the Lipschitz modulus of the bijective function $g \circ f^{-1} : X \to X$ is less than or equal to $\frac{\alpha}{1 - \kappa}$, namely

$$\| g(f^{-1}(u)) - g(f^{-1}(v)) \| \leq \frac{\alpha}{1 - \kappa} \| u - v \| \quad \forall u, v \in X.$$

When $S = X$, $f$ is the identity operator and $g$ is a linear function, Campanato results boils down to the well-known Neumann’s Lemma [8, Lemma 5.1.6], see also [2] for more details and extensions:

**Lemma 1.1 (Neumann’s lemma).** Let $X$ be a real Banach space, and $A$ be a bounded linear operator on $X$, such that

$$\exists \, \alpha > 0, \, 0 \leq \kappa < 1 : \| Id - \alpha A \| \leq \kappa \| Id \|,$$

where $\| \cdot \|$ is the operator norm defined by

$$\| A \| := \inf \{ c \geq 0 : \| A(x) \| \leq c \| x \|, \forall x \in X \}.$$

Then $A$ is invertible, and $\| A^{-1} \| \leq \frac{\alpha}{1 - \kappa}$.

A strongly related notion to the Campanato nearness is the Birkoff-James orthogonality in a locally convex space (see, [9, Definition 1.1]). Given a vector space $Y$ equipped with a family of semi-norms

$$\mathcal{P} := \{ p_i : Y \to \mathbb{R} : i \in I \},$$

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we say that the vector \( u \in Y \) is \textit{Birkhoff-James orthogonal} to \( v \in Y \) on \((Y, P)\) iff
\[
p_\iota(u) \leq p_\iota(u - t v) \quad \forall t \in \mathbb{R}, \; \iota \in I.
\]

Of course, when \( Y \) is a normed space, \( P \) contains only one element, namely the norm of \( Y \), and we retrieve the original Birkhoff’s definition (see [3]; the reader is also referred to the very complete survey [1]).

An abstract notion of the Campanato nearness can be defined as follows: given two vectors \( u \) and \( v \) in a locally convex space \((Y, P)\), we say that \( u \) is Campanato near \( v \) iff there exist two real constants \( \alpha > 0 \) and \( \kappa \in [0, 1) \) such that
\[
p_\iota(u - \alpha v) \leq \kappa p_\iota(u) \quad \forall \iota \in I.
\]

The original Campanato nearness is obtained for the particular case of the locally convex space \((X^S, P)\), where \( X^S \) denotes the set of all functions from \( S \) to \( X \), and the set \( P \) contains all semi-norms of the form
\[
p_{s_1, s_2} : X^S \to \mathbb{R}, \quad p_{s_1, s_2}(f) := \|f(s_1) - f(s_2)\|, \quad \forall f \in X^S,
\]
for all the points \( s_1, s_2 \in S \) such that \( s_1 \neq s_2 \). As no confusion risks to occur, we will drop, in the remaining part of this note, the wording “Birkhoff-James” and “Campanato”, and simply speak of orthogonality and nearness.

The main object of this note is to extend the definitions of orthogonality and nearness from the original case of single-valued functions, to the case of set-valued mappings. As many operators of interest in non-smooth optimisation are set-valued - like the subdifferential of a convex function, to pick an example out of many - a correct definition of nearness covering the set-valued case should be a valuable tool in establishing set-valued generalizations of the Neumann and Campanato results.

This note is organized as follows. Section 2 addresses several very simple attempts to define set-valued orthogonality and nearness. Their coherence and possible uses is discussed in the light of several examples.

As a consequence of this analysis, we propose in Section 3 a new definition of orthogonality and nearness in the set-valued setting.

2. THREE ATTEMPTS OF DEFINING SET-VALUED ORTHOGONALITY AND NEARNESS

This section is devoted to the study of three attempts to define orthogonality and nearness for set-valued mappings. The first one is based on the notion of Pompeiu-Hausdorff distance in a metric space \( X \), while the two others are stated in terms of the selections of the two involved set-valued mappings. The limitations of these three attempted definitions are highlighted by some elementary examples.

2.1. Distance-based nearness : first attempt of a definition. Let us consider the Pompeiu-Hausdorff distance in a metric space \( X \), defined between two subsets of \( X \) by the following formula:
\[
d(A, B) := \max(\sup_{v \in B} \inf_{u \in A} \|u - v\|, \sup_{u \in A} \inf_{v \in B} \|u - v\|) \quad \forall A, B \subset X.
\]

Let us recall that by a set-valued mapping (multifunction, correspondance, point-to-set, in some other terminologies) \( F \), we mean a function between \( S \) and the set \( P(X) \) of all the subsets (possibly empty) of \( X \). Throughout the paper, we will use the notation \( F : S \rightrightarrows X \).
A simple transposition to the set-valued setting of the original definition of orthogonality and nearness leads us to the following tentative definition.

