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# Bar codes of persistent cohomology and Arrhenius law for $p$ -forms

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February 14, 2020

## Abstract

This article shows that counting or computing the small eigenvalues of the Witten Laplacian in the semi-classical limit can be done without assuming that the potential is a Morse function as the authors did in [LNV]. In connection with persistent cohomology, we prove that the rescaled logarithms of these small eigenvalues are asymptotically determined by the lengths of the bar code of the function  $f$ . In particular, this proves that these quantities are stable in the  $C^0$  topology on the space of functions. Additionally, our analysis provides a general method for computing the subexponential corrections in a large number of cases.

**MSC2010:** 57N65, 58J32, 58J37, 81Q10, 81Q20

**Keywords:** Exponentially small eigenvalues, Witten Laplacians, Arrhenius Law, Persistence.

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# 1 Introduction

## 1.1 Motivations

Since its discovery in the late nineteenth century, Arrhenius law (see [Arr]) is one of the most robust laws of chemistry or physics. Actually, its range of applications has increased over decades and is now also commonly used in biology or social sciences as an empirical law whose parameters can be figured out rather easily, even when the microscopic or individual mechanisms are not well understood. Its early interpretations were done within the thermodynamical or statistical physics framework. They are now formulated in the modern and general language of stochastic processes, more specifically of the Brownian motion of a particle evolving in a gradient field. At low temperature  $h > 0$  in some dimensionless scaling, the lifetime  $\tau_{\alpha,h}$  of the state  $\alpha$  is exponentially large with

$$\log \tau_{\alpha,h} \sim \frac{\ell_{\alpha}}{h}, \quad (1)$$

where  $\ell_{\alpha}$  is the energy variation between a local minimum and the lowest saddle point that we need to cross to reach a state of lower energy. Practically and as an illustration of the robustness of Arrhenius law, it is neither necessary to know the energy landscape nor the configuration space: in the end only the  $\ell_{\alpha}$ 's are important and they are determined experimentally, e.g. in chemistry kinetics. A general justification of (1) was proposed by Freidlin and Wentzell in [VeFr1, VeFr2] relying on large deviation arguments (see also [FrWe] and [Ber] for a wider overview).

In an energy landscape described by the function  $2f : M \rightarrow \mathbb{R}$ , those lifetimes are generically the inverses of eigenvalues of the operator  $-h\Delta + 2\nabla f \cdot \nabla$  in  $L^2(M, e^{-\frac{2f}{h}} dx)$ , where  $e^{-\frac{2f}{h}} dx$  is the associated invariant measure (it exists e.g. when  $M$  is a compact Riemannian manifold without boundary). After a conjugation by  $e^{\frac{f}{h}}$  and a multiplication by  $h$  (corresponding to a change of time scale), it becomes the operator

$$\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h(\Delta f)(x) = d_{f,h}^* d_{f,h} \quad \text{acting in } L^2(M, dx),$$

where  $d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}}$  is the Witten differential and  $d_{f,h}^*$  its adjoint. This operator acts on general differential forms as the Witten Laplacian, a deformation of the Hodge Laplacian:

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 = \bigoplus_{p=0}^{\dim M} \Delta_{f,h}^{(p)},$$

where the direct sum separates the degrees. When  $f$  is a Morse function, Witten in [Wit] (see also [CFKS]) proved that as  $h$  goes to zero, the eigenvalues of  $\Delta_{f,h}$  are divided into two groups, given in our scaling as one bounded from below by  $C_f h$  for some  $C_f > 0$ , and one being of the order  $o(h)$ . The small (here  $o(h)$ ) eigenvalues of  $\Delta_{f,h}^{(p)}$  correspond to critical points of index  $p$ : this is intuitively to be expected, since the eigenfunctions should concentrate in the region where  $|\nabla f|$  is small, that is near the critical points of  $f$ . This argument provided an analytical proof of Morse inequalities, in the line of several results relating topological quantities and spectral analysis, one of the earliest being the Atiyah-Patodi-Singer proof of the index theorem (see [APS]).

In [HeSj4], Helffer and Sjöstrand gave a rigorous proof of Witten's claims and proved that those small eigenvalues were actually exponentially small, without specifying their size. This was later extended to Morse-Bott functions by Bismut and Helffer-Sjöstrand (see [Bis] and [HeSj6]). After this, many applications of Witten Laplacians or more general Witten deformations were used to study various global topological invariants of manifolds or fibre bundles by counting the small eigenvalues of such operators (see e.g. [BiZh, Zha, ChLi]).

When  $f$  is a Morse function, the Arrhenius law in degree 0 says that the  $o(h)$  eigenvalues of  $\Delta_{f,h}^{(0)}$  satisfy

$$\log \lambda_{\alpha,h}^{(0)} \sim -2 \frac{f(y_\alpha) - f(x_\alpha)}{h} \quad \text{as } h \rightarrow 0^+, \quad (2)$$

where  $x_\alpha$  is a local minimum and  $y_\alpha$  is an associated saddle point. Already around 1935, Eyring and Kramers (see [Eyr, Kra]), motivated by the theory of the activated complex in chemistry, proposed a more accurate version which reads here

$$\lambda_{\alpha,h}^{(0)} \sim \frac{h}{\pi} C_\alpha e^{-2 \frac{f(y_\alpha) - f(x_\alpha)}{h}} \quad \text{as } h \rightarrow 0^+, \quad (3)$$

where the constant  $C_\alpha$  depends on the Hessians at the non degenerate critical points  $x_\alpha$  and  $y_\alpha$ ,  $x_\alpha$  is a local minimum (here a critical point of index 0), and  $y_\alpha$  a saddle point (here a critical point of index 1).

The first mathematical proof of the Eyring-Kramers formula was performed in degree 0 in [BEGK, BGK] by using potential theoretic and capacity arguments, and in [HKN] by improving Helffer-Sjöstrand's semiclassical analysis for  $\Delta_{f,h}^{(0)}$  (see also e.g. the prior works [HKS, Micl] for results less precise than (3) but more precise than (2)). These results were proved under the assumption that  $f$  is a Morse function with simple local minima and simple saddle points (a Morse function has simple critical values or critical points if every critical value is the image of a single critical point), and with distinct lengths : the real numbers  $\ell_\alpha = 2(f(y_\alpha) - f(x_\alpha))$  are all distinct. The pairing between local minima  $x_\alpha$  and saddle points  $y_\alpha$  (critical points with index 1) was done by extending the intuitive picture of basins of attraction, more precisely by considering the connected components of sublevel sets of  $f$ . Note that this differs from the instantonic picture, associated with curves which are intersections of stable and unstable manifolds of  $-\nabla f$ , which is in some sense local and would lead to a complicated analysis of cancellations while computing precisely the  $\lambda_{\alpha,h}^{(0)}$ 's. This pairing relies on global topological considerations which are robust with respect to the  $C^0$  perturbations of the energy profile  $2f$ . By making use of the min-max principle, it is actually not difficult to start from the analysis done in [HKN] for Morse functions and to recover (2) and the results of [VeFr1, VeFr2, HKS, Micl] in cases where the local minima are degenerate.

The situation is completely different for general differential forms of degree  $p$ . In [LNV], we proved an Eyring-Kramers law (and therefore an Arrhenius law) by assuming again that the function  $f$  was a generic Morse function with simple critical values and such that the difference between critical values were all distinct. Here the problem is to understand which critical values  $f(x_\alpha)$  and  $f(y_\alpha)$  are paired in order to compute the exponential factors. This pairing is obtained topologically by using a refinement of Barannikov's presentation of Morse theory. This can be restated in modern terms with the bar code of  $f$ , denoted  $\mathcal{B}_f = ([a_\alpha^*, b_\alpha^{*+1}]_{\alpha \in A^*})$ , associated with the Morse function  $f$  on  $M$ , with the notation  $a_\alpha^{(p)} = f(x_\alpha)$  and  $b_\alpha^{(p+1)} = f(y_\alpha)$ , where the critical point  $x_\alpha$  has index  $p$  and  $y_\alpha$  has index  $p+1$ . Later, it was noticed in [UsZh, PoSh] that those bar codes were nothing but the bar codes of persistent homology, developed since the beginning of the 21st century (see [EdHa] for a historical review). An important feature of the Barannikov complex, and hence of persistent homology, is the stability result which says in the latter framework

$$d_{bot}(\mathcal{B}_f, \mathcal{B}_g) \leq \|f - g\|_{C^0},$$

where the bottleneck distance  $d_{bot}$  estimates the variations of the lengths of the bars.

But the bar code of a function is defined for any continuous function, except the bars are now infinitely many, with the property that for any  $\varepsilon_0 > 0$ , only finitely many are greater than  $\varepsilon_0$ . It is then natural to state the following conjecture.

**Main Conjecture :** Consider a  $C^\infty$  (or even Lipschitz) function  $f$  on a compact manifold  $M$  with bar code  $\mathcal{B}_f$ . We denote by  $A^{(p)}(\ell)$  the set of bars in  $\mathcal{B}_f$  of the type  $[a_\alpha^{(p)}, b_\alpha^{(p+1)}[$  with  $b_\alpha^{(p+1)} - a_\alpha^{(p)} > \ell$ , and  $A_c^{(p-1)}(\ell)$  the set of bars in  $\mathcal{B}_f$  of the type  $[a_\alpha^{(p-1)}, b_\alpha^{(p)}[$  with  $b_\alpha^{(p+1)} - a_\alpha^{(p)} > \ell$  and  $b_\alpha^{(p)} < +\infty$ . Then, there exists  $\varepsilon_0 > 0$  such that, for every  $\varepsilon \in ]0, \varepsilon_0]$ ,  $\Delta_{f,h}^{(p)}$  admits  $\sharp(A^{(p)}(\ell) \cup A_c^{(p-1)}(\ell))$  eigenvalues  $\lambda_{\alpha,h}^{(p)}$  smaller than  $e^{-2\frac{\ell+\varepsilon}{h}}$  (with multiplicity), where  $\alpha \in A^{(p)}(\ell) \cup A_c^{(p-1)}(\ell)$ . They can moreover be labelled such that

$$\forall \alpha \in A^{(p)}(\ell) \cup A_c^{(p-1)}(\ell), \quad \log \lambda_{\alpha,h}^{(p)} \sim -2 \frac{b_\alpha^{(p+1)} - a_\alpha^{(p)}}{h} \quad \text{as } h \rightarrow 0^+.$$

The goal of this paper is to prove this conjecture under the assumption that  $f$  has a finite number of critical values.

Note that we do not assume in the Main Conjecture (as well as in our theorems) that  $f$  is Morse. One important consequence of the Main Conjecture (and hence of our main theorems) is that the decay rate of the eigenvalues is continuous in  $f$  for the  $C^0$  topology. This is not the case for subexponential factors, since they usually depend on the eigenvalues of the Hessian of  $f$  at the critical points.

In the case  $p = 0$  of functions, the Eyring-Kramers law (3) has been extended in the form  $\lambda_{\alpha,h}^{(0)} \sim C_\alpha(f) h^{\nu_\alpha(f)} e^{-2\frac{f(y_\alpha) - f(x_\alpha)}{h}}$  when  $f$  is not a Morse function or when  $f$  is a Morse function with multiple critical values (i.e. the preimage of a critical value may contain several critical points), the latter appearing in practical situations with natural symmetries. We refer for example to [BeGe, BeDu, Mic, DLLN2, LeNe1, LeNe2], whence it appears that the exponent  $\nu_\alpha(f)$ , or the constant  $C_\alpha(f)$  in the subexponential factor, may be discontinuous when a general function  $f$  is approximated by a sequence of generic Morse functions. On the other hand, it will follow from our results that the  $\ell_\alpha = 2(f(y_\alpha) - f(x_\alpha))$  are stable. Understanding how the eigenvalues  $\lambda_{\alpha,h}(f)$  or the lifetimes  $\tau_{\alpha,h}(f)$  depend on  $f$  is also important for applications to acceleration of stochastic algorithms (see [LeNi, DLLN1, DLLN2, LeNe1, LeNe2] and references therein). This leads to the

**Main Question :** *Is there a way to analyze how the subexponential factor of Eyring-Kramers law for  $p$ -forms varies when  $f$  is changed ? In particular, does it explain the observed discontinuities ?*

Again, the answer is yes. Our presentation of Arrhenius law for  $p$ -forms provides a very general result. The method actually completely separates the determination of the exponential scales  $e^{-\frac{\ell_\alpha}{h}}$ , related with global algebraic topological objects, from the determination of the subexponential factors, which rely on some local analysis. Many applications with various discontinuous effects will be presented at the end of this text. Actually, the discontinuities w.r.t. the energy landscape  $f$  of the leading term for the subexponential factor  $C_\alpha(f) h^{\nu_\alpha(f)}$  are easily understood on the simple example of the Laplace integrals

$$I(\delta, h) = \int_{\mathbb{R}} e^{-\frac{x^4/4 - \delta x^2/2 + 1_{\mathbb{R}^+}(\delta)\delta^2/4}{h}} dx,$$

$$\begin{aligned} \text{which satisfy} \quad & I(\delta, h) \stackrel{h \rightarrow 0}{\sim} C_\delta h^{1/2} && \text{when } \delta \neq 0, \\ \text{and} \quad & I(\delta, h) \stackrel{h \rightarrow 0}{\sim} C h^{1/4} && \text{when } \delta = 0. \end{aligned}$$

## 1.2 General assumptions and notations

**The manifold  $M$ :** The Riemannian manifold  $(M, g)$  is assumed compact without boundary with  $\dim_{\mathbb{R}} M = d$  and non necessarily oriented. Some non compact manifold will be considered

in Subsection 8.2. In the non-orientable case, the Hodge star operator,  $\star$ , sends  $\Lambda T^*M = \bigoplus_{p=0}^d \Lambda^p T^*M$  to  $\Lambda T^*M \otimes_M \text{or}_M$ , where  $\text{or}_M$  is the orientation (line) bundle, which is of course locally trivial. When  $N \subset M$  is a regular hypersurface admitting a global unit normal (or conormal) vector the orientation twist  $\text{or}_N$  is the restriction of  $\text{or}_M$ .

In local coordinates the metric will be written  $g = g_{ij}(x)dx^i dx^j$  with  $g^{-1} = g^{ij}(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$  and the musical isomorphisms  $\sharp : T^*M \rightarrow TM$  and  $\flat : TM \rightarrow T^*M$  are given by

$$(\omega_i dx^i)^\sharp = g^{ij} \omega_j \frac{\partial}{\partial x^i} \quad \text{and} \quad (X^i \frac{\partial}{\partial x^i})^\flat = g_{ij} X^j dx^i.$$

The differential  $d$  acts on  $\mathcal{C}^\infty(M; \Lambda T^*M \otimes_M \mathbb{C})$  or  $\mathcal{D}'(M; \Lambda T^*M \otimes_M \mathbb{C})$  and augments the degree of forms by 1. The codifferential  $d^* = (-1)^{\deg} \star^{-1} d \star$  acts on  $\mathcal{C}^\infty(M; \Lambda T^*M \otimes_M \mathbb{C})$  and  $\mathcal{D}'(M; \Lambda T^*M \otimes_M \mathbb{C})$  and decreases the degree by 1. In the sequel and unless otherwise specified, we always consider complex valued differential forms and the tensorization by  $\mathbb{C}$  will be omitted in the notation. The duality bracket  $\langle \cdot, \cdot \rangle$  between  $\mathcal{D}'(M; \Lambda^p T^*M \otimes_M \mathbb{C})$  and  $\mathcal{C}^\infty(M; \Lambda^{d-p} T^*M)$  (where  $\mathcal{D}'$  and  $\mathcal{C}^\infty$  can be interchanged) is assumed  $\mathbb{C}$ -antilinear on the left-hand side and  $\mathbb{C}$ -linear on the right-hand side. Stokes's formula then implies that  $d^*$  is the formal adjoint of  $d$  according to

$$0 = \int_M d(\bar{\omega} \wedge \star \eta) = \int_M d\bar{\omega} \wedge (\star \eta) + (-1)^{\deg \omega} \bar{\omega} \wedge d(\star \eta) = \langle d\omega, \eta \rangle - \langle \omega, d^* \eta \rangle$$

for  $\omega, \eta \in \mathcal{C}^\infty(M; \Lambda^{p-1} T^*M)$ .

**Functional spaces:** The  $L^2$ -norm of sections of  $\Lambda T^*M$  is the one given by the metric  $g$  and we recall

$$\int_M \langle \omega, \eta \rangle_{\Lambda T^*_q M} d\text{vol}_g(q) = \int_M \bar{\omega} \wedge \star \eta.$$

We use the notation  $W^{s,p}$  for the Sobolev space with  $s$  derivatives in  $L^p$ . In particular,  $W^{s,2}$  corresponds to the standard Hilbertian Sobolev spaces while  $W^{1,\infty}$  will be used for the set of Lipschitz functions. For an open domain  $\Omega \subset M$  and for  $s \in \mathbb{R}$ , the notation  $W^{s,2}(\bar{\Omega}; \Lambda T^*M)$  denotes the set of restrictions to  $\Omega$  of  $W^{s,2}$ -sections in  $M$ , and when there is no ambiguity or necessity, we shall use the short version  $W^{s,2}(\bar{\Omega})$ . The same definition holds for  $\mathcal{C}^\infty(\bar{\Omega}; \Lambda T^*M)$ . We recall that when  $\Omega$  is a regular domain, that is when  $\partial\Omega$  is a  $\mathcal{C}^\infty$  hypersurface,  $W^{s,2}(\bar{\Omega}; \Lambda T^*M)$  coincides with  $W^{s,2}(\Omega; \Lambda T^*M)$  by interpolation and duality from the special cases of  $s \in \mathbb{N}$  (see e.g. [ChPi]). In such a case, the trace theorem holds from  $W^{s,2}(\Omega; \Lambda T^*M)$  to  $W^{s-1/2,2}(\Omega; \Lambda T^* \partial\Omega)$  for  $s > \frac{1}{2}$ . The local regularity theory is not affected when sections of  $\Lambda T^*M \otimes \text{or}_M$  and  $\Lambda T^*M \otimes \text{or}_{\partial\Omega}$  are considered and we shall use the short notation  $W^{s,2}(\bar{\Omega})$  or  $W^{s,2}(\partial\Omega)$  indifferently for sections of the trivial and orientation line bundles, unless we need to distinguish the global behaviour. Other functional spaces will be introduced later in our analysis.

**Witten differential and Witten Laplacian:** The Witten differential and the Witten Laplacian are deformations of the differential  $d$  and the Hodge Laplacian  $dd^* + d^*d$  associated with a real valued function  $f$  and a positive parameter  $h > 0$  in the asymptotics  $h \rightarrow 0$ .

**Definition 1.1.** *Let  $f$  be a real valued function on  $M$ . For  $a \in \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ , we use the notations*

$$\begin{aligned} f^a &= \{x \in M, f(x) < a\} \quad , \quad f^{\leq a} = \{x \in M, f(x) \leq a\} \quad , \\ f_a &= \{x \in M, f(x) > a\} \quad , \quad f_{\geq a} = \{x \in M, f(x) \geq a\} \quad , \end{aligned}$$

with all the combinations like  $f_a^b = \{x \in M, a < f(x) < b\}$ .

Although weaker regularity assumptions for the function  $f$  will be discussed later, the following simple hypothesis will be convenient for us.

**Hypothesis 1.2.** *The function  $f$  on  $(M, g)$  is assumed to be Lipschitz with a finite number  $N$  of values  $c_1, \dots, c_{N_f}$  such that:*

- $f \in C^\infty(M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\}); \mathbb{R})$
- $\forall x \in M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\}), \quad |\nabla f(x)| \neq 0.$

When  $f \in C^\infty(M; \mathbb{R})$ , the above assumption simply says that  $f$  has a finite number  $\leq N_f$  of critical values. For a Lipschitz function, we count also “fake” critical values allowing singularities of  $f$  at those values. We nevertheless call  $c_1, \dots, c_{N_f}$  the “critical values” of  $f$  and use the notation

$$M_{reg} = \{x \in (M \setminus \text{suppsing } f), \nabla f(x) \neq 0\} \subset M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\}).$$

When  $M$  is a real analytic manifold, Hypothesis 1.2 may be replaced by the following simpler natural assumption.

**Hypothesis 1.3.** *On the real analytic compact Riemannian manifold  $M$ ,  $f$  is a Lipschitz subanalytic function.*

Actually, the proof of the main result, Theorem 6.3, will hold under Hypothesis 1.2 or under some milder assumptions which are more technical and will appear as consequences of Hypothesis 1.3 in Subsection 8.3. We also refer to Subsection 8.3 for more material on Lipschitz subanalytic functions.

Under Hypothesis 1.2 or more generally for a Lipschitz function  $f$  and for  $h > 0$ , the differential operators  $d_{f,h}$ ,  $d_{f,h}^*$  and  $\Delta_{f,h}$  are defined by:

$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df \wedge, \quad d_{f,h} \circ d_{f,h} = 0, \quad (4)$$

$$d_{f,h}^* = e^{\frac{f}{h}}(hd^*)e^{-\frac{f}{h}} = hd^* + \mathbf{i}_{\nabla f}, \quad d_{f,h}^* \circ d_{f,h}^* = 0, \quad (5)$$

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^* = h^2 \Delta_{0,1} + |\nabla f(x)|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*). \quad (6)$$

The above identities make sense when considering  $d_{f,h}$  and  $d_{f,h}^*$  as operators from  $W^{1,2}(M)$  to  $L^2(M)$  or from  $L^2(M)$  to  $W^{-1,2}(M)$ , and for the compositions of two of them and for  $\Delta_{f,h}$ , as operators from  $W^{1,2}(M)$  to  $W^{-1,2}(M)$ . We shall be more precise on requirements for domains once we add the boundary conditions.

**Convention for closed operators and quadratic forms:**

We shall consider various closed realizations in  $L^2$  spaces of the above differential operators  $d_{f,h}$ ,  $d_{f,h}^*$ , and  $\Delta_{f,h}$ , which will be denoted  $d_{f,\bullet,h}$ ,  $d_{f,\bullet,h}^*$ , and  $\Delta_{f,\bullet,h}$ , where the subscript  $\bullet$  will specify the realization. When  $A$  is a closed operator in a Hilbert space (resp. when  $Q$  is a closed quadratic form), writing  $Au$  (resp.  $Q(u)$  or  $Q(u, v)$  for the associated sesquilinear form) means that  $u$  belongs to the domain of  $A$  (resp.  $u$  or  $u, v$  belong to the domain of  $Q$ ). For example  $d_{f,\bullet,h}\omega = \alpha \in L^2$  means in particular  $\omega \in D(d_{f,\bullet,h})$ , possibly imposing boundary conditions.

**Comparing exponential scales:**

**Definition 1.4.** *For two functions  $F, G : ]0, h_0[ \rightarrow \mathbb{C}$ , one says*

- $F(h) = \tilde{O}(G(h))$  if:

$$\forall \varepsilon > 0, \exists h_\varepsilon, C_\varepsilon > 0, \forall h \in ]0, h_\varepsilon[, \quad |F(h)| \leq C_\varepsilon |G(h)| e^{\frac{\varepsilon}{h}};$$

- $F(h) = \tilde{o}(G(h))$  if:

$$\exists \varepsilon, h_\varepsilon, C_\varepsilon > 0, \forall h \in ]0, h_\varepsilon[, \quad |F(h)| \leq C_\varepsilon |G(h)| e^{-\frac{\varepsilon}{h}};$$



- $F(h) \stackrel{\log}{\sim} G(h)$  if:

$$|F(h)| = \tilde{O}(|G(h)|) \quad \text{and} \quad |G(h)| = \tilde{O}(|F(h)|).$$

When  $|F|, |G| > 0$ , the above three conditions can be written respectively

$$\begin{aligned} \limsup_{h \rightarrow 0} h \log \left( \frac{|F(h)|}{|G(h)|} \right) &\leq 0, \\ \limsup_{h \rightarrow 0} h \log \left( \frac{|F(h)|}{|G(h)|} \right) &< 0 \\ \lim_{h \rightarrow 0} h \log \left( \frac{|F(h)|}{|G(h)|} \right) &= 0. \end{aligned}$$

In the two first definitions, the constant  $C_\varepsilon$  can be fixed to 1 by changing  $h_\varepsilon$  (and  $\varepsilon$  in the second definition).

When  $F : X \times ]0, h[ \rightarrow \mathbb{C}$ , the statements “ $F(x, h) = \tilde{O}(G(h))$  (or  $F(x, h) = \tilde{o}(G(h))$ ) (locally) uniformly” are used when the above definitions make sense for the corresponding suprema  $\sup_x F(x, h)$ .

### Bar code:

Although a more precise definition and construction will be recalled especially in Appendix B, we can start with a short definition.

**Definition 1.5.** Under Hypothesis 1.2, a (persistence cohomology) bar code associated with  $f$  is a finite family  $\mathcal{B} = ([a_\alpha, b_\alpha])_{\alpha \in A}$  with  $-\infty < a_\alpha < b_\alpha \leq +\infty$ ,  $a_\alpha \in \{c_1, \dots, c_{N_f}\}$ ,  $b_\alpha \in \{c_2, \dots, c_{N_f}, +\infty\}$ , with the following properties:

- it is graded according to  $A = \sqcup_{p=0}^d A^{(p)}$ ,  $[a_\alpha, b_\alpha[ = [a_\alpha^{(p)}, b_\alpha^{(p+1)}[$  when  $\alpha \in A^{(p)}$ ;
- for any pair  $a, b$ ,  $a < b$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$ , there exists a basis of the relative homology vector space  $H^p(f^b, f^a)$  indexed by the bars of degree  $p$  with a unique endpoint lying in  $]a, b[$ . In particular, the relative Betti number is given by:

$$\beta^p(f^b, f^a) = \dim H^p(f^b, f^a) = \# \left\{ \alpha \in A^{(p)}, \# \left\{ a_\alpha^{(p)}, b_\alpha^{(p+1)} \right\} \cap ]a, b[ = 1 \right\}.$$

For a general Lipschitz function, such a finite bar code is well defined under the following assumption (see Subsection 8.3.1 and Appendix B).

**Hypothesis 1.6.** The function  $f : M \rightarrow \mathbb{R}$  is a Lipschitz function and there exists a finite number of values  $c_1 < c_2 < \dots < c_{N_f}$  such that for any  $a \in \mathbb{R} \setminus \{c_1, \dots, c_{N_f}\}$ , the following property holds along  $f^{-1}(\{a\})$ :

For any  $x_0 \in f^{-1}(\{a\})$ , there exists a neighborhood  $U_{x_0}$  of  $x_0$  in  $M$ , a local coordinate system  $x = (x^1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ , and a constant  $C_{x_0}$  such that

$$\forall x = (x^1, x'), y = (y^1, x') \in U_{x_0}, \quad \frac{1}{C_{x_0}} |x^1 - y^1| \leq |f(x^1, x') - f(y^1, x')|.$$

This notion of bar code, and especially the identification of two bar codes, after possibly adding empty intervals, is better understood after associating with a bar code  $\mathcal{B}_A = ([a_\alpha, b_\alpha])_{\alpha \in A}$  the constructible sheaf  $\oplus_{\alpha \in A} \mathbb{K}_{[a_\alpha, b_\alpha[}$  of  $\mathbb{K}$ -vector spaces, on  $\mathbb{R}$ . Then, a persistence bar code associated with a function  $f$  satisfying Hypothesis 1.2 is essentially unique and then denoted  $\mathcal{B}(f)$ .

After possibly adding empty bars such that  $a_\alpha = b_\alpha$  or  $c_\beta = d_\beta$ , two different bar codes  $\mathcal{B}_A = ([a_\alpha, b_\alpha])_{\alpha \in A}$  and  $\mathcal{B}_B = ([c_\beta, d_\beta])_{\beta \in B}$  can be assumed with the same cardinality,  $\#A = \#B$ . The bottleneck distance is then defined by

$$d_{\text{bot}}(\mathcal{B}_A, \mathcal{B}_B) = \inf_{j: A \xrightarrow{\text{bij}} B} \max_{\alpha \in A} \max(|a_\alpha - c_{j(\alpha)}|, |b_\alpha - d_{j(\alpha)}|),$$

with the convention  $|(+\infty) - (+\infty)| = 0$ .

The stability theorem for persistent (co)homology (see e.g. [CEH, KaSc]) says that for two functions  $f_1, f_2$  which satisfy Hypothesis 1.2 or Hypothesis 1.6,

$$d_{\text{bot}}(\mathcal{B}(f_2), \mathcal{B}(f_1)) \leq \|f_2 - f_1\|_{C^0}.$$

### 1.3 Simple results

The method presented in this text leads to several results and can actually be extended to other cases. Essentially, we show that the usual generic assumption that the function  $f$  is a Morse function can be replaced by a very general one, after replacing the algebraic topological information in terms of Morse indices by the ones given by the persistent cohomology bar code associated with  $f$ . The following simple statements illustrate what can be obtained.

**Theorem 1.7.** *Assume that  $f$  satisfies Hypothesis 1.2 and let  $\Delta_{f,M,h}$  be the self-adjoint Witten Laplacian defined with  $D(\Delta_{f,M,h}) = \{\omega \in W^{1,2}(M), \Delta_{f,h}\omega \in L^2(M)\}$  and  $\Delta_{f,M,h}\omega = \Delta_{f,h}\omega$  according to (6), and  $\Delta_{f,M,h} = \bigoplus_{0 \leq p \leq d} \Delta_{f,M,h}^{(p)}$ . Let  $\mathcal{B}(f)$  be a persistent cohomology bar code associated with  $f$ . Then, there is a bijection between  $A^{(p)} \sqcup \{\alpha \in A^{(p-1)}, b_\alpha^{(p)} \neq +\infty\}$  and the  $\tilde{o}(1)$  eigenvalues counted with multiplicities of  $\Delta_{f,M,h}^{(p)}$ . Precisely, there exists  $\varepsilon_0 > 0$  small enough such that for all  $\varepsilon \in ]0, \varepsilon_0[$ , there exists  $h_\varepsilon > 0$  such that the  $\tilde{O}(e^{-\frac{\varepsilon}{h}})$ -eigenvalues of  $\Delta_{f,h}^{(p)}$  counted with multiplicity for  $h \in ]0, h_\varepsilon[$  are given by  $\lambda_\alpha^{(p)}(h)$ ,  $\alpha \in A^{(p)}$  or  $(\alpha \in A^{(p-1)}$  and  $b_\alpha^{(p)} \neq +\infty)$ , with*

$$\begin{aligned} \text{either } b_\alpha^{(p+1)} = +\infty, & \quad \text{and then} & \quad \lambda_\alpha^{(p)}(h) = 0, \\ \text{or } b_\alpha^{*+1} < +\infty, & \quad \text{and then} & \quad \lim_{h \rightarrow 0} -h \log \lambda_\alpha^{(p)}(h) = 2(b_\alpha^{*+1} - a_\alpha^*). \end{aligned}$$

Obviously, the multiplicity of the 0-eigenvalue of  $\Delta_{f,M,h}^{(p)}$ , the dimension of its kernel, equals the  $p^{\text{th}}$  Betti number of  $M$ ,  $\#\{\alpha \in A^{(p)}, b_\alpha^{(p+1)} = +\infty\} = \beta^{(p)}(M)$ , and does not depend on the function  $f$ .

To summarize the situation, the logarithms of the exponentially small eigenvalues of  $\Delta_{f,M,h}^{(p)}$  are given by the lengths of the bars  $b_\alpha^{*+1} - a_\alpha^*$  of which one endpoint in  $\mathbb{R}$  is of degree  $p$ , the eigenvalues associated with infinite lengths being identically 0 for  $h$  small enough. A direct application of the stability results of persistent cohomology then gives the variations of the exponentially small spectrum when the function  $f$  is perturbed.

**Corollary 1.8.** *Assume that  $f$  satisfies Hypothesis 1.2, let  $\mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A}, a_\alpha < b_\alpha, A = \sqcup_{0 \leq p \leq d} A^{(p)})$ , be a persistent bar code associated with  $f$ , and set  $\ell_{\min} = \min\{b_\alpha - a_\alpha, \alpha \in A\}$ . For any other function  $g$  which satisfy Hypothesis 1.2 with  $\|g - f\|_{C^0} < \frac{\ell_{\min}}{4}$ , the  $\tilde{O}(e^{-\frac{\ell_{\min}}{h}})$  eigenvalues of  $\Delta_{g,M,h}^{(p)}$  can be labelled with multiplicities*

$$\lambda_\alpha(g, h), \quad \alpha \in A^{(p)} \text{ or } \alpha \in A^{(p-1)}, b_\alpha^{(p)} \neq +\infty,$$

with

$$\begin{aligned} \lambda_\alpha(g, h) = 0 & \quad \text{if } b_\alpha^{(p+1)} = +\infty \\ \text{or } 2(b_\alpha - a_\alpha) + 4\|g - f\|_{C^0} & \geq \lim_{h \rightarrow 0} -h \log(\lambda_\alpha(g, h)) \geq 2(b_\alpha - a_\alpha) - 4\|g - f\|_{C^0} > \ell_{\min}. \end{aligned}$$

One rapidly realizes that we make no normal form assumption for  $f$  near the ‘‘critical values’’  $c_1, \dots, c_{N_f}$  of  $f$ . Even if we work with  $C^\infty$  functions, any closed set  $K$  of  $M$  can be the global

minimum of  $f \in \mathcal{C}^\infty(M)$  by taking a non negative  $\mathcal{C}^\infty$  function vanishing only on  $K$  after Whitney's extension theorem. Having a finite number of critical values restricts the possible sets  $K$  which still make a very large class. Hence, no algebraic behaviour with respect to  $h$  of the leading terms of the subexponential factors can be expected as it is the case when  $f$  is assumed to be a Morse function. Theorem 1.7 simply says that exponentially small eigenvalues and their exponential scales are given by the algebraic topology without specifying a possible subexponential factor. Among other results, we will obtain similar things for  $\Delta_{f,f^{-1}([a,b]),h}$ ,  $-\infty \leq a < b \leq +\infty$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$ , when considering the proper boundary conditions on  $f^{-1}(\{a\})$  (Dirichlet type) and  $f^{-1}(\{b\})$  (Neumann type). Actually, this leads us to the presentation of our strategy which passes through local problems on  $\mathbb{R} = f(M)$  and a recurrence argument on the number  $N$  of "critical values" lying in  $[a, b]$ .

## 1.4 Strategy and outline of the article

Proving a result like Theorem 1.7, even in this simplified form, is a rather long process which is clearly split into various steps.

- A general presentation of bar codes in persistent (co)homology as well as properties of Hodge Laplacians on weakly regular domains are recalled in Appendix B and in Appendix A.
- In Section 2, we set up the functional analysis framework, the relevant boundary conditions for Witten Laplacians, the corresponding integration by parts formulas, as well as weighted integration techniques *à la* Agmon, in order to obtain exponential decay estimates. We especially consider self-adjoint realizations of Witten Laplacians  $\Delta_{f,h}$  in the domain  $f^{-1}([a, b])$  when  $a < b$  are not critical values, always with Dirichlet boundary conditions along  $f^{-1}(\{a\})$ , the lowest level set, and Neumann boundary conditions along  $f^{-1}(\{b\})$ , the highest level set. Those self-adjoint realizations will be denoted by  $\Delta_{f,f^{-1}([a,b]),h}$ , and possibly  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  when specifying the degree. Remember the intuitive picture for functions: Dirichlet (resp. Neumann) boundary conditions are associated with a potential  $-\infty$  (resp.  $+\infty$ ). Such boundary conditions are actually the natural ones in order to avoid boundary layer phenomena along the boundaries in the spectral analysis. For further applications, this analysis is done in a weak regularity framework, and the long series of works by Mitrea *et al.* were instrumental in setting up the proper framework. The end of this section gathers repeatedly used technical lemmas, deduced from the exponential decay and weighted resolvent estimates for boundary Witten Laplacians.
- Once the geometrical issues in the weak regularity case are solved, the rest of the analysis becomes essentially one dimensional on  $\mathbb{R} \supset f(M)$ , as suggested by the bar code structure. The first step consists in understanding what happens when there is a single critical value in the energy interval  $[a, b]$ ,  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1\}$ . In this specific case, the bar code of  $f$  has no bar compactly included in  $]a, b[$ . Accordingly,  $\Delta_{f,f^{-1}([a,b]),h}$  should not have any non zero exponentially small eigenvalue. This is the main result of Section 3.2, formulated in Proposition 3.2, which states that all the  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  are equal to 0. After preliminary notations related with variations of the min-max principle, the core of the proof is done in Subsection 3.2, and follows in some sense Carleman's general scheme for uniqueness results of PDE, along the energy interval  $[a, b] \subset \mathbb{R}$ . Resolvent estimates and other corollaries are listed afterwards. Section 2 and Proposition 3.2 provide in particular the number of  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  counted with multiplicities in this setting: it equals the relative Betti number

$$\beta(f^b, f^a; \mathbb{R}) = \dim \ker(\Delta_{0,f^{-1}([a,b]),1}^{(p)}) = \dim \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)}).$$

- Only in Section 4 really starts the relationship between the bar code  $\mathcal{B}_f$  of  $f$  and the spectral properties of  $\Delta_{f,f^{-1}([a,b]),h}$ . It contains an enumeration of the non zero  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}$  in terms of bars compactly embedded in  $]a,b[$ , while the dimension  $\dim \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) = \beta^{(p)}(f^b, f^a; \mathbb{R})$  is also expressed in terms of  $\mathcal{B}_f$ . This section ends with Proposition 4.5 which proves the rough lower bound  $e^{-2\frac{b-a}{h}}$  for the non zero  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}$  (see Proposition 4.5).
- An important step elucidated in [HKN], and used in many forthcoming articles, consisted in the trivial observation that the eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}$ , restricted to some spectral compact segment, are the square of the singular values of the restricted differential  $d_{f,f^{-1}([a,b]),h}$ . Singular values are much more flexible spectral quantities than eigenvalues. One of their advantage is that, in many situations, the approximation errors appear as relative ones for all the singular values, a property which is not fulfilled by eigenvalues. We gather several functional analysis preliminary results in Section 5, which elaborates in a functional abstract setting how various matricial error estimates propagate nicely to singular values estimates.
- The core of the proof of Theorem 1.7 is done in Section 6. It is a rather sophisticated proof by induction on the number  $N$  of critical values contained in the energy interval  $[a,b]$ ,  $\{c_1, \dots, c_{N_f}\} \cap [a,b] = \{\tilde{c}_1, \dots, \tilde{c}_N\}$ . This recurrence is initiated by Section 3.2 for the case  $N = 1$ . Although it contains several steps, the induction from  $N$  to  $N + 1$  mimics in some way the proof of Mayer-Vietoris' Theorem. The main result of this section is Theorem 6.3, which can be considered as the central result of this text, while Proposition 3.2 proves the simplest non trivial particular case. This induction contains many intermediate results, which lead in particular in Section 7 to Theorems 7.1 and 7.6, which generalize respectively Theorem 1.7 and Corollary 1.8 to the boundary Witten Laplacian  $\Delta_{f,f^{-1}([a,b]),h}$ .
- Section 8 is devoted to various generalizations of Theorem 6.3 and of its spectral corollaries. The first one concerns results for some domains which are not bounded by level sets, e.g. for (non necessarily) small deformations  $N_t$  and  $N_n$  of the level sets  $f^{-1}(\{a\})$  and  $f^{-1}(\{b\})$  for which the conditions  $\partial_n f|_{N_t} < 0$  and  $\partial_n f|_{N_n} > 0$  are still valid, and for which all the conclusions of Theorem 6.3 and of its corollaries hold true. The second generalization is about noncompact manifolds like  $\mathbb{R}^d$ , for which the results still hold provided we make some assumptions on  $M$  and  $f$  at infinity. The most technical one concerns the extension to a general subanalytic Lipschitz function on a real analytic manifold (see Hypothesis 1.3). Even when  $f$  is a subanalytic real Lipschitz function, it is possible to define self-adjoint realizations  $\Delta_{f,f^{-1}([a,b]),h}$ , critical values and finite bar codes, but there is an extra difficulty to establish Agmon's type estimates to accurately control the exponential decay estimates. This problem is solved in Subsections 8.3.2 and 8.3.3 by modelling a collection of solutions to Hamilton-Jacobi equations associated with some natural stratification of the subanalytic graph of  $f$  in  $M \times \mathbb{R}$ .
- Finally, Section 9 answers precisely our Main Question in various explicit cases. We return to our results of [LNV], where Morse functions with simple critical values (one critical point for every critical value) were considered. It was too rapidly conjectured in [LNV] that some topological constant  $\kappa^2$  appearing in the subexponential factor was equal to 1. It is true in the case of oriented surfaces (see [Lep2]), but examples are now provided with a constant  $\kappa^2$  equal to any  $n^2$ ,  $n \in \mathbb{N}^*$ , the first example with  $\kappa^2 = 4$  arising in the case of a Morse function on  $\mathbb{R}P^2$ . Additionally, in the case of Morse functions with multiple critical values, the constant  $\kappa$  has to be replaced by an "incidence matrix",  $\boldsymbol{\kappa}$ , related with the bar code. Various examples, including non Morse functions, show that the accurate computation of the prefactors now results from two well separated analyses: one for the global topology of the sublevel sets relying on the bar code, and one for the local asymptotic expansions of Laplace integrals.

## 2 Boundary Witten Laplacians

In this section we specify the domain of various operators involved in our analysis and review the basic exponential decay estimates. The general assumptions and notations have been set in Subsection 1.2 and in particular the function  $f$  satisfies Hypothesis 1.2 or Hypothesis 1.6. We shall give the definition of Dirichlet and Neumann boundary conditions for Witten Laplacians on strongly Lipschitz domains  $\overline{\Omega}$ . Most of the time in the sequel, these domains will be level set domains  $\overline{\Omega} = f^{-1}([a, b])$  with  $a, b \notin \{c_1, \dots, c_{N_f}\}$ . The required Agmon's type or exponential decay estimates will be proved under Hypothesis 1.2. We are unable to prove these estimates in the general setting of Hypothesis 1.6 but will prove them for subanalytic Lipschitz functions (see Subsection 8.3).

### 2.1 Tangential and normal traces

#### 2.1.1 Smooth case

**Definition 2.1.** *Let  $N \subset M$  is a  $C^\infty$ -hypersurface of  $M$ ,  $n$  a unit normal vector and  $n^\flat$  the associated covector, defined locally. When  $\omega \in W^{s,2}(M; \Lambda T^*M)$ ,  $s > \frac{1}{2}$ , the tangential and normal traces denoted  $\mathbf{t}_N\omega$  and  $\mathbf{n}_N\omega$  are defined by*

$$\mathbf{t}_N\omega = \mathbf{i}_n(n^\flat \wedge \omega)|_N \quad \text{and} \quad \mathbf{n}_N = n^\flat \wedge (\mathbf{i}_n\omega)|_N.$$

Before we extend this definition to more singular forms, let us make explicit this definition in coordinate systems (see e.g. [Sch]):

- When  $n$  is a normalized normal vector to  $N$ , any vector field in  $X = T_N M$  can be decomposed into  $X = X_T \oplus X_n n$ . The traces  $\mathbf{t}_N\omega$  and  $\mathbf{n}_N\omega$  are then equal to

$$\mathbf{t}_N\omega(X_1, \dots, X_p) = \omega|_N(X_{1,T}, \dots, X_{p,T}) \quad \text{and} \quad \mathbf{n}_N\omega = \omega|_N - \mathbf{t}_N\omega.$$

- With local coordinates  $(x^1, \dots, x^d) = (x', x^d) \in \mathbb{R}^d$  in a neighborhood  $U_{x_0}^M$  in  $M$  of  $x_0 \in N$ , such that  $N \cap U_{x_0}^M = \{(x', x^d) \in U_0^{\mathbb{R}^d}, x^d = 0\}$ ,  $g = \sum_{ij < d} g_{i,j}(x', x^d) dx^i dx^j + (dx^d)^2$ ,  $n = \frac{\partial}{\partial x^d}$  and  $n^\flat = dx^d$ , and when a differential form is written

$$\omega = \sum_{\#I'=p, d \notin I'} \omega_{I'}(x', x^d) dx^{I'} + \sum_{\#J'=p-1, d \notin J'} \omega_{J'}(x', x^d) dx^{J'} \wedge dx^d,$$

with  $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_{\#I}}$ ,  $i_1 < \dots < i_{\#I}$ ,  $I = \{i_1, \dots, i_{\#I}\}$ ,

the tangential and normal traces are given by

$$\mathbf{t}_N\omega = \sum_{\#I'=p, d \notin I'} \omega_{I'}(x', 0) dx^{I'} \quad \text{and} \quad \mathbf{n}_N\omega = \sum_{\#J'=p-1, d \notin J'} \omega_{J'}(x', 0) dx^{J'} \wedge dx^d.$$

- From those formulas one gets at once  $\star \mathbf{t}_N = \mathbf{n}_N \star$ , where  $\star$  denotes the Hodge  $\star$  operator on  $(M, g)$ . The possible orientation twist  $\text{or}_M$  is locally trivial so that the orientability of  $M$  is not required.
- When restricted to the tangent space to  $N$ ,  $\mathbf{t}_N\omega$  coincides with  $j_N^* \omega$  where  $j_N : N \rightarrow M$  is the natural imbedding. Therefore  $\mathbf{t}_N d = dt_N$  and therefore  $\mathbf{n}_N d^* = d^* \mathbf{n}_N$ . Note also that  $\mathbf{t}_N$  and  $\mathbf{n}_N$  commute with multiplications by functions.

### 2.1.2 Lipschitz domains

The typical case which will be considered is when  $N = \partial\Omega$  is the boundary of a Lipschitz domain of  $M$  (strongly Lipschitz according to the terminology of [GMM]). This means that  $\Omega$  is locally the hypograph of a Lipschitz function in a proper coordinate system. For the notations,  $\Omega$  is an open domain in  $M$  and its closed version is  $\overline{\Omega} = \Omega \sqcup N$  with  $N = \partial\Omega$ . Precisely we consider the following situation.

**Hypothesis 2.2.** *The domain  $\overline{\Omega} = \Omega \sqcup N \subset M$  is a Lipschitz domain with  $N = N_t \sqcup N_n$  made of two disjoint closed hypersurfaces.*

When  $\Omega$  is a regular domain, with  $C^\infty$  boundaries  $N_t$  and  $N_n$ , the unit normal vector field  $n$  to  $N = \partial\Omega$  is globally defined so that the hypersurface measure  $d\sigma$ , the orientation twist  $\text{or}_N$  and the Hodge  $\star$  operation on  $N = \partial\Omega$  are deduced from  $d\text{Vol}_g$  and  $\text{or}_M$  and the Hodge  $\star$  on  $M$ . In the general case when the domain  $\Omega$  has only the assumed Lipschitz regularity, the same things hold except that the normal vector is defined  $d\sigma$ -almost everywhere along  $N_t \sqcup N_n$ ,  $d\sigma$  being the  $\mathcal{H}^{d-1}$ -Hausdorff measure.

For two forms  $u, v \in W^{1,2}(\Omega, \Lambda T^*M)$ , the Green formula yields

$$\begin{aligned} \langle du, v \rangle_{L^2(\Omega)} - \langle u, d^*v \rangle_{L^2(\Omega)} &= \int_{\Omega} d(\overline{u} \wedge \star v) = \int_N \mathbf{t}_N(\overline{u} \wedge \star v) \\ &= \int_N \langle u, \mathbf{i}_n v \rangle_{\Lambda T^*_\sigma M} d\sigma = \int_N \langle \mathbf{t}_N u, \mathbf{i}_n v \rangle_{\Lambda T^*_\sigma M} d\sigma, \end{aligned} \quad (7)$$

$$= \int_N \langle n^\flat \wedge u, v \rangle_{\Lambda T^*_\sigma M} d\sigma = \int_N \langle n^\flat \wedge u, \mathbf{n}_N v \rangle_{\Lambda T^*_\sigma M} d\sigma \quad (8)$$

while the decomposition  $N = N_t \sqcup N_n$  into two disjoint closed hypersurfaces clearly implies

$$\begin{aligned} (\mathbf{t}_{N_t} u = 0) &\Leftrightarrow (\text{supp } n^\flat \wedge u \subset N_n) \\ \text{and } (\mathbf{n}_{N_n} v = 0) &\Leftrightarrow (\text{supp } \mathbf{i}_n v \subset N_t). \end{aligned}$$

Moreover according to [JMM], when  $\omega \in L^2(\Omega; \Lambda T^*M)$  and  $d\omega \in L^2(\Omega; \Lambda T^*M)$ , the above Green formulas provide the duality needed to define  $n^\flat \wedge \omega|_N \in W^{-\frac{1}{2},2}(N; \Lambda T^*M)$  by

$$\forall g \in W^{\frac{1}{2},2}(N), \quad \langle n^\flat \wedge \omega, g \rangle_{W^{-\frac{1}{2},2}(N), W^{\frac{1}{2},2}(N)} = \langle d\omega, G \rangle_{L^2(\Omega)} - \langle \omega, d^*G \rangle_{L^2(\Omega)}, \quad (9)$$

where  $G$  is any form in  $W^{1,2}(\Omega; \Lambda T^*M)$  such that  $G|_N = g \in W^{\frac{1}{2},2}(N; \Lambda T^*M)$ . Similarly, when  $\omega$  and  $d^*\omega$  belong to  $L^2(\Omega; \Lambda T^*M)$ , one can define  $\mathbf{i}_n \omega|_N \in W^{-\frac{1}{2},2}(N; \Lambda T^*M)$  by

$$\forall g \in W^{\frac{1}{2},2}(N), \quad \langle \mathbf{i}_n \omega, g \rangle_{W^{-\frac{1}{2},2}(N), W^{\frac{1}{2},2}(N)} = \langle \omega, dG \rangle_{L^2(\Omega)} - \langle d^*\omega, G \rangle_{L^2(\Omega)}. \quad (10)$$

In particular, when  $O$  is an open subset of  $N$  and when the trace  $n^\flat \wedge \omega|_O$  defined in the sense of (9) (resp. of (10)) belongs to  $L^2(O; \Lambda T^*M)$ , the tangential (resp. normal) trace  $\mathbf{t}_O \omega$  (resp.  $\mathbf{n}_O \omega$ ) is well defined on  $O$  by the standard formula from Definition 2.1:

$$\mathbf{t}_O \omega = \mathbf{i}_n(n^\flat \wedge \omega)|_O \quad (\text{resp. } \mathbf{n}_O \omega = n^\flat \wedge (\mathbf{i}_n \omega)|_O).$$

We may thus make sense of the boundary condition  $\mathbf{t}_{N_t} \omega = 0$  (resp.  $\mathbf{n}_{N_n} \omega = 0$ ), which is equivalent to  $\text{supp } n^\flat \wedge \omega|_N \subset N_n$  (resp.  $\text{supp } \mathbf{i}_n \omega|_N \subset N_t$ ), for any  $\omega \in L^2(\Omega; \Lambda T^*M)$  such that  $d\omega \in L^2(\Omega; \Lambda T^*M)$  (resp.  $d^*\omega \in L^2(\Omega; \Lambda T^*M)$ ).

According to [JMM, Proposition 3.1],  $C_0^\infty(\Omega \cup N_n; \Lambda T^* M)$  (resp.  $C_0^\infty(\Omega \cup N_t; \Lambda T^* M)$ ) is dense in

$$\mathcal{T} = \{\omega \in L^2(\Omega; \Lambda T^* M), d\omega \in L^2(\Omega; \Lambda T^* M), \mathbf{t}_{N_t}\omega = 0\} \quad (11)$$

$$\text{(resp. in } \mathcal{N} = \{\omega \in L^2(\Omega; \Lambda T^* M), d^*\omega \in L^2(\Omega; \Lambda T^* M), \mathbf{n}_{N_n}\omega = 0\} \text{)} \quad (12)$$

endowed with the norm  $\|\omega\|_{L^2(\Omega)} + \|d\omega\|_{L^2(\Omega)}$  (resp.  $\|\omega\|_{L^2(\Omega)} + \|d^*\omega\|_{L^2(\Omega)}$ ). Theorem 3.4 of [JMM] also says that when  $u, v \in L^2(\Omega)$  with  $du \in L^2(\Omega; \Lambda^{p+1}T^*M)$ ,  $d^*v \in L^2(\Omega; \Lambda^p T^*M)$ , and

$$\text{supp } \mathbf{i}_n v \subset \Gamma \quad \text{or} \quad \text{supp } (n^b \wedge u) \subset \Gamma$$

with  $\Gamma = N_t$  or  $\Gamma = N_n$ , the following Green formulas

$$\begin{aligned} \langle du, v \rangle_{L^2(\Omega)} - \langle u, d^*v \rangle_{L^2(\Omega)} &= \int_{\Gamma} \langle n^b \wedge u, n^b \wedge (\mathbf{i}_n v) \rangle_{T_\sigma^* \Omega} d\sigma \\ &= \int_{\Gamma} \langle \mathbf{i}_n (n^b \wedge u), \mathbf{i}_n v \rangle_{T_\sigma^* \Omega} d\sigma \end{aligned} \quad (13)$$

make sense with a r.h.s. interpreted in general in a weak form specified in [JMM, Proposition 3.3]. Notice that under Hypothesis 2.2, the geometric assumptions concerned with  $\Gamma$  in [JMM] are trivially satisfied without any locally mixed boundary conditions. Additionally, when  $\mathbf{i}_n v$  and  $n^b \wedge u$  belong to  $L^2(\Gamma)$ , the r.h.s. of (13) are standard integrals along the boundary.

**Definition 2.3.** Let  $\Omega$  be a Lipschitz domain of  $M$  with  $\bar{\Omega} = \Omega \sqcup N$ ,  $N = N_t \sqcup N_n$  like above, and let  $\mathcal{T}, \mathcal{N}$  be the spaces defined in (11)(12).

The space

$$W(\Omega; \Lambda T^* M) = \{\omega \in L^2(\Omega; \Lambda T^* M); d\omega \in L^2(\Omega; \Lambda T^* M); d^*\omega \in L^2(\Omega; \Lambda T^* M)\}$$

is endowed with its natural Hilbert space norm given by

$$\|\omega\|_{W(\Omega)}^2 := \|\omega\|_{L^2(\Omega)}^2 + \|d\omega\|_{L^2(\Omega)}^2 + \|d^*\omega\|_{L^2(\Omega)}^2. \quad (14)$$

The closed subspace  $\mathcal{T} \cap \mathcal{N}$  of  $W(\Omega; \Lambda T^* M)$  will be denoted  $W_\partial(\Omega; \Lambda T^* M)$  and the restriction of the  $W(\Omega; \Lambda T^* M)$ -norm  $\|\cdot\|_{W_\partial(\Omega)}$ .

**Remark 2.4.** i) By interior elliptic regularity, note that

$$W_\partial(\Omega; \Lambda T^* M) \subset W(\Omega; \Lambda T^* M) \subset W_{loc}^{1,2}(\Omega; \Lambda T^* M)$$

with continuous embeddings. However it is known that  $W(\Omega; \Lambda T^* M)$ , and even  $W_\partial(\Omega; \Lambda T^* M)$  if we add boundary conditions, differs from  $W^{1,2}(\Omega; \Lambda T^* M)$  for a general Lipschitz domain (see e.g. [MTV][MMMMT]). An easy counter example is  $u = r^{\frac{\pi}{\theta_0}-1} \cos(\frac{\pi}{\theta_0}\theta)dr - r^{\frac{\pi}{\theta_0}-1} \sin(\frac{\pi}{\theta_0}\theta)d\theta$  in the sector  $0 < \theta < \theta_0$  of  $\mathbb{R}^2$  near  $r = 0$ . It satisfies  $\mathbf{n}u = 0$ ,  $du = 0$  and  $d^*u \in L^2$  near  $r = 0$  while  $u \notin W^{1,2}$  near  $r = 0$  when  $\theta_0 > \pi$ .

ii) The space  $W(\Omega; \Lambda T^* M)$  and its subspace  $W_\partial(\Omega; \Lambda T^* M)$  are Lipschitz-module: for any  $\varphi \in W^{1,\infty}(\Omega; \mathbb{R})$  and  $\omega \in W(\Omega; \Lambda T^* M)$ ,  $\varphi\omega$  belongs to  $W(\Omega; \Lambda T^* M)$  and the mapping  $\omega \in W(\Omega; \Lambda T^* M) \mapsto \varphi\omega \in W(\Omega; \Lambda T^* M)$  is continuous. Moreover, for any bounded sequence  $(\varphi_n)_{n \in \mathbb{N}}$  of  $W^{1,\infty}(\Omega; \mathbb{R})$  such that  $\varphi_n \rightarrow \varphi$  a.e. and  $d\varphi_n \rightarrow d\varphi$  a.e., the convergence  $\varphi_n\omega \rightarrow \varphi\omega$  holds for the  $W(\Omega; \Lambda T^* M)$ -norm for every  $\omega \in W(\Omega; \Lambda T^* M)$ .

iii) In our case it is proven in [MMMMT] and it is extended in [JMM] that  $W_\partial(\Omega; \Lambda T^* M)$  is embedded in  $W^{1/2,2}(\Lambda T^* M)$ . Again the exponent  $\frac{1}{2}$  cannot be improved for a general strongly Lipschitz domain  $\Omega$ .



iv) For a different approach on regularity issues for Lipschitz domains and relying on a generalization of Bogovskii and Poincaré type integrals, we refer to [CoMcI], [Mit] and [MiMo].

**Proposition 2.5.** *Let  $W_\partial(\Omega; \Lambda T^*M)$  be the space of Definition 2.3.*

*Every  $\omega \in W_\partial(\Omega; \Lambda T^*M)$  belongs to  $W^{\frac{1}{2},2}(\Omega; \Lambda T^*M)$  and has, in the sense of (9) and (10), tangential and normal traces  $\mathbf{t}_N\omega$  and  $\mathbf{n}_N\omega$  which actually belong to  $L^2(N; \Lambda T^*M)$ . Moreover, there exists  $C > 0$  such that*

$$\forall \omega \in W_\partial(\Omega; \Lambda T^*M), \quad \|\omega\|_{W^{\frac{1}{2},2}(\Omega)}^2 + \|\omega|_N\|_{L^2(N)}^2 \leq C\|\omega\|_{W_\partial(\Omega)}^2,$$

where  $\omega|_N := \mathbf{t}_N\omega + \mathbf{n}_N\omega \in L^2(N; \Lambda T^*M)$  is the total trace of  $\omega$ .

Finally, in the case where  $\overline{\Omega}$  is a smooth domain, Gaffney's inequality holds:

$$W_\partial(\Omega; \Lambda T^*M) = \{\omega \in W^{1,2}(\Omega; \Lambda T^*M), \mathbf{t}_{N_t}\omega = 0, \mathbf{n}_{N_n}\omega = 0\}$$

and there exists  $C \geq 1$  such that

$$\forall \omega \in W_\partial(\Omega; \Lambda T^*M), \quad C^{-1}\|\omega\|_{W^{1,2}(\Omega)}^2 \leq \|\omega\|_{W_\partial(\Omega)}^2 \leq C\|\omega\|_{W^{1,2}(\Omega)}^2.$$

*Proof.* The first part of the statement is an immediate consequence of the analysis led in [JMM] (see e.g. Theorem 1.1 there), but our setting is actually simpler since no locally mixed boundary conditions appear.

For Gaffney's inequality when the domain  $\overline{\Omega}$  is smooth, consider first

$$\omega \in W'(\Omega; \Lambda T^*M) := \{u \in W^{1,2}(\Omega; \Lambda T^*M), \mathbf{t}_{N_t}u = 0, \mathbf{n}_{N_n}u = 0\}$$

and a function  $\chi \in C_0^\infty(\Omega \cup N_t; [0, 1])$  such that  $\chi \equiv 1$  in a neighborhood of  $N_t$ , and decompose  $\omega$  as  $\omega = \chi\omega + (1 - \chi)\omega = \omega_1 + \omega_2$ . For any differential operator  $L$  of order  $\leq 1$ , note then the relation  $\|L\omega_j\|_{L^2} \leq C_{\chi,L,j}\|\omega\|_{L^2} + \|L\omega\|_{L^2}$ ,  $j = 1, 2$ . Now,  $\omega_1 = \chi\omega \in W^{1,2}(\Omega; \Lambda T^*M)$  satisfies  $\mathbf{t}_{\partial\Omega}\omega_1 = 0$  and  $\omega_2 = (1 - \chi)\omega \in W^{1,2}(\Omega; \Lambda T^*M)$  satisfies  $\mathbf{n}_{\partial\Omega}\omega_2 = 0$ . Gaffney's inequality for Dirichlet boundary conditions then says

$$\|\omega_1\|_{W^{1,2}}^2 \leq C_1 [\|\omega_1\|_{L^2}^2 + \|d\omega_1\|_{L^2}^2 + \|d^*\omega_1\|_{L^2}^2]$$

for some  $C_1$  independent of  $\omega_1$ , while Gaffney's inequality for Neumann boundary conditions says

$$\|\omega_2\|_{W^{1,2}}^2 \leq C_2 [\|\omega_2\|_{L^2}^2 + \|d\omega_2\|_{L^2}^2 + \|d^*\omega_2\|_{L^2}^2]$$

for some  $C_2$  independent of  $\omega_2$  (these two different boundary conditions have been treated separately in [Sch]). Adding the above two inequalities then leads to

$$\forall \omega \in W'(\Omega; \Lambda T^*M), \quad \|\omega\|_{W^{1,2}}^2 \leq C [\|\omega\|_{L^2}^2 + \|d\omega\|_{L^2}^2 + \|d^*\omega\|_{L^2}^2].$$

In order to achieve the proof of Proposition 2.5, it then suffices to show that  $W'(\Omega; \Lambda T^*M)$  equals  $W_\partial(\Omega; \Lambda T^*M)$ . We can forget the boundary conditions. With a regular boundary, a simple local reflexion after identifying the domain with a half space, leads to the problem on  $\mathbb{R}^d$  with a Lipschitz riemannian metric, asking if a compactly supported form in  $\omega \in L^2_{comp}(\mathbb{R}^d)$  such  $d\omega \in L^2(\mathbb{R}^d)$  and  $d^*\omega \in L^2(\mathbb{R}^d)$  belongs to  $H^1_{comp}(\mathbb{R}^d)$ . It is a straightforward application of Lax-Milgram's theorem.  $\square$

## 2.2 Witten's deformation

The function  $f$  is assumed to be a Lipschitz function and the domain  $\overline{\Omega}$  satisfies Hypothesis 2.2. Improved regularity results are stated when  $f$  and  $\overline{\Omega}$  are more regular.



**Definition 2.6.** Assume  $f \in W^{1,\infty}(M; \mathbb{R})$ ,  $h > 0$ , and Hypothesis 2.2 for  $\overline{\Omega} = \Omega \sqcup N = \Omega \sqcup N_t \sqcup N_n$ . The operators  $d_{f,\overline{\Omega},h}$  and  $d_{f,\overline{\Omega},h}^*$  are defined by

$$D(d_{f,\overline{\Omega},h}) := \{\omega \in L^2(\Omega; \Lambda T^*M), d_{f,h}\omega \in L^2(\Omega; \Lambda T^*M), \mathbf{t}_{N_t}\omega = 0\} = \mathcal{T}$$

and  $D(d_{f,\overline{\Omega},h}^*) := \{\omega \in L^2(\Omega; \Lambda T^*M), d_{f,h}^*\omega \in L^2(\Omega; \Lambda T^*M), \mathbf{n}_{N_n}\omega = 0\} = \mathcal{N}$ ,

where  $\mathcal{T}$  and  $\mathcal{N}$  are the spaces defined in (11) and (12), and we recall that

$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df \wedge \quad \text{and} \quad d_{f,h}^* = e^{\frac{f}{h}}(hd^*)e^{-\frac{f}{h}} = hd^* + \mathbf{i}_{\nabla f}$$

according to (4) and (5).

A particular case that we will study extensively is when  $\overline{\Omega} = f^{-1}([a, b])$ ,  $N_t = f^{-1}(\{a\})$ ,  $N_n = f^{-1}(\{b\})$ , and  $a < b$  do not belong to  $\{c_1, \dots, c_{N_f}\}$  under Hypothesis 1.2 (in this case  $\Omega = f_a^b$  according to Definition 1.1). With such an  $f$ -dependent domain, it will be useful to consider  $d_{0,f^{-1}([a,b]),h}$  and  $d_{f,f^{-1}([a,b]),h}$ .

**Proposition 2.7.** In the framework of Definition 2.6, the operator  $d_{f,\overline{\Omega},h}$  (resp.  $d_{f,\overline{\Omega},h}^*$ ) is densely defined, closed, and  $\text{Ran } d_{f,\overline{\Omega},h} \subset \ker d_{f,\overline{\Omega},h}$  (resp.  $\text{Ran } d_{f,\overline{\Omega},h}^* \subset \ker d_{f,\overline{\Omega},h}^*$ ). Its adjoint is  $d_{f,\overline{\Omega},h}^*$  (resp.  $d_{f,\overline{\Omega},h}$ ). The subspace  $\mathcal{C}_0^\infty(\Omega \cup N_n; \Lambda T^*M)$  (resp.  $\mathcal{C}_0^\infty(\Omega \cup N_t; \Lambda T^*M)$ ) is dense in  $D(d_{f,\overline{\Omega},h})$  (resp.  $D(d_{f,\overline{\Omega},h}^*)$ ). Finally, the identity

$$D(d_{f,\overline{\Omega},h}) \cap D(d_{f,\overline{\Omega},h}^*) = W_\partial(\Omega; \Lambda T^*M),$$

holds true when  $W_\partial(\Omega; \Lambda T^*M)$  is the space of Definition 2.3.

*Proof.* The operators  $d_{f,\overline{\Omega},h}$  and  $d_{f,\overline{\Omega},h}^*$  having respective domains  $\mathcal{T}$  and  $\mathcal{N}$ , with  $\mathcal{T} \cap \mathcal{N} = W_\partial(\Omega; \Lambda T^*M)$  by Definition 2.3, they are clearly densely defined, and they are bounded perturbations of  $hd_{0,\overline{\Omega},1}$  and  $hd_{0,\overline{\Omega},1}^*$  owing to  $d_{f,h} = hd + df \wedge$  and  $d_{f,h}^* = hd_{0,\overline{\Omega},1} + \mathbf{i}_{\nabla f}$ . The operators  $d_{0,\overline{\Omega},1}$  and  $d_{0,\overline{\Omega},1}^*$  are moreover closed with the density properties, according to the presentation around (9)–(12).

As bounded perturbations, the adjoint of  $d_{f,\overline{\Omega},h}$  equals  $d_{f,\overline{\Omega},h}^*$  because the adjoint of  $d_{0,\overline{\Omega},1}$  is  $d_{0,\overline{\Omega},1}^*$  while the adjoint of the bounded perturbation  $df \wedge$  is  $\mathbf{i}_{\nabla f}$ . Actually  $\omega$  belongs to the domain of the adjoint of  $d_{0,\overline{\Omega},1}$  iff

$$\exists C > 0, \forall u \in \mathcal{C}_0^\infty(\Omega \cup N_n; \Lambda T^*M), \quad |\langle du, \omega \rangle| \leq C \|u\|_{L^2}.$$

Taking any  $u \in \mathcal{C}_0^\infty(\Omega; \Lambda T^*M)$  implies  $d^*\omega \in L^2(\Omega; \Lambda T^*M)$  and therefore  $\mathbf{i}_n \omega|_N$  is well defined in  $W^{-1/2,2}(N; \Lambda T^*M)$ . Using afterwards Green's formula (10) with a general  $u \in \mathcal{C}_0^\infty(\Omega \cup N_n; \Lambda T^*M)$  leads to  $\mathbf{i}_n \omega|_{N_n} = 0$ . Thus the domain of the adjoint of  $d_{0,\overline{\Omega},1}$  is included in  $D(d_{0,\overline{\Omega},1}^*)$ , which is enough to conclude.

It remains to check  $\text{Ran } d_{f,\overline{\Omega},h} \subset \ker d_{f,\overline{\Omega},h}$  and  $\text{Ran } d_{f,\overline{\Omega},h}^* \subset \ker d_{f,\overline{\Omega},h}^*$ . The identities (4) and (5) already say that  $d_{f,h}d_{f,\overline{\Omega},h}\omega = 0$  in  $\mathcal{D}'(\Omega, \Lambda T^*M)$  (resp.  $d_{f,h}^*d_{f,\overline{\Omega},h}^*\omega = 0$ ) when  $\omega \in D(d_{f,\overline{\Omega},h})$  (resp.  $\omega \in D(d_{f,\overline{\Omega},h}^*)$ ). We can conclude that  $d_{f,\overline{\Omega},h}\omega \in \ker d_{f,\overline{\Omega},h}$  (resp.  $d_{f,\overline{\Omega},h}^*\omega \in \ker d_{f,\overline{\Omega},h}^*$ ) if  $\mathbf{t}_{N_t}d_{f,h}\omega = 0$  (resp.  $\mathbf{n}_{N_n}d_{f,h}^*\omega = 0$ ) or more precisely, with the weak formulation of the trace defined in (9) (resp. in (10)), if  $\text{supp } n^\flat \wedge (d_{f,h}\omega)|_N \subset N_n$  (resp.  $\text{supp } \mathbf{i}_n(d_{f,h}^*\omega)|_N \subset N_t$ ). For  $\omega \in \mathcal{C}_0^\infty(\Omega \cup N_n; \Lambda T^*M)$  (resp.  $\omega \in \mathcal{C}_0^\infty(\Omega \cup N_t; \Lambda T^*M)$ ) the weakly defined trace  $n^\flat \wedge (d_{f,h}\omega)|_{N_t}$  (resp.  $\mathbf{i}_n(d_{f,h}^*\omega)|_{N_n}$ ) obviously vanishes because  $N_t \cap \text{supp } d_{f,h}\omega = \emptyset$  (resp.  $N_n \cap \text{supp } d_{f,h}^*\omega = \emptyset$ ).

By the density of  $\mathcal{C}_0^\infty(\Omega \cup N_n; \Lambda T^*M)$  (resp.  $\mathcal{C}^\infty(\Omega \cup N_t; \Lambda T^*M)$ ) in  $D(d_{f,\bar{\Omega},h})$  (resp.  $D(d_{f,\bar{\Omega},h}^*)$ ), we deduce

$$\begin{aligned} \forall \omega \in D(d_{f,\bar{\Omega},h}), \quad n^\flat \wedge (d_{f,\bar{\Omega},h}\omega)|_{N_t} = 0 \quad \text{in } W^{-1/2,2}(N_t) \\ \text{(resp. } \forall \omega \in D(d_{f,\bar{\Omega},h}^*), \quad \mathbf{i}_n d_{f,\bar{\Omega},h}^* \omega|_{N_n} = 0 \quad \text{in } W^{-1/2,2}(N_n)). \end{aligned}$$

This ends the proof.  $\square$

We now apply results of the abstract Hodge theory reviewed in Appendix A to our specific framework.

**Proposition 2.8.** *Assume Hypothesis 2.2 for  $\bar{\Omega} = \Omega \sqcup N_t \sqcup N_n$ ,  $f \in W^{1,\infty}(\Omega; \mathbb{R})$  and let  $W_\partial(\Omega; \Lambda T^*M)$  be the space of Definition 2.3.*

1. The operator  $d_{f,\bar{\Omega},h} + d_{f,\bar{\Omega},h}^*$  with domain

$$D(d_{f,\bar{\Omega},h}) \cap D(d_{f,\bar{\Omega},h}^*) = W_\partial(\Omega; \Lambda T^*M)$$

is self-adjoint and has a compact resolvent.

2. The operator  $\Delta_{f,\bar{\Omega},h} := d_{f,\bar{\Omega},h} d_{f,\bar{\Omega},h}^* + d_{f,\bar{\Omega},h}^* d_{f,\bar{\Omega},h}$  with domain

$$D(\Delta_{f,\bar{\Omega},h}) = \{u \in D(d_{f,\bar{\Omega},h}) \cap D(d_{f,\bar{\Omega},h}^*) \text{ s.t. } d_{f,h}u \in D(d_{f,\bar{\Omega},h}^*) \text{ and } d_{f,h}^*u \in D(d_{f,\bar{\Omega},h})\}$$

is a self-adjoint operator with a compact resolvent. It is the Friedrichs extension associated with the (closed) quadratic form  $Q_{f,\bar{\Omega},h}(\omega) = \|d_{f,h}\omega\|_{L^2}^2 + \|d_{f,h}^*\omega\|_{L^2}^2$  with domain  $D(d_{f,\bar{\Omega},h}) \cap D(d_{f,\bar{\Omega},h}^*)$ .

3. The ranges of  $d_{f,\bar{\Omega},h}$  and  $d_{f,\bar{\Omega},h}^*$  are closed and the following Hodge decompositions hold in  $L^2$ :

$$L^2(\Omega; \Lambda T^*M) = \underbrace{\text{Ran}(d_{f,\bar{\Omega},h})}_{\ker(d_{f,\bar{\Omega},h})} \oplus \overbrace{\ker(\Delta_{f,\bar{\Omega},h})}^{\ker(d_{f,\bar{\Omega},h}^*)} \oplus \text{Ran}(d_{f,\bar{\Omega},h}^*)$$

4. For any  $z \in \mathbb{C} \setminus \sigma(\Delta_{f,\bar{\Omega},h})$ , one has for any compactly supported and bounded measurable function  $\chi$  on  $\mathbb{R}$  and for any  $\omega \in D(\mathbf{d})$ , where  $\mathbf{d} = d_{f,\bar{\Omega},h}$  or  $\mathbf{d} = d_{f,\bar{\Omega},h}^*$ ,

$$\mathbf{d}(z - \Delta_{f,\bar{\Omega},h})^{-1}\omega = (z - \Delta_{f,\bar{\Omega},h})^{-1}\mathbf{d}\omega \quad \text{and} \quad \mathbf{d} \circ \chi(\Delta_{f,\bar{\Omega},h})\omega = \chi(\Delta_{f,\bar{\Omega},h}) \circ \mathbf{d}\omega.$$

5. When  $\bar{\Omega}$  is smooth and  $f \in \mathcal{C}^2(\bar{\Omega}; \mathbb{R})$ , the domain of  $\dot{\Delta}_{f,\bar{\Omega},h}$  equals

$$D(\Delta_{f,\bar{\Omega},h}) = \left\{ \omega \in W^{2,2}(\Omega; \Lambda T^*M), \quad \begin{array}{ll} \mathbf{t}_{N_t}\omega = 0, & \mathbf{n}_{N_n}\omega = 0, \\ \mathbf{t}_{N_t}d_{f,h}^*\omega = 0, & \mathbf{n}_{N_n}d_{f,h}\omega = 0 \end{array} \right\}.$$

*Proof.* The identification of  $D(d_{f,\bar{\Omega},h}) \cap D(d_{f,\bar{\Omega},h}^*)$  is done in Proposition 2.7. The statements 1), 2), 3) are then straightforward applications of Proposition A.1 in Appendix A. The first identity of the statement 4) is an application of the general relation (155) in Appendix A. The second identity then comes from the functional calculus for self-adjoint operators.

Finally, for 5), it suffices to notice that  $\Delta_{f,h} = -h^2\Delta_{0,1} + |\nabla f|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*)$  and that  $\Delta_{f,\bar{\Omega},h}$  is a bounded perturbation of  $h^2\Delta_{0,\bar{\Omega},1}$  when  $f \in \mathcal{C}^2(\bar{\Omega}; \mathbb{R})$ . But the elliptic analysis made in [Sch][MMMT] (see also [LNV] for the combination of Dirichlet on  $N_t$  and Neumann on  $N_n$  boundary conditions) ensures that the domain of  $\Delta_{0,\bar{\Omega},1} = dd^* + d^*d$  is

$$D(\Delta_{0,\bar{\Omega},1}) = \left\{ \omega \in W^{2,2}(\Omega; \Lambda T^*M), \quad \begin{array}{ll} \mathbf{t}_{N_t}\omega = 0, & \mathbf{n}_{N_n}\omega = 0, \\ \mathbf{t}_{N_t}d_{f,h}^*\omega = 0, & \mathbf{n}_{N_n}d_{f,h}\omega = 0 \end{array} \right\}.$$

$\square$

**Remark 2.9.** *Let us complete the statements of Propositions 2.7 and 2.8 with some remarks when  $f$  satisfies Hypothesis 1.2 or Hypothesis 1.6.*

- *The domain  $D(d_{f,\overline{\Omega},h})$  does not contain any other regularity assumption than  $\omega \in L^2(\Omega)$ ,  $d_{f,h}\omega \in L^2(\Omega)$ , and does not contain any condition on  $N_n$ . In particular, when  $a' \leq a < b$  do not belong to  $\{c_1, \dots, c_{N_f}\}$  according to Hypothesis 1.2, the domain  $f_a^b$  (resp.  $f_{a'}^b$ ) equals to  $f^{-1}([a, b])$  and satisfies Hypothesis 2.2 with  $N_t = f^{-1}(\{a\})$  (resp.  $N_t = f^{-1}(\{a'\})$ ) and  $N_n = f^{-1}(\{b\})$ . This a consequence of implicit functions theorem which is the classical  $C^1$ -version under Hypothesis 1.2 and still holds in a Lipschitz version under the more general Hypothesis 1.6 (see Subsection 8.3.1). The density of  $C_0^\infty(f_a^b \cup f^{-1}(\{b\}); \Lambda T^*M)$  in  $D(d_{f,f^{-1}([a,b]),h})$  provides the following extension result:*

$$\forall \omega \in D(d_{f,f^{-1}([a,b]),h}), \quad \tilde{\omega} \in D(d_{f,f^{-1}([a',b]),h}), \quad \text{where } \tilde{\omega}|_{f_a^b} = \omega \text{ and } \tilde{\omega}|_{f_{a'}^b} \equiv 0. \quad (15)$$

- *Hodge decomposition in Proposition 2.8-3) says that*

$$\ker(\Delta_{f,\overline{\Omega},h}) \simeq \ker(d_{f,\overline{\Omega},h})/\text{Ran}(d_{f,\overline{\Omega},h}) \simeq \ker(d_{0,\overline{\Omega},1})/\text{Ran}(d_{0,\overline{\Omega},1}).$$

*From the usual Hodge theory on the manifold with boundary  $\overline{\Omega}$ , the dimension of  $\ker(\Delta_{f,\overline{\Omega},h}^{(p)})$  is thus the relative Betti number  $\dim H^p(\overline{\Omega}, N_t)$  and is independent of  $h > 0$ . In particular, when  $\Omega = f_a^b$  and  $a < b$  are not in  $\{c_1, \dots, c_{N_f}\}$ , it is*

$$\dim \ker(\Delta_{f,f^{-1}([a,b]),h}) = \dim H^p(f^b, f^a) =: \beta^{(p)}(f^b, f^a).$$

*If moreover  $[c, d] \subset [a, b]$  and  $([a, b] \setminus ]c, d]) \cap \{c_1, \dots, c_{N_f}\} = \emptyset$ , then for every  $a' \in [a, c]$  and  $b' \in [d, b]$ , the dimensions  $\dim H^p(f^b, f^a)$  and  $\dim H^p(f^{b'}, f^{a'})$  are equal and then*

$$\dim \ker(\Delta_{f,f^{-1}([a,b]),h}) = \dim \ker(\Delta_{f,f^{-1}([a',b']),h}). \quad (16)$$

- *When  $s \geq 0$ , the commutation of  $d_{f,\overline{\Omega},h}$  with  $1_{[0,s]}(\Delta_{f,\overline{\Omega},h})$  ensures that the restricted differential  $\delta_{[0,s]} = 1_{[0,s]}(\Delta_{f,\overline{\Omega},h})d_{f,\overline{\Omega},h}$  defines a finite dimensional complex with Betti numbers  $\dim H^p(\overline{\Omega}, N_t)$ :*

$$0 \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} F_{[0,s]}^{(0)} \dots F_{[0,s]}^{(p-1)} \begin{array}{c} \xrightarrow{\delta_{[0,s]}^{(p-1)}} \\ \xleftarrow{\delta_{[0,s]}^{(p-1)*}} \end{array} F_{[0,s]}^{(p)} \begin{array}{c} \xrightarrow{\delta_{[0,s]}^{(p)}} \\ \xleftarrow{\delta_{[0,s]}^{(p)*}} \end{array} F_{[0,s]}^{(p+1)} \dots F_{[0,s]}^{(d)} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 0 \quad (17)$$

*where  $F_{[0,s]}^{(p)} = \text{Ran} 1_{[0,s]}(\Delta_{f,\overline{\Omega},h}^{(p)})$ . This will be studied more carefully when  $\Omega = f_a^b$ , with the notations  $F_{[0,s],[a,b],h}$  and  $\delta_{[0,s],[a,b],h}$  in order to handle various intervals  $[a, b]$ .*

## 2.3 Agmon's type estimates

We review a series of exponential decay estimates which are adapted from [DiSj][HeSj2], and [LNV] for Witten Laplacians with boundary conditions. Those are standard when the function  $f$  satisfy Hypothesis 1.2 but only a part of them can be proved when  $f$  is a general Lipschitz function which satisfies Hypothesis 1.6.

### 2.3.1 Weighted integration by parts formulas

We present here weighted integration by parts formulas with low regularity assumptions. These formulas will be used in the sequel, after optimizing the weights, in order to prove different exponential decay estimates. Under Hypothesis 1.2, the regular case, this will lead to the usual Agmon estimates presented in the next section. A variation of these arguments will be developed in Section 8.3 under Hypothesis 1.3 (subanalytic case) and will require the low regularity results listed below.

