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# Extending Drawings of Graphs to Arrangements of Pseudolines

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## 1 Abstract

In the recent study of crossing numbers, drawings of graphs that can be extended to an arrangement of pseudolines (pseudolinear drawings) have played an important role as they are a natural combinatorial extension of rectilinear (or straight-line) drawings. A characterization of the pseudolinear drawings of  $K_n$  was found recently. We extend this characterization to all graphs, by describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the pseudolines when it is possible.

## 2012 ACM Subject Classification

**Keywords and phrases** graphs, graph drawings, geometric graph drawings, arrangements of pseudolines, crossing numbers, stretchability.

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**Lines** 500

## 9 1 Introduction

Since 2004, geometric methods have been used to make impressive progress for determining the crossing number of (certain classes of drawings of) the complete graph  $K_n$ . In particular, drawings that extend to straight lines, or, more generally, arrangements of pseudolines, have been central to this work, spurring interest in such drawings for arbitrary graphs, not just complete graphs [2, 5, 6, 7, 12].

In particular, for pseudolinear drawings, it is now known that, for  $n \geq 10$ , a pseudolinear drawing of  $K_n$  has more than

$$H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings [1, 13]. The number  $H(n)$  is conjectured by Harary and Hill to be the smallest number of crossings over all topological drawings of  $K_n$ ; that is, the crossing number  $\text{cr}(K_n)$  is conjectured to be  $H(n)$ .

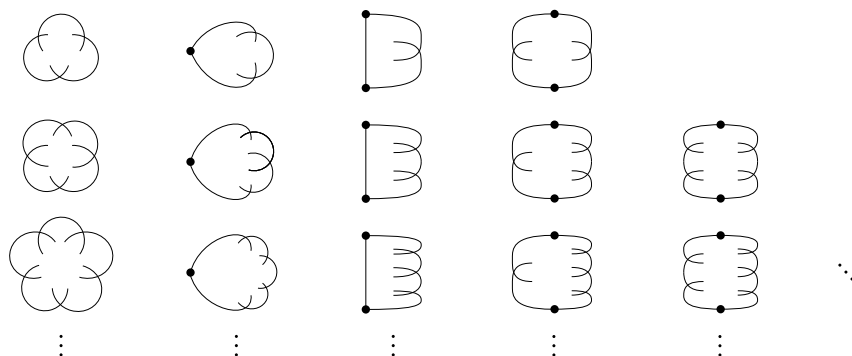
A *pseudoline* is the image  $\ell$  of a continuous injection from the real numbers  $\mathbb{R}$  to the plane  $\mathbb{R}^2$  such that  $\mathbb{R}^2 \setminus \ell$  is not connected. An *arrangement of pseudolines* is a set  $\Sigma$  of pseudolines



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48 ■ **Figure 1** Obstructions to pseudolinearity.

20 such that, if  $\ell, \ell'$  are distinct elements of  $\Sigma$ , then  $|\ell \cap \ell'| = 1$  and the intersection is a crossing  
21 point. More on pseudolines and their importance for studying geometric drawings of graphs  
22 can be found in [10, 11].

23 A drawing  $D$  of a graph  $G$  is *pseudolinear* if there is an arrangement of pseudolines  
24 consisting of a different pseudoline  $\ell_e$  for each edge  $e$  of  $G$  and such that  $D[e] \subseteq \ell_e$ .

25 In the study of crossing numbers, restricting the drawing to either straight lines or  
26 pseudolines yields the rectilinear crossing number  $\overline{\text{cr}}(K_n)$  or the pseudolinear crossing number  
27  $\tilde{\text{cr}}(K_n)$ , respectively. Clearly  $\overline{\text{cr}}(K_n) \geq \tilde{\text{cr}}(K_n)$  and the geometric methods prove that  
28  $\tilde{\text{cr}}(K_n) > H(n)$ , for  $n \geq 10$ .

29 A *good drawing* is one where no edge self-intersects and any two edges share at most  
30 one point—either a crossing or a common end point—and no three edges share a common  
31 crossing. One somewhat surprising result is from Aichholzer et al.: *a good drawing of  $K_n$   
32 in the plane is homeomorphic to a pseudolinear drawing if and only if it does not contain  
33 a non-planar drawing of  $K_4$  whose crossing is incident with the unbounded face of the  $K_4$   
34 [2].* There are equivalent characterizations in [5, 6]. These conditions can be shown to be  
35 equivalent to not containing the *B-configuration* depicted as the third drawing of the first  
36 row of Figure 1.

37 Twenty-five years earlier, Thomassen proved a similar theorem for drawings in which  
38 each edge is crossed only once [16]. The *B-* and *W-*configurations are shown as the third  
39 and fourth drawings in the first row of Figure 1. Thomassen's theorem is: *if  $D$  is a planar  
40 drawing of a graph  $G$  in which each edge is crossed at most once, then  $D$  is homeomorphic  
41 to a rectilinear drawing of  $G$  if and only if  $D$  contains no *B-* or *W-*configuration.*

42 Thomassen presented in [16] the *clouds* (first column in Figure 1) as an infinite family of  
43 drawings that are minimally non-pseudolinear.

44 Shortly after Thomassen's paper, Bienstock and Dean proved that if  $\text{cr}(G) \leq 3$ , then  
45  $\overline{\text{cr}}(G) = \text{cr}(G)$  [8]. They also exhibited examples based on overlapping *W-*configurations to  
46 show the result fails for  $\text{cr}(G) = 4$ ; such graphs can have arbitrarily large rectilinear crossing  
47 number.

48 Despite the existence of infinitely many obstructions to pseudolinearity, we characterize  
49 them all.

51 ► **Theorem 1.** A good drawing of a graph  $G$  is pseudolinear if and only if it does not contain  
52 one of the infinitely many obstructions shown in Figure 1.

53 The drawings in Figure 1 are obtained from the *clouds* (first column) by replacing at most  
54 two crossings by vertices. The formal statement of Theorem 1 is Theorem 15 in Section 6;

also a more general version of this statement, Theorem 2, is discussed below. That there is a result such as ours is somewhat surprising, because stretching an arrangement of pseudolines to a rectilinear drawing has been shown by Mněv [14, 15] to be  $\exists\mathbb{R}$ -hard. In particular, recognizing a drawing as being homeomorphic to a rectilinear drawing is NP-hard.

The natural setting for our characterization is strings embedded in the plane. An *arc*  $\sigma$  is the image  $f([0, 1])$  of the compact interval  $[0, 1]$  under a continuous map  $f : [0, 1] \rightarrow \mathbb{R}^2$ . Let  $S(\sigma) = \{p \in \sigma : |f^{-1}(p)| \geq 2\}$  be the set of self-intersections of  $\sigma$ . A *string* is an arc  $\sigma$  for which  $S(\sigma)$  is finite. If  $S(\sigma) = \emptyset$ , then  $\sigma$  is *simple*.

An intersection point between of two strings  $\sigma$  and  $\sigma'$  is *ordinary* if it is either an endpoint of  $\sigma$  or  $\sigma'$ , or is a *crossing* (a crossing is a non-tangential intersection point in  $\sigma \cap \sigma'$  that is not an end of  $\sigma$  or  $\sigma'$ ). A set  $\Sigma$  of strings is *ordinary* if  $\Sigma$  is finite and any two strings in  $\Sigma$  have only finitely many intersections, all of which are ordinary. All the sets of strings considered in this paper are ordinary.