**Definition 1.** Given two set-valued mappings \( F, G : S \rightarrow X \), we say that \( F \) is orthogonal to \( G \) iff
\[
d(F(s_1) - tG(s_1), F(s_1) - tG(s_2)) \geq d(F(s_1), F(s_2)) \quad \forall t \in \mathbb{R}, s_1, s_2 \in S,
\]
and \( F \) is near \( G \) iff there are two real constants \( \alpha > 0 \) and \( \kappa \in [0, 1) \) such that
\[
d(F(s_1) - \alpha G(s_1), F(s_1) - \alpha G(s_2)) \leq \kappa d(F(s_1), F(s_2)), \quad \forall s_1, s_2 \in S.
\]

However, within the setting of this definition, there are set-valued mappings which are not near themselves, as proved by the following result.

**Proposition 2.1.** Let \( S \) be a set containing at least two points, and \( X \) be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping \( F : S \rightarrow X \) which is not near itself, according to Definition 1.

**Proof of Proposition 2.1.** Let \( s_1 \) and \( s_2 \) two different elements from \( S \), and consider the set-valued mapping \( F : S \rightarrow X \) given by
\[
F(s) = \begin{cases} 
B_X & \text{if } s = s_1 \\
\{0\} & \text{if } s \neq s_1
\end{cases},
\]
where \( B_X \) is the closed unit ball in \( X \).

Let us pick a positive real number \( t \). Since \( X \) contains at least a non-null element, the Pompeiu-Hausdorff distance between the sets \( tB_X \) and \( \{0\} \) amounts to \( t \). Accordingly, the Pompeiu-Hausdorff distance between the two sets
\[
F(s_1) - \alpha F(s_1) = (1 + \alpha) B_X, \quad F(s_2) - \alpha F(s_2) = \{0\}
\]
equals to \( 1 + \alpha \), and we deduce that
\[
d(F(s_1) - \alpha G(s_1), F(s_2) - \alpha G(s_2)) = 1 + \alpha \geq \kappa d(F(s_1), F(s_2))
\]
for any two real constants \( \alpha > 0 \) and \( \kappa \in [0, 1) \).

Let us remark that the existence of set-valued applications which are not near themselves is still achieved even when the Pompeiu-Hausdorff distance is replaced by one of the numerous distances between sets available in the mathematical literature.

It is easy to verify that, if \( \delta : 2^X \times 2^X \rightarrow \mathbb{R}_+ \) is any of the distances described in the survey paper (\[7\]), then the function
\[
e : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad e(s) := \delta(\{0\}, sB_X) \quad \forall s \geq 0
\]
in increasing. Accordingly, Proposition 2.1 holds true even if the Pompeiu-Hausdorff distance is replaced by another distance between sets. It appears thus that no valid definition of the set-valued orthogonality and nearness can be achieved by using distance functions.
2.2. Selections-based nearness: second attempt of a definition. Given $F : S \rightrightarrows X$, we call selection of $F$ any function $\sigma_F : S \to X$ such that $\sigma_F(s) \in F(s)$ for any $s \in S$.

Our second very simple attempt to define a correct set-valued orthogonality and nearness is based on the analysis of all the selections of the two set-valued mappings involved.

**Definition 2.** Let $F, G : S \rightrightarrows X$. Then $F$ is orthogonal to (respectively near) $G$ iff any selection of $F$ is orthogonal to (respectively near) any selection of $G$.

Once again, this definition leads to the existence of set-valued mappings which are not near themselves.

**Proposition 2.2.** Let $S$ be a set containing at least two points, and $X$ be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is not near itself, according to Definition 2.

**Proof of Proposition 2.2.** Let us prove that the constant set-valued mapping $F : S \rightrightarrows X$, 

$$F(s) := B_X \quad \forall s \in S,$$

is not near itself according to Definition 2.

As the vector space $X$ contains non-null vectors, it follows that there is $u$, a vector of norm equal to 1. Let $s_1$ and $s_2$ two different elements of $S$. The functions

$$\sigma_1 : S \to X \quad \sigma_1(s) := \begin{cases} u, & \text{if } s = s_1 \\ -u, & \text{if } s \neq s_1 \end{cases},$$

and

$$\sigma_2 : S \to X \quad \sigma_2(s) := \begin{cases} u, & \text{if } s \neq s_1 \\ -u, & \text{if } s = s_1 \end{cases},$$

are thus two selections of the set-valued mapping $F$. But

$$\| (\sigma_1(s_1) - \alpha \sigma_2(s_1)) - (\sigma_2(s_2) - \alpha \sigma_2(s_2)) \| = 2(1 + \alpha) > 2\kappa = \kappa \| \sigma_1(s_1) - \sigma_1(s_2) \|$$

for any two real constants $\alpha > 0$ and $\kappa \in [0, 1)$. \hfill $\Box$

2.3. Selections-based nearness: third attempt of a definition. The first two attempts to define set-valued nearness were too restrictive. In both cases, set-valued mappings not being near themselves have been identified. Accordingly, in this subsection we address a selection-based definition which is considerably broader than Definitions 1 and 2.