**Lemma 2.10.** *Assume Hypothesis 2.2 for  $\bar{\Omega} = \Omega \sqcup N_t \sqcup N_n$ . Let  $f, \varphi \in W^{1,\infty}(M; \mathbb{R})$ ,  $\Delta_{f, \bar{\Omega}, h}$  be the self-adjoint operator defined in Proposition 2.8, and  $\sum_{j=1}^J \chi_j^2 = 1$  be a smooth partition of unity in  $\bar{\Omega}$ . For any  $\omega \in D(Q_{f, \bar{\Omega}, h}) = W_{\partial}(\Omega; \Lambda T^*M)$  (see (14) and the lines below), with the notation*

$$\tilde{\omega} = e^{\frac{\varphi}{h}} \omega,$$

the following identities hold true:

$$\operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) = \|d_{f, \bar{\Omega}, h} \tilde{\omega}\|^2 + \|d_{f, \bar{\Omega}, h}^* \tilde{\omega}\|^2 - \langle \tilde{\omega}, |\nabla \varphi|^2 \tilde{\omega} \rangle, \quad (18)$$

$$\text{and } \operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) = \sum_{j=1}^J \operatorname{Re} Q_{f, \bar{\Omega}, h}(\chi_j \omega, e^{\frac{2\varphi}{h}} \chi_j \omega) - h^2 \sum_{j=1}^J \|\nabla \chi_j \tilde{\omega}\|^2. \quad (19)$$

Moreover, when in addition  $f \in C^2(M)$ , the identity (18) writes also

$$\begin{aligned} \operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) &= h^2 \|d\tilde{\omega}\|_{L^2}^2 + h^2 \|d^* \tilde{\omega}\|_{L^2}^2 \\ &\quad + \langle \tilde{\omega}, (|\nabla f|^2 - |\nabla \varphi|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}_{\nabla \varphi}^*) \tilde{\omega} \rangle \\ &\quad + h \left( \int_{N_n} - \int_{N_t} \right) \langle \tilde{\omega}, \tilde{\omega} \rangle_{\Lambda T_{\sigma}^* \Omega} \left( \frac{\partial f}{\partial n} \right) (\sigma) d\sigma. \end{aligned} \quad (20)$$

Lastly, when  $f \in W^{1,\infty}(M; \mathbb{R})$  and  $\varphi \in C^2(M)$ , the above quantity can be written

$$\begin{aligned} \operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) &= Q_{f-\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) \\ &\quad + \langle \tilde{\omega}, (2\nabla f \cdot \nabla \varphi - 2|\nabla \varphi|^2 + h\mathcal{L}_{\nabla \varphi} + h\mathcal{L}_{\nabla \varphi}^*) \tilde{\omega} \rangle \\ &\quad + h \left( \int_{N_n} - \int_{N_t} \right) \langle \tilde{\omega}, \tilde{\omega} \rangle_{\Lambda T_{\sigma}^* \Omega} \left( \frac{\partial \varphi}{\partial n} \right) (\sigma) d\sigma. \end{aligned} \quad (21)$$

*Proof.* We recall that according to Remark 2.4,  $W_{\partial}(\Omega; \Lambda T^*M)$  is a Lipschitz-module. For the first statement (18), simply write

$$\begin{aligned} \operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) &= \operatorname{Re} Q_{f, \bar{\Omega}, h}(e^{-\frac{\varphi}{h}} \tilde{\omega}, e^{\frac{\varphi}{h}} \tilde{\omega}) \\ &= \operatorname{Re} \langle (d_{f, h} - d\varphi \wedge) \tilde{\omega}, (d_{f, h} + d\varphi \wedge) \omega \rangle \\ &\quad + \operatorname{Re} \langle (d_{f, h}^* + \mathbf{i}_{\nabla \varphi}) \tilde{\omega}, (d_{f, h}^* - \mathbf{i}_{\nabla \varphi}) \tilde{\omega} \rangle \\ &= \|d_{f, h} \omega\|^2 + \|d_{f, h}^* \omega\|^2 - \langle d\varphi \wedge \tilde{\omega}, d\varphi \wedge \tilde{\omega} \rangle - \langle \mathbf{i}_{\nabla \varphi} \tilde{\omega}, \mathbf{i}_{\nabla \varphi} \tilde{\omega} \rangle \\ &= \|d_{f, h} \omega\|^2 + \|d_{f, h}^* \omega\|^2 - \langle \tilde{\omega}, \underbrace{(\mathbf{i}_{\nabla \varphi}(d\varphi \wedge) + (d\varphi \wedge) \mathbf{i}_{\nabla \varphi})}_{=|\nabla \varphi|^2} \tilde{\omega} \rangle. \end{aligned}$$

For (19), we start from (18) after noticing that  $\chi_j \tilde{\omega} \in W_{\partial}(\Omega; \Lambda T^*M)$  when  $\omega \in W_{\partial}(\Omega; \Lambda T^*M)$ . We compute

$$\begin{aligned} \|d_{f, h} \chi_j \tilde{\omega}\|^2 + \|d_{f, h}^* \chi_j \tilde{\omega}\|^2 &= \|\chi_j d_{f, h} \tilde{\omega}\|^2 + \|\chi_j d_{f, h}^* \tilde{\omega}\|^2 \\ &\quad + 2\operatorname{Re} \langle \chi_j d_{f, h} \tilde{\omega}, (hd\chi_j \wedge) \tilde{\omega} \rangle - 2\operatorname{Re} \langle \chi_j d_{f, h}^* \tilde{\omega}, h\mathbf{i}_{\nabla \chi_j} \tilde{\omega} \rangle \\ &\quad + h^2 [\langle d\chi_j \wedge \tilde{\omega}, d\chi_j \wedge \tilde{\omega} \rangle + \langle \mathbf{i}_{\nabla \chi_j} \tilde{\omega}, \mathbf{i}_{\nabla \chi_j} \tilde{\omega} \rangle] \\ &= \|\chi_j d_{f, h} \tilde{\omega}\|^2 + \|\chi_j d_{f, h}^* \tilde{\omega}\|^2 \\ &\quad + \operatorname{Re} \langle d_{f, h} \tilde{\omega}, (hd\chi_j^2 \wedge) \tilde{\omega} \rangle - \operatorname{Re} \langle d_{f, h}^* \tilde{\omega}, h\mathbf{i}_{\nabla \chi_j^2} \tilde{\omega} \rangle \\ &\quad + h^2 \langle \tilde{\omega}, \underbrace{(\mathbf{i}_{\nabla \chi_j}(d\chi_j \wedge) + (d\chi_j \wedge) \mathbf{i}_{\nabla \chi_j})}_{=|\nabla \chi_j|^2} \tilde{\omega} \rangle. \end{aligned}$$

Summing w.r.t  $j \in \{1, \dots, J\}$  leads to

$$Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) - \sum_{j=1}^J Q_{f, \bar{\Omega}, h}(\chi_j \omega, e^{\frac{2\varphi}{h}} \chi_j \omega) = -h^2 \sum_{j=1}^J \|\nabla \chi_j \tilde{\omega}\|^2.$$

Let us now assume that  $f \in \mathcal{C}^2(M)$ . According to (18), the identity

$$\operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) = Q_{f, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) - \langle \tilde{\omega}, |\nabla \varphi|^2 \tilde{\omega} \rangle$$

holds true and it suffices to prove the formula (20) when  $\varphi = 0$ . To this end, one first writes for  $\omega \in D(Q_{f, \bar{\Omega}, h})$ ,

$$\begin{aligned} \|d_{f, h} \omega\|_{L^2}^2 + \|d_{f, h}^* \omega\|_{L^2}^2 &= h^2 \|d\omega\|_{L^2}^2 + h^2 \|d^* \omega\|_{L^2}^2 + \|df \wedge \omega\|_{L^2}^2 \\ &\quad + \|\mathbf{i}_{\nabla f} \omega\|_{L^2}^2 + h(\langle df \wedge \omega, d\omega \rangle_{L^2} + \langle d\omega, df \wedge \omega \rangle \\ &\quad + \langle d^* \omega, \mathbf{i}_{\nabla f} \omega \rangle_{L^2} + \langle \mathbf{i}_{\nabla f} \omega, d^* \omega \rangle) \\ &= h^2 \|d\omega\|_{L^2}^2 + h^2 \|d^* \omega\|_{L^2}^2 + \|\nabla f \omega\|_{L^2}^2 \\ &\quad + h\langle \omega, (\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*) \omega \rangle_{L^2} + h(\langle df \wedge \omega, d\omega \rangle_{L^2} \\ &\quad - \langle d^*(df \wedge \omega), \omega \rangle_{L^2} - \langle d\mathbf{i}_{\nabla f} \omega, \omega \rangle_{L^2} + \langle \mathbf{i}_{\nabla f} \omega, d^* \omega \rangle_{L^2}), \end{aligned}$$

where the last equality holds thanks to the relations  $(df \wedge)^* = \mathbf{i}_{\nabla f}$ ,

$$\mathcal{L}_{\nabla f} = d \circ \mathbf{i}_{\nabla f} + \mathbf{i}_{\nabla f} \circ d \quad \text{and} \quad \mathcal{L}_{\nabla f}^* = (df \wedge) \circ d^* + d^* \circ (df \wedge).$$

The relation (20) follows using in addition the generalized Green formula (13) which gives here, since  $\omega \in D(Q_{f, \bar{\Omega}, h})$  and hence admits a total trace on  $N$ , and  $df \wedge \omega, \mathbf{i}_{\nabla f} \omega \in \{v \in L^2, dv \in L^2, d^*v \in L^2\}$ :

$$\begin{aligned} \langle df \wedge \omega, d\omega \rangle_{L^2} - \langle d^*(df \wedge \omega), \omega \rangle_{L^2} &= \int_{N_n} \langle n^b \wedge \omega, n^b \wedge \mathbf{i}_n(df \wedge \omega) \rangle_{T_\sigma^* \Omega} d\sigma \\ &= \int_{N_n} \langle \omega, \mathbf{i}_n(n^b \wedge \mathbf{i}_n(df \wedge \omega)) \rangle_{T_\sigma^* \Omega} d\sigma \\ &= \int_{N_n} \langle \omega, \mathbf{i}_n(df \wedge \omega) \rangle_{T_\sigma^* \Omega} d\sigma \\ &= \int_{N_n} (\partial_n f \langle \omega, \omega \rangle_{T_\sigma^* \Omega} - \underbrace{\langle \omega, df \wedge \mathbf{i}_n \omega \rangle}_{=0})_{T_\sigma^* \Omega} d\sigma \\ &= \int_{N_n} \partial_n f \langle \omega, \omega \rangle_{T_\sigma^* \Omega} d\sigma \end{aligned}$$

as well as

$$\begin{aligned} \langle \mathbf{i}_{\nabla f} \omega, d^* \omega \rangle_{L^2} - \langle d\mathbf{i}_{\nabla f} \omega, \omega \rangle_{L^2} &= - \int_{N_t} \langle n^b \wedge \mathbf{i}_{\nabla f} \omega, n^b \wedge \mathbf{i}_n \omega \rangle_{T_\sigma^* \Omega} \\ &= - \int_{N_t} \partial_n f \langle \omega, \omega \rangle_{T_\sigma^* \Omega} d\sigma d\sigma. \end{aligned}$$

Lastly, let us prove the relation (21). By direct expansion with  $f$  and  $\varphi$  Lipschitz continuous and

$$d_{f-\varphi, h} = d_{f, h} - (d\varphi \wedge) = hd + (df \wedge) - (d\varphi \wedge) \quad \text{and} \quad d_{f-\varphi, h}^* = d_{f, h}^* - \mathbf{i}_{\nabla \varphi} = hd^* + \mathbf{i}_{\nabla f} - \mathbf{i}_{\nabla \varphi},$$

we obtain

$$\begin{aligned}
Q_{f-\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) &= Q_{f, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) \\
&\quad - 2\operatorname{Re} (\langle df \wedge \tilde{\omega}, d\varphi \wedge \tilde{\omega} \rangle + \langle \mathbf{i}_{\nabla f} \tilde{\omega}, \mathbf{i}_{\nabla \varphi} \tilde{\omega} \rangle) \\
&\quad - 2h\operatorname{Re} (\langle d\tilde{\omega}, d\varphi \wedge \tilde{\omega} \rangle + \langle d^* \tilde{\omega}, \mathbf{i}_{\nabla \varphi} \tilde{\omega} \rangle) \\
&\quad + \|d\varphi \wedge \tilde{\omega}\|^2 + \|\mathbf{i}_{\nabla \varphi} \tilde{\omega}\|^2.
\end{aligned}$$

By adding this relation for the pairs  $(f, \varphi)$  and  $(0, -\varphi)$ , we obtain

$$\begin{aligned}
Q_{f-\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) + Q_{\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) &= Q_{f, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) + Q_{0, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) \\
&\quad - 2\operatorname{Re} (\underbrace{\langle df \wedge \tilde{\omega}, d\varphi \wedge \tilde{\omega} \rangle + \langle \mathbf{i}_{\nabla f} \tilde{\omega}, \mathbf{i}_{\nabla \varphi} \tilde{\omega} \rangle}_{=\langle \tilde{\omega}, (\nabla f \cdot \nabla \varphi) \tilde{\omega} \rangle}) \\
&\quad + 0 \\
&\quad + 2\|\nabla \varphi| \tilde{\omega}\|^2.
\end{aligned}$$

Finally, using the relation (18) gives

$$\begin{aligned}
\operatorname{Re} Q_{f, \bar{\Omega}, h}(\omega, e^{\frac{2\varphi}{h}} \omega) &= Q_{f, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) - \|\nabla \varphi| \tilde{\omega}\|^2 \\
&= Q_{f-\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) + 2\langle \tilde{\omega}, (\nabla f \cdot \nabla \varphi - |\nabla \varphi|^2) \tilde{\omega} \rangle \\
&\quad + Q_{\varphi, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) - Q_{0, \bar{\Omega}, h}(\tilde{\omega}, \tilde{\omega}) - \|\nabla \varphi| \tilde{\omega}\|^2.
\end{aligned}$$

When in addition  $\varphi \in \mathcal{C}^2(M)$ , using (18) and (20) with  $f = \varphi$  leads to the relation (21).  $\square$

**Remark 2.11.** *Alternatively, one could first prove the relation (21) for  $f, \varphi \in \mathcal{C}^2(M)$ , and then approximate a general  $f \in W^{1, \infty}(M)$  by a sequence in  $\mathcal{C}^2(M)$  as in Remark 2.4.*

### 2.3.2 Exponential decay estimates

Under Hypothesis 1.2, these estimates rely on the integration by parts formula (20) of Lemma 2.10. They will be replaced by a new hypothesis for more general Lipschitz function  $f$ , which will be ultimately verified when  $f$  is Lipschitz subanalytic in Subsection 8.3.

**Definition 2.12.** *Assume Hypothesis 1.2 for  $f$  and remember*

$$M_{reg} = \{x \in (M \setminus \operatorname{suppsing} f), \nabla f(x) \neq 0\} \subset M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\}).$$

*The Agmon distance  $d_{Ag}$  on  $M$  associated with  $f \in \mathcal{C}^\infty(M)$  is the geodesic pseudodistance associated with the degenerate metric  $1_{M_{reg}} |\nabla f|^2 g$ , namely*

$$d_{Ag}(x, y) = \inf_{\substack{\gamma \in \mathcal{C}^1([0, 1]; M), \\ \gamma(0) = x, \gamma(1) = y}} \int_0^1 1_{M_{reg}}(\gamma(t)) |\nabla f(\gamma(t))| |\gamma'(t)| dt.$$

Because  $f \in W^{1, \infty}(M) \cap \mathcal{C}^\infty(M_{reg})$ , we know  $d_{Ag}(x, y) \leq \|\nabla f\|_{L^\infty} d_g(x, y)$  where  $d_g$  is the geodesic distance and  $d_{Ag}$  is a Lipschitz function of  $(x, y) \in M \times M$ . Moreover when  $x, y$  belong to the same connected component of  $M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\})$  any  $\mathcal{C}^1$  curve  $\gamma$  staying in this connected component satisfies

$$\int_0^1 |\nabla f(\gamma(t))| |\gamma'(t)| dt \geq \left| \int_0^1 \nabla f(\gamma(t)) \cdot \gamma'(t) dt \right| = |f(y) - f(x)|.$$

For a general  $\gamma \in \mathcal{C}^1([0, 1]; M)$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ ,  $\{f(\gamma(t)), t \in [0, 1]\}$  is a compact interval. Therefore, bounding from below the integral  $\int_0^1 \dots dt$  by a sum of integrals on intervals  $]t_k, t'_k[$ , where  $f(\gamma(t)) \notin \{c_1, \dots, c_{N_f}, \max(f \circ \gamma), \min(f \circ \gamma)\}$ , leads to

$$\int_0^1 1_{M_{reg}}(\gamma(t)) |\nabla f(\gamma(t))| |\gamma'(t)| dt \geq \max_{t \in [0, 1]} f(\gamma(t)) - \min_{t \in [0, 1]} f(\gamma(t)) \geq |f(y) - f(x)|.$$

We obtain

$$\forall x, y \in M, \quad \|\nabla f\|_{L^\infty} d_g(x, y) \geq d_{Ag}(x, y) \geq |f(y) - f(x)|. \quad (22)$$

When  $f$  is a  $\mathcal{C}^\infty$  Morse function, more details about the more general broken geodesic curves, which do not hold anymore with our general assumption and which we do not need, are given in [HeSj4].

**Proposition 2.13.** *Assume Hypothesis 1.2 for  $f$  and Hypothesis 2.2 for  $\overline{\Omega} = \Omega \sqcup N_t \sqcup N_n$  with*

$$\partial\Omega = N_t \sqcup N_n \subset M_{reg}, \quad \frac{\partial f}{\partial n}|_{N_t} < 0, \quad \frac{\partial f}{\partial n}|_{N_n} > 0. \quad (23)$$

Let  $\Delta_{f, \overline{\Omega}, h}$  be the self-adjoint operator defined in Proposition 2.8 and let  $U$  denote the compact subset of  $\Omega$ ,  $U = (M \setminus M_{reg}) \cap \Omega$ . All families  $(\lambda_h)_{h>0}$  in  $\mathbb{C}$ ,  $(r_h)_{h>0}$  in  $L^2(\Omega)$  and  $(\omega_h)_{h>0}$  in  $D(\Delta_{f, \overline{\Omega}, h}) \subset W_\partial(\Omega; \Lambda T^*M)$  such that

$$(\Delta_{f, \overline{\Omega}, h} - \lambda_h)\omega_h = r_h, \quad \text{supp } r_h \subset K, \quad \lim_{h \rightarrow 0} \lambda_h = 0,$$

where  $K$  is a fixed compact subset of  $\overline{\Omega}$ , satisfy the estimate (see (14) and the lines below)

$$\|e^{\frac{d_{Ag}(\cdot, U \cup K)}{h}} \omega_h\|_{W_\partial(\Omega)} = \tilde{O}(1) \times (\|r_h\|_{L^2(\Omega)} + t_U \|\omega_h\|_{L^2(\Omega)}),$$

where  $t_U = 1$  if  $U \neq \emptyset$  and  $t_U = 0$  if  $U = \emptyset$ .

*Proof.* For  $\varepsilon \in ]0, 1[$ , one introduces  $K_\varepsilon = \{y \in \overline{\Omega}, d_{Ag}(y, U \cup K) \leq \varepsilon\}$  and  $\chi_1 = \chi_{1, \varepsilon}, \chi_2 = \chi_{2, \varepsilon} \in \mathcal{C}^\infty(\overline{\Omega}, [0, 1])$  such that  $\chi_1 \equiv 0$  when  $U = \emptyset$  and  $\chi_1 = 1$  near  $U$  else,  $\text{supp } \chi_1 \subset K_\varepsilon \cap \Omega$ , and  $\chi_1^2 + \chi_2^2 \equiv 1$ . Let us also introduce  $\varphi_\varepsilon : x \mapsto (1 - \varepsilon)d_{Ag}(x, K_\varepsilon) \in W^{1, \infty}(\Omega)$ , so that  $\varphi_\varepsilon$  satisfies  $|\nabla \varphi_\varepsilon| \leq (1 - \varepsilon)|\nabla f|$  almost everywhere in  $\Omega$ . Setting  $\tilde{\omega}_h := e^{\frac{\varphi_\varepsilon}{h}} \omega_h$  and applying (19) with  $\varphi_\varepsilon = 0$  on  $K_\varepsilon$ ,  $\text{supp } \chi_1, \text{supp } r_h \subset K_\varepsilon$ , we obtain

$$\begin{aligned} \langle r_h, \omega_h \rangle_{L^2} + \lambda_h \|\tilde{\omega}_h\|_{L^2}^2 &= \text{Re } Q_{f, \overline{\Omega}, h}(\omega_h, e^{2\frac{\varphi_\varepsilon}{h}} \omega_h) \\ &\geq \text{Re } Q_{f, \overline{\Omega}, h}(\chi_2 \omega_h, \chi_2 e^{2\frac{\varphi_\varepsilon}{h}} \omega_h) + Q_{f, \overline{\Omega}, h}(\chi_1 \omega_h, \chi_1 \omega_h) - c_\varepsilon h^2 \|\tilde{\omega}_h\|_{L^2}^2. \end{aligned}$$

Then, applying (20) of Lemma 2.10 with a  $\mathcal{C}^2$ -extension to  $M$  of  $f|_{\text{supp } \chi_2}$ , with  $|\nabla f|^2 \geq C_\varepsilon$  on  $\text{supp } \chi_2$  and the sign condition (23) leads to

$$\begin{aligned} \|\omega_h\|_{L^2} \|r_h\|_{L^2} &\geq Q_{f, \overline{\Omega}, h}(\chi_1 \omega_h, \chi_1 \omega_h) + h^2 (\|d\chi_2 \tilde{\omega}_h\|_{L^2}^2 + \|d^* \chi_2 \tilde{\omega}_h\|_{L^2}^2) \\ &\quad + (C_\varepsilon - \lambda_h - c_\varepsilon h^2) \|\chi_2 \tilde{\omega}_h\|_{L^2}^2 - t_U (\lambda_h + c_\varepsilon h^2) \|\chi_1 \omega_h\|_{L^2}^2 \\ &\geq Q_{f, \overline{\Omega}, h}(\chi_1 \omega_h, \chi_1 \omega_h) + C'_\varepsilon h^2 \|\chi_2 \tilde{\omega}_h\|_W^2 - t_U \|\omega_h\|_{L^2}^2, \end{aligned} \quad (24)$$

where we recall from Definition 2.3 that  $\|\omega\|_W = \|\omega\|_{L^2} + \|d\omega\|_{L^2} + \|d^* \omega\|_{L^2}$ .

Since  $Q_{f, \overline{\Omega}, h}(\chi_1 \omega_h, \chi_1 \omega_h) \geq 0$  and  $\|\omega_h\|_{L^2} \leq C \|r_h\|_{L^2} + t_U \|\omega_h\|_{L^2}$  (this is obvious when  $U \neq \emptyset$  and apply (20) of Lemma 2.10 with  $\varphi = 0$  else), we obtain the estimate

$$\|\chi_2 \tilde{\omega}_h\|_{W_\partial(\Omega)} \leq \frac{C''_\varepsilon}{h} (\|r_h\|_{L^2} + t_U \|\omega_h\|_{L^2}). \quad (25)$$

This ends the proof when  $U = \emptyset$ .  
When  $U \neq \emptyset$ , the relations (24) and

$$\begin{aligned} Q_{f,\bar{\Omega},h}(\chi_1\omega_h, \chi_1\omega_h) &= \|(hd + df \wedge)\chi_1\omega_h\|_{L^2}^2 + \|(hd^* + \mathbf{i}\nabla f)\chi_1\omega_h\|_{L^2}^2 \\ &\geq \frac{h^2}{2}(\|d\chi_1\omega_h\|_{L^2}^2 + \|d^*\chi_1\omega_h\|_{L^2}^2) - C\|\chi_1\omega_h\|_{L^2}^2 \end{aligned}$$

lead, since  $\varphi_\varepsilon = 0$  on  $\text{supp } \chi_1$ , to

$$\|\chi_1\tilde{\omega}_h\|_{W_\partial(\Omega)} \leq \frac{C'}{h}(\|r_h\|_{L^2} + \|\omega_h\|_{L^2}). \quad (26)$$

The statement of Proposition 2.13 then follows from (25) and (26), by using again the IMS localization formula (19) with now  $\varphi = f = 0$  but  $\omega$  replaced by  $\tilde{\omega}$ .  $\square$

Following [HeSj2][DiSj] we extend the definition of  $\tilde{O}$  to the kernels of bounded operators from  $L^2$  to  $W$ , which appears to be more natural than  $W^{1,2}$  in our setting (see indeed Definition 2.3 and Proposition 2.5). For more flexibility, boundary conditions do not appear in the following definition and the full space  $W(\Omega; \Lambda T^*M)$  of Definition 2.3 is used.

**Definition 2.14.** *Let the domain  $\bar{\Omega}$  satisfy Hypothesis 2.2. Let the operator  $A_h$  act continuously from  $L^2(\Omega; \Lambda T^*M)$  to  $W(\Omega; \Lambda T^*M)$  and let  $\Phi \in C^0(\bar{\Omega} \times \bar{\Omega}; \mathbb{R})$ . We say that the kernel  $A_h(x, y)$  of  $A_h$  is  $\tilde{O}(e^{-\frac{\Phi(x,y)}{h}})$  if, for all  $x_0, y_0 \in \Omega$  and  $\varepsilon > 0$ , there exist neighborhoods  $U_\varepsilon, V_\varepsilon$  in  $M$  of  $y_0$  and  $x_0$  and constants  $h_\varepsilon$  such that*

$$\forall h \in ]0, h_\varepsilon[, \forall \chi \in C_0^\infty(V_\varepsilon), \exists C_{\chi,\varepsilon} > 0, \forall u \in L^2(\Omega) \text{ s.t. } \text{supp } u \subset U_\varepsilon, \\ \|\chi A_h u\|_{W(\Omega)} \leq C_{\chi,\varepsilon} e^{-\frac{\Phi(x_0,y_0)-\varepsilon}{h}} \|u\|_{L^2}.$$

For a finite family  $(\Phi_k)_{k \in \{1, \dots, K\}}$  in  $C^0(\bar{\Omega} \times \bar{\Omega}; \mathbb{R})$ , the kernel  $A_h(x, y)$  of  $A_h$  is said to be  $\tilde{O}(\sum_{k=1}^K e^{-\frac{\Phi_k(x,y)}{h}})$  when it is  $\tilde{O}(e^{-\frac{\min_{1 \leq k \leq K} \Phi_k(x,y)}{h}})$ .

When  $A_h(x, y) = \tilde{O}(e^{-\frac{\Phi(x,y)}{h}})$  and  $B_h(x, y) = \tilde{O}(e^{-\frac{\Psi(x,y)}{h}})$  and  $D_h$  is a differential operator of order  $\leq 1$  which vanishes in a fixed (independent of  $h$ ) neighborhood of  $\partial\Omega$  (remember  $W(\Omega; \Lambda T^*M) \subset W_{loc}^{1,2}(\Omega; \Lambda T^*M)$ ), with  $\|D_h\|_{\mathcal{L}(W^{1,2}; L^2)} = \tilde{O}(1)$ , then  $(A_h D_h B_h)(x, y) = \tilde{O}(e^{-\frac{\Theta(x,y)}{h}})$  with  $\Theta(x, y) = \min_{z \in \bar{\Omega}} \Phi(x, z) + \Psi(z, y)$ .

If  $A_h(x, y) = \tilde{O}(e^{-\frac{\Phi(x,y)}{h}})$  and  $\psi \in C^0(\bar{\Omega})$ ,  $\varphi \in W^{1,\infty}(\bar{\Omega})$  satisfy  $\varphi(x) \leq \Phi(x, y) - \psi(y)$  for all  $y \in \bar{\Omega}$ , then  $\sup_{u \in L^2(\Omega)} \frac{\|e^{\frac{\varphi}{h}} A_h u\|_{W^{1,2}}}{\|e^{\frac{\psi}{h}} u\|_{L^2}} = \tilde{O}(1)$ .

An easy application concerns the case when the gradient of  $f$  does not vanish in  $\bar{\Omega} \subset M_{reg}$ , under Hypothesis 1.2.

**Proposition 2.15.** *Assume Hypothesis 1.2 for  $f$ , and Hypothesis 2.2 for  $\bar{\Omega} = \Omega \sqcup N_t \sqcup N_n$  with now*

$$\bar{\Omega} \subset M_{reg} \quad , \quad \frac{\partial f}{\partial n}|_{N_t} < 0 \quad , \quad \frac{\partial f}{\partial n}|_{N_n} > 0, \quad (27)$$

where we recall that  $f \in C^\infty(M_{reg})$  has a non vanishing gradient. The self-adjoint operator  $\Delta_{f,\bar{\Omega},h}$  defined in Proposition 2.8 is bounded from below by  $c_{\Omega,f,h_1} > 0$ . when  $h \in ]0, h_1[$  with  $h_1 > 0$  small enough. If  $\lim_{h \rightarrow 0} \rho(h) = 0^+$ , then the resolvent  $(\Delta_{f,\bar{\Omega},h} - z)^{-1}$ ,  $|z| \leq \rho(h)$ , well defined for  $h \in ]0, h_0[$ ,  $h_0 > 0$  small enough, satisfies

$$(\Delta_{f,\bar{\Omega},h} - z)^{-1}(x, y) = \tilde{O}(e^{-\frac{d_{A_g}(x,y)}{h}}) \leq \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h}}),$$

according to Definition 2.14 and uniformly with respect to  $z$ ,  $|z| \leq \rho(h)$ .



*Proof.* The lower bound and the definition of the resolvent is deduced from (20) in Lemma 2.10 applied with  $\varphi \equiv 0$ ,  $|\nabla f(x)| \geq c > 0$  for all  $x \in \overline{\Omega}$  and where the condition (27) ensures the positivity of the boundary terms. The estimate of the kernel is then a straightforward consequence of Proposition 2.13 with here  $U = \emptyset$ .  $\square$

We cannot prove Proposition 2.13 and Proposition 2.15 for a general Lipschitz function even under Hypothesis 1.6. We replace it by an assumption which is proved to be fulfilled by subanalytic Lipschitz functions in Subsection 8.3

**Hypothesis 2.16.** *For a Lipschitz function which satisfy Hypothesis 1.6 with the “critical values”  $c_1 < \dots < c_{N_f}$ , we assume that Proposition 2.13 and Proposition 2.15 hold true after replacing  $M_{reg}$  by  $M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\})$ ,  $d_{Ag}(x, y)$  by the pseudodistance  $|f(x) - f(y)|$ , and by restricting to the case  $\overline{\Omega} = f^{-1}([a, b])$ ,  $a < b$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$ .*

### 2.3.3 Adjusting boundary conditions

Another consequence of Agmon estimates is the following lemma which will be used to correct boundary conditions and to extend solutions to  $d_{f,h}\omega = 0$  to a wider domain with suitably small errors. Under Hypothesis 1.2, it is stated in the more general framework of Proposition 2.15 with  $\overline{\Omega} \subset M_{reg}$ , although it will be applied essentially when  $\overline{\Omega} = f^{-1}([a, b])$  with  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \emptyset$ . For a more general Lipschitz function we work directly in the framework of Hypothesis 2.16.

**Lemma 2.17.** *Assume Hypothesis 1.2 for  $f$  and Hypothesis 2.2 for  $\overline{\Omega} = \Omega \sqcup N_t \sqcup N_n$  with  $\overline{\Omega} \subset M_{reg}$  and the sign conditions  $\frac{\partial f}{\partial n}|_{N_t} < 0$ ,  $\frac{\partial f}{\partial n}|_{N_n} > 0$ . Consider the operator  $\Delta_{f,\overline{\Omega},h}$  of Proposition 2.8. There exists  $c > 0$  and  $h_0 > 0$  determined by  $f$  and  $\overline{\Omega}$  and for any pair of cut-off functions  $\chi, \tilde{\chi} \in C^\infty(\overline{\Omega}; [0, 1])$  which satisfies  $d\chi, d\tilde{\chi} \in C_0^\infty(\Omega)$ , with  $\tilde{\chi} \equiv 1$  in a neighborhood of  $\text{supp } d\chi$ , a constant  $C_{\chi,\chi'} > 0$  such that the following holds. When  $\omega \in W(\Omega; \Lambda T^*M)$ , the forms*

$$\eta_1 = d_{f,\overline{\Omega},h}^*(\Delta_{f,\overline{\Omega},h})^{-1}((hd\chi) \wedge \omega) \quad \text{and} \quad \eta_2 = d_{f,\overline{\Omega},h}(\Delta_{f,\overline{\Omega},h})^{-1}(h\mathbf{i}_{\nabla\chi}\omega)$$

both belong to

$$D(\Delta_{f,\overline{\Omega},h}) \subset W_\partial(\Omega; \Lambda T^*M) \subset W(\Omega; \Lambda T^*M)$$

and satisfy the following inequality with convention  $d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi) = +\infty$  when  $\tilde{\chi}$  is the constant function 1:

$$\begin{aligned} \|\eta_1\|_{L^2} &\leq \frac{1}{\sqrt{c}} \|(hd\chi) \wedge \omega\|_{L^2} \quad \text{and} \quad \|\eta_2\|_{L^2} \leq \frac{1}{\sqrt{c}} \|(h\mathbf{i}_{\nabla\chi})\omega\|_{L^2}, \\ \|d_{f,h}(\chi\omega - \tilde{\chi}\eta_1)\|_{L^2} &\leq \frac{1}{\sqrt{c}} \|(hd\chi) \wedge d_{f,h}\omega\|_{L^2} + \|\chi d_{f,h}\omega\|_{L^2} \\ &\quad + \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi)}{h}}\right) \|(hd\chi) \wedge \omega\|_{L^2}, \\ \|d_{f,h}^*(\tilde{\chi}\eta_1)\|_{L^2} &\leq \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi)}{h}}\right) \|(hd\chi) \wedge \omega\|_{L^2}, \\ \|d_{f,h}^*(\chi\omega - \tilde{\chi}\eta_2)\|_{L^2} &\leq \frac{1}{\sqrt{c}} \|h\mathbf{i}_{\nabla\chi}d_{f,h}^*\omega\|_{L^2} + \|\chi d_{f,h}^*\omega\|_{L^2} \\ &\quad + \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi)}{h}}\right) \|h\mathbf{i}_{\nabla\chi}\omega\|_{L^2}, \end{aligned}$$

$$\|d_{f,h}(\tilde{\chi}\eta_2)\|_{L^2} \leq \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi)}{h}}\right) \|h\mathbf{i}_{\nabla\chi}\omega\|_{L^2},$$

$$\begin{aligned} \left( \begin{array}{c} \|d_{f,h}(\chi\omega - \tilde{\chi}(\eta_1 + \eta_2))\|_{L^2} \\ + \\ \|d_{f,h}^*(\chi\omega - \tilde{\chi}(\eta_1 + \eta_2))\|_{L^2} \end{array} \right) &\leq C_{\chi, \tilde{\chi}} [\|d_{f,h}\omega\|_{L^2} + \|d_{f,h}^*\omega\|_{L^2}] \\ &+ \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\tilde{\chi}, \text{supp } d\chi)}{h}}\right) \|\omega\|_{L^2(\text{supp } d\chi)}. \end{aligned}$$

When  $f$  is a Lipschitz function which satisfies Hypothesis 1.6 and Hypothesis 2.16 the results are the same when  $\bar{\Omega} = f^{-1}([a, b])$ ,  $c_n < a < b < c_{n+1}$ , and  $d_{Ag}(K, K')$  is replaced by  $\inf_{x \in K, y \in K'} |f(x) - f(y)|$ .

**Remark 2.18.** Note that  $\omega$  is not assumed to belong to the domain of  $d_{f, \bar{\Omega}, h}$ ,  $d_{f, \bar{\Omega}, h}^*$  or  $\Delta_{f, \bar{\Omega}, h}$  (no boundary conditions) and the same holds in general for  $\chi\omega$ . Accordingly, we used the notations  $d_{f,h}$  and  $d_{f,h}^*$  for the differential operators. In some applications  $\chi$  will be chosen such that  $\chi\omega$  and therefore  $\chi\omega - \tilde{\chi}(\eta_1 + \eta_2)$  belong to one of these domains. Example given, if  $\chi\omega \in D(\Delta_{f, \bar{\Omega}, h})$ , the last inequality then provides a good estimate of  $Q_{f, \bar{\Omega}, h}(\chi\omega - \tilde{\chi}(\eta_1 + \eta_2))$  when  $\text{supp } \chi$  and  $\text{supp } \tilde{\chi}$  are well chosen.

*Proof.* Proposition 2.15 under Hypothesis 1.2, or Hypothesis 2.16 with  $\Omega = f^{-1}([a, b])$  in the more general case, ensures  $\Delta_{f, \bar{\Omega}, h} \geq c > 0$  for  $h \in ]0, h_0[$ . When  $\Delta_{f, \bar{\Omega}, h} u = v \in L^2(\Omega)$ , it implies first  $\|u\| \leq \frac{1}{c} \|v\|$ . We apply (18) with  $\varphi = 0$ :

$$\|d_{f, \bar{\Omega}, h} u\|^2 + \|d_{f, \bar{\Omega}, h}^* u\|^2 = \text{Re} \langle u, \Delta_{f, \bar{\Omega}, h} u \rangle \leq \|u\| \|v\| \leq \frac{1}{c} \|v\|^2.$$

This proves the two first inequalities for  $\|\eta_1\|_{L^2}$  and  $\|\eta_2\|_{L^2}$ . Moreover, the equality

$$d_{f,h}(\chi\omega) = \chi(d_{f,h}\omega) + (hd\chi) \wedge \omega$$

implies

$$0 = d_{f,h}[\chi(d_{f,h}\omega)] + d_{f,h}[(hd\chi) \wedge \omega] = (hd\chi) \wedge (d_{f,h}\omega) + d_{f,h}[(hd\chi) \wedge \omega]. \quad (28)$$

Our assumptions ensure  $(hd\chi) \wedge \omega \in D(d_{f, \bar{\Omega}, h})$  and  $\eta_1 \in D(\Delta_{f, \bar{\Omega}, h}) \subset D(d_{f, \bar{\Omega}, h})$ . By using  $\Delta_{f, \bar{\Omega}, h} = d_{f, \bar{\Omega}, h} d_{f, \bar{\Omega}, h}^* + d_{f, \bar{\Omega}, h}^* d_{f, \bar{\Omega}, h}$  and the commutation relation stated in Proposition 2.8-4), compute:

$$\begin{aligned} d_{f, \bar{\Omega}, h} \eta_1 &= d_{f, \bar{\Omega}, h} d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} (hd\chi \wedge \omega) \\ &= (hd\chi) \wedge \omega - d_{f, \bar{\Omega}, h}^* d_{f, \bar{\Omega}, h} (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge \omega) \\ &= (hd\chi) \wedge \omega - d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} (d_{f, \bar{\Omega}, h} [(hd\chi) \wedge \omega]) \\ &\stackrel{(28)}{=} (hd\chi) \wedge \omega + d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge d_{f,h}\omega). \end{aligned}$$

With  $d_{f,h}(\tilde{\chi}\eta_1) = \tilde{\chi}(d_{f,h}\eta_1) + (hd\tilde{\chi}) \wedge \eta_1$  and  $\tilde{\chi}d\chi \equiv d\chi$ , this implies:

$$d_{f,h}(\tilde{\chi}\eta_1) = (hd\chi) \wedge \omega + \tilde{\chi} d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge d_{f,h}\omega) + (hd\tilde{\chi}) \wedge d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge \omega).$$

We have proved

$$\begin{aligned} d_{f,h}(\chi\omega - \tilde{\chi}\eta_1) &= \chi(d_{f,h}\omega) - \underbrace{\tilde{\chi} d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge d_{f,h}\omega)}_{(I)} \\ &\quad - \underbrace{(hd\tilde{\chi}) \wedge d_{f, \bar{\Omega}, h}^* (\Delta_{f, \bar{\Omega}, h})^{-1} ((hd\chi) \wedge \omega)}_{(II)}. \end{aligned}$$

Since  $\|d_{f,\bar{\Omega},h}^*(\Delta_{f,\bar{\Omega},h})^{-1}\| \leq \frac{1}{\sqrt{c}}$  for  $h$  small enough, it follows

$$\|(I)\|_{L^2} \leq \frac{1}{\sqrt{c}} \|(hd\chi) \wedge d_{f,h}\omega\|_{L^2}.$$

For the last term, Proposition 2.15 under Hypothesis 1.2 says

$$\|(II)\|_{L^2} = \tilde{O}\left(e^{-\frac{d_{Ag}(\text{supp } d\bar{\chi}, \text{supp } d\chi)}{h}}\right) \|(hd\chi) \wedge \omega\|_{L^2},$$

while Hypothesis 2.16 with  $\bar{\Omega} = f^{-1}([a, b])$  in the more general case gives

$$\|(II)\|_{L^2} = \tilde{O}\left(e^{-\frac{\min_{x \in \text{supp } d\bar{\chi}, y \in \text{supp } d\chi} |f(x) - f(y)|}{h}}\right) \|(hd\chi) \wedge \omega\|_{L^2},$$

Meanwhile the identities  $d_{f,h}^*(\tilde{\chi}\eta_1) = \tilde{\chi}d_{f,h}^*\eta_1 + h\mathbf{i}_{\nabla\tilde{\chi}}\eta_1$  and  $d_{f,h}^*\eta_1 = 0$  lead to

$$d_{f,h}^*(\tilde{\chi}\eta_1) = h\mathbf{i}_{\nabla\tilde{\chi}}\eta_1 = h\mathbf{i}_{\nabla\tilde{\chi}}d_{f,\bar{\Omega},h}^*(\Delta_{f,\bar{\Omega},h})^{-1}((hd\chi) \wedge \omega).$$

which yields the fourth inequality.

Working with  $\eta_2$  is completely symmetric by exchanging the role of  $d_{f,h}$  and  $d_{f,h}^*$ , after starting with

$$\begin{aligned} d_{f,h}^*(\chi\omega) &= \chi(d_{f,h}^*\omega) + h\mathbf{i}_{\nabla f}\omega \\ \text{and } 0 &= d_{f,h}^*(d_{f,h}^*\chi\omega) = h\mathbf{i}_{\nabla\chi}(d_{f,h}^*\omega) + d_{f,h}^*[h\mathbf{i}_{\nabla\chi}d_{f,h}^*\omega]. \end{aligned}$$

The last inequality is obtained by summation.  $\square$

### 2.3.4 Resolvent estimates

From this paragraph and until the end of Section 6, the analysis becomes essentially one dimensional along  $\mathbb{R} \supset f(M)$ . Accordingly we now work specifically with  $\bar{\Omega} = f^{-1}([a, b])$ ,  $N_t = f^{-1}(a)$ ,  $N_n = f^{-1}(b)$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$  or possibly  $\bar{\Omega} = \sqcup_{n=1}^N f^{-1}([a_n, b_n])$ ,  $a_n, b_n \notin \{c_1, \dots, c_{N_f}\}$ , under Hypothesis 1.2 for  $f$ , or by assuming Hypothesis 1.6 and Hypothesis 2.16 for a more general Lipschitz function  $f$ .

Also the upper bounds  $\tilde{O}\left(e^{-\frac{d_{Ag}(K, K')}{h}}\right)$  in Proposition 2.13, Proposition 2.15 and Lemma 2.17 are replaced by their weaker form  $\tilde{O}\left(e^{-\frac{\inf_{x \in K, y \in K'} |f(x) - f(y)|}{h}}\right)$  which is the one given in Hypothesis 2.16.

We present here resolvent kernel estimates when  $[a, b]$  contains one or a fixed number  $N$  of ‘‘critical values’’ of  $f$ . It assumes some spectral localization, in (29) and (31), which is not yet proved. It will be done in the next sections with increasing complexity and precision: first for  $N = 1$  in Section 3 and then for a general  $N$  in Section 4, followed by the accurate version for  $N \geq 1$  in Section 6. It is also presented in a more general form where actually the  $N$  critical values may be replaced by  $N$  clusters of critical values for further applications.

Let us first consider the case when  $[a, b]$  contains one cluster of ‘‘critical values’’.

**Proposition 2.19.** *Assume Hypothesis 1.2, or more generally Hypothesis 1.6 and Hypothesis 2.16, for  $f$  and let  $a < c < b$  and  $\varepsilon_0 \in ]0, \min(b - c, c - a)[$  be such that*

$$[a, b] \cap \{c_1, \dots, c_{N_f}\} \subset ]c - \frac{\varepsilon_0}{16}, c + \frac{\varepsilon_0}{16}[.$$

*Assume also that  $\Delta_{f, f^{-1}([a, b]), h}$ , the self-adjoint operator in  $f^{-1}([a, b]) \subset M$  given in Proposition 2.8 with  $N_t = \{f = a\}$  and  $N_n = \{f = b\}$  satisfies:*

$$\exists h_0 > 0, \forall h \in ]0, h_0[, \quad \sigma(\Delta_{f, f^{-1}([a, b]), h}) \cap [0, e^{-\frac{\varepsilon_0}{h}}] \subset [0, e^{-\frac{4\varepsilon_0}{h}}]. \quad (29)$$

Then the estimate

$$(\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}(x,y) = \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + \frac{3\varepsilon_0}{h}})$$

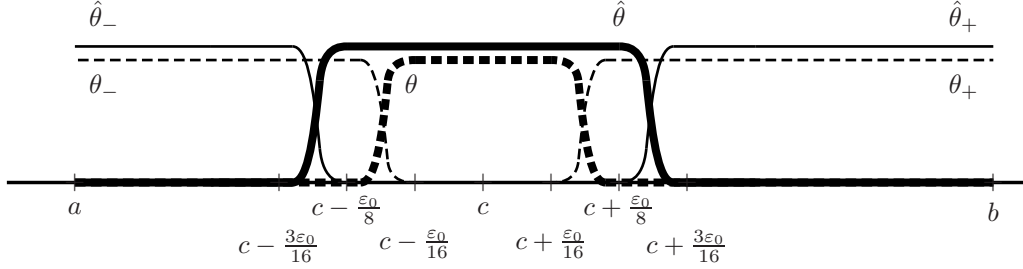
holds, according to Definition 2.14, uniformly with respect to  $z$ ,  $|z| = e^{-\frac{2\varepsilon_0}{h}}$ .

*Proof.* We prove Proposition 2.19 by adapting the analysis made in [DiSj, pp. 57–58]. Let us consider the self-adjoint realizations  $\Delta_{f,f^{-1}([a,c-\frac{\varepsilon_0}{16}],h}$  and  $\Delta_{f,f^{-1}([c+\frac{\varepsilon_0}{16},b],h}$  for which Proposition 2.15 says

$$(\Delta_{f,f^{-1}([a,c-\frac{\varepsilon_0}{16}],h} - z)^{-1}(x,y) \text{ and } (\Delta_{f,f^{-1}([c+\frac{\varepsilon_0}{16},b],h} - z)^{-1}(x,y) \text{ are } \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h}}), \quad (30)$$

uniformly with respect to  $z \in \mathbb{C}$ ,  $|z| = e^{-\frac{2\varepsilon_0}{h}}$ . Let moreover  $\theta$  and  $\hat{\theta}$  be two cut-off functions such that  $\theta \in C_0^\infty(f^{-1}([c - \frac{\varepsilon_0}{8}, c + \frac{\varepsilon_0}{8}]); [0, 1])$ ,  $\theta \equiv 1$  around  $f^{-1}([c - \frac{\varepsilon_0}{16}, c + \frac{\varepsilon_0}{16}])$ , and  $\hat{\theta} \in C_0^\infty(f^{-1}([c - \frac{3\varepsilon_0}{16}, c + \frac{3\varepsilon_0}{16}]); [0, 1])$ ,  $\hat{\theta} \equiv 1$  around  $f^{-1}([c - \frac{\varepsilon_0}{8}, c + \frac{\varepsilon_0}{8}])$ . Let us also define  $\theta_-, \hat{\theta}_- \in C^\infty(f^{-1}(-\infty, c]); [0, 1])$  and  $\theta_+, \hat{\theta}_+ \in C^\infty(f^{-1}(c, +\infty]); [0, 1])$  such that

$$\theta_- + \theta + \theta_+ = 1 \quad \text{and} \quad \hat{\theta}_- + \hat{\theta} + \hat{\theta}_+ = 1.$$



**Figure 1:** Positions of the cut-off functions,  $\theta_-$ ,  $\theta$ ,  $\theta_+$ ,  $\hat{\theta}_-$ ,  $\hat{\theta}$ ,  $\hat{\theta}_+$ .

The support conditions imply the following resolvent identity:

$$\begin{aligned} (\Delta_{f,f^{-1}([a,b]),h} - z)^{-1} &= (\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\hat{\theta} + \theta_- (\Delta_{f,f^{-1}([a,c-\frac{\varepsilon_0}{16}],h} - z)^{-1}\hat{\theta}_- \\ &\quad - (\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\hat{\theta}[\Delta_{f,h},\theta_-](\Delta_{f,f^{-1}([a,c-\frac{\varepsilon_0}{16}],h} - z)^{-1}\hat{\theta}_- \\ &\quad + \theta_+ (\Delta_{f,f^{-1}([c+\frac{\varepsilon_0}{16},b],h} - z)^{-1}\hat{\theta}_+ \\ &\quad - (\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\hat{\theta}[\Delta_{f,h},\theta_+](\Delta_{f,f^{-1}([c+\frac{\varepsilon_0}{16},b],h} - z)^{-1}\hat{\theta}_+). \end{aligned}$$

Since moreover  $\|(\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\|_{\mathcal{L}(L^2,L^2)} \leq 2e^{\frac{2\varepsilon_0}{h}}$  for  $|z| = e^{-\frac{2\varepsilon_0}{h}}$ , because the hypothesis ensures  $\text{dist}_{\mathbb{C}}(z, \sigma(\Delta_{f,f^{-1}([a,b]),h})) \leq \frac{e^{-\frac{2\varepsilon_0}{h}}}{2}$  for  $h > 0$  small enough, applying Proposition 2.13 to

$$(\Delta_{f,f^{-1}([a,b]),h} - z)\omega_h = r_h = \hat{\theta}\hat{r}_h$$

with  $\text{supp } \hat{\theta} \subset f^{-1}([c - \frac{3\varepsilon_0}{16}, c + \frac{3\varepsilon_0}{16}])$  first yields

$$[(\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\hat{\theta}](x,y) = \tilde{O}(e^{-\frac{|f(x)-c|-3\varepsilon_0/16}{h} + 2\frac{\varepsilon_0}{h}})$$

and then

$$\begin{aligned} [(\Delta_{f,f^{-1}([a,b]),h} - z)^{-1}\hat{\theta}](x,y) &= \tilde{O}(e^{-\frac{|f(x)-c|-3\varepsilon_0/16}{h} + 2\frac{\varepsilon_0}{h}}) \tilde{O}(e^{-\frac{|f(y)-c|-3\varepsilon_0/16}{h}}) \\ &= \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + 3\frac{\varepsilon_0}{h}}). \end{aligned}$$

By using (30),  $\|[\Delta_{f,h}, \theta_{\pm}]\|_{\mathcal{L}(W^{1,2}; L^2)} = \tilde{O}(1)$ ,  $[\Delta_{f,h}, \theta_{\pm}]$  vanishing in a neighborhood of  $f^{-1}(\{a, b\})$ , and the latter estimate for all the left factors concerned in the above resolvent identity, we obtain

$$\begin{aligned} (\Delta_{f, f^{-1}([a,b]), h} - z)^{-1}(x, y) &= \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + \frac{3\varepsilon_0}{h}}) + \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h}}) \\ &= \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + \frac{3\varepsilon_0}{h}}). \end{aligned}$$

□

**Proposition 2.20.** *Assume Hypothesis 1.2, or more generally Hypothesis 1.6 and Hypothesis 2.16 for  $f$ . Let  $a < b$  belong to  $\mathbb{R} \setminus \{c_1, \dots, c_{N_f}\}$  and let  $\bar{\Omega} = f^{-1}([a, b])$  with  $N_t = f^{-1}(\{a\})$ ,  $N_n = f^{-1}(\{b\})$ . Assume that there exist  $a = \tilde{c}_0 < \tilde{c}_1 < \dots < \tilde{c}_N < \tilde{c}_{N+1} = b$  and  $\varepsilon_0 \in ]0, \frac{\min_{1 \leq n \leq N+1}(\tilde{c}_n - \tilde{c}_{n-1})}{16}[$  such that*

$$]a, b[ \cap \{c_1, \dots, c_{N_f}\} \subset \sqcup_{n=1}^N ]\tilde{c}_n - \frac{\varepsilon_0}{16}, \tilde{c}_n + \frac{\varepsilon_0}{16}[.$$

The operator  $\Delta_{f, f^{-1}([a,b]), h}$  is the self-adjoint realization of the Witten Laplacian given in Proposition 2.8 and accordingly  $\Delta_n = \Delta_{f, f^{-1}([\tilde{c}_{n-1} + (1-\delta_{n,1})\varepsilon_0, \tilde{c}_{n+1} - (1-\delta_{n,N})\varepsilon_0]), h}$  is defined for  $1 \leq n \leq N$  where  $\delta_{m,n}$  is the Kronecker symbol. We assume

$$\forall n \in \{1, \dots, N\}, \quad \sigma(\Delta_n) \cap [0, e^{-\frac{\varepsilon_0}{h}}] \subset [0, e^{-\frac{4\varepsilon_0}{h}}]. \quad (31)$$

Then every  $z \in \mathbb{C}$  such that  $|z| = e^{-\frac{2\varepsilon_0}{h}}$  belongs to the resolvent set of  $\Delta_{f, f^{-1}([a,b]), h}$  provided that  $h \in ]0, h_0[$  with  $h_0 > 0$  small enough. Moreover, there exists a constant  $N_0 \in \mathbf{N}^*$ , determined by  $b - a$  and  $\min_{2 \leq n \leq N} \tilde{c}_n - \tilde{c}_{n-1}$ , such that

$$(\Delta_{f, f^{-1}([a,b]), h} - z)^{-1}(x, y) = \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + 3N_0 \frac{\varepsilon_0}{h}})$$

holds, according to Definition 2.14 uniformly with respect to  $z$ ,  $|z| = e^{-\frac{2\varepsilon_0}{h}}$ .

*Proof.* We prove Proposition 2.20 by adapting the analysis made in [DiSj, pp. 58–59]. Call  $\eta_0 = \min_{2 \leq n \leq N} \frac{\tilde{c}_n - \tilde{c}_{n-1}}{2}$  and take  $\varepsilon_0 \in ]0, \min_{1 \leq n \leq N+1} \frac{\tilde{c}_n - \tilde{c}_{n-1}}{16}[$ ,  $\varepsilon_0 \leq \frac{\eta_0}{8}$  as stated. For  $n \in \{1, \dots, N\}$ , let us introduce  $\theta_n \in C_0^\infty(f^{-1}([\tilde{c}_n - \frac{\varepsilon_0}{8}, \tilde{c}_n + \frac{\varepsilon_0}{8}]); [0, 1])$  such that  $\theta_n \equiv 1$  in a neighborhood of  $f^{-1}([\tilde{c}_n - \frac{\varepsilon_0}{16}, \tilde{c}_n + \frac{\varepsilon_0}{16}])$ , and

$$\chi_n := \left(1 - \sum_{m \neq n} \theta_m\right) \Big|_{f^{-1}([\tilde{c}_{n-1}, \tilde{c}_{n+1}])} = \left(1 - \theta_{n-1} - \theta_{n+1}\right) \Big|_{f^{-1}([\tilde{c}_{n-1}, \tilde{c}_{n+1}])}.$$

Here, we use the convention  $\theta_{-1} = \theta_{N+1} = 0$ . We also need another partition of unity  $1 = \sum_{n=1}^N \tilde{\chi}_n$ ,  $0 \leq \tilde{\chi}_n \leq 1$ , such that

$$\begin{aligned} \tilde{\chi}_n &\equiv 1 \text{ on } f^{-1}([\tilde{c}_n - \eta_0/2, \tilde{c}_n + \eta_0/2]) \quad \text{for } 1 \leq n \leq N, \\ \tilde{\chi}_n &\in C_0^\infty(f^{-1}([\tilde{c}_{n-1} + \eta_0/2, \tilde{c}_{n+1} - \eta_0/2])) \quad \text{for } 2 \leq n \leq N-1 \\ \text{and } \tilde{\chi}_1 &\equiv 0 \text{ on } f^{-1}([\tilde{c}_2 - \eta_0/2, b]) \quad \tilde{\chi}_N \equiv 0 \text{ on } f^{-1}([a, \tilde{c}_{N-1} + \eta_0/2]). \end{aligned}$$

Note in particular that our conditions,  $\varepsilon_0 \leq \frac{\eta_0}{8}$  and  $\text{supp } \theta_n \subset f^{-1}([\tilde{c}_n - \frac{\varepsilon_0}{8}, \tilde{c}_n + \frac{\varepsilon_0}{8}])$ , ensure  $\chi_n \equiv 1$  on  $\text{supp } \tilde{\chi}_n$ .

We now set for every  $z \in \mathbb{C}$ ,  $|z| = e^{-2\frac{\varepsilon_0}{h}}$ :

$$R_0(z) := \sum_{n=1}^N \chi_n (\Delta_n - z)^{-1} \tilde{\chi}_n, \quad (32)$$

where we recall  $\Delta_n = \Delta_{f, f^{-1}([\tilde{c}_{n-1} + (1 - \delta_{n,1})\varepsilon_0, \tilde{c}_{n+1} - \delta_{N,n}\varepsilon_0]), h}$ . Because the boundary conditions are satisfied, a simple computation shows

$$(\Delta_{f, f^{-1}([a, b]), h} - z)R_0 = I - K, \quad (33)$$

with

$$K = \sum_{n=1}^N \sum_{m \in \{n-1, n+1\}} [\Delta_{f, h, \theta_m}]|_{f^{-1}([\tilde{c}_{n-1}, \tilde{c}_{n+1}])} (\Delta_n - z)^{-1} \tilde{\chi}_n. \quad (34)$$

Moreover Proposition 2.19 applied to every  $\Delta_n$  and (34) combined with the support conditions of  $\theta_m, \tilde{\chi}_n$  imply

$$\|K\|_{\mathcal{L}(L^2, L^2)} = \tilde{O}(e^{-\frac{C}{h} + \frac{3\varepsilon_0}{h}}),$$

where

$$C := \min_{\substack{n \in \{1, \dots, N\} \\ m \in \{n-1, n+1\}}} \left( \min_{\substack{y \in \text{supp } \tilde{\chi}_n \\ x \in \text{supp } \theta_m}} |f(x) - f(y)| \right) \geq \frac{\eta_0}{2} - \frac{\varepsilon_0}{8},$$

and  $\varepsilon_0 \leq \frac{\eta_0}{8}$ , yields

$$\|K\|_{\mathcal{L}(L^2, L^2)} = \tilde{O}(e^{-\frac{\eta_0/2 - 25\varepsilon_0/8}{h}}) = \tilde{O}(e^{-\frac{7\eta_0}{64h}}).$$

For  $h > 0$  small enough,  $I - K : L^2 \rightarrow L^2$  is then invertible and the resolvent set of  $\Delta_{f, f^{-1}([a, b]), h}$  contains  $\{z \in \mathbb{C}, |z| = e^{-\frac{2\varepsilon_0}{h}}\}$ .

Let us now consider the exponential decay estimate. Write first

$$(\Delta_{f, f^{-1}([a, b]), h} - z)^{-1} = R_0(z) \sum_{\ell \in \mathbb{N}} K^\ell = R_0(z) \sum_{\ell=0}^{N_0-1} K^\ell + R_0(z)K_{N_0}, \quad (35)$$

and choose  $N_0 \in \mathbb{N}^*$  such that  $N_0 \times \frac{7\eta_0}{64} \geq (b - a)$  and

$$\|K_{N_0}\|_{\mathcal{L}(L^2, L^2)} = \left\| \sum_{\ell \geq N_0} K^\ell \right\|_{\mathcal{L}(L^2, L^2)} = \tilde{O}(e^{-\frac{b-a}{h}}) = \tilde{O}(e^{-\frac{\max_{x, y \in f^{-1}([a, b])} |f(x) - f(y)|}{h}}). \quad (36)$$

By referring again to Proposition 2.19 and from the definition (32) or  $R_0(z)$ , we know:

$$R_0(z)(x, y) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h} + 3\frac{\varepsilon_0}{h}}). \quad (37)$$

The relation (37) together with (36) implies that

$$(R_0 \circ K_{N_0})(x, y) = \tilde{O}(e^{-\frac{\min_{z \in M} |f(x) - f(z)| + b - a}{h} + 3\frac{\varepsilon_0}{h}}) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h} + 3\frac{\varepsilon_0}{h}}). \quad (38)$$

Moreover, the relation (37) together with

$$K(x, y) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h} + 3\frac{\varepsilon_0}{h}}),$$

which follows as well from Proposition 2.19, implies that for every  $\ell \in \mathbb{N}$ , one has:

$$(R_0(z) \circ K^\ell)(x, y) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h} + 3(\ell+1)\frac{\varepsilon_0}{h}}). \quad (39)$$

One finally deduces from (35) and from (38), (39) that the estimate

$$(\Delta_{f, f^{-1}([a, b]), h} - z)^{-1}(x, y) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h} + 3N_0\frac{\varepsilon_0}{h}}),$$

holds uniformly with respect to  $z \in \mathbb{C}, |z| = e^{-2\frac{\varepsilon_0}{h}}$ . This concludes the proof of Proposition 2.20.  $\square$

### 3 Local problems

In this section we shall use Agmon type estimates to study carefully the case when there is a unique “critical value” of  $f$  in  $]a, b[$ ,  $-\infty \leq a < b \leq +\infty$ .

**Hypothesis 3.1.** *The function  $f$  is assumed to satisfy Hypothesis 1.2, or Hypothesis 1.6 and Hypothesis 2.16, and the values  $a, b$ ,  $-\infty \leq a < b \leq +\infty$ , are chosen such that*

$$[a, b] \cap \{c_1, \dots, c_{N_f}\} = ]a, b[ \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1\}.$$

The domain is  $\overline{\Omega} = f^{-1}([a, b])$ , with  $N_t = f^{-1}(\{a\})$  and  $N_n = f^{-1}(\{b\})$ , and the operator  $\Delta_{f, f^{-1}([a, b]), h}$  is the one defined in Proposition 2.8.

With this assumption all the exponential decay estimates of Section 2.3 can be used with the pseudodistance  $|f(x) - f(y)|$ . The main result of this section says that, in this framework, the only possible exponentially small eigenvalue of  $\Delta_{f, f^{-1}([a, b]), h}$  is 0.

**Proposition 3.2.** *Under Hypothesis 3.1, the spectrum of the operator  $\Delta_{f, f^{-1}([a, b]), h}$  satisfies*

$$\forall \varepsilon > 0, \exists h_\varepsilon > 0, \forall h \in ]0, h_\varepsilon[, \sigma(\Delta_{f, f^{-1}([a, b]), h}) \cap [0, e^{-\frac{\varepsilon}{h}}] \subset \{0\}.$$

Proposition 3.2 will be proved in several steps. Consequences e.g. for resolvent estimates will be given afterwards.

#### 3.1 Useful quantities and notations

Let us first recall the following notion of distance between (spectral) subspaces which is convenient for spectral analysis (see e.g. [DiSj, pp. 59–61]).

**Definition 3.3.** *For  $E, F$  two closed subspaces of a Hilbert space  $\mathcal{H}$ , the non symmetric distance  $\vec{d}(E, F)$  is defined as*

$$\vec{d}(E, F) = \sup_{x \in E, \|x\|=1} d_{\mathcal{H}}(x, F) = \|\Pi_E - \Pi_F \Pi_E\| = \|\Pi_E - \Pi_E \Pi_F\|,$$

where  $\Pi_E, \Pi_F$  are the orthogonal projection on  $E, F$ .

This distance satisfies:

- $\vec{d}(E, F) = 0$  iff  $E \subset F$ ;
- $\vec{d}(E, G) \leq \vec{d}(E, F) + \vec{d}(F, G)$ ;
- $\vec{d}(E, F) < 1$  if and only if  $\Pi_F|_E : E \rightarrow F$  is one-to-one with a continuous left-inverse, and  $\Pi_E|_F : F \rightarrow E$  is onto in this case;
- ( $\vec{d}(E, F) < 1$  and  $\vec{d}(F, E) < 1$ ) if and only if  $\Pi_F|_E : E \rightarrow F$  and  $\Pi_E|_F : F \rightarrow E$  are bijections with continuous inverses. In this case, the equality  $\vec{d}(E, F) = \vec{d}(F, E)$  holds true;
- if we know a priori  $\dim E = \dim F < +\infty$  then

$$(\vec{d}(E, F) < 1) \Leftrightarrow (\vec{d}(E, F) < 1 \quad \text{and} \quad \vec{d}(F, E) < 1) \Leftrightarrow (\vec{d}(F, E) < 1).$$

We will use a variation of the min-max principle associated with the quantities  $\gamma(\alpha, [a, b], h)$  and  $\Gamma(\alpha, [a, b], h)$  defined below. Remember that  $Q_{f, f^{-1}([a, b]), h}^{(p)}$  is the quadratic form associated with  $\Delta_{f, f^{-1}([a, b]), h}$  (see the second item of Proposition 2.8).

**Definition 3.4.** For  $p \in \{0, \dots, d\}$ ,  $s \geq 0$ , let  $F_{[0,s],[a,b],h}^{(p)}$  denote the range of the spectral projection  $1_{[0,s]}(\Delta_{f,f^{-1}([a,b]),h}^{(p)})$ , with in particular  $F_{\{0\},f^{-1}([a,b]),h}^{(p)} = \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)})$ . For  $\alpha > 0$ , the quantities  $\gamma^{(p)}(\alpha, [a, b], h)$  and  $\Gamma^{(p)}(\alpha, [a, b], h)$  are defined by

$$\begin{aligned} \gamma^{(p)}(\alpha, [a, b], h) &= \vec{d}(F_{[0,e^{-\frac{\alpha}{h}}],[a,b],h}^{(p)}, F_{\{0\},[a,b],h}^{(p)}) = \vec{d}(F_{[0,e^{-\frac{\alpha}{h}}],[a,b],h}^{(p)}, \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)})) \\ &= \sup_{\omega_h \in F_{[0,e^{-\frac{\alpha}{h}}],[a,b],h}^{(p)} \setminus \{0\}} \frac{\text{dist}_{L^2}(\omega_h, \ker \Delta_{f,f^{-1}([a,b]),h}^{(p)})}{\|\omega_h\|_{L^2}}, \end{aligned} \quad (40)$$

$$\Gamma^{(p)}(\alpha, [a, b], h) = \sup_{\|\omega_h\|_{L^2}=1 : Q_{f,f^{-1}([a,b]),h}^{(p)}(\omega_h) \leq e^{-\frac{\alpha}{h}}} \text{dist}_{L^2}(\omega_h, \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)})). \quad (41)$$

Those quantities satisfy simple properties:

- The quantities  $\gamma^{(p)}(\alpha, [a, b], h)$  and  $\Gamma^{(p)}(\alpha, [a, b], h)$  are decreasing w.r.t  $\alpha$  and, since

$$F_{[0,e^{-\frac{\alpha}{h}}],[a,b],h}^{(p)} \subset \{\omega \in D(Q_{f,f^{-1}([a,b]),h}^{(p)}) \text{ s.t. } Q_{f,f^{-1}([a,b]),h}^{(p)}(\omega_h) \leq e^{-\frac{\alpha}{h}}\},$$

they satisfy

$$0 \leq \gamma^{(p)}(\alpha, [a, b], h) \leq \Gamma^{(p)}(\alpha, [a, b], h).$$

It says in particular:

$$\left( \lim_{h \rightarrow 0} \Gamma^{(p)}(\alpha, [a, b], h) = 0 \right) \Rightarrow \left( \lim_{h \rightarrow 0} \gamma^{(p)}(\alpha, [a, b], h) = 0 \right).$$

- Since  $\Delta_{f,f^{-1}([a,b]),h}$  is self-adjoint, the spectral theorem implies:

$$\gamma^{(p)}(\alpha, [a, b], h) = 0 \quad \text{iff} \quad \sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \cap [0, e^{-\frac{\alpha}{h}}] \subset \{0\}$$

and

$$\gamma^{(p)}(\alpha, [a, b], h) = 1 \quad \text{else.}$$

In particular, it provides the expression

$$\gamma^{(p)}(\alpha, [a, b], h) = \sup_{\|\omega_h\|=1 : \begin{cases} \Delta_{f,f^{-1}([a,b]),h}^{(p)} \omega_h = \lambda_h \omega_h \\ \lambda_h \leq e^{-\frac{\alpha}{h}} \end{cases}} \text{dist}_{L^2}(\omega_h, \ker \Delta_{f,f^{-1}([a,b]),h}^{(p)})$$

and the convergence  $\lim_{h \rightarrow 0} \gamma^{(p)}(\alpha, [a, b], h) = 0$  means precisely that:

$$\exists h_\alpha > 0, \forall h \in ]0, h_\alpha[ , \sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \cap [0, e^{-\frac{\alpha}{h}}] \subset \{0\}. \quad (42)$$

- The spectral theorem also implies

$$\Gamma^{(p)}(\alpha, [a, b], h) = 1 \quad \text{iff} \quad \sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \cap ]0, e^{-\frac{\alpha}{h}}] \neq \emptyset$$

and

$$(\Gamma^{(p)}(\alpha, [a, b], h))^2 \in [0, \frac{e^{-\frac{\alpha}{h}}}{\min(\sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \setminus \{0\})}] \subset [0, 1[ \quad \text{else.} \quad (43)$$

Actually,  $\sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \cap ]0, e^{-\frac{\alpha}{h}}] \neq \emptyset$  implies  $\Gamma^{(p)}(\alpha, [a, b], h) \geq \gamma^{(p)}(\alpha, [a, b], h) \geq 1$  and obviously  $\Gamma^{(p)}(\alpha, [a, b], h) = 1$ .



Reciprocally when  $\sigma(\Delta_{f,f^{-1}([a,b]),h}) \cap ]0, e^{-\frac{\alpha}{h}}] = \emptyset$  and for any  $\omega_h$  which satisfies the inequality  $Q_{f,f^{-1}([a,b]),h}^{(p)}(\omega_h) \leq e^{-\frac{\alpha}{h}} \|\omega_h\|_{L^2}^2$ , the spectral decomposition

$$\omega_h = 1_{\{0\}}(\Delta_{f,f^{-1}([a,b]),h}^{(p)})\omega_h + 1_{[\min(\sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \setminus \{0\}), +\infty[}(\Delta_{f,f^{-1}([a,b]),h}^{(p)})\omega_h$$

leads to

$$\begin{aligned} \text{dist}_{L^2}^2(\omega_h, \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)})) &= \|1_{[\min(\sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \setminus \{0\}), +\infty[}(\Delta_{f,f^{-1}([a,b]),h}^{(p)})\omega_h\|_{L^2}^2 \\ &\leq \frac{e^{-\frac{\alpha}{h}}}{\min(\sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \setminus \{0\})} \|\omega_h\|_{L^2}^2. \end{aligned}$$

- We deduce from (42) and (43) that

$$\left( \lim_{h \rightarrow 0} \gamma^{(p)}(\alpha', [a, b], h) = 0 \right) \Rightarrow \left( \forall \alpha > \alpha', \Gamma^{(p)}(\alpha, [a, b], h) \leq e^{-\frac{\alpha - \alpha'}{2h}} \xrightarrow{h \rightarrow 0} 0 \right).$$

Up to an arbitrary small change of the positive parameter  $\alpha$ , working with  $\gamma^{(p)}$  or  $\Gamma^{(p)}$  is then essentially equivalent.

### 3.2 Exponentially small eigenvalues are zero

This section is devoted to the proof of Proposition 3.2. First of all, we can assume  $\tilde{c}_1 = 0$  if  $f$  is replaced by  $f - \tilde{c}_1$ . The proof will be done in three steps connected by the remarks on  $\gamma^{(p)}$  and  $\Gamma^{(p)}$  from the previous subsection.

**Step 1:** Assume  $[a, b] = [-\varepsilon, \varepsilon]$  with  $\varepsilon > 0$  (and  $\tilde{c}_1 = 0$ ). We prove here that

$$\forall \alpha' = 2\varepsilon + c > 2\varepsilon, \quad \lim_{h \rightarrow 0} \gamma^{(p)}(\alpha', [-\varepsilon, \varepsilon], h) = 0,$$

where, owing to the monotonicity of  $\gamma^{(p)}(\alpha, [-\varepsilon, \varepsilon], h)$  w.r.t  $\alpha$ , we can focus on  $c \in ]0, \varepsilon[$ . According to (42), it amounts to show there exists  $h_c > 0$  such that

$$\left( \lambda_h \in \sigma(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) \cap [0, e^{-\frac{2\varepsilon+c}{h}}] \right) \Rightarrow (\forall h \in ]0, h_c[, \quad \lambda_h = 0).$$

Take then  $\omega_h \in D(\Delta_{f,f^{-1}([-\varepsilon, \varepsilon]),h}^{(p)})$  satisfying

$$\|\omega_h\|_{L^2} = 1 \quad \text{and} \quad \Delta_{f,f^{-1}([-\varepsilon, \varepsilon]),h}^{(p)}\omega_h = \lambda_h\omega_h \quad \text{with} \quad 0 \leq \lambda_h \leq e^{-\frac{2\varepsilon+c}{h}}$$

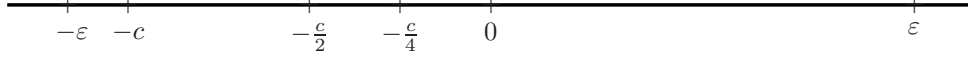
(the result is obvious for the  $h$ 's for which the existence of  $\omega_h$  fails). The exponential decay estimates of Proposition 2.13 (or Hypothesis 2.16 for a general Lipschitz function) applied with  $N_\varepsilon = f^{-1}(\{-\varepsilon\})$  and  $N_\varepsilon = f^{-1}(\{\varepsilon\})$ ,  $K = \emptyset$ ,  $U = f^{-1}(\{0\})$ ,  $d_{Ag}(x, U) \geq |f(x)|$ , and  $r_h = 0$  writes:

$$\int_{f^{-1}([-\varepsilon, \varepsilon])} e^{\frac{2|f(x)|}{h}} |\omega_h(x)|^2 dx \leq \|e^{\frac{|f|}{h}} \omega_h\|_{W(f^{-1}([-\varepsilon, \varepsilon]))}^2 = \tilde{O}(1). \quad (44)$$

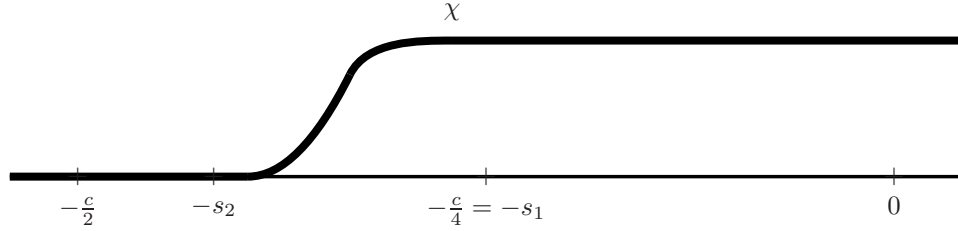
Hence the mass of the probability measure with density  $|\omega_h|^2(x)$  concentrates on  $U = f^{-1}(\{0\})$  as  $h \rightarrow 0$ . We deduce the a priori estimate

$$\forall \delta \in ]0, \varepsilon[, \exists h_\delta > 0, \forall h \in ]0, h_\delta[, \quad \|e^{\frac{f}{h}} 1_{f^{-1}(-\delta)}(x)\omega_h\|_{L^2} \geq \frac{e^{-\frac{\delta}{h}}}{2}.$$

Once the parameter  $c \in ]0, \varepsilon[$  is fixed, introduce  $s_1 = \frac{c}{4}$  and  $s_2 \in (\frac{c}{4}, \frac{c}{2})$  and take  $\chi \in \mathcal{C}^\infty(M; [0, 1])$  such that  $\chi \equiv 0$  near  $\overline{f^{-s_2}}$  which contains a neighborhood of  $\overline{f^{-\frac{c}{2}}}$  and  $\chi \equiv 1$  near  $\overline{f^{-s_1}} = \overline{f^{-\frac{c}{4}}}$ .



**Figure 2:** Positions in the interval  $[-\varepsilon, \varepsilon]$ .



**Figure 3:** Cut-off function  $\chi$  in  $[-\varepsilon, \varepsilon]$ .

Since

$$d(\chi e^{\frac{f}{h}} \omega_h) = \chi d(e^{\frac{f}{h}} \omega_h) + d\chi \wedge (e^{\frac{f}{h}} \omega_h),$$

we deduce

$$\|d(\chi e^{\frac{f}{h}} \omega_h)\|_{L^2}^2 \leq 2\|\chi d(e^{\frac{f}{h}} \omega_h)\|_{L^2}^2 + 2\|d\chi \wedge (e^{\frac{f}{h}} \omega_h)\|_{L^2}^2. \quad (45)$$

The estimate

$$Q_{f, f^{-1}([- \varepsilon, \varepsilon]), h}(\omega_h) = \|e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} \omega_h\|_{L^2}^2 + \|e^{\frac{f}{h}}(hd^*)e^{-\frac{f}{h}} \omega_h\|_{L^2}^2 \leq e^{-\frac{2\varepsilon+c}{h}}$$

with  $f \leq \varepsilon$  then implies that the first term in the r.h.s. of (45) is of order  $\tilde{O}(e^{-\frac{\varepsilon}{h}})$ . Meanwhile  $\text{supp}(d\chi) \subset \overline{f^{-\frac{c}{4}}}$  and the exponential decay estimate (44) imply that the second term in the r.h.s. of (45) is of order  $\tilde{O}(e^{-\frac{2c}{4h}})$ . Adding the boundary conditions  $\mathbf{n}_{f=\varepsilon} \omega_h = 0$  and  $\mathbf{n}_{f=\varepsilon} d_{f,h} \omega_h = 0$ , i.e.  $\mathbf{n}_{f=\varepsilon}(e^{\frac{f}{h}} \omega_h) = 0$  and  $\mathbf{n}_{f=\varepsilon} d(e^{\frac{f}{h}} \omega_h) = 0$ , we have thus proved that

$$\begin{cases} \chi e^{\frac{f}{h}} \omega_h \in D(\Delta_{0, f^{-1}([-s_2, \varepsilon]), 1}), \\ \|d_{0, f^{-1}([-s_2, \varepsilon]), 1}(\chi e^{\frac{f}{h}} \omega_h)\|_{L^2}^2 = \tilde{O}(e^{-\frac{\varepsilon}{h}}), \\ \lim_{h \rightarrow 0} h \log \|\chi e^{\frac{f}{h}} \omega_h\|_{L^2} = 0. \end{cases}$$

Set  $u_h = \frac{\chi e^{\frac{f}{h}} \omega_h}{\|\chi e^{\frac{f}{h}} \omega_h\|_{L^2}}$  so that  $\|u_h\|_{L^2} = 1$ ,  $u_h \in D(\Delta_{0, f^{-1}([-s_2, \varepsilon]), 1})$  and  $\|du_h\|_{L^2}^2 = \tilde{O}(e^{-\frac{\varepsilon}{h}})$ .

By using the Hodge decomposition (see Proposition 2.8) and  $\sigma(\Delta_{0, f^{-1}([-s_2, \varepsilon]), 1}) \setminus \{0\} \subset [\mu_1, +\infty) \subset \mathbb{R}^{+*}$ , with  $\mu_1$  fixed by  $\varepsilon > 0$  and  $s_2 > 0$ , we obtain the decomposition of  $u_h$ :

$$u_h = \Pi_{\ker d_{0, f^{-1}([-s_2, \varepsilon]), 1}} u_h + d_{0, f^{-1}([-s_2, \varepsilon]), 1}^* u_{2,h},$$

where  $d_{0, f^{-1}([-s_2, \varepsilon]), 1}^* u_{2,h}$  in  $(\ker \Delta_{0, f^{-1}([-s_2, \varepsilon]), 1})^\perp = \text{Ran } 1_{\{[\mu_1, +\infty)\}}(\Delta_{0, f^{-1}([-s_2, \varepsilon]), 1})^{(p)}$ . Writing shortly  $\mathbf{d} = d_{0, f^{-1}([-s_2, \varepsilon]), 1}$  and  $\mathbf{d}^* = d_{0, f^{-1}([-s_2, \varepsilon]), 1}^*$ , it follows that

$$\tilde{O}(e^{-\frac{\varepsilon}{h}}) = \|\mathbf{d}u_h\|_{L^2}^2 = \|\mathbf{d}\mathbf{d}^*u_{2,h}\|_{L^2}^2 = Q_{0, f^{-1}([-s_2, \varepsilon]), 1}^{(p)}(\mathbf{d}^*u_{2,h}) \geq \mu_1 \|\mathbf{d}^*u_{2,h}\|_{L^2}^2.$$

We deduce  $\text{dist}_{L^2}(u_h, \ker d_{0, f^{-1}([-s_2, \varepsilon], 1)}) = \tilde{O}(e^{-\frac{c}{2h}})$  and then the existence of a form  $\eta_h \in \ker(d_{0, f^{-1}([-s_2, \varepsilon], 1)})$  such that

$$\|\chi e^{\frac{f}{h}} \omega_h - \eta_h\|_{L^2(f^{-1}([-s_2, \varepsilon]))} = \tilde{O}(e^{-\frac{c}{2h}}).$$

By the first item of Remark 2.9, the extension  $\tilde{\eta}_h$  of  $\eta_h$  by 0 in  $f_{-\varepsilon}^{-s_2}$  belongs to  $\ker(d_{0, f^{-1}([- \varepsilon, \varepsilon], 1)})$  with  $\text{supp } \tilde{\eta}_h \subset \overline{f_{-\varepsilon}^{\varepsilon}}$  and  $\|\chi e^{\frac{f}{h}} \omega_h - \tilde{\eta}_h\|_{L^2} = \tilde{O}(e^{-\frac{c}{2h}})$ .

After multiplying by  $e^{-\frac{f}{h}} = \mathcal{O}(e^{\frac{s_2}{h}})$  in  $f_{-\varepsilon}^{\varepsilon}$ , we obtain

$$\begin{cases} \|\chi \omega_h - e^{-\frac{f}{h}} \tilde{\eta}_h\|_{L^2} = \tilde{O}(e^{-\frac{c}{2h} + \frac{s_2}{h}}) & , \quad \frac{c}{2} > s_2, \\ e^{-\frac{f}{h}} \tilde{\eta}_h \in \ker(d_{f, f^{-1}([- \varepsilon, \varepsilon], h)}). \end{cases}$$

We conclude with  $\|\chi \omega_h - \omega_h\|_{L^2} = \tilde{O}(e^{-\frac{c'}{4h}})$  (since  $\text{supp } (1 - \chi) \subset \overline{f^{-s_1}} = \overline{f^{-\frac{c}{4}}}$ ) that

$$\text{dist}(\omega_h, \ker(d_{f, f^{-1}([- \varepsilon, \varepsilon], h)})) = O(e^{-\frac{c'}{h}}) \quad \text{for some } c' > 0.$$

The duality consists in replacing  $f$  by  $-f$  (which does not change  $[-\varepsilon, \varepsilon]$ ), the differential form  $\omega_h \in W(f^{-1}([- \varepsilon, \varepsilon], 1); \Lambda^p T^* M)$  by  $\star \omega_h \in W(f^{-1}([- \varepsilon, \varepsilon], 1); \Lambda^{d-p} T^* M \otimes_{\text{or}_M})$  where the orientation twist does not change the analysis,  $\mathbf{t}$  by  $\mathbf{n}$  (and conversely),  $\star$  and  $\star^{-1}$ , and  $d_{f, h}$  by  $d_{-f, h}^*$  (and conversely). This leads to

$$\text{dist}(\star \omega_h, \ker(d_{-f, f^{-1}([- \varepsilon, \varepsilon], h)})) = \text{dist}(\omega_h, \ker(d_{f, f^{-1}([- \varepsilon, \varepsilon], h)}^*)) = O(e^{-\frac{c'}{h}}).$$

Assume by contradiction that  $\lambda_h \neq 0$ . Since  $\omega_h = \lambda_h^{-1} \Delta_{f, f^{-1}([- \varepsilon, \varepsilon], h)}^{(p)} \omega_h \in (\ker \Delta_{f, f^{-1}([- \varepsilon, \varepsilon], h)}^{(p)})^\perp$ , the Hodge decomposition (see Proposition 2.8) leads to

$$\omega_h = \Pi_{\ker d_{f, f^{-1}([- \varepsilon, \varepsilon], h)}} \omega_h + \Pi_{\ker d_{f, f^{-1}([- \varepsilon, \varepsilon], h)}^*} \omega_h.$$

The squared norm  $1 = \|\omega_h\|^2$  thus equals

$$\text{dist}_{L^2}^2(\omega_h, \ker(d_{f, f^{-1}([- \varepsilon, \varepsilon], h)})) + \text{dist}_{L^2}^2(\omega_h, \ker(d_{f, f^{-1}([- \varepsilon, \varepsilon], h)}^*)) = \tilde{O}(e^{-\frac{c'}{h}}),$$

which is impossible for  $0 < h < h_\varepsilon$ ,  $h_\varepsilon > 0$  small enough. It follows that  $\sigma(\Delta_{f, f^{-1}([a, b], h)}^{(p)}) \cap [0, e^{-\frac{2\varepsilon+c}{h}}] \subset \{0\}$  for  $h$  small enough, which implies  $\lim_{h \rightarrow 0} \gamma^{(p)}(\alpha', [-\varepsilon, \varepsilon], h) = 0$  according to the comments following Definition 3.4.