If  $\Sigma$  is an ordinary set of strings, then its *planarization*  $G(\Sigma)$  is the plane graph obtained from  $\Sigma$  by inserting vertices at each crossing between strings and also at the endpoints of every string in  $\Sigma$ . To keep track of the information given by the strings, we will always assume that each string  $\Sigma$  has a different color and that each edge in  $G(\Sigma)$  inherits the color of the string including it.

If  $\Sigma$  is an ordinary set of strings, then, for a cycle  $C$  in  $G(\Sigma)$  (which is a simple closed curve in  $\mathbb{R}^2$ ) and a vertex  $v \in V(C)$ ,  $v$  is a *rainbow* for  $C$  if all the edges incident with  $v$  and drawn in the closed disk bounded by  $C$  (including the two edges of  $C$  at  $v$ ) have different colours. The reader can verify that, for each drawing in Figure 1, if we let  $\Sigma$  be the edges of the drawing, then the unique cycle in  $G(\Sigma)$  has at most two rainbows. Our main result characterizes these cycles as the only possible obstructions:

► **Theorem 2.** *An ordinary set of strings  $\Sigma$  can be extended to an arrangement of pseudolines if and only if every cycle  $C$  of  $G(\Sigma)$  has at least three rainbows.*

Henceforth, we define any cycle  $C$  in  $G(\Sigma)$  with at most two rainbows as an *obstruction*. A set of strings is *pseudolinear* if it has an extension to an arrangement of pseudolines.

Theorem 2 is our main contribution. In the next section, we show that the presence of an obstruction implies the set of ordinary strings is not pseudolinear. The converse is proved in Section 4 by extending, one small step at a time, the strings in  $\Sigma$  to get closer to an arrangement of pseudolines. After each extension, we must show that no obstruction has been introduced. This involves dealing with cycles in  $G(\Sigma)$  that have precisely three rainbows (that we refer as *near-obstructions*). In Section 3 we show the key lemma that if  $G$  has two such near-obstructions that intersect nicely at a vertex  $v$ , then  $G$  has an obstruction. In Section 5 we present a polynomial-time algorithm for detecting obstructions and we argue why the proof of Theorem 2 implies a polynomial-time algorithm for extending a pseudolinear set of strings. Finally, in Section 6, we show how Theorem 1 follows from Theorem 2 and we present some concluding remarks.

## 2 A set of strings with an obstruction is not extendible

Let us start by showing the easy direction of Theorem 2:

► **Lemma 3.** *If the underlying graph  $G(\Sigma)$  of a set  $\Sigma$  of strings has an obstruction, then  $\Sigma$  is not pseudolinear.*

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Suppose that  $C$  is a cycle of  $G(\Sigma)$  for some set of strings  $\Sigma$ . We define  $\delta(C)$  as the set of vertices of  $C$  for which their two incident edges in  $C$  have different colours. In a set  $\Sigma$  of simple strings where no two intersect twice,  $|\delta(C)| \geq 3$  for every cycle  $C$  of  $G(\Sigma)$ .

► **Lemma 4.** *Let  $\Sigma$  be a set of simple strings where every pair intersect at most once. Suppose that  $C$  is an obstruction with  $|\delta(C)|$  as small as possible. Let  $S = x_0, x_1, \dots, x_\ell$  be a path of  $G(\Sigma)$  representing a subsegment of some string  $\sigma \in \Sigma$  such that  $x_0 x_1 \in E(C)$ ,  $x_1 \in \delta(C)$  and  $x_1$  is not a rainbow of  $C$ . Then  $V(C) \cap V(S) = \{x_0, x_1\}$ .*

**Proof.** By way of contradiction, suppose that there is a vertex  $x_r \in V(C) \cap V(S)$  with  $r \geq 3$ . Assume that  $r \geq 3$  is as small as possible. Let  $P$  be the subpath of  $S$  connecting  $x_1$  to  $x_r$ . Since  $x_0 x_1 \in E(C)$  and  $x_1 \in \delta(C)$  and  $P \subseteq \sigma$ ,  $x_1 x_2 \notin E(C)$ . Because  $x_1$  is not a rainbow for  $C$  and no two strings tangentially intersect at  $x_1$ , the edge  $x_1 x_2$  is drawn in the closed disk bounded by  $C$ . By choice of  $r$ ,  $P$  is an arc connecting  $x_1$  to  $x_r$  in the interior of  $C$ .

Let  $C_1$  and  $C_2$  be the cycles obtained from the union of  $P$  and one of the two  $xy$ -subpaths in  $C$ . We may assume that  $x_0 x_1 \in E(C_1)$ . Let  $\rho(C)$  be either  $\delta(C)$  or the set of rainbows in  $C$ . For  $i = 1, 2$ , let  $Q_i = V(C_i) \setminus V(P)$ . Then  $\rho(C) \cap Q_i = \rho(C_i) \cap Q_i$ . We see that  $\rho(C_1) \setminus Q_1 \subseteq \{x_r\}$  and  $\rho(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$ .

For  $\rho = \delta$ ,  $|\delta(C_2)| \geq 3$ , so  $|\delta(C) \cap Q_2| \geq 1$ . Since  $x_1 \notin \delta(C_1)$ ,  $|\delta(C_1)| \leq |\delta(C_1) \cap Q_2| + |\{x_r\}| \leq |\delta(C)| - 2 + |\{x_r\}| < |\delta(C)|$ . Likewise,  $|\delta(C) \cap Q_1| \geq 2$  and  $x_1 \in \delta(C) \cap \delta(C_2)$ . Therefore,  $|\delta(C_2)| \leq |\delta(C)| - 2 + |\{x_r\}| < |\delta(C)|$ . Thus, neither  $C_1$  nor  $C_2$  is an obstruction.

Now taking  $\rho$  to be the set of rainbows, the preceding paragraph shows  $|\rho(C_1)| \geq 3$  and  $|\rho(C_2)| \geq 3$ . Therefore,  $|\rho(C) \cap Q_1| = |\rho(C_1) \cap Q_1| \geq 2$  and  $|\rho(C) \cap Q_2| = |\rho(C_2) \cap Q_2| \geq 1$ . Thus,  $|\rho(C)| \geq 3$ , a contradiction. ◀

**Proof of Lemma 3.** By way of contradiction, suppose that  $\Sigma$  is pseudolinear and that  $G(\Sigma)$  has an obstruction  $C$ .

Consider an extension of  $\Sigma$  to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings  $\Sigma'$  extending  $\Sigma$ , and in which every pair of strings in  $\Sigma'$  cross once. In  $G(\Sigma')$ , there is a cycle  $C'$  that represents the same simple closed curve as  $C$ . Because  $C'$  is obtained from subdividing some edges of  $C$  and the colours of a subdivided edge are the same,  $C'$  has fewer than three rainbows. Therefore, we may assume that  $\Sigma = \Sigma'$  and  $C = C'$ . Now, the ends of every string in  $\Sigma$  are degree-1 vertices in the outer face of  $G(\Sigma)$ .

As every string in  $\Sigma$  is simple and no two strings intersect more than once,  $|\delta(C)| \geq 3$ . We will assume that  $C$  is chosen to minimize  $|\delta(C)|$ .