**Definition 3.** A set-valued mapping $F$ is orthogonal to (respectively near) $G$ iff any selection of $F$ is orthogonal (respectively near) to at least one selection of $G$, and for any selection of $G$, there is at least one selection of $F$ which is orthogonal to (respectively near) it.

It is now obvious that any set-valued mapping is near itself, in the sense of Definition 3, so the main difficulty plaguing Definitions 1 and 2 is now removed. However, Definition 3 is not satisfactory since, unlike the single-valued case, basic algebraic properties are not inherited from $F$ to $G$ in the case when $F$ is near $G$, as proved by the following result.
Proposition 2.3. Let $S := \{s_1, s_2\}$ be a set containing two points, and $X$ be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \ni X$ which is injective, in the sense that $F(s_1) \neq F(s_2)$, and which is near a constant set-valued mapping $G$, according to Definition 3.

Proof of Proposition 2.3. Let $u$ be a non-null vector from $X$, and define
$$F : S \ni X, \quad F(s_1) := \{-u, u\}, \quad F(s_2) := \{0, u\}$$
and
$$G : S \ni X, \quad G(s_1) = G(s_2) := \{0, u\}.$$ Each of the two set-valued mappings $F$ and $G$ has exactly four selections:
$$\sigma_{F,1}(s_1) := -u, \quad \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,2}(s_1) := -u, \quad \sigma_{F,2}(s_2) := u$$
$$\sigma_{F,3}(s_1) := u, \quad \sigma_{F,1}(s_2) := 0, \quad \sigma_{F,4}(s_1) := u, \quad \sigma_{F,2}(s_2) := u$$
are the selections of $F$, while the selections of $G$ are
$$\sigma_{G,1}(s_1) := 0, \quad \sigma_{G,1}(s_2) := 0, \quad \sigma_{G,2}(s_1) := 0, \quad \sigma_{G,2}(s_2) := u$$
$$\sigma_{G,3}(s_1) := u, \quad \sigma_{G,1}(s_2) := 0, \quad \sigma_{G,4}(s_1) := u, \quad \sigma_{G,2}(s_2) := u.$$

It is straight-forward to prove that $\sigma_{F,1}$ is near $\sigma_{G,1}$, that $\sigma_{F,2}$ is near $\sigma_{G,2}$, that $\sigma_{F,3}$ is near $\sigma_{G,3}$, that $\sigma_{F,4}$ is near $\sigma_{G,4}$, and that $\sigma_{F,4}$ is near $\sigma_{G,4}$. So the set-valued mappings $F$ and $G$ satisfy conditions of Definition 3. Yet, $F$ is injective, while $G$ is constant.

3. Nearness for set-valued functions

In view of the detailed analysis achieved in Section 2, we conclude the correct definition of set-valued orthogonality and nearness should be broader than Definition 2, but more restrictive than Definition 3. In order to attain this objective, let us introduce the following notations. Given $F, G : X \ni X$, we say that a binary relation $\mathcal{R}$ on $X$ is $(F,G)$-compatible if
$$\forall s \in S, \forall u \in F(s), \exists v \in G(s) \quad \text{s.t.} \quad u \mathcal{R} v,$$
and
$$\forall s \in S, \forall v \in G(s), \exists u \in F(s) \quad \text{s.t.} \quad u \mathcal{R} v.$$ Moreover, a selection $\sigma_G$ of the set-valued mapping $G$ is said $\mathcal{R}$-compatible with one selection $\sigma_F$ de $F$, if $\sigma_F(s) \mathcal{R} \sigma_G(s)$ for any $s \in S$.

We are now ready to state the main notion of this note.

Definition 4. A set-valued mapping $F$ is said to be orthogonal (respectively near) to $G$ iff there exists a $(F,G)$-compatible binary relation $\mathcal{R}$ on $X$ such that any selection $\sigma_F$ of $F$ is orthogonal (respectively near) to any $\mathcal{R}$-compatible selection $\sigma_G$ of $G$.

It is now easy to see that any set-valued function $F$ is near itself (just take for $\mathcal{R}$ the identity relation, $x \mathcal{R} y \Leftrightarrow x = y$, remark that $\mathcal{R}$ is $(F,F)$-compatible, and notice that any selection $\sigma_F$ of $F$ is $\mathcal{R}$-compatible only with itself).

In a future article, we will investigate the properties of the set-valued mapping $F$ which are inherited by any set-valued mapping $G$ such that $F$ is near $G$. 
References


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