**Step 2:** From Step 1, we know  $\lim_{h \rightarrow 0} \gamma^{(p)}(\alpha', [-\varepsilon, \varepsilon], h) = 0$  for any  $\alpha' > 2\varepsilon$  and the comparison of the quantities  $\gamma^{(p)}$  and  $\Gamma^{(p)}$  in the previous subsection leads to

$$\forall \alpha > 2\varepsilon, \quad \lim_{h \rightarrow 0} \Gamma^{(p)}(\alpha, [-\varepsilon, \varepsilon], h) = 0.$$

Working with  $\Gamma^{(p)}$  brings the flexibility to use some restriction argument from  $f_a^b$  to  $f_{-\varepsilon}^\varepsilon$ , which of course does not send eigenvectors onto eigenvectors.

**Step 3:** For the general case  $a < 0 = \tilde{c}_1 < b$ , we now prove

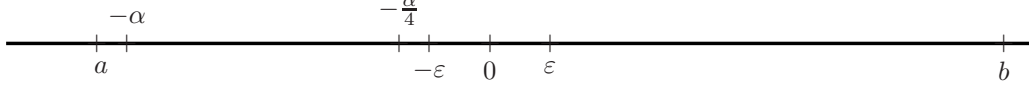
$$\forall \alpha > 0, \quad \sigma(\Delta_{f, f^{-1}([a, b], h)}^{(p)}) \cap [0, e^{-\frac{\alpha}{h}}] \subset \{0\},$$

where, by monotonicity w.r.t  $\alpha$ , it is sufficient to consider  $\alpha \leq \min(-a, b)$ . Let us then assume that  $\omega_h$  satisfies  $\Delta_{f, f^{-1}([a, b], h)}^{(p)} \omega_h = \lambda_h \omega_h$  with  $\|\omega_h\|_{L^2} = 1$  and  $0 \leq \lambda_h \leq e^{-\frac{\alpha}{h}}$ . Take  $\varepsilon \in ]0, \frac{\alpha}{4}[$  and consider  $f_{-\varepsilon}^\varepsilon \subset f_a^b$ . We know that

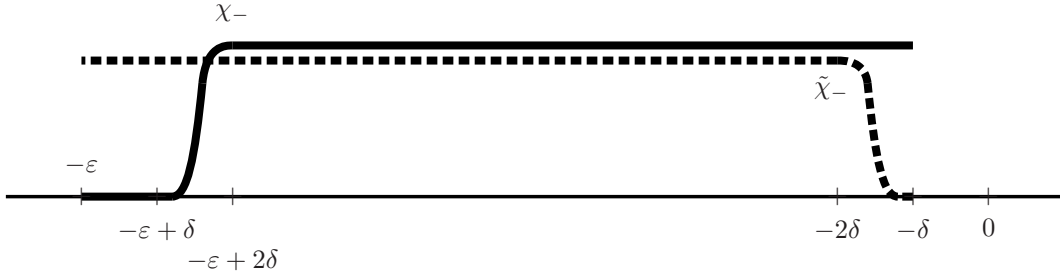
$$\|d_{f, h} \omega_h\|_{L^2(f_{-\varepsilon}^\varepsilon)}^2 + \|d_{f, h}^* \omega_h\|_{L^2(f_{-\varepsilon}^\varepsilon)}^2 \leq \|d_{f, f^{-1}([a, b], h)} \omega_h\|_{L^2(f_a^b)}^2 + \|d_{f, f^{-1}([a, b], h)}^* \omega_h\|_{L^2(f_a^b)}^2 \leq e^{-\frac{\alpha}{h}},$$

although  $\omega_h|_{f^{-\varepsilon}}$  a priori does not belong neither to  $D(\Delta_{f,f^{-1}([- \varepsilon, \varepsilon]), h}^{(p)})$  nor to  $D(Q_{f,f^{-1}([- \varepsilon, \varepsilon]), h}^{(p)})$ . We now use Lemma 2.17 in the two subsets  $f^{-1}([- \varepsilon, -\delta])$  and  $f^{-1}([\delta, \varepsilon])$  for some  $\delta \in ]0, \frac{\varepsilon}{4}[$  which will be fixed later.

Consider  $\bar{\Omega} = f^{-1}([- \varepsilon, -\delta])$  (the other case is symmetric) and take the cut-off  $\chi_-, \tilde{\chi}_- \in C^\infty(f^{-1}[- \varepsilon, -\delta]; [0, 1])$  with  $\text{supp } \chi_- \subset f^{-1}(-\varepsilon + \delta, -\delta)$ ,  $\chi_- \equiv 1$  in  $f^{-1}([- \varepsilon + 2\delta, -\delta])$ , and  $\text{supp } \tilde{\chi}_- \subset f^{-1}([- \varepsilon, -\delta])$ ,  $\tilde{\chi}_- \equiv 1$  in  $f^{-1}([- \varepsilon, -2\delta])$ .



**Figure 4:** Positions in the interval  $[a, b]$ .



**Figure 5:** Cut-off functions  $\chi_-$  and  $\tilde{\chi}_-$  in  $[-\varepsilon, 0] \subset [a, b]$ .

The form  $\eta_{1,-}$  and  $\eta_{2,-}$  in  $D(\Delta_{f,f^{-1}([- \varepsilon, -\delta]), h})$  are defined by

$$\begin{aligned} \eta_{1,-} &= d_{f,[- \varepsilon, -\delta], h}^* (\Delta_{f,f^{-1}([- \varepsilon, -\delta]), h})^{-1} ((hd\chi_-) \wedge \omega_h) \\ \eta_{2,-} &= d_{f,[- \varepsilon, -\delta], h} (\Delta_{f,f^{-1}([- \varepsilon, -\delta]), h})^{-1} (h\mathbf{i}_{\nabla\chi_-} \omega_h). \end{aligned}$$

Lemma 2.17 combined with  $d_{Ag}(x, y) \geq |f(x) - f(y)|$  implies

$$\begin{aligned} &\|d_{f,h}(\chi_- \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}))\|_{L^2(f^{-\varepsilon})} + \|d_{f,h}^*(\chi_- \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}))\|_{L^2(f^{-\varepsilon})} \\ &\leq \tilde{O}(e^{-\frac{\varepsilon-4\delta}{h}}) \|\omega_h\|_{L^2(f^{-\varepsilon+2\delta})} + C_{\chi_-} \left[ \|d_{f,h}\omega_h\|_{L^2(f^{-\varepsilon})} + \|d_{f,h}^*\omega_h\|_{L^2(f^{-\varepsilon})} \right]. \end{aligned}$$

Because  $\Delta_{f,f^{-1}([a,b], h)}\omega_h = \lambda_h\omega_h$  with  $\|\omega_h\|_{L^2} = 1$ , the Agmon estimate of Proposition 2.13 (or Hypothesis 2.16 for a general Lipschitz function), applied with  $N_t = f^{-1}(\{a\})$  and  $N_n = f^{-1}(\{b\})$ ,  $K = \emptyset$ ,  $U = f^{-1}(\{0\})$ ,  $d_{Ag}(x, U) \geq |f(x)|$ , and  $r_h = 0$  implies

$$\|\omega_h\|_{L^2(f^{-\varepsilon+2\delta})} = \tilde{O}(e^{-\frac{\varepsilon-2\delta}{h}}), \quad (46)$$

while we know

$$\|d_{f,h}\omega_h\|_{L^2(f^{-\varepsilon})}^2 + \|d_{f,h}^*\omega_h\|_{L^2(f^{-\varepsilon})}^2 \leq \|d_{f,h}\omega_h\|_{L^2(f_a^b)}^2 + \|d_{f,h}^*\omega_h\|_{L^2(f_a^b)}^2 \leq e^{-\frac{\alpha}{h}}.$$

With  $\frac{\alpha}{4} > \varepsilon > 4\delta$ , we have thus

$$\|d_{f,h}(\chi_- \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}))\|_{L^2(f^{-\varepsilon})} + \|d_{f,h}^*(\chi_- \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}))\|_{L^2(f^{-\varepsilon})} = \tilde{O}(e^{-\frac{2\varepsilon-6\delta}{h}}). \quad (47)$$

A symmetric construction provides two cut-off functions  $\chi_+, \tilde{\chi}_+ \in C^\infty(f^{-1}([\delta, \varepsilon]))$  such that

$$\text{supp } \chi_+ \subset f^{-1}([\delta, \varepsilon - \delta]) \quad \text{and} \quad \chi_+ \equiv 1 \quad \text{in} \quad f^{-1}([\delta, \varepsilon - 2\delta]),$$

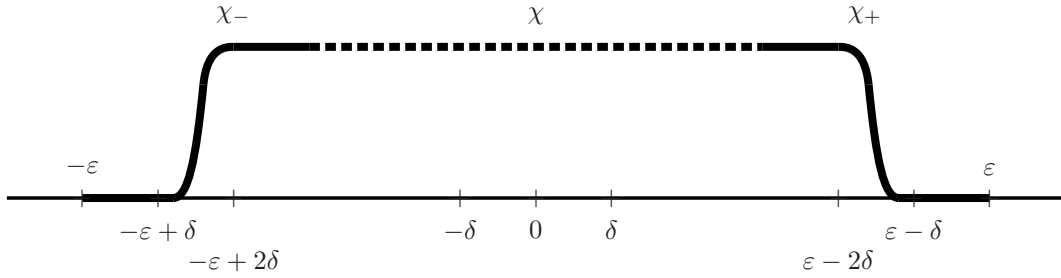
$$\text{supp } \tilde{\chi}_+ \subset f^{-1}(] \delta, \varepsilon]) \quad \text{and} \quad \tilde{\chi}_+ \equiv 1 \quad \text{in} \quad f^{-1}([2\delta, \varepsilon]),$$

and then two forms  $\eta_{1,+}, \eta_{2,+} \in D(\Delta_{f, f^{-1}([\delta, \varepsilon]), h})$  such that

$$\|d_{f,h}(\chi_+ \omega_h - \tilde{\chi}_+(\eta_{1,+} + \eta_{2,+}))\|_{L^2(f_\varepsilon^\varepsilon)} + \|d_{f,h}^*(\chi_+ \omega_h - \tilde{\chi}_+(\eta_{1,+} + \eta_{2,+}))\|_{L^2(f_\varepsilon^\varepsilon)} = \tilde{O}(e^{-\frac{2\varepsilon - 6\delta}{h}}). \quad (48)$$

Take now  $\chi \in C_0^\infty(f^{-1}(]-\varepsilon + \delta, \varepsilon - \delta]; [0, 1]))$  which equals 1 in  $f^{-1}([-\varepsilon + 2\delta, \varepsilon - 2\delta])$  and coincides with  $\chi_-$  (resp.  $\chi_+$ ) in  $f_{-\varepsilon+2\delta}^{-\varepsilon}$  (resp. in  $f_{\varepsilon-2\delta}^{\varepsilon-\delta}$ ) and set

$$v_h = \chi \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}) - \tilde{\chi}_+(\eta_{1,+} + \eta_{2,+}).$$



**Figure 6:** Cut-off function  $\chi$  in  $[-\varepsilon, \varepsilon]$ .

This form is close to  $\omega_h|_{f_{-\varepsilon}^\varepsilon}$ . In fact, write

$$v_h - \omega_h|_{f_{-\varepsilon}^\varepsilon} = (\chi - 1)\omega_h|_{f_{-\varepsilon}^\varepsilon} - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}) - \tilde{\chi}_+(\eta_{1,+} + \eta_{2,+}),$$

where, according to Lemma 2.17 and to the exponential decay estimate (46) (and its symmetric version on  $[\varepsilon - 2\delta, \varepsilon - \delta]$ ),

$$\|\tilde{\chi}_\pm \eta_{i,\pm}\|_{L^2} = \mathcal{O}(\|\omega_h\|_{\text{supp } d\chi_\pm}) = \tilde{O}(e^{-\frac{\varepsilon - 2\delta}{h}}) \quad \text{for } i \in \{1, 2\}$$

and

$$\|(\chi - 1)\omega_h\|_{L^2(f_{-\varepsilon}^\varepsilon)} = \tilde{O}(e^{-\frac{\varepsilon - 2\delta}{h}}),$$

which implies

$$\|v_h - \omega_h\|_{L^2(f_{-\varepsilon}^\varepsilon)} = \tilde{O}(e^{-\frac{\varepsilon - 2\delta}{h}}).$$

The form  $v_h$  also satisfies, for  $\mathbf{d} = d_{f,h}$  or  $\mathbf{d} = d_{f,h}^*$ ,

$$\mathbf{d}v_h = [\mathbf{d}(\chi_- \omega_h - \tilde{\chi}_-(\eta_{1,-} + \eta_{2,-}))]|_{f_{-\varepsilon}^{-\delta}} + [\mathbf{d}\omega_h]|_{f_\delta^\delta} + [\mathbf{d}(\chi_+ \omega_h - \tilde{\chi}_+(\eta_{1,+} + \eta_{2,+}))]|_{f_\varepsilon^\varepsilon}.$$

Then, since  $v_h$  belongs to  $D(\Delta_{f, f^{-1}([-\varepsilon, \varepsilon]), h})$  by construction, it satisfies, by (47) and (48),

$$\|d_{f, f^{-1}([-\varepsilon, \varepsilon])} v_h\|^2 + \|d_{f, f^{-1}([-\varepsilon, \varepsilon])}^* v_h\|^2 = \tilde{O}(e^{-\frac{4\varepsilon - 12\delta}{h}}).$$

We finally take  $\delta = \frac{\varepsilon}{12}$  for which the r.h.s. of the above relation is  $\tilde{O}(e^{-\frac{3\varepsilon}{h}})$ , with  $3\varepsilon > 2\varepsilon$ . By Step 2, this implies

$$\lim_{h \rightarrow 0} \frac{\text{dist}_{L^2}(v_h, \ker(\Delta_{f, f^{-1}([-\varepsilon, \varepsilon]), h}))}{\|v_h\|_{L^2}} = 0.$$

But the Agmon estimates of Proposition 2.13 or Hypothesis 2.16 also imply

$$\|\omega_h\|_{L^2(f_\varepsilon^-)} = 1 + \tilde{O}(e^{-\frac{\varepsilon}{h}}) \quad \text{and then} \quad \|v_h\|_{L^2(f_\varepsilon^-)} = 1 + \tilde{O}(e^{-\frac{\varepsilon-2\delta}{h}}).$$

Denoting by  $F \subset L^2(f_a^b)$  the subspace  $\ker(\Delta_{f,f^{-1}([- \varepsilon, \varepsilon]), h}^{(p)})$  extended by 0 in  $f_a^{-\varepsilon} \sqcup f_\varepsilon^b$ , it then follows from the preceding analysis that

$$\lim_{h \rightarrow 0} \text{dist}_{L^2}(\omega_h, F) = 0.$$

Since  $\dim F$  is finite and does not depend on  $h > 0$  (see the second item in Remark 2.9), there exists  $h_\alpha > 0$  such that for every  $h \in ]0, h_\alpha[$ ,

$$\dim F_{[0, e^{-\frac{\alpha}{h}}], [a, b], h}^{(p)} \leq \dim F = \dim \ker(\Delta_{f, f^{-1}([- \varepsilon, \varepsilon]), h}^{(p)}) = \dim \ker(\Delta_{f, f^{-1}([a, b]), h}^{(p)}),$$

where the last equality follows from  $[- \varepsilon, \varepsilon] \subset [a, b]$  and  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1 = 0\}$  (see (16)). This implies that  $\sigma(\Delta_{f, f^{-1}([a, b]), h}^{(p)}) \cap [0, e^{-\frac{\alpha}{h}}] \subset \{0\}$  for  $h \in ]0, h_\alpha[$  and this ends the proof.

### 3.3 Consequences

We still work under Hypothesis 3.1:  $f$  admits a unique ‘‘critical value’’  $\tilde{c}_1 \in [a, b]$ ,  $a < \tilde{c}_1 < b$ . With the information of Proposition 3.2, the resolvent estimates of Subsection 2.3.1 lead easily to similar estimates for spectrally defined operators. Finally we deduce other properties which will be used in the induction process in terms of the number  $N$  of ‘‘critical values’’.

#### 3.3.1 Estimates for spectral operators

For a Borel set  $I \subset \mathbb{R}$  we introduce the notation:

$$\Pi_{I, [a, b], h} = 1_I(\Delta_{f, f^{-1}([a, b]), h}). \quad (49)$$

**Proposition 3.5.** *Under Hypothesis 3.1 the spectral projection on the kernel  $\Pi_{\{0\}, [a, b], h}$  satisfies*

$$\Pi_{\{0\}, [a, b], h}(x, y) = \tilde{O}(e^{-\frac{|f(x) - f(y)|}{h}})$$

according to Definition 2.14.

*Proof.* It suffices to use the formula

$$\Pi_{\{0\}, [a, b], h} = \frac{1}{2i\pi} \int_{\gamma_h} (z - \Delta_{f, f^{-1}([a, b]), h})^{-1} dz$$

for the suitable contour  $\gamma_h$  such that  $1 = \tilde{O}(\text{dist}(\gamma_h, \sigma(\Delta_{f, f^{-1}([a, b]), h})))$ , and then to apply Proposition 2.19 with  $\varepsilon_0 > 0$  arbitrarily small. Such a contour is chosen as follows. For  $n \in \mathbb{N}$ , Proposition 3.2 says

$$\exists h_n > 0, \forall h \in ]0, h_n[, \quad \sigma(\Delta_{f, f^{-1}([a, b]), h}) \cap [0, e^{-\frac{1}{2(n+1)h}}] = \{0\},$$

and the condition  $h_{n+1} < h_n$  can be added. Take simply  $\gamma_h = \left\{ z \in \mathbb{C}, |z| = e^{-\frac{1}{(n+1)h}} \right\}$  for  $h \in [h_{n+1}, h_n[$ .  $\square$

The final result of this paragraph extends the exponential decay estimates of Proposition 2.13 (or Hypothesis 2.16), when  $f$  admits a single singular value  $\tilde{c}_1$ , under orthogonality conditions. It will be referred to as the ‘‘orthogonality lemma’’.

Because  $\Delta_{f,f^{-1}([a,b]),h}$  has a discrete spectrum, the operator

$$\Delta_{f,f^{-1}([a,b]),h} \Big|_{\ker(\Delta_{f,f^{-1}([a,b]),h})^\perp} : \ker(\Delta_{f,f^{-1}([a,b]),h})^\perp \rightarrow \ker(\Delta_{f,f^{-1}([a,b]),h})^\perp,$$

is invertible. We now define  $(\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1}$  by extension by 0 on  $\ker(\Delta_{f,f^{-1}([a,b]),h})$ :

$$(\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1} = \underbrace{0}_{\ker(\Delta_{f,f^{-1}([a,b]),h})} \oplus \underbrace{(\Delta_{f,f^{-1}([a,b]),h} \Big|_{\ker(\Delta_{f,f^{-1}([a,b]),h})^\perp})^{-1}}_{\ker(\Delta_{f,f^{-1}([a,b]),h})^\perp} \quad (50)$$

Thus, the equality  $\omega_h = (\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1} r_h$  simply means that  $\omega_h$  is the unique solution in  $\ker(\Delta_{f,f^{-1}([a,b]),h})^\perp \cap D(\Delta_{f,f^{-1}([a,b]),h})$  to

$$\Delta_{f,f^{-1}([a,b]),h} \omega_h = (1 - \Pi_{\{0\},[a,b],h}) r_h.$$

**Lemma 3.6.** *Under Hypothesis 3.1, the operator defined by (50) satisfies*

$$(\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1}(x, y) = \tilde{O}\left(e^{-\frac{|f(x)-f(y)|}{h}}\right)$$

in the sense of Definition 2.14.

*Proof.* With  $A = \Delta_{f,f^{-1}([a,b]),h}$  and  $\Pi_{\{0\},[a,b],h} = 1_{\{0\}}(A)$  write simply  $\omega_h = (\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1} r_h$  as

$$\begin{aligned} \omega_h &= (1 - \Pi_{\{0\},[a,b],h}) \omega_h = -\frac{1}{2i\pi} \int_{\gamma_h} \frac{A}{z(z-A)} \omega_h dz \\ &= -\frac{1}{2i\pi} \int_{\gamma_h} \frac{1}{z(z-A)} (1 - \Pi_{\{0\},[a,b],h}) r_h dz, \end{aligned}$$

where  $\gamma_h$  is the contour introduced in the proof of Proposition 3.5. To conclude, it then suffices to combine the resolvent estimates of Proposition 2.19 with  $\varepsilon_0 > 0$  arbitrarily small, as used in the proof of Proposition 3.5, and the result of Proposition 3.5.  $\square$

### 3.3.2 Changing the interval $[a, b]$

For further applications, it is useful to specify the effect of changing  $b$  in  $f_a^b$ . Rough estimates after a change of  $a$  and  $b$  are followed by more accurate estimates after a change of  $b$  only.

Remember that we work under Hypothesis 3.1 which contains Hypothesis 1.2 or for a more general Lipschitz function Hypothesis 1.6 and Hypothesis 2.16.

**Proposition 3.7.** *Assume Hypothesis 3.1 and  $a < a' < \tilde{c}_1 < b' < b$ . The kernels  $F_{\{0\},[\alpha,\beta],h} = \ker(\Delta_{f,f^{-1}([\alpha,\beta]),h}) = \text{Ran } \Pi_{\{0\},[\alpha,\beta],h}$ ,  $\alpha \in \{a, a'\}$ ,  $\beta \in \{b, b'\}$  satisfy*

$$\vec{d}(F_{\{0\},[a',b'],h}, F_{\{0\},[a,b],h}) = \vec{d}(F_{\{0\},[a,b],h}, F_{\{0\},[a',b'],h}) = \tilde{O}\left(e^{-\frac{\min\{b'-\tilde{c}_1, \tilde{c}_1-a'\}}{h}}\right),$$

where the second inclusion of  $F_{\{0\},[a',b'],h} \subset L^2(f_{a'}^{b'}) \subset L^2(f_a^b)$  is implemented by the extension by 0 on  $f_a^{a'} \cup f_{b'}^b$ .

*Proof.* We already know  $\dim F_{\{0\},[a,b],h}^{(p)} = \dim F_{\{0\},[a',b'],h}^{(p)} = \beta^{(p)}(f^b, f^a)$  for  $p \in \{0, \dots, d\}$ . From the remarks following Definition 3.3, it then suffices to prove

$$\vec{d}(F_{\{0\},[a,b],h}, F_{\{0\},[a',b'],h}) = \tilde{O}(e^{-\frac{\min\{b'-\tilde{c}_1, \tilde{c}_1-a'\}}{h}}).$$

For a normalized vector  $\psi \in F_{\{0\},[a,b],h}$  the exponential decay estimate of Proposition 2.13 (or Hypothesis 2.16 for a more general Lipschitz function  $f$ ) with  $r_h = 0$  and  $\lambda_h = 0$  says

$$\|e^{\frac{|f(x)-\tilde{c}_1|}{h}}\psi\|_{W(f_a^b)} = \tilde{O}(1).$$

For any  $\varepsilon > 0$  small enough, take  $\chi \in C_0^\infty(f_{a'+\varepsilon}^{b'-\varepsilon}; [0, 1])$  such that  $\chi \equiv 1$  in a neighborhood of  $f^{-1}([a'+2\varepsilon, b'-2\varepsilon])$ . The form  $\chi\psi$  then belongs to  $D(\Delta_{f, f^{-1}([a', b']), h})$  with  $d_{f,h}\psi = (hd\chi) \wedge \psi$ ,  $d_{f,h}^*\psi = -h\mathbf{i}_{\nabla\chi}\psi$ , and therefore

$$\langle \chi\psi, \Delta_{f, f^{-1}([a', b']), h}(\chi\psi) \rangle = \|d_{f,h}(\chi\psi)\|_{L^2}^2 + \|d_{f,h}^*(\chi\psi)\|_{L^2}^2 = \tilde{O}(e^{-2\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}-2\varepsilon}{h}})$$

$$\text{and } \|\psi - \chi\psi\|_{L^2}^2 = \tilde{O}(e^{-2\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}-2\varepsilon}{h}}).$$

Because 0 is the only exponentially small eigenvalue of  $\Delta_{f, f^{-1}([a', b']), h}$ , this implies

$$\text{dist}_{L^2}(\chi\psi, F_{\{0\},[a',b'],h}) = \tilde{O}(e^{-\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}-2\varepsilon}{h}}).$$

If  $F = F_{\{0\},[a',b'],h}$  is considered as a subspace of  $L^2(f_a^b)$  after extension by 0 on  $f_a^{a'} \cup f_{b'}^b$ , the orthogonal projection  $\Pi_F : L^2(f_a^b) \rightarrow F$  is given by  $\Pi_F u = \Pi_{\{0\},[a',b'],h}(u|_{f_a^{b'}})$  again extended by 0 on  $f_a^{a'} \cup f_{b'}^b$ .

From  $\|\Pi_{\{0\},[a',b'],h}\| \leq 1$  and the exponential decay estimates for  $\psi$ , we deduce, by setting  $E = F_{\{0\},[a,b],h}$ ,

$$\begin{aligned} \|(\Pi_E - \Pi_F \Pi_E)\psi\| &= \|\psi - \Pi_{\{0\},[a',b'],h}(\psi|_{f_a^{b'}})\|_{L^2(f_a^b)} \\ &\leq \|\psi - \chi\psi\|_{L^2(f_a^b)} + \|\chi\psi - \Pi_{\{0\},[a',b'],h}(\chi\psi)\|_{L^2(f_a^{b'})} + \|\chi\psi - \psi|_{f_a^{b'}}\|_{L^2(f_a^{b'})} \\ &\leq \tilde{O}(e^{-\frac{\min(\tilde{c}_1-a', b'-\tilde{c}_1)-2\varepsilon}{h}}). \end{aligned}$$

Since this holds for all  $\psi \in E$ ,  $\|\psi\| = 1$ , this proves  $\vec{d}(E, F) = \tilde{O}(e^{-\frac{\min(\tilde{c}_1-a', b'-\tilde{c}_1)-2\varepsilon}{h}})$ , and we conclude by taking  $\varepsilon > 0$  arbitrarily small.  $\square$

The above result implies that the mapping  $A_h : F_{\{0\},[a,b],h} \rightarrow F_{\{0\},[a',b'],h} \subset L^2(f_a^b)$  defined by  $A_h\psi = \Pi_{\{0\},[a',b'],h}(\psi|_{f_a^{b'}})$  satisfies

$$\|A_h^* A_h - 1\|_{\mathcal{L}(F_{\{0\},[a,b],h})} = \tilde{O}(e^{-\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}}{h}})$$

and then

$$\|A_h^* A_h - 1\|_{\mathcal{L}(F_{\{0\},[a,b],h})} + \|A_h A_h^* - 1\|_{\mathcal{L}(F_{\{0\},[a',b'],h})} = \tilde{O}(e^{-\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}}{h}}).$$

A more accurate version can be given when  $a = a'$ . Actually  $\tilde{O}(e^{-\frac{\min\{\tilde{c}_1-a', b'-\tilde{c}_1\}}{h}})$  is easily replaced by  $\tilde{O}(e^{-\frac{b'-\tilde{c}_1}{h}})$  but additionally a small change of  $A_h$  allows to improve the estimates in  $f_a^{\tilde{c}_1}$ .



**Proposition 3.8.** *Keep the same assumptions and conventions as in Proposition 3.7 with now  $a = a'$ . There exists a linear mapping  $A_h : F_{\{0\},[a,b],h} \rightarrow F_{\{0\},[a,b'],h}$  such that*

$$\|e^{\frac{b'-f(x)+b'-\tilde{c}_1}{h}}[\psi - A_h\psi]\|_{W(f_a^{\tilde{c}_1})} = \tilde{O}(1)\|\psi\|_{L^2}$$

holds for all  $\psi \in F_{\{0\},[a,b],h}$  and

$$\|A_h^*A_h - 1\|_{\mathcal{L}(F_{\{0\},[a,b],h})} + \|A_hA_h^* - 1\|_{\mathcal{L}(F_{\{0\},[a',b'],h})} = \tilde{O}(e^{-\frac{b'-\tilde{c}_1}{h}}). \quad (51)$$

*Proof.* The proof is modelled on Lemma 2.17.

Let  $\varepsilon \in ]0, \frac{b'-\tilde{c}_1}{4}[$ , and let  $\chi, \tilde{\chi} \in \mathcal{C}^\infty(f^{-1}([a, b'])); [0, 1]$  satisfy

$$\begin{aligned} \chi &\equiv 1 \text{ in } f_a^{b'-2\varepsilon} \quad , \quad \chi \equiv 0 \text{ in } f_{b'-\varepsilon}^{b'} \quad , \\ \tilde{\chi} &\equiv 0 \text{ in } f_a^{\tilde{c}_1+\varepsilon} \quad , \quad \tilde{\chi} \equiv 1 \text{ in } f_{\tilde{c}_1+2\varepsilon}^{b'} \quad . \end{aligned}$$

A form  $\psi \in F_{\{0\},[a,b],h} = \ker(\Delta_{f,f^{-1}([a,b]),h})$ ,  $\|\psi\|_{L^2} = 1$ , satisfies  $d_{f,h}\psi = 0$  and  $d_{f,h}^*\psi = 0$  in  $f_a^{b'}$  but has not to belong to  $D(\Delta_{f,f^{-1}([a,b']),h})$ . We introduce

$$\tilde{\psi}_\varepsilon = \chi\psi - \tilde{\chi}(\eta_1 + \eta_2),$$

where

$$\begin{aligned} \eta_1 &= d_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)}^*(\Delta_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)})^{-1}(hd\chi \wedge \psi) \\ &= (\Delta_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)})^{-1}[d_{f,h}^*(hd\chi \wedge \psi)] \end{aligned}$$

and

$$\begin{aligned} \eta_2 &= -d_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)}(\Delta_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)})^{-1}(h\mathbf{i}_{\nabla\chi}\psi) \\ &= -(\Delta_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)})^{-1}[d_{f,h}(h\mathbf{i}_{\nabla\chi}\psi)]. \end{aligned}$$

Note that the last equality in each of the two above relations follows from the intertwining relations of Proposition 2.8-4). This implies in particular that  $\eta_1, \eta_2$  both belong to the domain  $D(\Delta_{f,f^{-1}([\tilde{c}_1+\varepsilon,b'],h)})$  and hence satisfy the boundary conditions at  $\{f = b'\}$ . Since moreover  $\psi \in D(\Delta_{f,f^{-1}([a,b]),h})$  satisfies the boundary conditions at  $\{f = a\}$ ,  $\tilde{\psi}_\varepsilon$  then belongs to  $D(\Delta_{f,f^{-1}([a,b']),h})$ .

Besides, the exponential decay estimates on  $\psi$  given by Proposition 2.13 (or Hypothesis 2.16) imply

$$\|\psi\|_{W(f_{b'-2\varepsilon}^{b'})} = \tilde{O}(e^{-\frac{b'-\tilde{c}_1-2\varepsilon}{h}})$$

and therefore

$$\|d_{f,h}^*(hd\chi \wedge \psi)\|_{L^2} = \tilde{O}(e^{-\frac{b'-\tilde{c}_1-2\varepsilon}{h}}) \quad , \quad \|d_{f,h}(h\mathbf{i}_{\nabla\chi}\psi)\|_{L^2} = \tilde{O}(e^{-\frac{b'-\tilde{c}_1-2\varepsilon}{h}}).$$

The exponential decay estimates stated in Proposition 2.15 (or Hypothesis 2.16) then imply

$$\|e^{\frac{b'-f(x)+b'-\tilde{c}_1-4\varepsilon}{h}}\eta_1\|_{W(f_{\tilde{c}_1+\varepsilon}^{b'})} + \|e^{\frac{b'-f(x)+b'-\tilde{c}_1-4\varepsilon}{h}}\eta_2\|_{W(f_{\tilde{c}_1+\varepsilon}^{b'})} = \tilde{O}(1).$$

Set  $\omega_h = \tilde{\psi}_\varepsilon - \Pi_{\{0\},[a,b'],h}\tilde{\psi}_\varepsilon \in D(\Delta_{f,f^{-1}([a,b']),h}) \cap \ker(\Delta_{f,f^{-1}([a,b']),h})^\perp$  and compute

$$\begin{aligned} d_{f,f^{-1}([a,b']),h}\omega_h &= d_{f,f^{-1}([a,b']),h}\tilde{\psi}_\varepsilon \stackrel{d_{f,h}\psi=0}{=} -hd\tilde{\chi} \wedge (\eta_1 + \eta_2) \\ d_{f,f^{-1}([a,b']),h}^*\omega_h &= d_{f,f^{-1}([a,b']),h}^*\tilde{\psi}_\varepsilon \stackrel{d_{f,h}^*\psi=0}{=} h\mathbf{i}_{\nabla\tilde{\chi}}(\eta_1 + \eta_2) \\ \Delta_{f,f^{-1}([a,b']),h}\omega_h &= r_h = (1 - \Pi_{\{0\},[a,b'],h})r_h \\ \|e^{\frac{b'-f(x)+b'-\tilde{c}_1-4\varepsilon}{h}}r_h\|_{L^2(f_a^{b'})} &= \tilde{O}(1). \end{aligned}$$

The ‘‘orthogonality lemma’’ (Lemma 3.6) with  $\omega_h = \tilde{\psi}_\varepsilon - \Pi_{\{0\},[a,b'],h}\tilde{\psi}_\varepsilon$  yields

$$\|e^{\frac{b'-f(x)+b'-\tilde{c}_1-4\varepsilon}{h}}[\tilde{\psi}_\varepsilon - \Pi_{\{0\},[a,b'],h}\tilde{\psi}_\varepsilon]\|_{W(f_a^{b'})} = \tilde{O}(1).$$

By defining  $A_h^\varepsilon\psi := \Pi_{\{0\},[a,b'],h}\tilde{\psi}_\varepsilon \in F_{\{0\},[a,b'],h} \subset L^2(f_a^b)$ , it then follows from the latter relation and from the relation  $\psi \equiv \tilde{\psi}_\varepsilon$  in  $f_a^{\tilde{c}_1+\varepsilon}$  that

$$\|e^{\frac{b'-f(x)+b'-\tilde{c}_1}{h}}[\psi - A_h^\varepsilon\psi]\|_{W(f_a^{\tilde{c}_1})} = \tilde{O}(e^{\frac{4\varepsilon}{h}})$$

and

$$\begin{aligned} \|\psi - A_h^\varepsilon\psi\|_{L^2(f_a^b)} &\leq \|\psi - \tilde{\psi}_\varepsilon\|_{L^2(f_a^b)} + \|\tilde{\psi}_\varepsilon - A_h^\varepsilon\psi\|_{L^2(f_a^{b'})} \\ &\leq \|(1-\chi)\psi\|_{L^2(f_a^{b'})} + \|\tilde{\chi}(\eta_1 + \eta_2)\|_{L^2(f_a^{b'})} + \|\psi\|_{L^2(f_a^{b'})} + \|\tilde{\psi}_\varepsilon - A_h^\varepsilon\psi\|_{L^2(f_a^{b'})} \\ &= \tilde{O}(e^{-\frac{b'-\tilde{c}_1-4\varepsilon}{h}}). \end{aligned}$$

In order to conclude, it thus just remains to choose  $\varepsilon$  depending on  $h \in ]0, h_0[$  in a proper way. To do so, note that when  $\varepsilon = \frac{1}{n+1}$  with  $n \in \mathbb{N}$  large enough to ensure  $\varepsilon \in ]0, \frac{b'-\tilde{c}_1}{4}[$ , there exists  $h_n > 0$  such that for every  $h \in ]0, h_n[$ ,

$$\|e^{\frac{b'-f(x)+b'-\tilde{c}_1}{h}}[\psi - A_h^\varepsilon\psi]\|_{W(f_a^{\tilde{c}_1})} \leq e^{\frac{5}{(n+1)h}} \quad \text{and} \quad \|\psi - A_h^\varepsilon\psi\|_{L^2(f_a^b)} \leq e^{\frac{5}{(n+1)h}} e^{-\frac{b'-\tilde{c}_1}{h}}.$$

The sequence  $(h_n)_{n \in \mathbb{N}}$  can be chosen decreasing and it then suffices to define  $A_h := A_h^{\frac{1}{n+1}}$  when  $h \in [h_{n+1}, h_n[$ .  $\square$

### 3.3.3 Interactions of solutions to $d_{f,h}\omega = 0$ with local spectral problems

We conclude this section with a result which will be used in the construction and analysis of global quasimodes (see Section 6). It provides information about solutions to  $d_{f,h}\omega = 0$  in  $f^{\tilde{c}_1}$ , in particular how the exponential decay can be combined with local spectral information.

**Proposition 3.9.** *Assume Hypothesis 3.1 and  $a_0 \leq a < \tilde{c}_1 < b' < b$ . Let  $\delta(h) > 0$  satisfy  $\lim_{h \rightarrow 0} \delta(h) = 0$  and let the family  $(\omega_h)_{h \in ]0, h_0[}$  satisfy  $\omega_h \in W(f_a^{\tilde{c}_1-\delta(h)}; \Lambda T^*M)$  and  $d_{f,h}\omega_h = 0$  in  $f_{a_0}^{\tilde{c}_1-\delta(h)}$  with*

$$\|e^{\frac{f(x)-a_0}{h}}\omega_h\|_{W(f_a^{\tilde{c}_1-\delta(h)})} = \tilde{O}(1).$$

Take any cut-off function  $\chi \in \mathcal{C}_0^\infty(f^{-1}([a, \tilde{c}_1]; [0, 1]))$  such that  $\chi \equiv 1$  in a neighborhood of  $\{f = a\}$  and assume that  $h > 0$  is small enough so that  $\text{supp } \chi \subset [a, \tilde{c}_1 - \delta(h)[$ .

- i) The form  $\Pi_{\{0\},[a,b],h}[d_{f,h}(\chi\omega_h)] = \Pi_{\{0\},[a,b],h}[(hd\chi) \wedge \omega_h]$  does not depend on the choice of the cut-off function  $\chi$ .
- ii) If  $\Pi_{\{0\},[a,b],h}[d_{f,h}(\chi\omega_h)] = 0$ , then there exists a family of similar cut-off functions  $\chi_h$  such that  $\tilde{\omega}_h = \chi_h\omega_h - d_{f,f^{-1}[a,b],h}^*(\Delta_{f,f^{-1}[a,b],h}^\perp)^{-1}[(hd\chi_h) \wedge \omega_h]$ , where, in the r.h.s.,  $\chi_h$  in the first term is extended by 1 and the second term is extended by 0 in  $f_{a_0}^a$ , satisfies

$$\begin{aligned} \tilde{\omega}_h &\equiv \omega_h \quad \text{in } f_{a_0}^a, \\ d_{f,h}\tilde{\omega}_h &= 0 \quad \text{in } f_{a_0}^b, \end{aligned}$$

$$\text{and} \quad \|e^{\frac{f(x)-a_0}{h}}\tilde{\omega}_h\|_{W(f_a^b)} = \tilde{O}(1).$$

- iii) If  $A_h : F_{\{0\},[a,b],h} \rightarrow F_{\{0\},[a,b'],h}$  is the operator introduced in Proposition 3.8, then for any  $\psi \in F_{\{0\},[a,b],h}$ , the quantity  $\langle d_{f,h}(\chi\omega_h), \psi - A_h\psi \rangle$  does not depend on the choice of  $\chi$  and

$$\forall \psi \in F_{\{0\},[a,b],h}, \quad \langle d_{f,h}(\chi\omega_h), \psi - A_h\psi \rangle = \tilde{O}(e^{-\frac{b'-a_0+b'-\tilde{c}_1}{h}})\|\psi\|_{L^2}.$$

*Proof. i)* Let  $\chi_1, \chi_2$  be two cut-off functions like  $\chi$  in our assumptions. Then  $\chi_1\omega_h - \chi_2\omega_h$  belongs to  $D(d_{f,f^{-1}[a,b],h})$  and

$$d_{f,f^{-1}[a,b],h}(\chi_1\omega_h - \chi_2\omega_h) = d_{f,h}(\chi_1\omega_h) - d_{f,h}(\chi_2\omega_h).$$

We simply conclude with the commutation

$$\Pi_{\{0\},[a,b],h}d_{f,f^{-1}([a,b]),h} = d_{f,f^{-1}([a,b]),h}\Pi_{\{0\},[a,b],h} = 0.$$

**ii)** When  $\Pi_{\{0\},[a,b],h}[d_{f,h}(\chi\omega_h)] = 0$ , **i)** ensures that the latter relation is also satisfied if we replace  $\chi$  by  $\chi_\varepsilon$  with  $\chi_\varepsilon \equiv 1$  in  $f_a^{a'-\varepsilon}$  and  $\chi_\varepsilon = 0$  in  $f_a^{a'+\varepsilon}$  for  $a' = \frac{a+\tilde{c}_1}{2}$  and some  $\varepsilon \in ]0, \frac{\tilde{c}_1-a}{2}[$ . The a priori estimates on  $\omega_h$  and  $\text{supp}(hd\chi_\varepsilon) \wedge \omega_h \subset f^{-1}([a' - \varepsilon, a' + \varepsilon])$  imply

$$\|((hd\chi_\varepsilon) \wedge \omega_h)\|_{L^2(f_a^b)} = \tilde{O}(e^{-\frac{a'-a_0-\varepsilon}{h}}).$$

The orthogonality lemma, Lemma 3.6, then implies that

$$\eta_\varepsilon = d_{f,f^{-1}([a,b]),h}^*(\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1}[(hd\chi_\varepsilon) \wedge \omega_h]$$

(is well defined and) satisfies

$$\|e^{\frac{|f(x)-a'|-\varepsilon}{h}}\eta_\varepsilon\|_{L^2(f_a^b)} = \tilde{O}(e^{-\frac{a'-a_0-\varepsilon}{h}}).$$

Since moreover  $d_{f,h}(\chi_\varepsilon\omega_h) = (hd\chi_\varepsilon) \wedge \omega_h = (1 - \Pi_{\{0\},[a,b],h})[(hd\chi_\varepsilon) \wedge \omega_h]$  belongs to  $D(d_{f,f^{-1}([a,b],h)})$ , we can write

$$d_{f,f^{-1}([a,b]),h}\eta_\varepsilon = \Delta_{f,f^{-1}([a,b]),h}(\Delta_{f,f^{-1}([a,b]),h}^\perp)^{-1}((hd\chi_\varepsilon) \wedge \omega_h) = (hd\chi_\varepsilon) \wedge \omega_h.$$

Using in addition  $d_{f,f^{-1}([a,b]),h}^*\eta_\varepsilon = 0$ , we deduce

$$\|e^{\frac{|f(x)-a'|-\varepsilon}{h}}\eta_\varepsilon\|_{W(f_a^b)} = \tilde{O}(e^{-\frac{a'-a_0-\varepsilon}{h}}).$$

If  $\eta_\varepsilon$  denotes the extension by 0 in  $f_{a_0}^a$  of  $\eta_\varepsilon \in D(d_{f,f^{-1}([a,b],h)})$ , it still belongs to  $D(d_{f,f^{-1}([a_0,b],h)})$  and solves  $d_{f,h}\eta_\varepsilon = (hd\chi_\varepsilon) \wedge \omega_h$  in  $f_{a_0}^a \cup f_a^b$ . We have thus proved that  $\tilde{\omega}_\varepsilon := \chi_\varepsilon\omega_h - \eta_\varepsilon$  satisfies

$$d_{f,h}\tilde{\omega}_\varepsilon = 0 \text{ in } f_{a_0}^b \text{ and } \|e^{\frac{f(x)-a_0}{h}}\tilde{\omega}_\varepsilon\|_{W(f_a^b)} = \tilde{O}(e^{\frac{2\varepsilon}{h}}).$$

We then end the proof by choosing conveniently  $\varepsilon$  depending on  $h \in ]0, h_0[$  as we did at the end of the proof of Proposition 3.8: when  $\varepsilon = \frac{1}{n+1}$ , take  $h_n > 0$  such that

$$\forall h \in ]0, h_n[, \quad \|e^{\frac{f(x)-a_0}{h}}\tilde{\omega}_\varepsilon\|_{W(f_a^b)} \leq e^{\frac{3}{(n+1)h}}$$

with  $(h_n)_{n \in \mathbb{N}}$  decreasing, and choose  $\chi_h := \chi_{\frac{1}{n+1}}$  when  $h \in [h_{n+1}, h_n[$ .

**iii)** Since

$$\langle d_{f,h}(\chi\omega_h), \psi - A_h\psi \rangle = \langle \Pi_{\{0\},[a,b],h}[d_{f,h}(\chi\omega_h)], \psi \rangle - \langle \Pi_{\{0\},[a,b'],h}[d_{f,h}(\chi\omega_h)], A_h\psi \rangle$$

does not depend on  $\chi$ , we may take the preceding  $\chi = \chi_\varepsilon$ . Owing to Proposition 3.8, we deduce

$$\begin{aligned} |\langle d_{f,h}(\chi_\varepsilon\omega_h), \psi - A_h\psi \rangle| &\leq \|((hd\chi_\varepsilon) \wedge \omega_h)\|_{L^2(f_{a'-\varepsilon}^{a'+\varepsilon})} \|\psi - A_h\psi\|_{L^2(f_{a'-\varepsilon}^{a'+\varepsilon})} \\ &= \tilde{O}(e^{-\frac{a'-a_0-\varepsilon}{h}}) \times \tilde{O}(e^{-\frac{b'-a'-\varepsilon+b'-\tilde{c}_1}{h}}) \|\psi\|_{L^2}. \end{aligned}$$

Since this holds for every  $\varepsilon > 0$  small enough, this yields the result.  $\square$

## 4 Rough estimates for several “critical values”

In this section, we give first estimates for the exponentially small eigenvalues of  $\Delta_{f,f^{-1}([a,b],h)}$ . We work under the following assumption which, like Hypothesis 3.1 in Section 3, gathers Hypothesis 1.2 or (Hypothesis 1.6 and Hypothesis 2.16), and specify some notations.

**Hypothesis 4.1.** *The function  $f$  satisfies Hypothesis 1.2, or more generally Hypothesis 1.6 and Hypothesis 2.16, and we choose  $\eta_f$  such that*

$$0 < \eta_f < \frac{1}{2} \min_{1 < n \leq N_f} |c_n - c_{n-1}|.$$

In addition,  $a, b, -\infty \leq a < b \leq +\infty$ , are not “critical values” of  $f$ :  $a, b \notin \{c_1, \dots, c_{N_f}\}$ .

### 4.1 Bar code associated with $f$

We refer to Appendix B for details and simply recall the useful notations. We already mentioned in Subsection 1.2 that Hypothesis 1.6 implies Hypothesis B.1 in the beginning of Appendix B (this is actually proved in Subsection 8.3).

Under the assumption that  $M$  is compact and  $f$  has a finite number of “critical values”  $c_1 < \dots < c_{N_f}$ , there is a bar code  $\mathcal{B} = \mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A})$  where  $A$  is finite,  $-\infty < a_\alpha < b_\alpha \leq +\infty$ ,  $a_\alpha \in \{c_1, \dots, c_{N_f}\}$ ,  $b_\alpha \in \{c_2, \dots, c_{N_f}, +\infty\}$ . The set  $A$  is graded according to  $A = \sqcup_{p=0}^{\dim M} A^{(p)}$  so that, for  $\alpha \in A^{(p)}$ , the grading of endpoints of the corresponding bar is given by  $[a_\alpha, b_\alpha] = [a_\alpha^{(p)}, b_\alpha^{(p+1)}]$ . It contains all the information about the relative cohomology groups  $H(f^b, f^a; \mathbb{R})$  when  $a < b$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$ .

More precisely here is the situation when  $a < b$  are not “critical values”. We forget the bars with no end point in  $]a, b[$ , and among the remaining ones we distinguish the ones with two endpoints in  $]a, b[$ :

$$A^*(a, b) = \{\alpha \in A^*, [a_\alpha^*, b_\alpha^{*+1}[\cap]a, b[ \neq \{\emptyset, ]a, b[\}\}, \quad (52)$$

$$A_c^*(a, b) = \{\alpha \in A^*(a, b), [a_\alpha^*, b_\alpha^{*+1}[\cap]a, b[ \text{ relatively compact in } ]a, b[\}, \quad (53)$$

$$\alpha \in A^*(a, b) \Leftrightarrow a < a_\alpha^* < b \text{ or } a < b_\alpha^{*+1} < b,$$

$$\alpha \in A_c^*(a, b) \Leftrightarrow a < a_\alpha^* < b_\alpha^{*+1} < b.$$

We now partition the endpoints of the bars, multiple value being distinguished by the index  $\alpha \in A(a, b)$ , according to

$$\mathcal{X}^*(a, b) = \{(\alpha, a_\alpha^*), \alpha \in A_c^*(a, b)\} \quad (54)$$

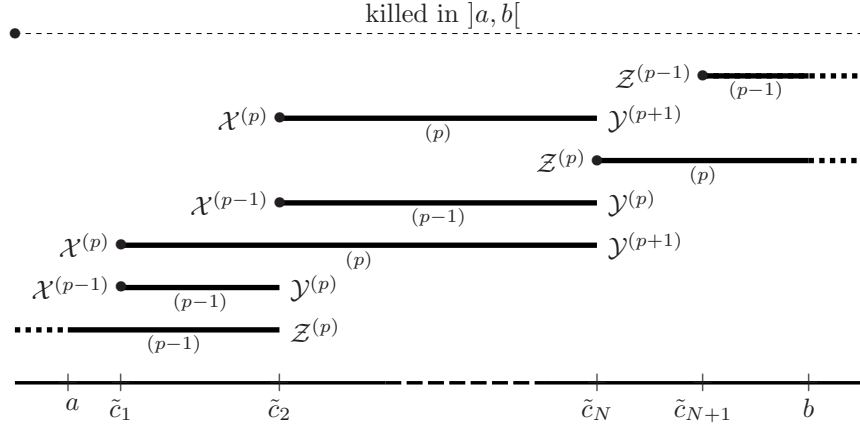
$$\mathcal{Y}^*(a, b) = \{(\alpha, b_\alpha^*), \alpha \in A_c^{*-1}(a, b)\} \quad (55)$$

$$\mathcal{Z}^*(a, b) = \{(\alpha, a_\alpha^*), \alpha \in A^*(a, b) \setminus A_c^*(a, b), a < a_\alpha^* < b\} \quad (56)$$

$$\sqcup \{(\alpha, b_\alpha^*), \alpha \in A^{*-1}(a, b) \setminus A_c^{*-1}(a, b), a < b_\alpha^* < b\},$$

$$\mathcal{J}^*(a, b) = \mathcal{X}^*(a, b) \sqcup \mathcal{Y}^*(a, b) \sqcup \mathcal{Z}^*(a, b). \quad (57)$$

Those definitions are illustrated in Figure 7: the degrees of the bars and of the corresponding endpoints are indicated. The bars in  $A_c(a, b)$  are the ones with two endpoints in  $]a, b[$  and the critical values lying in  $]a, b[$  are relabelled  $\tilde{c}_1 < \dots < \tilde{c}_N$ .



**Figure 7:**  $\mathcal{X}^* = \mathcal{X}^*(a, b)$  (lower),  $\mathcal{Y}^* = \mathcal{Y}^*(a, b)$  (upper),  $\mathcal{Z}^* = \mathcal{Z}^*(a, b)$  (lonely)

Then the relative Betti number are given by

$$\beta^{(p)}(f^b, f^a) = \dim H^p(f^b, f^a; \mathbb{R}) = \dim F_{\{0\}, [a, b], h} = \#\mathcal{Z}^{(p)}(a, b), \quad (58)$$

which counts the number of degree  $p$  endpoints of the bar code lying lonely in  $]a, b[$ . The rest of this section shows that there are exactly  $\#\mathcal{J}^{(p)}(a, b)$  exponentially small eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}^{(p)}$ , and provides a priori estimates on the size of the non zero ones.

## 4.2 Counting exponentially small eigenvalues

**Proposition 4.2.** *Under Hypothesis 4.1 and with the notations of Subsection 4.1, the exponentially small eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}$  are counted according to:*

$$\dim \ker(\Delta_{f, f^{-1}([a, b]), h}^{(p)}) = \#\mathcal{Z}^{(p)}(a, b) \quad (59)$$

$$\dim F_{[0, \tilde{o}(1)], [a, b], h}^{(p)} = \#\mathcal{J}^{(p)}(a, b) = \#\mathcal{X}^{(p)}(a, b) + \#\mathcal{Y}^{(p)}(a, b) + \#\mathcal{Z}^{(p)}(a, b), \quad (60)$$

where the second quality holds for  $h \in ]0, h_\varepsilon[$  when  $\tilde{o}(1)$  is replaced by  $e^{-2\frac{\eta_f - 2\varepsilon}{h}}$  for  $\varepsilon \in ]0, \frac{\eta_f}{2}[$ .

Note that the right-hand side of (60) is nothing but the total number of degree  $p$  endpoints of the bar code lying in  $]a, b[$ . This counting also says that the  $\tilde{o}(1)$  eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}^{(p)}$  are actually  $\tilde{O}(e^{-\frac{2\eta_f}{h}})$ .

*Proof.* Equality (59), which was already stated in Subsection 4.1, is proved in Appendix B. Equality (60) relies on exponential decay estimates and on the result in the case  $\{c_1, \dots, c_{N_f}\} \cap [a, b] = \{\tilde{c}_1\} \subset ]a, b[$  stated in Proposition 3.2.

The “critical values” of  $f$  lying in  $]a, b[$  are relabelled as  $a < \tilde{c}_1 < \dots < \tilde{c}_N < b$  according to

$$]a, b[ \cap \{c_1, \dots, c_{N_f}\} = [a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_N\}.$$

Consider the disjoint union  $\overline{\Omega}$ :

$$\overline{\Omega} = \bigsqcup_{j=1}^N f^{-1}([\tilde{c}_j - \eta_f, \tilde{c}_j + \eta_f] \cap [a, b])$$

for which the associated boundary Witten Laplacian is

$$\Delta_{f,\overline{\Omega},h} = \bigoplus_{j=1}^N \Delta_{f,f^{-1}([a,b] \cap [\tilde{c}_j - \eta_f, \tilde{c}_j + \eta_f]),h} \quad (61)$$

By Proposition 3.2, we know that the  $\tilde{o}(1)$  eigenvalues of  $\Delta_{f,\overline{\Omega},h}$  are equal to 0. For  $\varepsilon \in ]0, \eta_f/2[$ , take  $\chi \in C^\infty(\overline{\Omega}; [0, 1])$  such that  $\chi(x) = 1$  if  $\min_{1 \leq j \leq N} |f(x) - \tilde{c}_j| \leq \eta_f - \varepsilon$  and  $\chi(x) = 0$  if  $\min_{1 \leq j \leq N} |f(x) - \tilde{c}_j| \geq \eta_f - \varepsilon/2$ . For any  $\omega \in \ker(\Delta_{f,\overline{\Omega},h}^{(p)})$ ,  $\|\omega\|_{L^2} = 1$ , Proposition 2.13 (or Hypothesis 2.16) gives

$$d_{f,h}(\chi\omega) = (hd\chi) \wedge \omega = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}}) \quad \text{and} \quad d_{f,h}^*(\chi\omega) = h\mathbf{i}_{\nabla\chi}\omega = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}}).$$

Meanwhile our choice of  $\chi$  ensures  $\chi\omega \in D(\Delta_{f,f^{-1}([a,b]),h}^{(p)})$  with now

$$\|d_{f,f^{-1}([a,b]),h}(\chi\omega)\|_{L^2}^2 + \|d_{f,f^{-1}([a,b]),h}^*(\chi\omega)\|_{L^2}^2 \leq \tilde{O}(e^{-2\frac{\eta_f - \varepsilon}{h}}). \quad (62)$$

Since  $\|\chi\omega - \omega\|_{L^2} = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}})$ , the spectral decomposition of  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  ensures

$$\vec{d}(\ker(\Delta_{f,\overline{\Omega},h}^{(p)}), F_{[0, e^{-2\frac{\eta_f - 2\varepsilon}{h}}], [a,b], h}^{(p)}) = \tilde{O}(e^{-\frac{\varepsilon}{h}})$$

and then (see indeed the lines following Definition 3.3)

$$\dim(\ker(\Delta_{f,\overline{\Omega},h}^{(p)})) \leq \dim F_{[0, e^{-2\frac{\eta_f - 2\varepsilon}{h}}], [a,b], h}^{(p)}, \quad (63)$$

for  $h \in ]0, h_\varepsilon[$  with  $h_\varepsilon > 0$  small enough.

Reciprocally, when  $\omega \in F_{[0, e^{-\frac{\varepsilon}{h}}], [a,b], h}^{(p)}$ , the exponential decay estimates of Proposition 2.13 (or Hypothesis 2.16) lead again to

$$(hd\chi) \wedge \omega = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}}) \quad \text{and} \quad h\mathbf{i}_{\nabla\chi}\omega = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}})$$

and then to

$$\|d_{f,h}(\chi\omega)\|_{L^2}^2 + \|d_{f,h}^*(\chi\omega)\|_{L^2}^2 \leq \tilde{O}(e^{-\frac{\varepsilon}{h}})$$

with now  $\chi\omega \in D(\Delta_{f,\overline{\Omega},h}^{(p)})$ . Again, with  $\|\chi\omega - \omega\|_{L^2} = \tilde{O}(e^{-\frac{\eta_f - \varepsilon}{h}})$ , the spectral decomposition of  $\Delta_{f,\overline{\Omega},h}^{(p)}$ , with  $1_{[0, e^{-\frac{\varepsilon}{2h}}]}(\Delta_{f,\overline{\Omega},h}^{(p)}) = 1_{\{0\}}(\Delta_{f,\overline{\Omega},h}^{(p)})$ , leads to

$$\vec{d}(F_{[0, e^{-\frac{\varepsilon}{h}}], [a,b], h}, \ker(\Delta_{f,\overline{\Omega},h}^{(p)})) = \tilde{O}(e^{-\frac{\varepsilon}{4h}})$$

and then to

$$\dim F_{[0, e^{-\frac{\varepsilon}{h}}], [a,b], h}^{(p)} \leq \dim \ker \Delta_{f,\overline{\Omega},h}^{(p)} \leq \dim F_{[0, e^{-2\frac{\eta_f - 2\varepsilon}{h}}], [a,b], h}^{(p)},$$

for  $h \in ]0, h_\varepsilon[$ ,  $h_\varepsilon > 0$  small enough, where the last inequality follows from (63).

In particular, we deduce that for every  $\varepsilon > 0$  small enough:

$$F_{[0, e^{-\frac{\varepsilon}{h}}], [a,b], h}^{(p)} = F_{[0, e^{-2\frac{\eta_f - 2\varepsilon}{h}}], [a,b], h}^{(p)} \quad (64)$$

and

$$\dim F_{[0, e^{-2\frac{\eta_f - 2\varepsilon}{h}}], [a,b], h}^{(p)} = \dim \ker \Delta_{f,\overline{\Omega},h}^{(p)}.$$

We conclude with

$$\begin{aligned} \dim \ker \Delta_{f, \bar{\Omega}, h}^{(p)} &= \sum_{j=1}^N \beta^{(p)}(f^{\min(b, \tilde{c}_j + \eta_f)}, f^{\max(a, \tilde{c}_j - \eta_f)}) \\ &= \sum_{j=1}^N \#\mathcal{Z}^{(p)}(\max(a, \tilde{c}_j - \eta_f), \min(b, \tilde{c}_j + \eta_f)) = \#\mathcal{J}^{(p)}(a, b), \end{aligned}$$

the total number of degree  $p$  endpoints of the bar code lying in  $]a, b[$ .  $\square$

We have also proved the following result.

**Proposition 4.3.** *In the framework of Proposition 4.2 and when  $\Delta_{f, \bar{\Omega}, h}$  is the operator defined in (61), the following inequality holds:*

$$\vec{d}(F_{[0, \tilde{\sigma}(1)], h}^{(p)}, \ker(\Delta_{f, \bar{\Omega}, h}^{(p)})) + \vec{d}(\ker(\Delta_{f, \bar{\Omega}, h}^{(p)}), F_{[0, \tilde{\sigma}(1)], h}^{(p)}) = \tilde{O}(e^{-\frac{\eta_f}{h}}).$$

*Proof.* By (64) we know that for  $\varepsilon > 0$  small enough

$$\vec{d}(\ker(\Delta_{f, \bar{\Omega}, h}^{(p)}), F_{[0, e^{-\frac{\varepsilon}{h}}], h}^{(p)}) = \vec{d}(\ker(\Delta_{f, \bar{\Omega}, h}^{(p)}), F_{[0, e^{-\frac{2\eta_f - 2\varepsilon}{h}}], h}^{(p)}),$$

while we are in cases with  $\vec{d}(A, B) = \vec{d}(B, A) < 1$  by the result of Proposition 4.2. From (62) we deduce

$$\vec{d}(\ker(\Delta_{f, \bar{\Omega}, h}^{(p)}), F_{[0, e^{-\frac{\varepsilon}{h}}], h}^{(p)}) = \tilde{O}(e^{-\frac{\eta_f - 3\varepsilon/2}{h}}),$$

which yields the result.  $\square$

The result of Proposition 4.2 can be translated in terms of singular values of  $d_{f, f^{-1}([a, b]), h}$ . Remember that  $d_{f, f^{-1}([a, b]), h}$  and  $d_{f, f^{-1}([a, b]), h}^*$  are endomorphisms of  $F_{[0, C], [a, b], h}$  such that

$$\Delta_{f, f^{-1}([a, b]), h} \Big|_{F_{[0, C], [a, b], h}} = \delta_{[0, C], [a, b], h} \delta_{[0, C], [a, b], h}^* + \delta_{[0, C], [a, b], h}^* \delta_{[0, C], [a, b], h}$$

$$\text{with } \delta_{[0, C], [a, b], h} = d_{f, f^{-1}([a, b]), h} \Big|_{F_{[0, C], [a, b], h}}.$$

**Proposition 4.4.** *Under Hypothesis 4.1 and with the notations of Subsection 4.1, the number of  $\tilde{\sigma}(1)$  non zero singular values of  $\delta_{[0, \tilde{\sigma}(1)], [a, b], h} = d_{f, f^{-1}([a, b]), h} \Big|_{F_{[0, \tilde{\sigma}(1)], [a, b], h}}$  is  $\#\mathcal{A}_c(a, b)$  for  $h > 0$  small enough. More precisely “ $h > 0$  small enough” means  $h \in ]0, h_\varepsilon[$  for some  $h_\varepsilon > 0$  when  $\tilde{\sigma}(1)$  is replaced by  $e^{-\frac{\varepsilon}{h}}$ ,  $\varepsilon \in ]0, \frac{\eta_f}{2}[$ .*

*Proof.* Eigenvalues and singular values are counted with multiplicities. The non zero singular values of  $\underline{\delta} = \delta_{[0, e^{-\frac{\varepsilon}{h}}], [a, b], h}$  are the square roots of the non zero eigenvalues of  $\underline{\delta}^* \underline{\delta}$  and coincide with the non zero singular values of  $\underline{\delta} \underline{\delta}^*$ , i.e. the square roots of the non zero eigenvalues of  $\underline{\delta} \underline{\delta}^*$ . By Hodge decomposition, the number of non zero eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h} \Big|_{F_{[0, e^{-\frac{\varepsilon}{h}}], [a, b], h}} = \underline{\delta} \underline{\delta}^* + \underline{\delta}^* \underline{\delta}$  is twice the number of non zero singular values of  $\underline{\delta}$ . For  $h \in ]0, h_\varepsilon[$ , Proposition 4.2 gives

$$\begin{aligned} \dim F_{[0, e^{-\frac{\varepsilon}{h}}], [a, b], h} &= \#\mathcal{X}(a, b) + \#\mathcal{Y}(a, b) + \#\mathcal{Z}(a, b) \\ &= 2\#\mathcal{A}_c(a, b) + \dim(\ker(\Delta_{f, f^{-1}([a, b]), h})), \end{aligned}$$

which ends the proof.  $\square$

### 4.3 Rough exponential estimates

The upper bound on the  $\tilde{o}(1)$  eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}$  contained in Proposition 4.2 can be completed by a rough lower bound for the non zero ones.

**Proposition 4.5.** *Assume Hypothesis 4.1 and denote  $a < \tilde{c}_1 \dots < \tilde{c}_N < b$  the “critical values” of  $f$  in  $]a, b[$ . There exist  $r(h) > 0$  satisfying  $e^{-2\frac{\max\{b-\tilde{c}_1, \tilde{c}_N-a\}}{h}} = \tilde{O}(r(h))$  and  $R(h) = \tilde{O}(e^{-2\frac{\eta f}{h}})$  such that the  $\tilde{o}(1)$  non zero eigenvalues  $\lambda(h)$  of  $\Delta_{f,f^{-1}([a,b]),h}$  all belong to  $[r(h), R(h)]$  for  $h \in ]0, h_0[$ ,  $h_0 > 0$  small enough.*

*Proof.* The upper bound  $R(h) = \tilde{O}(e^{-2\frac{\eta f}{h}})$  is given by Proposition 4.2.

For the lower bound, it suffices to check that if  $\lambda(h) \in \sigma(\Delta_{f,f^{-1}([a,b]),h})$  satisfies  $\lambda(h) \leq e^{-2\frac{\max\{b-\tilde{c}_1, \tilde{c}_N-a\}+c}{h}}$  for some fixed  $c \in ]0, \min\{\tilde{c}_1 - a, b - \tilde{c}_N\}[$ , then there exists  $h_c > 0$  such that  $\lambda(h) = 0$  for all  $h \in ]0, h_c[$ . The proof follows the same arguments as those of Step 1 in Subsection 3.2.

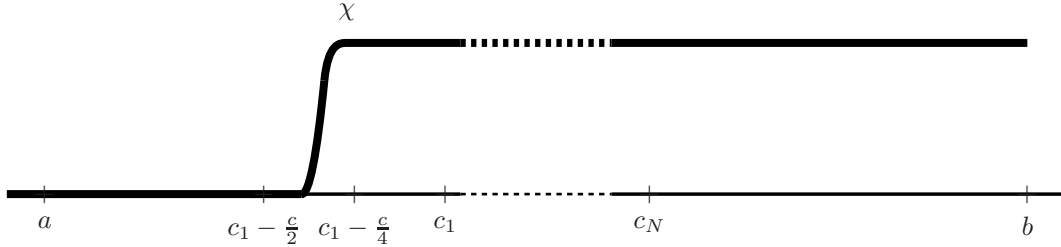
Let us proceed by contradiction and assume that there exists a decreasing sequence  $(h_n)_{n \in \mathbb{N}}$  tending to 0 such that, for every  $n \in \mathbb{N}$ ,  $\Delta_{f,f^{-1}([a,b]),h_n}$  admits an eigenvalue  $\lambda(h_n)$  in the interval  $]0, e^{-2\frac{\max\{b-\tilde{c}_1, \tilde{c}_N-a\}+c}{h_n}}]$ . Let then, for every  $n \in \mathbb{N}$ ,  $\omega_n \in D(\Delta_{f,f^{-1}([a,b]),h_n})$  satisfy  $\|\omega_n\|_{L^2} = 1$  and  $\Delta_{f,f^{-1}([a,b]),h_n}\omega_n = \lambda(h_n)\omega_n$ . From the Agmon estimates of Proposition 2.13 (or Hypothesis 2.16) with  $U \subset f^{-1}(\{\tilde{c}_1, \dots, \tilde{c}_N\})$ , we know that

$$\forall \delta > 0, \exists h_\delta > 0, \forall h_n \in ]0, h_\delta[, \quad \|e^{\frac{f-\tilde{c}_1}{h_n}} \omega_n\|_{L^2(f_{\tilde{c}_1-\delta}^b)} \geq \frac{e^{-\frac{\delta}{h_n}}}{2}$$

$$\text{while} \quad \|d_{f,h_n}\omega_n\|_{L^2(f_{\tilde{c}_1-\delta}^b)}^2 + \|d_{f,h_n}^*\omega_n\|_{L^2(f_{\tilde{c}_1-\delta}^b)}^2 \leq e^{-\frac{2(b-\tilde{c}_1)+c}{h_n}}.$$

By setting  $\tilde{\omega}_n = e^{\frac{f-\tilde{c}_1}{h_n}} \chi \omega_n$ , with  $\chi \in C^\infty(f^{-1}([a,b]); [0,1])$ ,  $\chi \equiv 1$  in  $f_{\tilde{c}_1-\frac{c}{4}}^b$  and  $\chi \equiv 0$  in  $f_a^{\tilde{c}_1-c'}$  with  $c' \in (\frac{c}{4}, \frac{c}{2})$ , we get, for every  $n \in \mathbb{N}$ ,

$$\begin{cases} \tilde{\omega}_n \in D(\Delta_{0,f^{-1}([\tilde{c}_1-c',b]),1}), \\ \|d_{0,f^{-1}([\tilde{c}_1-c',b]),1}\tilde{\omega}_n\|_{L^2}^2 = \tilde{O}(e^{-\frac{c}{h_n}}) \\ \liminf_{n \rightarrow +\infty} h_n \log \|\tilde{\omega}_n\|_{L^2} \geq 0. \end{cases}$$



**Figure 8:** The cut-off  $\chi$  in the interval  $[a, b]$ .

Besides, the Agmon estimates of Proposition 2.13 (or Hypothesis 2.16) with  $U \subset f^{-1}(\{\tilde{c}_1, \dots, \tilde{c}_N\})$  also imply

$$\limsup_{n \rightarrow +\infty} h_n \log \|\tilde{\omega}_n\|_{L^2} \leq \tilde{c}_N - \tilde{c}_1.$$

Hence, by extracting, we can assume that there exists  $\ell \in [0, 2(\tilde{c}_N - \tilde{c}_1)]$  such that

$$\lim_{n \rightarrow +\infty} h_n \log \|\tilde{\omega}_n\|_{L^2} = \frac{\ell}{2}.$$



The normalized form  $u_n = \frac{\tilde{\omega}_n}{\|\tilde{\omega}_n\|_{L^2}}$  thus belongs to  $D(\Delta_{0,f^{-1}([\tilde{c}_1-c',b]),1})$  and  $\|du_n\|_{L^2}^2 = \tilde{O}(e^{-\frac{c+\ell}{h_n}})$ . By Hodge decomposition (see Step 1 in Subsection 3.2 for details), this implies that  $\eta_n$  belongs to  $\ker(d_{0,f^{-1}([\tilde{c}_1-c',b]),1})$  and

$$\|u_n - \eta_n\|_{L^2(f_{\tilde{c}_1-c'})} = \tilde{O}(e^{-\frac{c+\ell}{2h_n}}).$$

Moreover, extending  $\eta_n$  by 0 in  $f_a^{\tilde{c}_1-c'}$  gives  $\eta_n \in D(d_{0,f^{-1}([a,b]),1})$  and therefore  $e^{-\frac{f-\tilde{c}_1}{h_n}}\eta_n \in \ker(d_{f,f^{-1}([a,b]),h_n})$  with

$$\|\chi\omega_n - \|\tilde{\omega}_n\|_{L^2} e^{-\frac{f-\tilde{c}_1}{h_n}} \eta_n\|_{L^2(f_a^b)} = \tilde{O}(e^{-\frac{c/2-c'}{h_n}}).$$

With  $\|\omega_n - \chi\omega_n\|_{L^2} = \tilde{O}(e^{-\frac{c}{4h_n}})$  and  $c'' = \min\{c/2 - c', c/4\}$ , we deduce

$$\text{dist}_{L^2}(\omega_h, \ker(d_{f,f^{-1}([a,b]),h_n})) = \tilde{O}(e^{-\frac{c''}{h_n}}) \xrightarrow{h \rightarrow 0} 0.$$

By duality, starting from  $\|d_{f,h}^* \omega_{h_n}\|_{L^2}^2 \leq e^{-\frac{2(\tilde{c}_N - a) + c}{h_n}}$  and extracting again, we also get

$$\lim_{h \rightarrow 0} \text{dist}_{L^2}(\omega_{h_n}, \ker(d_{f,f^{-1}([a,b]),h_n}^*)) = 0$$

and Hodge decomposition implies  $\lambda(h_n) = 0$  for  $n$  large enough (see indeed the end of Step 1 in Subsection 3.2), which leads to a contradiction and achieves the proof of Proposition 4.5.  $\square$

**Remark 4.6.** *The lower bound for the non zero eigenvalues is not optimal at this level. Actually, generalizing Step 3 of Subsection 3.2 requires the propagation of exponential decay through “critical values”, which is not true in general. This will be refined into  $e^{-2\frac{\tilde{c}_N - \tilde{c}_1}{h}} = \tilde{O}(r(h))$  at the end, when global quasimodes for  $d_{f,f^{-1}([a,b],h)}$  will have been constructed by induction on  $N$ . Like e.g. in [HKN, HeNi, Lep1, LNV], we follow the strategy which consists in studying carefully the singular values of  $d_{f,f^{-1}([a,b],h)}$ , which brings more flexibility than studying the tricky problem of interacting wells for  $\Delta_{f,f^{-1}([a,b],h)}$  in the spirit of [HeSj2, HeSj3].*

**Proposition 4.7.** *Assume Hypothesis 4.1, let  $a < \tilde{c}_1, \dots, \tilde{c}_N < b$  be the “critical values” of  $f$  in  $]a, b[$  and let  $R(h)$  be the function of  $h \in ]0, h_0[$  given by Proposition 4.5 such that  $\sigma(\Delta_{f,f^{-1}([a,b],h)} \cap [0, \tilde{o}(1)]) \subset [0, R(h)]$ . The projection  $\Pi_{[0,R(h)], [a,b], h} = 1_{[0,R(h)]}(\Delta_{f,f^{-1}([a,b],h)})$  satisfies*

$$\Pi_{[0,R(h)], [a,b], h} = \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h}})$$

in the sense of Definition 2.14.

*Proof.* By Proposition 4.5, we know that  $R(h) = \tilde{O}(e^{-\frac{2\eta_f}{h}})$ . Set  $\tilde{c}_0 = a$  and  $\tilde{c}_{N+1} = b$  and take any  $\varepsilon_0 \in ]0, \frac{\eta_f}{8}[$ , where  $\eta_f$  is defined in Hypothesis 4.1. Here the first assumption of Proposition 2.20 is obviously satisfied:

$$]a, b[ \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_N\} \subset \sqcup_{n=1}^N \tilde{c}_n - \frac{\varepsilon_0}{16}, \tilde{c}_n + \frac{\varepsilon_0}{16}].$$

For  $\Delta_n = \Delta_{f,f^{-1}([\tilde{c}_{n-1} + (1-\delta_{n,1})\varepsilon_0, \tilde{c}_{n+1} - (1-\delta_{n,N})\varepsilon_0])}$ ,  $n \in \{1, \dots, N\}$ , we know moreover that

$$\sigma(\Delta_n) \cap [0, e^{-\frac{\varepsilon_0}{h}}] \subset \{0\} \subset [0, e^{-\frac{4\varepsilon_0}{h}}],$$

owing to Proposition 3.2 because we are in the case  $[\tilde{c}_{n-1} + (1-\delta_{n,1})\varepsilon_0, \tilde{c}_{n+1} - (1-\delta_{n,N})\varepsilon_0] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_n\}$ .

Then Proposition 2.20 says: for some  $N \in \mathbb{N}^*$ ,

$$(\Delta_{f,f^{-1}([a,b],h)} - z)^{-1}(x, y) = \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + \frac{3N\varepsilon_0}{h}})$$

uniformly w.r.t  $z$ ,  $|z| = e^{-\frac{2\varepsilon_0}{h}}$ . But our choice of  $\varepsilon_0$ ,  $\varepsilon_0 > 0$  and  $4\varepsilon_0 \leq \frac{\eta_f}{2}$ , and

$$\sigma(\Delta_{f,f^{-1}([a,b]),h}) \cap [0, e^{-\frac{\varepsilon_0}{h}}] \subset [0, R(h)] \subset [0, e^{-\frac{\eta_f}{h}}] \subset [0, e^{-\frac{4\varepsilon_0}{h}}]$$

for  $h \in ]0, h_0[$ ,  $h_0$  small enough, imply

$$\Pi_{[0,R(h)],[a,b],h} = \frac{1}{2i\pi} \int_{|z|=e^{-\frac{2\varepsilon_0}{h}}} (z - \Delta_{f,f^{-1}([a,b]),h})^{-1} dz.$$

This proves

$$\Pi_{[0,R(h)],[a,b],h}(x, y) = \tilde{O}(e^{-\frac{|f(x)-f(y)|}{h} + \frac{3N\varepsilon_0}{h}}),$$

and we conclude by choosing  $\varepsilon_0 > 0$  arbitrarily small.  $\square$

## 5 Singular values

Singular values of compact operators are much more flexible than eigenvalues because they allow to work with two different orthonormal bases instead of one. Ky Fan inequalities recalled below provide uniform multiplicative errors for all the singular values after perturbing the orthonormal bases or moving the initial and final spaces. We recall those facts in a convenient way and complete those results by some refined analysis of additive error terms. This is a better rewriting of techniques already used e.g. in [HKN, HeNi, Lep1, LNV]

The singular values of a compact operator  $B : E \mapsto F$ ,  $E$  and  $F$  Hilbert spaces, are the square roots of the eigenvalues of  $B^*B$  (and  $BB^*$ ) and they are labelled in the decreasing order  $\mu_1(B) = \|B\| \geq \dots \geq \mu_\ell(B) \geq \mu_{\ell+1}(B) \dots$  with  $\lim_{\ell \rightarrow \infty} \mu_\ell(B) = 0$  after possibly completing the sequence by a sequence of 0's. They satisfy  $\mu_\ell(B) = \mu_\ell(B^*)$ . With this order, the min-max principle becomes a max-min principle applied to  $B^*B$  and gives:

$$\mu_\ell(B) = \min_{\dim V = \ell - 1} \max_{u \in V^\perp \setminus \{0\}} \frac{\|Bu\|}{\|u\|}. \quad (65)$$

Note also that the definition also provides the existence of two Hilbert bases  $(\varphi_j)_{j \in \mathcal{J}}$ ,  $\mathcal{J} \supset \mathcal{J}_1 = \{\ell \in \mathbb{N} \setminus \{0\}, \mu_\ell(B) > 0\}$ , of  $E$ , and  $(\psi_k)_{k \in \mathcal{K}}$  of  $F$ , and a one-to-one mapping  $j \in \mathcal{J}_1 \rightarrow k(j) \in \mathcal{K}$  such that

$$\begin{aligned} B\varphi_\ell = \mu_\ell(B)\psi_{k(\ell)} \quad \text{and then} \quad \mu_\ell(B) &= \|B\varphi_\ell\| = \langle \psi_{k(\ell)}, B\varphi_\ell \rangle & \text{if } \ell \in \mathcal{J}_1 \\ B\varphi_j &= 0 & \text{if } j \in \mathcal{J} \setminus \mathcal{J}_1. \end{aligned}$$

When  $E, F, G$  are three Hilbert spaces and  $A : E \rightarrow F$ ,  $B : F \rightarrow F$ , and  $C : F \rightarrow G$ , the singular values of  $B$  also satisfy

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \mu_\ell(CBA) \leq \|C\| \mu_\ell(B) \|A\|.$$

In order to handle accumulated multiplicative errors, it is convenient to use the function

$$\tau : \prod_{n=1}^{\infty} [0, 1[ \rightarrow ]0, +\infty[ \quad , \quad \tau(\varepsilon_1, \dots, \varepsilon_n) = \prod_{k=1}^n \frac{1 + \varepsilon_k}{1 - \varepsilon_k}. \quad (66)$$

In particular we have the implications **i**)  $\Rightarrow$  **ii**)  $\Rightarrow$  **iii**) for

- i)**  $\max(\|CC^* - \text{Id}_G\|, \|C^*C - \text{Id}_F\|) \leq \varepsilon_1 < 1$   
and  $\max(\|AA^* - \text{Id}_F\|, \|A^*A - \text{Id}_E\|) \leq \varepsilon_2 < 1$ ;
- ii)**  $\max(\|C\|, \|C^{-1}\|) \leq \tau(\varepsilon_1)^{1/2}$  ,  $\max(\|A\|, \|A^{-1}\|) \leq \tau(\varepsilon_2)^{1/2}$ ;
- iii)**  $\forall j \in \mathbb{N} \setminus \{0\}$  ,  $\tau(\varepsilon_1, \varepsilon_2)^{-1/2} \mu_j(B) \leq \mu_j(CBA) \leq \tau(\varepsilon_1, \varepsilon_2)^{1/2} \mu_j(B)$ .

The first implication is a consequence of the following operator inequalities

$$(1 - \varepsilon \leq |A|^2 = A^*A \leq 1 + \varepsilon) \Rightarrow \left( \tau(\varepsilon)^{-1/2} \leq (1 - \varepsilon)^{1/2} \leq |A| \leq (1 + \varepsilon)^{1/2} \leq \tau(\varepsilon)^{1/2} \right).$$

**Definition 5.1.** Let  $\mathcal{H}, \mathcal{H}'$  be two Hilbert spaces and let  $\varepsilon \in [0, 1[$ . An operator  $A : \mathcal{H} \rightarrow \mathcal{H}'$  will be said  $\varepsilon$ -unitary if it satisfies the condition

$$\max(\|A^*A - \text{Id}_{\mathcal{H}}\|, \|AA^* - \text{Id}_{\mathcal{H}'}\|) \leq \varepsilon,$$

used in **i**) just above.

A family of vectors  $(v_j)_{j \in \mathcal{J}}$  is an  $\varepsilon$ -orthonormal basis of  $\mathcal{H}$  if

- it is total in  $\mathcal{H}$ ,  $\overline{\text{Span}(v_j, j \in \mathcal{J})} = \mathcal{H}$ ,
- $\|(\langle v_j, v_k \rangle)_{j, k \in \mathcal{J}} - \text{Id}_{\ell^2(\mathcal{J})}\|_{\mathcal{L}(\ell^2(\mathcal{J}))} \leq \varepsilon$ .

Two closed subspaces  $\mathcal{H}_1, \mathcal{H}_2$  of  $\mathcal{H}$  provide an  $\varepsilon$ -orthogonal decomposition of  $\mathcal{H}$  if  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  and  $\|\Pi_{\mathcal{H}_1} \Pi_{\mathcal{H}_2}\| \leq \varepsilon$ .

Before we review applications to singular values, notice the following properties.

**Lemma 5.2.** Let  $\mathcal{H}, \mathcal{H}'$  be Hilbert spaces and let  $\varepsilon \in [0, 1[$ .

- a) For an operator  $A : \mathcal{H} \rightarrow \mathcal{H}'$ , the condition  $\|A^*A - \text{Id}_{\mathcal{H}}\| \leq \varepsilon$  is satisfied iff  $|A| : \mathcal{H} \rightarrow \mathcal{H}$  is  $\varepsilon$ -unitary and iff  $\text{Id}_{\mathcal{H}} : (\mathcal{H}, \langle \cdot, \cdot \rangle) \rightarrow (\mathcal{H}, \langle \cdot, |A|^2 \cdot \rangle)$  is  $\varepsilon$ -unitary.
- b) An operator  $A : \mathcal{H} \rightarrow \mathcal{H}'$  is  $\varepsilon$ -unitary iff

$$\|A^*A - \text{Id}_{\mathcal{H}}\| \leq \varepsilon \quad \text{and} \quad \overline{\text{Ran } A} = \mathcal{H}'.$$

- c) A family  $(v_j)_{j \in \mathcal{J}}$  is an  $\varepsilon$ -orthonormal basis of  $\mathcal{H}'$  iff the linear map  $A : \ell^2(\mathcal{J}) \rightarrow \mathcal{H}'$  given by  $A((a_j)_{j \in \mathcal{J}}) = \sum_{j \in \mathcal{J}} a_j v_j$  is  $\varepsilon$ -unitary.
- d) If the decomposition  $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2$  is  $\varepsilon$ -orthogonal and  $(\varphi_{j'})_{j' \in \mathcal{J}'}$  and  $(\varphi_{j''})_{j'' \in \mathcal{J}''}$  are orthonormal bases of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively, then  $(\varphi_j)_{j \in \mathcal{J}' \cup \mathcal{J}''}$  is an  $\varepsilon$ -orthonormal basis of  $\mathcal{H}'$ . Additionally, the identity map induces an  $\varepsilon$ -unitary map from  $\mathcal{H}' = \mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2$  to  $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2$ , where the first space is endowed with the scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2}$  making  $(\varphi_j)_{j \in \mathcal{J}' \cup \mathcal{J}''}$  orthonormal, i.e. defined by

$$\forall u_1, v_1 \in \mathcal{H}_1, \forall u_2, v_2 \in \mathcal{H}_2, \quad \langle u_1 + u_2, v_1 + v_2 \rangle_{\mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2} := \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle.$$

*Proof.* **a)** The first statement is a consequence of  $|A|^* = |A|$  and  $|A|^2 = A^*A$ . The second one is deduced from  $\text{Id}^* = |A|^2$  when the identity operator maps  $\mathcal{H}$  with the scalar product  $\langle u, v \rangle$  to itself with the scalar product  $\langle u, |A|^2 v \rangle$ .