Since  $C$  is an obstruction, there exists  $x_1 \in \delta(C)$  such that  $x_1$  is not a rainbow in  $C$ . Consider a neighbour  $x_0$  of  $x_1$  in  $C$ . Let  $S = x_0, x_1, \dots, x_\ell$  be the path obtained by traversing the string  $\sigma$  extending  $x_0 x_1$ , such that  $x_\ell$  is an end of  $\sigma$ . By Observation 4,  $V(S) \cap V(C) = \{x_0, x_1\}$ , and because  $x_\ell$  is in the outer face of  $C$ , the segment of  $\sigma$  from  $x_1$  to  $x_\ell$  has its relative interior in the outer face of  $C$ .

However, since  $x_1$  is not a rainbow, there exists a string  $\sigma' \in \Sigma$  including two edges at  $x_1$  drawn in the disk bounded by  $C$ . Thus,  $\sigma$  and  $\sigma'$  tangentially intersect at  $x_1$ , a contradiction. ◀

### 3 The key lemma

In this section we present the key lemma used in the proof of Theorem 2.

A plane graph  $G$  is *path-partitioned* if for  $m \geq 1$ , there exists a colouring  $\chi : E(G) \rightarrow \{1, \dots, m\}$  such that for each  $i \in \{1, \dots, m\}$ , the edges in  $\chi^{-1}(i)$  induce a path  $P_i \subseteq G$  where

any two distinct paths  $P_i$  and  $P_j$  do not tangentially intersect. Indeed, every underlying planar graph  $G(\Sigma)$  of a set of simple strings  $\Sigma$  is path-partitioned. Moreover, every path-partitioned plane graph can be obtained by subdividing a planarization of an ordinary set of simple strings. To extend the previously introduced notation we refer to each  $P_i$  as a string. The concepts of rainbow and obstruction naturally extend to the context of path-partitioned plane graphs.

Suppose that  $G$  is a path-partitioned plane graph. Given  $v \in V(G)$ , a *near-obstruction* at  $v$  is a cycle  $C$  with at most three rainbows and such that  $v$  is a rainbow of  $C$ . Understanding how near-obstructions behave is the key ingredient needed in the proof of Theorem 2:

► **Lemma 5.** *Let  $G$  be a path-partitioned plane graph and let  $v \in V(G)$ . Suppose that  $C_1$  and  $C_2$  are two near-obstructions at  $v$  such that the union of the closed disks bounded by  $C_1$  and  $C_2$  contains a small open ball centered at  $v$ . Suppose that one of the following two holds:*

1. *no obstruction of  $G$  contains  $v$ ; or*
2. *the two edges of  $C_1$  incident with  $v$  are the same as the two edges of  $C_2$  incident with  $v$ .*

*Then  $G$  has an obstruction not including  $v$ .*

Given a plane graph  $G$ , a cycle  $C \subseteq G$  and a vertex  $v \in V(C)$ , the edges at  $v$  inside  $C$  are the edges of  $G$  incident with  $v$  drawn in the disk bounded by  $C$ .

► **Useful Fact.** *Let  $G$  be planar path-partitioned graph. Suppose that for two cycles  $C$  and  $C'$ ,  $v \in V(C) \cap V(C')$  is a vertex such that the edges at  $v$  inside  $C'$  are also edges at  $v$  inside  $C$ . If  $v$  is a rainbow for  $C$ , then  $v$  is a rainbow for  $C'$ .*

**Proof of Lemma 5.** By way of contradiction, suppose that  $G$  has no obstruction not including  $v$ . The “small ball” hypothesis implies that  $v$  is not in the outer face of the subgraph  $C_1 \cup C_2$ .

We claim that  $|V(C_1) \cap V(C_2)| \geq 2$ . Suppose not. Then  $C_1$  and  $C_2$  are edge-disjoint and  $V(C_1) \cap V(C_2) = \{v\}$ . For  $i = 1, 2$ , let  $e_i$  and  $f_i$  be the edges of  $C_i$  at  $v$  and let  $\Delta_i$  be the closed disk bounded by  $C_i$ . From the “small ball” hypothesis it follows that (i)  $\Delta_1$  contains the edges  $e_2$  and  $f_2$ ; and (ii) the points near  $v$  in the exterior of  $\Delta_2$  are contained in  $\Delta_1$ . These two properties imply that the path  $C_2 - \{e_2, f_2\}$  intersects  $C_1$  at least twice, and hence,  $|V(C_1) \cap V(C_2)| \geq 2$ .

From the last paragraph we know that  $C_1 \cup C_2$  is 2-connected, and hence the outer face of  $C_1 \cup C_2$  is bounded by a cycle  $C_{out}$ . We will assume that

- (\*) the cycles  $C_1$  and  $C_2$  satisfying the hypothesis of Lemma 5 are chosen so that the number of vertices of  $G$  in the disk bounded by  $C_{out}$  is minimal.

The Useful Fact applied to  $C = C_{out}$  and to each  $C' \in \{C_1, C_2\}$ , shows that every vertex that is a rainbow in  $C_{out}$  is also a rainbow in each of the cycles in  $\{C_1, C_2\}$  containing it. We can assume that  $C_{out}$  is not an obstruction or else we are done. We may relabel  $C_1$  and  $C_2$  so that two of the rainbows of  $C_{out}$ , say  $p$  and  $q$ , are also rainbows in  $C_1$ . Neither  $p$  nor  $q$  is  $v$  because  $v \notin V(C_{out})$ . Because  $C_1$  is a near-obstruction,  $p$ ,  $q$  and  $v$  are the only rainbows of  $C_1$ .

Since  $v \notin V(C_{out})$ , by following  $C_1$  in the two directions starting at  $v$ , we find a path  $P_v \subseteq C_1$  containing  $v$  in which only the ends  $u$  and  $w$  of  $P_v$  are in  $C_{out}$  (note that  $u \neq v$  because  $\{p, q\} \subseteq V(C_1) \cap V(C_{out})$ ). As  $v$  is in the interior face of  $C_{out}$ ,  $P_v$  is also in the interior of  $C_{out}$ . Let  $Q_{out}^1, Q_{out}^2$  be the  $uw$ -paths of  $C_{out}$ . One of the two closed disks bounded

186 by  $P_v \cup Q_{out}^1$  and  $P_v \cup Q_{out}^2$  contains  $C_1$ . By symmetry, we may assume that  $C_1$  is contained  
 187 in the first disk. Since  $C_{out} \subseteq C_1 \cup C_2$ , this implies that  $Q_{out}^2$  is a subpath of  $C_2$ .

188 Our desired contradiction will be to find three rainbows in  $C_2$  distinct from  $v$ . We  
 189 find the first: let  $C_1 - (P_v)$  be the  $uv$ -path in  $C_1$  distinct from  $P_v$ . The disk bounded  
 190 by  $(C_1 - (P_v)) \cup Q_{out}^2$  contains the one bounded by  $C_1$ . The Useful Fact applied to  $C =$   
 191  $(C_1 - (P_v)) \cup Q_{out}^2$  and  $C' = C_1$  implies that each vertex in  $C_1 - (P_v)$  that is rainbow in  
 192  $(C_1 - (P_v)) \cup Q_{out}^2$  is also rainbow in  $C_1$ . Since  $C_1$  has at most two rainbows in  $C_1 - (P_v)$ ,  
 193 namely  $p$  and  $q$ ,  $(C_1 - (P_v)) \cup Q_{out}^2$  has a third rainbow  $r_1$  in the interior of  $Q_{out}^2$  (else  
 194  $(C_1 - (P_v)) \cup Q_{out}^2$  is an obstruction and we are done). Note that  $r_1$  is also a rainbow for  $C_2$ .