**b)** It suffices to notice that the condition  $\|A^*A - \text{Id}_{\mathcal{H}}\| \leq \varepsilon$  implies

$$\forall u \in \mathcal{H}, \quad \sqrt{1 - \varepsilon} \|u\| \leq \|Au\| \leq \sqrt{1 + \varepsilon} \|u\|.$$

Thus  $A$  is one-to-one with a closed range which has to be  $\mathcal{H}'$  by the second assumption and  $A, A^*$ , and  $AA^*$  are invertible. Hence the spectrum of  $AA^*$  coincides with the spectrum of  $A^*A$  by  $A^*(AA^* - \lambda \text{Id}_{\mathcal{H}'}) = (A^*A - \lambda \text{Id}_{\mathcal{H}})A^*$  for  $\lambda \in \mathbb{C}$ . The spectral theorem yields  $\|AA^* - \text{Id}_{\mathcal{H}'}\| \leq \varepsilon$ . **c)** is a particular case of **b)** if we notice that  $\|A^*A - \text{Id}_{\mathcal{H}}\| = \|(\langle v_j, v_k \rangle)_{j, k \in \mathcal{J}} - \text{Id}_{\ell^2(\mathcal{J})}\|$  with  $\mathcal{H} = \ell^2(\mathcal{J})$ , while the condition  $\overline{\text{Ran } A} = \mathcal{H}'$  becomes equivalent to the totality of the family  $(v_j)_{j \in \mathcal{J}}$ .

**d)** The family  $(\varphi_j)_{j \in \mathcal{J}' \cup \mathcal{J}''}$  is clearly total in  $\mathcal{H}'$  and, defining the map  $A : \mathcal{H} \rightarrow \mathcal{H}'$  with  $\mathcal{H} = \ell^2(\mathcal{J})$  as in **c)**, we get

$$A^*A - \text{Id}_{\ell^2(\mathcal{J})} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \quad \text{with} \quad B = (\langle \varphi_k, \varphi_j \rangle)_{j \in \mathcal{J}'', k \in \mathcal{J}'}$$

To prove that  $(\varphi_j)_{j \in \mathcal{J}' \cup \mathcal{J}''}$  is an  $\varepsilon$ -orthonormal basis of  $\mathcal{H}'$ , it is then enough to prove that  $\|B\|_{\mathcal{L}(\ell^2(\mathcal{J}''), \ell^2(\mathcal{J}'))} \leq \varepsilon$ , which follows from the observation that  $B$  is unitarily equivalent to  $\Pi_{\mathcal{H}_1}|_{\mathcal{H}_2} : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ .

For the last statement, it suffices to note that the mapping

$$u \in \left( \mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2} \right) \longmapsto \left( \langle \varphi_j, u \rangle_{\mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2} \right)_{j \in \mathcal{J}' \cup \mathcal{J}''} \in \ell^2(\mathcal{J}' \cup \mathcal{J}'')$$

is unitary and to apply **c**). □

Below are consequences of those notions on singular values.

**Proposition 5.3.** *Let  $E, F, G$  be three closed subspaces of a Hilbert space  $\mathcal{H}$  and assume  $\vec{d}(E, F) + \vec{d}(F, E) = \varepsilon_1 < 1$  and  $\vec{d}(F, G) + \vec{d}(G, F) = \varepsilon_2 < 1$ . Let  $B : F \rightarrow F$  be a bounded operator and let  $\Pi_F, \Pi_G$  be the orthogonal projections on  $F$  and  $G$ . The operator  $\tilde{B} = \Pi_G B \Pi_F|_E : E \rightarrow G$  is compact iff  $B$  is compact and in this case:*

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \tau(\varepsilon_1^2, \varepsilon_2^2)^{-1/2} \mu_\ell(\tilde{B}) \leq \mu_\ell(B) \leq \mu_\ell(\tilde{B}) \tau(\varepsilon_1^2, \varepsilon_2^2)^{1/2}.$$

*Proof.* Call  $A_{FE} = \Pi_F \Pi_E + (1 - \Pi_F)(1 - \Pi_E)$ , with  $1 = \text{Id}_{\mathcal{H}}$ , and compute

$$A_{FE}^* A_{FE} - 1 = \Pi_E \Pi_F + \Pi_F \Pi_E - \Pi_E - \Pi_F.$$

We deduce that for all  $u \in \mathcal{H}$ ,

$$\begin{aligned} \langle u, (A_{FE}^* A_{FE} - 1)u \rangle &= 2 \operatorname{Re} \langle \Pi_E u, \Pi_F u \rangle - \|\Pi_E u\|^2 - \|\Pi_F u\|^2 \\ &= -\|(\Pi_E - \Pi_F)u\|^2 \\ &\geq -2\|(\Pi_E - \Pi_F)\Pi_E u\|^2 - 2\|(\Pi_E - \Pi_F)(1 - \Pi_E)u\|^2 \\ &\geq -2\|(\Pi_E - \Pi_F \Pi_E)u\|^2 - 2\|(\Pi_F - \Pi_F \Pi_E)u\|^2 \\ &\geq -2(\vec{d}(E, F)^2 + \vec{d}(F, E)^2)\|u\|^2. \end{aligned}$$

Since  $0 \leq \varepsilon_1 < 1$ , we know that  $\vec{d}(E, F) = \vec{d}(F, E) = \frac{\varepsilon_1}{2}$  (see indeed the lines following Definition 3.3) and we have thus proved the operator inequalities

$$0 \leq (\text{Id}_{\mathcal{H}} - A_{FE}^* A_{FE}) \leq \varepsilon_1^2 \text{Id}_{\mathcal{H}}.$$

Owing to the spectral theorem, it follows

$$\|A_{FE}^* A_{FE} - \text{Id}_{\mathcal{H}}\| \leq \varepsilon_1^2,$$

and by symmetry, since  $A_{FE}^* = A_{EF}$ , we also get  $\|A_{FE} A_{FE}^* - \text{Id}\| \leq \varepsilon_1^2$ . The operator  $A_{FE}$  is thus  $\varepsilon_1^2$ -unitary, and similarly  $A_{GF}$  is  $\varepsilon_2^2$ -unitary.

Finally,  $\tilde{B} = \Pi_G B \Pi_F|_E : E \rightarrow G$  is nothing but the nonzero diagonal block of

$$A_{GF} B A_{FE} : \mathcal{H} = E \overset{\perp}{\oplus} E^\perp \longrightarrow \mathcal{H} = G \overset{\perp}{\oplus} G^\perp.$$

It is thus compact if and only if  $B$  is compact. Moreover, up to some additional irrelevant zeros, the singular values of  $\tilde{B}$  are the ones of  $A_{GF} B A_{FE}$  and the result follows from the general statement **i**)  $\Rightarrow$  **iii**) above. □

**Proposition 5.4.** *Let  $E, F$  be two Hilbert spaces,  $B : E \rightarrow F$  be a bounded operator and let  $\varepsilon_1, \varepsilon_2 \in ]0, 1[$ .*

- a) When  $(\varphi_j)_{j \in \mathcal{J}}$  is an  $\varepsilon_1$ -orthonormal basis in  $E$  and  $(\psi_k)_{k \in \mathcal{K}}$  is an  $\varepsilon_2$ -orthonormal in basis  $F$ , let  $\tilde{B} : \ell^2(\mathcal{J}) \rightarrow \ell^2(\mathcal{K})$  be defined by  $\tilde{B}\delta_j = \sum_{k \in \mathcal{K}} \langle \psi_k, B\varphi_j \rangle \delta_k$ . Then  $\tilde{B}$  is compact iff  $B$  is compact, and in this case their singular values satisfy

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \tau(\varepsilon_1, \varepsilon_2)^{-1/2} \mu_\ell(\tilde{B}) \leq \mu_\ell(B) \leq \mu_\ell(\tilde{B}) \tau(\varepsilon_1, \varepsilon_2)^{1/2}. \quad (67)$$

- b) Assume that  $E = E' \oplus E''$  is an  $\varepsilon_1$ -orthogonal decomposition and  $F = F' \oplus F''$  is an  $\varepsilon_2$ -orthogonal decomposition such that  $BE' \subset F'$  and  $BE'' \subset F''$ , then the relation (67) holds with  $\tilde{B} = \Pi_{F'} B|_{E'} \oplus \Pi_{F''} B|_{E''} : E' \oplus E'' \rightarrow F' \oplus F''$ .
- c) Assume that  $B$  is compact and that  $E = E' \oplus E''$  is an  $\varepsilon_1$ -orthogonal decomposition, and set  $F' = \overline{BE'}$ ,  $F'' = (F')^\perp$ . Assume moreover that

$$\nu = \inf \left( \{ \mu_\ell(B|_{E'}), \ell \in \mathbb{N} \setminus \{0\} \} \cap ]0, +\infty[ \right) \geq \frac{\|B|_{E''}\|}{(1 - \varepsilon_1)^{\frac{1}{2}} \varepsilon_2}.$$

Then, the operator  $\tilde{B} = B|_{E'} \oplus \Pi_{F''} B|_{E''} : E' \oplus E'' \rightarrow F' \oplus F''$  satisfies

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \tau(\varepsilon_1, \varepsilon_2)^{-1} \mu_\ell(\tilde{B}) \leq \mu_\ell(B) \leq \mu_\ell(\tilde{B}) \tau(\varepsilon_1, \varepsilon_2).$$

*Proof.* **a)** This item simply follows from the general statement **i)**  $\Rightarrow$  **iii)** above and from the relation  $\tilde{B} = \Psi_F^* B \Phi_E$ , where  $\Phi_E : \ell^2(\mathcal{J}) \rightarrow E$  and  $\Psi_F : \ell^2(\mathcal{K}) \rightarrow F$  are defined by  $\Phi_E((u_j)_{j \in \mathcal{J}}) = \sum_{j \in \mathcal{J}} u_j \varphi_j$  and  $\Psi_F((v_k)_{k \in \mathcal{K}}) = \sum_{k \in \mathcal{K}} v_k \psi_k$ , and are thus respectively  $\varepsilon_1$ - and  $\varepsilon_2$ -unitary according to item **c)** in Lemma 5.2.

**b)** Let  $(\varphi_j)_{j \in \mathcal{J}'}$  and  $(\varphi_j)_{j \in \mathcal{J}''}$  be two Hilbert bases of  $E'$  and  $E''$ , so that  $(\varphi_j)_{j \in \mathcal{J}' \cup \mathcal{J}''}$  is an  $\varepsilon_1$ -orthonormal basis of  $E$  according to item **d)** in Lemma 5.2. An  $\varepsilon_2$ -orthonormal basis  $(\psi_k)_{k \in \mathcal{K}' \cup \mathcal{K}''}$  of  $F$  is constructed in a similar way. It also follows from item **d)** in Lemma 5.2 that the identity

$\text{Id}_E : E = E' \oplus E'' \rightarrow E = E' \oplus E''$  is  $\varepsilon_1$ -unitary and, similarly,  $\text{Id}_F$  is  $\varepsilon_2$ -unitary. We conclude by applying the general statement **i)**  $\Rightarrow$  **iii)** above to the relation  $\tilde{B} = \text{Id}_F^* B \text{Id}_E$ .

**c)** If  $\|B|_{E''}\| = 0$  there is nothing to do. Actually this is a particular case of **b)** with  $BE'' = \{0\} \subset F''$ ,  $\varepsilon_2 = 0$  and of course  $\tau^{1/2} \leq \tau$ . If  $\|B|_{E''}\| > 0$ , then there exists  $\ell_1 \in \mathbb{N} \setminus \{0\}$  such that  $\nu = \mu_{\ell_1}(B|_{E'})$  and  $\text{rank}(B|_{E'}) = \ell_1$ . In particular, we can find two Hilbert bases  $(\varphi_j)_{j \in \mathcal{J}'}$  of  $E'$  and  $(\psi_k)_{k \in \mathcal{K}'}$  of  $F$  such that  $\mathcal{J}' \cap \mathcal{K} \supset \{1, \dots, \ell_1\}$  and

$$\forall j \in \{1, \dots, \ell_1\}, \quad B\varphi_j = \mu_j(B|_{E'}) \psi_j.$$

Set  $F' = \text{Span}(\psi_j, j \in \{1, \dots, \ell_1\}) = \text{Ran} B|_{E'}$ , and  $F'' = (F')^\perp$ , and introduce the map  $R : E \rightarrow E$  defined by

$$R|_{E'} = 0, \\ \forall u \in E'', \quad Ru = \sum_{j=1}^{\ell_1} \frac{\langle \psi_j, Bu \rangle}{\mu_j(B|_{E'})} \varphi_j.$$

The norm of  $R$  is not greater than  $\varepsilon_2$  since for every  $u = u' + u'' \in E = E' \oplus E''$ ,

$$\|Ru\|^2 = \|Ru''\|^2 = \sum_{j=1}^{\ell_1} \frac{|\langle \psi_j, Bu'' \rangle|^2}{\mu_j^2(B|_{E'})} \leq \frac{\|B|_{E''}\|^2}{\mu_{\ell_1}^2(B|_{E'})} \|u''\|^2 \leq (1 - \varepsilon_1) \varepsilon_2^2 \|u''\|^2 \leq \varepsilon_2^2 \|u\|^2,$$

where the last inequality follows from the last statement of Lemma 5.2. We deduce

$$\|\text{Id}_E - R\| \leq 1 + \varepsilon_2 \leq \tau(\varepsilon_2), \\ \|(\text{Id}_E - R)^{-1}\| \leq (1 - \varepsilon_2)^{-1} \leq \tau(\varepsilon_2),$$

and for every  $\ell \in \mathbb{N} \setminus \{0\}$ , using the above general statement **ii**)  $\Rightarrow$  **iii**),

$$\tau(\varepsilon_2)^{-1} \mu_\ell(B(\text{Id}_E - R)) \leq \mu_\ell(B) \leq \mu_\ell(B(\text{Id}_E - R))\tau(\varepsilon_2).$$

Moreover, the operator  $B_1 = B(1 - R)$  clearly sends  $E'$  into  $F'$ , and also sends  $E''$  into  $F'' = (F')^\perp$  according to

$$\forall u \in E'', \quad BRu = \sum_{j=1}^{\ell_1} \langle \psi_j, Bu \rangle \psi_j = \Pi_{F'} Bu.$$

Since in addition  $E' \oplus E''$  is a  $\varepsilon_1$ -orthogonal decomposition of  $E$  and  $F = F' \overset{\perp}{\oplus} F''$ , a direct application of **b**) (with  $\varepsilon_2 = 0$ ) says that the singular values of  $B_1 : E \rightarrow F$  and  $\tilde{B}_1 = \Pi_{F'} B_1|_{E'} \overset{\perp}{\oplus} \Pi_{F''} B_1|_{E''}$  are related by

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \tau(\varepsilon_1)^{-1/2} \mu_\ell(\tilde{B}_1) \leq \mu_\ell(B_1) \leq \mu_\ell(\tilde{B}_1)\tau(\varepsilon_1)^{1/2}.$$

We conclude with

$$\begin{aligned} \Pi_{F'} B_1|_{E'} &= \Pi_{F'} [B|_{E'} - \underbrace{BR|_{E'}}_{=0}] = B|_{E'}; \\ \Pi_{F''} B_1|_{E''} &= \Pi_{F''} B|_{E''} - \underbrace{\Pi_{F''} BR|_{E''}}_{=0} = \Pi_{F''} B|_{E''}. \end{aligned}$$

□

**Remark 5.5. 1)** *In the sequel, Propositions 5.3 and 5.4 will be used and combined with spaces  $E^h, F^h, G^h, E'^h, E''^h, F'^h, F''^h$ , operators  $B^h, \tilde{B}^h$ , and bases  $(\varphi_j^h)_{j \in \mathcal{J}}$  and  $(\psi_k^h)_{k \in \mathcal{K}}$  which depend on a small parameter  $h > 0$  and such that the hypotheses are satisfied with*

$$\lim_{h \rightarrow 0} \varepsilon_1(h) = \lim_{h \rightarrow 0} \varepsilon_2(h) = 0.$$

*More generally, note that when  $N$  parameters  $\varepsilon_1(h), \dots, \varepsilon_N(h)$  are involved and satisfy  $0 \leq \varepsilon_n(h) \leq \varrho(h)$  for  $n \in \{1, \dots, N\}$  with  $\lim_{h \rightarrow 0} \varrho(h) = 0$ , then for any  $\alpha \geq 0$ , the estimate*

$$\tau(\varepsilon_1(h), \dots, \varepsilon_N(h))^\alpha = 1 + O(\varrho(h))$$

*holds uniformly in the sense that there exist  $h_{\alpha, N, \varrho}, C_{\alpha, N} > 0$  independent of  $\varepsilon_1, \dots, \varepsilon_N$  such that*

$$\forall h \in ]0, h_{\varrho, N, \alpha}[, \quad \tau(\varepsilon_1(h), \dots, \varepsilon_N(h))^\alpha - 1 \leq C_{N, \alpha} \varrho(h).$$

*Several applications of the previous results in this setting will lead to estimates of the type*

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \mu_\ell(B^h) = \mu_\ell(\tilde{B}^h)(1 + O(\varrho(h))).$$

**2)** *A case is especially easy to handle: when  $E^h, F^h, G^h$  are finite dimensional with dimension bounded by a common number  $n_F$ . In this case, one can use any norm  $\| \cdot \|_{n_F^2}$  on  $\mathcal{M}_{n_F, n_F}(\mathbb{C})$  in order to check the  $O(\varepsilon_{1,2}(h))$ -orthonormality of the bases. The constants in the  $O(\varrho(h))$ -estimates are then fixed when  $n_F$ , the norm  $\| \cdot \|_{n_F^2}$  and possibly the above  $N \in \mathbb{N}$  and  $\alpha \geq 0$  are fixed.*

*Additionally, we recall that in this case,  $\vec{d}(E^h, F^h) = \vec{d}(F^h, E^h) < 1$  is equivalent to  $\vec{d}(E^h, F^h) < 1$  and  $\dim E^h = \dim F^h$ .*

The following lemma will be useful in the sequel.

**Lemma 5.6.** Let  $B^h : D(B^h) \rightarrow \mathcal{H}$ ,  $D(B^h) \subset \mathcal{H}$ , be a closed unbounded operator and assume that the closed subspaces  $E^h, F^h, G^h$  and the operator  $B^h$  satisfy

- $E^h \subset D(B^h)$ ;
- the restriction  $B^h|_{E^h}$  is a left multiple of  $\Pi_{F^h} B^h|_{E^h}$ :

$$\begin{array}{ccc} E^h & \xrightarrow{B^h} & \mathcal{H} \\ & \searrow & \uparrow C^h \\ & \Pi_{F^h} B^h|_{E^h} & F^h \end{array} ;$$

- the distance between  $F^h$  and  $G^h$  satisfies

$$\left[ \bar{d}(F^h, G^h) + \bar{d}(G^h, F^h) \right] \|C^h\| = O(\varrho(h)) \quad \text{with} \quad \lim_{h \rightarrow 0} \varrho(h) = 0.$$

Then  $\Pi_{G^h} B^h|_{E^h} = (\text{Id}_{\mathcal{H}} + O(\varrho(h))) \Pi_{F^h} B^h|_{E^h}$  and the restriction  $B^h|_{E^h}$  is also a left multiple of  $\Pi_{G^h} B^h|_{E^h}$ :

$$\begin{array}{ccc} E^h & \xrightarrow{B^h} & \mathcal{H} \\ & \searrow & \uparrow \tilde{C}^h \\ & \Pi_{G^h} B^h & G^h \end{array} ;$$

with  $\tilde{C}^h = C^h(\text{Id}_{\mathcal{H}} + O(\varrho(h)))$ . The roles of  $F^h$  and  $G^h$  are therefore symmetric.

*Proof.* Note first that the relation  $B^h|_{E^h} = C^h \Pi_{F^h} B^h|_{E^h}$  implies

$$\|B^h|_{E^h}\| \leq \|C^h\| \|\Pi_{F^h}\| \|B^h|_{E^h}\|$$

and then  $\|C^h\| \geq 1$  (except when  $B^h|_{E^h} = 0$ , in which case the statement of Lemma 5.6 is trivial). Consider now the difference in  $\mathcal{L}(E^h; \mathcal{H})$ :

$$\Pi_{G^h} B^h|_{E^h} - \Pi_{G^h} \Pi_{F^h} B^h|_{E^h} = (\Pi_{G^h} - \Pi_{G^h} \Pi_{F^h}) C^h \Pi_{F^h} B^h|_{E^h}.$$

By introducing the operator

$$C_{G^h F^h} = \Pi_{G^h} \Pi_{F^h} + (1 - \Pi_{G^h})(1 - \Pi_{F^h}) = \text{Id}_{\mathcal{H}} + O\left(\frac{\varrho(h)^2}{\|C^h\|^2}\right) = \text{Id}_{\mathcal{H}} + O(\varrho(h)^2)$$

like in the proof of Proposition 5.3, we obtain

$$\Pi_{G^h} B^h|_{E^h} = [C_{G^h F^h} + (\Pi_{G^h} - \Pi_{G^h} \Pi_{F^h}) C^h] \Pi_{F^h} B^h|_{E^h} = [\text{Id}_{\mathcal{H}} + O(\varrho(h))] \Pi_{F^h} B^h|_{E^h}.$$

We get  $\Pi_{F^h} B^h|_{E^h} = [\text{Id}_{\mathcal{H}} + O(\varrho(h))]^{-1} \Pi_{G^h} B^h|_{E^h}$  and we take  $\tilde{C}^h = C^h [\text{Id}_{\mathcal{H}} + O(\varrho(h))]^{-1}$ .  $\square$

We now consider additive error terms which arise in our applications.

**Proposition 5.7.** Let  $B_1^h, B_2^h : E^h \rightarrow F^h$  be two compact operators parametrized by  $h > 0$ , like possibly the Hilbert spaces  $E^h, F^h$ . Fix  $\ell_0 \in \mathbb{N} \setminus \{0\}$  and let  $\varrho(h) > 0$  satisfy  $\lim_{h \rightarrow 0} \varrho(h) = 0$ .

**a)** When  $\|B_2^h - B_1^h\| = O(\varrho(h)) \max(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h))$ , the singular values are related by

$$\forall \ell \in \{1, \dots, \ell_0\}, \quad \mu_{\ell}(B_2^h) = \mu_{\ell}(B_1^h)(1 + O(\varrho(h))).$$

b) The two following statements are equivalent:

$$\min(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) + \|B_2^h - B_1^h\| = O(\varrho(h) \max(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h)))$$

and

$$\max(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) + \|B_2^h - B_1^h\| = O(\varrho(h) \min(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h))).$$

*Proof.* The two results are simple consequences of the max-min principle.

**a)** Assume  $\|B_2^h - B_1^h\| \leq \varepsilon \mu_{\ell_0}(B_1^h)$  with  $\varepsilon < 1$ . For  $\ell \in \{1, \dots, \ell_0\}$  and  $V \subset E^h$ ,  $\dim V = \ell - 1$ , we write

$$\forall u \in V^\perp, \quad \|B_1^h u\| - \varepsilon \mu_{\ell_0}(B_1^h) \|u\| \leq \|B_2^h u\| \leq \|B_1^h u\| + \varepsilon \mu_{\ell_0}(B_1^h) \|u\|$$

and then, using  $\mu_{\ell_0}(B_1^h) \leq \mu_\ell(B_1^h)$ ,

$$\forall u \in V^\perp, \quad \frac{\|B_2^h u\|}{\|u\|} \leq \max_{v \in V^\perp \setminus \{0\}} \frac{\|B_1^h v\|}{\|v\|} + \varepsilon \mu_\ell(B_1^h)$$

$$\frac{\|B_1^h u\|}{\|u\|} - \varepsilon \mu_\ell(B_1^h) \leq \max_{v \in V^\perp \setminus \{0\}} \frac{\|B_2^h v\|}{\|v\|}.$$

Therefore, for every  $\ell \in \{1, \dots, \ell_0\}$ , we deduce

$$\max_{u \in V^\perp \setminus \{0\}} \frac{\|B_1^h u\|}{\|u\|} - \varepsilon \mu_\ell(B_1^h) \leq \max_{u \in V^\perp \setminus \{0\}} \frac{\|B_2^h u\|}{\|u\|} \leq \max_{u \in V^\perp \setminus \{0\}} \frac{\|B_1^h u\|}{\|u\|} + \varepsilon \mu_\ell(B_1^h)$$

for any subspace  $V$  such that  $\dim V = \ell - 1$ . Continuing by taking the minimum w.r.t  $V$  finally leads to

$$\forall \ell \in \{1, \dots, \ell_0\}, \quad \mu_\ell(B_1^h)(1 - \varepsilon) \leq \mu_\ell(B_2^h) \leq (1 + \varepsilon) \mu_\ell(B_1^h).$$

The  $h$ -dependent assumption and the symmetry  $B_1^h \leftrightarrow B_2^h$  in the above proof yield the result.

**b)** First, since  $\min \leq \max$ , the second condition obviously implies the first one. Moreover, the first condition implies  $\|B_2^h - B_1^h\| = O(\varrho(h) \max(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h)))$  and we deduce from **a)**  $\max(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h)) = O(\min(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h)))$ . We have then to show that the second condition is implied by

$$\min(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) + \|B_2^h - B_1^h\| = O(\varrho(h) \min(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h))). \quad (68)$$

But assuming this and reasoning as in the proof of **a)** with  $V \subset E^h$ ,  $\dim V = \ell_0$ , and using now  $\|B_2^h - B_1^h\| = O(\varrho(h) \mu_{\ell_0}(B_1^h))$ , leads to

$$\max_{u \in V^\perp \setminus \{0\}} \frac{\|B_2^h u\|}{\|u\|} = \max_{u \in V^\perp \setminus \{0\}} \frac{\|B_1^h u\|}{\|u\|} + O(\varrho(h) \mu_{\ell_0}(B_1^h))$$

and then, by taking the minimum w.r.t  $V$ , to

$$\mu_{\ell_0+1}(B_2^h) = \mu_{\ell_0+1}(B_1^h) + O(\varrho(h) \mu_{\ell_0}(B_1^h)).$$

It follows that

$$\max(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) = \min(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) + O(\varrho(h) \mu_{\ell_0}(B_1^h)).$$

Then, since  $\mu_{\ell_0}(B_1^h) = O(\min(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h)))$ , (68) leads to

$$\max(\mu_{\ell_0+1}(B_1^h), \mu_{\ell_0+1}(B_2^h)) + \|B_2^h - B_1^h\| = O(\varrho(h) \min(\mu_{\ell_0}(B_1^h), \mu_{\ell_0}(B_2^h))),$$

which concludes the proof.  $\square$



The final result of this section combines multiplicative and additive error estimates.

**Proposition 5.8.** *Let  $(B^h, D(B^h))$  be a densely defined closed operator in  $\mathcal{H}$ . Let  $E^h$ ,  $F^h$ , and  $G^h$  be finite dimensional subspaces of  $\mathcal{H}$  and let  $\varrho(h) > 0$  satisfy  $\lim_{h \rightarrow 0} \varrho(h) = 0$ . Assume that both  $E^h$  and  $F^h$  are contained in  $D(B^h)$  and that the space  $E^h$  admits the  $\varrho(h)$ -orthogonal decomposition  $E^h = E'^h \oplus E''^h$ , such that:*

1.  $\Pi_{F^h} B^h = B^h \Pi_{F^h}$  on  $D(B^h)$ ;
2.  $\Pi_{F^h} B^h \Pi_{F^h}$  has a fixed finite rank  $\ell_0 \in \mathbb{N}$ ;
3.  $B^h|_{E'^h}$  is a left multiple of  $\Pi_{F^h} B^h \Pi_{F^h}|_{E'^h} = \Pi_{F^h} B^h|_{E'^h}$ :

$$\begin{array}{ccc} E'^h & \xrightarrow{B^h} & \mathcal{H} \\ & \searrow & \uparrow C^h \\ \Pi_{F^h} B^h \Pi_{F^h} & & F^h \end{array}$$

4. with the convention  $\mu_0(A) = +\infty$  for any compact operator  $A$  and when  $\ell_1^h$  denotes the rank of  $\Pi_{G^h} B^h|_{E'^h}$ , the following inequalities are satisfied:

$$\vec{d}(E^h, F^h) + \vec{d}(F^h, E^h) + \|C^h\| \left( \vec{d}(F^h, G^h) + \vec{d}(G^h, F^h) \right) = O(\varrho(h)), \quad (69)$$

$$\|B^h|_{E''^h}\| \left[ \frac{1}{\mu_{\ell_1^h}(\Pi_{G^h} B^h|_{E'^h})} + \frac{\|C^h\| (\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h))}{\max(\mu_{\ell_0}(\Pi_{G^h} B^h|_{E^h}), \mu_{\ell_0}(B^h|_{F^h}))} \right] = O(\varrho(h)). \quad (70)$$

Then, the  $\ell_0$  first singular values of  $\Pi_{F^h} B^h \Pi_{F^h}$  and  $\Pi_{G^h} B^h \Pi_{E^h}$  satisfy

$$\forall \ell \in \{1, \dots, \ell_0\}, \quad \underbrace{\mu_\ell(\Pi_{G^h} B^h \Pi_{E^h})}_{=\mu_\ell(\Pi_{G^h} B^h|_{E^h})} = \underbrace{\mu_\ell(\Pi_{F^h} B^h \Pi_{F^h})}_{=\mu_\ell(B^h|_{F^h})} (1 + O(\varrho(h))). \quad (71)$$

Moreover, the  $\ell_0 + 1$ -th singular value of  $\Pi_{G^h} B^h \Pi_{E^h}$  satisfies

$$\frac{\mu_{\ell_0+1}(\Pi_{G^h} B^h \Pi_{E^h})}{\mu_{\ell_0}(\Pi_{G^h} B^h \Pi_{E^h})} = \frac{\mu_{\ell_0+1}(\Pi_{G^h} B^h|_{E^h})}{\mu_{\ell_0}(\Pi_{G^h} B^h|_{E^h})} \underset{h \rightarrow 0}{\sim} \frac{\mu_{\ell_0+1}(\Pi_{G^h} B^h|_{E^h})}{\mu_{\ell_0}(B^h|_{F^h})} = O(\varrho(h)). \quad (72)$$

*Proof.* Since the statement is trivial when  $\ell_0 = 0$ , we assume here that  $\ell_0 \geq 1$ . The assumptions 1. and 3. then imply  $\|C^h\| \geq 1$  because

$$\|B^h|_{E'^h}\| \leq \|C^h\| \|\Pi_{F^h}\| \|B^h|_{E'^h}\|$$

(except when  $B^h|_{E'^h} = 0$ , in which case one chooses  $C^h = \Pi_{F^h}$  so that  $\|C^h\| = 1$ ). Therefore, the first estimate (69) of 4. implies  $\dim E^h = \dim F^h = \dim G^h < \infty$  as well as  $\vec{d}(E^h, F^h) = \vec{d}(F^h, E^h)$  and  $\vec{d}(F^h, G^h) = \vec{d}(G^h, F^h)$ .

About dimensions, the assumptions 1. and 3. also imply

$$\text{rank}(\Pi_{F^h} B^h|_{E'^h}) = \text{rank}(B^h|_{E'^h}) = \ell_1^h \leq \ell_0.$$

This rank  $\ell_1^h$ , which is not assumed to be independent of  $h$ , will be proved to be equal to  $\text{rank}(\Pi_{G^h} B^h|_{E'^h})$ .

**Multiplicative estimates:** By replacing  $E^h$  by  $E'^h$  in Lemma 5.6, we get

$$\Pi_{G^h} B^h|_{E'^h} = [\text{Id}_{\mathcal{H}} + O(\varrho(h))] \Pi_{F^h} B^h|_{E'^h}$$

and therefore

$$\forall \ell \in \{1, \dots, \dim E'^h\}, \quad \mu_\ell(\Pi_{G^h} B^h|_{E'^h}) = \mu_\ell(\Pi_{F^h} B^h \Pi_{F^h}|_{E'^h})(1 + O(\varrho(h))). \quad (73)$$

In particular,

$$\text{rank}(\Pi_{G^h} B^h|_{E'^h}) = \text{rank}(\Pi_{F^h} B^h|_{E'^h}) = \ell_1^h = \text{rank}(B^h|_{E'^h}).$$

An accurate information about the orthogonal projections on  $F'^h := \text{Ran } \Pi_{F^h} B^h|_{E'^h}$  and on  $G'^h := \text{Ran } \Pi_{G^h} B^h|_{E'^h}$  is achieved as follows. There exist two orthonormal systems  $(\varphi_j^h)_{1 \leq j \leq \ell_1^h}$  in  $E'^h$  and  $(\psi_j^h)_{1 \leq j \leq \ell_1^h}$  in  $F'^h \subset F^h$  such that

$$\forall j \in \{1, \dots, \ell_1^h\}, \quad \Pi_{F^h} B^h \varphi_j^h = \mu_j^h \psi_j^h, \quad \text{where } \mu_j^h = \mu_j(\Pi_{F^h} B^h|_{E'^h}) > 0.$$

By computing

$$\begin{aligned} \psi_j^h - \frac{1}{\mu_j^h} \Pi_{G^h} B^h \varphi_j^h &= \frac{1}{\mu_j^h} [\Pi_{F^h} B^h \varphi_j^h - \Pi_{G^h} B^h \varphi_j^h] \\ &= \frac{1}{\mu_j^h} (\Pi_{F^h} - \Pi_{G^h})(C^h \Pi_{F^h} B^h \Pi_{F^h} \varphi_j^h) \\ &= (\Pi_{F^h} - \Pi_{F^h} \Pi_{G^h}) C^h \psi_j^h + (\Pi_{F^h} \Pi_{G^h} - \Pi_{G^h}) C^h \psi_j^h, \end{aligned}$$

we obtain the estimates:

$$\forall j \in \{1, \dots, \ell_1^h\}, \quad \|\psi_j^h - \frac{1}{\mu_j^h} \Pi_{G^h} B^h \varphi_j^h\| \leq \|C^h\| (\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h)) \underbrace{=}_{(69)} O(\varrho(h)).$$

Since moreover  $\text{rank } \Pi_{G^h} B^h|_{E'^h} = \dim G'^h = \ell_1^h \leq \ell_0$ , it follows that  $(\frac{1}{\mu_j^h} \Pi_{G^h} B^h \varphi_j^h)_{1 \leq j \leq \ell_1^h}$  is an  $O(\|C^h\|(\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h)))$ -orthonormal basis of  $G'^h$  (see Definition 5.1) and then that

$$\|\Pi_{F'^h} - \Pi_{G'^h}\| = O(\|C^h\|(\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h))) = O(\varrho(h)).$$

By calling  $F''^h$  the orthogonal of  $F'^h$  in  $F^h$  and  $G''^h$  the orthogonal of  $G'^h$  in  $G^h$ , the equality

$$\begin{aligned} \Pi_{F''^h} - \Pi_{G''^h} &= (1 - \Pi_{F'^h}) \Pi_{F^h} - (1 - \Pi_{G'^h}) \Pi_{G^h} \\ &= (1 - \Pi_{F'^h})(\Pi_{F^h} - \Pi_{G^h} \Pi_{F^h}) - (\Pi_{F'^h} - \Pi_{G'^h}) \Pi_{G^h} \Pi_{F^h} \\ &\quad - (1 - \Pi_{G'^h})(\Pi_{G^h} - \Pi_{G^h} \Pi_{F^h}) \end{aligned}$$

now implies (using also  $\|C^h\| \geq 1$ )

$$\|\Pi_{F''^h} - \Pi_{G''^h}\| = O(\|C^h\|(\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h))) = O(\varrho(h)). \quad (74)$$

The separation between the  $\ell_1^h$  first singular values and the smaller ones is obtained by applying Proposition 5.4-c) to  $B = \Pi_{F^h} B^h|_{E^h} : E^h \rightarrow F^h$  and to  $B = \Pi_{G^h} B^h|_{E^h} : E^h \rightarrow G^h$  with: the  $\varrho(h)$ -orthogonal decomposition  $E^h = E'^h \oplus E''^h$ ,

$$\begin{aligned} \mu_{\ell_1^h}(\Pi_{F^h} B^h|_{E'^h}) &= \mu_{\ell_1^h}(\Pi_{G^h} B^h|_{E'^h})(1 + O(\varrho(h))), \\ \text{and } \frac{\|\Pi_{F^h} B^h|_{E''^h}\|}{\mu_{\ell_1^h}(\Pi_{F^h} B^h|_{E'^h})} + \frac{\|\Pi_{G^h} B^h|_{E''^h}\|}{\mu_{\ell_1^h}(\Pi_{G^h} B^h|_{E'^h})} &\leq C \frac{\|B^h|_{E''^h}\|}{\mu_{\ell_1^h}(\Pi_{G^h} B^h|_{E'^h})} \underbrace{=}_{(70)} O(\varrho(h)). \end{aligned}$$

This implies that the singular values of  $\Pi_{F^h} B^h|_{E^h}$  and of  $\Pi_{G^h} B^h|_{E^h}$  satisfy

$$\forall \ell \in \{1, \dots, \ell_1^h\}, \quad \mu_\ell(\Pi_{F^h} B^h|_{E^h}) = \mu_\ell(\Pi_{F^h} B^h|_{E'^h})(1 + O(\varrho(h))), \quad (75)$$

$$\mu_\ell(\Pi_{G^h} B^h|_{E^h}) = \mu_\ell(\Pi_{G^h} B^h|_{E'^h})(1 + O(\varrho(h))), \quad (76)$$

and, for every  $k \geq 1$ ,

$$\mu_{\ell_1^h+k}(\Pi_{F^h} B^h|_{E^h}) = \mu_k(\Pi_{F''^h} B^h|_{E''^h})(1 + O(\varrho(h))) = O(\mu_{\ell_1^h}(\Pi_{F^h} B^h|_{E^h})\varrho(h)), \quad (77)$$

$$\mu_{\ell_1^h+k}(\Pi_{G^h} B^h|_{E^h}) = \mu_k(\Pi_{G''^h} B^h|_{E''^h})(1 + O(\varrho(h))) = O(\mu_{\ell_1^h}(\Pi_{G^h} B^h|_{E^h})\varrho(h)). \quad (78)$$

Besides, using  $\vec{d}(E^h, F^h) + \vec{d}(F^h, E^h) = O(\varrho(h))$  and the commutation  $\Pi_{F^h} B^h|_{E^h} = \Pi_{F^h} B^h \Pi_{F^h}|_{E^h}$ , a direct application of Proposition 5.3 with  $B = \Pi_{F^h} B^h|_{F^h} : F^h \rightarrow F^h$  and  $\tilde{B} = \Pi_{F^h} B^h \Pi_{F^h}|_{E^h} = \Pi_{F^h} B^h|_{E^h} : E^h \rightarrow F^h$  leads to:

$$\forall \ell \in \mathbb{N} \setminus \{0\}, \quad \mu_\ell(\Pi_{F^h} B^h|_{F^h}) = \mu_\ell(\Pi_{F^h} B^h|_{E^h})(1 + O(\varrho(h)^2)). \quad (79)$$

**Additive estimates:** When  $\ell_0 = \ell_1^h$ , the statement of Proposition 5.8 follows from the equations (73) and (75)–(79) and, when  $\ell_0 > \ell_1^h$ , these equations reduce the problem to the comparison of the singular values  $\mu_k$ ,  $1 \leq k \leq \ell_0 - \ell_1^h$ , of the two operators

$$\Pi_{G''^h} B^h|_{E''^h} \quad \text{and} \quad \Pi_{F''^h} B^h|_{E''^h}.$$

By (74) and (70), we know that

$$\frac{\|B^h|_{E''^h}\| \|\Pi_{G''^h} - \Pi_{F''^h}\|}{\max(\mu_{\ell_0 - \ell_1^h}(\Pi_{G''^h} B^h|_{E''^h}), \mu_{\ell_0 - \ell_1^h}(\Pi_{F''^h} B^h|_{E''^h}))} = O(\varrho(h)).$$

The first result (71) is thus an application of Proposition 5.7-a) with

$$B_1^h = \Pi_{F''^h} B^h|_{E''^h} \quad \text{and} \quad B_2^h = \Pi_{G''^h} B^h|_{E''^h}$$

while replacing  $\ell_0$  by  $\ell_0 - \ell_1^h$ .

Lastly, the definition of  $\ell_0$  in 2. implies

$$\min(\mu_{\ell_0 - \ell_1^h + 1}(\Pi_{G''^h} B^h|_{E''^h}), \mu_{\ell_0 - \ell_1^h + 1}(\Pi_{F''^h} B^h|_{E''^h})) = \mu_{\ell_0 - \ell_1^h + 1}(\Pi_{F''^h} B^h|_{E''^h}) = 0.$$

The remaining statement (72) is then given by Proposition 5.7-b).  $\square$

## 6 Accurate analysis with $N$ “critical values”

This section is the core and the most technical part of our analysis. It combines: i) the exponential decay estimates of eigenvectors solving  $\Delta_{f, f^{-1}([a, b]), h} \omega_h = \lambda_h \omega_h$ ,  $\lambda_h \xrightarrow{h \rightarrow 0} 0$ , and all the properties of solutions to  $d_{f, h} \omega_h = 0$  stated in Sections 2 and 3; ii) the information on local problems, that is when  $\sharp([a, b] \cap \{c_1, \dots, c_{N_f}\}) = 1$ , from Section 3; iii) the rough estimates when  $\sharp([a, b] \cap \{c_1, \dots, c_{N_f}\}) = N$  of Section 4. Finally, the recurrence analysis with respect to  $N$  is modelled on linear algebra lemmas about singular values given in Section 5. In the first paragraph, we review and complete previous useful notations before stating a general result which leads easily to Theorem 1.7, variations of which will be given afterwards. It is about the construction of global quasimodes for  $\Delta_{f, f^{-1}([a, b]), h}$ , and more precisely of a suitable basis of widely extended solutions to  $d_{f, h} \omega_h = 0$ , which, contrarily to the eigenfunctions of  $\Delta_{f, f^{-1}([a, b]), h}$ , provide a high flexibility when changing the geometrical domain, in particular the values  $a, b$ . After specifying the framework in the first paragraph, we check in Subsection 6.2 the first step of the recursive construction of such global quasimodes and presents the strategy of our iterative method, developed in the other paragraphs.

## 6.1 Assumptions, notations and main result

We assume Hypothesis 4.1 which is: The function  $f$  has a finite number of “critical values”,  $c_1 < \dots < c_{N_f}$ , according to Hypothesis 1.2 or Hypothesis 1.6, while Hypothesis 2.16 is assumed for a general Lipschitz function  $f$ , and we choose

$$\eta_f \in ]0, \frac{1}{2} \min_{1 < n \leq N_f} |c_n - c_{n-1}|[. \quad (80)$$

Moreover, the values  $a, b$ ,  $-\infty \leq a < b \leq +\infty$ , are not “critical values” of  $f$ .

Like in Sections 3 and 4, we use the the space  $W(f_a^b; \Lambda T^*M)$  of Definition 2.3. We recall that it coincides with  $W^{1,2}(f_a^b; \Lambda T^*M)$  under Hypothesis 1.2, while we only know  $W(f_a^b; \Lambda T^*M) \subset W_{loc}^{1,2}(f_a^b; \Lambda T^*M)$  in general (when  $a, b \notin \{c_1, \dots, c_{N_f}\}$ ). According to this remark, when  $E = \sqcup_{k=1}^K ]a_k, b_k[, a_k, b_k \notin \{c_1, \dots, c_{N_f}\}$ , the space  $W(f^{-1}(E); \Lambda T^*M)$  is nothing but the direct sum  $\oplus_{k=1}^K W(f_{a_k}^{b_k}; \Lambda T^*M)$ , which is included in  $W_{loc}^{1,2}(f^{-1}(E); \Lambda T^*M)$ .

The set of “critical values” lying in  $[a, b]$  are relabelled according to

$$[a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_N\} \quad , \quad a < \tilde{c}_1 < \dots < \tilde{c}_N < b.$$

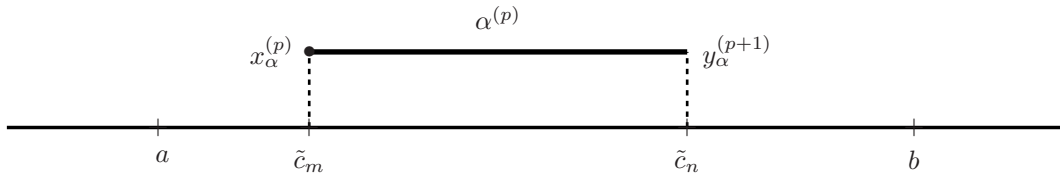
The bar code associated with  $f$  is still denoted by  $\mathcal{B} = \mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A}$ . We keep the notation  $A^*(a, b)$ ,  $A_c^*(a, b)$  given in (52),(53), while the endpoints of the bars with a non trivial intersection with  $]a, b[$  are partitionned into

$$\mathcal{J}^*(a, b) = \mathcal{X}^*(a, b) \sqcup \mathcal{Y}^*(a, b) \sqcup \mathcal{Z}^*(a, b),$$

where the definition of those sets are given in (54),(55),(56), and (57). Remember that an element  $j \in \mathcal{J}^{(p)}(a, b)$  is a pair  $j = (\alpha, \tilde{c})$  with  $\alpha \in A^*(a, b)$  and  $\tilde{c} \in \{\tilde{c}_1, \dots, \tilde{c}_N\}$ ,  $\tilde{c} = x_\alpha^{(p)}$ ,  $y_\alpha^{(p)}$ , or  $z_\alpha^{(p)}$ , depending on wether:

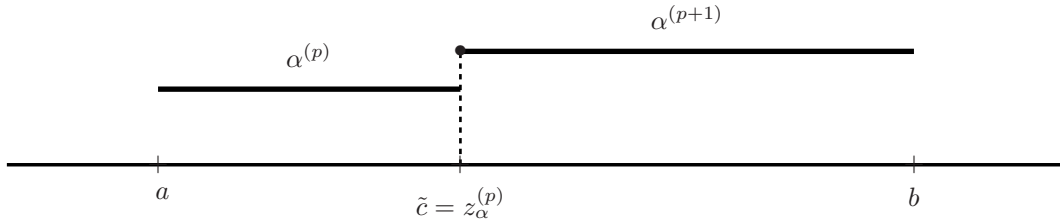
- $j \in \mathcal{X}^{(p)}(a, b)$ , which means  $\alpha \in A_c^{(p)}(a, b)$  and  $\tilde{c} = x_\alpha^{(p)}$ ,
- $j \in \mathcal{Y}^{(p)}(a, b)$ , which means  $\alpha \in A_c^{(p-1)}(a, b)$  and  $\tilde{c} = y_\alpha^{(p)}$ ,
- or  $j \in \mathcal{Z}^{(p)}(a, b)$ , which means  $\alpha \in A^*(a, b) \setminus A_c^*(a, b)$  and  $\tilde{c} = z_\alpha^{(p)}$ .

Below are figures which summarize the three different cases.



**Figure 9:** A bar  $[x_\alpha^{(p)}, y_\alpha^{(p+1)}[ = [\tilde{c}_m, \tilde{c}_n[, \alpha^{(p)} \in A_c^{(p)}(a, b)$ .

There are two extreme points  $j = (\alpha^{(p)}, \tilde{c}_m) \in \mathcal{X}^{(p)}(a, b)$  and  $j' = (\alpha^{(p)}, \tilde{c}_n) \in \mathcal{Y}^{(p+1)}(a, b)$ .



**Figure 10:** An extreme point  $j = (\alpha, \tilde{c}) \in \mathcal{Z}^{(p)}(a, b)$ .

After restriction to  $[a, b]$ , this represents the two possible equivalent ways of having

$$j = (\alpha^*, z_\alpha^{(p)}) \in \mathcal{Z}^{(p)}(a, b).$$

We recall that

$$\begin{aligned} \delta_{[0, e^{-\varepsilon/h}], [a, b], h}^{(p)} &= \mathbf{1}_{[0, e^{-\varepsilon/h}]}(\Delta_{f, f^{-1}([a, b]), h}^{(p+1)}) d_{f, f^{-1}([a, b]), h}^{(p)} \mathbf{1}_{[0, e^{-\varepsilon/h}]}(\Delta_{f, f^{-1}([a, b]), h}^{(p)}) \\ \text{and } F_{[0, e^{-\varepsilon/h}], [a, b], h}^{(p)} &= \text{Ran } \mathbf{1}_{[0, e^{-\varepsilon/h}]}(\Delta_{f, f^{-1}([a, b]), h}^{(p)}) \end{aligned}$$

do not depend on  $\varepsilon \in ]0, \varepsilon_0[$ , provided that  $h_\varepsilon > 0$  ( $h \in ]0, h_\varepsilon[$ ) is chosen small enough when  $\varepsilon$  is fixed. We then use the notation

$$\delta_{[0, \tilde{\alpha}(1)], [a, b], h}^{(p)} = \delta_{[0, e^{-\varepsilon/h}], [a, b], h}^{(p)} \quad \text{and} \quad F_{[0, \tilde{\alpha}(1)], [a, b], h}^{(p)} = F_{[0, e^{-\varepsilon/h}], [a, b], h}^{(p)} \quad (81)$$

without mentioning  $\varepsilon > 0$ .

The exponent  $^{(p)}$  is forgotten when the direct sum w.r.t  $p \in \{0, \dots, \dim M\}$  is considered.

The distance between vector spaces  $\tilde{d}(E, F)$  is the one defined in Subsection 3.1 (see Definition 3.3) and used in Section 5. We also recall that for  $\varepsilon > 0$ , an  $\tilde{O}(e^{-\frac{\varepsilon}{h}})$ -orthonormal family of vectors  $(\varphi_\ell^h)_{1 \leq \ell \leq L}$  in a Hilbert space  $\mathcal{H}$  is a family such that  $|\langle \varphi_\ell^h, \varphi_{\ell'}^h \rangle - \delta_{\ell, \ell'}| = \tilde{O}(e^{-\frac{\varepsilon}{h}})$  according to Definition 5.1.

With the family  $\mathcal{J}^*(a, b)$  of endpoints of bars with a non trivial intersection with  $]a, b[$ , we will associate an  $\tilde{\alpha}(1)$ -orthonormal family of solutions to  $d_{f, h} \omega_h = 0$  in the proper range.

**Definition 6.1.** *Under Hypothesis 4.1 and for  $\delta_1 \in ]0, \frac{\eta f}{8}]$ , let*

$$S_{\delta_1} := \{\tilde{c}_n - \delta_1, \tilde{c}_n + \delta_1, \quad 1 \leq n \leq N\}. \quad (82)$$

A family  $(\varphi_j^{*, h})_{j \in \mathcal{J}^*(a, b)}$ ,  $\varphi_j^{*, h} = \varphi_j^{(p), h}$  when  $j \in \mathcal{J}^{(p)}(a, b)$ , is called a  $\delta_1$ -family of quasimodes if there exists  $\gamma : ]0, h_0[ \rightarrow ]0, +\infty[$  with  $\lim_{h \rightarrow 0} \gamma(h) = 0$  such that:

- $(\varphi_j^{(p), h})_{j \in \mathcal{J}^{(p)}(a, b)}$  is a linearly independent family of  $D(d_{f, f^{-1}([a, b]), h}^{(p)})$  for all  $p \in \{0, \dots, d\}$ ;
- by setting  $j = (\alpha, \tilde{c})$  and  $I_j^h = [x_\alpha^{(p)} - \delta_1, y_\alpha^{(p+1)} - \gamma(h)] = [\tilde{c} - \delta_1, y_\alpha^{(p+1)} - \gamma(h)]$  when  $j \in \mathcal{X}^{(p)}(a, b)$ , and  $I_j^h = [\tilde{c} - \delta_1, b]$  when  $j \in \mathcal{Y}^{(p)}(a, b) \cup \mathcal{Z}^{(p)}(a, b)$ :

$$\text{supp } \varphi_j^{(p), h} \subset f^{-1}((I_j^h + [0, \gamma(h)/2]) \cap [a, b]), \quad (83)$$

$$\|e^{\frac{f-\tilde{c}}{h}} \varphi_j^{(p), h}\|_{W(f^{-1}([a, b]) \setminus S_{\delta_1})} = \tilde{O}(1), \quad (84)$$

$$d_{f, h} \varphi_j^{(p), h} \equiv 0 \quad \text{in } f^{-1}(I_j^h \cap [a, b]) \quad \text{and then in } f^{-1}([a, \tilde{c} - \delta_1] \cup (I_j^h \cap [a, b])). \quad (85)$$

For such a family of quasimodes, we will use the notation:

$$\mathcal{V}^{(p), h} = \text{Span}(\varphi_j^{(p), h}, j \in \mathcal{J}^{(p)}(a, b)) \quad , \quad \mathcal{V}^h = \bigoplus_{p=0}^d \mathcal{V}^{(p), h}.$$

The idea is that the quasimode associated with the endpoint  $j = (\alpha, \tilde{c}) \in \mathcal{J}^*(a, b)$  is supported in  $[\tilde{c} - \delta_1, b]$ , decays exponentially away from  $\tilde{c}$ , and solves  $d_{f, h} \varphi_j^{*, h} = 0$  in a region essentially covered by the bar indexed by  $\alpha$ . Global quasimodes for  $d_{f, h}$  are constructed by climbing along the values of  $f$ . The reason why  $W$ -estimates fail in a neighborhood of  $f^{-1}(S_{\delta_1})$  will appear in the construction of such a family (see in particular Remark 6.11 about the values  $\tilde{c}_n + \delta_1$ ).

The following definition specifies how such quasimodes are truncated around the upper endpoints  $y_\alpha^{(p+1)}$  when  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b)$ . This truncation operator preserves the spaces  $W(f^{-1}(I))$  for  $I \subset [a, b]$  and  $D(d_{f, f^{-1}([a, b]), h})$  with its boundary conditions.

**Definition 6.2.** In the framework of Definition 6.1 and for  $\delta_2 \in ]0, \frac{\eta_f}{8}]$ , let

$$\chi_{\tilde{c}_n, \delta_2}(x) = \chi \left( \frac{f(x) - \tilde{c}_n}{\delta_2} \right) \quad (86)$$

for  $n \in \{2, \dots, N\}$  and a fixed  $\chi \in C^\infty(\mathbb{R}; [0, 1])$  such that  $\chi \equiv 1$  on  $] -\infty, -2]$  and  $\text{supp } \chi \subset ] -\infty, -1[$ .

The operator  $T_{\delta_2}$  is defined on  $\mathcal{V}^h$  by

$$T_{\delta_2} \varphi_j^{(p), h} = \begin{cases} \chi_{y_\alpha^{(p+1)}, \delta_2} \varphi_j^{(p), h} & \text{if } j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b) \\ \varphi_j^{(p), h} & \text{if } j \in \mathcal{Y}^{(p)}(a, b) \cup \mathcal{Z}^{(p)}(a, b). \end{cases} \quad (87)$$

**Theorem 6.3.** Assume Hypothesis 4.1 with  $\eta_f$  given by (80).

**a)** For any  $p \in \{0, \dots, \dim M\}$ , the  $\tilde{o}(1)$  non zero singular values of  $d_{f, f^{-1}([a, b]), h}^{(p)}$ , that is the non zero singular values of  $\delta_{[0, \tilde{o}(1)], [a, b], h}^{(p)}$ , can be labelled by the family  $(\mu_j^h)_{j \in \mathcal{X}^{(p)}(a, b)}$  (with possible multiplicities) with

$$\mu_j^h \stackrel{\text{log}}{\sim} e^{-\frac{y_\alpha^{(p+1)} - x_\alpha^{(p)}}{h}}, \quad j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b).$$

**b)** For any  $\delta_1 \in ]0, \frac{\eta_f}{8}]$ , there exists a  $\delta_1$ -family  $(\varphi_j^{*, h})_{j \in \mathcal{J}^*(a, b)}$  of quasimodes in the sense of Definition 6.1 which is  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal in  $L^2(f_a^b)$ . The vector space  $\mathcal{V}^h$  spanned by those quasimodes satisfies:

$$\forall p \in \{0, \dots, \dim M\}, \quad \vec{d}(\mathcal{V}^{(p), h}, F_{[0, \tilde{o}(1)], [a, b], h}^{(p)}) + \vec{d}(F_{[0, \tilde{o}(1)], [a, b], h}^{(p)}, \mathcal{V}^{(p), h}) = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

**c)** If  $T_{\delta_2}$  is the truncation operator of Definition 6.2 for  $\delta_2 \in ]0, \frac{\eta_f}{8}]$ , then the map  $d_{f, f^{-1}([a, b]), h}^{(p)} T_{\delta_2} : \mathcal{V}^{(p), h} \rightarrow L^2(f^{-1}([a, b]))$  is a left multiple of  $\delta_{[0, \tilde{o}(1)], [a, b], h}^{(p)} T_{\delta_2} :$

$$\begin{array}{ccc} \mathcal{V}^{(p), h} & \xrightarrow{d_{f, f^{-1}([a, b]), h}^{(p)} T_{\delta_2}} & L^2(f^{-1}([a, b])) \\ & \searrow & \uparrow C^h \\ & \underbrace{\delta_{[0, \tilde{o}(1)], [a, b], h}^{(p)} T_{\delta_2}}_{(81)} & F_{[0, \tilde{o}(1)], [a, b], h}^{(p)} \end{array} \quad (88)$$

$$\text{with } \|C^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}}).$$

The proof will be done in several steps, by induction on the number of ‘‘critical values’’  $N$ . Because the graduation w.r.t  $p \in \{0, \dots, \dim M\}$  is associated with an obvious orthogonal decomposition of  $F_{[0, \tilde{o}(1)], [a, b], h}$  and  $\delta_{[0, \tilde{o}(1)], [a, b], h}$ , and clear partitions of the sets of indices for bars and endpoints,  $A(a, b) = \sqcup_{p=0}^d A^{(p)}(a, b)$ ,  $\mathcal{J}(a, b) = \sqcup_{p=0}^d \mathcal{J}^{(p)}(a, b)$ , etc., we can treat globally  $F_{[0, \tilde{o}(1)], [a, b], h}$  and  $\delta_{[0, \tilde{o}(1)], [a, b], h}$  and forget the degree  $p$ .

## 6.2 Initialisation and outline of the recurrence

**The result holds true for  $N = 1$ :** According to Proposition 3.2, we know that  $\mathcal{J}(a, b) = \mathcal{Z}(a, b)$  and that the  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}$ , and therefore the  $\tilde{o}(1)$ -singular values of  $\delta_{[0, \tilde{o}(1)], [a, b], h}$ , all vanish. This proves **a)**. To prove **b)**, it suffices to take an orthonormal basis  $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$  of  $\ker(\Delta_{f, f^{-1}([\max(a, \tilde{c}_1 - \delta_1), b]), h})$ , extended by 0 on  $f_a^{\tilde{c}_1 - \delta_1}$  when  $a < \tilde{c}_1 - \delta_1$ . Note

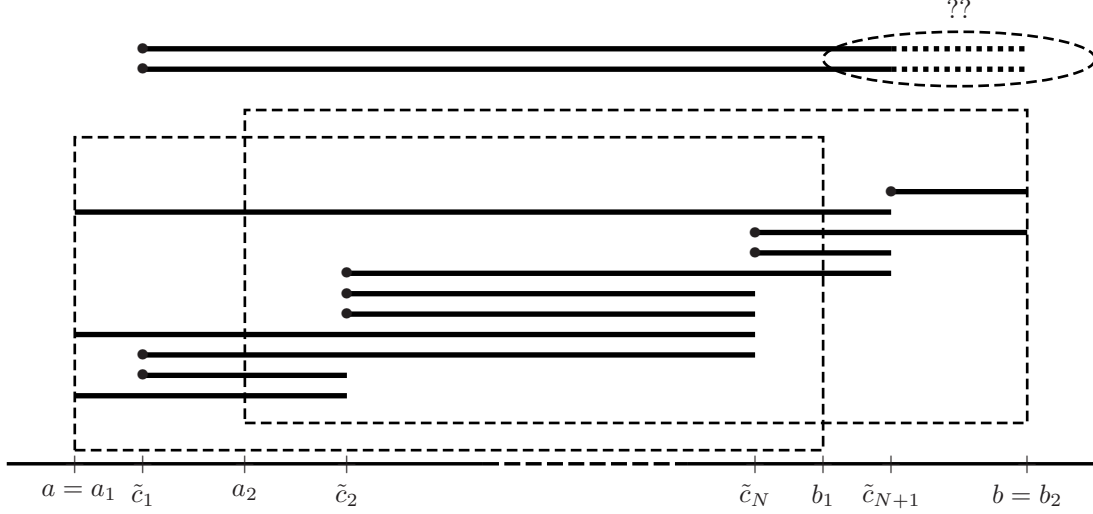
that in the latter case, the extended family  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  is still included in  $D(d_{f,f^{-1}([a,b],h)})$ , and actually in  $\ker(d_{f,f^{-1}([a,b],h)})$ . The exponential decay estimate (84) comes from the exponential decay estimates on the  $\varphi_j^h \in \ker(\Delta_{f,f^{-1}([\max(a,\tilde{c}_1-\delta_1),b],h)})$  given by Proposition 2.13 or Hypothesis 2.16 applied with  $\overline{\Omega} = f^{-1}([\max(a,\tilde{c}_1-\delta_1),b])$ ,  $r_h = 0$ ,  $\lambda_h = 0$ ,  $U = f^{-1}(\{\tilde{c}_1\})$  and  $d_{Ag}(x,y) \geq |f(x) - f(y)|$ . The distance between  $\mathcal{V}^h$  and  $F_{[0,\tilde{\sigma}(1)], [a,b], h}$  is also deduced from the exponential decay estimates on the  $\varphi_j^h \in \ker(\Delta_{f,f^{-1}([\max(a,\tilde{c}_1-\delta_1),b],h)})$  as we did in the proofs of Propositions 4.2 and 4.3. The statement **c)** is obvious in this case because

$$\begin{aligned} T_{\delta_2} &= \text{Id}_{\mathcal{V}^h}, \\ d_{f,f^{-1}([a,b],h)}|_{\mathcal{V}^h} &= 0, \\ \text{and } \delta_{[0,\tilde{\sigma}(1)], [a,b], h}|_{\mathcal{V}^h} &= \Pi_{[0,\tilde{\sigma}(1)], [a,b], h} d_{f,f^{-1}([a,b],h)}|_{\mathcal{V}^h} = 0. \end{aligned}$$

### Strategy of the proof by induction:

1. Already while checking the initial step  $N = 1$  or when proving e.g. Proposition 3.2 in Subsection 3.2, it was convenient to work with different values of  $a$  and  $b$ . From this point of view, the construction of  $\delta_1$ -quasimodes in the sense of Definition 6.1, which are some specific solutions to  $d_{f,h}\omega_h = 0$ , is more flexible than working with spectral eigenvectors of  $\Delta_{f,f^{-1}([a,b],h)}$ . Note that even though the extension by 0 in  $f^a$  of  $\varphi \in \ker(d_{f,f^{-1}([a,b],h)})$  does not belong to  $W(f_a^b)$  for  $a' < a$ , it belongs to  $\ker(d_{f,f^{-1}([a',b],h)})$ . This provides a way to extend the quasimodes in the area of the lower values of  $f$ . The extension to  $f_a^{b'}$  with  $b < b'$  will be done with a repeated use of Proposition 3.9. Note for example that if there is no “critical value” in  $[b, b']$ , a solution to  $d_{f,f^{-1}([a,b],h)}\varphi_h = 0$ , which satisfies some exponential decay estimates of the type  $\|e^{\frac{f(a)}{h}}\varphi_h\|_{W(f_a^b)} \leq \tilde{O}(C_h)$ , can be “extended” to a solution to  $d_{f,f^{-1}([a,b'],h)}\tilde{\varphi}_h = 0$ , with the same decay estimates in  $W(f_a^{b'} \setminus f^{-1}(\{b-\delta\}))$  for some  $\delta > 0$  small enough. To prove this, it suffices to consider  $b$  as an artificial new “critical value”  $\tilde{c}$  and to apply Proposition 3.9-i) with  $a_0, a, \tilde{c}_1, b$  there replaced by  $a, b-\delta, \tilde{c} = b, b'$ . Note that with this extension procedure,  $\tilde{\varphi}_h$  fails in general to belong to  $W$  in a neighborhood of  $f^{-1}(\{b-\delta\})$  (see (84) in this connection). If there is a “critical value”  $\tilde{c}_n \in ]b, b'[$ , then one has to study more carefully the orthogonality condition of Proposition 3.9-ii).
2. Now Theorem 6.3 will be assumed to be true in the case of  $N$  “critical values” in  $[a, b]$ , we can deduce several consequences. The aforementioned flexibility of a family of quasimodes in  $\mathcal{V}^h$ , as compared to a family of eigenvectors for the initial space  $F_{[0,\tilde{\sigma}(1)], [a,b], h}$ , can be completed by replacing the arrival space  $F_{[0,\tilde{\sigma}(1)], [a,b], h}$  in the diagram (88) by a more flexible approximation. Moreover, the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonality of the  $\delta_1$ -family of quasimodes can be preserved while ensuring true orthogonality properties on the images  $d_{f,h}T_{\delta_2}\varphi_j^h$ . This will be done in Subsection 6.3. The corresponding results will be used in the rest of the proof and for other constructions later.
3. Let us now explain how we pass from the case of  $N$  critical values  $\tilde{c}_1 < \dots < \tilde{c}_N$  to the case of  $N+1$  critical values  $\tilde{c}_1 < \dots < \tilde{c}_{N+1}$  in  $[a, b]$ . To do so, introduce  $a_2 \in ]\tilde{c}_1, \tilde{c}_2[$  and  $b_1 \in ]\tilde{c}_N, \tilde{c}_{N+1}[$ , set  $a_1 = a$ ,  $b_2 = b$ , and apply the result valid for  $N$  “critical values” to  $a_1 = a < \tilde{c}_1 < \dots < \tilde{c}_N < b_1$  and to  $a_2 < \tilde{c}_2 < \dots < \tilde{c}_{N+1} < b_2 = b$ . From the  $\delta_1$ -families of quasimodes for the intervals  $[a_1, b_1]$  and  $[a_2, b_2]$ , we can extract a partial  $\delta_1$ -family of quasimodes for  $[a, b]$  which satisfies the required properties for all bars of length strictly smaller than  $\tilde{c}_{N+1} - \tilde{c}_1$ . This construction, and all the information coming from step  $N$  in the intervals  $[a_1, b_1]$  and  $[a_2, b_2]$ , is collected in Subsection 6.4. After this, in Subsection 6.5, the construction of  $\delta_1$ -quasimodes associated with bars  $j = (\alpha, x_\alpha) \in \mathcal{X}(a, b)$  with  $x_\alpha = \tilde{c}_1$  and  $y_\alpha = \tilde{c}_{N+1}$  must be specified. This leads to the definition of “intermediate  $\delta_1$ -family of quasimodes” (see Definition 6.12) which, comparatively to Definition 6.1, does not

yet elucidate the interaction with the local spectral problems around the “critical value”  $\tilde{c}_{N+1}$ . This strategy is summarized in Figure 11 below. It is related to Mayer-Vietoris type arguments in algebraic topology, but handling and propagating all the estimates on exponentially small quantities requires some care. From this point of view, the inspiration is also taken from the standard techniques for handling exponential decay estimates, and several up and down inductions on  $n \in \{1, \dots, N+1\}$  are used.



**Figure 11:** Positions of the bars while the interval  $[a, b]$  is covered by  $[a_1, b_1] \cup [a_2, b_2]$ .

We will use the recurrence hypothesis at step  $N$  first in the interval  $]a_2, b_2[$  and then in the interval  $]a_1, b_1[$ , where the corresponding proper bars (not equal to  $]a_i, b_i[$ ) are collected in dashed rectangles. Quasimodes in  $]a_2, b_2[$  are extended by 0 in  $f_a^{a_2}$ , while the extension of quasimodes in  $]a_1, b_1[$  to  $f_{b_1}^b$  requires more care.

Once the latter “intermediate  $\delta_1$ -family of quasimodes” is constructed, it is used in order to prove Theorem 6.3-a) in Subsection 6.6. Like in the proof of Proposition 3.2 for  $N = 1$ , we have to play with different values of  $a, b$  such that  $a < \tilde{c}_1 < \dots < \tilde{c}_{N+1} < b$ . Using Proposition 4.5, we deduce firstly a lower bound  $r(h) \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}}$  when  $a = \tilde{c}_1 - \delta_1$  and  $b = \tilde{c}_{N+1} + \delta_3$ , and translate it in the various variations of the operator  $\delta_{[0, \tilde{\sigma}(1)], [a, b], h} T_{\delta_2}$  that we have introduced. Secondly, we study the effect of changing  $a$  and  $b$  while keeping  $N+1$  “critical values” in  $[a, b]$  as it was done in Subsection 3.3.2 for the case of one “critical value” in  $[a, b]$ . Thirdly, and only after proving Theorem 6.3-a), we can construct in Subsection 6.7 the  $\delta_1$ -family of quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$  at step  $N+1$ , and check all the conditions stated in the items **b)** and **c)** of Theorem 6.3.

### 6.3 Consequences of Theorem 6.3 at step $N$

We assume in this section that Theorem 6.3 holds true at step  $N$ . We refer in particular to the Definition 6.1 of  $\delta_1$ -quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$  and of  $\mathcal{V}^h = \text{Span}(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$ , and to the Definition 6.2 of the truncation  $T_{\delta_2} : \mathcal{V}^h \rightarrow D(d_{f, f^{-1}([a, b]), h})$ , for  $\delta_1, \delta_2 \in ]0, \frac{\eta f}{8}]$ .

While keeping the initial space  $\mathcal{V}^h$  for  $d_{f, f^{-1}([a, b]), h} T_{\delta_2}$ , we replace the arrival space  $F_{[0, \tilde{\sigma}(1)], [a, b], h}$ , and therefore the left-multiplying projection  $\Pi_{[0, \tilde{\sigma}(1)], [a, b], h}$ , by a more flexible space  $G^h$  and a projection  $\Pi_{G^h}$ . In view of Lemma 5.6 and of the general analysis of singular values led in Section 5, consider

$$G^h = \ker(\Delta_{f, \bar{\sigma}, h}) \quad \text{and} \quad F^h = F_{[0, \tilde{\sigma}(1)], [a, b], h}, \quad (89)$$



where  $\Delta_{f,\bar{\Omega},h}$  is the operator introduced in (61) with

$$\bar{\Omega} = \bigsqcup_{n=1}^N f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a, b]). \quad (90)$$

We recall that according to Proposition 4.3,

$$\begin{aligned} \vec{d}(G^h, F^h) + \vec{d}(F^h, G^h) &= \tilde{O}(e^{-\frac{\eta_f}{h}}) \\ \text{and } \dim G^h &= \dim F^h. \end{aligned}$$

The interest of the space  $G^h$  is that it is defined in terms of local spectral problems, actually kernels of local Witten Laplacians, around the ‘‘critical values’’  $\tilde{c}_1, \dots, \tilde{c}_N$ .

**Proposition 6.4.** *Assume that Theorem 6.3 holds true at step  $N$  and let  $G^h$  be defined by (89). The operator*

$$\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} = \Pi_{G^h} d_{f,h} T_{\delta_2} : \mathcal{V}^h \rightarrow L^2(\Omega) \subset L^2(f_a^b)$$

does not depend on  $\delta_2 \in ]0, \frac{\eta_f}{8}]$  for  $h > 0$  small enough. Namely, for two different choices  $\delta_2, \delta'_2 \in ]0, \frac{\eta_f}{8}]$ , there exists  $h_{\delta_2, \delta'_2} > 0$  such that the equality  $\Pi_{G^h} d_{f,h} T_{\delta_2} = \Pi_{G^h} d_{f,h} T_{\delta'_2}$  holds for all  $h \in ]0, h_{\delta_2, \delta'_2}[$ .

Its singular values satisfy:

$$\forall \ell \in \{1, \dots, \dim F^h\}, \quad \mu_\ell(\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} |_{\mathcal{V}^h}) = \mu_\ell(\delta_{[0, \tilde{\sigma}(1)], [a,b], h})(1 + \tilde{O}(e^{-\frac{\delta_1}{h}})). \quad (91)$$

Its kernel equals

$$\ker(\Pi_{G^h} d_{f,h} T_{\delta_2}) = \text{Span}(\varphi_j^h, j \in \mathcal{Y}(a, b) \cup \mathcal{Z}(a, b)). \quad (92)$$

In particular, when the non zero singular values of  $\delta_{[0, \tilde{\sigma}(1)], [a,b], h}$  are labelled as  $(\mu_j^h)_{j \in \mathcal{X}(a,b)}$  with  $\mu_j^h \stackrel{\log}{\sim} e^{-\frac{y_\alpha - x_\alpha}{h}}$  for  $j = (\alpha, x_\alpha)$ , the same result holds for the  $\delta_2$ -independent operator  $\Pi_{G^h} d_{f,h} T_{\delta_2}$ .

*Proof.* The Definition 6.1 of  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  and the Definition 6.2 of  $T_{\delta_2}$  give

$$d_{f,h} T_{\delta_2} \varphi_j^h = d_{f,h} \varphi_j^h = 0 \quad \text{if } j \in \mathcal{Y}(a, b) \cup \mathcal{Z}(a, b),$$

and

$$d_{f,h} T_{\delta_2} \varphi_j^h = 0 \quad \text{in } f^{-1}([a, y_\alpha - 2\delta_2]) \cup f^{-1}([y_\alpha - \delta_2, b]) \quad \text{if } j = (\alpha, x_\alpha) \in \mathcal{X}(a, b).$$

We deduce firstly

$$\ker(\Pi_{G^h} d_{f,h} T_{\delta_2}) \supset \text{Span}(\varphi_j^h, j \in \mathcal{Y}(a, b) \cup \mathcal{Z}(a, b)).$$

In the case  $j = (\alpha, x_\alpha) \in \mathcal{X}(a, b)$ , the equality  $\Pi_{G^h} d_{f,h} T_{\delta_2} = \Pi_{G^h} d_{f,h} T_{\delta'_2}$  for  $h > 0$  small enough is secondly a direct consequence of Proposition 3.9-i) applied around the ‘‘critical value’’  $y_\alpha$ , owing to  $\text{supp } d_{f,h} T_{\delta_2} \varphi_j^h \subset f^{-1}([y_\alpha - \eta_f, y_\alpha])$  and to

$$\Pi_{G^h} d_{f,h} T_{\delta_2} \varphi_j^h = \Pi_{\{0, [y_\alpha - \eta_f, y_\alpha + \eta_f] \cap [a,b], h\}} d_{f,h} T_{\delta_2} \varphi_j^h.$$

The result (91) on singular values implies  $\dim \ker(\Pi_{G^h} d_{f,h} T_{\delta_2}) = \#\mathcal{Y}(a, b) \cup \mathcal{Z}(a, b)$  and yields the equality (92). Let us now prove (91).

Consider the initial vector space  $E^h = T_{\delta_2} \mathcal{V}^h = \text{Span}(T_{\delta_2} \varphi_j^h, j \in \mathcal{J}(a, b))$  and the mapping  $B^h = d_{f,f^{-1}([a,b]),h} : E^h \rightarrow L^2(\Omega) \subset L^2(f_a^b)$ . The distance to  $\mathcal{V}^h$  is estimated by

$$\vec{d}(E^h, \mathcal{V}^h) + \vec{d}(\mathcal{V}^h, E^h) = \tilde{O}(e^{-\frac{\eta_f}{h}}) \leq \tilde{O}(e^{-\frac{\delta_1}{h}}). \quad (93)$$

With the factorization (88) stated in Theorem 6.3-c) and  $\vec{d}(G^h, F^h) + \vec{d}(F^h, G^h) = \tilde{O}(e^{-\frac{\eta_f}{h}})$  with  $2\delta_2 < \frac{\eta_f}{2} < \eta_f$ , we are exactly in the framework of Lemma 5.6 with  $\varrho(h) = \tilde{O}(e^{\frac{2\delta_2 - \eta_f}{h}})$ . Therefore,  $d_{f,f^{-1}([a,b]),h}|_{E^h}$  is a left multiple of  $\Pi_{G^h} d_{f,f^{-1}([a,b]),h}|_{E^h}$ ,

$$\begin{array}{ccc} E^h & \xrightarrow{B^h} & L^2(f_a^b) \\ & \searrow \Pi_{G^h} B^h & \uparrow \tilde{C}^h \\ & & G^h \end{array} \quad (94)$$

with  $\tilde{C}^h = C^h(\text{Id}_{L^2(f_a^b)} + \tilde{O}(e^{\frac{2\delta_2 - \eta_f}{h}}))$ , and

$$\Pi_{G^h} d_{f,f^{-1}([a,b]),h}|_{E^h} = (\text{Id}_{L^2(f_a^b)} + \tilde{O}(e^{\frac{2\delta_2 - \eta_f}{h}})) \underbrace{\Pi_{F^h} d_{f,f^{-1}([a,b]),h}|_{E^h}}_{=:\delta_{[0,\tilde{\sigma}(1)],[a,b],h} \Pi_{F^h}|_{E^h}}.$$

Using additionally Proposition 5.3 and the relation  $\vec{d}(E^h, F^h) + \vec{d}(F^h, E^h) = \tilde{O}(e^{-\frac{\delta_1}{h}})$  arising from Theorem 6.3-b) and (93), this leads to

$$\forall \ell \in \{1, \dots, \dim F^h\}, \quad \mu_\ell(\Pi_{G^h} d_{f,f^{-1}([a,b]),h}|_{E^h}) = \mu_\ell(\delta_{[0,\tilde{\sigma}(1)],[a,b],h})(1 + r(h)),$$

where  $r(h) = \max(\tilde{O}(e^{\frac{2\delta_2 - \eta_f}{h}}), \tilde{O}(e^{-\frac{2\delta_1}{h}})) \leq \tilde{O}(e^{-\frac{\delta_1}{h}})$ . The comparison (91) for  $\Pi_{G^h} d_{f,h} T_{\delta_2}|_{\mathcal{V}^h}$  is then a consequence of

$$\|T_{\delta_2}^* T_{\delta_2} - \text{Id}_{\mathcal{V}^h}\| + \|T_{\delta_2} T_{\delta_2}^* - \text{Id}_{E^h}\| = \tilde{O}(e^{-\frac{\eta_f}{h}}) = \tilde{O}(e^{-\frac{\delta_1}{h}}). \quad (95)$$

□

Below are details about a useful block decomposition of the operator  $\Pi_{G^h} d_{f,h} T_{\delta_2} : \mathcal{V}^h \rightarrow L^2(\Omega)$ . Of course, there is the orthogonal block decomposition with respect to the degree  $p$  according to  $\Pi_{G^h} d_{f,h}^{(p)} T_{\delta_2} : \mathcal{V}^{(p),h} \rightarrow L^2(\Omega; \Lambda^{p+1} T^* M)$ . But we consider here a block decomposition according to the length of the bars, which correspond to clusters of singular values. Again, we forget the degree  $p$  here. We need some notations. Let

$$\mathcal{X}_n(a, b) = \{j = (\alpha, x_\alpha) \in \mathcal{X}(a, b), y_\alpha = \tilde{c}_n\}, \quad 2 \leq n \leq N, \quad (96)$$

$$\mathcal{X}_{m,n}(a, b) = \{j = (\alpha, x_\alpha) \in \mathcal{X}_n(a, b), x_\alpha = \tilde{c}_m\}, \quad 1 \leq m < n \leq N, \quad (97)$$

and consider the following  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal decompositions:

$$\mathcal{V}_{m,n}^h = \text{Span}(\varphi_j^h, j \in \mathcal{X}_{m,n}(a, b)) \quad \text{for } 1 \leq m < n \leq N, \quad (98)$$

$$\mathcal{V}_n^h = \bigoplus_{m=1}^{n-1} \mathcal{V}_{m,n}^h = \text{Span}(\varphi_j^h, j \in \mathcal{X}_n(a, b)) \quad \text{for } 2 < n \leq N, \quad (99)$$

$$\mathcal{V}_+^h = \bigoplus_{n=2}^N \mathcal{V}_n^h, \quad (100)$$

$$\mathcal{V}_0^h = \text{Span}(\varphi_j^h, j \in \mathcal{Y}(a, b) \cup \mathcal{Z}(a, b)) = \ker(\Pi_{G^h} d_{f,h} T_{\delta_2}|_{\mathcal{V}^h}), \quad (101)$$

$$\mathcal{V}^h = \mathcal{V}_+^h \oplus \mathcal{V}_0^h, \quad (102)$$

with  $\Pi_{G^h} d_{f,h} T_{\delta_2} \mathcal{V}_n^h \subset \ker(\Delta_{f,f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a,b]),h})$ ,

$$\text{while } G^h = \bigoplus_{n \in \{1, \dots, N\}} \underbrace{\ker(\Delta_{f,f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a,b]),h})}_{=: G_n^h}. \quad (103)$$

**Proposition 6.5.** *Under the assumptions of Proposition 6.4 and with the notations (98)–(103), the operator  $\Pi_{G^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_{m,n}^h} : \mathcal{V}_{m,n}^h \rightarrow G_n^h$  is, for  $1 \leq m < n \leq N$ , one to one, and, when*

*$\dim \mathcal{V}_{m,n}^h \neq 0$ , its singular values all satisfy  $\mu^h \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}$ .*

*Moreover, the non zero singular values of  $\Pi_{G^h} d_{f,h} T_{\delta_2} : \mathcal{V}^h \rightarrow L^2(\Omega)$  (resp. of  $\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_n^h} : \mathcal{V}_n^h \rightarrow L^2(\Omega)$ , where  $n \in \{2, \dots, N\}$  is fixed) are obtained by collecting all those non zero singular values for  $1 \leq m < n \leq N$  (resp. for  $1 \leq m < n$ ), with an  $\tilde{O}(e^{-\frac{\delta_1}{h}})$  relative error.*

*Proof.* For every  $1 \leq m < n \leq N$  such that  $\mathcal{X}_{m,n}(a,b) \neq \emptyset$ , the composition of the exponential decay estimates on the  $\varphi_j^h$  given in (84),  $j \in \mathcal{X}_{m,n}(a,b)$ , and on the elements of any orthonormal basis  $(\psi_k^h)_{1 \leq k \leq K_n}$  of  $G_n^h = \ker(\Delta_{f,f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a,b]),h})$  leads to

$$\forall u \in \mathcal{V}_{m,n}^h, \quad \|\Pi_{G^h} d_{f,h} T_{\delta_2} u\| = \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}) \|u\|. \quad (104)$$

Let us now prove by reductio ad absurdum that

$$\forall 1 \leq m < n \leq N \text{ such that } \mathcal{X}_{m,n}(a,b) \neq \emptyset, \quad \forall u \in \mathcal{V}_{m,n}^h, \quad \|u\| = \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}) \|\Pi_{G^h} d_{f,h} T_{\delta_2} u\|.$$

Let us then assume that there exist  $\varepsilon_1 > 0$ ,  $1 \leq m_0 < n_0 \leq N$ , a strictly decreasing sequence  $(h_k)_{k \in \mathbb{N}}$  converging to 0 and, for every  $k \in \mathbb{N}$ ,  $u_{h_k} \in \mathcal{V}_{m_0, n_0}^{h_k} \setminus \{0\}$  such that

$$\|\Pi_{G^{h_k}} d_{f,f^{-1}([a,b]),h_k} T_{\delta_2} u_{h_k}\| \leq e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_{m_0} + \varepsilon_1}{h_k}} \|u_{h_k}\|. \quad (105)$$

Without restriction, we choose the pair  $(m_0, n_0)$  among the pairs for which (105) holds such that  $\lambda_0 := \tilde{c}_{n_0} - \tilde{c}_{m_0}$  is minimal. Set

$$\ell := \#\{(m,n) \in \mathcal{X}(a,b), \tilde{c}_n - \tilde{c}_m \leq \lambda_0\}.$$

Theorem 6.3-a) says that the  $\ell$ -th singular value of  $\delta_{[0,\tilde{\sigma}(1)],[a,b],h}$  and therefore, with (91), the  $\ell$ -th singular value of  $\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}^h}$  satisfy

$$\lim_{h \rightarrow 0} -h \log \mu_\ell(\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}^h}) = \lim_{h \rightarrow 0} -h \log \mu_\ell(\delta_{[0,\tilde{\sigma}(1)],[a,b],h}) = \lambda_0.$$

By using in addition the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal decomposition

$$\mathcal{V}^h = \mathcal{V}_+^h \oplus \mathcal{V}_0^h \quad \text{with} \quad \mathcal{V}_0^h = \ker(\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}^h}),$$

applying Proposition 5.4-b) gives

$$\lim_{h \rightarrow 0} -h \log \mu_\ell(\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}_+^h}) = \lim_{h \rightarrow 0} -h \log \mu_\ell(\delta_{[0,\tilde{\sigma}(1)],[a,b],h}) = \lambda_0.$$

Because  $\mathcal{V}_+^h$  is finite dimensional,  $\dim \mathcal{V}_+^h = \#\mathcal{X}(a,b)$ , the max-min principle implies

$$\mu_\ell(\Pi_{G^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_+^h}) = \min_{\dim W = \#\mathcal{X}(a,b) - \ell + 1} \max_{v \in W \setminus \{0\}} \frac{\|\Pi_{G^h} d_{f,h} T_{\delta_2} v\|}{\|v\|}.$$

We obtain a contradiction by considering

$$W = \left( \tilde{c}_n - \tilde{c}_m > \lambda_0 \oplus \mathcal{V}_{m,n}^{h_k} \right) \oplus \mathbb{C} u_{h_k}.$$

This ends the proof of the first statement.

By applying again Proposition 5.4-b) with now  $B = \Pi_{G^h} d_{f,h} T_{\delta_2}$  acting on  $\mathcal{V}_+^h$ , the singular values

of  $\Pi_{G^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_+^h}$  are obtained, modulo some  $\tilde{O}(e^{-\frac{\delta_1}{h}})$  relative error, by collecting all the singular values of  $\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_n^h}$ ,  $n \in \{2, \dots, N\}$ . Actually,  $G_n^h \perp G_{n'}^h$  and  $\mathcal{V}_n^h$  and  $\mathcal{V}_{n'}^h$  are  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal for  $n \neq n'$ . This reduces the problem to the computation of the singular values of  $\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_n^h}$ . For  $n \in \{2, \dots, N\}$ , we solve it by reverse induction on  $m \in \{1, \dots, n-1\}$  by considering  $\oplus_{m \leq m' < n} \mathcal{V}_{m',n}^h$ . Simply apply Proposition 5.4-c) with

$$E'^h = \bigoplus_{m \leq m' < n} \mathcal{V}_{m',n}^h, \quad \mu_{\dim E'^h}(\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{E'^h}) \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}},$$

$$E''^h = \mathcal{V}_{m-1,n}^h, \quad \|\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{E''^h}\| = \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_{m-1}}{h}}) \leq \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_m + 2\eta_f}{h}}) \leq \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_m + \delta_1}{h}}),$$

by starting from the first case when  $\dim E'^h \neq 0$ . This implies that the non zero singular values of  $\Pi_{G_n^h} d_{f,h} T_{\delta_2} : \oplus_{m-1 \leq m' < n} \mathcal{V}_{m',n}^h$  are obtained, modulo some  $\tilde{O}(e^{-\frac{\delta_1}{h}})$  relative error, by collecting the non zero singular values of  $\Pi_{G_n^h} d_{f,h} T_{\delta_2} \big|_{\mathcal{V}_{m',n}^h}$  for  $m-1 \leq m' < n$ . This ends the proof of the second statement.  $\square$

**Proposition 6.6.** *Assume that Theorem 6.3 holds true at step  $N$  and let  $G^h$  be defined by (89). There exists a basis  $(\phi_j^h)_{j \in \mathcal{J}(a,b)}$  of  $\mathcal{V}^h$  such that the  $\phi_j^h$ 's satisfy the same properties as the  $\varphi_j^h$ 's, that is the ones of Definition 6.1 and of Theorem 6.3, as well as the additional following one:*

$$\Psi_j^h \perp \Psi_{j'}^h \quad \text{for } j \neq j' \quad (106)$$

$$\text{where } \Psi_j^h = \Pi_{G^h} d_{f,h} T_{\delta_2} \phi_j^h. \quad (107)$$

In particular, according to Proposition 6.5, the singular values of  $\Pi_{G^h} d_{f,h} T_{\delta_2} : \mathcal{V}^h \rightarrow L^2(\Omega)$  are given by the numbers  $\|\Psi_j^h\|_{L^2}(1 + \tilde{O}(e^{-\frac{\delta_1}{h}}))$ ,  $j \in \mathcal{J}(a,b)$ , where  $\|\Psi_j^h\|_{L^2} \stackrel{\log}{\sim} e^{-\frac{y_\alpha - x_\alpha}{h}}$  when  $j = (\alpha, x_\alpha) \in \mathcal{X}(a,b)$ .

*Proof.* We keep  $\phi_j^h = \varphi_j^h$  if  $j \in \mathcal{Y}(a,b) \cup \mathcal{Z}(a,b)$ . Because  $G_n^h \perp G_{n'}^h$  for  $n \neq n'$  and  $\Pi_{G^h} d_{f,h} T_{\delta_2} \mathcal{V}_n^h \subset G_n^h$  for  $2 \leq n \leq N$ , it suffices to construct the family  $(\phi_j^h)_{j \in \mathcal{X}_n(a,b)}$  for any  $n \in \{2, \dots, N\}$ . Take some fixed  $n \in \{2, \dots, N\}$ . While keeping the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal decomposition

$$\mathcal{V}_n^h = \bigoplus_{1 \leq m < n} \mathcal{V}_{m,n}^h,$$

the first result of Proposition 6.5 says that, for a fixed pair  $(m,n)$ , the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal basis  $(\varphi_j^h)_{j \in \mathcal{X}_{m,n}(a,b)}$  can be replaced by an orthonormal one  $(\tilde{\varphi}_j^h)_{j \in \mathcal{X}_{m,n}(a,b)}$  such that

$$\Pi_{G_n^h} d_{f,h} T_{\delta_2} \tilde{\varphi}_j^h \perp \Pi_{G_n^h} d_{f,h} T_{\delta_2} \tilde{\varphi}_{j'}^h \quad \text{for } j \neq j', \quad j, j' \in \mathcal{X}_{m,n}(a,b)$$

$$\text{and } \|\Pi_{G_n^h} d_{f,h} T_{\delta_2} \tilde{\varphi}_j^h\| \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}} \quad \text{for } j \in \mathcal{X}_{m,n}(a,b).$$

Because the change of basis  $P_{m,n}^h \in \mathcal{L}(\mathcal{V}_{m,n}^h)$  given by  $\tilde{\varphi}_j^h = P_{m,n}^h \varphi_j^h$  satisfies

$$\|(P_{m,n}^h)^* P_{m,n}^h - \text{Id}_{\mathcal{V}_{m,n}^h}\| = \tilde{O}(e^{-\frac{\delta_1}{h}}),$$

the new family  $(\tilde{\varphi}_j^h)_{j \in \mathcal{X}_{m,n}(a,b)}$  keeps all the properties of the initial one  $(\varphi_j^h)_{j \in \mathcal{X}_{m,n}(a,b)}$ . In Theorem 6.3 at step  $N$ , nothing is changed when the  $\varphi_j^h$ ,  $j \in \mathcal{X}_{m,n}(a,b)$ , are replaced by the  $\tilde{\varphi}_j^h$ ,  $j \in \mathcal{X}_{m,n}(a,b)$ , and this can be done for all pairs  $(m,n)$  and with any initial guess of the family  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$ .

Thus, it suffices to construct the family  $(\phi_j^h)_{j \in \mathcal{X}_n(a,b)}$  such that (106) and (107) hold when  $j \in \mathcal{X}_{m_1,n}(a,b)$ ,  $j' \in \mathcal{X}_{m_2,n}(a,b)$ ,  $m_1 \neq m_2$ . Like at the end of the previous proof, we do it by reverse induction on  $m \in \{1, \dots, n-1\}$ .

- For  $m = n - 1$ , simply take  $\phi_j^h = \tilde{\varphi}_j^h$  and set

$$\mathcal{W}_{n-1,n}^h = \text{Span}(\phi_j^h, h \in \mathcal{X}_{n-1,n}(a, b)) = \mathcal{V}_{n-1,n}^h.$$

- Assume that the  $\phi_j^h$ 's have been constructed for  $j \in \mathcal{X}_{m',n}(a, b)$ , for all  $m' \in \{m, \dots, n-1\}$ , with  $\mathcal{W}_{m',n}^h = \text{Span}(\phi_j^h, j \in \mathcal{X}_{m',n}(a, b))$  and the equality of the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal decompositions

$$\bigoplus_{m \leq m' < n} \mathcal{W}_{m',n}^h = \bigoplus_{m \leq m' < n} \mathcal{V}_{m',n}^h.$$

Set, for  $j \in \mathcal{X}_{m-1,n}(a, b)$ ,

$$\hat{\varphi}_j^h = \varphi_j^h - \sum_{j' \in \sqcup_{m \leq m' < n} \mathcal{X}_{m',n}(a, b)} \frac{\langle \Psi_{j'}^h, \Pi_{G_n^h} d_{f,h} T_{\delta_2} \varphi_j^h \rangle}{\|\Psi_{j'}^h\|^2} \phi_{j'}^h,$$

and define

$$\mathcal{W}_{m-1,n}^h := \text{Span}(\hat{\varphi}_j^h, j \in \mathcal{X}_{m-1,n}(a, b)).$$

We have clearly

$$\Pi_{G_n^h} d_{f,h} T_{\delta_2} \hat{\varphi}_j^h \perp \text{Span}(\Psi_{j'}^h, j' \in \mathcal{X}_{m',n}(a, b), m \leq m' < n)$$

and

$$\bigoplus_{m-1 \leq m' < n} \mathcal{V}_{m',n}^h = \mathcal{W}_{m-1,n}^h \oplus \left( \bigoplus_{m \leq m' < n} \mathcal{V}_{m',n}^h \right).$$

All the properties of Theorem 6.3 at step  $N$  are verified for the  $\delta_1$ -family of quasimodes given by the  $\hat{\varphi}_j^h$ ,  $j \in \mathcal{X}_{m-1,n}(a, b)$ , and the  $\phi_j^h$ ,  $j \in \mathcal{X}_{m',n}(a, b)$ . The estimates on  $\hat{\varphi}_j^h$ ,  $j \in \mathcal{X}_{m-1,n}(a, b)$ , are consequences of:

$$\Psi_{j'}^h = \Pi_{G_n^h} d_{f,h} \phi_{j'}^h \in \ker(\Delta_{f,f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a, b]), h}) \quad \text{for } j' \in \sqcup_{m \leq m' < n} \mathcal{X}_{m',n}(a, b),$$

where  $\|\Psi_{j'}^h\|_{L^2} \stackrel{\text{log}}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_{m'}}{h}}$  when  $j' \in \mathcal{X}_{m',n}(a, b)$

and  $\frac{\langle \Psi_{j'}^h, \Pi_{G_n^h} d_{f,h} T_{\delta_2} \varphi_j^h \rangle}{\|\Psi_{j'}^h\|^2} = \tilde{O}(e^{\frac{\tilde{c}_n - \tilde{c}_{m'}}{h}}) \times \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_{m-1}}{h}}) = \tilde{O}(e^{-\frac{\tilde{c}_{m'} - \tilde{c}_{m-1}}{h}}),$

$$\|e^{\frac{|f - \tilde{c}_{m'}|}{h}} \phi_{j'}^h\|_{W(f^{-1}([a, b]) \setminus S_{\delta_1})} = \tilde{O}(1),$$

$$\tilde{c}_{m'} - \tilde{c}_{m-1} \geq 2\eta_f \geq \delta_1.$$

Hence, the vectors  $\hat{\varphi}_j^h$ ,  $j \in \mathcal{X}_{m-1,n}(a, b)$ , satisfy

$$\|e^{\frac{|f - \tilde{c}_{m-1}|}{h}} \hat{\varphi}_j^h\|_{W(f^{-1}([a, b]) \setminus S_{\delta_1})} = \tilde{O}(1).$$

Note in particular that the total space  $\mathcal{V}^h$  is not changed, so the statement of Theorem 6.3-**b**) and the factorization in Theorem 6.3-**c**) are obviously true.

Once we have the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal decomposition

$$\mathcal{V}_+^h = \left( \bigoplus_{m-1 \leq m' < n} \mathcal{W}_{m',n}^h \right) \oplus \left( \bigoplus_{1 \leq m' < m-2} \mathcal{V}_{m',n}^h \right),$$

we just apply our first argument with  $\mathcal{V}_{m-1,n}^h$  now replaced by  $\mathcal{W}_{m-1,n}^h$ , which permits to replace the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal basis  $(\hat{\varphi}_j^h)_{j \in \mathcal{X}_{m-1,n}(a, b)}$  of  $\mathcal{W}_{m-1,n}^h$  by an orthonormal basis  $(\tilde{\varphi}_j^h)_{j \in \mathcal{X}_{m-1,n}(a, b)}$  such that

$$\Pi_{G_n^h} d_{f,h} T_{\delta_2} \tilde{\varphi}_j^h \perp \Pi_{G_n^h} d_{f,h} T_{\delta_2} \tilde{\varphi}_{j'}^h \quad \text{for } j \neq j', j, j' \in \mathcal{X}_{m-1,n}(a, b).$$

We finally define  $\phi_j^h = \tilde{\varphi}_j^h$  for  $j \in \mathcal{X}_{m-1,n}(a, b)$ . □

## 6.4 $N \rightarrow N + 1$ : Collecting the information from step $N$

We assume that Theorem 6.3 holds at step  $N$ , i.e. when  $\sharp([a, b] \cap \{c_1, \dots, c_{N_f}\}) = N$ , and we consider the case

$$[a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_{N+1}\}.$$

Define

$$a_1 = a \quad , \quad b_1 = \tilde{c}_N + \eta_f \quad \text{and} \quad a_2 = \tilde{c}_2 - \eta_f \quad , \quad b_2 = b.$$

We can use Theorem 6.3 and its consequences given in Subsection 6.3 for

$$[a_1, b_1] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_N\} \quad \text{and} \quad [a_2, b_2] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_2, \dots, \tilde{c}_{N+1}\}.$$

Let us start with the interval  $[a_2, b_2]$ . Consider  $\Delta_{f, \overline{\Omega}_2, h}$  and let  $G_2^h$  and  $F_2^h$  be defined like  $G^h$  and  $F^h$  in (90) and (89) while replacing  $(a, b)$  by  $(a_2, b_2)$ , with

$$G_2^h = \bigoplus_{2 \leq n \leq N+1}^\perp G_{2,n}^h = \bigoplus_{2 \leq n \leq N+1} \ker(\Delta_{f, f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a, b]), h}).$$

For this interval  $[a_2, b_2]$ , the family of quasimodes  $(\phi_{2,j}^h)_{j \in \mathcal{J}(a_2, b_2)}$  is given by Proposition 6.6 with the orthogonality condition (106), (107), and we set

$$\mathcal{W}_{m,n}^h(a_2, b_2) = \text{Span}(\phi_{2,j}^h, j \in \mathcal{X}_{m,n}(a_2, b_2)), \quad 2 \leq m < n \leq N + 1.$$

For the interval  $[a_1, b_1]$ , we use similar notations  $\Delta_{f, \overline{\Omega}_1, h}$ ,  $G_1^h$ ,  $F_1^h$  with now

$$G_1^h = \bigoplus_{1 \leq n \leq N}^\perp G_{1,n}^h = \bigoplus_{1 \leq n \leq N} \ker(\Delta_{f, f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [a, b]), h}).$$

We start with a family of quasimodes

$$(\varphi_{0,j}^h)_{j \in \mathcal{J}(a_1, b_1)} \tag{108}$$

given by Theorem 6.3 and merge this family with  $(\phi_{2,j}^h)_{j \in \mathcal{J}(a_2, b_2) \cap \mathcal{J}(a_1, b_1)}$ , after considering the restrictions  $\phi_{2,j}^h|_{f^{-1}([a_2, b_1])}$  extended by 0 in  $f_{a_1}^{a_2} = a$ , according to the following procedure:

$$\begin{aligned} \varphi_{1,j}^h &= \phi_{2,j}^h \quad \text{if } j \in (\alpha, \tilde{c}) \in \mathcal{J}(a_1, b_1), \quad \tilde{c} \geq \tilde{c}_2, \\ \varphi_{1,j}^h &= \varphi_{0,j}^h \quad \text{if } j \in (\alpha, \tilde{c}_1) \in \mathcal{Z}(a_1, b_1), \\ \varphi_{1,j}^h &= \varphi_{0,j}^h - \sum_{j' \in \mathcal{X}(a_2, b_2) \cap \mathcal{X}(a_1, b_1)} \frac{\langle \Psi_{2,j'}^h, \Pi_{G_2^h} d_{f,h} T_{\delta_2} \varphi_{0,j}^h \rangle}{\|\Psi_{2,j'}^h\|^2} \phi_{2,j'}^h \quad \text{if } j = (\alpha, \tilde{c}_1) \in \mathcal{X}(a_1, b_1), \\ \text{with } \Psi_{2,j'}^h &= \Pi_{G_2^h} d_{f,h} T_{\delta_2} \phi_{2,j'}^h = \Pi_{G_1^h} d_{f,h} T_{\delta_2} \varphi_{1,j'}^h \quad \text{for } j' \in \mathcal{X}(a_2, b_2) \cap \mathcal{X}(a_1, b_1), \end{aligned}$$

where we recall that  $j = (\alpha, \tilde{c}) \in \mathcal{X}(a_2, b_2) \cap \mathcal{X}(a_1, b_1)$  means  $\tilde{c}_2 \leq x_\alpha < y_\alpha \leq \tilde{c}_N$ .

**Remark 6.7.** Assume that  $\gamma_1(h), (\varphi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  and  $\gamma_2(h), (\varphi_{2,j}^h)_{j \in \mathcal{J}(a_2, b_2)}$  are given by Theorem 6.3 and Definition 6.1 at step  $N$ , respectively in  $[a_1, b_1]$  and in  $[a_2, b_2]$ . Let us then define  $\gamma(h) := \max(\gamma_1(h), \gamma_2(h))$  and, for  $i \in \{1, 2\}$ ,

$$\tilde{\varphi}_{i,j}^h := \begin{cases} \varphi_{i,j}^h & \text{when } j \in \mathcal{Y}(a_i, b_i) \cup \mathcal{Z}(a_i, b_i), \\ \chi_{y_\alpha^{(p+1)}, \gamma(h)} \varphi_{i,j}^h & \text{when } j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a_i, b_i), \quad p \in \{0, \dots, N-1\}, \end{cases}$$

where  $\chi_{y_\alpha^{(p+1)}, \gamma(h)}$  is defined by (86) in Definition 6.2. Then, the families  $(\tilde{\varphi}_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  and  $(\tilde{\varphi}_{2,j}^h)_{j \in \mathcal{J}(a_2, b_2)}$  both satisfy the properties of Theorem 6.3 and Definition 6.1, respectively in  $[a_1, b_1]$  and in  $[a_2, b_2]$ , but now with the same  $\gamma(h)$ . Hence, we will assume here that the properties of the families  $(\phi_{2,j}^h)_{j \in \mathcal{J}(a_2, b_2)}$  and  $(\varphi_{0,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  are satisfied with the same  $\gamma(h)$ .

The spaces generated by those quasimodes are denoted by

$$\mathcal{V}^h(a_1, b_1) = \text{Span}(\varphi_{1,j}^h, j \in \mathcal{J}(a_1, b_1)) \quad \text{and} \quad \mathcal{V}^h(a_2, b_2) = \text{Span}(\phi_{2,j}^h, j \in \mathcal{J}(a_2, b_2)),$$

and the same rule applies for  $\mathcal{V}_{m,n}^h$ ,  $\mathcal{V}_n^h$ ,  $1 \leq m < n \leq N+1$ ,  $\mathcal{V}_+^h$ ,  $\mathcal{V}_0^h$  defined in (98)–(101), while writing  $\mathcal{W}_{m,n}^h(a_2, b_2)$  instead of  $\mathcal{V}_{m,n}^h(a_2, b_2)$  refers to the additional orthogonality property of Proposition 6.6.

**Proposition 6.8.** *The family  $(\varphi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  satisfies all the properties of Theorem 6.3 at step  $N$ . Moreover, the family  $(\phi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  deduced from  $(\varphi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  in Proposition 6.6 can be constructed such that*

$$\forall j \in \mathcal{X}(a_1, b_1) \cap \mathcal{X}(a_2, b_2) \quad , \quad \phi_{1,j}^h = \phi_{2,j}^h.$$

*Proof.* By construction (and Remark 6.7), the family  $(\varphi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  is a  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal  $\delta_1$ -family of quasimodes, and  $G_{1,n}^h = G_{2,n}^h$  for  $2 \leq n \leq N$  and

$$\begin{aligned} \vec{d}(\text{Span}(\varphi_{0,j}^h, j \in \mathcal{J}(a_1, b_1)), F_1^h) + \vec{d}(F_1^h, \text{Span}(\varphi_{0,j}^h, j \in \mathcal{J}(a_1, b_1))) &= \tilde{O}(e^{-\frac{\delta_1}{h}}), \\ \vec{d}(\mathcal{V}^h(a_2, b_2), F_2^h) + \vec{d}(F_2^h, \mathcal{V}^h(a_2, b_2)) &= \tilde{O}(e^{-\frac{\delta_1}{h}}), \\ \vec{d}(F_1^h, G_1^h) + \vec{d}(G_1^h, F_1^h) &= \tilde{O}(e^{-\frac{\eta f}{h}}) \leq \tilde{O}(e^{-\frac{\delta_1}{h}}), \\ \text{and} \quad \vec{d}(F_2^h, G_2^h) + \vec{d}(G_2^h, F_2^h) &= \tilde{O}(e^{-\frac{\eta f}{h}}) \leq \tilde{O}(e^{-\frac{\delta_1}{h}}) \end{aligned}$$

ensure the validity of the last statement of **b)** in Theorem 6.3, that is

$$\vec{d}(\mathcal{V}^h(a_1, b_1), F_1^h) + \vec{d}(F_1^h, \mathcal{V}^h(a_1, b_1)) = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

The exponential decay estimates on the  $\varphi_{1,j}^h$ ,  $j = (\alpha, \tilde{c}_1) \in \mathcal{X}(a_1, b_1)$ , are actually obtained like in the proof of Proposition 6.6 by noticing that

$$\forall j \in \mathcal{X}_{1,n}(a_1, b_1), \quad \varphi_{1,j}^h = \varphi_{0,j}^h - \sum_{j' \in \bigsqcup_{2 \leq m' < n \leq N} \mathcal{X}_{m',n}(a_2, b_2)} \frac{\langle \Psi_{2,j'}^h, \Pi_{G_{2,n}^h} d_{f,h} T_{\delta_2} \varphi_{0,j}^h \rangle}{\|\Psi_{2,j'}^h\|^2} \phi_{2,j'}^h,$$

where  $G_{1,n}^h = G_{2,n}^h$  for  $3 \leq n \leq N$  and

$$\Psi_{2,j'}^h = \Pi_{G_{2,n}^h} d_{f,h} T_{\delta_2} \phi_{2,j'}^h = \Pi_{G_{1,n}^h} d_{f,h} T_{\delta_2} \varphi_{1,j'}^h \quad \text{for } j' \in \mathcal{X}_{m',n}(a_1, b_1), \quad 2 \leq m' < n \leq N.$$

We still have to check the factorization of Theorem 6.3-c), namely

$$d_{f,f^{-1}([a_1, b_1])h} T_{\delta_2} \Big|_{\mathcal{V}^h(a_1, b_1)} = C^h \Pi_{F_1^h} d_{f,f^{-1}([a_1, b_1])h} T_{\delta_2} \Big|_{\mathcal{V}^h(a_1, b_1)} \quad \text{with } \|C^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}}).$$

We will do it by first considering the operator  $\Pi_{G_1^h} d_{f,h} T_{\delta_2}$ .

From the properties of the  $\varphi_{1,j}^h$ ,  $j \in \mathcal{J}(a_1, b_1)$ , we already know that (see indeed (104))

$$\|\Pi_{G_1^h} d_{f,h} T_{\delta_2} \Big|_{\mathcal{V}_{m,n}^h(a_1, b_1)}\| = \tilde{O}(e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}) \quad \text{and} \quad \mathcal{V}_0^h(a_1, b_1) \subset \ker(d_{f,h} T_{\delta_2}).$$

We now check that  $\Pi_{G_1^h} d_{f,h} T_{\delta_2} \Big|_{\mathcal{V}_{m,n}^h(a_1, b_1)}$  is one to one and that its singular values, which thus do not vanish, all satisfy  $\mu_h \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}$  for every  $1 \leq m < n \leq N$  such that  $\mathcal{X}_{m,n}(a_1, b_1) \neq \emptyset$ :

- Since the vectors  $\Psi_{2,j}^h = \Pi_{G_1^h} d_{f,h} T_{\delta_2} \phi_{2,j}^h = \Pi_{G_2^h} d_{f,h} T_{\delta_2} \phi_{2,j}^h$  are, according to Proposition 6.6 applied in  $[a_2, b_2]$ , mutually orthogonal with  $\|\Psi_{2,j}^h\| \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}$  when  $j \in \mathcal{X}_{m,n}(a_1, b_1)$ ,  $2 \leq m < n \leq N$ , the result holds for  $m \geq 2$ .

- Case  $m = 1$ : as in the proof of Proposition 6.5, assume by reductio ad absurdum that there exist  $2 \leq n \leq N$ , a strictly decreasing sequence  $(h_k)_{k \in \mathbb{N}}$  converging to 0 and, for every  $k \in \mathbb{N}$ ,  $u_{h_k} \in \mathcal{V}_{1,n}^{h_k}(a_1, b_1) \setminus \{0\}$  such that

$$\|\Pi_{G_1^{h_k}} d_{f,h_k} T_{\delta_2} u_{h_k}\| = \tilde{o}(e^{-\frac{\tilde{c}_n - \tilde{c}_1}{h_k}}) \|u_{h_k}\|,$$

and let  $n_0 \in \{2, \dots, N\}$  be the smallest  $n$  such that the above holds. Consider then

$$E''^{h_k} = (\mathbb{C}u_{h_k}) \oplus \mathcal{V}_0^{h_k}(a_1, b_1) \oplus \left( \tilde{c}_n - \tilde{c}_m > \tilde{c}_{n_0} - \tilde{c}_1 \mathcal{V}_{m,n}^{h_k}(a_1, b_1) \right),$$

$$\text{so that } \dim \mathcal{V}^{h_k}(a_1, b_1) - \dim(E''^{h_k}) = \# \left( \tilde{c}_n - \tilde{c}_m \leq \tilde{c}_{n_0} - \tilde{c}_1 \mathcal{X}_{m,n}(a_1, b_1) \right) - 1 =: \ell_0 - 1.$$

Owing to the exponential decay estimates on the quasimodes, we obtain

$$\|d_{f,h_k} T_{\delta_2}|_{E''^{h_k}}\| = \tilde{O}(e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_1 - 2\delta_2}{h_k}}) \quad (109)$$

and (see (104))

$$\|\Pi_{G_1^{h_k}} d_{f,h_k} T_{\delta_2}|_{E''^{h_k}}\| = \tilde{o}(e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_1}{h_k}}).$$

Since moreover  $\|\Pi_{F_1^h} - \Pi_{F_1^h} \Pi_{G_1^h}\| = \tilde{O}(e^{-\frac{\eta_f}{h}})$ , we deduce  $\|\Pi_{F_1^{h_k}} d_{f,h_k} T_{\delta_2}|_{E''^{h_k}}\| = \tilde{o}(e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_1}{h_k}})$  and then, applying the max-min principle as in the proof of Proposition 6.5 with here  $W = E''^{h_k}$ ,

$$\mu_{\ell_0}(\Pi_{F_1^{h_k}} d_{f,h_k} T_{\delta_2}|_{\mathcal{V}^{h_k}(a_1, b_1)}) = \tilde{o}(e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_1}{h_k}}).$$

Hence, since  $T_{\delta_2}$  is  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -unitary (see (95)) and  $\vec{d}(F_1^h, T_{\delta_2} \mathcal{V}^h(a_1, b_1)) + \vec{d}(T_{\delta_2} \mathcal{V}^h(a_1, b_1), F_1^h) = \tilde{O}(e^{-\frac{\delta_1}{h}})$  (see (93)), it follows from Proposition 5.3 that

$$\mu_{\ell_0}(\Pi_{[0, \tilde{o}(1)], [a_1, b_1], h_k} d_{f, f^{-1}([a_1, b_1]), h_k}) = \tilde{o}(e^{-\frac{\tilde{c}_{n_0} - \tilde{c}_1}{h_k}}) \text{ with } \ell_0 = \# \left( \tilde{c}_n - \tilde{c}_m \leq \tilde{c}_{n_0} - \tilde{c}_1 \mathcal{X}_{m,n}(a_1, b_1) \right),$$

in contradiction with Theorem 6.3-a) in  $[a_1, b_1]$ .

Because the spaces  $\mathcal{V}_{m,n}^h(a_1, b_1)$  have mutually orthogonal images by  $\Pi_{G_1^h} d_{f,h} T_{\delta_2}$ , i.e.

$$\Pi_{G_1^h} d_{f,h} T_{\delta_2} \mathcal{V}_{m_1, n_1}^h(a_1, b_1) \perp \Pi_{G_1^h} d_{f,h} T_{\delta_2} \mathcal{V}_{m_2, n_2}^h(a_1, b_1) \quad \text{for } (m_1, n_1) \neq (m_2, n_2),$$

we can conclude like at the end of the proof of Proposition 6.6 that there exists a basis  $(\phi_{1,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  such that (106) and (107) hold, and in which nothing needs to be changed when  $j \in \mathcal{X}(a_2, b_2)$ .

It follows from the above analysis that  $\Pi_{G_1^h} d_{f,h} T_{\delta_2}|_{\mathcal{V}_+^h(a_1, b_1)}$  is one to one, and the factorization  $d_{f,h} T_{\delta_2} = \tilde{C}^h \Pi_{G_1^h} d_{f,h} T_{\delta_2}$  is then satisfied with  $\tilde{C}^h : G_1^h \rightarrow L^2(f_{a_1}^{b_1})$  defined by  $\tilde{C}^h = 0$  on the orthogonal complement of  $\Pi_{G_1^h} d_{f,h} T_{\delta_2}(\mathcal{V}_+^h(a_1, b_1))$  in  $G_1^h$  and

$$\forall j \in \mathcal{X}(a_1, b_1), \quad \tilde{C}^h \Psi_{1,j}^h = d_{f,h} T_{\delta_2} \phi_{1,j}^h.$$

Moreover, the relation  $\|\tilde{C}^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$  follows from the orthogonality of the family  $(\Psi_{1,j}^h)_{j \in \mathcal{X}(a_1, b_1)}$  and from  $\|\Psi_{1,j}^h\| \stackrel{\log}{\sim} e^{\frac{y_\alpha - x_\alpha}{h}}$  and  $\|d_{f,h} T_{\delta_2} \phi_{1,j}^h\| = \tilde{O}(e^{-\frac{y_\alpha - x_\alpha - 2\delta_2}{h}})$  for  $j = (\alpha, \tilde{c}) \in \mathcal{X}(a_1, b_1)$  (see (109)).

Finally, applying the symmetric version of Lemma 5.6, that is exchanging  $F^h$  and  $G^h$ , yields the factorization  $d_{f,h} T_{\delta_2} = C^h \Pi_{F_1^h} d_{f,h} T_{\delta_2} : \mathcal{V}^h(a_1, b_1) \rightarrow L^2(f_{a_1}^{b_1})$  stated in Theorem 6.3-c).  $\square$



We have now spaces  $\mathcal{W}_{m,n}^h(a_1, b_1)$ ,  $1 \leq m < n \leq N$ , and  $\mathcal{W}_{m,n}^h(a_2, b_2)$ ,  $2 \leq m < n \leq N+1$ , such that

$$\mathcal{W}_{m,n}^h(a_1, b_1) = \mathcal{W}_{m,n}^h(a_2, b_2) \quad \text{when } 2 \leq m < n \leq N.$$

We now work in the interval  $[a, b]$  and we consider  $\Delta_{f, \bar{\Omega}, h}$ ,  $G^h$ , and  $F^h$  according to (90) and (89), after replacing  $N$  by  $N+1$  and  $\{\tilde{c}_1, \dots, \tilde{c}_N\}$  by  $\{\tilde{c}_1, \dots, \tilde{c}_{N+1}\}$ . We set

$$\mathcal{W}_{m,n}^h(a, b) = \begin{cases} \mathcal{W}_{m,n}^h(a_2, b_2) = \text{Span}(\phi_{2,j}^h, j \in \mathcal{X}_{m,n}(a_2, b_2)) & \text{for } 2 \leq m < n \leq N+1, \\ \mathcal{W}_{1,n}^h(a_1, b_1) = \text{Span}(\phi_{1,j}^h, j \in \mathcal{X}_{1,n}(a_1, b_1)) & \text{for } 1 = m < n \leq N, \end{cases} \quad (110)$$

$$\mathcal{V}_0^h(a, b) = \text{Span}(\phi_{2,j}^h, j \in \mathcal{Y}(a_2, b_2) \cup \mathcal{Z}(a_2, b_2)), \quad (111)$$

$$\text{and } \mathcal{V}^h(a, b) = \underbrace{\left( \bigoplus_{0 < n-m \leq N-1} \mathcal{W}_{m,n}^h(a, b) \right)}_{\mathcal{V}_+^h} \oplus \mathcal{V}_0^h. \quad (112)$$

Accordingly, we introduce

$$\mathcal{J}'_+(a, b) = \mathcal{X}(a_2, b_2) \sqcup \left( \bigcup_{2 < n \leq N} \mathcal{X}_{1,n}(a_1, b_1) \right) = \bigcup_{0 < n-m \leq N-1} \mathcal{X}_{m,n}(a, b), \quad (113)$$

$$\mathcal{J}'_0(a, b) = \mathcal{Y}(a_2, b_2) \sqcup \mathcal{Z}(a_2, b_2) \quad \text{and} \quad \mathcal{J}'(a, b) = \mathcal{J}'_+(a, b) \sqcup \mathcal{J}'_0(a, b), \quad (114)$$

$$\varphi_j^h = \phi_j^h = \begin{cases} \phi_{2,j}^h & \text{if } j = (\alpha, \tilde{c}) \in \mathcal{J}'_+(a, b), \tilde{c}_2 \leq \tilde{c}, \\ \phi_{2,j}^h & \text{if } j \in \mathcal{J}'_0(a, b), \\ \phi_{1,j}^h & \text{if } j = (\alpha, \tilde{c}_1) \in \mathcal{J}'_+(a, b). \end{cases} \quad (115)$$

In the perspective of applying Proposition 5.8, we now consider the space  $E'^h = T_{\delta_2} \mathcal{V}^h$ .

**Proposition 6.9.** *With the notation (112), consider  $E'^h = T_{\delta_2} \mathcal{V}^h(a, b)$ ,  $E_0'^h = T_{\delta_2} \mathcal{V}_0^h(a, b)$ , and let  $G^h$  be defined by (89) with  $N$  replaced by  $N+1$ . The operator  $\Pi_{G^h} d_{f, f^{-1}([a, b])_h} \big|_{E'^h}$  satisfies*

$$\text{rank}(\Pi_{G^h} d_{f, h} \big|_{E'^h}) = \#\mathcal{J}'_+(a, b) =: \ell_1$$

$$\text{and } \ker(\Pi_{G^h} d_{f, f^{-1}([a, b])_h} \big|_{E'^h}) = E_0'^h,$$

and its non zero singular values can be written  $(\mu_j^h)_{j \in \mathcal{J}'_+(a, b)}$  with

$$\mu_j^h \stackrel{\log}{\sim} e^{-\frac{\mu \alpha - x \alpha}{h}} \quad \text{for every } j = (\alpha, x_\alpha) \in \mathcal{J}'_+(a, b).$$

In particular, its  $\ell_1$ -th singular value satisfies

$$e^{-\frac{\max(\tilde{c}_{N+1} - \tilde{c}_2, \tilde{c}_N - \tilde{c}_1)}{h}} = \tilde{O}(\mu_{\ell_1}(\Pi_{G^h} d_{f, f^{-1}([a, b])_h} \big|_{E'^h})).$$

Moreover, the operator  $d_{f, f^{-1}([a, b])_h} \big|_{E'^h}$  is a left multiple of  $\Pi_{G^h} d_{f, f^{-1}([a, b])_h} \big|_{E'^h}$ :

$$\begin{array}{ccc} E'^h & \xrightarrow{d_{f, f^{-1}([a, b])_h}} & L^2(f^{-1}([a, b])) \\ & \searrow \Pi_{G^h} d_{f, f^{-1}([a, b])_h} & \uparrow \tilde{C}^h \\ & & G^h \end{array}$$

with  $\|\tilde{C}^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$ .

Finally, the same results hold when  $G^h$  is replaced by  $F^h = F_{[0, \tilde{\alpha}(1)], [a, b], h}$ .

*Proof.* The basis  $(\phi_j^h)_{j \in \mathcal{J}'(a,b)}$  of  $\mathcal{V}^h(a,b)$  defined in (115) (note that the inclusion  $\mathcal{J}'(a,b) \subset \mathcal{J}(a,b)$  is strict in general) has been constructed so that it is a partial  $\delta_1$ -family of quasimodes in the sense of Definition 6.1, with the additional orthogonality property (106),(107). Moreover, we know that (see indeed Proposition 6.6)

$$\|\Psi_j^h\| = \|\Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \phi_j^h\|_{L^2} \stackrel{\log}{\sim} e^{-\frac{y_\alpha - x_\alpha}{h}} \quad \text{when } j = (\alpha, x_\alpha) \in \mathcal{J}'_+(a,b) \quad (116)$$

$$\text{and} \quad \Psi_j^h = \Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \phi_j^h = 0 \quad \text{when } j \in \mathcal{J}'_0(a,b). \quad (117)$$

Again, with (see (95))

$$\|T_{\delta_2} T_{\delta_2}^* - \text{Id}_{E'^h}\| + \|T_{\delta_2}^* T_{\delta_2} - \text{Id}_{\mathcal{V}^h}\| = \tilde{O}(e^{-\frac{\eta_f}{h}}),$$

this proves the results about the rank, the kernel, and the singular values of  $\Pi_{G^h} d_{f,f^{-1}([a,b]),h} \big|_{E'^h}$ . Moreover, reasoning with the orthogonality of the family  $(\Psi_j^h)_{j \in \mathcal{J}'(a,b)}$  and (116),(117), like at the end of the proof of Proposition 6.8, leads to the factorization

$$\tilde{C}^h \Pi_{G^h} d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}^h(a,b)} = d_{f,f^{-1}([a,b]),h} T_{\delta_2} \big|_{\mathcal{V}^h(a,b)}$$

with  $\|\tilde{C}^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$ . We conclude with the invertibility of  $T_{\delta_2} : \mathcal{V}^h(a,b) \rightarrow E'^h$ . Finally, replacing  $G^h$  by  $F^h$  simply relies on Lemma 5.6 used as we did around (94).  $\square$

## 6.5 $N \rightarrow N + 1$ : Handling the bars containing $[\tilde{c}_1, \tilde{c}_{N+1}[$

We continue in the framework of the previous paragraph with

$$[a, b] \cap \{c_1, \dots, c_{N_f}\} = \{\tilde{c}_1, \dots, \tilde{c}_{N+1}\}$$

and

$$a_1 = a \quad , \quad b_1 = \tilde{c}_N + \eta_f \quad , \quad a_2 = \tilde{c}_2 - \eta_f \quad , \quad b_2 = b.$$

We use the partition

$$\mathcal{J}(a, b) = \mathcal{J}'(a, b) \sqcup \mathcal{J}''(a, b),$$

where  $\mathcal{J}'(a, b)$  is defined in (114) and

$$\begin{aligned} \mathcal{J}''(a, b) &= \{j = (\alpha, \tilde{c}_1) \in \mathcal{X}(a, b), y_\alpha = \tilde{c}_{N+1}\} \sqcup \{j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a, b)\} \\ &= \{j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a_1, b_1)\} = \mathcal{J}(a_1, b_1) \setminus (\mathcal{J}(a_1, b_1) \cap \mathcal{J}'(a, b)). \end{aligned} \quad (118)$$

If we remember that  $(\alpha, \tilde{c}) \in \mathcal{Z}(a, b)$  can be represented by the bar  $[\tilde{c}, b[$ , the set  $\mathcal{J}''(a, b)$  actually collects the lower endpoints (which are multiple copies of  $\tilde{c}_1$ ) of bars containing  $[\tilde{c}_1, \tilde{c}_{N+1}[$ . Thus, the partition of  $\mathcal{J}(a, b)$  and the identifications of  $\mathcal{J}''(a, b)$  are clear. In the preceding section, we started with a  $\delta_1$ -family of quasimodes  $(\varphi_{0,j}^h)_{j \in \mathcal{J}(a_1, b_1)}$  in the interval  $[a_1, b_1] = [a, \tilde{c}_N + \eta_f]$  (see (108)), and only used for the construction of  $E'^h$  in Proposition 6.9, among the corresponding  $j \in \mathcal{J}(a_1, b_1)$ , the indexes  $j \in \mathcal{J}(a_1, b_1) \cap \mathcal{J}'(a, b)$  (see (115)). We now use the vectors  $\varphi_{0,j}^h$  for  $j \in \mathcal{J}''(a, b)$ .

**Proposition 6.10.** *The vectors  $\varphi_{0,j}^h$ ,  $j \in \mathcal{J}''(a, b)$ , introduced in (108), where  $b_1 = \tilde{c}_N + \eta_f$ , can be “extended” to  $f^{-1}([a, b_2])$  into vectors  $\varphi_j^h \in D(d_{f,f^{-1}([a, b_2]),h})$  such that  $\varphi_j^h|_{f_a^{\tilde{c}_N + \delta_1}} = \varphi_{0,j}^h|_{f_a^{\tilde{c}_N + \delta_1}}$  and such that all the properties of Definition 6.1 hold on the interval  $[a, b] = [a_1, b_2]$  with  $I_j^h = [\tilde{c}_1 - \delta_1, \tilde{c}_{N+1} - \gamma''(h)]$ ,  $\lim_{h \rightarrow 0} \gamma''(h) = 0$ .*

*Proof.* For  $j \in \mathcal{J}''(a, b)$ ,  $j$  has the form  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a_1, b_1)$  and the vector  $\varphi_{0,j}^h$  then satisfies the support condition (83) (that is more precisely  $\text{supp } \varphi_{0,j}^h \subset f^{-1}([\tilde{c}_1 - \delta_1, b_1] \cap [a_1, b_1])$ ),

the exponential decay estimate (84) with  $a, b$  replaced by  $a_1 = a, b_1 = \tilde{c}_N + \eta_f$ , and  $\varphi_{0,j}^h \in \ker(d_{f,f^{-1}([a_1, b_1]), h})$ . For any  $\gamma \in ]0, \eta_f/2[$ , we consider the domain  $[\tilde{c}_N + \delta_1, \tilde{c}_{N+1} - \gamma]$  and we consider  $\tilde{c}_N + \eta_f$  as a new artificial “critical value”, for which we know

$$\ker(\Delta_{f,f^{-1}([\tilde{c}_N + \delta_1, \tilde{c}_{N+1} - \gamma]), h}) = \{0\}.$$

We then apply Proposition 3.9-ii) with  $a_0, a, \tilde{c}_1, b$  there replaced here by  $\tilde{c}_1, \tilde{c}_N + \delta_1, \tilde{c}_N + \eta_f, \tilde{c}_{N+1} - \gamma$  and  $\omega_h$  replaced by  $\varphi_{0,j}^h$ . This provides us a new  $\tilde{\omega}_{j,h} \in D(d_{f,f^{-1}([a, \tilde{c}_{N+1} - \gamma]), h})$  which satisfies (83)–(85), now on  $[a, \tilde{c}_{N+1} - \gamma]$  with  $I_j^h = [\tilde{c}_1 - \delta_1, \tilde{c}_{N+1} - \gamma]$ . With the cut-off  $\chi_{\tilde{c}_{N+1}, \gamma}$  defined like in Definition 6.2, set

$$\varphi_j^{\gamma, h} = \chi_{\tilde{c}_{N+1}, \gamma} \tilde{\omega}_{j,h} \in D(d_{f,f^{-1}([a, b_2]), h}).$$

It does satisfy, on the interval  $[a, b_2]$ , the conditions (83)–(85) with  $I_j^h$  and  $\gamma(h)$  there replaced by  $[a, \tilde{c}_{N+1} - 2\gamma]$  and  $2\gamma$ .

For  $n \in \mathbb{N}$ , take  $\gamma = \frac{1}{n+1}$ . The estimate  $B_h = \tilde{O}(A_h)$  implies  $B_h \leq e^{\frac{1}{(n+1)h}} A_h$  for  $h \in ]0, h_n[$ , and  $(h_n)_{n \in \mathbb{N}}$  can be chosen to be strictly decreasing. We then adjust  $\gamma''(h) = 2\gamma = \frac{2}{n+1}$  for  $h \in [h_{n+1}, h_n[$  as we did at the end of the proof of Proposition 3.8. This ends the proof.  $\square$

**Remark 6.11.** *In the construction of Proposition 3.9-ii), we used the extension by 0, here on  $f_a^{\tilde{c}_N + \delta_1}$ , of*

$$d_{f,f^{-1}([\tilde{c}_N + \delta_1, \tilde{c}_{N+1} - \gamma]), h}^* (\Delta_{f,f^{-1}([\tilde{c}_N + \delta_1, \tilde{c}_{N+1} - \gamma]), h})^{-1} (hd\chi_h \wedge \varphi_{0,j}^h).$$

Because of this, the point  $\tilde{c}_N + \delta_1$  must be included in the set  $S_{\delta_1}$  introduced in Definition 6.1.

When the family  $(\varphi_j^h)_{j \in \mathcal{J}''(a,b)}$  is given by Proposition 6.10, the operator  $T_{\delta_2}$  is defined on  $\text{Span}(\varphi_j^h, j \in \mathcal{J}''(a,b))$  by

$$\forall j \in \mathcal{J}''(a,b), \quad T_{\delta_2} \varphi_j^h = \chi_{\tilde{c}_{N+1}, \delta_2} \varphi_j^h,$$

like in Definition 6.2 when  $j \in \mathcal{X}(a,b)$ . Moreover, following the procedure of Remark 6.7, we can assume without loss of generality that  $\gamma''(h)$ , given by Proposition 6.10, equals  $\gamma(h)$ , considered in Section 6.4 (see Remark 6.7). Now, the orthogonalization process of Proposition 6.6 can be continued by setting

$$\forall j \in \mathcal{J}''(a,b), \quad \hat{\varphi}_j^h = \varphi_j^h - \sum_{j' \in \bigsqcup_{2 \leq m' \leq N} \mathcal{X}_{m', N+1}(a,b)} \frac{\langle \Psi_{2,j'}^h, \Pi_{G_{2,N+1}^h} d_{f,h} T_{\delta_2} \varphi_j^h \rangle}{\|\Psi_{2,j'}^h\|^2} \phi_{2,j'}^h, \quad (119)$$

where  $\phi_{2,j'}^h = \varphi_{j'}^h$  (see (115)) and  $\Pi_{G_{2,N+1}^h} = \Pi_{G_{N+1}^h}$ . Moreover, without knowing the singular values of  $\Pi_{G_{N+1}^h} d_{f,h} T_{\delta_2} \big|_{\text{Span}(\hat{\varphi}_j^h, j \in \mathcal{J}''(a,b))}$ , we can replace the basis  $(\hat{\varphi}_j^h)_{j \in \mathcal{J}''(a,b)}$  by an orthonormal basis  $(\phi_j^h)_{j \in \mathcal{J}''(a,b)}$  such that  $\Pi_{G_{N+1}^h} d_{f,h} T_{\delta_2} \phi_j^h = \Psi_j^h$ , with  $\Psi_j^h \perp \Psi_{j'}^h$  when  $j \neq j'$ ,  $j, j' \in \mathcal{J}''(a,b)$ , without changing its characteristic properties.

The construction of the new quasimode basis at step  $N+1$  is almost achieved, except that the family  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  is not exactly a  $\delta_1$ -family of quasimodes in the sense of Definition 6.1. In fact, we have not distinguished the endpoints of bars  $j \in \mathcal{X}_{1,N+1}(a,b)$  from the endpoints  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b)$  in (118). For this reason, we prefer to introduce a different notation.

**Definition 6.12.** *The family  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a,b)}$ , where we keep the notation  $\varphi_j^h = \tilde{\varphi}_j^h$  for  $j \in \mathcal{J}'(a,b)$ , is called an intermediate  $\delta_1$ -family of quasimodes if the following conditions are satisfied:*

1. It is  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal like in Theorem 6.3 and all the properties of  $\delta_1$ -quasimodes in Definition 6.1 are verified, with the only difference that  $I_j^h = [\tilde{c}_1 - \delta_1, \tilde{c}_{N+1} - \gamma(h)]$  for all  $j \in \mathcal{J}''(a, b)$ . For such a family, we set  $\tilde{\mathcal{V}}^h(a, b) = \text{Span}(\tilde{\varphi}_j^h, j \in \mathcal{J}(a, b))$ , and the operator  $T_{\delta_2} : \tilde{\mathcal{V}}^h(a, b) \rightarrow D(d_{f, f^{-1}([a, b])}, h)$  keeps the same definition  $T_{\delta_2} \tilde{\varphi}_j^h = T_{\delta_2} \varphi_j^h$  as in Definition 6.2 for  $j \in \mathcal{J}'(a, b)$ , while

$$T_{\delta_2} \tilde{\varphi}_j^h = \chi_{\tilde{c}_{N+1}, \delta_2} \tilde{\varphi}_j^h \quad \text{for } j \in \mathcal{J}''(a, b).$$

2. The space  $\tilde{\mathcal{V}}^h(a, b)$  is  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -close to  $F^h = F_{[0, \tilde{\sigma}(1)], [a, b], h}$ :

$$\tilde{d}(\tilde{\mathcal{V}}^h(a, b), F_{[0, \tilde{\sigma}(1)], [a, b], h}) + \tilde{d}(F_{[0, \tilde{\sigma}(1)], [a, b], h}, \tilde{\mathcal{V}}^h(a, b)) = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

3. When  $\mathcal{V}^h(a, b) = \text{Span}(\varphi_j^h, j \in \mathcal{J}'(a, b))$ ,  $\mathcal{V}_0^h(a, b) = \text{Span}(\varphi_j^h, j \in \mathcal{J}_0'(a, b))$ ,  $\mathcal{V}_+^h(a, b) = \text{Span}(\varphi_j^h, j \in \mathcal{J}_+^h(a, b))$ , all the properties of Proposition 6.9 hold true.

If  $G^h$  is defined like in (89), we use the notation  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a, b)}$  and  $\phi_j^h = \tilde{\phi}_j^h$  for  $j \in \mathcal{J}'(a, b)$  when the following additional orthogonality property holds:

$$\tilde{\Psi}_j^h \perp \tilde{\Psi}_{j'}^h \quad \text{for } j \neq j' \quad (120)$$

$$\text{with } \tilde{\Psi}_j^h = \Pi_{G^h} d_{f, h} T_{\delta_2} \tilde{\phi}_j^h. \quad (121)$$

When  $1 \leq m < n \leq N + 1$  and  $\tilde{c}_n - \tilde{c}_m < \tilde{c}_{N+1} - \tilde{c}_1$ , the corresponding spaces will be denoted

$$\mathcal{V}_{m, n}^h(a, b) = \text{Span}(\varphi_j^h, j \in \mathcal{X}_{m, n}(a, b)) \quad , \quad \mathcal{W}_{m, n}^h(a, b) = \text{Span}(\phi_j^h, j \in \mathcal{X}_{m, n}(a, b)),$$

while

$$\tilde{\mathcal{V}}_{1, N+1}^h(a, b) = \text{Span}(\tilde{\varphi}_j^h, j \in \mathcal{J}''(a, b)) \quad , \quad \tilde{\mathcal{W}}_{1, N+1}^h(a, b) = \text{Span}(\tilde{\phi}_j^h, j \in \mathcal{J}''(a, b)).$$

Our construction, and especially Proposition 6.10, provides such a family  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a, b)}$ . More precisely, according to (119) and the lines below, and since  $G_n^h \perp G_{n'}^h$  for  $1 \leq n < n' \leq N + 1$ , our construction actually provides a family  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a, b)}$ , that is satisfying in addition (120) and (121). Note that, like in Proposition 6.4, the operator

$$\Pi_{G^h} d_{f, h} T_{\delta_2} : \mathcal{V}^h(a, b) \rightarrow L^2(f_a^b)$$

does not depend on  $\delta_2 \in ]0, \eta_f[$ .

In the remaining steps, we will consider various values of  $a$  and  $b$  and the above properties, especially the ones involving  $G^h$  and  $G_{N+1}^h = \ker(\Delta_{f, f^{-1}([\tilde{c}_{N+1} - \eta_f, \min(b, \tilde{c}_{N+1} + \eta_f)])}, h)$ , which depend on  $b$ . More precisely, an intermediate  $\delta_1$ -family of quasimodes in the sense of Definition 6.12, and constructed for the pair  $a < b$ , will have to be conveniently adapted for another pair  $a' < b'$  so that it satisfies Definition 6.12 for this new pair.

## 6.6 Lower bound for non zero singular values at step $N + 1$

This paragraph will end with the proof of Theorem 6.3-a) at step  $N + 1$ . We are in the case

$$\{c_1, \dots, c_{N_f}\} \cap [a, b] = \{c_1, \dots, c_{N_f}\} \cap a, b[ = \{\tilde{c}_1, \dots, \tilde{c}_{N+1}\}. \quad (122)$$

The notations  $\mathcal{J}'_+(a, b)$ ,  $\mathcal{J}'_0(a, b)$ ,  $\mathcal{J}'(a, b)$ , and  $\mathcal{J}''(a, b)$  are the ones introduced in (113), (114), and (118), and the spaces  $\mathcal{V}_{m, n}^h(a, b)$ ,  $\mathcal{W}_{m, n}^h(a, b)$ ,  $\tilde{c}_n - \tilde{c}_m < \tilde{c}_{N+1} - \tilde{c}_1$ ,  $\tilde{\mathcal{V}}_{1, N+1}^h(a, b)$ ,  $\tilde{\mathcal{W}}_{1, N+1}^h(a, b)$ ,  $\mathcal{V}_0^h(a, b)$ ,  $\mathcal{V}_+^h(a, b)$ ,  $\mathcal{V}^h(a, b)$ , are the ones of Definition 6.12. We set

$$\ell_0 := \#\mathcal{X}(a, b) = \#A_c(a, b) = \text{rank } \delta_{[0, \tilde{\sigma}(1)], [a, b], h},$$

where the last equality was proved in Proposition 4.4 and  $\sharp A_c(a, b) = \sharp \mathcal{X}(a, b)$  since the number of bars  $\alpha$  such that in  $a < x_\alpha < y_\alpha < b$  equals the number of their lower endpoints. Meanwhile, we set

$$\ell_1 := \ell_0 - \sharp \mathcal{X}_{1,N+1}(a, b) = \sharp \{j = (\alpha, x_\alpha) \in \mathcal{X}(a, b), y_\alpha - x_\alpha < \tilde{c}_{N+1} - \tilde{c}_1\} = \dim \mathcal{V}_+^h(a, b).$$

**Proposition 6.13.** *Consider the case  $\tilde{c}_1 - \delta_1 \leq a < \tilde{c}_1$ ,  $\tilde{c}_{N+1} < b \leq \tilde{c}_{N+1} + \delta_3$ , and assume  $\delta_1, \delta_2, \delta_3 \in ]0, \frac{\eta f}{8}]$ . Let  $G^h$  be given by (89), define  $\mathcal{V}^h(a, b)$ ,  $\tilde{\mathcal{V}}_{1,N+1}^h(a, b)$ , and  $T_{\delta_2}$  like in Definition 6.12, and consider*

$$E^h := T_{\delta_2} \tilde{\mathcal{V}}^h(a, b) = T_{\delta_2} [\mathcal{V}^h(a, b) \oplus \tilde{\mathcal{V}}_{1,N+1}^h(a, b)].$$

Then, the  $\ell_0$ -th singular value of  $\Pi_{G^h} d_{f,h}|_{E^h}$  is bounded from below by

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} \leq e^{-\frac{\max(b - \tilde{c}_1, \tilde{c}_{N+1} - a)}{h}} = \tilde{O}(\mu_{\ell_0}(\Pi_{G^h} d_{f,h}|_{E^h})).$$

*Proof.* With our choice  $\tilde{c}_1 - \delta_1 \leq a < \tilde{c}_1$  and  $\tilde{c}_{N+1} < b \leq \tilde{c}_{N+1} + \delta_3$ , Proposition 4.5 says

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} \leq e^{-\frac{\max(b - \tilde{c}_1, \tilde{c}_{N+1} - a)}{h}} = \tilde{O}(\mu_{\ell_0}(\delta_{[0, \tilde{\sigma}(1)], [a, b], h})), \quad (123)$$

$$\text{with } \delta_{[0, \tilde{\sigma}(1)], [a, b], h} = \Pi_{F^h} d_{f, f^{-1}([a, b]), h}|_{F^h},$$

where we recall  $F^h = F_{[0, \tilde{\sigma}(1)], [a, b], h}$ .

Write

$$E'^h = T_{\delta_2} \mathcal{V}^h(a, b) \quad \text{and} \quad E''^h = T_{\delta_2} \tilde{\mathcal{V}}_{1,N+1}^h(a, b).$$

The assumed exponential decay and the definition of  $T_{\delta_2}$  in Definition 6.12 yield

$$\vec{d}(E^h, \tilde{\mathcal{V}}^h(a, b)) + \vec{d}(\tilde{\mathcal{V}}^h(a, b), E^h) = \tilde{O}(e^{-\frac{\eta f}{h}}) \leq \tilde{O}(e^{-\frac{\delta_1}{h}})$$

and therefore

$$\vec{d}(F^h, E^h) + \vec{d}(E^h, F^h) = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

Moreover, the decomposition  $E^h = E'^h \oplus E''^h$  is  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal and we know that

$$E^h \subset D(d_{f, f^{-1}([a, b]), h}) \quad , \quad d_{f, f^{-1}([a, b]), h}|_{E^h} = d_{f, h}|_{E^h}$$

and  $\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h) = \tilde{O}(e^{-\frac{\eta f}{h}}).$

In addition, Proposition 6.9, whose properties are ensured by the condition 3 of Definition 6.12, provides the factorization

$$d_{f, f^{-1}([a, b]), h}|_{E'^h} = C^h \Pi_{F^h} d_{f, f^{-1}([a, b]), h}|_{E'^h}$$

$$\text{with } \|C^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}}) \quad \text{and then} \quad \|C^h\| \left[ \vec{d}(F^h, G^h) + \vec{d}(G^h, F^h) \right] = \tilde{O}(e^{\frac{2\delta_2 - \eta f}{h}}) \leq \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

So, Hypotheses 1,2,3, and the inequality (69) of Hypothesis 4 in Proposition 5.8 are satisfied with  $B^h = d_{f, f^{-1}([a, b]), h}$  and  $\varrho(h) = \tilde{O}(e^{-\frac{\delta_1}{h}})$  when  $\delta_1, \delta_2, \delta_3 \in ]0, \frac{\eta f}{8}]$ . Moreover, we know from Proposition 6.9 that

$$\begin{aligned} \text{rank}(\Pi_{G^h} d_{f, f^{-1}([a, b]), h}|_{E'^h}) &= \ell_1 = \dim \mathcal{V}_+^h(a, b) = \sharp \mathcal{J}'_+(a, b) \\ \text{and} \quad e^{-\frac{\max(\tilde{c}_N - \tilde{c}_1, \tilde{c}_{N+1} - \tilde{c}_2)}{h}} &= \tilde{O}(\mu_{\ell_1}(\Pi_{G^h} d_{f, f^{-1}([a, b]), h}|_{E'^h})), \\ \text{with} \quad \max(\tilde{c}_N - \tilde{c}_1, \tilde{c}_{N+1} - \tilde{c}_2) &\leq \tilde{c}_{N+1} - \tilde{c}_1 - 2\eta f. \end{aligned}$$

With  $B_h = d_{f,f^{-1}([a,b]),h}$ , the upper bound  $\|d_{f,f^{-1}([a,b]),h}|_{E''h}\| = \tilde{O}(e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1-2\delta_2}{h}})$  (see (109)), and (123), the inequality (70) of Hypothesis 4 is deduced from

$$\begin{aligned} \|B^h|_{E''h}\| & \left[ \frac{1}{\mu_{\ell_1}(\Pi_{G^h} B^h|_{E''h})} + \frac{\|C^h\|(\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h))}{\max(\mu_{\ell_0}(\Pi_{G^h} B^h|_{E^h}), \mu_{\ell_0}(B^h|_{F^h}))} \right] \\ & = \tilde{O}(e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1-2\delta_2}{h}}) \times \left[ \tilde{O}(e^{\frac{\tilde{c}_{N+1}-\tilde{c}_1-2\eta_f}{h}}) + \underbrace{\frac{\tilde{O}(e^{\frac{2\delta_2-\eta_f}{h}})}{\mu_{\ell_0}(B^h|_{F^h})}}_{\text{see (123)}} \right] \\ & = \tilde{O}(e^{-\frac{\eta_f}{h}}) + \tilde{O}(e^{\frac{4\delta_2+\max(\delta_1, \delta_3)-\eta_f}{h}}) = \tilde{O}(e^{-\frac{\delta_1}{h}}), \end{aligned}$$

if  $\delta_1, \delta_2, \delta_3 \in ]0, \frac{\eta_f}{8}]$ .

The first result of Proposition 5.8 then implies

$$\forall \ell \in \{1, \dots, \ell_0\}, \quad \mu_\ell(\Pi_{G^h} d_{f,f^{-1}([a,b]),h}|_{E^h}) = \mu_\ell(\delta_{[0, \bar{\alpha}(1)], [a,b], h})(1 + \tilde{O}(e^{-\delta_1/h})),$$

which yields in particular (see (123))

$$e^{-\frac{\max(b-\tilde{c}_1, \tilde{c}_{N+1}-a)}{h}} = \tilde{O}(\mu_{\ell_0}(\Pi_{G^h} d_{f,f^{-1}([a,b]),h}|_{E^h})).$$

□

In the spirit of the proof of Proposition 3.2, and in particular of Step 3 in Subsection 3.2, we transfer our estimates from  $[\tilde{c}_1 - \delta_1, \tilde{c}_{N+1} + \delta_3]$  to a generally wider interval  $[a, b]$ .

**Proposition 6.14.** *Assume  $\delta_1, \delta_2 \in ]0, \frac{\eta_f}{8}]$ , let  $a, b$  satisfy (122), and let  $G^h$  be defined by (89). There exists an intermediate  $\delta_1$ -family of quasimodes in the sense of Definition 6.12 such that*

$$e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1+\delta_1}{h}} = \tilde{O}(\mu_{\ell_0}(\Pi_{G^h} d_{f,h}|_{E^h})) \quad \text{with } \ell_0 = \#\mathcal{X}(a, b),$$

holds true by defining  $E^h = T_{\delta_2} \tilde{\mathcal{V}}^h(a, b) = T_{\delta_2} \text{Span}(\tilde{\varphi}_j^h, j \in \mathcal{J}(a, b))$ .

*Proof.* Let  $\delta_1, \delta_2 \in ]0, \frac{\eta_f}{8}]$ . When  $\tilde{c}_1 - \delta_1 \leq a < \tilde{c}_1$  and  $\tilde{c}_{N+1} < b \leq \tilde{c}_{N+1} + \delta_1$ , the statement of Proposition 6.14 is an immediate consequence of Proposition 6.13. Moreover, when  $a < \tilde{c}_1 - \delta_1$  and  $\tilde{c}_{N+1} < b \leq \tilde{c}_{N+1} + \delta_1$ , the statement of Proposition 6.14 simply follows after extending the quasimodes by 0 on  $f_a^{a'}$ . We thus focus on the case  $b > \tilde{c}_{N+1} + \delta_1$ . Let then  $\delta_3 \in ]\delta_1, \frac{\eta_f}{8}]$  be such that  $b' := \tilde{c}_{N+1} + \delta_3 < b$  and set  $a' := \max(\tilde{c}_1 - \delta_1, a)$ .

We start from an intermediate  $\delta_1$ -family of quasimodes  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a', b')}$ , for the interval  $[a', b']$ , with the orthogonality property (120), (121). When  $a < \tilde{c}_1 - \delta_1 = a'$ , these quasimodes are extended by 0 on  $f_a^{a'}$ . We will use the spaces

$$E^h(a', b') = \underbrace{T_{\delta_2} \mathcal{V}_0^h(a', b') \oplus \left( \bigoplus_{1 \leq n-m \leq N-1} T_{\delta_2} \mathcal{W}_{m,n}^h(a', b') \right)}_{E^h(a', b')} \oplus \underbrace{T_{\delta_2} \tilde{\mathcal{W}}_{1, N+1}^h(a', b')}_{E''h(a', b')}$$

and, for  $(\bar{a}, \bar{b}) = (a', b')$  or  $(\bar{a}, \bar{b}) = (a, b)$ ,

$$G^h(\bar{a}, \bar{b}) = \bigoplus_{1 \leq n \leq N+1}^\perp \underbrace{\ker(\Delta_{f, f^{-1}([\tilde{c}_n - \eta_f, \tilde{c}_n + \eta_f] \cap [\bar{a}, \bar{b}], h))}_{G_n^h(\bar{a}, \bar{b})}.$$

According to (104) and to Propositions 6.9 and 6.13, we know that

$$\begin{aligned} T_{\delta_2} \mathcal{V}_0^h(a', b') &\subset \ker(\Pi_{G^h(a', b')} d_{f, h} \big|_{E^h(a', b')}), \\ \|\Pi_{G^h(a', b')} d_{f, h} \big|_{E'^h(a', b')}\| &= \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1}{h}}), \\ \text{and } e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} &= \tilde{O}(\mu_{\ell_0}(\Pi_{G^h(a', b')} d_{f, h} \big|_{E^h(a', b')})). \end{aligned}$$

Comparing the singular values of  $\Pi_{G^h(a', b')} d_{f, h} T_{\delta_2} \big|_{V^h(a', b')}$  and of  $\Pi_{G^h(a', b')} d_{f, h} \big|_{E^h(a', b')}$  is straightforward owing to

$$\|T_{\delta_2} T_{\delta_2}^* - \text{Id}_{E^h(a', b')}\| + \|T_{\delta_2}^* T_{\delta_2} - \text{Id}_{V^h(a', b')}\| = \tilde{O}(e^{-\frac{\eta_f}{h}}).$$

Meanwhile, the spaces  $\Pi_{G^h(a', b')} d_{f, h} (T_{\delta_2} \mathcal{W}_{m, n}^h(a', b'))$  are mutually orthogonal and orthogonal to  $\Pi_{G^h(a', b')} d_{f, h} (T_{\delta_2} \tilde{\mathcal{W}}_{1, N+1}^h(a', b'))$ , thanks to the orthogonality property (120), (121). Owing to Proposition 5.4-b), the non zero singular values of  $\Pi_{G^h(a', b')} d_{f, h} \big|_{E^h(a', b')}$  are then obtained by collecting the ones of  $\Pi_{G_n^h(a', b')} d_{f, h} \big|_{T_{\delta_2} \mathcal{W}_{m, n}^h(a', b')}$ ,  $1 \leq n - m \leq N - 1$ , and of  $\Pi_{G_{N+1}^h(a', b')} d_{f, h} \big|_{T_{\delta_2} \tilde{\mathcal{W}}_{1, N+1}^h(a', b')}$ .

Moreover, since the family  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}'(a', b')}$  satisfies Definition 6.12, and thus the statement of Proposition 6.9, the singular values of  $\Pi_{G_n^h(a', b')} d_{f, h} T_{\delta_2} \big|_{\mathcal{W}_{m, n}^h(a', b')}$  satisfy  $\mu_h \stackrel{\log}{\sim} e^{-\frac{\tilde{c}_n - \tilde{c}_m}{h}}$  when  $n - m < N - 1$  (see indeed (116)), while we know that the ones of  $\Pi_{G_{N+1}^h(a', b')} d_{f, h} T_{\delta_2} \big|_{\tilde{\mathcal{W}}_{1, N+1}^h(a', b')}$  satisfy, for  $\ell \leq \#\mathcal{X}_{1, N+1}(a', b') = \#\mathcal{X}_{1, N+1}(a, b)$ ,

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} = \tilde{O}(\mu_\ell(\Pi_{G_{N+1}^h(a', b')} d_{f, h} T_{\delta_2} \big|_{\tilde{\mathcal{W}}_{1, N+1}^h(a', b')})) = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1}{h}}).$$

Let us now construct the family  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a, b)}$  for the interval  $[a, b]$ .

- For the  $j = (\alpha, \tilde{c}_{N+1}) \in \mathcal{J}'_0(a, b)$ , we take an orthonormal basis  $(\tilde{\varphi}_j^h)_{j=(\alpha, \tilde{c}_{N+1}) \in \mathcal{J}'_0(a, b)}$  of  $\ker(\Delta_{f, f^{-1}([\tilde{c}_{N+1} - \delta_1, b]), h})$  (extended by 0 on  $f_a^{\tilde{c}_{N+1} - \delta_1}$ ).
- For  $j = (\alpha, \tilde{c}) \in \mathcal{J}'_0(a, b)$  with  $\tilde{c} < \tilde{c}_{N+1}$ , we “extend” the quasimode  $\tilde{\phi}_j^h$  as a solution to  $d_{f, h} \tilde{\varphi}_j^h = 0$  in  $[a, b]$ , as we did in Proposition 6.10 by referring to Proposition 3.9-ii), with the new artificial “critical value”  $b' = \tilde{c}_{N+1} + \delta_3 > \tilde{c}_{N+1} + \delta_1$ , in the interval  $[\tilde{c}_{N+1} + \delta_1, b]$ .
- For  $j \in \mathcal{X}_{m, n}(a, b)$  with  $1 \leq m < n \leq N$ , we simply keep  $\tilde{\varphi}_j^h = \tilde{\phi}_j^h$ .
- For the  $j = (\alpha, x_\alpha) \in \mathcal{X}(a, b)$  such that  $y_\alpha = \tilde{c}_{N+1}$  and the  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a, b)$ , the construction is detailed below after comparing, for  $m_0 \in \{1, \dots, N\}$ , the two maps

$$\Pi_{G_{N+1}^h(a', b')} d_{f, h} T_{\delta_2} \big|_{V_{m_0, N+1}^h} \quad \text{and} \quad \Pi_{G_{N+1}^h(a, b)} d_{f, h} T_{\delta_2} \big|_{V_{m_0, N+1}^h} = \Pi_{G_{N+1}^h(a', b)} d_{f, h} T_{\delta_2} \big|_{V_{m_0, N+1}^h},$$

with

$$V_{m_0, N+1}^h = \left( \bigoplus_{\max\{2, m_0\} \leq m < N+1} \mathcal{W}_{m, N+1}^h(a', b') \right) \underbrace{\oplus \tilde{\mathcal{W}}_{1, N+1}^h(a', b')}_{\text{if } m_0=1}.$$

We recall that

$$\begin{aligned} \dim \mathcal{W}_{m, N+1}^h(a', b') &= \#\mathcal{X}_{m, N+1}(\bar{a}, \bar{b}) \quad \text{when } 2 \leq m < N + 1 \\ \text{and } \dim \tilde{\mathcal{W}}_{1, N+1}^h(a', b') &= \#\mathcal{X}_{1, N+1}(\bar{a}, \bar{b}) \sqcup \#\{j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(\bar{a}, \bar{b})\}, \end{aligned}$$

where  $(\bar{a}, \bar{b}) = (a', b')$  or  $(\bar{a}, \bar{b}) = (a, b)$ , and we set, for  $m_0 \in \{1, \dots, N\}$ ,

$$J_{m_0, N+1} = \left( \bigsqcup_{m_0 \leq m < N+1} \mathcal{X}_{m, N+1}(a, b) \right) \underbrace{\sqcup \{j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a, b)\}}_{\text{if } m_0=1}.$$

Since the  $\tilde{\Psi}_j^h = \Pi_{G_{N+1}^h(a', b')} d_{f,h} T_{\delta_2} \tilde{\phi}_j^h$ ,  $j \in J_{m_0, N+1}$ , are mutually orthogonal and owing to the information on the singular values, there exists an orthonormal basis  $(\psi_k)_{1 \leq k \leq \dim G_{N+1}^h(a', b')}$  of  $G_{N+1}^h(a', b')$  such that the matrix

$$\begin{aligned} M^h &= \left( \langle \psi_k, \Pi_{G_{N+1}^h(a', b')} d_{f,h} T_{\delta_2} \tilde{\phi}_j^h \rangle \right)_{1 \leq k \leq \dim G_{N+1}^h(a', b'), j \in J_{m_0, N+1}} \\ &= \left( \langle \psi_k, d_{f,h} T_{\delta_2} \tilde{\phi}_j^h \rangle \right)_{1 \leq k \leq \dim G_{N+1}^h(a', b'), j \in J_{m_0, N+1}} \end{aligned}$$

has the following block diagonal structure:

- When  $m_0 > 1$ :

$$M^h = \begin{pmatrix} D^h \\ 0 \end{pmatrix}, \quad D^h = \text{diag}(\lambda_j^h, j \in J_{m_0, N+1}),$$

$$\text{where } \lambda_j^h \stackrel{\text{log}}{\sim} e^{-\frac{\tilde{c}_{N+1} - x_\alpha}{h}} \text{ for } j = (\alpha, x_\alpha) \in J_{m_0, N+1}.$$

- When  $m_0 = 1$ :

$$M^h = \begin{pmatrix} D^h & 0 \\ 0 & R^h \end{pmatrix}, \quad D^h = \text{diag}(\lambda_j^h, j = (\alpha, x_\alpha) \in J_{1, N+1}, x_\alpha \geq \tilde{c}_2),$$

$$\text{where } \lambda_j^h \stackrel{\text{log}}{\sim} e^{-\frac{\tilde{c}_{N+1} - x_\alpha}{h}} \text{ for } j = (\alpha, x_\alpha) \in J_{1, N+1}, x_\alpha \geq \tilde{c}_2,$$

$$\text{and } \|R^h\| = \tilde{O}\left(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1}{h}}\right), \quad (124)$$

while, for  $\ell'_0 = \sharp(J_{1, N+1} \cap \mathcal{X}(a, b))$ , the  $\ell'_0$ -th singular value is bounded from below by

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} = \tilde{O}(\mu_{\ell'_0}(M^h)).$$

Proposition 3.9-iii) provides an isomorphism  $A_h : G_{N+1}^h(a, b) \rightarrow G_{N+1}^h(a', b') = G_{N+1}^h(a, b')$  such that

$$\left\| A_h^* A_h - \text{Id}_{G_{N+1}^h(a, b)} \right\| + \|A_h A_h^* - \text{Id}_{G_{N+1}^h(a', b')}\| = \tilde{O}\left(e^{-\frac{\delta_3}{h}}\right)$$

$$\forall j = (\alpha, \tilde{c}) \in J_{m_0, N+1}, \forall \psi \in G_{N+1}^h(a, b),$$

$$\langle d_{f,h} T_{\delta_2} \phi_j^h, \psi - A_h \psi \rangle = \tilde{O}\left(e^{-\frac{\tilde{c}_{N+1} - \tilde{c} + 2\delta_3}{h}}\right) \|\psi\|. \quad (125)$$

By using the  $\tilde{O}\left(e^{-\frac{\delta_1}{h}}\right)$ -orthonormal basis  $(\tilde{\phi}_j^h)_{j \in J_{m_0, N+1}}$  of  $V_{m_0, N+1}^h$  and the  $\tilde{O}\left(e^{-\frac{\delta_3}{h}}\right)$ -orthonormal basis  $(A_h^{-1} \psi_k^h)_{1 \leq k \leq \dim G_{N+1}^h(a, b)}$  of  $G_{N+1}^h(a, b)$ , the singular values of the matrix

$$M'^h = \left( \langle A_h^{-1} \psi_k^h, d_{f,h} T_{\delta_2} \tilde{\phi}_j^h \rangle \right)_{1 \leq k \leq \dim G_{N+1}^h(a, b), j \in J_{m_0, N+1}}$$

coincide modulo a  $\tilde{O}\left(e^{-\frac{\min(\delta_1, \delta_3)}{h}}\right)$ -relative error with the ones of  $\Pi_{G^h(a, b)} d_{f,h} T_{\delta_2} \big|_{V_{m_0, N+1}^h}$  according to Proposition 5.4-a). With the above inequality (125), the  $j$ -th columns of  $M'^h$  and of  $M^h$ , for  $j = (\alpha, x_\alpha) \in J_{m_0, N+1}$ ,  $x_\alpha \geq \tilde{c}_2$ , differ by a  $\tilde{O}\left(\lambda_j^h \times e^{-\frac{2\delta_3}{h}}\right)$ . When  $m_0 = 1$  and  $j = (\alpha, \tilde{c}_1) \in J_{1, N+1}$ , the  $j$ -th columns of  $M'^h$  and of  $M^h$  differ by a  $\tilde{O}\left(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + 2\delta_3}{h}}\right)$  error. Hence, we can write

$$M'^h = (\text{Id} + \tilde{O}\left(e^{-\frac{2\delta_3}{h}}\right)) M^h + \underbrace{\tilde{O}\left(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + 2\delta_3}{h}}\right)}_{\text{if } m_0=1}.$$



When  $m_0 > 1$ , the singular values of  $M''^h$  coincide with the ones of  $M'''^h := (\text{Id} + \tilde{O}(e^{-\frac{2\delta_3}{h}}))M^h$  with a  $\tilde{O}(e^{-\frac{2\delta_3}{h}})$ -relative error.

When  $m_0 = 1$ , the  $\ell'_0$ -th singular value of  $M'''^h := (\text{Id} + \tilde{O}(e^{-\frac{2\delta_3}{h}}))M^h$  satisfies

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \max(\delta_1, \delta_3)}{h}} = \tilde{O}(\mu_{\ell'_0}(M'''^h)).$$

Hence, we get

$$M''^h = M'''^h + \tilde{O}(e^{-\frac{2\delta_3 - \max(\delta_1, \delta_3)}{h}}) \mu_{\ell'_0}(M'''^h).$$

Since  $\delta_1 < \delta_3$ , Proposition 5.7 implies:

$$\forall \ell \in \{1, \dots, \ell'_0\}, \quad \mu_\ell(M''^h) = \mu_\ell(M'''^h)(1 + \tilde{O}(e^{-\frac{\delta_1}{h}})).$$

We have thus proved that for all  $m_0 \in \{1, \dots, N\}$ :

$$\forall \ell \in \{1, \dots, \min(\#J_{m_0, N+1}, \ell'_0)\},$$

$$\mu_\ell(\Pi_{G_{N+1}^h(a,b)} d_{f,h} T_{\delta_2} \big|_{V_{m_0, N+1}^h}) = \mu_\ell(\Pi_{G_{N+1}^h(a',b')} d_{f,h} T_{\delta_2} \big|_{V_{m_0, N+1}^h})(1 + \tilde{O}(e^{-\frac{\delta_1}{h}})).$$

In particular, since

$$\forall \delta_3 \in ]\delta_1, \min(\frac{\eta_f}{8}, b - \tilde{c}_{N+1})[, \quad e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_3}{h}} = \tilde{O}(\mu_{\ell'_0}(\Pi_{G_{N+1}^h(a,b)} d_{f,h} T_{\delta_2} \big|_{V_{1, N+1}^h})),$$

and the right-hand side in the latter equality does not depend on  $\delta_3$ , we get

$$e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}} = \tilde{O}(\mu_{\ell'_0}(\Pi_{G_{N+1}^h(a,b)} d_{f,h} T_{\delta_2} \big|_{V_{1, N+1}^h})). \quad (126)$$

We now finish the presentation of our quasimodes  $(\tilde{\varphi}_j^h)_{j \in J_{1, N+1}}$ . Like in the proof of Proposition 6.6, we construct by reverse induction from  $m_0 = N$  to  $m_0 = 1$ , starting from the family  $(\tilde{\phi}_j^h)_{j \in J_{1, N+1}}$ , a basis  $(\tilde{\varphi}_j^h)_{j \in J_{m_0, N+1}}$  of  $V_{m_0, N+1}^h$  and an orthonormal basis of  $G_{N+1}^h(a, b)$ , independent of  $m_0$ , such that the matrix of  $\Pi_{G_{N+1}^h(a,b)} d_{f,h} T_{\delta_2} \big|_{V_{m_0, N+1}^h}$  in these bases is diagonal (add possibly lines or columns of zeros to make it square). Since this process preserves the flag  $(V_{m_0, N+1}^h)_{1 \leq m_0 < N+1}$ , the support condition and the exponential decay estimates are valid for this new basis of  $V_{1, N+1}^h$ . The  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormality of the full new family  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a,b)}$  and the  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -proximity to  $F_{[0, \tilde{\sigma}(1)], [a,b], h}$  hold true, especially with our choice for  $j = (\alpha, \tilde{c}_{N+1}) \in \mathcal{Z}(a, b)$ . This proves the conditions 1 and 2 of Definition 6.12. For the third condition, we notice that the spaces  $\mathcal{V}_+^h(a, b)$  and  $\mathcal{V}_+^h(a', b')$  are equal, like the spaces  $G_n^h(a, b)$  and  $G_n^h(a', b')$  when  $2 \leq n \leq N$ , while  $T_{\delta_2}$  is not changed. Moreover, in the case  $n = N+1$ , the above orthogonalization process until  $V_{2, N+1}^h$  and the asymptotics of the singular values of  $\Pi_{G_{N+1}^h(a,b)} d_{f,h} T_{\delta_2} \big|_{V_{2, N+1}^h}$  finish the verification of the properties stated in Proposition 6.9 for  $E''^h = T_{\delta_2} \mathcal{V}^h(a, b)$  with  $G^h = G^h(a, b)$ .