195 To find another rainbow in  $C_2$ , consider the edge  $e_u$  of  $C_2$  incident to  $u$  and not in  $Q_{out}^2$ .  
 196 We claim that either  $u$  is a rainbow in  $C_2$  or that  $e_u$  is not included in the closed disk  
 197 bounded by  $P_v \cup Q_{out}^2$ . Seeking a contradiction, suppose that  $u$  is not a rainbow of  $C_2$  and  
 198 that  $e_u$  is included in the disk. Then we can find two edges in the rotation at  $u$ , included in  
 199 the disk bounded by  $P_v \cup Q_{out}^2$ , that belong to the same string  $\sigma$ . The vertex  $u$  is a rainbow  
 200 in  $C_1$ , as else, we would find a string  $\sigma'$  with two edges inside  $Q_{out}^1 \cup P_v$ , showing that  $\sigma$  and  
 201  $\sigma'$  tangentially intersect at  $u$ . As  $p$  and  $q$  are the only rainbows of  $C_1$  in  $C_{out}$ ,  $u$  is one of  $p$   
 202 and  $q$ . Therefore  $u$  is a rainbow in  $C_{out}$ , and hence, a rainbow in  $C_2$ , a contradiction.

203 If  $u$  is a rainbow in  $C_2$ , then this is the desired second one. Otherwise,  $e_u$  is not in the  
 204 closed disk bounded by  $P_v \cup Q_{out}^2$ . Let  $P_u \subseteq C_2$  be the path starting at  $u$ , continuing on  $e_u$   
 205 and ending on the first vertex  $u'$  in  $P_v$  that we encounter. Let  $C_u$  be the cycle consisting of  
 206  $P_u$  and the  $uu'$ -subpath  $uP_vu'$  of  $P_v$ .

207  $\triangleright$  Claim 6. If  $P_u$  does not have a rainbow of  $C_u$  in its interior, then either  $C_u$  is an  
 208 obstruction not containing  $v$  or:

- 209 (a)  $C_u$  and  $C_2$  are near-obstructions at  $v$  satisfying the same conditions as  $C_1$  and  $C_2$  in  
 210 Lemma 5; and
- 211 (b) the closed disk bounded by the outer cycle of  $C_u \cup C_2$  contains fewer vertices than the  
 212 disk bounded by  $C_{out}$ .

213 **Proof.** Suppose that all the rainbows of  $C_u$  are located in  $uP_vu'$ . If  $z$  is a rainbow of  $C_u$ ,  
 214 then  $z \in \{u, v, u'\}$ , as otherwise  $z$  is a rainbow of  $C_1$  distinct from  $p, q$  and  $v$ , a contradiction.  
 215 Thus, if  $v \notin V(C_u)$ , then  $C_u$  is the desired obstruction. We may assume that  $v \in V(C_u)$ .

216 If  $u' = w$ , then  $C_2 = P_u \cup Q_{out}^2$ , violating the assumption that  $v \in V(C_2)$ . Thus  $u' \neq w$ .  
 217 If  $u' = v$ , then the rainbows of  $C_u$  are included in  $\{u, u'\}$ , and hence  $C_u$  is an obstruction.  
 218 However, the existence of  $C_u$  shows that both alternatives (1) and (2) in Lemma 5 fail:  
 219 condition (1) fails because  $C_u$  contains  $v$  and (2) fails because the edge of  $P_u$  incident with  $v$   
 220 is in  $E(C_2) \setminus E(C_1)$ . Thus  $u' \neq v$ .

221 The previous two paragraphs show that  $C_u$  is a near-obstruction at  $v$  with rainbows  $u$ ,  
 222  $v$  and  $u'$ . Since the interior of  $C_u$  near  $v$  is the same as the interior of  $C_1$  near  $v$ , the pair  
 223  $(C_u, C_2)$  satisfies the “small ball” hypothesis. Thus, (a) holds.

224 Let  $C'_{out}$  be the outer cycle of  $C_u \cup C_2$ . From the fact that  $C_u \cup C_2 \subseteq C_1 \cup C_2$  it follows  
 225 that the disk bounded by  $C_{out}$  includes the disk bounded by  $C'_{out}$ .

226 Since  $p, q \in V(C_{out})$ ,  $p$  and  $q$  are in the disk bounded by  $C_{out}$ . If both  $p$  and  $q$  are in  
 227  $C_2$ , then  $p, q$  and  $r_1$  are rainbows in  $C_2$ , and also distinct from  $v$ , contradicting that  $C_2$  is a  
 228 near-obstruction for  $v$ . If, say  $p \notin V(C_2)$ , then  $p$  is not in the disk bounded by  $C'_{out}$ , which  
 229 implies (b).  $\blacktriangleleft$

230 From Claim 6(b) and assumption (\*) either  $C_u$  is the desired obstruction or  $P_u$  contains  
 231 a rainbow  $r_2$  of  $C_2$  in its interior. We assume the latter as else we are done.



232 In the same way, the last rainbow  $r_3$  comes by considering the edge of  $C_2 - Q_{out}^2$  incident  
 233 with  $w$ . It follows that  $v, r_1, r_2$  and  $r_3$  are four different rainbows in  $C_2$ , contradicting the  
 234 fact that  $C_2$  is a near-obstruction. ◀

## 235 **4 Proof of Theorem 2**

236 In this section we prove that a set of strings with no obstructions can be extended to an  
 237 arrangement of pseudolines.

238 **Proof of Theorem 2.** It was shown in Observation 3 that the existence of obstructions  
 239 implies non-extendibility. For the converse, suppose that  $\Sigma$  is a set of strings for which  $G(\Sigma)$   
 240 has no obstructions.

241 We start by reducing to the case where the point set  $\bigcup \Sigma$  is connected: iteratively add a  
 242 new string in a face of  $\bigcup \Sigma$  connecting two connected components of  $\bigcup \Sigma$ . No obstruction is  
 243 introduced at each step (obstructions are cycles), and, eventually, the obtained set  $\bigcup \Sigma$  is  
 244 connected. An extension of the new set of strings contains an extension for the original set,  
 245 thus we may assume that  $\bigcup \Sigma$  is connected.

246 Our proof is algorithmic, and consists of repeatedly applying one of the three steps  
 247 described below.

- 248 ■ **Disentangling Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in  $G(\Sigma)$ ,  
 249 then we slightly extend the  $a$ -end of  $\sigma$  into one of the faces incident with  $a$ .
- 250 ■ **Face-Escaping Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree 1 in  $G(\Sigma)$ , and is  
 251 incident with an interior face, then we extend the  $a$ -end of  $\sigma$  until it intersects some point  
 252 in the boundary of this face.
- 253 ■ **Exterior-Meeting Step.** Assuming that all the strings in  $\Sigma$  have their two ends in  
 254 the outer face and these ends have degree 1 in  $G(\Sigma)$ , we extend the ends of two disjoint  
 255 strings so that they meet in the outer face.