Finally, it then follows from (124) and (126) that

$$\mu_{\ell_0}(\Pi_{G^h(a,b)} d_{f,h} \big|_{T_{\delta_2} \mathcal{V}^h(a,b)=E^h}) = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1}{h}}) \quad (127)$$

$$\text{and } e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}} = \tilde{O}(\mu_{\ell_0}(\Pi_{G_{N+1}^h(a,b)} d_{f,h} \big|_{T_{\delta_2} \mathcal{V}^h(a,b)=E^h})). \quad (128)$$

□

**Remark 6.15.** *Although we used the notation  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a,b)}$ , notice that we obtain at the end of the proof an intermediate  $\delta_1$ -family of quasimodes  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a,b)}$  which satisfies the orthogonality property (120),(121) in the interval  $[a, b]$ . It was actually more important in the proof to put the stress on this property for the initial family given for the interval  $[a', b'] = [\tilde{c}_1 - \delta_1, \tilde{c}_{N+1} + \delta_3]$ . However, the orthogonalization process can always be carried out afterwards.*

*Proof of Theorem 6.3-a).* Let  $a, b$  satisfy (122) and take  $\delta_1, \delta_2 \in ]0, \frac{\eta f}{8}]$ . We reconsider the proof of Proposition 6.13 for the pair  $(a, b)$  with the new lower bound of Proposition 6.14:

$$e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1+\delta_1}{h}} = \tilde{O}(\mu_{\ell_0}(\Pi_{G^h} d_{f,h}|_{E^h})) \quad \text{with} \quad \ell_0 = \#\mathcal{X}(a, b).$$

We then set  $E^h = E'^h \oplus E''^h$ ,

$$E'^h = T_{\delta_2} \mathcal{V}^h(a, b) \quad , \quad E''^h = T_{\delta_2} \tilde{\mathcal{V}}_{1, N+1}^h(a, b),$$

where  $\mathcal{V}^h(a, b)$  and  $\tilde{\mathcal{V}}_{1, N+1}^h(a, b)$  are associated with the intermediate  $\delta_1$ -family of quasimodes  $(\tilde{\varphi}_j^h)_{j \in \mathcal{J}(a, b)}$  provided by Proposition 6.14. In particular, the verification of the inequality (70) in Proposition 5.8 now becomes:

$$\begin{aligned} \|B^h|_{E''^h}\| & \left[ \frac{1}{\mu_{\ell_1}(\Pi_{G^h} B^h|_{E', h})} + \frac{\|C^h\|(\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h))}{\max(\mu_{\ell_0}(\Pi_{G^h} B^h|_{E^h}), \mu_{\ell_0}(B^h|_{F^h}))} \right] \\ & = \tilde{O}(e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1-2\delta_2}{h}}) \times \left[ \tilde{O}(e^{\frac{\tilde{c}_{N+1}-\tilde{c}_1-2\eta f}{h}}) + \tilde{O}(e^{\frac{2\delta_2-\eta f}{h}}) \times \tilde{O}(e^{\frac{\tilde{c}_{N+1}-\tilde{c}_1+\delta_1}{h}}) \right] \\ & = \tilde{O}(e^{-\frac{\eta f}{h}}) + \tilde{O}(e^{\frac{4\delta_2+\delta_1-\eta f}{h}}) = \tilde{O}(e^{-\frac{\delta_1}{h}}), \end{aligned}$$

with  $\delta_1, \delta_2 \leq \frac{\eta f}{8}$ .

The conclusion of Proposition 5.8 is then

$$\begin{aligned} \forall \ell \in \{1, \dots, \ell_0\}, \quad \mu_\ell(\delta_{[0, \bar{\sigma}(1)], [a, b], h}) & = \mu_\ell(\Pi_{G^h} d_{f, f^{-1}([a, b]), h}|_{E^h})(1 + \tilde{O}(e^{-\frac{\delta_1}{h}})), \\ \text{and} \quad \mu_{\ell_0+1}(\Pi_{G^h} d_{f, f^{-1}([a, b]), h}|_{E^h}) & = \tilde{O}(e^{-\frac{\delta_1}{h}}) \mu_{\ell_0}(\delta_{[0, \bar{\sigma}(1)], [a, b], h}). \end{aligned}$$

In particular, we obtain

$$e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1+\delta_1}{h}} = \tilde{O}(\mu_{\ell_0}(\delta_{[0, \bar{\sigma}(1)], [a, b], h}))$$

and therefore, since the right-hand side of the latter equality does not depend on  $\delta_1$ ,

$$e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1}{h}} = \tilde{O}(\mu_{\ell_0}(\delta_{[0, \bar{\sigma}(1)], [a, b], h})).$$

Using in addition (127) (together with Proposition 6.9) leads to the statement of Theorem 6.3-a) at step  $N + 1$ .  $\square$

We also proved

$$\mu_{\ell_0+1}(\Pi_{G^h} d_{f, f^{-1}([a, b]), h}|_{E^h}) = \tilde{O}(e^{-\frac{\delta_1}{h}}) \mu_{\ell_0}(\delta_{[0, \bar{\sigma}(1)], [a, b], h}) = \tilde{O}(e^{-\frac{\tilde{c}_{N+1}-\tilde{c}_1+\delta_1}{h}}). \quad (129)$$

Moreover, according to the comments made around (119), one can choose the intermediate  $\delta_1$ -family  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a, b)}$  such that the orthogonality property (120), (121) holds, and then such that

$$\|\tilde{\Psi}_j^h\| \stackrel{\log}{\sim} e^{-\frac{\eta \alpha - x_\alpha}{h}} \quad \text{for every} \quad j = (\alpha, x_\alpha) \in \mathcal{X}(a, b). \quad (130)$$

## 6.7 Construction of the family $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$ at step $N + 1$

We now end the proof of Theorem 6.3 at step  $N + 1$  by finishing the construction of the  $\delta_1$ -family of quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$ . The statements **b)** and **c)** in Theorem 6.3 will be easily checked at the end.

Let  $a, b$  satisfy (122), let  $G^h$  be defined by (89), and let  $\delta_1, \delta_2 \in ]0, \frac{\eta f}{8}]$ . We start with an intermediate  $\delta_1$ -family of quasimodes for the interval  $[a, b]$  which satisfies the orthogonality condition

(120),(121) and the estimates (129) and (130).

We first work in the interval  $[a', b]$  with  $a' = \max(a, \tilde{c}_1 - \delta_1)$ . Note that, since the quasimodes are all supported in  $[a', b]$  and  $G_n^h(a, b) = G_n^h(a', b)$  for every  $2 \leq n \leq N+1$ , the family  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a, b) = \mathcal{J}(a', b)}$  is still, for the interval  $[a', b]$ , an intermediate  $\delta_1$ -family of quasimodes which satisfies the orthogonality condition (120),(121) and the estimates (129) and (130).

The quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a', b)}$  are not changed, i.e.

$$\varphi_j^h = \tilde{\phi}_j^h,$$

when

$$\begin{aligned} j &\in \mathcal{J}'_0(a', b) = \mathcal{Y}(a', b) \sqcup \{j = (\alpha, \tilde{c}) \in \mathcal{Z}(a', b), \tilde{c} > \tilde{c}_1\} \\ \text{or } j &\in \mathcal{X}(a', b) = \bigsqcup_{1 \leq m < n \leq N+1} \mathcal{X}_{m, n}(a', b). \end{aligned}$$

We must now construct the remaining quasimodes  $\varphi_j^h$ ,  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b)$ , in order to ensure

$$\varphi_j^h \in \ker(d_{f, f^{-1}([a', b]), h}) \quad \text{for every } j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b),$$

while we only know for the moment that, for those  $j$ , (129) implies

$$\|\tilde{\Psi}_j^h\| = \|\Pi_{G^h} d_{f, h} T_{\delta_2} \tilde{\phi}_j^h\| = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}}).$$

We recall that those quasimodes  $\tilde{\phi}_j^h$ ,  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b)$ , were until now considered in the space  $\tilde{\mathcal{W}}_{1, N+1}^h(a', b)$ , together with the quasimodes  $\tilde{\phi}_j^h$ ,  $j \in \mathcal{X}_{1, N+1}(a', b)$ . Let us also recall that the rank of  $\delta_{[0, \tilde{\sigma}(1)][a', b], h}$  satisfies (see Proposition 4.4):

$$\text{rank } \delta_{[0, \tilde{\sigma}(1)][a', b], h} = \ell_0 = \#\mathcal{X}(a', b). \quad (131)$$

**Proposition 6.16.** *For  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b)$ , where  $a' = \max(a, \tilde{c}_1 - \delta_1)$ , there exists  $(\alpha_{j, j'}^h)_{j' \in \mathcal{X}(a', b)}$  such that*

$$\tilde{\phi}_j^h - \sum_{j' \in \mathcal{X}(a', b)} \frac{\alpha_{j, j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h$$

belongs to  $\ker(\delta_{[0, \tilde{\sigma}(1)][a', b], h} T_{\delta_2})$  with, for every  $j' \in \mathcal{X}(a', b)$ ,

$$\alpha_{j, j'}^h = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}}).$$

*Proof.* For every  $j' \in \mathcal{X}(a', b)$ , we set

$$\psi_{j'}^h := \frac{\tilde{\Psi}_{j'}^h}{\|\tilde{\Psi}_{j'}^h\|},$$

so that, when  $j' = (\alpha, x_\alpha) \in \mathcal{X}(a', b)$ ,

$$\Pi_{G^h} d_{f, h} T_{\delta_2} \tilde{\phi}_{j'}^h = \|\tilde{\Psi}_{j'}^h\| \psi_{j'}^h, \quad \|\tilde{\Psi}_{j'}^h\| \stackrel{\text{log}}{\sim} e^{-\frac{\gamma_\alpha - x_\alpha}{h}}$$

and  $(\psi_{j'}^h)_{j' \in \mathcal{X}(a', b)}$  is an orthonormal system in  $G^h$ .

By writing, for  $j' \in \mathcal{X}(a', b)$ ,

$$\begin{aligned} \delta_{[0, \tilde{\sigma}(1)][a', b], h} T_{\delta_2} \tilde{\phi}_{j'}^h &= \Pi_{F^h} d_{f, h} T_{\delta_2} \tilde{\phi}_{j'}^h \\ &= \Pi_{G^h} d_{f, h} T_{\delta_2} \tilde{\phi}_{j'}^h - (\Pi_{G^h} - \Pi_{F^h} \Pi_{G^h}) d_{f, h} T_{\delta_2} \tilde{\phi}_{j'}^h \\ &\quad + (\Pi_{F^h} - \Pi_{F^h} \Pi_{G^h}) d_{f, h} T_{\delta_2} \tilde{\phi}_{j'}^h \end{aligned} \quad (132)$$

with  $F^h = F_{[0,\tilde{\sigma}(1)],[a',b],h}$ ,  $\tilde{d}(F^h, G^h) + \tilde{d}(G^h, F^h) = \tilde{O}(e^{-\frac{\eta f}{h}})$ , and (see (109))

$$\|d_{f,h}T_{\delta_2}\tilde{\phi}_{j'}^h\| = \|\tilde{\Psi}_{j'}^h\|\tilde{O}(e^{\frac{2\delta_2}{h}}),$$

we deduce from (132) that the family made of the

$$\theta_{j'}^h = \frac{\delta_{[0,\tilde{\sigma}(1)],[a',b],h}T_{\delta_2}\tilde{\phi}_{j'}^h}{\|\tilde{\Psi}_{j'}^h\|}, \quad j' \in \mathcal{X}(a', b),$$

defines an  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal system of  $R^h := \text{Ran } \delta_{[0,\tilde{\sigma}(1)],[a',b],h}$ . Owing to (131), the family  $(\theta_{j'}^h)_{j' \in \mathcal{X}(a', b)}$  is thus an  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal basis of  $R^h$ . Denoting now by  $(\hat{\theta}_{j'}^h)_{j' \in \mathcal{X}(a', b)}$  the dual basis of  $(\theta_{j'}^h)_{j' \in \mathcal{X}(a', b)}$  in  $R^h$ , that is the unique family satisfying

$$\forall j'_1, j'_2 \in \mathcal{X}(a', b), \quad \hat{\theta}_{j'_1}^h \in R_h \quad \text{and} \quad \langle \hat{\theta}_{j'_1}^h, \theta_{j'_2}^h \rangle = \delta_{j'_1, j'_2},$$

the family  $(\hat{\theta}_{j'}^h)_{j' \in \mathcal{X}(a', b)}$  is also an  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal basis of  $R^h$  and the orthogonal projection on  $R^h$  is given by

$$\forall u \in F^h, \quad \Pi_{R^h} u = \sum_{j' \in \mathcal{X}(a', b)} \langle \hat{\theta}_{j'}^h, u \rangle \theta_{j'}^h = \sum_{j' \in \mathcal{X}(a', b)} \frac{\langle \hat{\theta}_{j'}^h, u \rangle}{\|\tilde{\Psi}_{j'}^h\|} \delta_{[0,\tilde{\sigma}(1)],[a',b],h} T_{\delta_2} \tilde{\phi}_{j'}^h.$$

For  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b)$ , the same decomposition as (132) with now  $\|\tilde{\Psi}_j^h\| = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}})$  and  $\|d_{f,h}T_{\delta_2}\tilde{\phi}_j^h\| = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 - 2\delta_2}{h}})$  leads to

$$\|\delta_{[0,\tilde{\sigma}(1)],[a',b],h}T_{\delta_2}\tilde{\phi}_j^h\| = \tilde{O}(e^{-\frac{\tilde{c}_{N+1} - \tilde{c}_1 + \delta_1}{h}}).$$

The statement of Proposition 6.16 follows easily by taking, for every  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a', b)$  and  $j' \in \mathcal{X}(a', b)$ ,

$$\alpha_{j,j'}^h = \langle \hat{\theta}_{j'}^h, \delta_{[0,\tilde{\sigma}(1)],[a',b],h}T_{\delta_2}\tilde{\phi}_j^h \rangle.$$

□

The following statement finishes the proof of Theorem 6.3.

**Proposition 6.17.** *Assume that  $a, b$  satisfy (122), let  $\delta_1, \delta_2 \in ]0, \frac{\eta f}{8}]$ , and set  $a' = \max(a, \tilde{c}_1 - \delta_1)$ . The family  $(\varphi_j^h)_{j \in \mathcal{J}(a, b)}$  defined by*

$$\varphi_j^h = \tilde{\phi}_j^h \quad \text{when } j \in \mathcal{X}(a, b) \sqcup \mathcal{Y}(a, b) \sqcup \{(\alpha, \tilde{c}) \in \mathcal{Z}(a, b), \tilde{c} > \tilde{c}_1\}$$

and

$$\varphi_j^h = 1_{f_a^b} \times \Pi_{[0,\tilde{\sigma}(1)],[a',b],h}T_{\delta_2} \left( \tilde{\phi}_j^h - \sum_{j' \in \mathcal{X}(a, b)} \frac{\alpha_{j,j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h \right) \quad \text{when } j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a, b),$$

where the coefficients  $\alpha_{j,j'}^h$  are given by Proposition 6.16, fulfills all the conditions of Theorem 6.3 at step  $N + 1$ .

*Proof.* We use here the notations  $a' = \max(a, \tilde{c}_1 - \delta_1)$  and, in order to avoid confusions,

$$\begin{aligned} \tilde{\mathcal{W}}^h(a, b) &= \text{Span}(\tilde{\phi}_j^h, j \in \mathcal{J}(a, b)) \\ \text{and} \quad \tilde{\mathcal{W}}_+^h(a, b) &= \text{Span}(\tilde{\phi}_j^h, j \in \mathcal{X}(a, b)), \end{aligned}$$

where  $(\tilde{\phi}_j^h)_{j \in \mathcal{J}(a,b)}$  is the intermediate  $\delta_1$ -family of quasimodes we started with.

From the estimates  $\alpha_{j,j'}^h = \tilde{O}(e^{-\frac{\tilde{c}_1 + 1 - \tilde{c}_1 + \delta_1}{h}})$  (see Proposition 6.16) and  $\|\tilde{\Psi}_{j'}^h\| \stackrel{\log}{\sim} e^{-\frac{y_\alpha - x_\alpha}{h}}$  for  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b)$  and  $j' = (\alpha, x_\alpha) \in \mathcal{X}(a,b)$ , we deduce that

$$\forall j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b), \quad \left\| \sum_{j' \in \mathcal{X}(a,b)} \frac{\alpha_{j,j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h \right\|_{L^2} = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

Since in addition  $\vec{d}(F_{[0,\tilde{\sigma}(1)], [a',b],h}, \tilde{\mathcal{W}}^h(a,b)) + \vec{d}(\tilde{\mathcal{W}}^h(a,b), F_{[0,\tilde{\sigma}(1)], [a',b],h}) = \tilde{O}(e^{-\frac{\delta_1}{h}})$ , it follows that

$$\|\varphi_j^h - \tilde{\phi}_j^h\| = \tilde{O}(e^{-\frac{\delta_1}{h}}) \quad \text{for } j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b),$$

and the family  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  is thus  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal. Moreover, the exponential decay estimates on the  $\tilde{\phi}_{j'}^h$ ,  $j' \in \mathcal{X}(a,b)$ , lead to

$$\forall j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b), \quad \left\| e^{\frac{|f-\tilde{c}_1|}{h}} \left( \sum_{j' \in \mathcal{X}(a,b)} \frac{\alpha_{j,j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h \right) \right\|_{W(f^{-1}([a',b] \setminus S_{\delta_1}))} = \tilde{O}(e^{-\frac{\delta_1}{h}}).$$

This implies, together with Proposition 4.7, the required exponential decay estimates on the  $\varphi_j^h$ ,  $j = (\alpha, \tilde{c}_1) \in \mathcal{Z}(a,b)$ . Besides, Proposition 6.16 gives

$$d_{f,h} \Pi_{[0,\tilde{\sigma}(1)], [a',b],h} T_{\delta_2} \left( \tilde{\phi}_j^h - \sum_{j' \in \mathcal{X}(a,b)} \frac{\alpha_{j,j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h \right) = \delta_{[0,\tilde{\sigma}(1)], [a',b],h} T_{\delta_2} \left( \tilde{\phi}_j^h - \sum_{j' \in \mathcal{X}(a,b)} \frac{\alpha_{j,j'}^h}{\|\tilde{\Psi}_{j'}^h\|} \tilde{\phi}_{j'}^h \right) = 0.$$

All those properties are preserved after extending those quasimodes by 0 on  $f_a^{a'}$  when  $a < a'$ . Therefore, the family  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  satisfies all the conditions of Definition 6.1 and is thus a  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthonormal  $\delta_1$ -family of quasimodes. Since in addition

$$\vec{d}(F_{[0,\tilde{\sigma}(1)], [a',b],h}, F_{[0,\tilde{\sigma}(1)], [a,b],h}) + \vec{d}(F_{[0,\tilde{\sigma}(1)], [a,b],h}, F_{[0,\tilde{\sigma}(1)], [a',b],h}) = \tilde{O}(e^{-\frac{\delta_1}{h}}),$$

the statement **b)** of Theorem 6.3 is also satisfied.

It only remains to check the factorization stated in Theorem 6.3-c). Since

$$d_{f,h} T_{\delta_2} \varphi_j^h = d_{f,h} \varphi_j^h = 0 \quad \text{for every } j \notin \mathcal{X}(a,b),$$

it suffices to prove the existence of  $C^h$  such that

$$\begin{array}{ccc} \mathcal{V}_+^h(a,b) = \tilde{\mathcal{W}}_+^h(a,b) & \xrightarrow{d_{f,f^{-1}([a,b]),h} T_{\delta_2}} & L^2(f^{-1}([a,b])) \\ & \searrow \Pi_{[0,\tilde{\sigma}(1)], [a,b],h} d_{f,h} T_{\delta_2} & \uparrow C^h \\ & & F_{[0,\tilde{\sigma}(1)], [a,b],h} \end{array}$$

with  $\|C^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$ . Since  $\Pi_{C^h} d_{f,h} T_{\delta_2} \tilde{\phi}_j^h = \tilde{\Psi}_j^h$  with  $\|\tilde{\Psi}_j^h\| \stackrel{\log}{\sim} e^{-\frac{y_\alpha - x_\alpha}{h}}$  when  $j = (\alpha, x_\alpha) \in \mathcal{X}(a,b)$  with the orthogonality property (120),(121), reasoning as at the ends of the proofs of Propositions 6.8 and 6.9, we obtain the diagram

$$\begin{array}{ccc} \mathcal{V}_+^h(a,b) = \tilde{\mathcal{W}}_+^h(a,b) & \xrightarrow{d_{f,f^{-1}([a,b]),h} T_{\delta_2}} & L^2(f^{-1}([a,b])) \\ & \searrow \Pi_{C^h} d_{f,h} T_{\delta_2} & \uparrow \tilde{C}^h \\ & & G^h \end{array}$$

with  $\|\tilde{C}^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$ . We conclude by applying Lemma 5.6 with  $B^h = d_{f,f^{-1}([a,b]),h}T_{\delta_2}$ ,  $F^h = F_{[0,\tilde{\sigma}(1)],[a,b],h}$ , and

$$\vec{d}(F^h, G^h) + \vec{d}(G^h, F^h) = \tilde{O}(e^{-\frac{\eta_f}{h}}).$$

□

## 7 Corollaries of Theorem 6.3

The statement of Theorem 6.3 is much more flexible than its illustrative statement, Theorem 1.7, given in the introduction. Actually, even its proof, and especially the intermediate propositions of Subsection 6.3, have easily derived consequences which are listed here. Subsection 7.1 reviews consequences on the eigenvalues and eigenvectors of the Witten Laplacian  $\Delta_{f,f^{-1}([a,b]),h}$  when  $f$  is fixed. Subsection 7.2 studies how the logarithms of the singular values of  $d_{f,f^{-1}([a,b]),h}$  vary when  $f$  is changed. It contains a generalization of Corollary 1.8. Remember that Theorem 6.3 is proved under Hypothesis 4.1 which gathers Hypothesis 1.2 or (Hypothesis 1.6 and Hypothesis 2.16) for a more general Lipschitz function  $f$ . Hypothesis 1.2 or Hypothesis 1.6 ensure that  $f$  has finitely many “critical values”  $c_1 < \dots < c_{N_f}$ .

When  $a, b \notin \{c_1, \dots, c_{N_f}\}$ ,  $\Delta_{f,f^{-1}([a,b]),h}$  is the self-adjoint Witten Laplacian in  $f_a^b$ , with Dirichlet boundary conditions on  $f^{-1}(\{a\})$  and Neumann boundary conditions on  $f^{-1}(\{b\})$ , according to Section 2.

Finally, the bar code associated with  $f$ , under Hypothesis 1.2 or Hypothesis 1.6 (see Subsection 8.3.1), is  $\mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A})$ , defined in Subsection 4.1 and in Appendix B. The restricted bar code  $\mathcal{B}(f; a, b)$ , and the set of endpoints  $\mathcal{J}(a, b)$ ,  $\mathcal{X}(a, b)$ ,  $\mathcal{Y}(a, b)$ ,  $\mathcal{Z}(a, b)$ , all graded according to the degree  $p \in \{0, \dots, d\}$ , are the ones introduced in Subsection 4.1.

### 7.1 Spectral results

The first result generalizes Theorem 1.7.

**Theorem 7.1.** *Assume Hypothesis 1.2 or (Hypothesis 1.6 and Hypothesis 2.16) for a more general Lipschitz function  $f$ . Let  $a, b \notin \{c_1, \dots, c_{N_f}\}$  with  $a < b$  and let  $\Delta_{f,f^{-1}([a,b]),h} = \bigoplus_{p=0}^d \Delta_{f,f^{-1}([a,b]),h}^{(p)}$  be defined like in Proposition 2.8 with  $N_t = f^{-1}(\{a\})$  and  $N_n = f^{-1}(\{b\})$ . The number of  $\tilde{\sigma}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  equals  $\#\mathcal{J}^{(p)}(a, b)$ , while*

$$\dim \ker(\Delta_{f,f^{-1}([a,b]),h}) = \beta^{(p)}(f^b, f^a) = \#\mathcal{Z}^{(p)}(a, b).$$

Moreover, the non zero  $\tilde{\sigma}(1)$ -eigenvalues of  $\Delta_{f,f^{-1}([a,b]),h}^{(p)}$  counted with multiplicity can be labelled  $\lambda_\alpha^{(p)}(h)$ ,  $\alpha \in A_c^{(p)}(a, b) \sqcup A_c^{(p-1)}(a, b)$ , with

$$\lambda_\alpha^{(p)}(h) \stackrel{\log}{\sim} e^{-2\frac{y_\alpha^* + 1 - x_\alpha^*}{h}} \quad , \quad \alpha \in A_c^{(p)}(a, b) \sqcup A_c^{(p-1)}(a, b).$$

With the usual supersymmetric argument which was already recalled in Proposition 4.4, it is a straightforward consequence of Theorem 6.3-a).

The above result can be completed by some information on the eigenvectors. We start with the link between the singular values of  $\delta_{f,f^{-1}([a,b]),h}$ , and their approximation via the introduction of a basis made of quasimodes, and the spectral elements of the operator  $\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^* \delta_{[0,\tilde{\sigma}(1)],[a,b],h}$ . The spectral elements of

$$\Pi_{[0,\tilde{\sigma}(1)],[a,b],h} \Delta_{f,f^{-1}([a,b]),h} = \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^* \delta_{[0,\tilde{\sigma}(1)],[a,b],h} + \delta_{[0,\tilde{\sigma}(1)],[a,b],h} \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^*$$

will be described afterwards by referring to Hodge decomposition and to duality.

**Proposition 7.2.** *Keep the same assumptions as in Theorem 7.1 and define  $\eta_f > 0$  like in Hypothesis 4.1. Let  $\delta_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)}$  denote the restriction of  $d_{f,f^{-1}([a,b]),h}$  to  $F_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)}$ ,  $\delta_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)} : F_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)} \rightarrow F_{[0,\bar{\sigma}(1)],[a,b],h}^{(p+1)}$ , according to (81), and set*

$$L^{(p)} = \left\{ b_\alpha^{(p+1)} - a_\alpha^{(p)}, \alpha \in A_c^{(p)}(a, b) \right\},$$

$$\delta_f = \min\left(\frac{\eta_f}{8}, \frac{|\ell - \ell'|}{8}, \ell \neq \ell' \in L^{(p)} > 0\right).$$

Take the  $\delta_1$ -family of quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  given by Theorem 6.3 with  $\delta_1 = \frac{\eta_f}{8}$  (and with any  $\delta_2 \in ]0, \frac{\eta_f}{8}]$ ) and define, for  $\ell \in L^{(p)}$ ,

$$\mathcal{U}_\ell^{(p),h} := \text{Span}\left(\varphi_j^h, j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b), y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell\right),$$

and

$$\mathcal{U}_{+\infty}^{(p),h} := \text{Span}(\varphi_j^h, j \in \mathcal{Y}^{(p)}(a, b) \sqcup \mathcal{Z}^{(p)}(a, b)).$$

Then, for every  $\ell \in L^{(p)} \sqcup \{+\infty\}$  and  $p \in \{0, \dots, d\}$ , the distance between  $\mathcal{U}_\ell^{(p),h}$  and  $F_\ell^{(p),h}$  is estimated by

$$\vec{d}(\mathcal{U}_\ell^{(p),h}, F_\ell^{(p),h}) + \vec{d}(F_\ell^{(p),h}, \mathcal{U}_\ell^{(p),h}) = \tilde{O}(e^{-\frac{\delta_f}{h}}),$$

where  $F_\ell^{(p),h} \subset F_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)} \subset L^2(\mathcal{J}_a^b; \Lambda^p T^* M)$  is the spectral subspace of  $\delta_{[0,\bar{\sigma}(1)],[a,b],h}^{(p),*} \delta_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)}$  for the spectral range  $[e^{-2\frac{\ell+\delta_f}{h}}, e^{-2\frac{\ell-\delta_f}{h}}]$ .

*Proof.* With our choice  $\delta_1 = \frac{\eta_f}{8}$ , the basis  $(\varphi_j^h)_{j \in \mathcal{J}^{(p)}(a,b)}$  is a  $\tilde{O}(e^{-\frac{\eta_f}{8h}})$ -orthonormal family such that, according to Theorem 6.3-b) and to the definition of  $T_{\delta_2}$  (see Definition 6.2),

$$\forall j \in \mathcal{J}^{(p)}(a, b), \quad \|\Pi_{[0,\bar{\sigma}(1)],[a,b],h} T_{\delta_2} \varphi_j^h - \varphi_j^h\| = \tilde{O}(e^{-\frac{\eta_f}{8h}}). \quad (133)$$

For  $j \in \mathcal{Y}^{(p)}(a, b) \sqcup \mathcal{Z}^{(p)}(a, b)$ , the equality

$$\delta_{[0,\bar{\sigma}(1)],[a,b],h} \Pi_{[0,\bar{\sigma}(1)],[a,b],h} T_{\delta_2} \varphi_j^h = \Pi_{[0,\bar{\sigma}(1)],[a,b],h} d_{f,h} \varphi_j^h = 0$$

then implies that  $(\Pi_{[0,\bar{\sigma}(1)],[a,b],h} T_{\delta_2} \varphi_j^h)_{j \in \mathcal{Y}^{(p)}(a,b) \sqcup \mathcal{Z}^{(p)}(a,b)}$  is a  $\tilde{O}(e^{-\frac{\eta_f}{8h}})$ -orthonormal basis of

$$\ker(\delta_{[0,\bar{\sigma}(1)],[a,b],h}^{(p)}) = F_{+\infty}^{(p),h}.$$

This leads to the result for  $\ell = +\infty$  and initializes the decreasing induction with respect to  $\ell$ . Assume now that for all  $\ell > \ell_0$  in  $L^{(p)}$ , we have proved

$$\vec{d}(\mathcal{U}_\ell^{(p),h}, F_\ell^{(p),h}) + \vec{d}(F_\ell^{(p),h}, \mathcal{U}_\ell^{(p),h}) = \tilde{O}(e^{-\frac{\delta_f}{h}}).$$

Let us check that it is still true for  $\ell = \ell_0$ . Like in Subsection 6.3, we introduce  $G^h$  defined by (89),(90),  $G_n^h = \ker(\Delta_{f,f^{-1}([\bar{c}_n - \eta_f, \bar{c}_n + \eta_f] \cap [a,b]),h})$  defined in (103), and the spaces  $\mathcal{V}_{m,n}^h$  defined in (98) by

$$\mathcal{V}_{m,n}^{(p),h} = \text{Span}(\varphi_j^h, j \in \mathcal{X}_{m,n}^{(p)}(a, b)).$$

In particular, we have

$$\mathcal{U}_{\ell_0}^{(p),h} = \bigoplus_{\bar{c}_n - \bar{c}_m = \ell_0} \mathcal{V}_{m,n}^{(p),h},$$

while  $\Pi_{G^{h,(p+1)}} d_{f,h}^{(p)} T_{\delta_2} (\mathcal{V}_{m,n}^{(p),h}) \subset G_n^{h,(p+1)}$  with  $G_n^{h,(p+1)} \perp G_{n'}^{h,(p+1)}$  for  $n \neq n'$ . From Proposition 6.4, we know that the mapping  $\Pi_{G^{h,(p+1)}} d_{f,h}^{(p)} T_{\delta_2} : \mathcal{U}_{\ell_0}^{(p),h} \rightarrow G^{h,(p+1)}$  does not depend

on  $\delta_2 \in ]0, \frac{\eta f}{8}]$ , while Proposition 6.5 and Proposition 5.4-**b**) ensure that it is one to one with (only non zero) singular values all satisfying  $\mu_h \stackrel{\log}{\sim} e^{-\frac{\ell_0}{h}}$ . Moreover, following the analysis made in the proof of Proposition 6.4, the factorization (94) holds with here  $E^h = T_{\delta_2} \mathcal{U}_{\ell_0}^{(p),h}$ ,  $B_h = d_{f,f^{-1}([a,b]),h}^{(p)}$ , and  $\|\tilde{C}^h\| = \tilde{O}(e^{\frac{2\delta_2}{h}})$ . Hence, using Lemma 5.6 with the relation

$$\tilde{d}(G^{h,(p+1)}, F_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p+1)}) + \tilde{d}(F_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p+1)}, G^{h,(p+1)}) = \tilde{O}(e^{-\frac{\eta f}{h}})$$

leads to

$$\Pi_{G^{h,(p+1)}} d_{f,f^{-1}([a,b]),h}^{(p)} \Big|_{E^h} = (\text{Id}_{L^2(f_a^b)} + \tilde{O}(e^{\frac{2\delta_2 - \eta f}{h}})) \underbrace{\Pi_{[0,\tilde{\sigma}(1)],[a,b],h} d_{f,f^{-1}([a,b]),h}^{(p)} \Big|_{E^h}}_{=\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)} \Big|_{E^h}}.$$

Thus, since  $T_{\delta_2} : \mathcal{U}_{\ell_0}^{(p),h} \rightarrow E^h$  is  $\tilde{O}(e^{-\frac{\eta f}{h}})$ -unitary, the operator  $\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)} : T_{\delta_2} \mathcal{U}_{\ell_0}^{(p),h} \rightarrow F_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p+1)}$  is, as  $\Pi_{G^{h,(p+1)}} d_{f,h}^{(p)} T_{\delta_2} : \mathcal{U}_{\ell_0}^{(p),h} \rightarrow G^{h,(p+1)}$ , one to one with singular values all logarithmically equivalent to  $e^{-\frac{\ell_0}{h}}$ . In particular, for all  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b)$  such that  $y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell_0$ , we must have

$$\|\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)} \Pi_{[0,\tilde{\sigma}(1)],[a,b],h} T_{\delta_2} \varphi_j^h\| \stackrel{\log}{\sim} e^{-\frac{\ell_0}{h}}.$$

From the previous estimates, the new family of vectors  $(u_j^h)$  defined by

$$u_j^h = (1 - \sum_{\ell > \ell_0} \Pi_{F_\ell^{(p),h}}) \Pi_{[0,\tilde{\sigma}(1)],[a,b],h} T_{\delta_2} \varphi_j^h$$

and indexed by  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b)$ ,  $y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell_0$  satisfies

$$\langle u_j^h, \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{*,(p)} \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)} u_j^h \rangle \stackrel{\log}{\sim} e^{-2\frac{\ell_0}{h}}, \quad (134)$$

$$u_j^h \perp \text{Ran } 1_{[0, e^{-2\frac{\ell_0 + \delta_f}{h}}]} (\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{*,(p)} \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)}), \quad (135)$$

$$\text{and } \|u_j^h - \varphi_j^h\| = \tilde{O}(e^{-\frac{\delta_f}{h}}). \quad (136)$$

Note that (134) and (135) follow easily from the definition of the family  $(u_j^h)$ , while (136), which also implies the  $\tilde{O}(e^{-\frac{\delta_f}{h}})$ -orthonormality of the family  $(u_j^h)$ , follows from (133) together with the estimate, for  $\ell > \ell_0$  and  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b)$ ,  $y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell_0$ ,

$$\begin{aligned} \Pi_{F_\ell^{(p),h}} \varphi_j^h &= (\Pi_{F_\ell^{(p),h}} - \Pi_{F_\ell^{(p),h}} \Pi_{\mathcal{U}_\ell^{(p),h}}) \varphi_j^h + \Pi_{F_\ell^{(p),h}} \Pi_{\mathcal{U}_\ell^{(p),h}} \varphi_j^h \\ &= \tilde{O}(e^{-\frac{\delta_f}{h}}) + \tilde{O}(e^{-\frac{\eta f}{8h}}) \leq \tilde{O}(e^{-\frac{\delta_f}{h}}), \end{aligned}$$

where the last line follows from the induction hypothesis and from the  $\tilde{O}(e^{-\frac{\eta f}{8h}})$ -orthonormality of the family  $(\varphi_j^h)_{j \in \mathcal{J}^{(p)}(a,b)}$ . The relations (134) and (136) imply that the vector

$$v_j^h = 1_{[0, e^{-2\frac{\ell_0 - \delta_f}{h}}]} (\delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{*,(p)} \delta_{[0,\tilde{\sigma}(1)],[a,b],h}^{(p)} u_j^h)$$

satisfies

$$\|v_j^h - u_j^h\| = \tilde{O}(e^{-\frac{\delta_f}{h}}) \quad \text{and thus} \quad \|v_j^h - \varphi_j^h\| = \tilde{O}(e^{-\frac{\delta_f}{h}}),$$

while (135) yields

$$v_j^h \in F_{\ell_0}^{(p),h}.$$



Hence, we have proved  $\vec{d}(\mathcal{U}_{\ell_0}^{(p),h}, F_{\ell_0}^{(p),h}) = \tilde{O}(e^{-\frac{\delta_f}{h}})$  and thus, using

$$\dim \mathcal{U}_{\ell_0}^{(p),h} = \# \left\{ j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b), y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell_0 \right\} = \dim F_{\ell_0}^{(p),h},$$

implies  $\vec{d}(F_{\ell_0}^{(p),h}, \mathcal{U}_{\ell_0}^{(p),h}) + \vec{d}(\mathcal{U}_{\ell_0}^{(p),h}, F_{\ell_0}^{(p),h}) = \tilde{O}(e^{-\frac{\delta_f}{h}})$ . This ends the proof of Proposition 7.2.  $\square$

Now quasimodes have been constructed for  $d_{f, f^{-1}([a,b]), h}$ , the dual version can be given. Remember that

$$d_{f,h}^* = (-1)^{\deg} \star^{-1} e^{\frac{f}{h}} (hd) e^{-\frac{f}{h}} \star$$

and the construction of  $\delta_1$ -quasimodes for  $d_{f, f^{-1}([a,b]), h}^*$  is equivalent to the construction of  $\delta_1$ -quasimodes for  $d_{-f, (-f)^{-1}([-b, -a]), h}$ , where the fiber bundle  $\Lambda T^*M$  is replaced by  $\Lambda T^*M \otimes_{\text{or}M}$ . Accordingly, the degree  $p$  is changed into  $d - p$ , the order of critical values is reversed and, in the interval  $[a, b]$ , the role of lower and upper endpoints in the sets  $\mathcal{X}^*(a, b)$  and  $\mathcal{Y}^*(a, b)$  are interchanged.

**Definition 7.3.** Under Hypothesis 4.1 and with  $\delta_1 \in ]0, \frac{\eta_f}{8}]$ , a dual  $\delta_1$ -family of quasimodes denoted by  $\left( \widehat{\varphi}_j^{*,h} \right)_{j \in \mathcal{J}(a,b)}$  is defined like the family  $\left( \varphi_j^{*,h} \right)_{j \in \mathcal{J}(a,b)}$  in Definition 6.1 after replacing:

- $d_{f, f^{-1}([a,b]), h}$  by  $d_{f, f^{-1}([a,b]), h}^*$ ,
- $I_j^h = [x_\alpha^{(p)} - \delta_1, y_\alpha^{(p+1)} - \gamma(h)]$  when  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a, b)$  by

$$\widehat{I}_j^h = [x_\alpha^{(p-1)} + \gamma(h), y_\alpha^{(p)} + \delta_1] \quad \text{when } j = (\alpha, y_\alpha^{(p)}) \in \mathcal{Y}^{(p)}(a, b),$$

- and  $I_j^h = [\tilde{c} - \delta_1, b]$  when  $j = (\alpha, \tilde{c}) \in \mathcal{Y}^{(p)}(a, b) \sqcup \mathcal{Z}^{(p)}(a, b)$  by

$$\widehat{I}_j^h = [a, \tilde{c} + \delta_1] \quad \text{when } j = (\alpha, \tilde{c}) \in \mathcal{X}^{(p)}(a, b) \sqcup \mathcal{Z}^{(p)}(a, b).$$

Finally, the truncation operator  $T_{\delta_2}$  introduced for  $\delta_2 \in ]0, \frac{\eta_f}{8}]$  in Definition 6.2 has to be replaced by  $\widehat{T}_{\delta_2}$  defined by

$$\widehat{T}_{\delta_2} \widehat{\varphi}_j^{(p),h} = \begin{cases} \chi_{x_\alpha^{(p-1)}, \delta_2} \widehat{\varphi}_j^{(p),h} & \text{if } j = (\alpha, y_\alpha^{(p)}) \in \mathcal{Y}^{(p)}(a, b) \\ \varphi_j^{(p),h} & \text{if } j \in \mathcal{X}^{(p)}(a, b) \cup \mathcal{Z}^{(p)}(a, b), \end{cases}$$

where  $\widehat{\chi}_{\tilde{c}, \delta_2}(x) = \widehat{\chi} \left( \frac{f(x) - \tilde{c}}{\delta_2} \right)$ ,

for a fixed  $\widehat{\chi} \in C^\infty(\mathbb{R}; [0, 1])$  such that  $\widehat{\chi} \equiv 1$  on  $[2, +\infty[$  and  $\text{supp } \widehat{\chi} \subset ]1, +\infty[$ .

**Theorem 7.4.** Like in Theorem 7.1, assume Hypothesis 1.2 or (Hypothesis 1.6 and Hypothesis 2.16) for a more general  $f$ , which is equivalent to Hypothesis 4.1 when the definition of  $\eta_f > 0$  is added. Let  $a, b \notin \{c_1, \dots, c_{N_f}\}$  and let  $\Delta_{f, f^{-1}([a,b]), h} = \bigoplus_{p=0}^d \Delta_{f, f^{-1}([a,b]), h}^{(p)}$  be defined like in Proposition 2.8 with  $N_t = f^{-1}(\{a\})$  and  $N_n = f^{-1}(\{b\})$ . We set, like in Proposition 7.2,

$$L^{(p)} = \left\{ b_\alpha^{(p+1)} - a_\alpha^{(p)}, \alpha \in A_c^{(p)}(a, b) \right\}$$

$$\text{and } \delta_f = \min\left(\frac{\eta_f}{8}, \frac{|\ell - \ell'|}{8}, \ell \neq \ell' \in L^{(p)}\right) > 0.$$

The  $\delta_1$ -family of quasimodes  $(\varphi_j^{*,h})_{j \in \mathcal{J}(a,b)}$  is given by Theorem 6.3 with  $\delta_1 = \frac{\eta f}{8}$ , and its dual version  $(\widehat{\varphi_j^{*,h}})_{j \in \mathcal{J}(a,b)}$  by Definition 7.3. For  $\ell \in L^{(p)}$ , we define lastly

$$\begin{aligned} \overline{\mathcal{U}}_\ell^{(p),h} &:= \mathcal{U}_\ell^{(p),h} \oplus \widehat{\mathcal{U}}_\ell^{(p),h}, \\ \text{where } \mathcal{U}_\ell^{(p),h} &= \text{Span} \left( \varphi_j^h, j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a,b), y_\alpha^{(p+1)} - x_\alpha^{(p)} = \ell \right) \\ \text{and } \widehat{\mathcal{U}}_\ell^{(p),h} &= \text{Span} \left( \widehat{\varphi_j^h}, j = (\alpha, y_\alpha^{(p)}) \in \mathcal{Y}^{(p)}(a,b), y_\alpha^{(p)} - x_\alpha^{(p-1)} = \ell \right). \end{aligned}$$

Then, for every  $\ell \in L^{(p)}$ , the space  $\overline{\mathcal{U}}_\ell^{(p),h}$  is close to  $F_{[e^{-2\frac{\ell+\delta f}{h}}, e^{2\frac{\ell-\delta f}{h}}], [a,b], h}^{(p)}$  according to

$$\vec{d} \left( \overline{\mathcal{U}}_\ell^{(p),h}, F_{[e^{-2\frac{\ell+\delta f}{h}}, e^{2\frac{\ell-\delta f}{h}}], [a,b], h}^{(p)} \right) + \vec{d} \left( F_{[e^{-2\frac{\ell+\delta f}{h}}, e^{2\frac{\ell-\delta f}{h}}], [a,b], h}^{(p)}, \overline{\mathcal{U}}_\ell^{(p),h} \right) = \tilde{O}(e^{-\frac{\delta f}{h}}).$$

*Proof.* Let us first recall the relation

$$\Delta_{f, f^{-1}([a,b]), h}^{(p)} \Pi_{[0, \tilde{o}(1)], [a,b], h}^{(p)} = \underbrace{\delta_{[0, \tilde{o}(1)], [a,b], h}^{(p-1)}}_A \delta_{[0, \tilde{o}(1)], [a,b], h}^{(p-1),*} + \underbrace{\delta_{[0, \tilde{o}(1)], [a,b], h}^{(p),*}}_B \delta_{[0, \tilde{o}(1)], [a,b], h}^{(p)},$$

where  $A$  and  $B$  are self-adjoint and satisfy  $AB = BA = 0$ . We deduce from this observation and from the Hodge decomposition that, for  $\lambda_h \neq 0$ ,  $\lambda_h = \tilde{o}(1)$ ,

$$\ker(\Delta_{f, f^{-1}([a,b]), h} - \lambda_h) = \ker(A - \lambda_h) \oplus \ker(B - \lambda_h).$$

Moreover Proposition 7.2 says

$$\vec{d}(\mathcal{U}_\ell^{(p),h}, F_\ell^{(p),h}) + \vec{d}(F_\ell^{(p),h}, \mathcal{U}_\ell^{(p),h}) = \tilde{O}(e^{-\frac{\delta f}{h}}),$$

where

$$F_\ell^{(p)} = \bigoplus_{e^{-2\frac{\ell+\delta f}{h}} \leq \lambda_h \leq e^{-2\frac{\ell-\delta f}{h}}} \ker(B - \lambda_h).$$

The proximity of  $\widehat{\mathcal{U}}_\ell^{(p),h}$  to  $\bigoplus_{e^{-2\frac{\ell+\delta f}{h}} \leq \lambda_h \leq e^{-2\frac{\ell-\delta f}{h}}} \ker(A - \lambda_h)$  is the dual version.  $\square$

**Remark 7.5.** The last result about the eigenvectors of  $\Delta_{f, f^{-1}([a,b]), h}$  arouses several comments.

- When there is a single bar  $\alpha \in A_c^{(p)}(a,b)$  with length  $\ell$ , then  $\Delta_{f, f^{-1}([a,b]), h}^{(p)}$  (resp.  $\Delta_{f, f^{-1}([a,b]), h}^{(p+1),h}$ ) has one eigenvector associated with the eigenvalue  $\lambda_h \stackrel{\log}{\sim} e^{-\frac{2\ell}{h}}$  localized around  $f^{-1}(x_\alpha^{(p)})$  (resp.  $f^{-1}(y_\alpha^{(p+1)})$ ) and  $\tilde{O}(e^{-\frac{\delta f}{h}})$ -close to the corresponding quasimode  $\varphi_j^{(p),h}$  (resp.  $\varphi_j^{(p+1),h}$ ) with  $j = (\alpha, x_\alpha^{(p)}) \in \mathcal{X}^{(p)}(a,b)$  (resp.  $j = (\alpha, y_\alpha^{(p+1)}) \in \mathcal{Y}^{(p+1)}(a,b)$ ).
- Once we have approximated the eigenvectors associated with the non zero eigenvalues by the quasimodes  $\varphi_j^h$  or  $\widehat{\varphi_j^h}$ , one can recover an approximate description of  $\ker(\Delta_{f, f^{-1}([a,b]), h})$  by considering a basis of  $\text{Span}(\varphi_j^h, j \in \mathcal{Y}(a,b) \sqcup \mathcal{Z}(a,b))$  whose elements are  $\tilde{O}(e^{-\frac{\delta_1}{h}})$ -orthogonal to all the  $\widehat{\varphi_{j'}^h}$ ,  $j' \in \mathcal{Y}(a,b)$ .
- Actually, the description of the eigenvectors with a  $\tilde{O}(e^{-\frac{\delta f}{h}})$  error in the  $L^2$ -norm is much less precise than what we were able to do with the quasimodes  $\varphi_j^{*,h}$ , with a wide range control of the exponential decay estimates. We also know from the proof of Theorem 6.3, and this is again illustrated in the proof of Proposition 7.2, that working with the family

of quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  is much more flexible and informative than working with the eigenvectors of  $\Delta_{f,f^{-1}([a,b]),h}$ . Note specifically, in the proof of Proposition 7.2, the use of the orthogonality  $G_n^h \perp G_{n'}^h$  for  $n \neq n'$  in the separation of the different exponential scales associated with the different lengths of bars. This really relies on the fact that  $G^h$  is made of kernels of separated local problems. Such an exact property is completely lost if we use instead the full spectral space  $F_{[0,\bar{\sigma}(1)],[a,b],h}$ .

- From the modeling interpretation, it is interesting to note that the quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  carry the same heuristic as the true eigenvectors for small times although they do not belong to  $D(\Delta_{f,f^{-1}([a,b]),h})$ . For simplicity, assume that there is a single bar  $\alpha \in A_c^{(p)}(a,b)$  with length  $\ell$ . Then  $\varphi_j^h$ ,  $j = (\alpha, x_\alpha) \in \mathcal{X}(a,b)$ , satisfies

$$\begin{aligned} \|e^{-t\Delta_{f,f^{-1}([a,b]),h}} \varphi_j^h - e^{-t\lambda_h} \varphi_j^h\| &= \|(e^{-t\Delta_{f,f^{-1}([a,b]),h}} - e^{-t\lambda_h})(\varphi_j^h - u_h)\| \\ &\leq 2\|\varphi_j^h - u_h\| = \tilde{O}(e^{-\frac{\delta f}{h}}), \end{aligned}$$

where  $u_h$  is the unitary eigenvector associated with the eigenvalue  $\lambda_h \stackrel{\log}{\sim} e^{-2\frac{\ell}{h}}$ . In particular,  $e^{-t\Delta_{f,f^{-1}([a,b]),h}} \varphi_j^h \sim e^{-t\lambda_h} \varphi_j^h$  makes sense for times longer than the lifetime  $\frac{1}{\lambda_h} \stackrel{\log}{\sim} e^{2\frac{\ell}{h}}$  of the metastable state  $u_h$  as  $h \rightarrow 0$ .

## 7.2 Stability theorem

The following stability theorem, of which a simple version, Corollary 1.8, was given in the introduction, is a direct consequence of Theorem 7.1 and of the topological stability result

$$d_{\text{bot}}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_{C^0}$$

recalled in Appendix B.3.

**Theorem 7.6.** *In the framework of Theorem 7.1, namely Hypothesis 1.2, or (Hypothesis 1.6 and Hypothesis 2.16) for a more general Lipschitz function  $f$ , and  $a, b \notin \{c_1, \dots, c_{N_f}\}$ , set*

$$\ell_{\min} := \min \left( \{y_\alpha - x_\alpha, \alpha \in A_c(a,b)\} \cup \{|c_n - b|, |c_n - a|, 1 \leq n \leq N_f\} \right),$$

where  $A_c(a,b) = A_c(f; a,b)$  is the set defined in (53) for the function  $f$ , that is indexing the bars of  $f$  with two endpoints in  $]a,b[$ .

Let moreover  $g$  be any other function satisfying Hypothesis 1.2, or (Hypothesis 1.6 and Hypothesis 2.16), as well as

$$\|g - f\|_{C^0} < \frac{\ell_{\min}}{4},$$

and such that  $a, b$  do not belong to the set  $\{c'_1, \dots, c'_{N_g}\}$  made of its “critical values”.

Then, the  $\tilde{O}(e^{-\frac{\ell_{\min}}{h}})$  non zero eigenvalues of  $\Delta_{g,g^{-1}([a,b]),h}^{(p)}$  can be labelled

$$\lambda_\alpha^{(p)}(g; h) \quad , \quad \alpha \in A_c^{(p)}(a,b) \sqcup A_c^{(p-1)}(a,b),$$

with, for every  $\alpha \in A_c^{(p)}(a,b) \sqcup A_c^{(p-1)}(a,b)$ ,

$$\ell_{\min} < 2(y_\alpha^{*+1} - x_\alpha^*) - 4\|g - f\|_{C^0} \leq \lim_{h \rightarrow 0} -h \log(\lambda_\alpha^{(p)}(g, h)) \leq 2(y_\alpha^{*+1} - x_\alpha^*) + 4\|g - f\|_{C^0}.$$

Meanwhile, for  $\mathbf{f} = f$  or  $\mathbf{f} = g$ , the dimension  $\dim \ker(\Delta_{\mathbf{f},\mathbf{f}^{-1}([a,b]),h}^{(p)})$  equals  $\beta^{(p)}(\mathbf{f}^b, \mathbf{f}^a)$ , and thus

$$\dim \ker(\Delta_{f,f^{-1}([a,b]),h}^{(p)}) = \ker(\Delta_{g,g^{-1}([a,b]),h}^{(p)}) \quad \text{if and only if} \quad \beta^{(p)}(f^b, f^a) = \beta^{(p)}(g^b, g^a).$$

*Proof.* After possibly adding empty bars, the bar codes associated with  $f$  and  $g$  can be written  $\mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A})$  and  $\mathcal{B}(g) = ([c_\alpha, d_\alpha]_{\alpha \in A})$ , where

$$\max\{|a_\alpha - c_\alpha|, |d_\alpha - b_\alpha|, \alpha \in A, b_\alpha < +\infty\} \leq d_{\text{bot}}(\mathcal{B}(g), \mathcal{B}(f)) \leq \|g - f\|_{C^0} < \frac{\ell_{\min}}{4}.$$

The definition

$$\ell_{\min} := \min(\{y_\alpha - x_\alpha, \alpha \in A_c(f; a, b)\} \cup \{|c_n - a|, |c_n - b|, 1 \leq n \leq N_f\})$$

implies that the number of bars  $\alpha \in A_c(g; a, b)$  such that  $y_\alpha - x_\alpha > \frac{\ell_{\min}}{2}$ , for the function  $g$ , is in bijection with the whole set of bars  $A_c(f; a, b)$  for the function  $f$ , which is made by assumption of bars not smaller than  $\ell_{\min}$ . The other potential bars of  $A_c(g; a, b)$  have a length strictly smaller than  $\frac{\ell_{\min}}{2}$ .

Moreover, for  $\alpha \in A_c(a, b)$ , the expression of  $\lim_{h \rightarrow 0} -h \log \lambda_\alpha^{(p)}(h)$  given in Theorem 7.1, respectively applied with  $g$  and  $f$ , provides the inequalities for the  $\tilde{O}(e^{-\frac{2(\ell_{\min}/2)}{h}})$  non zero eigenvalues of  $\Delta_{g, g^{-1}([a, b]), h}$ .

Finally, the last statement of Theorem 7.6 is a direct consequence of the comments made in the second item of Remark 2.9.  $\square$

## 8 Generalizations

Our proofs are definitely done under Hypothesis 1.2, while, for a more general Lipschitz function  $f$ , consequences of Hypothesis 1.6 have not yet been checked and the exponential decay estimates of Propositions 2.13 and 2.15 have simply been replaced by assumptions.

This framework was chosen in order to put the stress on the essentially one-dimensional analysis on  $\mathbb{R} \supset f(M)$ . Once this is well understood, it is rather easy to adapt the analysis and the results in order to consider more general domains, manifolds, or Lipschitz functions  $f$ . The first generalizations will be presented for the sake of simplicity in the framework of Hypothesis 1.2. Additionally, we will check that Hypothesis 1.6 and Hypothesis 2.16 hold true under the simple assumption that  $f$  is a subanalytic Lipschitz function (see Hypothesis 1.3), which describes, in some sense, a wider class of functions than Hypothesis 1.2 in a real analytic geometry.

### 8.1 More general domains

It is not difficult to adapt all the analysis to some simple cases when the geometrical domain  $\bar{\Omega}$  differs from  $f^{-1}([a, b])$  by tamed deformations of  $\partial\Omega$ .

**Proposition 8.1.** *Let  $(M, g)$  be a compact Riemannian manifold and let  $f$  satisfy Hypothesis 1.2. If there exist  $m_0, n_0 \in \{1, \dots, N_f\}$  such that  $m_0 < n_0$  and the boundary of the domain  $\bar{\Omega} = \Omega \sqcup N_t \sqcup N_n$  satisfies*

$$f(N_t) \subset ]c_{m_0}, c_{m_0+1}[ \quad , \quad f(N_n) \subset ]c_{n_0}, c_{n_0+1}[ ,$$

and  $\frac{\partial f}{\partial n}|_{N_t} < 0 \quad , \quad \frac{\partial f}{\partial n}|_{N_n} > 0 .$

then all the results of Theorem 6.3 hold true with  $\tilde{c}_1 = c_{m_0+1}$ ,  $\tilde{c}_N = c_{n_0}$  when  $\eta_f$  is chosen in the interval

$$0 < \eta_f < \frac{1}{2} \min_{1 < n \leq N_f} c_n - c_{n-1} ,$$

and  $\eta_f < \min_{x \in N_t} (c_{m_0+1} - f(x)) \quad , \quad \eta_f < \min_{x \in N_n} (f(x) - c_{n_0}) .$

*Proof.* All the proof of Theorem 6.3 relies on the construction of the  $\delta_1$ -family of quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  when  $\overline{\Omega} = f^{-1}([a,b])$ . We fix  $a = c_{m_0+1} - \eta_f = \tilde{c}_1 - \eta_f$  and  $b = c_{n_0} + \eta_f = \tilde{c}_N + \eta_f$ . Because the gradient lines provide a homotopy between the pairs  $(\overline{\Omega}, N_t)$  and  $(f^{-1}([a,b]), f^{-1}\{a\})$ , the bar code for  $f$  in  $\overline{\Omega}$  relatively to  $N_t$  can be identified with  $\mathcal{B}(f; [a,b])$ . Now, the quasimodes  $(\varphi_j^h)_{j \in \mathcal{J}(a,b)}$  are extended by 0 in  $f^a \cap \overline{\Omega}$  and, when  $j \in \mathcal{Y}(a,b) \cup \mathcal{Z}(a,b)$ , they are “extended” in  $f_b \cap \Omega$  as

$$\chi \varphi_j^h - d_{f,f^{-1}([c_{n_0} + \delta_1, +\infty[\cap \overline{\Omega}], h)}^* (\Delta_{f,f^{-1}([c_{n_0} + \delta_1, +\infty[\cap \overline{\Omega}], h)}^{-1} (hd\chi \wedge \varphi_j^h),$$

like in Proposition 3.9-ii), with  $\delta_1 \in ]0, \frac{\eta_f}{8}]$ ,  $\chi \in \mathcal{C}^\infty(M; [0,1])$ ,  $\chi \equiv 1$  in  $f^{b-\eta_f/2}$ ,  $\chi \equiv 0$  in  $f_{b-\eta_f/4}$ , and where Dirichlet (resp. Neumann) boundary conditions are put on  $f^{-1}(\{c_{n_0} + \delta_1\})$  (resp. on  $N_n$ ), for the domain  $f^{-1}([c_{n_0} + \delta_1, +\infty[\cap \overline{\Omega})$   $\square$

**Remark 8.2.** *Another interesting case is when the Neumann boundary conditions on  $N = N_n$ , where  $\frac{\partial f}{\partial n} > 0$ , are replaced by Dirichlet boundary conditions. Then, generalized critical values corresponding to critical values of  $f|_N$  appear following what is known for a Morse function  $f$  (see e.g. [ChLi, HeNi, Lep1, LeNi, Lau]). As a topological tool, bar codes make sense for boundary manifolds. But the analysis has to be reconsidered from the beginning, especially by introducing mixed Dirichlet-Neumann problems along the upper boundary of  $\overline{\Omega} \cap f^{\leq t}$ . We do not develop this point here (see however [DLLN1] where such conditions are considered).*

## 8.2 More general manifolds

The following generalization aims at including the particular case when  $M = \mathbb{R}^d$  is not compact and the gradient of  $f$  does not vanish outside a compact set. More specifically, we assume

**Hypothesis 8.3.** *Let  $(M, g)$  be a complete Riemannian manifold and assume  $f \in \mathcal{C}^\infty(M; \mathbb{R})$  for the sake of simplicity. We suppose that there exist  $-\infty < a_0 < b_0 < +\infty$  and  $\kappa > 0$  such that*

- $K_0 = f^{-1}([a_0, b_0])$  is compact,
- for all  $x \in M \setminus K_0$ ,  $|\nabla f(x)| \geq \kappa$ ,
- $f$  has a finite number of critical values  $c_1, \dots, c_{N_f}$  in  $[a_0, b_0]$  which belong to  $]a_0, b_0[$ .

Under this assumption, the definition of the bar code  $\mathcal{B}(f) = ([a_\alpha^*, b_\alpha^{*+1}])_{\alpha \in A}$  is essentially the same as in the compact case, except that bars with  $a_\alpha^* = -\infty$  and  $b_\alpha^{*+1} \in \mathbb{R}$  are possible, according to the topology of  $f^t$  as  $t \rightarrow -\infty$ . In such a case,  $b_\alpha^{*+1} \in \mathcal{Z}^{*+1}(a, b)$  for all  $a, b \in [-\infty, +\infty] \setminus \{c_1, \dots, c_{N_f}\}$  such that  $a < b_\alpha^{*+1} < b$ . The domain  $f^{-1}([a, b]) \subset M$  is actually  $f^{-1}([a, b] \cap ]-\infty, +\infty[)$  when  $a = -\infty$  or  $b = +\infty$ . Accordingly,  $\Delta_{f,f^{-1}([a,b]),h}$ ,  $d_{f,f^{-1}([a,b]),h}$ , and  $d_{f,f^{-1}([a,b]),h}^*$  do not include boundary conditions on the infinite end in the definition of their domains.

**Proposition 8.4.** *Under Hypothesis 8.3, all the results of Theorem 6.3 still hold.*

*Proof.* The completeness of the manifold ensures that the scalar Laplacian is essentially self-adjoint on  $\mathcal{C}_0^\infty(M)$ . Adapting the proof of Simader’s theorem ensures that  $\Delta_{f,h}$  is essentially self-adjoint on  $M$  and that  $\Delta_{f,f^{-1}([a,b]),h}$  is essentially self-adjoint on the subspace of  $\mathcal{C}_0^\infty(f^{-1}([a,b]); \Lambda T^*M)$  containing the boundary conditions, of Dirichlet type on  $f^{-1}(\{a\})$  when  $-\infty < a$  and of Neumann type on  $f^{-1}(\{b\})$  when  $b < +\infty$ .

Agmon estimates and the compactness of  $K_0 = f^{-1}([a_0, b_0])$  with  $|\nabla f| \geq \kappa > 0$  in  $M \setminus K_0$  implies that the solutions to  $\Delta_{f,f^{-1}([a,b]),h} \omega_h = \lambda_h \omega_h$  with  $\lambda_h \rightarrow 0$  as  $h \rightarrow 0$  must satisfy

$$\|e^{\frac{\kappa d_g(x, K_0)}{h}} \omega_h\|_{W^{1,2}} \leq \tilde{O}(1).$$

One can then localize the analysis of exponentially small eigenvalues to  $K'_0 = f^{-1}([a_0 - 1, b_0 + 1])$ , which amounts to the case of a compact manifold treated in Theorem 6.3.  $\square$

### 8.3 More general Lipschitz functions

We consider more accurately the situation of a general Lipschitz function  $f$ , while the analysis was presented under conjectural assumption. As a first step we recall in Subsection 8.3.1 how Hypothesis 1.6 implies Hypothesis B.1 of Appendix B and therefore provides a finite bar code  $\mathcal{B}_f$ .

Once this is clarified we prove that Hypothesis 1.6 and Hypothesis 2.16 are satisfied when  $f$  is a subanalytic Lipschitz function, after the suitable specification of the “critical values”,  $c_1 < \dots < c_{N_f}$ . It relies on the stratification of the subanalytic graph of  $f$ , of which the properties are recalled in Subsection 8.3.2. A variation of Agmon distance will also be constructed after solving the Hamilton-Jacobi equation  $|\nabla'\varphi| = |\nabla'f|$ , where  $\nabla'$  concerns only tangential partial derivatives in some tubular neighborhoods of every stratum. From this point of view, the analysis of this Lipschitz subanalytic case, via a stratification technique, takes some inspiration from [GeNi]. Finally in Subsubsection 8.3.3, Hypothesis 2.16 is checked to hold true, via some partition of unity adapted with the stratification.

#### 8.3.1 Hypothesis 1.6 and consequences

The manifold  $M$  is assumed to be compact without boundary although it could be extended to more general cases like in Subsection 8.2.

Let us first define the critical values of a Lipschitz function  $f$  or more exactly, its contrary.

**Definition 8.5.** *When  $f : M \rightarrow \mathbb{R}$  is a Lipschitz function a value  $a$  is not a critical value if for any  $x_0 \in f^{-1}(\{a\})$  there exists a neighborhood  $U_{x_0}$  of  $x_0$  and a local coordinate system  $x = (x^1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$  and a constant  $C_{x_0} > 0$  such that*

$$\forall x = (x^1, x'), y = (y^1, x') \in U_{x_0}, \quad \frac{1}{C_{x_0}} |x^1 - y^1| \leq |f(x^1, x') - f(y^1, x')|. \quad (137)$$

A critical value  $a \in f(M) \subset \mathbb{R}$  is a point where the above property fails.

Since the function  $f$  is continuous, the local condition condition (137) can be replaced by

$$\forall x = (x^1, x'), y = (y^1, x') \in U_{x_0}, \quad \frac{1}{C_{x_0}} (x^1 - y^1) \leq f(x^1, x') - f(y^1, x') \quad \text{when } x^1 > y^1.$$

Hypothesis 1.6 simply says that the Lipschitz function  $f$  has a finite number of critical values. But the set  $\{c_1, \dots, c_{N_f}\}$  of Hypothesis 1.6 may be strictly larger than the set of critical values as defined above, and this is a reason why the values  $c_1, \dots, c_{N_f}$  were called “critical values”. Actually this flexibility is especially useful when we consider subanalytic Lipschitz functions below.

The above definition ensures that the implicit functions theorem in the Lipschitz case can be applied locally around  $x \in f^{-1}(\{a, b\})$  with the following straightforward consequences for the domain  $f_a^b$  when  $a, b$  are not “critical values”:

- i)  $f_a^b$  is a strongly Lipschitz domain of  $M$  according to the terminology of [GMM], meaning that it is locally the hypograph of a Lipschitz function in the proper coordinate system.
- ii)  $\overline{f_a^b} = f^{-1}([a, b])$ .
- iii) When  $a = -\infty$ ,  $f^b$  with  $c < b < c'$  and no critical values in  $]c, c'[$ , is homotopic to  $\Omega$  a  $\mathcal{C}^\infty$  domain with  $\partial\Omega \subset f_c^{c'}$ .

The last statement can be checked by using finitely many local homotopies in coordinate systems, but one could also use the global construction of a smooth transverse vector field as proposed in [Ver]-Theorem 1.12-vi).

The above three properties were used in our analysis. In particular the finiteness of  $N_f$  and **iii)**

appear in Hypothesis B.1 which allows the introduction of a finite bar code  $\mathcal{B}_f$ . The properties **i)** and **ii)** are used in the definition of  $\Delta_{f,f^{-1}([a,b]),h}$  according to Proposition 2.8

Critical points and values can actually be defined in a coordinate free way, in terms of the standard notion in non smooth analysis of Clarke's generalized gradient and Clarke's critical points: In  $\mathbb{R}^d$  or locally in a coordinate system in  $M$ , a Lipschitz function admits a differential  $df(x)$  almost every where by Rademacher's theorem and the domain  $Dom(df)$  is the set of  $x$  where  $df(x)$  exists. Clarke's generalized gradient at  $x$  then equals the closed convex set

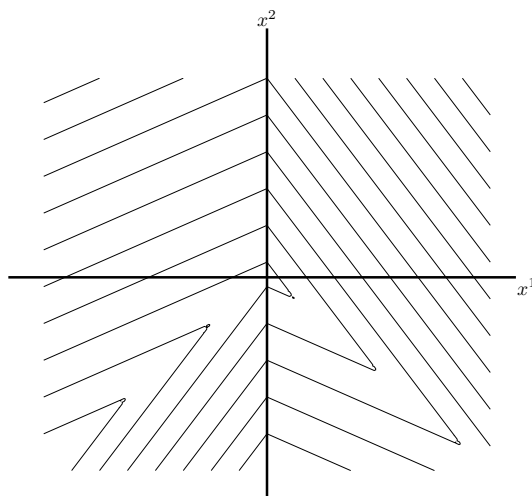
$$\partial^\circ f(x) = \text{co} \left\{ \zeta \in \mathbb{R}^d, \exists (x_n)_{n \in \mathbb{N}} \in Dom(df)^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} df(x_n) = \zeta \right\}$$

where  $\text{co}$  denotes the convex hull. A Clarke critical point  $x$  is a point where  $0 \subset \partial^\circ f(x)$  and a Clarke critical value of  $f$  is a value  $a$  where  $f^{-1}(\{a\})$  contains a critical points. In the case of subanalytic Lipschitz functions which will be considered more specifically in the other paragraphs, this definition actually coincides with the wavefront naturally introduced in [DeLe]. Staying at the local level the local condition (137) for  $x_0 \in f^{-1}\{a\}$ , actually means that for all  $x \in Dom(df) \cap U_{x_0}$ ,  $df(x)$  lies in the intersection of some closed salient ( $\zeta \neq 0$  and  $-\zeta$  cannot both belong to it) convex cone  $\mathcal{C}_{x_0}$  with a shell  $S_{x_0} = \{\zeta \in \mathbb{R}^d, r < |\zeta| \leq R\}$ ,  $0 < r < R < +\infty$ . This writing is equivalent to the fact that for all  $x \in f^{-1}(\{a\})$ , Clarke's generalized gradient is included in the intersection of a salient convex cone and a closed shell. This property is independent of the coordinate system and of the metric if we replace the differential  $df$  by the gradient  $\nabla f$ .

Even in the subanalytic setting, those critical values (according to Definition 8.5 or Clarke) may overestimate what the intuition and even the final result would retain. Warga's example carefully analyzed in [CzRi],

$$f(x^1, x^2) = ||x^1| + x^2| + \frac{1}{2}x^1,$$

with the level curves in the picture below, satisfies the above consequences **i),ii)** and **iii)** for any value  $b \in \mathbb{R}$  although 0 is a critical value of  $f$ . Note also that 0 will be a critical value of non well chosen regularizations of  $f$  and we refer to [CzRi] for a thorough discussion of this point.



**Figure 12:** Level curves of Warga's function  $f(x^1, x^2) = ||x^1| + x^2| + \frac{1}{2}x^1$ .



Actually in the subanalytic setting an even larger, but still finite, set of values  $\{c_1, \dots, c_{N_f}\}$  will be introduced in order to verify the second assumption, Hypothesis 2.16, used in our analysis.

### 8.3.2 Stratification of Lipschitz subanalytic functions

According to [BDLS] a Lipschitz subanalytic function has a finite number of critical values and Hypothesis 1.6 holds true. We also recalled in the previous paragraph that Clarke's gradient coincides with the wavefront set of subanalytic Lipschitz functions introduced in [DeLe]. However such a notion of gradient or wavefront above a point  $x \in M$ , is a wide closed convex set which contains all the convex combinations of limits of neighboring gradients without discriminating the information which can be deduced from the stratified structure. We specify the corresponding constructions when  $f$  is a real subanalytic Lipschitz function on a real analytic compact Riemannian manifold  $M$  according to Hypothesis 1.3.

Let us first remind the basic notions about subanalytic sets and functions. We refer the reader to the founding articles [Hardt][Hiro] and to [Loja][BiMi] for a panoramic or historical presentation. A part but not all of the material, presented or recalled here, may be found in [DeLe] for the specific case of subanalytic Lipschitz functions.

#### Review of subanalytic notions and results:

- In the real analytic category, the class of subanalytic sets is the one which contains the semianalytic sets, characterized by real analytic equations or inequalities, and which is stable by finite set operations (finite union, finite intersection and complement) and by proper real analytic projections. The name "subanalytic" was introduced by Hironaka and Hardt used the name "analytic shadow" in [Hardt] although they finally happened to describe the same class (see [Loja]).
- Any subanalytic set  $E$  of a real analytic manifold  $X$  admits a stratification, that is a locally finite partition in real analytic connected submanifold of  $X$  called strata  $E = \sqcup_{S \in \mathcal{S}} S$  such that  $S \cap \overline{S'} \neq \emptyset$ ,  $S \neq S'$ , implies  $S \subset \partial S'$  with  $\dim S < \dim S'$ , or equivalently because  $\mathcal{S}$  is a partition,  $S \cap \partial S' \neq \emptyset$ ,  $S \neq S'$  implies  $S \subset \partial S'$  with  $\dim S < \dim S'$ . Such a stratification can always be refined in order to satisfy Whitney's local condition B which reads in  $\mathbb{R}^n$  or in a coordinate system:

$$\left( (x_n)_{n \in \mathbb{N}} \in (S')^{\mathbb{N}} \quad , \quad \lim_{n \rightarrow \infty} x_n = x \in S \subset \overline{S'} \right) \Rightarrow (T_x S \subset \lim_{n \rightarrow \infty} T_{x_n} S').$$

When  $\mathcal{C}$  is a locally finite family of subanalytic sets, the stratification  $\mathcal{S}$  can also be chosen in order to be compatible with  $\mathcal{C}$ , which means that for all  $S \in \mathcal{S}$  and  $C \in \mathcal{C}$ , either  $S \cap C = \emptyset$  or  $S \subset C$ .

- A subanalytic function  $X \rightarrow Y$  is a function of which the graph is a subanalytic set of  $X \times Y$ .
- When  $f : X \rightarrow Y$  is a real analytic mapping, a stratification of  $f$  is made of two stratifications  $\mathcal{S}$  of  $X$  and  $\mathcal{F}$  of  $Y$  such that

$$\forall S \in \mathcal{S}, f(S) \in \mathcal{F} \quad , \quad \text{rank}(f|_S) = \dim f(S).$$

- Corollary 4.4 of [Hardt] assumes that  $f : X \rightarrow Y$  is real analytic and  $\mathcal{C}$  and  $\mathcal{D}$  are two locally finite families of subanalytic sets of  $X$  and  $Y$  and  $\Omega$  is a subanalytic open set such that  $f|_{\overline{\Omega}}$  is proper. It then says that there exists a stratification  $(\mathcal{S}, \mathcal{F})$  of  $f|_{\Omega}$  which is compatible with  $\mathcal{C}$  and  $\mathcal{D}$ .
- Famous Hironaka's desingularisation theorem says that any compact subanalytic set is the image of a compact real analytic manifold with same dimension by a real analytic mapping. We will not use it specifically.



When  $f : M \rightarrow \mathbb{R}$  is a Lipschitz subanalytic function we consider the two projections  $p_1 : M \times \mathbb{R} \rightarrow M$  and  $p_2 : M \times \mathbb{R} \rightarrow \mathbb{R}$ . From Hardt's result we know that there is a stratification of  $p_2 : M \times \mathbb{R} \rightarrow \mathbb{R}$  which is compatible with  $\mathcal{C} = \text{graph}(f) \sqcup (M \times \mathbb{R} \setminus \text{graph}(f))$  and  $\mathcal{D} = \mathbb{R}$ . From this we deduce that there is a stratification  $\tilde{\mathcal{S}}$  of  $\text{graph}(f)$  and a finite number of points  $\{c_1, \dots, c_{N_f}\} \in \mathbb{R}$  such that all  $\tilde{S} \in \tilde{\mathcal{S}}$  satisfies

- either  $p_2$  is constantly equal to some  $c_n$  along  $\tilde{S}$ ;
- or there exists  $n$  such that  $p_2(\tilde{S}) = ]c_n, c_{n+1}[$  and  $\text{rank}(p_2|_{\tilde{S}}) = 1$ .

**Definition 8.6.** For such a stratification of  $\text{graph}(f)$ , strata corresponding to the first case will be called horizontal strata.

Because  $f$  is a Lipschitz function the projection  $p_1 : M \times \mathbb{R} \rightarrow M$  makes a diffeomorphism from  $\tilde{S}$  to  $S = p_1(\tilde{S})$  which is a submanifold of  $M$ . The family  $\mathcal{S} = \{p_1(\tilde{S}), \tilde{S} \in \tilde{\mathcal{S}}\}$  is now a stratification of  $M$ . When  $\tilde{S}$  is a horizontal stratum, then  $f|_S$  is constant along  $S = p_1(\tilde{S})$ . On the contrary when  $\tilde{S}$  is not horizontal  $f|_S$  is a real analytic function with no critical point on  $S = p_1(\tilde{S})$ .

Whitney's condition B also has a nice interpretation. It simply says in a local coordinate system (which allows the local identification of  $T_y M$  with  $\mathbb{R}^d$  around any point  $x \in M$ )

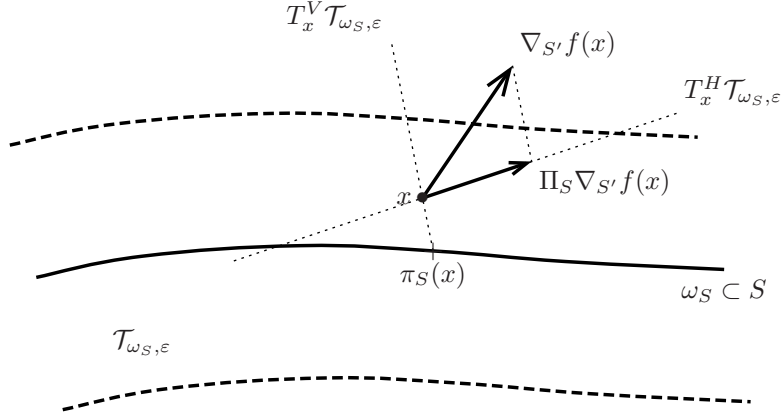
$$\left( (x_n)_{n \in \mathbb{N}} \in (S')^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x \right) \Rightarrow \left( \forall T \in T_x S \sim \mathbb{R}^d, \lim_{n \rightarrow \infty} (d(f|_{S'})_{x_n}[T] - d(f|_S)_x[T]) = 0 \right).$$

With the Riemannian structure it can be expressed in terms of gradients. More exactly for any relatively compact open subset  $\omega_S$  of the stratum  $S$ , and for  $\varepsilon \in ]0, \varepsilon_{\omega_S}[$ ,  $\varepsilon_{\omega_S} > 0$  small enough, the exponential map  $\exp(x, t) = \exp_x(t) \in M$  for  $(x, t) \in TM$  is a diffeomorphism from  $\{(x, t) \in N\omega_S, |t| < \varepsilon\}$ , where  $N\omega_S$  is the normal fiber bundle over  $\omega_S$ , to its range  $\mathcal{T}_{\omega_S, \varepsilon} \subset \{x \in M, d(x, \overline{\omega_S}) < \varepsilon\}$ , that we call a tubular neighborhood of  $\omega_S$ . We refer the reader to [Lee] where tubular neighborhoods of closed submanifold are introduced in this way and [Lan] for further details and generalizations with more general pseudo Riemannian structures. Another presentation using the embedding of  $M$  in some  $\mathbb{R}^{N_M}$  is given in [Hirs]. Such a tubular neighborhood  $\mathcal{T}_{\omega_S, \varepsilon} \subset M$  is an open subset of the fiber bundle  $\pi_S : N\omega_S \rightarrow \omega_S$  and is endowed with the metric  $g$  defined on  $M$ . Therefore the tangent bundle  $T_x \mathcal{T}_{\omega_S, \varepsilon} = T_x M$  for  $x \in \mathcal{T}_{\omega_S, \varepsilon}$ , has an orthonormal decomposition  $T_x M = T_x^V M \oplus^\perp T_x^H M$  where  $T_x^V M = \ker(d\pi_S) \sim N_{\pi_S(x)} \omega_S$ . For  $x \in \mathcal{T}_{\omega_S, \varepsilon}$  and  $t \in T_x M = T_x \mathcal{T}_{\omega_S, \varepsilon}$  we define  $\Pi_S t$  as the horizontal component of  $t$  in this decomposition. For  $x \in \mathcal{T}_{\omega_S, \varepsilon}$ , the function  $f_S(x) = f(\pi_S x)$  is a real analytic function of  $x \in \mathcal{T}_{\omega_S, \varepsilon}$ . Because  $f$  is a regular function along a stratum  $S' \in \mathcal{S}$  its gradient along  $S'$  (with the metric induced by  $g$ ) is denoted  $\nabla_{S'} f$ . With those notations the previous property can be written

$$\left( (x_n)_{n \in \mathbb{N}} \in (S' \cap \mathcal{T}_{\omega_S, \varepsilon})^{\mathbb{N}}, \lim_{n \rightarrow \infty} x_n = x \in \omega_S \right) \Rightarrow \left( \lim_{n \rightarrow \infty} |\Pi_S \nabla_{S'} f(x_n) - \nabla f_S(x_n)| = 0 \right).$$

Let us summarize our notations:

- $\omega_S$  is a relatively compact open set of the stratum  $S$ .
- $\mathcal{T}_{\omega_S, \varepsilon}$  is a tubular neighborhood of  $\omega_S$  diffeomorphic to  $\{(x, t) \in N\omega_S, |t| < \varepsilon\}$ . It will be convenient to extend the notation to  $\varepsilon = 0$  with the large inequality and  $\omega_S = S$ , namely  $\mathcal{T}_{S, 0} = S$ , which makes sense as  $S = \limsup_{\varepsilon \rightarrow 0} \mathcal{T}_{\omega_S, \varepsilon}$  where  $\omega_S, \varepsilon$  relatively compact in  $S$  is well chosen when  $\varepsilon > 0$  is small.
- When  $S'$  is a stratum  $\nabla_{S'} f$  is the gradient of  $f$  along  $S'$  and for  $x \in S' \cap \mathcal{T}_{\omega_S, \varepsilon}$ ,  $\Pi_S \nabla_{S'} f(x)$  is the horizontal component of  $\nabla_{S'} f(x)$  in the orthogonal decomposition  $T_x M = T_x^V \mathcal{T}_{\omega_S, \varepsilon} \oplus^\perp T_x^H \mathcal{T}_{\omega_S, \varepsilon}$ .
- Finally in  $\mathcal{T}_{\omega_S, \varepsilon}$ , which is diffeomorphic to a subset of  $N\omega_S$ , one defines the regular function  $f_S(x) = f(\pi_S x)$  where  $\pi_S$  is the natural projection  $\pi_S : N\omega_S \rightarrow \omega_S$ .



**Figure 13:** Picture of the projections  $\pi_S$  and  $\Pi_S$  when  $x \in \mathcal{T}_{\omega_S, \varepsilon} \cap S'$ .

With the compactness of  $\overline{\omega_S}$  in  $S$ , Whitney's condition B actually implies the following uniform convergence result.

**Lemma 8.7.** *Fix the relatively compact open set  $\omega_S$  of the stratum  $S$  and let  $\mathcal{T}_{\omega_S, \varepsilon}$  denote the tubular neighborhood defined for  $\varepsilon > 0$  small enough. Then the quantities*

$$\max_{S' \in \mathcal{S}} \sup_{x \in \mathcal{T}_{\omega_S, \varepsilon} \cap S'} |\Pi_S \nabla_{S'} f(x) - \nabla f_S(x)|$$

and

$$\sup_{x \in \mathcal{T}_{\omega_S, \varepsilon}} |\nabla f_S(x) - \nabla f_S(\pi_S x)|,$$

tend to 0 as  $\varepsilon \rightarrow 0^+$ .

*Proof.* Ad absurdum if there is a sequence  $(x_n)_{n \in \mathbb{N}}$  such that  $|\Pi_S \nabla_{S'} f(x_n) - \nabla f_S(x_n)| \geq \eta > 0$  while  $x_n \in \mathcal{T}_{\omega_S, \frac{1}{n}} \cap S'$ , then by the compactness of  $\overline{\omega_S}$  and the finiteness of  $\mathcal{S}$ , we can assume that  $S'$  is fixed and that  $\lim_{n \rightarrow \infty} x_n = x \in \overline{\omega_S}$ . The lower bound  $|\Pi_S \nabla_{S'} f(x_n) - \nabla f_S(x_n)| \geq \eta > 0$  while  $\lim_{n \rightarrow \infty} |\nabla f_S(x_n) - \nabla f_S(x)| = 0$ ,  $\nabla f_S(x) = \nabla_S f(x)$ , then contradicts Whitney's condition B.

Finally the last convergence is a consequence of the uniform continuity of  $\nabla f_S$  which can be defined on a compact neighborhood of  $\mathcal{T}_{\omega_S, \varepsilon}$  for  $\varepsilon \in ]0, \varepsilon_S[$ ,  $\varepsilon_S > 0$  small enough.  $\square$

**Proposition 8.8.** *When  $f$  is a Lipschitz subanalytic function on  $M$ , Hypothesis 1.6 is satisfied with  $c_1, \dots, c_{N_f} \in \mathbb{R}$  being the values associated with horizontal strata in the stratification of  $\text{graph}(f) \subset M \times \mathbb{R}$  described above.*

*Proof.* Let  $x_0 \in M \setminus f^{-1}(\{c_1, \dots, c_{N_f}\})$ . It belongs to a stratum  $S \in \mathcal{S}$  and we can find a relatively compact open set  $\omega_S \subset S$  such that  $x_0 \in \omega_S \subset S$ . The function  $f_S$  is a real analytic-function defined in the tubular open  $\mathcal{T}_{\omega_S, \varepsilon}$  for  $\varepsilon \in ]0, \varepsilon_{x_0}[$  with  $\varepsilon_{x_0} > 0$  small enough. For  $y \in \mathcal{T}_{\omega_S, \varepsilon} \cap S'$ , with  $S' \in \mathcal{S}$ ,  $\dim S' = d$ , we write

$$\nabla f_S(y) \cdot \nabla f(y) = |\Pi_S \nabla f(y)|^2 - (\nabla f_S(y) - \Pi_S \nabla f(y)) \cdot \nabla f(y)$$

and

$$|\nabla f_S(y) \cdot \nabla f(y) - |\nabla f_S(x_0)|^2| \leq ||\Pi_S \nabla f(y)|^2 - |\nabla f_S(x_0)|^2| + M_f |\Pi_S \nabla f(y) - \nabla f_S(y)|.$$

We know that  $|\nabla f_S(x_0)| > 0$  because  $S$  cannot be an horizontal stratum. By Lemma 8.7,  $\varepsilon \in ]0, \varepsilon_{x_0}[$  can be chosen such that the right-hand side is smaller than  $\frac{1}{2} |\nabla f_S(x_0)|^2$  for all  $S' \in \mathcal{S}$ , such that  $\dim S' = d$  and  $x_0 \in S \cap S'$ . We have found a tubular neighborhood  $U_{x_0}$  of  $x_0$  and a coordinate system  $(x^1, \dots, x^d)$  around  $x_0$  by taking  $x^1 = f_S(x)$  such that

$$\forall S' \in \mathcal{S}, \dim S' = d, \forall x \in U_{x_0} \cap S', \quad \partial_{x^1} f(x) \geq \frac{1}{C_{x_0}}.$$

This neighborhood  $U_{x_0}$  can then be reduced to

$$U_{x_0} = \{x = (x^1, x') = (x^1, x^2, \dots, x^d), |x^1 - x_0^1| < \delta, |x'| < \delta\}$$

for some  $\delta > 0$ . The set  $E = U_{x_0} \setminus (\cup_{\dim S' = d} S' \cap U_{x_0})$  has measure 0 and  $\nabla f(x)$  is well defined for all  $x \in U_{x_0} \setminus E$ . By Fubini's theorem the set of  $x', |x'| < \delta$ , such that  $\{(x^1, x'), |x^1 - x_0^1| < \delta\} \cap E$  has a non zero one dimensional measure, has Lebesgue's measure 0 and we can write for almost all  $x', |x'| < \delta$

$$\forall x^1, y^1 \in ]x_0^1 - \delta, x_0^1 + \delta[, \quad f(x^1, x') - f(y^1, x') = \int_0^1 (x^1 - y^1) \partial_{x^1} f(x^1 + t(y^1 - x^1)) dt$$

where the integrand is well defined for almost every  $t \in [0, 1]$  and bounded from below by  $\frac{1}{C_{x_0}}(x^1 - y^1)$  when  $x^1 > y^1$ . The continuity of  $f$  then implies

$$\forall (x^1, x'), (y^1, x') \in U_{x_0}, \quad \frac{1}{C_{x_0}} |x^1 - y^1| \leq |f(x^1, x') - f(y^1, x')|.$$

□

We will use open coverings of  $f^{-1}([a, b])$  when  $[a, b] \# \{c_1, \dots, c_{N_f}\} = \emptyset$ , made of tubes  $\mathcal{T}_{\omega_S, \varepsilon_S}$  with  $\varepsilon_{\omega_S} > 0$ . They will be constructed by induction on the dimensions of the strata. They will be associated with a family of parameters  $(\varepsilon_1, \dots, \varepsilon_d)$ , with  $\varepsilon_{\omega_S} = \varepsilon_{\dim S}$ . In the induction process we authorize  $\varepsilon_{\dim S} = 0$  for  $m < \dim S \leq d$ , in which case  $\omega_S = S$  for every stratum  $S$  of dimension  $\dim S > m$ .

**Definition 8.9.** *Let  $a < b$  belong to  $\mathbb{R}$  and set  $\mathcal{S}_{[a, b]} = \{S \in \mathcal{S}, S \cap f^{-1}([a, b]) \neq \emptyset\}$ . A tubular covering of  $f^{-1}([a, b])$  contains two data, a family  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d) \in [0, +\infty[^{d+1}$  and for every  $S \in \mathcal{S}_{[a, b]}$ , a subset  $\omega_S$  of  $S$  which is either open and relatively compact in  $S$  if  $\varepsilon_{\dim S} > 0$  or equal to  $S$  if  $\varepsilon_{\dim S} = 0$  such that for all  $m \leq d$*

$$f^{-1}([a, b]) \cap \left( \bigcup_{S \in \mathcal{S}_{[a, b]}, \dim S \leq m} S \right) \subset \bigcup_{S \in \mathcal{S}_{[a, b]}, \dim S \leq m} \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}, \quad (138)$$

$$\mathcal{T}_{\omega_{S_1}, \varepsilon_{m'}} \cap \mathcal{T}_{\omega_{S_2}, \varepsilon_{m'}} = \emptyset \quad \text{if } S_1 \neq S_2, \quad \dim S_1 = \dim S_2 = m' \leq m. \quad (139)$$

Such a tubular covering is said  $\varepsilon$ -adapted for  $\varepsilon \in ]0, 1]$ , if for any  $S, S' \in \mathcal{S}_{[a, b]}$ ,

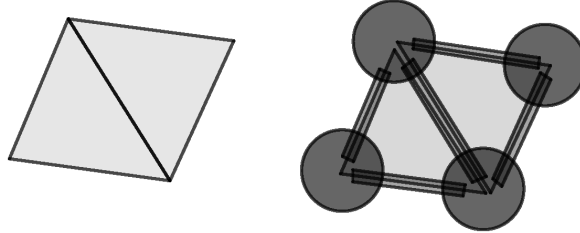
$$\sup_{x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}} \cap S'} |\Pi_S \nabla_{S'} f(x) - \nabla f_S(x)| \leq \varepsilon, \quad (140)$$

and

$$\sup_{x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}} |\nabla f_S(x) - \nabla f_S(\pi_S x)| \leq \varepsilon. \quad (141)$$

Such a covering is clearly an open covering when all the  $\varepsilon_i$ 's are positive.

We will summarize those situations by speaking of a (possibly "an  $\varepsilon$ -adapted") (possibly "open") tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_S})_{S \in \mathcal{S}_{[a, b]}}$  associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_d)$ .



**Figure 14:** A schematic example of an open covering: The stratification is on the left-hand side made of two triangles, the edges and the vertices; the open tubular covering with positive values for  $\varepsilon_0, \varepsilon_1, \varepsilon_2$  is on the right-hand side. The outside of the two triangles is forgotten or one can compactify by identifying opposite external edges

A trivial example is given by  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d) = (0, \dots, 0)$  and  $\omega_S = S$  for all  $S \in \mathcal{S}_{[a,b]}$ . When  $\mathcal{S}_{[a,b]}$  contains no stratum of dimension  $m$ , any value  $\varepsilon_m \geq 0$  can be used in the above definitions. When all the parameters  $\varepsilon_m$ ,  $0 \leq m \leq d$ , are positive, this provides an open covering of  $f^{-1}([a, b])$ . Note that when  $\varepsilon_m = 0$  and  $\omega_S = S$  for  $\dim S = m$ , the condition  $x \in \mathcal{T}_{\omega_S, \varepsilon_m} \cap S'$  actually implies  $x \in S = S'$  so that the condition (140) is void for strata  $S$  of dimension  $m$ . As a consequence, if  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  is a (resp. an  $\varepsilon$ -adapted) tubular covering of  $f^{-1}([a, b])$  associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_d)$ , then for any  $m$  replacing  $\varepsilon_{m'}$  by 0 for  $m' > m$ ,  $\omega_S$  by  $S$  if  $\dim S > m$ ,  $\varepsilon_m$  by  $\varepsilon'_m \in ]0, \varepsilon_m]$  and leaving the other data,  $\omega_S$  for  $\dim S \leq m$ ,  $\varepsilon_0, \dots, \varepsilon_{m-1}$ , unchanged give another (resp.  $\varepsilon$ -adapted) tubular covering. The following proposition implements the induction which leads to the construction of families of  $\varepsilon$ -adapted open tubular coverings of  $f^{-1}([a, b])$ , especially when  $[a, b]$  contains no “critical value”.

**Proposition 8.10.** *Assume first  $f^{-1}([a, b]) \cap \{c_1, \dots, c_{N_f}\} = \emptyset$  where  $c_1, \dots, c_{N_f}$  are values of  $f$  associated with horizontal strata. Then  $\mathcal{S}_{[a,b]}$  contains no 0-dimensional stratum and there exists a (resp. an  $\varepsilon$ -adapted) tubular covering associated with  $(\varepsilon_0, 0, \dots, 0)$  for any  $\varepsilon_0 > 0$ .*

*Assume that there exists a (resp. an  $\varepsilon$ -adapted) tubular covering associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_{m-1}, 0, \dots, 0)$  for  $1 \leq m \leq d$  with  $\varepsilon_0 > 0, \dots, \varepsilon_{m-1} > 0$ , then there exists  $\varepsilon_m^0 > 0$  and for any  $S \in \mathcal{S}_{[a,b]}$ ,  $\dim S = m$ , a subset  $\omega_S \subset S$  open and relatively compact  $S$  such that for all  $\varepsilon_m \in ]0, \varepsilon_m^0]$ , the family  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  associated with  $(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)$  and  $\omega_S$  unchanged if  $\dim S \leq m - 1$ , is another (resp.  $\varepsilon$ -adapted) tubular covering of  $f^{-1}([a, b])$ .*

*Proof.* Because  $\mathcal{S}_{[a,b]}$  contains no stratum of dimension 0 a tubular covering is given by  $\omega_S = S$  where all  $S \in \mathcal{S}_{[a,b]}$  satisfy  $\dim S \geq 1$  and any value of  $\varepsilon_0 > 0$  makes sense.

Additionally every  $S \in \mathcal{S}_{[a,b]}$  of dimension 1 satisfies  $\nabla_S f(x) \neq 0$  for every  $x \in S \cap f^{-1}([a, b])$  and hence  $f^{-1}([a, b]) \cap S$  is a compact subset of  $S$ . We can choose  $\omega_S$  open and relatively compact in  $S$  such that  $\omega_S \cap f^{-1}([a, b])$  is a neighborhood in  $S$  of  $S \cap f^{-1}([a, b])$ . This is done for every  $S \in \mathcal{S}_{[a,b]}$  such that  $\dim S = 1$ . We can then choose  $\varepsilon_1 > 0$  such that  $\varepsilon_1 < \frac{1}{2}d_g(\overline{\omega_{S_1}}, \overline{\omega_{S_2}})$  for any  $S_1, S_2 \in \mathcal{S}_{[a,b]}$ ,  $\dim S_1 = \dim S_2 = 1$ , in order to ensure  $\mathcal{T}_{\omega_{S_1}, \varepsilon_1} \cap \mathcal{T}_{\omega_{S_2}, \varepsilon_1} = \emptyset$  for  $S_1 \neq S_2$ .

Assume now that the result holds for a given  $m$ ,  $1 \leq m \leq d$ . For  $\dim S \leq m$ , the set  $\mathcal{T}_{\omega_S, \varepsilon_{\dim S}}$  is

an open set and  $\cup_{S \in \mathcal{S}_{[a,b], \dim S \leq m}} \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}$  is an of  $K_{[a,b], m} = f^{-1}([a, b]) \cap (\cup_{S \in \mathcal{S}_{[a,b], \dim S \leq m}} S)$ . Consider the compact subset  $K_{[a,b], m+1} = f^{-1}([a, b]) \cap (\cup_{S \in \mathcal{S}_{[a,b], \dim S = m+1}} S)$ . It is a compact set and so is  $K_{[a,b], m+1} \setminus (\cup_{S \in \mathcal{S}_{[a,b], \dim S \leq m}} \mathcal{T}_{\omega_S, \varepsilon_{\dim S}})$  which by the definition of the stratification  $\mathcal{S}$  can be decomposed into  $\cup_{S \in \mathcal{S}_{[a,b], \dim S = m+1}} K_S$  where  $K_S$  is a compact subset of  $S$ . We choose for  $\omega_S$ ,  $S \in \mathcal{S}_{[a,b]}$ ,  $\dim S = m+1$ , a relatively compact neighborhood of  $K_S$  and then fix  $\varepsilon_{m+1} > 0$  small enough such that  $\mathcal{T}_{\omega_{S_1}, \varepsilon_{m+1}} \cap \mathcal{T}_{\omega_{S_2}, \varepsilon_{m+1}} = \emptyset$  for any  $S_1, S_2 \in \mathcal{S}_{[a,b]}$ ,  $\dim S_1 = \dim S_2 = m+1$  like in the case  $m+1 = 1$ .

Following this induction and by assuming that  $(\mathcal{T}_{\omega_S, \varepsilon_S})_{S \in \mathcal{S}_{[a,b]}}$  is an  $\varepsilon$ -adapted tubular covering associated with  $(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)$ ,  $\varepsilon_0 \dots \varepsilon_m > 0$ ,  $\varepsilon_{m+1}^0 > 0$  can be chosen such that

$$\sup_{x \in \mathcal{T}_{\omega_S, \varepsilon_{m+1}^0} \cap S'} |\Pi_S \nabla_{S'} f(x) - \nabla f_S(x)| \leq \varepsilon$$

and

$$\sup_{x \in \mathcal{T}_{\omega_S, \varepsilon_{m+1}^0}} |\nabla f_S(x) - \nabla f_S(\pi_S x)| \leq \varepsilon,$$

for all  $S \in \mathcal{S}_{[a,b]}$ ,  $\dim S = m+1$ , and all  $S' \in \mathcal{S}_{[a,b]}$ . This still holds if  $\varepsilon_{m+1}^0$  is replaced by any  $\varepsilon_{m+1} \in ]0, \varepsilon_{m+1}^0]$ , without changing the  $\omega_S$ , and this ends the proof.  $\square$

**Definition 8.11.** Assume  $f^{-1}([a, b]) \cap \{c_1, \dots, c_{N_f}\} = \emptyset$  and let  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  be a tubular covering associated with the parameters  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d) \in [0, +\infty[^{d+1}$ . The functions  $\tilde{F}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $F_{(\varepsilon_0, \dots, \varepsilon_d)}$  are defined on  $f^{-1}([a, b])$  by

$$\tilde{F}_{(\varepsilon_0, \dots, \varepsilon_d)}(x) = \min_{x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}} \cap S'} |\Pi_S \nabla_{S'} f(x)|, \quad (142)$$

$$F_{(\varepsilon_0, \dots, \varepsilon_d)}(x) = \min_{x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}} |\nabla f_S(x)|, \quad (143)$$

where the minima are taken over  $S, S' \in \mathcal{S}_{[a,b]}$ .

On  $f^{-1}([a, b]) \times f^{-1}([a, b])$  the functions  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $G_{(\varepsilon_0, \dots, \varepsilon_d)}$  are given by

$$G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, y) = \inf_{\substack{\gamma \in \mathcal{C}^1([0, 1]; f^{-1}([a, b])) \\ \gamma(0) = x; \gamma(1) = y}} \int_0^1 F_{(\varepsilon_0, \dots, \varepsilon_d)}(\gamma(t)) |\gamma'(t)| dt \quad (144)$$

with the same definition for  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_d)}$ .

Before proving some results about those functions let us list some simple properties:

- Because  $\mathcal{S}_{[a,b]}$  is a finite collections of measurable sets, the functions  $\tilde{F}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $F_{(\varepsilon_0, \dots, \varepsilon_d)}$  are measurable and the functions  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $G_{(\varepsilon_0, \dots, \varepsilon_d)}$  are well defined.
- When  $\varepsilon_1 = \dots = \varepsilon_d = 0$ , the functions  $\tilde{F}_{(0, \dots, 0)}$  and  $F_{(0, \dots, 0)}$  are equal to

$$\tilde{F}_{(0, \dots, 0)}(x) = F_{(0, \dots, 0)}(x) = \sum_{x \in S} 1_S(x) |\nabla_S f(x)|,$$

which is a lower semicontinuous function on  $f^{-1}([a, b])$  due to Whitney's condition B and  $|\Pi_S \nabla_{S'} f(x)| \leq |\nabla_{S'} f(x)|$  for  $x \in S'$  close enough to  $S \subset \partial S'$ .

- Because  $f$  is a Lipschitz function, the function  $\tilde{F}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $F_{(\varepsilon_0, \dots, \varepsilon_d)}$  are uniformly bounded by  $M_f = 1 + \|\nabla f\|_{L^\infty}$  when  $\varepsilon \leq 1$  because of  $|\Pi_S \nabla_{S'} f(x)| \leq |\nabla_{S'} f(x)| \leq \|f\|_{W^{1, \infty}}$  and (140). Therefore the functions  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_d)}$  and  $G_{\varepsilon_0, \dots, \varepsilon_d}$  are  $M_f$ -Lipschitz pseudodistances on  $f^{-1}([a, b]) \times f^{-1}([a, b])$ .

- When  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  is an  $\varepsilon$ -adapted tubular covering of  $f^{-1}([a, b])$ , then

$$\sum_{x \in f^{-1}([a,b])} \left| \tilde{F}_{(\varepsilon_0, \dots, \varepsilon_d)}(x) - F_{(\varepsilon_0, \dots, \varepsilon_d)}(x) \right| \leq \varepsilon$$

and hence

$$\sup_{(x,y) \in f^{-1}([a,b])} \left| \tilde{G}_{(\varepsilon_0, \dots, \varepsilon_d)}(x, y) - G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, y) \right| \leq \varepsilon \times \text{diam}(f^{-1}([a, b])),$$

where  $\text{diam}$  is the diameter for the metric  $g$ .

- Let  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  be a tubular covering of  $f^{-1}([a, b])$  associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)$  with  $\varepsilon_0 \dots \varepsilon_m > 0$ ,  $1 \leq m \leq d$ . For any  $\varepsilon'_m \in ]0, \varepsilon_m]$ , one gets another tubular covering of  $f^{-1}([a, b])$  while keeping all the other data unchanged and for  $\varepsilon'_m = 0$  simply change  $\omega_S$  into  $S$  when  $\dim S = m$ . Then the functions  $H_{(\varepsilon_0, \dots, \varepsilon'_m, 0, \dots, 0)}$ , with  $H = \tilde{F}, F, \tilde{G}, G$ , are well defined for any  $\varepsilon'_m \in [0, \varepsilon_m]$  and they are decreasing with respect to  $\varepsilon'_m$ , i.e. increase as  $\varepsilon'_m$  decays.

**Lemma 8.12.** *In the framework of Definition 8.11, the function  $F_{0, \dots, 0}(x) = \tilde{F}_{(0, \dots, 0)}(x)$  is lower semi-continuous bounded by  $M_f = 1 + \|\nabla f\|_{L^\infty}$  and bounded from below by a positive constant  $m_{a,b,f} > 0$ . The function  $\tilde{G}_{(0, \dots, 0)}(x, y) = G_{(0, \dots, 0)}(x, y)$  is a pseudodistance (fulfilling the symmetry and the triangular inequality) which satisfies*

$$\forall x, y \in f^{-1}([a, b]), \quad |f(x) - f(y)| \leq G_{(0, \dots, 0)}(x, y) \leq M_f d_g(x, y),$$

where  $d_g$  is the geodesic distance between  $x$  and  $y$  in the metric  $g$ .

*Proof.* We already noticed that  $F_{(0, \dots, 0)} = \tilde{F}_{(0, \dots, 0)}$  is a lower semicontinuous function, bounded by  $\|\nabla f\|_{L^\infty}$ . Since  $f^{-1}([a, b])$  contains no horizontal stratum

$$F_{(0, \dots, 0)}(x) = \sum_{S \in \mathcal{S}_{[a,b]}} 1_S(x) |\nabla_S f(x)|$$

does not vanish. The achieved minimum  $m_{a,b,f}$  on the compact set  $f^{-1}([a, b])$  must be positive. With the estimate  $F_{(0, \dots, 0)}(x) \leq M_f$  for all  $x \in f^{-1}([a, b])$ , the fact that  $G_{(0, \dots, 0)}(x, y)$  defines a pseudodistance with the upper bound  $G_{(0, \dots, 0)}(x, y) \leq M_f d_g(x, y)$  is standard. For the lower bound because  $M$ -valued real analytic functions are dense in  $\mathcal{C}^1([0, 1]; M)$ , the function  $G_{(0, \dots, 0)}$  can be defined as

$$G_{(0, \dots, 0)}(x, y) = \inf_{\substack{\gamma \in \mathcal{C}^\omega([0, 1]; f^{-1}([a, b])) \\ \gamma(0) = x; \gamma(1) = y}} \int_0^1 F_{(0, \dots, 0)}(\gamma(t)) |\gamma'(t)| dt.$$

Let  $\gamma : [0, 1] \rightarrow f^{-1}([a, b]) \subset M$  be a real analytic function such that

$$G_{(0, \dots, 0)}(x, y) + \eta \geq \int_0^1 F_{(0, \dots, 0)}(\gamma(t)) |\gamma'(t)| dt \geq G_{(0, \dots, 0)}(x, y).$$

By using the recalled Hardt's result in [Hardt] about the stratification of real analytic mapping, now applied to  $\gamma$  from  $[0, 1]$  with the trivial stratification to  $M$  with the stratification  $\mathcal{S}$ , there exists a stratification of  $[0, 1]$ , that is a finite partition into open intervals and points  $[0, 1] = \sqcup_{I \in \mathcal{I}} I$  such that for any  $I \in \mathcal{I}$  there exists  $S_I \in \mathcal{S}$  such that  $\gamma(I) \subset S_I$ . Hence there exist

$N \in \mathbb{N}$ ,  $0 = t_0 < t_1 < \dots < t_N = 1$  and for any  $1 \leq n \leq N$  a stratum  $S_n \in \mathcal{S}_{[a,b]}$  such that  $\gamma(]t_{n-1}, t_n]) \subset S_n$ . We deduce

$$\begin{aligned} \int_0^1 F_{(0,\dots,0)}(\gamma(t))|\gamma'(t)| dt &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\nabla_{S_n} f(\gamma(t))||\gamma'(t)| dt \\ &\geq \sum_{n=1}^N |f(\gamma(t_n)) - f(\gamma(t_{n-1}))| \geq |f(x) - f(y)|. \end{aligned}$$

We have proved for all  $\eta > 0$  the lower bound

$$G_{(0,\dots,0)}(x, y) + \eta \geq |f(x) - f(y)|,$$

which ends the proof.  $\square$

**Proposition 8.13.** *Assume that  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \emptyset$ . For any  $\varepsilon \in ]0, 1[$  there exist parameters  $(\varepsilon_0, \dots, \varepsilon_d) \in ]0, +\infty[^{d+1}$  and an  $\varepsilon$ -adapted open tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_d)$ , such that the function  $G_{(\varepsilon_0, \dots, \varepsilon_d)}$  defined in Definition 8.11 satisfies the uniform estimates:*

$$\forall x, y \in f^{-1}([a, b]), \quad |f(x) - f(y)| - \varepsilon \leq G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, y) \leq M_f d_g(x, y) \quad (145)$$

where  $M_f = 1 + \|\nabla f\|_{L^\infty}$  and  $d_g$  is the geodesic distance on  $(M, g)$ .

For any  $\varepsilon' \in ]0, 1[$ , this tubular covering can be chosen, after taking  $\varepsilon > 0$  small enough, such that

$$\forall S \in \mathcal{S}_{[a,b]}, \nabla f(x) \cdot \nabla f_S(x) - (1 - \varepsilon')|\nabla f_S(x)|^2 \quad \text{for a.e. } x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}, \quad (146)$$

and

$$\forall S, S' \in \mathcal{S}_{[a,b]}, \forall x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}} \cap S', \quad |\nabla f_S(x)| \geq \frac{m_{f,a,b}}{2}, \quad |\Pi_S \nabla_{S'} f(x)| \geq \frac{m_{f,a,b}}{2}, \quad (147)$$

where  $m_{f,a,b} > 0$  was introduced in Lemma 8.12.

*Proof.* The diameter  $\text{diam}(f^{-1}([a, b]))$  for the geodesic distance on  $(M, g)$  is denoted by

$$\Delta_{a,b,f} = \text{diam}(f^{-1}([a, b])).$$

The proof is made by induction on  $m$ , where  $m$  is the maximal number such that  $\varepsilon_0 \dots \varepsilon_m > 0$ , while playing with the two functions  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}$  and  $G_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}$ . More precisely we will prove that for  $0 \leq m \leq d$ , there exists  $(\varepsilon_0, \dots, \varepsilon_m) \in ]0, +\infty[^{m+1}$  and an  $\frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}$ -adapted tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}([a,b])}$  associated with the parameters  $(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)$  such that

$$|f(x) - f(y)| - \frac{m\varepsilon}{d} \leq G_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}(x, y).$$

Notice that “ $\frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}$ -adapted” is stronger than “ $\varepsilon$ -adapted”.

The statement is clearly true for  $m = 0$  because our assumption says that  $\mathcal{S}_{[a,b]}$  contains no 0-dimensional stratum and  $G_{(\varepsilon_0, 0, \dots, 0)} = \tilde{G}_{(\varepsilon_0, 0, \dots, 0)}$  does not depend on  $\varepsilon_0 \in [0, +\infty[$ , while the lower bound  $G_{(0, \dots, 0)}(x, y) \geq |f(x) - f(y)|$  was proved in Lemma 8.12. Note additionally that the tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$ ,  $\mathcal{T}_{\omega_S, 0} = S$  for  $S \in \mathcal{S}_{[a,b]}$  is an  $\frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}$ -adapted tubular covering of  $f^{-1}([a, b])$ .

Assume now that we have found  $(\varepsilon_0, \dots, \varepsilon_m) \in ]0, +\infty[^{m+1}$  and an  $\frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}$ -adapted tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  such that

$$|f(x) - f(y)| - \frac{m\varepsilon}{d} \leq G_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}(x, y).$$

By Proposition 8.10  $\varepsilon_{m+1}^0 > 0$  can be chosen such that for any  $\varepsilon_{m+1} \in ]0, \varepsilon_{m+1}^0]$  there exists an  $\frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}$ -adapted tubular covering  $(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a,b]}}$  associated with  $(\varepsilon_1, \dots, \varepsilon_{m+1}, 0, \dots, 0)$ , with  $\omega_S$  independent of  $\varepsilon_{m+1} > 0$ . For any  $\varepsilon_{m+1} \in [0, \varepsilon_{m+1}^{(0)}]$  we deduce

$$\sup_{x, y \in f^{-1}([a, b])} |\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y) - G_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y)| \leq \frac{\varepsilon}{d(2\Delta_{a,b,f}+1)} \times \Delta_{a,b,f}.$$

Meanwhile we observed that  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}$  is the monotonous limit as  $\varepsilon_{m+1} \rightarrow 0^+$  of  $\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}$ , in the class of Lipschitz continuous functions on the compact set  $f^{-1}([a, b]) \times f^{-1}([a, b])$ . Dini's convergence theorem then ensures that this convergence is uniform and we can choose  $\varepsilon_{m+1} \in ]0, \varepsilon_{m+1}^0]$  such that

$$\sup_{x, y \in f^{-1}([a, b])} |\tilde{G}_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y) - \tilde{G}_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}(x, y)| \leq \frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}.$$

Gathering all those inequalities yields

$$\begin{aligned} |f(x) - f(y)| - \frac{m\varepsilon}{d} &\leq G_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}(x, y) \\ &\leq \tilde{G}_{(\varepsilon_0, \dots, \varepsilon_m, 0, \dots, 0)}(x, y) + \frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}\Delta_{a,b,f} \\ &\leq \tilde{G}_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y) + \frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}(\Delta_{a,b,f}+1) \\ &\leq G_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y) + \frac{\varepsilon}{d(2\Delta_{a,b,f}+1)}(2\Delta_{a,b,f}+1) \\ &\leq G_{(\varepsilon_0, \dots, \varepsilon_{m+1}, 0, \dots, 0)}(x, y) + \frac{\varepsilon}{d}. \end{aligned}$$

This ends the recurrence. The lower bound in (145) is finally proved when  $m = d$  is reached. For (147) it suffices to write

$$\begin{aligned} |\nabla f_S(x) - \nabla f_S(\pi_S x)| &\leq \varepsilon \quad , \quad |\nabla f_S(\pi_S x)| = G_{(0, \dots, 0)}(\pi_S x) \geq m_{f,a,b}, \\ |\Pi_S \nabla_{S'} f(x) - \nabla f_S(x)| &\leq \varepsilon, \end{aligned}$$

and then to choose  $\varepsilon \leq \frac{m_{f,a,b}}{4}$ .

Finally with  $S, S' \in \mathcal{S}_{[a,b]}$ ,  $\dim S' = d$ , and  $x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}$ , we have

$$\nabla f(x) \cdot \Pi_S \nabla f(x) - (1 - \frac{\varepsilon'}{2}) |\Pi_S \nabla f(x)|^2 = \frac{\varepsilon'}{2} |\Pi_S \nabla f(x)|^2 \geq \frac{\varepsilon' m_{f,a,b}^2}{8},$$

while  $\|\nabla f\|_{L^\infty} \leq M_f$  and

$$|\Pi_S \nabla f(x) - \nabla f_S(x)| \leq \varepsilon.$$

By choosing  $\varepsilon > 0$  small enough we obtain for all  $S, S' \in \mathcal{S}_{[a,b]}$ ,  $\dim S' = d$ , and all  $x \in S'$ ,

$$\nabla f(x) \cdot \nabla f_S(x) - (1 - \varepsilon') |\nabla f_S(x)|^2 \geq 0.$$

□

### 8.3.3 Agmon type estimate for Lipschitz subanalytic potential

Proposition 8.8 says that Hypothesis 1.6 is satisfied when  $f$  is a real analytic function on the compact Riemannian real analytic manifold  $M$  (Hypothesis 1.3), where the values  $c_1 < \dots < c_{N_f}$  are the values associated with horizontal strata of  $f$ .

We now prove that Hypothesis 2.16 is a consequence of Hypothesis 1.3 so that Theorem 6.3 and



its consequences in Section 7 hold true under Hypothesis 1.3.

Remember that Hypothesis 2.16 gathers the results of Proposition 2.13 and Proposition 2.15 adapted to a general Lipschitz function  $f$ . We will first prove the analogous of Proposition 2.15 in Proposition 8.14 and then deduce in Proposition 8.16 the analogous of Proposition 2.13.

**Proposition 8.14.** *Under Hypothesis 1.3 and when  $c_1 < \dots < c_{N_f}$  are the values associated with horizontal strata according to Proposition 8.8, choose  $a < b$  such that  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \emptyset$ . If  $\lim_{h \rightarrow 0} \lambda_h = 0$ , the resolvent kernel  $(\Delta_{f, f^{-1}([a, b]), h} - \lambda_h)^{-1}(x, y)$  is well defined and satisfies*

$$(\Delta_{f, f^{-1}([a, b]), h} - \lambda_h)^{-1}(x, y) = \tilde{O}\left(e^{-\frac{|f(x) - f(y)|}{h}}\right),$$

according to Definition 2.14.

*Proof.* This result relies on the stratification tools introduced in the previous paragraph. It is proved in several steps, the first one being a localization in suitable open subsets. Let us fix  $x_0 \in f^{-1}([a, b])$  with  $f(x_0) = t_0$  and we fix the neighborhood of  $x_0$  in  $f^{-1}([a, b])$  as

$$\mathcal{V}_{x_0} = f^{-1}([a, b]) \cap f^{-1}(]t_0 - \eta; t_0 + \eta)$$

where  $\eta > 0$  is a small parameter to be fixed at the end of the analysis.

We want to prove that for any  $\varepsilon > 0$ , any  $h \in ]0, h_\varepsilon[$ ,  $\Delta_{f, f^{-1}([a, b]), h} - \lambda_h$  is invertible and that for any  $r_h \in L^2(f_a^b)$  such that  $\text{supp } r_h \subset \mathcal{V}_{x_0}$ ,  $\omega_h = (\Delta_{f, f^{-1}([a, b]), h} - \lambda_h)^{-1} r_h$  satisfies

$$\|e^{\frac{|f(x) - f(x_0)|}{h}} \omega_h\|_{W_\partial(f_a^b)} = \tilde{O}(1) \|r_h\|.$$

It will be convenient to call  $a = t_1$  and  $b = t_2$  especially when the arguments gather the three levels  $t_k$ ,  $k = 0, 1, 2$ .

**i) Open covering of  $f^{-1}([a, b])$ :** Because  $[a, b] \cap \{c_1, \dots, c_{N_f}\} = \emptyset$ , for any  $x \in f^{-1}([a, b])$  there exist a neighborhood  $U_x$  of  $x$  in  $M$  and a smooth function  $\varphi_x$  on  $U_x$  and a constant  $C_x > 0$  such that

$$\nabla f(y) \cdot \nabla \varphi_x(y) \geq \frac{1}{C_x} \quad \text{and} \quad |\nabla \varphi_x(y)| \leq C_x \quad \text{for a.e. } y \in U_x.$$

Take for  $\varphi_x(y)$  the coordinate function  $\varphi_x(y) = y^1$  given in Hypothesis 1.6 (see also Proposition 8.8). By the compactness of  $f^{-1}(\{t_0, t_1, t_2\})$ , there exists a finite family  $(x_i)_{i \in I}$  and constant  $\kappa > 0$  small enough such that

$$\nabla f \cdot (\kappa \nabla \varphi_{x_i}(y)) \geq 2|\kappa \nabla \varphi_{x_i}(x)|^2 \geq 2\kappa^3 > 0 \quad \text{for a.e. } y \in U_{x_i}$$

and for all  $i \in I$ .

Once this open covering  $f^{-1}(\{t_k, k = 0, 1, 2\}) \subset \cup_{i \in I} U_{x_i}$  is fixed, we can choose the parameter  $\eta > 0$  such that

$$f^{-1}\left(\{t_k, k = 0, 1, 2\} + \right] - \frac{\eta}{2}, \frac{\eta}{2} [) \subset \cup_{i \in I} U_{x_i}.$$

Again when  $\eta > 0$  is fixed and the stratification  $\mathcal{S}_{[a, b]}$  is introduced as in Subsection 8.3.2, Proposition 8.13 provides us an open covering

$$(\mathcal{T}_{\omega_S, \varepsilon_{\dim S}})_{S \in \mathcal{S}_{[a, b]}}$$

such that the associated functions,  $f_S$ ,  $S \in \mathcal{S}_{[a, b]}$ , and  $G_{(\varepsilon_0, \dots, \varepsilon_d)}$  satisfy

$$\begin{aligned} \forall x, y \in f^{-1}([a, b]), \quad |f(x) - f(y)| - \eta &\leq G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, y) \leq M_f d(x, y), \\ \forall S \in \mathcal{S}_{[a, b]}, \nabla f(x) \cdot \nabla f_S(x) - (1 - \frac{\eta}{2}) |\nabla f_S(x)|^2 &\leq 0 \quad \text{for a.e. } x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}, \\ \forall S \in \mathcal{S}_{[a, b]}, \forall x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}, \quad |\nabla f_S(x)| &\geq \frac{m_{f, a, b}}{2}. \end{aligned}$$

We now choose our open covering  $f^{-1}([a, b]) \subset \cup_{j \in J} \Omega_j$ :

- $J = \mathcal{S}_{[a,b]} \cup I$ ;
- when  $j = S \in \mathcal{S}_{[a,b]}$ ,  $\Omega_j = \{x \in \mathcal{T}_{\omega_S, \varepsilon_{\dim S}}, |f(x) - t_k| > \frac{\eta}{4}, k = 0, 1, 2\}$  and  $\varphi_j = f_S$ ;
- when  $j = i \in I$ ,  $\Omega_j = U_{x_i} \cap f^{-1}(\{t_k, k = 0, 1, 2\} + ] - \frac{\eta}{2}, \frac{\eta}{2}[)$ , and  $\varphi_j = \kappa \varphi_{x_i}$ .

**ii) Choice of a global function  $\varphi$ :** Once the open covering  $f^{-1}([a, b]) \subset \cup_{j \in J} \Omega_j$  is fixed we choose

$$\varphi(x) = (1 - \eta) \inf_{\substack{\gamma \in \mathcal{C}^1([0, 1]; f^{-1}([a, b])) \\ \gamma(0) = x_0, \gamma(1) = x}} \int_0^1 \mathbf{1}_{[a, b] \setminus \cup_{k=0}^2 ]t_k - \eta, t_k + \eta[} (f(\gamma(t))) F_{(\varepsilon_0, \dots, \varepsilon_d)}(\gamma(t)) |\gamma'(t)| dt.$$

Because the integrand is 0 when  $f(\gamma(t)) \in ]t_k - \eta, t_k + \eta[$  the integral  $\int_0^1 [\dots] dt$  can be replaced by  $\int_{T_0}^{T_1} [\dots] dt$  where  $T_0 = \max\{t \in [0, 1], f(\gamma(t)) \in [t_0 - \eta, t_0 + \eta]\}$  and

$$\begin{aligned} T_1 &= \min\{t \in [0, 1], f(\gamma(t)) \geq f(b) - \eta\} \quad \text{if } f(x) > f(b) - \eta, b = t_2, \\ T_1 &= \min\{t \in [0, 1], f(\gamma(t)) \leq f(a) + \eta\} \quad \text{if } f(x) < f(a) + \eta, a = t_1. \end{aligned}$$

The comparison with  $G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, x_0)$  then gives

$$\frac{\varphi(x)}{1 - \eta} \geq G_{(\varepsilon_0, \dots, \varepsilon_d)}(x, x_0) - 2\eta \geq |f(x) - f(x_0)| - 3\eta$$

and

$$\forall x \in f^{-1}([a, b]), \quad \varphi(x) \geq |f(x) - f(x_0)| - (b - a + 3)\eta. \quad (148)$$

The function  $\varphi$  is a Lipschitz function of which the gradient can be estimated almost surely in any  $\Omega_j$ ,  $j \in J$ . The triangle inequality for a pseudodistance implies for all  $x, x' \in f^{-1}([a, b]) \cap \Omega_j$

$$\begin{aligned} \frac{|\varphi(x) - \varphi(x')|}{(1 - \eta)} &\leq \inf_{\substack{\gamma \in \mathcal{C}^1([0, 1]; f^{-1}([a, b])) \\ \gamma(0) = x, \gamma(1) = x'}} \int_0^1 \mathbf{1}_{[a, b] \setminus \cup_{k=0}^2 ]t_k - \eta, t_k + \eta[} (f(\gamma(t))) F_{(\varepsilon_0, \dots, \varepsilon_d)}(\gamma(t)) |\gamma'(t)| dt \\ &\leq \inf_{\substack{\gamma \in \mathcal{C}^1([0, 1]; f^{-1}([a, b]) \cap \Omega_j) \\ \gamma(0) = x, \gamma(1) = x'}} \int_0^1 |\nabla \varphi_j(\gamma(t))|(\gamma(t)) |\gamma'(t)| dt. \end{aligned}$$

We used that

$$\mathbf{1}_{[a, b] \setminus \cup_{k=0}^2 ]t_k - \eta, t_k + \eta[} (f(\gamma(t))) F_{(\varepsilon_0, \dots, \varepsilon_d)}(\gamma(t)) |\gamma'(t)|$$

is

- 0 and therefore bounded by  $|\nabla \varphi_j(\gamma(t))|$  when  $\gamma(t) \in \Omega_j \subset f^{-1}(\cup_{k=0}^2 ]t_k - \eta, t_k + \eta[)$  when  $j \in I$ ;
- bounded by  $|\nabla f_S(\gamma(t))| |\gamma'(t)|$  when  $\gamma(t) \in \Omega_j$  with  $j = S \in \mathcal{S}_{[a,b]}$ .

We deduce

$$\forall j \in J, \quad |\nabla \varphi(x)| \leq (1 - \eta) |\nabla \varphi_j(x)| \quad \text{for a.e. } x \in \Omega_j. \quad (149)$$

**iii) Partition of unity:** Let  $\sum_{j \in J} \chi_j^2 \equiv 1$  in a neighborhood of  $f^{-1}([a, b])$  be a partition of unity with  $\chi_j \in \mathcal{C}_0^\infty(\Omega_j; [0, 1])$  where  $f^{-1}([a, b]) \subset \cup_{j \in J} \Omega_j$  is the open covering introduced in **i)**. Accordingly the function  $\varphi \in W^{1, \infty}(f^{-1}([a, b]))$  is the one introduced in **ii)**. For any  $\omega \in W_{\partial}(f_a^b; \Lambda T^*M)$ , the relations (19) and (18) of Lemma 2.10 give

$$\begin{aligned} \operatorname{Re} Q_{f, f^{-1}([a, b]), h}(\omega, e^{\frac{2\varphi}{h}} \omega) &= \sum_{j \in J} \operatorname{Re} Q_{f, f^{-1}([a, b]), h}(\chi_j \omega, e^{\frac{2\varphi}{h}} \chi_j \omega) - h^2 \|\nabla \chi_j | \tilde{\omega} \|^2 \\ &= \sum_{j \in J} \left\| d_{f, f^{-1}([a, b]), h} \chi_j \tilde{\omega} \right\|^2 + \left\| d_{f, f^{-1}([a, b]), h}^* \chi_j \tilde{\omega} \right\|^2 \\ &\quad - \langle \chi_j \tilde{\omega}, |\nabla \varphi|^2 \chi_j \tilde{\omega} \rangle - h^2 \|\nabla \chi_j | \tilde{\omega} \|^2. \end{aligned}$$

With (149) we deduce

$$\begin{aligned} \operatorname{Re} Q_{f,f^{-1}([a,b]),h}(\omega, e^{\frac{2\varphi}{h}}\omega) &= \sum_{j \in J} \left\| d_{f,f^{-1}([a,b]),h} \chi_j \tilde{\omega} \right\|^2 + \left\| d_{f,f^{-1}([a,b]),h}^* \chi_j \tilde{\omega} \right\|^2 \\ &\quad - (1-\eta)^2 \langle \chi_j \tilde{\omega}, |\nabla \varphi_j|^2 \chi_j \tilde{\omega} \rangle - h^2 \|\nabla \chi_j \tilde{\omega}\|^2. \end{aligned}$$

Now  $\varphi_j$  can be extended to a  $C^\infty$  function away from a neighborhood of  $\operatorname{supp} \chi_j$  without changing the expression and using (18) and (21) with  $\omega_j = e^{-(1-\eta)\frac{\varphi_j}{h}} \chi_j \tilde{\omega} \in W_\partial(f_a^b; \Lambda T^*M)$  and  $\varphi$  replaced by  $(1-\eta)\varphi_j$ , we obtain

$$\begin{aligned} &\left\| d_{f,f^{-1}([a,b]),h} \chi_j \tilde{\omega} \right\|^2 + \left\| d_{f,f^{-1}([a,b]),h}^* \chi_j \tilde{\omega} \right\|^2 - (1-\eta)^2 \langle \chi_j \tilde{\omega}, |\nabla \varphi_j|^2 \chi_j \tilde{\omega} \rangle \\ &= Q_{f-(1-\eta)\varphi_j, f^{-1}([a,b]),h}(\chi_j \tilde{\omega}, \chi_j \tilde{\omega}) \\ &\quad + (1-\eta) \langle (2\nabla f \cdot \nabla \varphi_j - 2(1-\eta)|\nabla \varphi_j|^2 + h\mathcal{L}_{\nabla \varphi_j} + h\mathcal{L}_{\nabla \varphi_j}^*) \chi_j \tilde{\omega}, \chi_j \tilde{\omega} \rangle \\ &\quad + h(1-\eta) \left( \int_{f=b} - \int_{f=a} \right) \langle \chi_j \tilde{\omega}, \chi_j \tilde{\omega} \rangle_{\Lambda T_\sigma^* M} \left( \frac{\partial \varphi_j}{\partial n} \right) (\sigma) d\sigma. \end{aligned}$$

Because all the  $\varphi_j$  are  $C^\infty$  functions there exists  $C > 0$  such that

$$\left| \langle (\mathcal{L}_{\nabla \varphi_j} + \mathcal{L}_{\nabla \varphi_j}^*) \chi_j \tilde{\omega}, \chi_j \tilde{\omega} \rangle \right| \leq C \|\chi_j \tilde{\omega}\|^2.$$

We have proved

$$\operatorname{Re} Q_{f,f^{-1}([a,b]),h}(\omega, e^{\frac{2\varphi}{h}}\omega) = \sum_{j \in J} Q_{f-(1-\eta)\varphi_j}(\chi_j \tilde{\omega}, \chi_j \tilde{\omega}) \quad (150)$$

$$+ 2(1-\eta) \langle (\nabla f \cdot \nabla \varphi_j - (1-\eta)|\nabla \varphi_j|^2) \chi_j \tilde{\omega}, \chi_j \tilde{\omega} \rangle \quad (151)$$

$$+ h(1-\eta) \left( \int_{f=b} - \int_{f=a} \right) \langle \chi_j \tilde{\omega}, \chi_j \tilde{\omega} \rangle_{\Lambda T_\sigma^* M} \left( \frac{\partial \varphi_j}{\partial n} \right) (\sigma) d\sigma \quad (152)$$

$$+ R_h(\tilde{\omega})$$

where the constant  $C_\eta > 0$  in

$$|R_h(\tilde{\omega})| \leq C_\eta h \|\tilde{\omega}\|^2$$

depends on  $\eta > 0$  via the construction of the open covering  $f^{-1}([a,b]) \subset \cup_{j \in J} \Omega_j$ , the functions  $\varphi_j$  and the partition of unity  $\sum_{j \in J} \chi_j^2 \equiv 1$ .

**iv) Local lower bounds:** We give a lower bound for every individual  $j \in J$  for the three terms (150)(151) and (152). The first one (150) is obviously non negative according to

$$Q_{f-(1-\eta)\varphi_j}(\chi_j \tilde{\omega}, \chi_j \tilde{\omega}) = \left\| d_{f-(1-\eta)\varphi_j}(\chi_j \tilde{\omega}) \right\|^2 + \left\| d_{f-(1-\eta)\varphi_j}^*(\chi_j \tilde{\omega}) \right\|^2 \geq 0.$$

For the other terms we distinguish according to  $j \in I$  and  $j = S \in \mathcal{S}_{[a,b]}$ .

- $j \in I$ : In this case by recalling the choice  $\varphi_j = \kappa \varphi_{x_j}$ , we know

$$\nabla f \cdot \nabla \varphi_j \geq 2|\nabla \varphi_j|^2 \geq \kappa^2 > 0 \quad \text{for a.e. } x \in \Omega_j.$$

This implies

$$2(1-\eta) \left[ \nabla f \cdot \nabla \varphi_j - (1-\eta)|\nabla \varphi_j|^2 \right] \geq 2(1-\eta) \|\nabla \varphi_j\|^2 \geq (1-\eta)\kappa^2 \quad \text{for a.e. } x \in \Omega_j,$$

where the positive constant  $(1 - \eta)\kappa^2$  is uniform w.r.t  $j \in I$ .

Finally the condition  $\nabla f \cdot \nabla \varphi_j \geq 0$  makes sense almost surely along the boundary  $f^{-1}(\{a, b\})$  so that the integral terms (152) are non negative.

•  $j = S \in \mathcal{S}_{[a,b]}$  : Our choice of  $\Omega_j \subset \{x \in M, |f(x) - t_k| > \eta, k = 0, 1, 2\}$  implies that the boundary terms (152) vanish. Finally our choice  $\varphi_j = f_S$  in **i**) implies

$$\nabla f \cdot \nabla \varphi_j - (1 - \frac{\eta}{2}) |\nabla \varphi_j|^2 \geq 0 \quad \text{for a.e. } x \in \Omega_j,$$

We deduce

$$2(1 - \eta) \left[ \nabla f \cdot \nabla \varphi_j - (1 - \eta) |\nabla \varphi_j|^2 \right] \leq 2(1 - \eta) \frac{\eta}{2} |\nabla \varphi_j|^2 \geq (1 - \eta) \frac{m_{f,a,b}^2}{4}$$

almost every where in  $\Omega_j$  with the positive constant independent  $(1 - \eta) \frac{m_{f,a,b}^2}{4}$  independent of  $j = S \in \mathcal{S}_{[a,b]}$ .

**v) Gathering all the lower bounds and conclusion:**

We take  $\nu_\eta = (1 - \eta) \min \left\{ \kappa^2, \frac{m_{f,a,b}^2}{4} \right\}$  and summing the previous lower bound w.r.t  $j \in J$  leads to

$$\text{Re } Q_{f,f^{-1}([a,b]),h}(\omega, e^{\frac{2\varphi}{h}} \omega) - \lambda_h \|e^{\frac{\varphi}{h}} \omega\|^2 \geq (\nu_\eta - C_\eta h - \lambda_h) \|\tilde{\omega}\|^2 \geq \frac{\nu_\eta}{2} \|\tilde{\omega}\|^2$$

by taking  $h \in ]0, h_\eta[$  for some small enough  $h_\eta > 0$ .

Because  $\Delta_{f,f^{-1}([a,b]),h}$  is self-adjoint the inequality

$$\text{Re } \langle e^{\frac{2\varphi}{h}} \omega, (\Delta_{f,f^{-1}([a,b]),h} - \lambda_h) \omega \rangle \geq \frac{\nu_\eta}{2} \|\tilde{\omega}\|^2 \geq c_{\eta,h} \|\omega\|^2, \quad \tilde{\omega} = e^{\frac{\varphi}{h}} \omega,$$

valid for all  $\omega \in D(\Delta_{f,f^{-1}([a,b]),h})$  for some  $c_{\eta,h} > 0$ , implies that  $\lambda_h$  belongs to the resolvent set of  $\Delta_{f,f^{-1}([a,b]),h}$ .

When  $\omega_h$  solves  $(\Delta_{f,f^{-1}([a,b]),h} - \lambda_h) \omega_h = r_h$ , the same inequality with  $\varphi \equiv 0$  on  $\text{supp } r_h \subset f^{-1}([t_0 - \eta, t_0 + \eta])$ , gives

$$\|r_h\| \|\tilde{\omega}_h\| \geq \frac{\nu_\eta}{2} \|\tilde{\omega}_h\|^2,$$

and  $\|\tilde{\omega}_h\| \leq \frac{2}{\nu_\eta} \|r_h\|$ . By using again (18) we deduce

$$\begin{aligned} \frac{2}{\nu_\eta} \|r_h\|^2 &\geq \|r_h\| \|\tilde{\omega}_h\| \geq \text{Re } Q_{f,f^{-1}([a,b]),h}(\omega_h, e^{\frac{2\varphi}{h}} \omega_h) - \lambda_h \|\tilde{\omega}_h\|^2 \\ &\geq \|d_{f,h} \tilde{\omega}_h\|^2 + \|d_{f,h}^* \tilde{\omega}_h\|^2 - |\nabla \varphi|^2 \tilde{\omega}_h - \lambda_h \|\tilde{\omega}_h\|^2. \end{aligned}$$

And finally there exists a constant  $M_\eta > 0$  such that

$$\frac{M_\eta}{h^2} \|r_h\|^2 \geq \|\tilde{\omega}_h\|^2 + \|d\tilde{\omega}_h\|^2 + \|d^* \tilde{\omega}_h\|^2 = \|e^{\frac{\varphi}{h}} \omega_h\|_{W_\circ(f_a^b, \Lambda T^* M)},$$

with  $\varphi(x) \geq |f(x) - f(x_0)| - (b - a + 3)\eta$ .

We conclude by taking  $\eta > 0$ , on which all the construction depends, arbitrarily small, the limit  $h \rightarrow 0$  being taken for any fixed  $\eta > 0$ .  $\square$

**Remark 8.15.** *In this proof, we did not use the global solution  $\varphi$  to the inequation  $|\nabla \varphi|^2 - |\nabla f|^2 \leq 0$  provided in **ii**) because such a solution has no better regularity than the Lipschitz one. Instead we really introduce the partition of unity in the process of obtaining exponential decay estimates with all the functions  $\varphi_j$  which are regular enough and allow to use the various integration tricks of Lemma 2.10, used in particular in order to absorb the singularity of the term  $h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*)$ .*

**Proposition 8.16.** *Under Hypothesis 1.3 and when  $c_1 < \dots < c_{N_f}$  are the values associated with horizontal strata according to Proposition 8.8, choose  $a < b$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$  and call  $U$  the compact set  $f^{-1}(\{c_1, \dots, c_{N_f}\} \cap [a, b])$ . All families  $(\lambda_h)_{h>0} \in \mathbb{C}$ ,  $(r_h)_{h>0} \in L^2(f_a^b)$  and  $\omega_h \in D(\Delta_{f, f^{-1}([a, b]), h}) \subset W_{\partial}(f_a^b; \Lambda T^*M)$  such that*

$$(\Delta_{f, f^{-1}([a, b]), h} - \lambda_h)\omega_h = r_h \quad , \quad \text{supp } r_h \subset K \quad , \quad \lim_{h \rightarrow 0} \lambda_h = 0 \quad ,$$

where  $K$  is a fixed compact subset of  $f^{-1}([a, b])$ , satisfy the estimate

$$\|e^{\frac{\min_{y \in U \cup K} |f(\cdot) - f(y)|}{h}} \omega_h\|_{W_{\partial}(f_a^b)} = \tilde{O}(1) [\|r_h\|_{L^2(f_a^b)} + t_U \|\omega_h\|_{L^2(f_a^b)}] \quad ,$$

where  $t_U = 1$  if  $U \neq \emptyset$  and  $t_U = 0$  if  $U = \emptyset$ .

*Proof.* The case when  $U = \emptyset$  is contained in Proposition 8.14. Let us consider the case when  $U \neq \emptyset$ . First of all, the positivity of  $\Delta_{f, f^{-1}([a, b]), h}$  implies

$$\begin{aligned} \|d_{f, h} \omega_h\|^2 + \|d_{f, h}^* \omega_h\|^2 + (C - \text{Re } \lambda_h) \|\omega_h\|^2 &= \text{Re} \langle \omega_h, (\Delta_{f, f^{-1}([a, b]), h} + C - \lambda_h) \omega_h \rangle \\ &\leq \|r_h\| \|\omega_h\| + C \|\omega_h\|^2. \end{aligned}$$

By taking  $C > 2(1 + \|f\|_{W^{1, \infty}})$  we obtain

$$\|\omega_h\|_{W_{\partial}(f_a^b)} = \tilde{O}(1) (\|r_h\|_{L^2(f_a^b)} + \|\omega_h\|_{L^2(f_a^b)})$$

which provides  $W^{1, 2}$  estimates of  $\omega_h$  in any compact subset of  $f_a^b = f^{-1}([a, b])$ .

For  $\varepsilon > 0$  small enough, consider a cut-off function  $\chi_\varepsilon \in C^\infty(M; [0, 1])$  equal to 1 in  $K_\varepsilon = f^{-1}(\cup_{k=1}^{N_f} [c_k - \varepsilon, c_k + \varepsilon]) \cap [a, b])$  and to 0 in the complement of  $K_{2\varepsilon}$ . The form  $\chi_\varepsilon \omega_h$  solves

$$(\Delta_{f, h} - \lambda_h)((1 - \chi_\varepsilon)\omega_h) = (1 - \chi_\varepsilon)r_h + P_{\chi_\varepsilon} \omega_h \quad ,$$

where  $P_{\chi_\varepsilon}$  is a first order differential operator with coefficients supported in  $K_{2\varepsilon} \setminus K_\varepsilon$  and  $\chi_\varepsilon \omega_h \in \oplus_{k=1}^{N_f-1} \Delta_{f, f^{-1}([\max(c_k + \varepsilon, a), \min(c_{k+1} - \varepsilon, b)])}$ . The resolvent estimate of Proposition 8.14 applied to every  $\Delta_{f, f^{-1}([\max(c_k + \varepsilon, a), \min(c_{k+1} - \varepsilon, b)])}$  then implies

$$\|e^{\frac{\min_{y \in U \cup K} |f(\cdot) - f(y)|}{h}} \omega_h\|_{W_{\partial}(f_a^b)} \leq \tilde{O}(e^{\frac{10\varepsilon}{h}}) [\|r_h\|_{L^2(f_a^b)} + \|\omega_h\|_{L^2(f_a^b)}] \quad ,$$

and then we choose  $\varepsilon > 0$  arbitrarily small before taking the limit  $h \rightarrow 0$ .  $\square$

## 9 Applications

The spectral version of the stability theorem, Corollary 1.8 in the Introduction or Theorem 7.6 for a more general version, corresponds to what can be expected at the level of Arrhenius law identifying the exponential scales. It is a straightforward consequence of Theorem 6.3. But the construction of global quasimodes for Theorem 6.3 is actually much more informative. It allows to compute the subexponential factor, a la Eyring-Kramers, in many situations which lead to different kind of asymptotic behaviours. As it was discussed in the Introduction, no continuity with respect to  $f$  can be expected in the asymptotic leading term. Nevertheless some robust integral formulation allow to follow the effect of deformations of  $f$  on the spectral quantities and to explain the emerging discontinuities. Contrary to Theorem 6.3 and its consequences in Section 7, we do not have a satisfactory general formulation of this kind of refined stability property and we prefer to make explicit various examples, corresponding to interesting practical cases.

## 9.1 The generic Morse case

In this subsection, we recall the results of [LNV]. Although they were presented in the oriented case, those results hold in the more general case of non necessarily oriented compact Riemannian manifolds. The proofs are simply adapted by paying attention to the duality arguments, the Hodge  $\star$  operator sending the sections of  $\Lambda^p T^*M$  to sections of  $\Lambda^p T^*M \otimes \text{or}_M$ . The important assumption which was made in [LNV] concerns the simplicity of the critical values of the Morse function  $f$ : the latter function has distinct critical values, which allows in particular to identify critical points with critical values. In [LNV], the set  $\mathcal{U}$  of critical points was partitioned into lower  $\mathcal{U}_L = \cup_{p \in \{0, \dots, d\}} \mathcal{U}_L^{(p)}$ , upper  $\mathcal{U}_U = \cup_{p \in \{0, \dots, d\}} \mathcal{U}_U^{(p)}$ , and homological  $\mathcal{U}_H = \cup_{p \in \{0, \dots, d\}} \mathcal{U}_H^{(p)}$  critical points. This partition actually coincides with the partition of bar endpoints  $\mathcal{J} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$  in this order. In [LNV], we defined a boundary map  $\partial_{\mathcal{B}} : \mathcal{U}_U^{(p+1)} \rightarrow \mathcal{U}_L^{(p)}$  and  $\mathcal{U}_U \cup \mathcal{U}_H \subset \ker \partial_{\mathcal{B}}$ . It is exactly the dual version of the map  $\mathbf{d}_{\mathcal{B}}$  of Appendix B.2 defined by  $\mathbf{d}_{\mathcal{B}} : \mathcal{X}^{(p)} \rightarrow \mathcal{Y}^{(p+1)}$  and  $\mathcal{Y} \cup \mathcal{Z} \subset \ker \mathbf{d}_{\mathcal{B}}$ . Actually, in [LNV], we started with the homological point of view before we realized that working directly in terms of cohomology was more natural for this analysis. The link with relative cohomology groups of sublevel sets of  $f$ , which is detailed at the end of Appendix B.1, can be handled with elementary arguments under the assumptions of [LNV] (Morse function with distinct critical values). Note that this generic Morse situation is often used as a simple way to introduce persistent homology (see e.g. [EdHa]). Although it is an obvious bijection under the assumption that the Morse function  $f$  has simple critical values, we use the notations, when it is necessary,  $\underline{x}_\alpha, \underline{y}_\alpha$  or  $\underline{z}_\alpha$  for the critical points associated with values  $x_\alpha = f(\underline{x}_\alpha) \in \mathcal{X}$ ,  $y_\alpha = f(\underline{y}_\alpha) \in \mathcal{Y}$  and  $z_\alpha = f(\underline{z}_\alpha) \in \mathcal{Z}$ . As a comparison with the notations of Subsection 4.1, it is not necessary nor useful to distinguish  $x_\alpha = (a_\alpha, \alpha) \in \mathbb{R} \times A$  from the value  $a_\alpha = f(\underline{x}_\alpha)$ .

Finally note that the result of [LNV] can be recovered while combining Theorem 6.3 of the present text with the final computations of [LNV]-Section 4 which rely on local WKB approximations valid locally for any Morse function  $f$ .

Here is the main result of [LNV] with the above modified notations.

**Theorem 9.1.** *Assume that  $f$  is a Morse function with simple critical values. For any  $p \in \{0, \dots, d\}$ , there exists  $c > 0$  such that for every  $h > 0$  small enough, the spectrum of  $\Delta_{f,M,h}^{(p)}$  satisfies*

$$\sigma(\Delta_{f,M,h}^{(p)}) \cap [0, ch] = \sigma(\Delta_{f,h}^{(p)}) \cap [0, e^{-\frac{c}{h}}],$$

and the latter set consists in  $\text{card}(\mathcal{J}^{(p)})$  eigenvalues counted with multiplicity. For every  $h > 0$  small enough, there exists moreover a bijection  $j : \mathcal{J}^{(p)} \rightarrow \sigma(\Delta_{f,M,h}^{(p)}) \cap [0, ch]$ , where the latter set is counted with multiplicity, such that:

1. For every  $z_\alpha$  in  $\mathcal{Z}^{(p)}$ , the associated eigenvalue is

$$j(z_\alpha) = 0.$$

2. For every  $x_\alpha$  in  $\mathcal{X}^{(p)}$ ,  $x_\alpha$  being the lower endpoint of the bar  $[x_\alpha, y_\alpha[$ , and hence  $y_\alpha = \mathbf{d}_{\mathcal{B}}x_\alpha$ , there exists a homological constant  $\kappa_\alpha \in \mathbb{Q}^*$  such that

$$j(x_\alpha) = \kappa_\alpha^2 \frac{h}{\pi} \frac{|\lambda_1(\underline{y}_\alpha) \cdots \lambda_{p+1}(\underline{y}_\alpha)|}{|\lambda_1(\underline{x}_\alpha) \cdots \lambda_p(\underline{x}_\alpha)|} \frac{|\det \text{Hess } f(\underline{x}_\alpha)|^{\frac{1}{2}}}{|\det \text{Hess } f(\underline{y}_\alpha)|^{\frac{1}{2}}} e^{-2\frac{y_\alpha - x_\alpha}{h}} (1 + \mathcal{O}(h)),$$

where, for any critical point  $\underline{s}$  of  $f$  with index  $\ell$  and critical value  $s = f(\underline{s})$ ,  $\lambda_1(\underline{s}), \dots, \lambda_\ell(\underline{s})$  denote the negative eigenvalues of  $\text{Hess } f(\underline{s})$ .

3. And  $y_\alpha$  in  $\mathcal{Y}^{(p)}$ ,  $y_\alpha$  being the upper endpoint of the bar  $[x_\alpha, y_\alpha[$ , and hence  $y_\alpha = \mathbf{d}_{\mathcal{B}}x_\alpha$ , there exists a homological constant  $\kappa_\alpha \in \mathbb{Q}^*$  such that

$$j(y_\alpha) = \kappa_\alpha^2 \frac{h}{\pi} \frac{|\lambda_1(\underline{y}_\alpha) \cdots \lambda_p(\underline{y}_\alpha)|}{|\lambda_1(\underline{x}_\alpha) \cdots \lambda_{p-1}(\underline{x}_\alpha)|} \frac{|\det \text{Hess } f(\underline{x}_\alpha)|^{\frac{1}{2}}}{|\det \text{Hess } f(\underline{y}_\alpha)|^{\frac{1}{2}}} e^{-2\frac{y_\alpha - x_\alpha}{h}} (1 + \mathcal{O}(h)),$$

where, for any critical point  $\underline{s}$  of  $f$  with index  $\ell$  and critical value  $s = f(\underline{s})$ ,  $\lambda_1(\underline{s}), \dots, \lambda_\ell(\underline{s})$  denote the negative eigenvalues of  $\text{Hess } f(\underline{s})$ .

**Remark 9.2.** 1. Theorem 9.1 is a refinement of Theorem 1.7 in this generic Morse situation.

It extends Eyring-Kramers asymptotic formulas known in the case  $p = 0$ . The boundary version in  $f^{-1}([a, b])$  corresponding to Theorem 7.1 is also found in [LNV, Theorem 4.5]. In both papers, the general strategy consists in a recurrence with respect to the number of critical values, carried out by increasing the interval  $[a, b]$ . The setting in [LNV], was simpler because: a) the critical values were assumed to be simple while here they may be multiple or very degenerate; b) the subexponential factors of exponentially small quantities had explicit leading terms derived from the WKB approximations (this is not possible here).

<sup>1</sup> In this section, we will combine Theorem 6.3 with the local computations of [LNV]-Section 4 to provide a more general approach.

2. In [LNV], thanks to the Morse assumption, we could compute the subexponential factors using WKB and Laplace methods.. On the other hand, the exponential factors are given by global topological quantities: the lengths of the bar code. In the present paper we manage to compute the logarithmic equivalents of the small eigenvalues without any knowledge of the exponential factor.
3. The connexion between the local computation around the lower endpoint  $x_\alpha$  and the upper one  $y_\alpha = \mathbf{d}_B x_\alpha$  is implemented by an application of Stokes's formula. The boundary operator  $\partial$  for chains induces a linear application from  $H_{p+1}(f^{y_\alpha+\varepsilon}, f^{y_\alpha-\varepsilon})$  into  $H_p(f^{x_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon})$ . Under the generic Morse assumption, these spaces are 1-dimensional with natural bases given by the stable manifolds of  $\nabla f$ . This actually provides the coefficients  $\kappa_\alpha$  (see [LNV, Proposition 2.12]). When the critical values correspond to multiple critical points, such a construction has to be replaced by more general linear algebra (see Subsection 9.3).

As shown in [HKN], the homological constants  $\kappa_\alpha^2$  equal 1 when  $p = 0$ , and also when  $p = d$  and  $M$  is oriented by duality. In the case of oriented surfaces treated in [Lep2], a combination of these results together with simple duality and chain complex arguments then implies that these constants equal 1 for any  $p \in \{0, 1, 2\}$ . Nevertheless, contrary to this indication that it could be true in general, which was moreover our intuition when we wrote [LNV], this is not the case as soon as  $d \geq 3$  and even when  $d = 2$  in the non-oriented case. The simplest example comes from Morse theory on the projective plane. It is more generally related to the ‘‘open book picture’’ exhibited on the front cover of [LauB].

To be more specific, we shall prove the following result.

**Proposition 9.3.** *Let  $X$  be a  $d$ -dimensional manifold.*

1. If  $d = 1, 2$ , and  $X$  is orientable, then  $\kappa_\alpha^2 = 1$ ,
2. The coefficient  $\kappa_\alpha^2$  may be equal to 4 for some well chosen Morse functions on  $\mathbb{R}P^2$  and on  $\mathbb{R}P^3$ .
3. For  $d \geq 3$  and each integer  $n$ , there exists a manifold  $X_n$  of dimension  $d$  such that  $\kappa_\alpha^2 = n^2$ .
4. For  $d \geq 4$ , for any integer  $n$  and any closed manifold  $X$ , there is a function  $f_n$  on  $X$  such that  $\kappa_\alpha^2$  takes the value  $n^2$ .

---

<sup>1</sup>A small confusion occurred in the construction of accurate global quasimodes in [LNV, Section 4.2]: a version of Proposition 6.16 is missing and can be easily corrected.

*Proof.* The number  $\kappa_\alpha$  is obtained as follows: consider the sphere  $S^-(\underline{y}_\alpha)$  in the unit disc bundle of the descending manifold from  $\underline{y}_\alpha$ , the stable manifold of  $\nabla f$ . It is homologous to a multiple,  $\kappa_\alpha$  of the descending manifold from  $\underline{x}_\alpha$ ,  $W^-(\underline{x}_\alpha)$ , with  $\mathbf{d}_B x_\alpha = y_\alpha$ . But since the ascending manifold from  $\underline{x}_\alpha$ , the unstable manifold of  $\nabla f$ ,  $W^+(\underline{x}_\alpha)$  has intersection +1 with  $W^-(\underline{x}_\alpha)$ , the number  $\kappa_\alpha$  is the intersection number of  $S^-(\underline{y}_\alpha)$  and  $W^+(\underline{x}_\alpha)$ . We work here under the generic Morse-Smale assumption saying that all the stable and unstable manifolds are mutually transverse, which ensures the finiteness of  $\kappa_\alpha$ , within the construction of the Thom-Smale complex (see [LauB]). In homological terms, if we set  $x_\alpha = f(\underline{x}_\alpha)$ ,  $y_\alpha = f(\underline{y}_\alpha)$ , and  $\varepsilon > 0$  small enough, we have the maps

$$\begin{array}{ccc}
 & H_*(f^{y_\alpha+\varepsilon}, f^{y_\alpha-\varepsilon}) & \\
 & \downarrow \partial & \\
 H_{*-1}(f^{x_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon}) & \longrightarrow & H_{*-1}(f^{y_\alpha-\varepsilon}, f^{x_\alpha-\varepsilon}) \\
 & & \downarrow \\
 & & H_*(f^{y_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon})
 \end{array}$$

Now since  $H_*(f^{y_\alpha+\varepsilon}, f^{y_\alpha-\varepsilon}; \mathbb{Z})$  and  $H_*(f^{x_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon}; \mathbb{Z})$  are isomorphic to  $\mathbb{Z}$ , the  $\mathbb{R}$ -vector spaces  $H_*(f^{y_\alpha+\varepsilon}, f^{y_\alpha-\varepsilon})$  and  $H_*(f^{x_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon})$  have canonical generators (i.e. well defined, and not just up to a constant multiple).

But a generator on the left-hand side has its image zero in  $H_*(f^{y_\alpha+\varepsilon}, f^{x_\alpha-\varepsilon})$  by assumption. Therefore this generator has an image in  $H_{*-1}(f^{y_\alpha-\varepsilon}, f^{x_\alpha-\varepsilon})$  that lies in the image of  $\partial$ . It is thus equal to the image by  $\partial$  of  $\kappa(\alpha)$  times a generator.

Now consider the Morse function on  $\mathbb{R}P^2$  obtained by perturbing the following Morse-Bott function:

$$[x_0, x_1, x_2] \mapsto x_2^2$$

where  $[x_0, x_1, x_2]$  is the class of  $(x_0, x_1, x_2) \in S^2$  by the equivalence relation  $(x_0, x_1, x_2) \simeq (-x_0, -x_1, -x_2)$ . This Morse-Bott function has a point of index 2 at  $[0, 0, 1]$ , and a circle of index 0 at  $[\cos(\theta), \sin(\theta), 0]$  for  $\theta \in [0, \pi]$ . Perturbing this circle yields a pair of critical points of index 0 and 1, and the Thom-Smale complex is then

$$\partial \underline{z} = 2 \cdot \underline{y}, \partial \underline{y} = 0, \partial \underline{x} = 0$$

represented as

$$\begin{array}{c}
 \underline{z} \\
 \downarrow 2 \\
 \underline{y} \\
 \downarrow 0 \\
 \underline{x}
 \end{array}$$

The Barannikov complex (on  $\mathbb{Q}$  or  $\mathbb{R}$ ) is then

$$\begin{array}{c}
 z \\
 \downarrow \\
 y \\
 \\
 x
 \end{array}$$

But necessarily  $\kappa_\alpha = 2$ , hence  $\kappa_\alpha^2 = 4$ .



For  $\mathbb{R}P^3$ , which is orientable, we have the similar function  $[x_0, x_1, x_2, x_3] \mapsto x_3^2$  where  $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$  and we identify  $(x_0, x_1, x_2, x_3)$  and  $(-x_0, -x_1, -x_2, -x_3)$ . We then have a maximum  $x_3 = \pm 1$  of index 3, and an  $\mathbb{R}P^2$  Morse-Bott critical submanifold, which after perturbation yields a critical point of index 0, one of index 1 and one of index 2.

The Thom-Smale complex is then

$$\begin{array}{c} \underline{t} \\ \downarrow 0 \\ \underline{z} \\ \downarrow 2 \\ \underline{y} \\ \downarrow 0 \\ \underline{x} \end{array}$$

so again  $\kappa(z) = 2$ .

To obtain any squared integer, we can consider the lens space  $L(n, 1)$  quotient of  $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$  by

$$(z_0, z_1) \simeq (\omega z_0, \omega z_1)$$

where  $\omega$  is a primitive  $n$ -th root of unity. The function  $(z_0, z_1) \mapsto |z_0|^2$  has two critical circles a minimum and a maximum. After perturbation we get

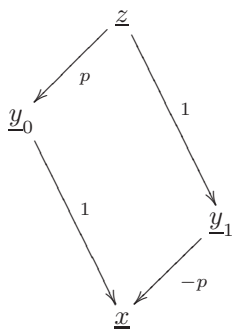
$$\begin{array}{c} \underline{t} \\ \downarrow 0 \\ \underline{z} \\ \downarrow n \\ \underline{y} \\ \downarrow 0 \\ \underline{x} \end{array}$$

and then  $|\kappa_\alpha| = n$ , since  $H_1(L(n, 1), \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ ,  $H_2(L(n, 1), \mathbb{Z}) = 0$ .

Now assume there is some function  $f$  on the manifold  $V$  with a given bar code  $\mathcal{B}_f$ , and we embed  $V$  into a manifold  $X$ . Consider the function  $g_\varepsilon(x) = d(x, V)^2 + \varepsilon \rho(d(x, V)^2) f(p(x))$  where  $\rho$  is nonnegative, equal to 1 near 0 and vanishes outside a neighbourhood of 0. then for  $\varepsilon > 0$  small enough, the lower part of the bar code of  $g_\varepsilon$  coincides with the bar code of  $f$ . As a result if there is a function with some  $\kappa_\alpha = n$  on  $V$ , the same holds for  $X$ . Consider the function  $f$  above on  $L(n, 1)$ , and normalize it so that the critical points are  $0, 1/3, 2/3, 1$ . Consider the subset  $\Lambda(n, 1) = \{x \in L(n, 1) \mid 1/4 \leq f(x) \leq 3/4\}$ . This is a Lens space with two punctures, hence embeds in  $\mathbb{R}^4$  as a subset of a compact hypersurface  $\Sigma_{n,1}$ : if  $\Lambda(n, 1)$  is contained in  $\{x \in \mathbb{R}^4 \mid \psi(x) = 0\}$  and extending  $\psi$  to a proper function having 0 as a regular value, we set  $\Sigma_{n,1} = \psi^{-1}(0)$ . Now we can extend  $f$  to a function  $\tilde{f}$  on  $\Sigma_{n,1}$  and its bar code contains  $\mathcal{B}_f$ . Applying the previous argument, we get a function close to  $d(x, \Sigma_{n,1})^2$  containing  $\mathcal{B}_f$  in its bar code. Since near infinity,  $d(x, \Sigma_{n,1})^2$  is close to  $|x|^2$ , we get a function  $F$  on the ball, with  $F \leq c$  and  $F = c$  near the boundary with arbitrary  $\kappa_\alpha$ . By embedding the ball in any 4-manifold  $M$ , we get a function on  $M$  with  $\kappa_\alpha = n$ . Again by embedding, this works on any manifold of dimension  $\geq 4$ . □

More generally if for some prime  $p$ , the homology mod  $p$  has rank different from the rational homology there must be a  $y$  such that  $p$  divides  $\kappa_\alpha$ .

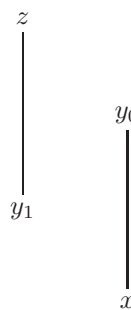
**Remark 9.4.** 1. *The converse does not hold, i.e. we may have  $\kappa_\alpha \neq \pm 1$  while the homology has the same rank for all fields. For example if we have a Morse complex containing the following diagram*



the corresponding homology vanishes and the rational Barannikov complex is

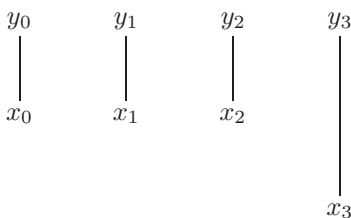


but mod  $p$ , we get



But in both cases the homology vanishes. Note however that if we look at the homology of sublevels, we can distinguish the two situations : if  $a < f(x) < f(y_1) < c < f(y_0)$  the rank of the homology  $H^*(f^c, f^a)$  depends on the coefficient field : for  $k = \mathbb{Q}$  we get 0 while mod  $p$ , we get 2.

2. When several critical values coincide, the numbers  $\kappa_\alpha$  are replaced by integral matrices. For example if we have the following bar code



and if  $a < x_3 < b < x_2 < c < y_2 < d$ , we have the map

$$\begin{array}{ccc}
& & H_*(f^{y_{\alpha+\varepsilon}}, f^{y_{\alpha-\varepsilon}}) \simeq \mathbb{Z}^4 \\
& & \downarrow \partial \\
H_{*-1}(f^{x_{\alpha+\varepsilon}}, f^{x_{\alpha-\varepsilon}}) \simeq \mathbb{Z}^3 & \longrightarrow & H_{*-1}(f^{y_{\alpha-\varepsilon}}, f^{x_{\alpha-\varepsilon}}) \\
& & \downarrow \\
& & H_*(f^{y_{\alpha+\varepsilon}}, f^{x_{\alpha-\varepsilon}}) \simeq \mathbb{Z}
\end{array}$$

hence we get a matrix  $\kappa \in M(4, 3, \mathbb{Z})$  such that  $\kappa \otimes \mathbb{R}$  is surjective. We can then consider the singular values of  $M$ , and we get three numbers  $\kappa_1, \kappa_2, \kappa_3$ , however these are not the homological constants that will yield the prefactor of the eigenvalues, since we must first compose with diagonal matrices depending on the Hessian at each critical point involved (see Proposition 9.10).

## 9.2 Simple critical values for non Morse functions

We consider here cases where changing the function  $f$  from  $f_1$  to  $f_2$  leads to explicit changes of the global quasimodes  $(\varphi_j^{(p)})_{j \in \mathcal{J}^{(p)}(a,b)}$  and provides accurate formulas, even for the subexponential factor, already known when  $f_1$  is a generic Morse function. It works especially well for functions, i.e. for  $p = 0$ , and although we are not considering Dirichlet boundary conditions at  $f^{-1}(\{b\})$  in  $f_a^b$ , like it is done in the study of quasi-stationary distributions, this sketches possible generalizations of the analyses made in [LeNi, DLLN1, DLLN2, LeNe1, LeNe2]. Note however that, though obtaining precise estimates on the low spectrum of the corresponding Witten Laplacians with Dirichlet boundary conditions is an important step in the studies made in [LeNi, DLLN1, DLLN2, LeNe1, LeNe2], these works actually focus on further issues such as the exit events or the concentration of the associated quasi-stationary distributions. In particular, in [DLLN1] are considered rare exit events, which are actually rather related with the low spectrum of appropriate Witten Laplacians with mixed Dirichlet–Neumann boundary conditions. Simple cases when  $p \neq 0$  will also be discussed afterwards.

### 9.2.1 Degenerate local minima

We consider a reference function  $f_1$  which is a generic Morse function like in Theorem 9.1 with a bar code  $\mathcal{B}_{f_1} = ([a_{1,\alpha}^*, b_{1,\alpha}^{*+1}]_{\alpha \in \mathcal{A}_1}$ . In particular in degree 0, there is one bar  $[a_{1,0}^{(0)}, +\infty[ = [x_{1,0}^{(0)}, y_{1,0}^{(1)}[$  associated with the global minimum  $a_{1,0}^{(0)}$  and the sublevel set  $\Omega_{1,0}^{(0)} = M = f_1^{+\infty}$ , and there are bars  $[x_{1,k}^{(0)}, y_{1,k}^{(1)}[ \in \mathcal{A}_{1,c}$ ,  $1 \leq k \leq K_0$  where  $y_{1,k}^{(1)}$  is the value of saddle point and  $x_{1,k}^{(0)}$  is the global minimum value of the newly created connected component  $\Omega_{1,k}^{(0)}$  of  $f_1^{y_{1,k}^{(1)}}$ , when we pass from the sublevel set  $f_1^{y_{1,k}^{(1)+0}}$  to  $f_1^{y_{1,k}^{(1)-0}}$ .

We take  $\ell_{min}^{(0)} < \min \left\{ y_{1,k}^{(1)} - x_{1,k}^{(0)}, |x_{1,k}^{(0)} - x_{1,k'}^{(0)}|, 0 \leq k < k' \leq K_0 \right\}$  and we assume that the function  $f_2$  satisfies Hypothesis 1.2 and coincides with  $f_1$  except around the local minima. The open set called

$$\omega_k^{(0)} = \Omega_{1,k}^{(0)} \cap f_1^{x_{1,k}^{(0)} + \frac{\ell_0}{2}},$$

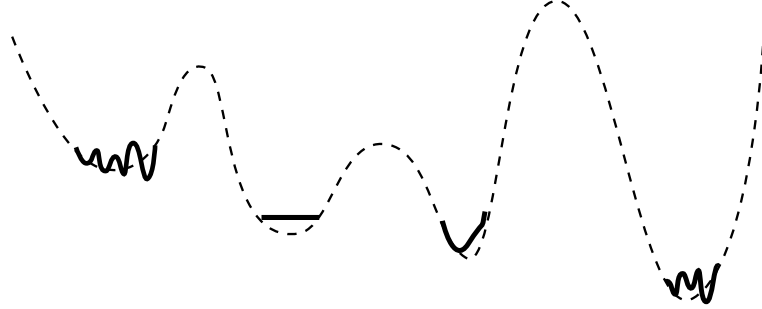
is a connected open neighborhood of  $x_{1,k}^{(0)}$  for all  $k = 0, \dots, K_0$ . The two functions  $f_1$  and  $f_2$  are compared by:

- i)  $f_1 \equiv f_2$  in a neighborhood of  $M \setminus (\sqcup_{k=0}^{K_0} \omega_k^{(0)})$ ;
- ii)  $\|f_1 - f_2\|_{C^0} \leq \frac{\ell_0}{4}$ .

Those two assumptions combined with the stability theorem

$$d_{\text{bot}}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_{C^0}$$

recalled in Appendix B.3, ensure that there are exactly  $K_0 + 1$  bars  $[x_{2,k}^{(0)}, y_{2,k}^{(0)}]$  of degree 0 and length larger than  $\frac{\ell_0}{2}$ , where the saddle points are not changed  $y_{2,k}^{(1)} = y_{1,k}^{(1)}$  for  $1 \leq k \leq K_0$ . Additionally and especially because with our choice of  $\ell_0 < \min \{|x_{1,k}^{(0)} - x_{1,k'}^{(0)}|, k < k'\}$  and **ii**), the associated connected component remain unchanged as well  $\Omega_{2,k}^{(0)} = \Omega_{1,k}^{(0)}$  for  $0 \leq k \leq K_0$ . We drop the index  $j = 1, 2$  for  $\Omega_k^{(0)}$  and  $y_k^{(1)}$ . Like in the previous Subsection, we use the notation



**Figure 15:** The function  $f_1$  is represented by dashed lines and the modification giving  $f_2$  by plain lines.

$\underline{s}$  for the point associated with the critical value  $s$ , when it is uniquely defined.