256 Each of these three steps either increases the number of pairs of strings that intersect, or  
 257 increase the number crossings (recall that a crossing between  $\sigma$  and  $\sigma'$  is a non-tangential  
 258 intersection point in  $\sigma \cap \sigma'$  that is not an end of  $\sigma$  or  $\sigma'$ ). Moreover, these steps can be  
 259 performed as long as not all the strings have their ends in the outer face and they are pairwise  
 260 crossing (in this case we extend their ends to infinity to obtain the desired arrangement  
 261 of pseudolines). Henceforth, we will show that, if performed correctly, none of these steps  
 262 introduces an obstruction. The proof for each step can be read independently.

263 ► **Lemma 7 (Disentangling Step).** *Suppose that  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in*  
 264  *$G(\Sigma)$ . Then we can extend the  $a$ -end of  $\sigma$  into one of the faces incident to  $a$  without creating*  
 265 *an obstruction.*

266 **Proof.** A pair of different edges  $f$  and  $f'$  in  $G(\Sigma)$  incident with  $a$  are *twins* if they belong to  
 267 the same string in  $\Sigma$ . The edge  $e \subseteq \sigma$  incident with  $a$  has no twin.

268 The fact that no pair of strings tangentially intersect at  $a$  tells us that if  $(f_1, f'_1)$  and  
 269  $(f_2, f'_2)$  are pairs of twins, then  $f_1, f_2, f'_1, f'_2$  occur in this cyclic order for either the clockwise  
 270 or counterclockwise rotation at  $a$ . Thus, we may assume that the counterclockwise rotation  
 271 at  $a$  restricted to the twins and  $e$  is  $e, f_1, \dots, f_t, f'_1, \dots, f'_t$ , where  $(f_i, f'_i)$  is a twin pair for  
 272  $i = 1, \dots, t$ .

273 To avoid tangential intersections, the extension of  $\sigma$  at  $a$  must be in the angle between  $f_t$   
 274 and  $f'_1$  not containing  $e$ . Let  $e_1, \dots, e_k$  be the counterclockwise ordered list of non-twin edges



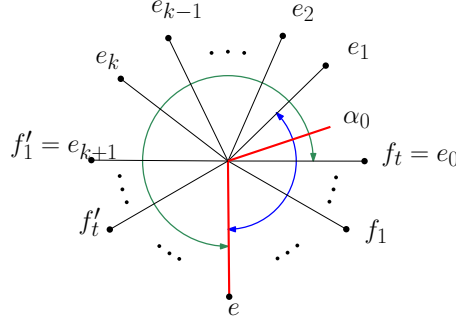


Figure 2 Substrings included in the disk bounded by  $C_0$ .

at  $a$  having an end in this angle (as depicted in Figure 2). We label  $e_0 = f_t$  and  $e_{k+1} = f'_1$ . If there are no twins, then let  $e_0 = e_{k+1} = e$ .

Let us consider all the possible extensions: for  $i \in \{0, \dots, k\}$ , let  $\Sigma_i$  be the set of strings obtained from  $\Sigma$  by slightly extending the  $a$ -end of  $\sigma$  into the face containing the angle between  $e_i$  and  $e_{i+1}$ . Let  $\alpha_i$  be the new edge at  $a$  extending  $\sigma$  in  $\Sigma_i$  (see  $\alpha_0$  in Figure 2).

Seeking a contradiction, suppose that, for each  $i \in \{0, \dots, k\}$ ,  $G(\Sigma_i)$  contains an obstruction  $C_i$ . Since  $\alpha_i$  contains a degree-1 vertex,  $\alpha_i$  is not in  $C_i$ . Hence  $C_i$  is a cycle of  $G(\Sigma)$ . Thus  $C_i$  is not an obstruction in  $G(\Sigma)$  that becomes one in  $G(\Sigma_i)$ . This conversion has a simple explanation: in  $G(\Sigma)$ ,  $C_i$  has exactly three rainbows, and one of them is  $a$ . After  $\alpha_i$  is added,  $a$  is not a rainbow in  $C_i$  (witnessed by the edges  $e$  and  $\alpha_i$  included in the new version of  $\sigma$ ).

Recall from Section 3 that a *near-obstruction* at  $a$  is a cycle with exactly three rainbows, and one of them is  $a$ . Each of  $C_0, C_1, \dots, C_k$  is a near-obstruction at  $a$  in  $G(\Sigma)$ .

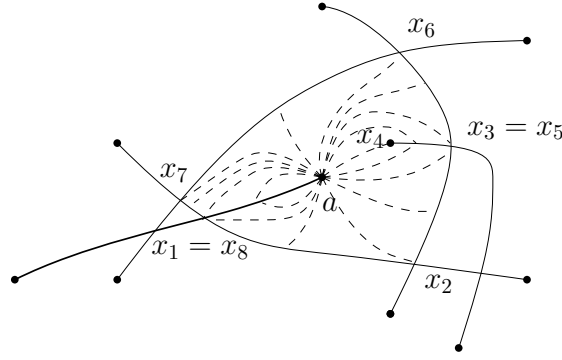
For a cycle  $C \subseteq G$ , let  $\Delta(C)$  denote the closed disk bounded by  $C$ . Both  $e$  and  $\alpha_0$  are in  $\Delta(C_0)$ . Thus, either  $\Delta(C_0) \supseteq \{e, f_1, f_2, \dots, f_t, e_1\}$  (blue bidirectional arrow in Figure 2) or  $\Delta(C_0) \supseteq \{f_t, e_1, \dots, e_k, f'_1, f'_2, \dots, f'_t, e\}$  (green bidirectional arrow). We rule out the latter situation as the second list contains  $f_t$  and  $f'_t$ , and this would imply that  $a$  is not a rainbow for  $C_0$  in  $G(\Sigma)$ .

We just showed that  $\{e, e_0, e_1\} \subseteq \Delta(C_0)$ . By symmetry,  $\{e_k, e_{k+1}, e\} \subseteq \Delta(C_k)$ . Consider the largest index  $i \in \{0, 1, \dots, k-1\}$  for which  $\{e, e_0, \dots, e_{i+1}\} \subseteq \Delta(C_i)$ . By the choice of  $i$ , and because  $\{e, \alpha_{i+1}\} \subseteq \Delta(C_{i+1})$ ,  $\{e, f'_t, \dots, f'_1, e_k, \dots, e_i\} \subseteq \Delta(C_{i+1})$ . However, by applying Lemma 5 to the pair  $C_i$  and  $C_{i+1}$ , we obtain that  $G(\Sigma)$  has an obstruction, a contradiction.  $\blacktriangleleft$

► **Lemma 8** (Face-Escaping Step). *Suppose that there is a string  $\sigma$  that has an end  $a$  with degree 1 in  $G(\Sigma)$ , and  $a$  is incident to an interior face  $F$ . Then there is an extension  $\sigma'$  of  $\sigma$  from its  $a$ -end to a point in the boundary of  $F$  such that the set  $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$  has no obstruction.*

**Proof.** Let  $W$  be the closed boundary walk  $(x_0, e_1, x_1, e_2, \dots, e_n, x_n)$  of  $F$  such that  $x_0 = x_n = a$  and  $F$  is to the left as we traverse  $W$  (see Figure 3 for an illustration with  $n = 9$ ). For  $i = 1, \dots, n$  we let  $m_i$  be a point in the relative interior of  $e_i$ , and let  $P$  be the list of non-necessarily distinct points  $m_1, x_1, m_2, x_2, \dots, m_n$ , which are the potential ends for all the different extensions. For each  $p \in P$ , let  $\Sigma_p$  be the set of strings obtained from  $\Sigma$  by extending the  $a$ -end of  $\sigma$  by adding an arc  $\alpha_p$  connecting  $a$  to  $p$  in  $F$  (see Figure 3). We assume that every two distinct arcs  $\alpha_p$  and  $\alpha_{p'}$  are internally disjoint.

Let  $f_p$  be the edge  $e_1 \cup \alpha_p$  in  $G(\Sigma_p)$ ;  $f_p$  has ends  $x_1$  and  $p$ . Also, let  $\sigma^p = \sigma \cup \alpha_p$ . Seeking a contradiction, suppose that each  $G(\Sigma_p)$  has an obstruction.



302 ■ **Figure 3** All possible extensions in the Face-Escaping Step.

312 ▷ **Claim 9.** Let  $p \in P$ . Then there exists an obstruction  $C_p$  in  $G(\Sigma_p)$  including  $f_p$ . Moreover,

- 313 (1) if  $p \in \sigma$ , then  $C_p$  can be chosen so that all its edges are included in  $\sigma^p$ ; and  
 314 (2) if  $p \notin \sigma$ , then every obstruction includes  $f_p$ .

315 **Proof.** First, if  $p \in \sigma$ , then the string  $\sigma^p$  self-intersects at  $p$ ; thus  $\sigma^p$  has a simple close curve  
 316 including  $f_p$ . In this case let  $C_p$  be the cycle in  $G(\Sigma_p)$  representing this simple closed curve  
 317 without rainbows, and thus (1) holds.

318 Second, assume that  $p \notin \sigma$  and let  $C_p$  be any obstruction of  $G(\Sigma_p)$ . For (2), we will show  
 319 that  $f_p \in E(C_p)$ .

320 Seeking a contradiction, suppose that  $f_p \notin E(C_p)$ .

321 If  $p = m_i$  for  $i \in \{1, \dots, n\}$ , since  $m_i$  is the only vertex whose rotation in  $G(\Sigma)$  differs  
 322 from its rotation in  $G(\Sigma_{m_i})$ ,  $m_i \in V(C_p)$ . Consider the cycle  $C$  of  $G(\Sigma)$  obtained from  $C_p$   
 323 by replacing the subpath  $(x_{i-1}, m_i, x_i)$  by the edge  $x_{i-1}x_i$ . For each vertex  $v \in V(C)$  the  
 324 colors of the edges of  $G(\Sigma)$  at  $v$  included in the disk bounded by  $C$  are the same as in  $G(\Sigma_p)$   
 325 for the disk bounded by  $V(C_p)$ . Thus,  $C$  is an obstruction for  $G(\Sigma)$ , a contradiction.

326 Suppose now that  $p$  is one of  $x_1, \dots, x_{n-1}$ . The only vertex in  $G(\Sigma)$  whose rotation is  
 327 different in  $G(\Sigma_p)$  is  $p$ . Therefore,  $p$  is a point that is a rainbow for  $C_p$  in  $G(\Sigma)$ , but not  
 328 a rainbow in  $G(\Sigma_p)$ , witnessed by two edges included in  $\sigma^p$ . Since at least one of the two  
 329 witnessing edges is in  $G(\Sigma)$ ,  $p \in \sigma$ . This contradicts the assumption that  $p \notin \sigma$ . Hence  
 330  $f_p \in E(C_p)$ . ◀

331 Henceforth we assume that, for  $p \in P$ ,  $C_p$  is an obstruction in  $G(\Sigma_p)$  as in Claim 9.

332 More can be said about the obstructions in  $G(\Sigma_p)$ , but for this we need some terminology.  
 333 If we orient an edge  $e$  in a plane graph, then the *sides* of  $e$  are either the points near  $e$  that  
 334 are to the right of  $e$ , or the points near  $e$  to the left of  $e$ . For any cycle  $C$  of  $G$  through  $e$ ,  
 335 exactly one side of  $e$  lies inside  $C$ . This is the side of  $e$  *covered* by  $C$ . For the next claim  
 336 and in the rest of the proof we will assume that for  $p \in P$ ,  $f_p$  is oriented from  $x_1$  to  $p$ .

337 ▷ **Claim 10.** For  $p \in P$  with  $p \notin \sigma$ , every obstruction in  $G(\Sigma_p)$  covers the same side of  $f_p$ .

338 **Proof.** Suppose that for  $p \in P$  there are obstructions  $C_p$  and  $C'_p$  covering both sides of  $f_p$ .  
 339 Let  $G'$  be the plane graph obtained from  $G(\Sigma_p)$  by subdividing  $f_p$ , and let  $v$  be the new  
 340 degree-2 vertex inside  $f_p$ .

341 We consider the edge-colouring  $\chi$  induced by the strings in  $\Sigma_p$ . Let  $\chi'$  be a new colouring  
 342 obtained from  $\chi$  by replacing the colour of the edge  $vp$  by a new colour not used in  $\chi$ . It is a

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routine exercise to verify that (i)  $\chi'$  induces a path-partition in  $G'$  (defined in Section 3); and (ii)  $C_p$  and  $C'_p$  are near-obstructions for  $v$  with respect to  $\chi'$ . By applying Lemma 5 to  $C_1 = C_p$  and  $C_2 = C'_p$ , we obtain an obstruction in  $G'$  not containing  $v$ . However, this implies the existence of an obstruction in  $G(\Sigma)$ , a contradiction.  $\blacktriangleleft$

Recall that the boundary walk of  $F$  is  $W = (x_0, e_1, \dots, e_n, x_n)$ , with  $x_0 = x_n = a$ . Since  $x_1$  and  $x_{n-1}$  are in  $\sigma$ , the extreme obstructions  $C_{x_1}$  and  $C_{x_2}$  cover the right of  $f_{x_1}$  and the left of  $f_{x_{n-1}}$ , respectively. Thus, there are two consecutive vertices  $x_{i-1}, x_i$  in  $W - a$ , such that the interior of  $C_{x_{i-1}}$  covers the right of  $f_{x_{i-1}}$  and the interior of  $C_{x_i}$  covers the left of  $f_{x_i}$ . Moreover, we may assume that the interior of  $C_{m_i}$  includes the left of  $f_{m_i}$  (otherwise we reflect our drawing).

The next claim (proved in the full version of this paper [4]) is the last ingredient to obtain a final contradiction.

▷ **Claim 11.** Exactly one of the following holds:

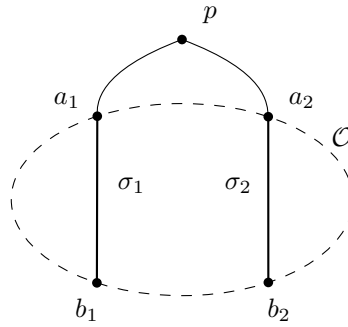
- (a)  $x_{i-1} \in \sigma$ ,  $m_i \notin \sigma$  and  $G(\Sigma_{m_i})$  has an obstruction covering the side of  $f_{m_i}$  not covered by  $C_{m_i}$ ; or
- (b)  $x_{i-1} \notin \sigma$  and  $G(\Sigma_{x_{i-1}})$  has an obstruction covering the side of  $f_{x_{i-1}}$  not covered by  $C_{x_{i-1}}$ .

Claims 10 and 11 contradict each other. Thus, for some  $p \in P$ ,  $G(\Sigma_p)$  has no obstructions.

► **Lemma 12 (Exterior-Meeting Step).** *If all the strings in  $\Sigma$  have their ends on the outer face of  $G(\Sigma)$  and the ends have degree 1 in  $G(\Sigma)$ , then we can extend a pair disjoint strings so that they intersect without creating an obstruction.*

**Proof.** By following the outer boundary of  $\bigcup \Sigma$ , we obtain a simple closed curve  $\mathcal{O}$  containing all the ends of the strings in  $\Sigma$ , but otherwise disjoint from  $\bigcup \Sigma$ .

Suppose  $\sigma_1, \sigma_2$  are two disjoint strings in  $\Sigma$ . For  $i = 1, 2$ , let  $a_i, b_i$  be the ends of  $\sigma_i$ ; since  $\sigma_1$  and  $\sigma_2$  do not cross, we may assume that these ends occur in the cyclic order  $a_1, b_1, b_2, a_2$ . We extend the  $a_i$ -ends of  $\sigma_1$  and  $\sigma_2$  so that they meet in a point  $p$  in the outer face, and so that all the ends of  $\sigma_1$  and  $\sigma_2$  remain incident with the outer face (Figure 4). Let  $\Sigma'$  be the obtained set of strings.



■ **Figure 4** Exterior-Meeting Step.

Seeking a contradiction, suppose that  $G(\Sigma')$  has an obstruction  $C$ . Since  $G(\Sigma)$  has no obstruction,  $p \in V(C)$ . Our contradiction will be to find three rainbows in  $C$ . Note that  $p$  is a rainbow. To obtain a second rainbow, traverse  $C$  starting from  $p$  towards  $a_1$ . Let  $d_1$  be the first vertex during our traversal that is not in the extended  $\sigma_1$ , and let  $c_1$  be its neighbour in  $\sigma_1$ , one step before we reach  $d_1$ . Since  $b_1$  has degree one,  $c_1 \neq b_1$ .

377  $\triangleright$  **Claim 13.** The cycle  $C$  has a rainbow included in the closed disk  $\Delta_1$  bounded by  $\sigma_1$  and  
 378 the  $a_1b_1$ -arc of  $\mathcal{O}$  disjoint from  $\sigma_2$ .

379 **Proof.** First, suppose that  $d_1 \notin \Delta_1$ . In this case,  $c_1$  is a rainbow because otherwise there  
 380 would be a string  $\sigma$  that tangentially intersects  $\sigma_1$  at  $c_1$ . Thus, if  $d_1 \notin \Delta_1$ , then  $c_1$  is the  
 381 desired rainbow.

382 Second, suppose that  $d_1 \in \Delta_1$ . Let  $P_1$  be the path of  $C$  starting at  $c_1$ , continuing on the  
 383 edge  $c_1d_1$ , and ending at the first vertex we encounter in  $\sigma_1$ . Since the cycle  $C'$  enclosed by  
 384  $P_1 \cup \sigma_1$  is not an obstruction, there is one rainbow of  $C'$  that is an interior vertex of  $P_1$ ; this  
 385 is the desired rainbow of  $C$ . This concludes the proof of Claim 13.  $\blacktriangleleft$

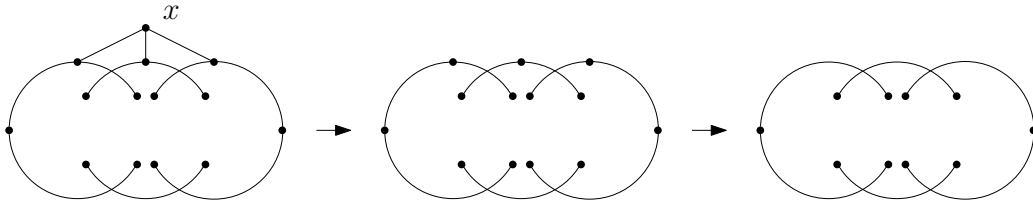
386 Considering  $\sigma_2$  instead of  $\sigma_1$ , Claim 13 yields a third rainbow in  $C$  inside an analogous  
 387 disk  $\Delta_2$  disjoint from  $\Delta_1$ , contradicting that  $C$  is an obstruction. Hence Lemma 12 holds.  $\blacktriangleleft$

388 We iteratively apply the Disentangling Step, Face-Escaping Step or Exterior-Meeting Step  
 389 without creating obstructions. Each step increases the number of pairwise intersecting strings  
 390 in  $\Sigma$  until we reach a stage where the strings are pairwise intersecting and all of them have  
 391 their two ends in the unbounded face. From this we extend them into an arrangement of  
 392 pseudolines. This concludes the proof of Theorem 2.  $\blacktriangleleft$

## 393 **5 Finding obstructions and extending strings in polynomial time**

394 We start this section by describing an algorithm to detect obstructions. Henceforth, we  
 395 assume that the input to the problem is the planarization  $G(\Sigma)$  of an ordinary set of  $s$  strings  
 396  $\Sigma$ . For the running-time analysis, we assume that  $n$  and  $m$  are the number of vertices and  
 397 edges in  $G(\Sigma)$ , respectively. Since  $G(\Sigma)$  is planar,  $m = O(n)$ . Moreover, if  $\Sigma$  is pseudolinear,  
 398 then  $n \leq \binom{s}{2} + 2s = \binom{s+2}{2} - 1$ . At the end of this section we explain how to extend  $\Sigma$  (if  
 399 possible) in polynomial time.

401 Recall that each string in  $\Sigma$  receives a different colour; this induces an edge-colouring on  
 402  $G(\Sigma)$  where each string is a monochromatic path. An *outer-rainbow* is a vertex  $x \in V(G(\Sigma))$   
 403 incident with the outer face and for which the edges incident with  $x$  have different colours.  
 Next we describe the basic operation in our obstruction-detecting algorithm.



400 **Figure 5** From  $\Sigma$  to  $\Sigma - x$ .

405 **Outer-rainbow deletion.** Given an outer-rainbow  $x \in V(G(\Sigma))$ , the instance  $G(\Sigma - x)$  is  
 406 defined by: first, removing  $x$  and the edges incident to  $x$ ; second, suppressing the degree-2  
 407 vertices incident with edges of the same colour; and third, removing remaining degree-0  
 408 vertices (Figure 5 illustrates this process). Edge colours are preserved.

409 It is easy to verify that  $G(\Sigma - x)$  is the planarization of an arrangement of strings. The  
 410 colours removed by this operation are those belonging to strings that are paths of length 1 in  
 411  $G(\Sigma)$  incident with  $x$ . Our obstruction-detecting algorithm relies on the following property:

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412 (\*\*) if  $x$  is an outer-rainbow of  $G(\Sigma)$ , then there is an obstruction in  $G(\Sigma)$  not including  $x$  if  
 413 and only if there is an obstruction in  $G(\Sigma - x)$ .

414 This property holds because cycles in  $G(\Sigma) - x$  and in  $G(\Sigma - x)$  are in 1-1 correspon-  
 415 dence: two cycles correspond to each other if they are the same simple closed curve. This  
 416 correspondence is obstruction-preserving.

417 Let us now describe the two subroutines in our algorithm. For this, we remark that an  
 418 outer-rainbow of  $G(\Sigma)$  is a rainbow for any cycle containing it.

419 **Subroutine 1.** *Detecting an obstruction through two outer-rainbows  $x$  and  $y$ .*

- 420 (1) Find a cycle  $C$  through  $x$  and  $y$  whose edges are incident with the outer face of  $G(\Sigma)$ . If  
 421 no such  $C$  exists, then output *No obstruction through  $x$  and  $y$* . Else, go to Step 2.
- 422 (2) Find whether there is a third outer-rainbow  $z \in V(C) \setminus \{x, y\}$ . If such  $z$  exists, update  
 423  $G(\Sigma) \leftarrow G(\Sigma - z)$  and go to Step 1. If no such  $z$  exists, output  $C$ .

424 *Correctness and running-time of Subroutine 1:* If an obstruction through  $x$  and  $y$  exists, then  
 425  $x$  and  $y$  are in the same block (some authors use the term ‘biconnected component’). Since  
 426  $x$  and  $y$  are incident with the outer face, the outer boundary of the block containing  $x$  and  $y$   
 427 is the cycle  $C$  from Step 1. This  $C$  can be found by considering outer boundary walk  $W$  of  
 428  $G(\Sigma)$  and then by finding whether  $x$  and  $y$  belong to the same non-edge block of  $W$ . Finding  
 429  $W$  is  $O(m)$  and computing the blocks of  $W$  via a DFS takes  $O(m)$  time.

430 In Step 2, if there is a third outer rainbow  $z$  in  $C$ , then no obstruction through  $x$  and  $y$   
 431 contains  $z$ . Property (\*\*) justifies the update that takes  $O(m)$  time.

432 A full run from Step 1 to Step 2 takes  $O(m)$ . Moving from Step 2 to Step 1 occurs  $O(n)$   
 433 times. Thus, the total time for Subroutine 1 is  $O(mn) = O(n^2)$ .

434 **Subroutine 2.** *Detecting an obstruction through a single outer-rainbow  $x$ .*

- 435 (1) Find a cycle  $C$  through  $x$  whose edges are incident with the outer face of  $G(\Sigma)$ . If no  
 436 such  $C$  exists, output *No obstruction through  $x$* . Else, go to Step 2.
- 437 (2) Find whether there is an outer-rainbow  $y$  in  $V(C) \setminus \{x\}$ . If no such  $y$  exists, output  $C$ .  
 438 Else, apply Subroutine 1 to  $x$  and  $y$ ; if there is an obstruction  $C'$  through  $x$  and  $y$ , then  
 439 output  $C'$ . Else, update  $G(\Sigma) \leftarrow G(\Sigma - y)$  and go to Step 1.

440 *Correctness and running-time of Subroutine 2:* If  $G(\Sigma)$  has an obstruction through  $x$ , then  
 441 there is a non-edge block in  $G(\Sigma)$  containing  $x$ . The outer boundary of this block is a cycle  
 442  $C$  through  $x$  having all edges incident with the outer face. As in Subroutine 1, Step 1 takes  
 443  $O(m)$  time.

444 Detecting the existence of  $y$  in Step 2 is  $O(m)$  because to detect rainbows in  $C$ , each edge  
 445 incident with a vertex in  $V(C)$  is verified at most twice. The update in Step 2 is justified  
 446 by Property (\*\*). Since Step 2 may use Subroutine 1, Step 2 takes  $O(n^2)$  time. As moving  
 447 from Step 2 to Step 1 occurs  $O(n)$  times, the total running-time for Subroutine 2 is  $O(n^3)$ .

448 We are now ready for the algorithm to detect obstructions.

449 **Algorithm 1:** *Detecting obstructions in  $G(\Sigma)$ .*

- 450 (1) Find a cycle  $C$  having all edges incident with the outer face. If no such  $C$  exists, output  
 451 *No obstruction*. Else, go to step 2.
- 452 (2) Find whether there is an outer rainbow  $x \in V(C)$ . If not, output  $C$ . Else apply Subroutine  
 453 2 to  $x$ ; if there is an obstruction  $C'$  through  $x$ , output  $C'$ . Else, update  $G(\Sigma) \leftarrow G(\Sigma - x)$   
 454 and go to Step 1.

*Correctness and running-time of Algorithm 1:* If  $G(\Sigma)$  has an obstruction, then it has a non-trivial block whose outer boundary is a cycle  $C$  as in Step 1. As before,  $C$  and  $x$  as in Step 2 can be found in  $O(m)$  steps. If  $C$  has not outer rainbow  $x$ , then  $C$  is an obstruction; Property (\*\*) justifies the update in Step 2.

Since Step 2 may use Subroutine 2, a full run of Steps 1 and 2 takes  $O(n^3)$  time. Since Step 2 goes to Step 1  $O(n)$  times, the running-time of Algorithm 1 is  $O(n^4)$ .

Algorithm 1 and the constructive proof of Theorem 2 imply the following result (proved in the full version of this paper [4]).

► **Theorem 14.** *There is a polynomial-time algorithm to recognize and extend an ordinary set of strings that are extendible to an arrangement of pseudolines.*

## 6 Concluding remarks

In this work we characterized in Theorem 2 sets of strings that can be extended into arrangements of pseudolines. Moreover, we showed that the obstructions to pseudolinearity can be detected in  $O(n^4)$  time, where  $n$  is the number of vertices in the planarization of the set of strings.

An easy consequence of Theorem 2 is the following (presented before as Theorem 1). We prove this result in the full version of this paper [4].

► **Theorem 15.** *Let  $D$  be a non-pseudolinear good drawing of a graph  $H$ . Then there is a subset  $S$  of edge-arcs in  $\{D[e] : e \in E(H)\}$ , such that each  $\sigma \in S$  has a substring  $\sigma' \subseteq \sigma$  for which  $\bigcup_{\sigma \in S} \sigma'$  is one of the drawings represented in Figure 1.*

Theorem 2 can also be applied to find a short proof that pseudolinear drawings of  $K_n$  are characterized by forbidding the  $B$ -configuration (see Theorem 2.5.1 in [3]). This implies the characterizations of pseudolinear drawings of  $K_n$  presented in [2, 5, 6].

A drawing is *stretchable* if it is homeomorphic to a rectilinear drawing. There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 6. Nevertheless, as an immediate consequence of Thomassen's main result in [16], pseudolinear and stretchable drawings are equivalent, under the assumption that every edge is crossed at most once.

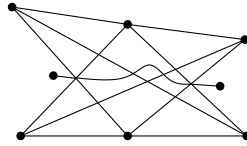


Figure 6 Non-Pappus configuration.

► **Corollary 16.** *A drawing  $D$  of a graph in which every edge is crossed at most once is stretchable if and only if it is pseudolinear.*

**Proof.** If a drawing  $D$  is stretchable then clearly it is pseudolinear. To show the converse, suppose that  $D$  is pseudolinear. Then  $D$  does not contain any obstruction, and in particular, neither of the  $B$ - and  $W$ -configurations in Figure 1 occurs in  $D$ . This condition was shown in [16] to be equivalent to being stretchable. ◀

One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-stretchable arrangements of pseudolines. However, such examples seem to inevitably have some edge with multiple crossings. This leads to a natural question.

▷ Question 17. Is it true that if  $D$  is a pseudolinear drawing in which every edge is crossed at most twice, then  $D$  is stretchable?

We believe that there are other instances where pseudolinearity characterizes stretchability of drawings. A drawing is *near planar* if the removal of one edge produces a planar graph. One instance, is the following result by Eades et al. that can be translated to the language of pseudolines:

► **Theorem 18.** [9] *A drawing of a near-planar graph is stretchable if and only if the drawing induced by the crossed edges is pseudolinear.*

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