**Proposition 9.5.** *Under the above assumptions and in particular the comparison i)ii) between  $f_1$  and  $f_2$ , the  $\tilde{o}(e^{-\frac{\ell_0}{h}})$  eigenvalues of  $\Delta_{f_2,h}^{(0)}$  are given by*

$$\frac{h|\lambda_1(\underline{y}_k^{(1)})|}{\pi \left| \det \text{Hess } f_1(\underline{y}_k^{(1)}) \right|^{1/2}} \frac{e^{-2\frac{y_k^{(1)} - x_{2,k}^{(0)}}{h}}}{(\pi h)^{-d/2} \int_{\Omega_k^{(0)}} e^{-2\frac{f_2(x) - x_{2,k}^{(0)}}{h}} dx} \times (1 + \mathcal{O}(h)) \quad (153)$$

as  $h \rightarrow 0$  for all  $k = 0, 1, \dots, K_0$  (it is exactly 0 for  $k = 0$ ).

With this formula it then suffices to apply the Laplace method for the integral  $\int_{\Omega_k^{(0)}} e^{-2\frac{f(x) - x_{2,k}^{(0)}}{h}} dx$  in order to exhibit various asymptotic behaviours as  $h \rightarrow 0$  of the subexponential factor. We refer in particular to [AGV] for the case when  $f$  is a multidimensional polynomial function.

*Proof.* When we work with functions, we are actually in the simpler framework of [HKN] for the generic Morse function  $f_1$ . The problem consists in computing the square modulus of the interaction  $\langle \psi_k^{(1),h}, d_{f,h} T_{\delta_2} \varphi_k^{(0),h} \rangle$  where  $\psi_k^{(1),h}$  is a local WKB-approximation of eigenvectors of  $\Delta_{f,h}^{(1)}$  around the point  $\underline{y}_k^{(1)}$  while  $\varphi_{1,k}^{(0),h}$  is a global quasimode associated with the bar  $[x_{1,k}^{(0)}, y_k^{(1)}]$ , solving  $d_{f,h} \varphi_{1,k}^h = 0$  in  $\Omega_k^{(0)} \cap f^{y_k^{(1)} - \delta(h)}$  with  $\lim_{h \rightarrow 0} \delta(h) = 0$ . The truncation  $T_{\delta_2}$  is a smooth truncation around the level  $y_k^{(1)} - \delta_2$  with  $\delta_2 > 0$  small. By Theorem 6.3 and Theorem 7.1 the same method holds by replacing the global quasimodes  $\varphi_{1,k}^{(0),h}$  by global quasimodes  $\varphi_{2,k}^{(0),h}$  constructed in Theorem 6.3. In details we refer more specifically to the consequences stated in Subsection 6.3. Moreover we can focus on the bars of length larger than  $\frac{\ell_0}{2}$  which are  $([x_{2,k}^{(0)}, y_k^{(1)}])_{k=0, \dots, K_0}$ . Since

those quasimodes satisfy  $d_{f_2, h} \varphi_{2, k}^{(0), h} = 0$  in  $\Omega_k^{(0)} \cap f y_k^{(1) - \delta(h)}$  they equal  $\sqrt{C_{k, h}} e^{-\frac{f_2(x) - x_{2, k}^{(0)}}{h}}$  where  $C_{k, h}$  is the normalization constant

$$C_{k, h} = \frac{1}{\int_{\Omega_k^{(0)} \cap f y_k^{(1) - \delta(h)}} e^{-2 \frac{f_2(x) - x_{2, k}^{(0)}}{h}} dx} = \frac{1 + \tilde{o}(1)}{\int_{\Omega_k^{(0)}} e^{-2 \frac{f_2(x) - x_{2, k}^{(0)}}{h}} dx}$$

which replaces

$$\frac{1}{\int_{\Omega_k^{(0)}} e^{-2 \frac{f_1(x) - x_{1, k}^{(0)}}{h}} dx} = (\pi h)^{-d/2} |\det \text{Hess } f(x_{1, k}^{(0)})|^{1/2} \times (1 + \mathcal{O}(h)).$$

Finally it suffices to notice that up to the normalization constant and the change of the length of the bar which brings another constant factor, the functions  $\varphi_{1, k}^{(0), h}$  and  $\varphi_{2, k}^{(0), h}$  coincide in the neighborhood of  $\underline{y}_k^{(1)}$  and the local computation of the interaction is not changed.  $\square$

The above formula shows a good stability when  $f_1$  is changed into  $f_2$  although such a stability may not appear when we make an explicit asymptotic expansion of the Laplace integral  $\int_{\Omega_k^{(0)}} e^{-2 \frac{f(x) - x_{2, k}^{(0)}}{h}} dx$ . Here is an example in dimension 1, that is for functions defined on  $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ . The function  $f_1$  is assumed to have four non degenerate critical points:

- at  $\underline{x}_{1, 1}^{(0)} = 0$  with value  $x_{1, 1}^{(0)} = 0$  and second derivative 1;
- at  $\underline{x}_{0, 1}^{(0)} = \pi$  with value  $x_{0, 1}^{(0)} = -1$ , the global minimum;
- at  $\underline{y}_{1, 1}^{(1)} = \frac{\pi}{2}$  with value  $y_{1, 1}^{(1)} = 1$  and the second derivative equal to  $-\lambda_1$ ;
- at  $\underline{y}_{0, 1}^{(1)} = \frac{3\pi}{2}$  with value  $y_{0, 1}^{(1)} = 2$ , the global maximum.

The modified function  $f_{2, \delta}$  parametrized by  $\delta \in \mathbb{R}$ ,  $\delta$  small, and consists in replacing  $f_1(x) = \frac{x^2}{2} + O(x^3)$  in a small neighborhood  $[-\varepsilon, \varepsilon]$  of  $\underline{x}_{1, 1}^{(0)} = 0$  by

$$f_{2, \delta}(x) = \frac{x^4 + 2\delta x^2 + 1_{(-\infty, 0]}(\delta)\delta^2}{4},$$

while  $f_{2, \delta} \equiv f_1$  outside  $[-2\varepsilon, 2\varepsilon]$ . Formula (153) then says that the  $\tilde{o}(e^{-\frac{1}{h}})$  non zero eigenvalue of  $\Delta_{f_{2, \delta}, \mathbb{S}^1, h}^{(0)}$  (for  $\delta > 0$  and  $\varepsilon > 0$  small enough) equals

$$\frac{h\sqrt{\lambda_1} e^{-\frac{2}{h}}}{\pi(\pi h)^{-1/2} \int_{\mathbb{R}} e^{-\frac{x^4 + 2\delta x^2 + 1_{(-\infty, 0]}(\delta)\delta^2}{2h}} dx} \times (1 + \mathcal{O}(h)).$$

It is equivalent as  $h \rightarrow 0$  to

$$\begin{aligned} & \frac{h\sqrt{\lambda_1} \delta e^{-\frac{2}{h}}}{\pi} && \text{when } \delta > 0 \\ & \frac{h^{5/4} \sqrt{\lambda_1} e^{-\frac{2}{h}}}{\sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{u^4}{2}} du} && \text{when } \delta = 0, \\ & \frac{h\sqrt{\lambda_1} |\delta| e^{-\frac{2}{h}}}{\sqrt{2\pi}} && \text{when } \delta < 0. \end{aligned}$$

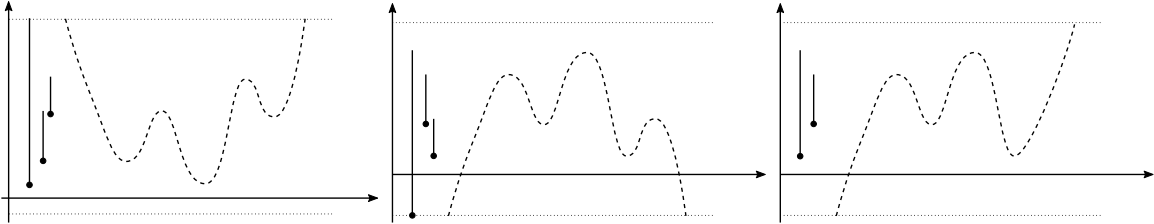
So the apparent discontinuity in the exponent of  $h$  at  $\delta = 0$  is a simple consequence of the discontinuity of the Laplace integral. Actually the stability of persistence homology has a stronger spectral counterpart than what is stated in Theorem 7.6: It does not concern only the exponential scales but also allows to study the deformations of the asymptotic subexponential factors provided that a robust formula can be proved for them. The rest of this section explores different cases for which we are able to prove such formulas.

## 9.2.2 Variations

In the previous paragraph we used a good enough knowledge of the global quasimodes  $\varphi_{2,k}^{(0),h} = \sqrt{C_{k,h}} e^{-\frac{f_2(\cdot) - x_{2,k}}{h}}$  in degree  $p = 0$ , in order to get the explicit change in the asymptotic formulas when we pass from the Morse function  $f_1$  with simple critical values to the function  $f_2$  with degenerate local minima. Such an analysis can be done in more general degree if we have explicit enough information on the local forms of quasimodes the global ones  $\varphi_k^{(p),h}$  and the local ones  $\psi_k^{(p+1),h}$ . By duality this is obviously true in dimension 1 and we start with this example. We then consider other possible extensions.

**The one dimensional case with degenerate critical values** Consider a  $C^\infty$  Morse function  $f_1$  on  $\mathbb{R}$  such that  $|\partial_x f_1| \geq c$  for some positive constant  $c$  when  $x \in \mathbb{R} \setminus [-R, R]$  for  $R > 0$  large enough. For  $-a = |a|$  and  $b = |b|$  large enough the bar code  $\mathcal{B}_{f_1}(a, b)$  does not change when  $a, b$  are changed, except for the value of the endpoints  $a, b$ , while for such a fixed pair  $(a, b)$  it can be viewed as a restricted bar code  $B_{\tilde{f}_1}(a, b)$  of a function  $\tilde{f}_1$  defined on  $\mathbb{S}_1$ . This solves the compactness problem in order to fit with our general framework. It can be checked easily that in all such cases the exponentially small eigenvalues of  $\Delta_{f_1, f_1^{-1}([a,b]), h}^{(p)}$  are close to the ones of  $\Delta_{f_1, \mathbb{R}, h}^{(p)}$  for  $p = 0, 1$  and even that the endpoints of the interval  $f_1^{-1}([a, b])$  can be moved as long as they do not meet the critical point without changing the final approximate spectral result (the same will be true for the function  $f_2$ ). So let us focus on  $f_1^{-1}([a, b])$  with  $-a = |a|$  and  $b = |b|$  large. The bar code is made of bars  $[x_{k,1}^{(0)}, y_{k,1}^{(1)}[$ ,  $k = 1, \dots, K$  with an additional bar:

- $[x_{0,1}^{(0)}, b[$  if  $f_1|_{f_1^{-1}([a,b])}$  admits an interior global minimum at  $x_{0,1}^{(0)} = f(\underline{x}_{0,1})$ ,  $\underline{x}_{0,1} \in f_1^{-1}([a, b])$ ;
- or  $[a, y_{0,1}^{(1)}[$  if  $f_1|_{f_1^{-1}([a,b])}$  admits an interior global maximum at  $y_{0,1}^{(1)} = f(\underline{y}_{0,1})$ ,  $\underline{y}_{0,1} \in f_1^{-1}([a, b])$ .



**Figure 16:** Three different cases for  $f_1$  between the level  $a$  and  $b$ , from left-hand side to right-hand side with an interior global minimum, an interior global maximum in the interior and none of them. The bar code in  $[a, b]$  is represented beside the  $y$ -axis.

Only in the first case, the Witten Laplacian  $\Delta_{f_1, f_1^{-1}([a,b])}^{(0)}$  has a non trivial kernel  $Ce^{-\frac{f_1(\cdot) - x_{0,1}^{(0)}}{h}}$ .

Only in the second case, the Witten Laplacian  $\Delta_{f_1, f_1^{-1}([a,b]), h}^{(1)}$  has a non trivial kernel  $Ce^{\frac{f_1(\cdot) - y_{0,1}^{(1)}}{h}} dx$ .

The two cases are exclusive and a third one is when the global minimum value of  $f_1|_{f_1^{-1}([a,b])}$  is  $a$  and the global maximum value is  $b$ . Depending on the cases  $f_1$  admits  $2K + 1$  or  $2K$  distinct critical values and their set in  $[a, b]$  is denoted  $\mathcal{C}$ .

In order to specify our modified function  $f_2$  we first choose  $\ell_0 < \min\{|c - c'|, c \neq c', c, c' \in \mathcal{C}\}$ .

The connected open set  $\Omega_{k,1}^{(0)}$  as the connected component of  $(f_1)^{y_{k,1}^{(1)}}$  which contains  $\underline{x}_{k,1}^{(0)}$  for

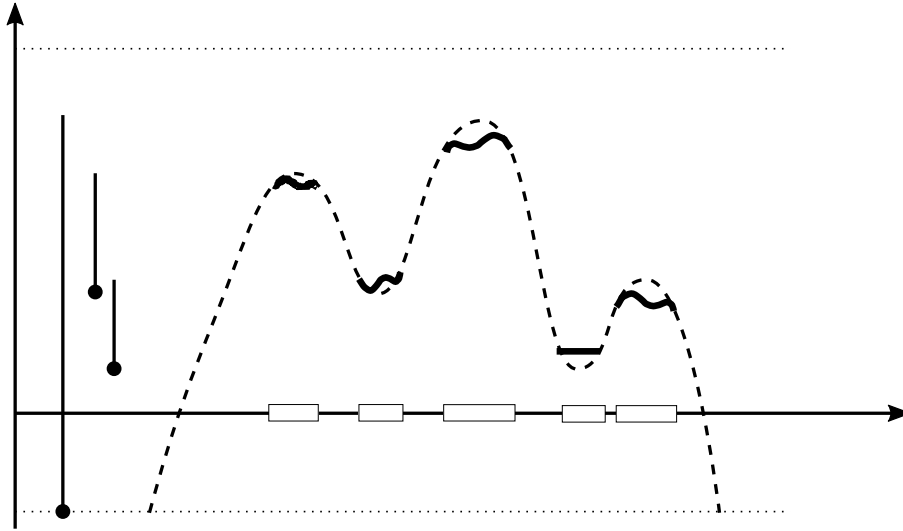
$1 \leq k \leq K$ , with  $\Omega_{0,1}^{(0)} = f_1^{-1}(]a, b])$  if the global minimum  $\underline{x}_{0,1}^{(0)} \in f_1^{-1}(]a, b])$  exists. By duality one defines  $\Omega_{k,1}^{(1)}$  as the connected component of  $(f_1)_{x_{k,1}^{(0)}}$  for  $1 \leq k \leq K$ , with  $\Omega_{0,1}^{(1)} = f_1^{-1}(]a, b])$  if the global maximum  $\underline{y}_{0,1}^{(1)} \in f_1^{-1}(]a, b])$  exists. Then the connected open sets  $\omega_k^{(0)}$  and  $\omega_k^{(1)}$  are defined by

$$\omega_k^{(0)} = \Omega_{k,1}^{(0)} \cap (f_1)^{x_{k,1}^{(0)} + \frac{\ell_0}{4}} \quad , \quad \omega_k^{(1)} = \Omega_{k,1}^{(1)} \cap (f_1)^{y_{k,1}^{(1)} - \frac{\ell_0}{4}} .$$

The function  $f_2$  satisfies Hypothesis 1.2 and

- $f_1 \equiv f_2$  in a neighborhood of  $\mathbb{R} \setminus (\sqcup_{0 \leq k \leq K} (\omega_k^{(0)} \sqcup \omega_k^{(1)}))$  where  $\omega_0^{(0)}$  and  $\omega_0^{(1)}$  are replaced by the empty set when they are not defined;
- $\|f_1 - f_2\| \leq \frac{\ell_0}{4}$ .

Note in particular  $f_1^{-1}(]a, b]) = f_2^{-1}(]a, b])$ .



**Figure 17:** The function  $f_1$  is represented by the dashed curve, the open sets  $\omega_k^{(p)}$  are materialized by the white rectangles along the  $x$ -axis and the modifications leading to  $f_2$  by the plain curve.

Owing to the stability theorem

$$d_{bot}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_{C^0}$$

the bars  $[x_{k,1}^{(0)}, y_{k,1}^{(1)}[$  are transformed into bars  $[x_{k,2}^{(0)}, y_{k,2}^{(1)}[$  of length  $y_{k,2}^{(1)} - x_{k,2}^{(0)} > \frac{\ell_0}{2}$  while all the other bars have length smaller than  $\frac{\ell_0}{2}$ . After those assumptions the spectral result take a nice simple form.

**Proposition 9.6.** *For the values  $a, b$  and the function  $f_2$  chosen like above, there are  $K$  non zero  $\tilde{o}(e^{-\frac{\ell_0}{h}})$  eigenvalues of  $\Delta_{f_2, f_2^{-1}(]a, b]), h}^{(0) \text{ or } (1)}$  which are equal to*

$$\frac{1 + \tilde{o}(1)}{(h^{-1} \int_{\omega_k^{(1)}} e^{2\frac{f(x)}{h}} dx) \times (h^{-1} \int_{\omega_k^{(0)}} e^{-2\frac{f(x)}{h}} dx)}, \quad k = 1, \dots, K .$$

*Proof.* By the usual supersymmetric arguments the non zero eigenvalues of  $\Delta_{f_2, f_2^{-1}([a,b]), h}^{(0)}$  and  $\Delta_{f_2, f_2^{-1}([a,b]), h}^{(1)}$  are the same in dimension 1 and we thus focus on  $\Delta_{f_2, f_2^{-1}([a,b]), h}^{(0)}$  or more precisely on the non zero singular values of the restricted differential. We follow the general method which consist in computing the interaction scalar product  $\langle \psi_k^{(1)}, d_{f,h} T_{\delta_2} \varphi_k^{(0)} \rangle$  where  $\psi_k^{(1)}$  is a local quasimode for  $\Delta_{f_2, h}$  in the neighborhood  $\omega_k^{(1)}$  around  $\underline{y}_{k,1}^{(1)}$  while  $\varphi_k^{(0)}$  is a global quasimode associated with the bar  $[x_{2,k}^{(0)}, y_{2,k}^{(1)}[$  solving  $d_{f,h} \varphi_k^{(0)} = 0$  in the connected component which contains  $\omega_k^{(0)}$  of  $f^{y_k^{(2)} - \delta(h)}$ , with  $\lim_{h \rightarrow 0} \delta(h) = 0$ . We work directly with the function  $f_2$  the global quasimode  $\varphi_k^{(0)}$  equals

$$\frac{1 + \tilde{o}(1)}{\sqrt{\int_{\omega_k^{(0)}} e^{-2 \frac{f_2(x) - x_{2,k}^{(0)}}{h}} dx}} e^{-\frac{f_2(\cdot) - x_{2,k}^{(0)}}{h}} \quad \text{in } \omega_k^{(0)}.$$

By noticing that  $\partial_n f_2|_{\partial \omega_k^{(1)}} = \partial_n f_1|_{\partial \omega_k^{(1)}} < 0$ , and by using Dirichlet boundary conditions on  $\partial \omega_k^{(1)}$  in degree  $p = 1$ , we find that  $\psi_k^{(1)}$  can be chosen as

$$\frac{1 + \tilde{o}(1)}{\sqrt{\int_{\omega_k^{(1)}} e^{2 \frac{f_2(x) - y_{2,k}^{(1)}}{h}} dx}} e^{\frac{f_2(x) - y_{2,k}^{(1)}}{h}} \quad \text{in } \omega_k^{(1)}.$$

A direct computation gives

$$\langle \psi_k^{(1)}, d_{f,h} T_{\delta_2} \varphi_k^{(0)} \rangle = \pm \frac{h e^{-\frac{y_{2,k}^{(1)} - x_{2,k}^{(0)}}{h}} (1 + \tilde{o}(1))}{\sqrt{\int_{\omega_k^{(1)}} e^{2 \frac{f_2(x) - y_{2,k}^{(1)}}{h}} dx} \times \sqrt{\int_{\omega_k^{(0)}} e^{-2 \frac{f_2(x) - x_{2,k}^{(0)}}{h}} dx}}$$

where the factor  $e^{-\frac{y_{2,k}^{(1)} - x_{2,k}^{(0)}}{h}}$  can be simplified. □

**Remark 9.7.** *Note that in this proof the result on the generic Morse function  $f_1$  is not used. The function  $f_1$  was introduced in order to have a simple formulation of the assumptions fulfilled by  $f_2$ . The result actually comes from a direct computation when we know well enough the local forms of the global ( $\varphi_k^{(0)}$ ) and local ( $\psi_k^{(1)}$ ) quasimodes. We have explicit form in dimension 1 and the computation is straightforward. It is not the same in the multidimensional case although Stokes's formula allows to perform the computation when local approximations of local and global quasimodes are well known.*

**Piecewise affine functions** In this paragraph we make more explicit the one dimensional result when  $f$  is a continuous piecewise affine function and discuss the possible extension to the multidimensional case. Let  $f$  be a piecewise affine function on  $\mathbb{R}$  such that:

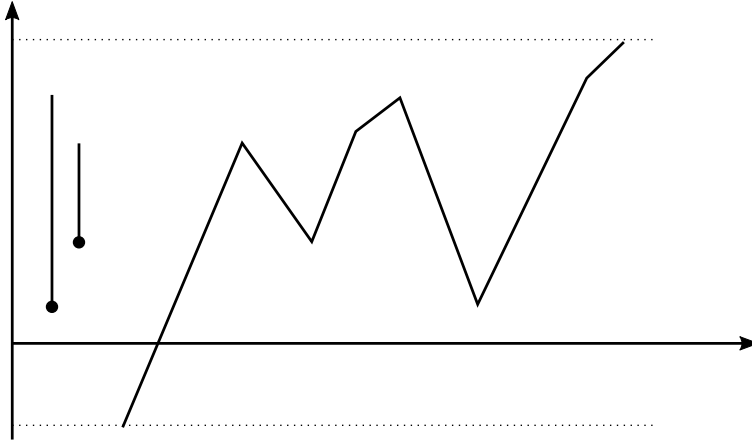
- the derivative  $f'$  does not vanish when it is defined;
- there exists  $R > 0$  such that the derivative  $f'$  is a constant on  $[R, +\infty)$  and on  $(-\infty, -R]$ ;
- the values  $f(x)$  of the points  $x$  where  $f'$  is discontinuous are all distinct.

Such a function  $f$  can be written as a function  $f_2$  of the previous paragraph (simply regularize locally the discontinuous change of slopes in order to get the Morse function  $f_1$ ).

The extension of Proposition 9.6 to  $a = -\infty$  and  $b = +\infty$  says that the  $\tilde{o}(1)$ -eigenvalues of  $\Delta_{f, \mathbb{R}, h}^{(0)}$  (and by duality of  $\Delta_{f, \mathbb{R}, h}^{(1)}$ ) are given by

$$H[|f'(\underline{y}_k^{(1)} + 0)|, f'(\underline{y}_k^{(1)} - 0)] H[f''(\underline{x}_k^{(0)} + 0), |f'(\underline{x}_k^{(0)} - 0)|] e^{-2 \frac{y_k^{(1)} - x_k^{(0)}}{h}} (1 + \tilde{o}(1)) \quad , \quad k = 1, \dots, K,$$





**Figure 18:** A piecewise affine potential in  $1D$  with distinct and some fake critical values.

where the finite length bars of  $\mathcal{B}_f$  are  $[x_k^{(0)}, y_k^{(1)}[$ ,  $k = 1, \dots, K$ ;  $x_k^{(0)}$  is the local strict minimal value around the point  $\underline{x}_k^{(0)}$ ;  $y_k^{(1)}$  is the local maximal value around the point  $\underline{y}_k^{(1)}$ ;  $f'(x+0)$  and  $f'(x-0)$  denote respectively the right and left derivative and  $H[s, t] = \frac{2st}{s+t}$  is the harmonic mean of  $s, t > 0$ .

The computation when  $f$  is constant on some intervals is also possible with a subexponential factor behaving like  $h$  or  $h^2$ , depending on the different cases (left to the reader).

Now let  $f$  be a piecewise affine function defined on a finite triangulation of  $\mathbb{R}^d = \sqcup_{1 \leq i \leq I} \mathcal{T}_i$  where  $\mathcal{T}_i$  is a  $d$ -dimensional non degenerate simplex with endpoints  $A_i^0, \dots, A_i^d$  and where non finite simplices are roughly taken into account by sending the first endpoint to infinity  $A_i^0 = \infty$  (a more precise description is not necessary here). We assume that  $\lim_{x \rightarrow \infty} f(x) = +\infty$ . The function  $f$  is a subanalytic function on  $\mathbb{R}^d$  of which the restriction to any ball  $B(0, R)$  can be viewed as the restriction to  $B(0, R)$  of a subanalytic function defined on  $\mathbb{S}^d$ . This solves the compactness problem or the questions about the topology at infinity (alternatively we could work on the  $d$ -dimensional flat torus). The function  $f$  has a finite number of horizontal strata according to the terminology of Definition 8.6, which contain all the critical values and the possible endpoints of the bar code  $\mathcal{B}_f$ . We may consider either  $\Delta_{f, \mathbb{R}^d, h}$  or by approximation  $\Delta_{f, f^{-1}([a, b]), h}$  with  $-a, b > 0$  large enough. According to our analysis in Subsection 8.3, in particular Proposition 8.8, Proposition 8.14 and Proposition 8.16, the results of Theorem 6.3 hold in this case and we know that the exponentially small eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}^{(p)}$  satisfy

$$\lambda_\alpha^{(p)}(h) \stackrel{\log}{\sim} e^{-2 \frac{y_\alpha^{*+1} - x_\alpha^*}{h}},$$

where  $\alpha$  belongs to  $A_c^{(p)}(a, b) \sqcup A_c^{(p-1)}(a, b)$ .

The question is whether it is possible to give algebraic formulas for the accurate asymptotic behaviour as this is done easily in the one dimensional case. For such a function  $f$ , the Witten Laplacian  $\Delta_{f, h}$  is a matricial Schrödinger operator with a singular potential. Many things are known on scalar Schrödinger operators with singular potentials (see e.g. [AGHKKH][BGP]), but little seems to be known for those Witten Laplacians, and especially when we think about the algebraic topology subtleties. We may also start directly, instead of  $\mathbb{R}^d$ , on a Lipschitz manifold made of glued simplexes, with a function  $f$  which has a constant gradient along every simplex. The functional analysis of Hodge Laplacian on Lipschitz manifold has been considered in [GMM, MMT]. An accurate analysis of the low lying spectrum of such Witten Laplacians

would provide a large family of discrete and easily encoded models, from the point of view of data and hopefully of results, which could be used as approximations of complicated realistic situations. It would be interesting to compare with the approach starting from purely discrete models on graphs as presented in [CdVPY].

**Critical submanifolds** This case is related with degenerate Witten Laplacians studied in connection with Bott-Morse inequalities (see e.g. [Bis, HeSj6]). We consider here simple examples where we have a critical submanifold instead of a critical point. We start with the mexican hat function  $f(r, \theta) = \frac{r^4}{4} - \frac{r^2}{2} + \frac{1}{4}$  in polar coordinates  $(r, \theta)$  in  $\mathbb{R}^2$  with the euclidean metric  $dr^2 + (rd\theta)^2$ , which admits a non degenerate maximum at  $r = 0$  with  $f(0_{\mathbb{R}^2}) = \frac{1}{4}$  and a degenerate minimum at  $r = 1$  with  $f(1, \theta) = 0$ .

The bar code of the function  $f$  is made of the bar  $[0, +\infty[$  in degree 0 and the bar  $[0, \frac{1}{4}[$  in degree 1. We compute the non zero exponentially small eigenvalue of  $\Delta_{f, \mathbb{R}^2, h}^{(p)}$  with  $p = 1$  or 2 by computing the interaction scalar product  $\langle \psi_1^{(2)}, d_{f, h}^{(1)} T_{\delta_2} \varphi_1^{(1)} \rangle$  where  $\varphi_1^{(1)}$  is a global quasimode 1-form associated with the bar  $[0, \frac{1}{4}[$  and  $\psi_1^{(2)}$  is a local quasimode 2-form around  $r = 0$ .

In this particular example we have explicit forms for  $\varphi_1^{(1)}$  and  $\psi_1^{(2)}$ :

- We take  $\nu > 0$  smaller than the truncation parameter  $\delta_2$ . Then a explicit normalized element of  $\ker(\Delta_{f, f^{-1}([-1, \frac{1}{4}-\nu]), h}^{(1)})$  is given by

$$\varphi_1^{(1)} = \frac{1}{\sqrt{\int_{f^{-1}[\frac{1}{4}-\nu]} e^{-\frac{r^4}{2}-r^2+\frac{1}{2}} r^{-2} dr (rd\theta)}} e^{-\frac{r^4}{4}-\frac{r^2}{2}+\frac{1}{4}} d\theta.$$

- For the local quasimode  $\psi_1^{(2)}$  defined around  $r = 0$ , we can use either a WKB approximation, or by duality the exact normalized element of  $\ker(\Delta_{f, f^{-1}([\frac{1}{4}-\delta]), h}^{(2)})$  ( $\delta > 0$  is small enough but bigger than  $2\delta_2$ ) given by equal to

$$\psi_1^{(2)} = \frac{1}{\sqrt{\int_{f^{-1}[\frac{1}{4}-\delta]} e^{\frac{r^4}{2}-r^2} dr (rd\theta)}} e^{\frac{r^4}{4}-\frac{r^2}{2}} dr \wedge (rd\theta).$$

The scalar product  $\langle \psi_1^{(2)}, d_{f, h}(T_{\delta_2} \varphi_1^{(1)}) \rangle$  is then equal to

$$\frac{1}{\sqrt{\int_{f^{-1}[\frac{1}{4}-\delta]} e^{\frac{r^4}{2}-r^2} dr (rd\theta)} \sqrt{\int_{f^{-1}[\frac{1}{4}-\nu]} e^{-\frac{r^4}{2}-r^2+\frac{1}{2}} r^{-2} dr (rd\theta)}} \langle dr \wedge (rd\theta), h\chi'_{\delta_2}(r) dr \wedge d\theta \rangle e^{-\frac{1}{4h}},$$

where

$$\langle dr \wedge (rd\theta), h\chi'_{\delta_2}(r) dr \wedge d\theta \rangle = \pm h \int_{r=\varrho} \frac{rd\theta}{r} = \pm 2\pi h$$

does not depend on the value  $\varrho > 0$  (This is an explicit illustration of Stokes's formula argument used in [LNV] when  $f$  is a Morse function).

Using the asymptotics of non degenerate Laplace integrals, the non zero exponentially small of  $\Delta_{f, \mathbb{R}^2, h}^{(p)}$ , for  $p = 1, 2$  equals

$$\frac{1 + O(h)}{\pi h} \times \frac{1 + O(h)}{\pi(2\pi h)^{1/2}} \times (2\pi h)^2 e^{-\frac{1}{2h}} = \frac{2\sqrt{2}h^{1/2} + O(h^{3/2})}{\sqrt{\pi}} e^{-\frac{1}{2h}}.$$

The subexponential factor  $Cte \times \sqrt{\frac{h}{\pi}}$  differs from the asymptotic behaviour  $Cte \times \frac{h}{\pi}$  obtained when  $f$  is a generic Morse function. Actually it is possible to study the transition from the Morse generic case to this degenerate case by taking  $f_\delta(r, \theta) = f(r, \theta) + \delta\gamma(r) \cos(\theta)$  where  $\gamma \in C^\infty(]0, +\infty[; [0, 1])$  equals 1 in a neighborhood of 1, and  $\delta \in \mathbb{R}$  is chosen small enough. Let us illustrate this in a larger framework. Note that the above formula is not changed if the metric  $dr^2 + r^2d\theta^2$  is replaced by  $dr^2 + d\theta^2$  in a neighborhood of  $r = 1$ . This will make the forthcoming analysis simpler.

We consider a  $C^\infty$  function  $f$  on the compact Riemannian manifold  $M$  with a finite number of critical values, which are all non degenerate and simple except the critical value fixed to be 0. We further assume:

- the critical set around the value 0 is a closed orientable submanifold  $M'$  of dimension  $p$ ;
- there is a tubular neighborhood of  $M'$  which is a product of two Riemannian manifolds  $M' \times M''$  with the metric  $g = g' \oplus g''$ ; a corresponding local coordinate system is written  $x = (x', x'')$ ;
- in the tubular neighborhood  $M' \times M''$  the function  $f$  is a function of  $x'' \in M''$  and has a unique minimum  $f(x''_0) = 0$ ;
- the bar code  $\mathcal{B}_f$  contains a unique bar  $[0, y_1^{(p+1)}[$  of degree  $p$  with lower endpoint 0 and upper endpoint  $y_1^{(p+1)} < +\infty$ ; the eigenvalues of the Hessian at the corresponding point  $\underline{y}_1^{(p+1)}$  are denoted  $-\lambda_1(\underline{y}_1^{(p+1)}), \dots, -\lambda_{p+1}(\underline{y}_1^{(p+1)})$  and  $\lambda_{p+2}(\underline{y}_1^{(p+1)}), \dots, \lambda_d(\underline{y}_1^{(p+1)})$ ;
- a local unstable (for  $-\nabla f$ ) closed cell around the non degenerate critical point  $\underline{y}_1^{(p+1)}$  is denoted  $e_1^{(p+1)}$  and its boundary in  $M$  which is a  $p$ -dimensional sphere is denoted by  $\partial e_1^{(p+1)}$ ;
- if  $\phi$  is  $C^\infty$  Morse function on  $M'$  with the maximal value 0 and  $\chi \in C_0^\infty(M''; [0, 1])$  is equal to 1 in a neighborhood of  $x''_0$  and such that  $f(x'') \geq c > 0$  on  $\text{supp } d\chi$ , the function  $f_\delta$  is defined as  $f_\delta = f + \delta\chi(x'')\phi(x')$ ;
- for the sake of simplicity we work in the energy interval  $[a, b]$  with  $a = -\varepsilon$  and  $b = y_1^{(p+1)} + \varepsilon$  where  $\varepsilon > 0$  is fixed so that the critical values of  $f$  in  $[a, b]$  are the ones contained in  $[0, y_1^{(p+1)}]$ .

**Proposition 9.8.** *Under the above assumptions, the boundary of the unstable cell  $\partial e_1^{(p+1)}$  is homologous to  $\kappa M'$ , for some constant  $\kappa$  and relatively to  $f^{-\varepsilon}$ .*

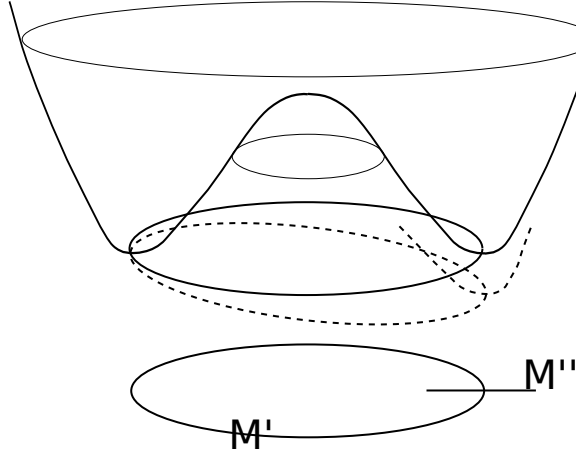
*For  $\delta \geq 0$  small enough, the bar code  $\mathcal{B}_{f_\delta}(a, b)$  admits the unique bar  $[0, y_1^{(p+1)}[$  of degree  $p$  and length  $y_1^{(p+1)}$ .*

*The corresponding eigenvalue of  $\Delta_{f_\delta, f_\delta^{-1}([a, b]), h}^{(p)}$  or  $(p+1)$  equals*

$$\frac{h}{\pi} \times \frac{|\lambda_1(\underline{y}_1^{(p+1)}) \cdots \lambda_{p+1}(\underline{y}_1^{(p+1)})|^{1/2}}{|\lambda_{p+2}(\underline{y}_1^{(p+1)}) \cdots \lambda_d(\underline{y}_1^{(p+1)})|^{1/2}} \times \frac{(\pi h)^{-p} \left( \kappa \int_{M'} e^{\frac{2\delta\phi(x')}{h}} dx' \right)^2}{(\pi h)^{-d/2} \int_{M' \times M''} e^{-2\frac{f - \delta\chi(x'')\phi(x')}{h}} dx} \times e^{-\frac{2y_1^{(p+1)}}{h}} \times (1 + \mathcal{O}(h)).$$

*Proof.* The first statement is due to the fact that the bar  $[0, y_1^{(p+1)}[$  of degree  $p$  provides a non null linear application from the relative homology vector space  $H_{p+1}(f y_1^{(p+1)+\varepsilon}; f y_1^{(p+1)-\varepsilon})$ , of which  $e_1^{(p+1)}$  is a representant, via the boundary map to  $H_p(f^\varepsilon; f^{-\varepsilon})$ , of which the cycle  $M'$  is a representant. Therefore there exists a constant  $\kappa$  such that  $\partial e_1^{(p+1)} - \kappa M'$  is a boundary relatively to  $f^{-\varepsilon}$ . In particular if  $\omega$  is a regular  $p$ -form in  $\ker d_{0, f^{-1}([-\varepsilon, +\infty]), 1}$  then

$$\int_{\partial e_1^{(p+1)}} \omega = \kappa \int_{M'} \omega. \quad (154)$$



**Figure 19:** Case of a critical submanifold (plain line) and its perturbation (dashed line): The above example is modelled on the mexican hat function  $\frac{r^4}{4} - \frac{r^2}{2}$  with the manifold  $M' = \mathbb{S}^1$  with the metric  $d\theta^2$  and  $M'' \sim \mathbb{R}$  (around  $r = 1$ ) with the metric  $dr^2$ . The function  $\Phi(\theta) = -1 - \cos(\theta)$  is a negative Morse function with maximum value 0 when  $\theta = \pi$ .

The fact that  $[0, y_1^{(p+1)}[$  remains the only bars of degree  $p$  and length  $y_1^{(p+1)}$  for  $\delta > 0$  small enough is a consequence of the stability theorem (Note that for  $\delta > 0$ ,  $f_\delta$  is a Morse function if  $x'' \mapsto f(x'')$  has a non degenerate minimum at  $x''_0$ .)

Let  $\varphi_1^{(p)}$  be a global quasimode and  $\psi_1^{(p+1)}$  be a local quasimode associated with the bar  $[0, y_1^{(p+1)}[$  and let us compute the scalar product

$$\langle \psi_1^{(p+1)}, d_{f_\delta, h} T_{\delta_2} \varphi_1^{(p)} \rangle.$$

Because we have a non degenerate critical point at  $\underline{y}_1^{(p+1)}$ , the computations of [LNV]-Section 4.3, which rely on the WKB approximation for  $\psi_1^{(p+1)}$  around  $\underline{y}_1^{(p+1)}$  and  $d_{f_\delta, h} \varphi_1^{(p)} \equiv 0$  in  $f_{\delta_2}^{y_1^{(p+1)} - \delta(h)} = f_{y_1^{(p+1)} - \delta(h)}$ , leads to

$$\begin{aligned} \langle \psi_1^{(p+1)}, d_{f_\delta, h} T_{\delta_2} \varphi_1^{(p)} \rangle &= \pm \left( \frac{h}{\pi} \right)^{1/2} \times \frac{|\lambda_1(\underline{y}_1^{(p+1)}) \cdots \lambda_{p+1}(\underline{y}_1^{(p+1)})|^{1/4}}{|\lambda_{p+2}(\underline{y}_1^{(p+1)}) \cdots \lambda_d(\underline{y}_1^{(p+1)})|^{1/4}} \times (\pi h)^{\frac{d}{4} - \frac{p}{2}} \\ &\quad \times \int_{\partial e_1^{(p+1)}} e^{\frac{f_\delta}{h} \varphi_1^{(p)}} \times e^{-\frac{y_1^{(p+1)}}{h}} \times (1 + \mathcal{O}(h)). \end{aligned}$$

Because  $d(e^{\frac{f_\delta}{h} \varphi_1^{(p+1)}}) \equiv 0$  in  $f_{y_1^{(p+1)} - \delta(h)}$  we may apply (154) with  $\omega = e^{\frac{f_\delta}{h} \varphi_1^{(p+1)}}$  and the integral  $\int_{\partial e_1^{(p+1)}}$  can be replaced by  $\kappa \int_{M'}$ . Thus it suffices to know  $\varphi_1^{(p)}$  in a neighborhood of  $M'$ . A good approximation is given by a normalized element of  $\ker(\Delta_{f_\delta, f_\delta^{-1}([- \varepsilon, \varepsilon], h)}^{(p)})$  which is exponentially close (in any Sobolev norm) to the  $p$ -form constructed by the separation of variables in  $M' \times M''$

$$\frac{1}{\left( \int_{M' \times M''} e^{-2\frac{f - \delta x(x')\phi(x')}{h}} dx \right)^{1/2}} e^{-\frac{f - \delta x(x')\phi(x')}{h}} |\det g'(x')|^{1/2} dx_1 \wedge \dots \wedge dx_p.$$

The final result follows by taking the square. □

When  $f(x'')$  near  $x''_0 \in M''$  and  $\phi(x')$ ,  $x' \in M'$ , are Morse functions, the above formula allows again to study the transition between the case when  $f$  is a Morse function on  $M$  for  $\delta > 0$  small and when 0 is a degenerate critical value with critical set  $M'$  for  $\delta = 0$ . We get the following asymptotic behaviour for the eigenvalue of  $\Delta_{f_\delta, f_\delta^{-1}([a,b]), h}^{(p) \text{ or } (p+1)}$  associated with the bar  $[0, y_1^{(p+1)}[$ :

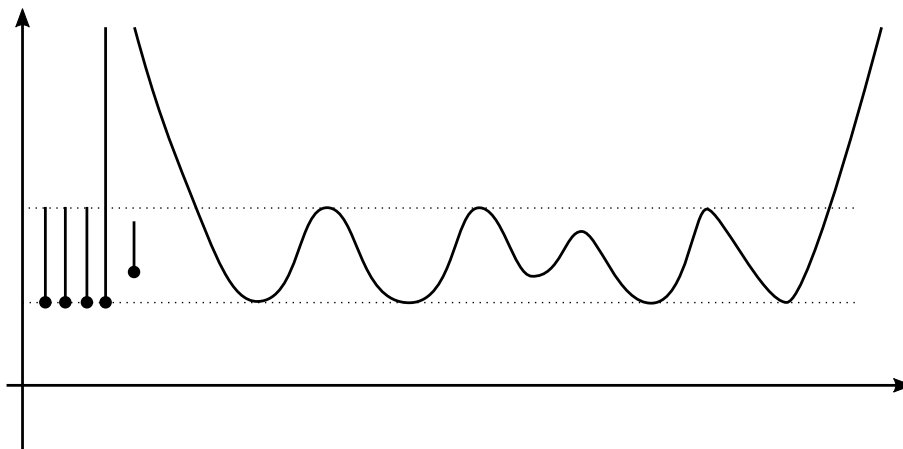
$$\begin{aligned} C_\delta \frac{h}{\pi} e^{-2\frac{y_1^{(p+1)}}{h}} (1 + \mathcal{O}(h)) & \quad \text{when } \delta > 0, \\ C_0 \frac{h}{\pi} (\pi h)^{-p/2} e^{-2\frac{y_1^{(p+1)}}{h}} (1 + \mathcal{O}(h)) & \quad \text{when } \delta = 0. \end{aligned}$$

In general degree  $p$  it is possible to have a good information on the local approximations of the global quasimodes  $\varphi_k^{(p)}$  either when the critical value is  $x_k^{(p)}$  is non degenerate via a WKB approximation or when we can use some separation of variables. Otherwise it is not clear that we could get a general robust integral formula for the subexponential factor. Note also that we used the fact that  $y_k^{(p+1)}$  is a non degenerate critical value when we reduced the computation of  $\langle \psi_1^{(p+1)}, d_{f,h} T_{\delta_2} \varphi_1^{(p)} \rangle$  to an integral along the explicit cycle  $\partial e_1^{(p+1)}$ . Again it is not clear that such a simple argument can be used when  $y_k^{(p+1)}$  is a degenerate critical value without some other specific assumptions.

### 9.3 More general Morse functions

We consider in this paragraph a Morse function  $f$  which may admit multiple critical values. For the sake of simplicity, we work in the following situation:

- $c < c'$ ,  $c, c' \in \{c_1, \dots, c_{N_f}\}$  are the only multiple critical values.
- All the critical points with critical value  $c$  (resp.  $c'$ ),  $\underline{x}_k^{(p)}$ ,  $1 \leq k \leq K$ , (resp.  $\underline{y}_{k'}^{(p+1)}$ ,  $1 \leq k' \leq K'$ ) have the index  $p$  (resp.  $p+1$ ).
- All the bars of  $\mathcal{B}_f$  with the lower (resp. upper) endpoint  $c$  (resp.  $c'$ ) have a length larger or equal to  $c' - c$ . The numbers of such bars of length equal to  $c' - c$  (the bar is a copy of  $[c, c']$ ), is denoted by  $K_0 \leq \min(K, K')$ .



**Figure 20:** A simple example in dimension 1 with  $K = 4$ ,  $K_0 = K' = 3$ .

- We will consider the energy interval  $[a, b]$  such that  $c$  (resp.  $c'$ ) is the smallest (resp. largest) critical value in  $[a, b]$ .

- When  $\underline{x}_k^{(p)}$ ,  $k = 1, \dots, K$  (resp.  $\underline{y}_{k'}^{(p+1)}$ ,  $k' = 1, \dots, K'$ ) denote the critical points for the value  $c$  (resp.  $c'$ ) the function  $\chi_k^{(p)} \in \mathcal{C}^\infty(M; [0, 1])$  (resp.  $\chi_{k'}^{(p+1)} \in \mathcal{C}^\infty(M; [0, 1])$ ) is supported in a neighborhood and equals 1 in a smaller neighborhood of  $\underline{x}_k^{(p)}$  (resp.  $\underline{y}_{k'}^{(p+1)}$ ) for  $k = 1, \dots, K$  (resp.  $k' = 1, \dots, K'$ ). Let  $t_k^{(p)}$ ,  $k = 1, \dots, K$ , (resp.  $t_{k'}^{(p+1)}$ ,  $k' = 1, \dots, K'$ ) be real numbers. For  $\delta \in \mathbb{R}$  small, we consider

$$f_\delta = f + \delta \left[ \sum_{k=1}^K t_k^{(p)} \chi_k^{(p)} + \sum_{k'=1}^{K'} t_{k'}^{(p+1)} \chi_{k'}^{(p+1)} \right].$$

Because  $f$  is a Morse function we may find  $\varepsilon > 0$  small enough such that the homology vector space  $H_p(f^{c+\varepsilon}, f^{c-\varepsilon}; \mathbb{R})$  (resp.  $H_{p+1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}; \mathbb{R})$ ) have a basis made of the descending (unstable of  $-\nabla f$ ) manifolds  $e_k^{(p)}$ ,  $1 \leq k \leq K$  (resp.  $e_{k'}^{(p+1)}$ ,  $1 \leq k' \leq K'$ ) restricted to  $f_{c-\varepsilon}$  (resp.  $f_{c'-\varepsilon}$ ). The boundary of  $e_k^{(p)}$  (resp.  $e_{k'}^{(p+1)}$ ) is a  $p-1$ -dimensional (resp.  $p$ -dimensional) sphere  $\partial e_k^{(p)}$  (resp.  $\partial e_{k'}^{(p+1)}$ ) lying in  $f^{-1}(\{c-\varepsilon\})$  (resp. in  $f^{-1}(\{c'-\varepsilon\})$ ).

On the Witten Laplacian side,  $\ker(\Delta_{f, f^{-1}([c-\varepsilon, c+\varepsilon]), h}^{(p)})$  (resp.  $\ker(\Delta_{f, f^{-1}([c'-\varepsilon, c'+\varepsilon]), h}^{(p+1)})$ ) may be approximated with a  $\tilde{O}(e^{-\frac{\varepsilon}{h}})$ -distance by  $\bigoplus_{1 \leq k \leq K} \mathbb{C} \psi_k^{(p)}$  (resp.  $\bigoplus_{1 \leq k' \leq K'} \mathbb{C} \psi_{k'}^{(p+1)}$ ), where  $\psi_k^{(p)}$  (resp.  $\psi_{k'}^{(p+1)}$ ) is a normalized ground state of  $\Delta_{f, k}^{(p)}$  (resp.  $\Delta_{f, k'}^{(p+1)}$ ), the Witten Laplacian in degree  $p$  (resp.  $p+1$ ) with full Dirichlet boundary conditions in  $B(\underline{x}_k^{(p)}, R\sqrt{\varepsilon})$  (resp.  $B(\underline{y}_{k'}^{(p+1)}, R\sqrt{\varepsilon})$ ) for  $R > 0$  chosen large enough. We refer to [Hel] and [HeSj4] and we recall that for the Witten Laplacian associated with a Morse function  $f$ , the local Agmon distance to a critical point  $s$ ,  $\phi$  solving  $|\nabla \phi|^2 = |\nabla f|^2$  and satisfying  $\phi(x) \geq |f(x) - f(s)|$ , behaves like the square of the geodesic distance to  $s$ . Additionally, the  $L^2$  estimate between  $\psi_k^{(p)}$  (resp.  $\psi_{k'}^{(p+1)}$ ) and its projection onto  $\ker \Delta_{f, f^{-1}([c-\varepsilon, c+\varepsilon]), h}^{(p)}$  (resp.  $\ker \Delta_{f, f^{-1}([c'-\varepsilon, c'+\varepsilon]), h}^{(p+1)}$ ) can be completed by a  $\tilde{O}(e^{-\frac{\varepsilon}{4h}})$  error estimate in any Sobolev norm on the open set  $f_{c-\frac{\varepsilon}{2}}^{c+\frac{\varepsilon}{2}} \cap B(\underline{x}_k^{(p)}, \frac{R}{2}\sqrt{\varepsilon})$  (resp.  $f_{c'-\frac{\varepsilon}{2}}^{c'+\frac{\varepsilon}{2}} \cap B(\underline{y}_{k'}^{(p+1)}, \frac{R}{2}\sqrt{\varepsilon})$ ). We also have WKB-approximations for all the  $\psi_k^{(p)}$  (resp.  $\psi_{k'}^{(p+1)}$ )  $1 \leq k \leq K$  (resp.  $1 \leq k' \leq K'$ ) in  $B(\underline{x}_k^{(p)}, \frac{R}{2}\sqrt{\varepsilon})$  (resp.  $B(\underline{y}_{k'}^{(p+1)}, \frac{R}{2}\sqrt{\varepsilon})$ ) which are valid in  $W^{s,2}$ -norm.

By the construction of Theorem 6.3 there is a  $\tilde{O}(e^{-\frac{\varepsilon}{h}})$ -orthonormal family of quasimodes  $\varphi_k^{(p)}$ ,  $1 \leq k \leq p$ , which are approximated by the  $\Pi_{\ker(\Delta_{f, f^{-1}([c-\varepsilon, c+\varepsilon]), h}^{(p)})} \psi_k^{(p)}$  and therefore by  $\psi_k^{(p)}$  or their WKB-approximation and which solve  $d_{f, h} \varphi_k^{(p)} = 0$  in  $f^{-1}([c-\varepsilon, c'-\frac{\varepsilon}{2}])$ , vanish in  $f^{c-\varepsilon}$  and satisfy the exponential decay property.

At the level  $c'$  the local quasimodes are  $\Pi_{\ker(\Delta_{f, f^{-1}([c'-\varepsilon, c'+\varepsilon]), h}^{(p+1)})} \psi_{k'}^{(p+1)}$  and are therefore close to  $\psi_{k'}^{(p+1)}$ .

For a generic choice of the coefficients  $t_k^{(p)}$  and  $t_{k'}^{(p+1)}$ , the perturbation  $f_\delta$  is a Morse function with simple critical values as soon as  $\delta \in \mathbb{R}$  is chosen small enough. Moreover the stability theorem says that the bars with endpoints  $c$  and  $c'$  are simply modified by  $\mathcal{O}(\delta)$  variations of the endpoints while all the other bars are not changed owing to our choice of  $f_\delta$ . We can even be more specific. The above parameter  $\varepsilon > 0$ ,  $R$  being fixed,  $\varepsilon$  small enough, we may take the cut-off function  $\chi_k^{(p)}$ ,  $k = 1, \dots, K$ , (resp.  $\chi_{k'}^{(p+1)}$ ,  $k' = 1, \dots, K'$ ) such that the equal 1 in  $B(\underline{x}_k^{(p)}, 2R\sqrt{\varepsilon})$  (resp.  $B(\underline{y}_{k'}^{(p+1)}, 2R\sqrt{\varepsilon})$ ). Finally  $\delta > 0$  is chosen small enough such that all the critical values of  $f_\delta$  close to  $c$  (resp.  $c'$ ) are in  $[c-\varepsilon/2, c+\varepsilon/2]$  (resp.  $[c'-\varepsilon/2, c'+\varepsilon/2]$ ). With this choice of  $f_\delta$ ,  $(e_k^{(p)})_{k=1, \dots, K}$  (resp.  $(e_{k'}^{(p+1)})_{k'=1, \dots, K'}$ ) defines a basis of  $H_p((f_\delta)^{c+\varepsilon}, (f_\delta)^{c-\varepsilon}; \mathbb{R})$  (resp.  $H_{p+1}((f_\delta)^{c'+\varepsilon}, (f_\delta)^{c'-\varepsilon}; \mathbb{R})$ ). The quasimodes  $\psi_k^{(p)}$ ,  $\psi_{k'}^{(p+1)}$ , and their WKB-approximations are not changed because we have just changed  $f$  by a constant in  $B(\underline{x}_k^{(p)}, R\sqrt{\varepsilon})$  (resp.  $B(\underline{y}_{k+1}^{(p+1)}, R\sqrt{\varepsilon})$ ).

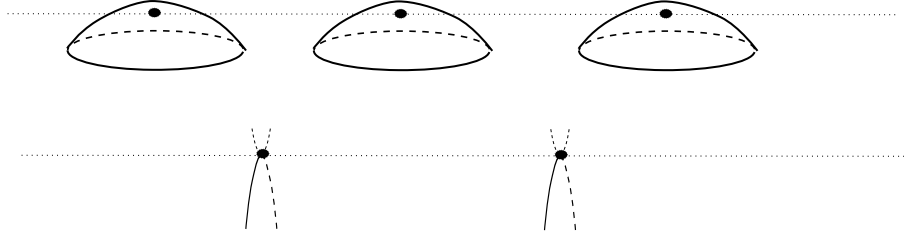
**Lemma 9.9.** *In the above framework and for  $\delta \in \mathbb{R}$  small enough the boundary map  $\partial$  :*

$H_{p+1}((f_\delta)^{c'+\varepsilon}, (f_\delta)^{c'-\varepsilon}; \mathbb{R}) \stackrel{\text{can.}}{\sim} \bigoplus_{k'=1}^{K'} \mathbb{R} e_{k'}^{(p+1)}$  induces a linear map to  $H_p((f_\delta)^{c+\varepsilon}, (f_\delta)^{c-\varepsilon}; \mathbb{R}) \stackrel{\text{can.}}{\sim} \bigoplus_{k=1}^K \mathbb{R} e_k^{(p)}$  of rank  $K_0$  which is written

$$\partial : e_{k'}^{(p+1)} \mapsto \sum_{k=1}^K \kappa_{k,k'} e_k^{(p)} .$$

The matrix  $\kappa$  does not depend on  $\delta$ .

*Proof.* When  $\delta = 0$ , the boundary map sends  $H_{p+1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}; \mathbb{R})$  to  $H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon}; \mathbb{R})$  of which a dual basis (in cohomology) is indexed by the  $K_0$  bars  $[c, c']$ ,  $k = 1, \dots, K_0$ . It suffices to follow the bars to the lower endpoint to define a linear map to  $H_p(f^{c+\varepsilon}, f^{c-\varepsilon}; \mathbb{R})$ . For a general  $\delta$  small enough,  $f_\delta$  differs from  $f$  only by a constant in each ball of radius  $R\sqrt{\varepsilon}$  around the critical points  $x_k^{(p)}, y_{k'}^{(p+1)}$ . Therefore, the gradient vector fields and the Morse models remain unchanged around these points. The homotopy becomes trivial by replacing locally the level set  $f^{-1}(\{c - \varepsilon\})$  (resp.  $f^{-1}(\{c' - \varepsilon\})$ ) by  $f_\delta^{-1}(\{c - \varepsilon\}) = f^{-1}(\{c - \varepsilon - \delta t_k^{(p)}\})$  (resp.  $f_\delta^{-1}(\{c' - \varepsilon\}) = f^{-1}(\{c' - \varepsilon - \delta t_{k'}^{(p+1)}\})$ ). Hence,  $(e_k^{(p)})_{k \in \{1, \dots, K\}}$  (resp.  $(e_{k'}^{(p+1)})_{k' \in \{1, \dots, K'\}}$ ) appears as a canonical basis of  $H_p((f_\delta)^{c+\varepsilon}, (f_\delta)^{c-\varepsilon}; \mathbb{R})$  (resp.  $H_{p+1}((f_\delta)^{c'+\varepsilon}, (f_\delta)^{c'-\varepsilon}; \mathbb{R})$ ) in which the matrix  $\kappa$  of the topological linear map  $\partial : H_{p+1}((f_\delta)^{c'+\varepsilon}, (f_\delta)^{c'-\varepsilon}; \mathbb{R}) \rightarrow H_p((f_\delta)^{c+\varepsilon}, (f_\delta)^{c-\varepsilon}; \mathbb{R})$  remains unchanged.  $\square$



**Figure 21:** In dimension 2 we have represented 3 critical points with index 2 at the value  $c'$  and 2 critical points with index 1 at the value  $c$ . The unstable (and stable manifold for the index 1) of  $-\nabla f$  are considered in the level sets  $f_{c'+\varepsilon}^{c'+\varepsilon}$  and  $f_{c-\varepsilon}^{c+\varepsilon}$ . The homotopy with respect to  $\delta$  consists simply to move separately up or down, the disconnected parts of this picture.

**Proposition 9.10.** *In the above framework with  $\delta$  small enough, the singular values  $\mu_h$  of  $d_{f_\delta, (f_\delta)^{-1}([a,b]), h}^{(p)}$  which satisfy  $\lim_{h \rightarrow 0} -h \log \mu_h = c - c' + \mathcal{O}(\delta)$  are equal to  $(1 + \mathcal{O}(h)) \times$  the non zero singular values of the  $K \times K'$  matrix*

$$\left(\frac{h}{\pi}\right)^{1/2} (D^{(p)})^{-1} \kappa D^{(p+1)}$$

where  $D^{(p)}$  (resp.  $D^{(p+1)}$ ) is the diagonal matrix with entries

$$\frac{|\lambda_1(\underline{x}_k^{(p)}) \cdots \lambda_p(\underline{x}_k^{(p)})|^{1/4}}{|\lambda_{p+1}(\underline{x}_k^{(p)}) \cdots \lambda_d(\underline{x}_k^{(p)})|^{1/4}} e^{-\frac{f_\delta(\underline{x}_k^{(p)})}{h}}, \quad k = 1, \dots, K,$$

resp.

$$\frac{|\lambda_1(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_{p+1}(\underline{y}_{k'}^{(p+1)})|^{1/4}}{|\lambda_{p+2}(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_d(\underline{y}_{k'}^{(p+1)})|^{1/4}} e^{-\frac{f_\delta(\underline{y}_{k'}^{(p+1)})}{h}}, \quad k' = 1, \dots, K'.$$

*Proof.* Set  $x_{k,\delta}^{(p)} = f_\delta(\underline{x}_k^{(p)}) = c + \delta t_k^{(p)}$  and  $y_{k',\delta}^{(p+1)} = f_\delta(\underline{y}_{k'}^{(p+1)}) = c' + \delta t_{k'}^{(p+1)}$ , for  $k = 1, \dots, K$  and  $k' = 1, \dots, K'$ . An orthonormal basis of  $\ker(\Delta_{f_\delta, f_\delta}^{(p+1)}([c - \varepsilon, c + \varepsilon], h))$  is well approximated by the local quasimodes  $\psi_{k'}^{(p+1)}$  which is the ground state of the full Dirichlet realization of  $\Delta_{f,h}^{(p+1)}$  in  $B(\underline{y}_{k'}^{(p)}, R\sqrt{\varepsilon})$  which do not depend on  $\delta$ . The same holds for  $\ker(\Delta_{f_\delta, f_\delta}^{(p)}([c - \varepsilon, c + \varepsilon], h))$  with the notation  $\psi_k^{(p)}$ ,  $k = 1, \dots, p$ . Hence  $\bigoplus_{k=1}^{\perp} \mathbb{C}\psi_k^{(p)}$  provides a good approximation in the energy interval  $[c - \varepsilon, c + \varepsilon]$  for  $f_\delta$  for the vector space of global quasimodes  $\varphi_{k,\delta}^{(p)}$  for  $f_\delta$  associated with the bars  $[x_{k,\delta}^{(p)}, y_{k,\delta}^{(p+1)}[$  for  $k = 1, \dots, K_0$  and  $[x_{k,\delta}^{(p)}, b[$  for  $k = K_0 + 1, \dots, K$ . Let us chose the basis  $(\varphi_{k,\delta}^{(p)})_{k=1, \dots, K}$  as an orthonormal basis such that  $\|\varphi_{k,\delta}^{(p)} - \psi_k^{(p)}\|_{L^2} = \tilde{o}(1)$ , while such a  $\tilde{o}(1)$  estimate also holds in any Sobolev norm in  $f_{c-\frac{\varepsilon}{2}}^{c+\frac{\varepsilon}{2}} \cap B(\underline{x}_{k,\delta}^{(p)}, \frac{R}{2}\sqrt{\varepsilon})$ . Those global quasimodes are assumed to solve  $d_{f_\delta, h} \varphi_{k,\delta}^{(p)} = 0$  in  $f_\delta^{-1}([a, c' - M\delta])$  for some  $M > 0$  large enough and we assume  $M\delta \ll \delta_2 \ll \varepsilon$ . We now compute the interaction  $K' \times K$  matrix  $\langle \psi_{k'}^{(p+1)}, d_{f,h} \chi_{\delta_2}(f_\delta) \varphi_{k,\delta}^{(p)} \rangle$  where  $\chi_{\delta_2}$  smoothly vanishes in  $[c' - \delta_2, b]$  and equals 1 in  $[a, c' - 2\delta_2]$  for all  $k = 1, \dots, K$ . Because  $d_{f,h} \varphi_{k,\delta}^{(p)} = 0$  in  $f_\delta^{-1}([b, c' - M\delta])$ , the local computation around  $y_{k'}^{(p+1)}$  done in [LNV]-Section 4 are the same and they say:

$$\begin{aligned} \langle \psi_{k'}^{(p+1)}, d_{f,h} T_{\delta_2} \varphi_{k,\delta}^{(p)} \rangle &= \pm \left( \frac{h}{\pi} \right)^{1/2} \times \frac{|\lambda_1(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_{p+1}(\underline{y}_{k'}^{(p+1)})|^{1/4}}{|\lambda_{p+2}(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_d(\underline{y}_{k'}^{(p+1)})|^{1/4}} \times (\pi h)^{\frac{d}{4} - \frac{p}{2}} \\ &\quad \times \int_{\partial e_{k'}^{(p+1)}} e^{\frac{f_\delta}{h}} \varphi_{k,\delta}^{(p)} \times e^{-\frac{y_{k',\delta}^{(p+1)}}{h}} \times (1 + \mathcal{O}(h)). \end{aligned}$$

By Stokes's formula applied with  $d[e^{\frac{f_\delta}{h}} \varphi_{k,\delta}^{(p)}] = 0$  in  $f_\delta^{-1}([b, c' - M\delta])$  we obtain

$$\begin{aligned} \langle \psi_{k'}^{(p+1)}, d_{f,h} T_{\delta_2} \varphi_{k,\delta}^{(p)} \rangle &= \pm \left( \frac{h}{\pi} \right)^{1/2} \times \frac{|\lambda_1(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_{p+1}(\underline{y}_{k'}^{(p+1)})|^{1/4}}{|\lambda_{p+2}(\underline{y}_{k'}^{(p+1)}) \cdots \lambda_d(\underline{y}_{k'}^{(p+1)})|^{1/4}} \times (\pi h)^{\frac{d}{4} - \frac{p}{2}} \\ &\quad \times \left[ \sum_{j=1}^K \kappa_{j,k'} \int_{e_j^{(p)}} e^{\frac{f_\delta}{h}} \varphi_{k,\delta}^{(p)} \right] \times e^{-\frac{y_{k',\delta}^{(p+1)}}{h}} \times (1 + \mathcal{O}(h)). \end{aligned}$$

By approximating  $\varphi_{k,\delta}^{(p)}$  by  $\psi_k^{(p)}$  and its WKB approximation in  $B(\underline{x}_k^{(p)}, \frac{R}{2}\sqrt{\varepsilon})$  we have

$$\begin{aligned} (\pi h)^{\frac{d}{4} - \frac{p}{2}} \int_{e_j^{(p)}} e^{\frac{f_\delta}{h}} \varphi_{k,\delta}^{(p)} &= (\pi h)^{\frac{d}{4} - \frac{p}{2}} \int_{e_j^{(p)}} e^{\frac{f_\delta}{h}} \psi_k^{(p)} \times (1 + \tilde{o}(1)) \\ &= \pm 1 \frac{|\lambda_{p+1}(\underline{x}_k^{(p)}) \cdots \lambda_d(\underline{x}_k^{(p)})|^{1/4}}{|\lambda_1(\underline{x}_k^{(p)}) \cdots \lambda_p(\underline{x}_k^{(p)})|^{1/4}} e^{\frac{x_{k,\delta}^{(p)}}{h}} \times (1 + \mathcal{O}(h)). \end{aligned}$$

The error terms actually occur as matricial products on the left-hand side for the approximation of  $\psi_{k'}^{(p+1)}$  and on the right-hand side for  $\varphi_{k,\delta}^{(p)}$ .

The interaction matrix  $\langle (\psi_{k'}^{(p+1)}, d_{f_\delta, h} \chi_{\delta_2} \varphi_{k,\delta}^{(p)})_{1 \leq k' \leq K', 1 \leq k \leq K}$  is thus equal to

$$\text{diag}(\pm 1 + \mathcal{O}(h)) \left( \frac{h}{\pi} \right)^{1/2} D^{(p+1)}(t\kappa) (D^{(p)})^{-1} \text{diag}(\pm 1 + \mathcal{O}(h)).$$

Its singular values are thus equal up to a  $\mathcal{O}(h)$ -relative error to the singular values of

$$\left( \frac{h}{\pi} \right)^{1/2} D^{(p+1)}(t\kappa) (D^{(p)})^{-1}$$



or equivalently of

$$\left(\frac{h}{\pi}\right)^{1/2} (D^{(p)})^{-1} \kappa D^{(p+1)}.$$

□

**Remark 9.11.** *The result of Propostion 9.10, in a specific case, show that it is possible to get a matricial robust accurate formula for the exponentially small eigenvalues of Witten Laplacians for general Morse potentials. This provides another stability property valid for the first term asymptotics of the subexponential factor, which allows to study the transition from the generic Morse case with simple critical values to the general case. Note that here the power of  $h$  in this subexponential factor is not changed. But discontinuities appear on the constants as it is shown in the next simple examples. Actually we have considered a simple case where only one multiple bar  $[c, c[$  has to be taken into account. A more general form would consist in following the induction scheme of Theorem 6.3 and would lead to some complicated linear matricial structure for which we do not have an elegant presentation at the moment. In the degree  $p = 0$ , L. Michel in [Mic] proposed an interpretation in terms of the spectrum of a discrete Laplacian on a finite graph with vertices given by the local minima and edges given by saddle points. This formulation is written for a fixed Morse function with possible multiple local minima and saddle points, the perturbative issue is not really clarified there. In our specific example, the discrete Laplacian proposed by L. Michel is actually the square*

$$\frac{h}{\pi} (D^{(0)})^{-1} \kappa D^{(1)} D^{(1),*} \kappa^* (D^{(0)})^{-1,*}.$$

It would be interesting to find such a general robust formulation, with several multiple critical values, in degree  $p > 0$ .

**Examples:**

1. Consider a Morse function  $f$  on  $[s, t]$  such that  $\min_{x \in [s, t]} f(x) = f(s) = a$ ,  $\max_{x \in [s, t]} f(x) = f(t) = b$ , with  $f'(s) > 0$  and  $f'(t) > 0$ , with two local maxima and two local minima  $s < \underline{y}_1^{(1)} < \underline{x}_1^{(0)} < \underline{y}_2^{(1)} < \underline{x}_2^{(0)} < t$ ,  $f(\underline{y}_1^{(1)}) = f(\underline{y}_2^{(1)}) = c'$  and  $f(\underline{x}_1^{(0)}) = f(\underline{x}_2^{(0)}) = c$ . For the perturbation of  $f$  we will consider the cases when  $(t_1^{(1)}, t_2^{(1)}) = (0, 0)$ ,  $(t_1^{(0)}, t_2^{(0)}) \in \{(0, 0), (0, -1), (-1, 0)\}$ . The matrix  $\kappa$  equals

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

while the matrices  $D^{(0)}$  and  $D^{(1)}$  are given by

$$D^{(0)} = \begin{pmatrix} |\lambda_1(\underline{x}_1^{(0)})|^{-1/4} e^{-\frac{c+\delta t_1^{(0)}}{h}} & 0 \\ 0 & |\lambda_1(\underline{x}_2^{(0)})|^{-1/4} e^{-\frac{c+\delta t_2^{(0)}}{h}} \end{pmatrix} = \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix} e^{-\frac{c}{h}},$$

$$D^{(1)} = \begin{pmatrix} |\lambda_1(\underline{y}_1^{(0)})|^{1/4} e^{-\frac{c'}{h}} & 0 \\ 0 & |\lambda_2(\underline{y}_2^{(0)})|^{1/4} e^{-\frac{c'}{h}} \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} e^{-\frac{c'}{h}}.$$

The singular values of the matrix  $(D^{(0)})^{-1} \kappa D^{(1)}$  are the square roots of the eigenvalues of the symmetric square matrix

$$\begin{pmatrix} \alpha_1^2(\beta_1^2 + \beta_2^2) & -\alpha_1 \alpha_2 \beta_2^2 \\ -\alpha_1 \alpha_2 \beta_2^2 & \alpha_2^2 \beta_2^2 \end{pmatrix} e^{-2\frac{c'-c}{h}}.$$

Those eigenvalues equal

$$\frac{[(\alpha_1^2(\beta_1^2 + \beta_2^2) + \alpha_2^2 \beta_2^2) \pm \sqrt{[\alpha_1^2(\beta_1^2 + \beta_2^2) - \alpha_2^2 \beta_2^2]^2 + 4\alpha_1^2 \alpha_2^2 \beta_2^4}]}{2} \times e^{-2\frac{c'-c}{h}}.$$

Depending on the three considered cases, we obtain:

$t_1^{(0)} = t_2^{(0)} = 0$ : The 2 exponentially small eigenvalues of  $\Delta_{f,[s,t],h}^{(0) \text{ or } (1)}$  have the form  $C_k \frac{h}{\pi} e^{-2\frac{e'-c}{h}} (1 + \mathcal{O}(h))$ ,  $k = 1, 2$ , where the constants  $C_1$  and  $C_2$  clearly depend on the four Hessians at the critical points.

$t_1^{(0)} = -1, t_2^{(1)} = 0$ : The 2 exponentially small eigenvalues of  $\Delta_{f,[s,t],h}^{(0) \text{ or } (1)}$  are equal to:

$$\begin{aligned} & \frac{h}{\pi} |\lambda_1(\underline{x}_2^{(0)})|^{1/2} |\lambda_1(\underline{y}_2^{(1)})|^{1/2} e^{-2\frac{e'-c}{h}} (1 + \mathcal{O}(h)) \\ & \frac{h}{\pi} |\lambda_1(\underline{x}_1^{(0)})|^{1/2} |\lambda_1(\underline{y}_1^{(1)})|^{1/2} e^{-2\frac{e'-c+\delta}{h}} (1 + \mathcal{O}(h)). \end{aligned}$$

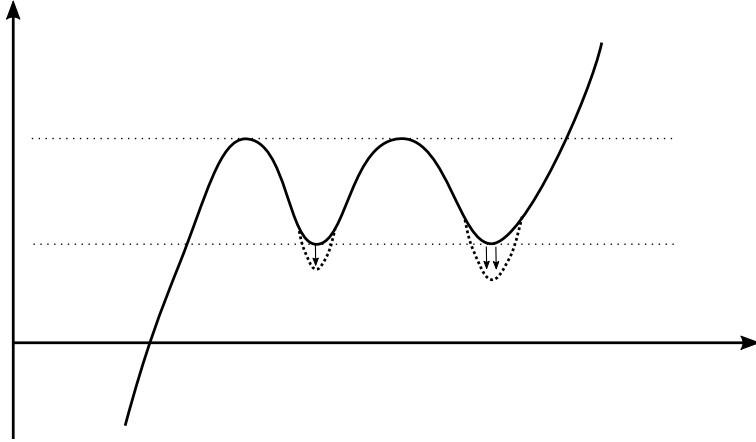
In particular the smallest one depends on the Hessians of  $f_\delta$  at the only points  $\underline{x}_1^{(0)}$  and  $\underline{y}_1^{(1)}$ .

$t_1^{(0)} = 0, t_2^{(0)} = -1$ : The 2 exponentially small eigenvalues of  $\Delta_{f,[s,t],h}^{(0) \text{ or } (1)}$  are equal to:

$$\begin{aligned} & 2 \frac{h}{\pi} |\lambda_1(\underline{x}_1^{(0)})|^{1/2} \frac{|\lambda_1(\underline{y}_1^{(1)})|^{1/2} + |\lambda_1(\underline{y}_2^{(1)})|^{1/2}}{2} e^{-2\frac{e'-c}{h}} (1 + \mathcal{O}(h)) \\ & \frac{1}{2} \frac{h}{\pi} |\lambda_1(\underline{x}_2^{(0)})|^{1/2} H(|\lambda_1(\underline{y}_1^{(1)})|^{1/2}, |\lambda_1(\underline{y}_2^{(1)})|^{1/2}) e^{-2\frac{e'-c+\delta}{h}} (1 + \mathcal{O}(h)), \end{aligned}$$

where  $H(u, v) = \frac{2uv}{u+v}$  denotes the harmonic mean.

In this case the smallest eigenvalue depends on the Hessians of  $f_\delta$  at the points  $\underline{x}_2^{(0)}$ ,  $\underline{y}_1^{(1)}$  and  $\underline{y}_2^{(1)}$ .



**Figure 22:** The three considered cases:  $(t_1, t_2) = (0, 0)$  plain line;  $(t_1, t_2) = (-1, 0)$  move the curve downward with  $(\downarrow)$ ,  $(t_1, t_2) = (0, -1)$  move the curve downward with  $(\downarrow \downarrow)$ .

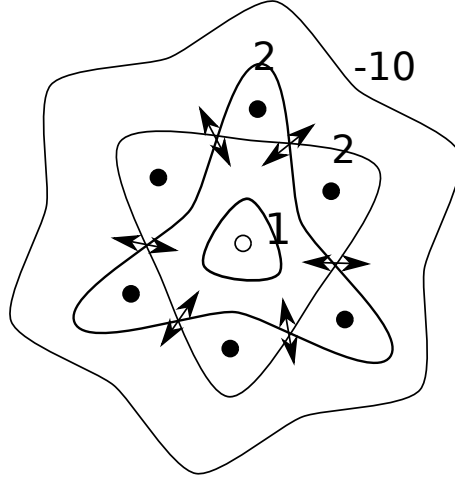
The general formula is again a robust formula which allow to follow the dependence on the parameter  $\delta$  of the asymptotic expressions although those at the end are not continuous with respect to  $\delta$ .

2. Consider in  $\mathbb{R}^d$  a function  $f$  with a unique local minimum at  $x_1^{(0)} = 0$  with  $f(0) = c$ , such that  $\lim_{x \rightarrow \infty} f(x) = -\infty$  and surrounded by  $K'$  saddle points, critical points with index 1, such that  $f(\underline{y}_{k'}^{(1)}) = c'$ , while all the other critical values are larger than  $c'$  with

an index  $p \geq 2$ . For the perturbation we will consider the two cases when  $t_1^{(0)} = 0$  and  $t_1^{(1)} \in \{0, -1\}$ . The matrix  $\kappa$  is the  $1 \times K'$  matrix  $(1, 1, \dots, 1)$ . Thus the smallest eigenvalue of  $\Delta_{f_\delta, (f_\delta)^{-1}([a,b]), h}^{(0)}$ , which is the unique exponentially small eigenvalue, equals

$$\begin{aligned} \frac{h}{\pi} |\det(\text{Hess}f(\underline{x}_1^{(0)}))|^{1/2} \sum_{k'=1}^{K'} \frac{|\lambda_1(\underline{y}_{k'}^{(1)})|^{1/2}}{|\lambda_2(\underline{y}_{k'}^{(1)}) \dots \lambda_d(\underline{y}_{k'}^{(1)})|^{1/2}} e^{-2\frac{c'-c}{h}} (1 + \mathcal{O}(h)) & \text{ if } \delta = 0, \\ \frac{h}{\pi} |\det(\text{Hess}f(\underline{x}_1^{(0)}))|^{1/2} \frac{|\lambda_1(\underline{y}_1^{(1)})|^{1/2}}{|\lambda_2(\underline{y}_1^{(1)}) \dots \lambda_d(\underline{y}_1^{(1)})|^{1/2}} e^{-2\frac{c'-c-\delta}{h}} (1 + \mathcal{O}(h)) & \text{ if } \delta > 0. \end{aligned}$$

Similar formulas are obtained for various configurations in [DLLN1, DLLN2, LeNe2, LeMi].



**Figure 23:** An example with  $K = 6$ . Level curves at the level 1, 2,  $-10$  are represented, the global minimum is denoted by  $\circ$ , the saddle points by  $\leftrightarrow$  and the local maximum by  $\bullet$ .

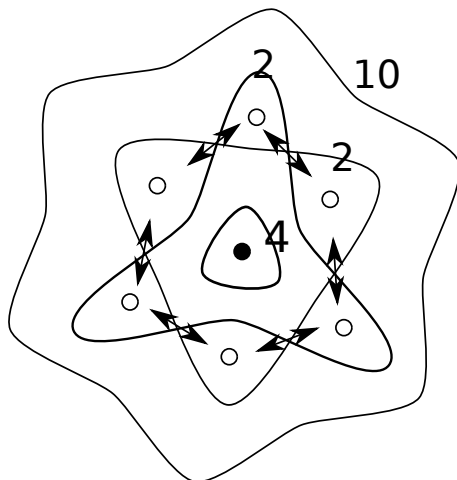
**3. A case with symmetries:** Consider in  $\mathbb{R}^2$  a Morse function  $f$  with a local maximum at  $\underline{y}_1^{(2)} = 0$ ,  $f(\underline{y}_1^{(2)}) = c_2$  surrounded by  $K$  saddle points at  $\underline{x}_k^{(1)} = \underline{y}_k^{(1)}$ ,  $k = 1 \dots K$ ,  $f(\underline{x}_k^{(1)}) = c_1$ , and  $K$  local minima at  $\underline{x}_k^{(0)}$ ,  $k = 1 \dots, K$ ,  $f(\underline{x}_k^{(0)}) = c_0$ ,  $c_0 < c_1 < c_2$ . We also assume that  $\lim_{x \rightarrow \infty} f(x) = +\infty$  and that  $f$  has no other critical points. When  $j \in \{1, 2\}$  or  $p \in \{0, 1\}$  are fixed  $\lambda_j(\underline{x}_k^{(p)}) = \lambda_j^{(p)}$  do not depend on  $k = 1, \dots, K$ . We study the eigenvalues of  $\Delta_{f, \mathbb{R}^2, h}^{(p)}$ ,  $p = 0, 1, 2$  by restricting to the case  $c_0 < a < c = c_1 < c' = c_2 < b$  for  $p = 2$  and to the case  $a < c = c_0 < c' = c_1 < b < c_2$  for  $p = 0$ . By supersymmetry, the non zero eigenvalues of  $\Delta_{f, \mathbb{R}^2, h}^{(1)}$  are obtained by gathering the ones of  $\Delta_{f, \mathbb{R}^2, h}^{(0)}$  and  $\Delta_{f, \mathbb{R}^2, h}^{(2)}$ .

**For  $p = 2$ ,  $c_0 < a < c = c_1 < c' = c_2 < b$ :** The matrix  $\kappa$  equals the  $K \times 1$  matrix

$$\kappa = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The smallest eigenvalue of  $\Delta_{f, f^{-1}([a,b]), h}^{(2)}$ , which is the only exponentially small one, then equals:

$$\frac{h}{\pi} |\det(\text{Hess}f(\underline{y}_1^{(2)}))|^{1/2} K \frac{|\lambda_2^{(1)}|^{1/2}}{|\lambda_1^{(1)}|^{1/2}} e^{-2\frac{c_2-c_1}{h}} (1 + \mathcal{O}(h)).$$



**Figure 24:** An example with  $K = 6$ . Level curves at the level 4, 2, 10 are represented, the local minima are denoted by  $\circ$ , the saddle points by  $\leftrightarrow$  and the global maxima by  $\bullet$ .

For  $p = 0$ ,  $a < c = c_0 < c' = c_1 < b < c_2$ : The matrix  $\kappa$  is the  $K \times K$  matrix

$$\kappa = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & -1 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

of which the singular values equal  $|1 - \omega^k|$ ,  $k = 1, \dots, K$ , where  $\omega^k = e^{2i\pi \frac{k}{K}}$  for  $k = 1, \dots, K$ . Owing to

$$(D^{(0)})^{-1} \kappa D^{(1)} = |\lambda_1^{(0)} \lambda_2^{(0)}|^{1/4} \frac{|\lambda_1^{(1)}|^{1/4}}{|\lambda_2^{(1)}|^{1/4}} e^{-\frac{c_1 - c_0}{h}} \kappa,$$

we deduce that the  $K$  exponentially small eigenvalues of  $\Delta_{f, f^{-1}([a, b]), h}^{(0)}$  are equal to

$$|\lambda_1^{(0)} \lambda_2^{(0)}|^{1/2} \frac{|\lambda_1^{(1)}|^{1/2}}{|\lambda_2^{(1)}|^{1/2}} |1 - \omega^k|^2 e^{-2\frac{c_1 - c_0}{h}} (1 + \mathcal{O}(h)), \quad k = 1, \dots, K.$$

This case with  $p = 0$  was considered by Michel in [Mic] for the Witten Laplacian and by Hérou-Hitrik-Sjöstrand in [HHS] for the non-self-adjoint Kramers-Fokker-Planck operator.

## 10 Broadening the scope

Our work provides a general method for analyzing the exponentially small eigenvalues of Witten Laplacians with a general potential function. However, it does not answer all the questions that arose along this analysis. Here is a short list of still open questions and of connections with closely related fields.

**a) General  $C^\infty$  potential:** A general  $C^\infty$ -function on a compact manifold  $M$  may have an

infinite number of critical values and bars in its bar code. Nevertheless, for any  $\varepsilon > 0$ , the set of bars of length larger than  $\varepsilon$  is finite. In order to realize this, take a covering  $[\min f, \max f] \subset \cup_{i=1}^{N_\varepsilon} [a_i, a_{i+1}]$ , where the  $a_i$ 's are not critical values and such that  $0 < a_{i+1} - a_i \leq \varepsilon$  for all  $i \in 1, \dots, N - 1$ . Any bar  $\alpha^{(p)}$  of degree  $p$  and length larger than  $\varepsilon$  has at most two endpoints lying in different intervals  $[a_i, a_{i+1}]$  and appearing as an element of  $\mathcal{Z}^{(p)}(a_i, a_{i+1})$  for the possible lower endpoint and an element of  $\mathcal{Z}^{(p+1)}(a_{i'}, a_{i'+1})$  for the possible upper endpoint with  $i \neq i'$ . Therefore, the set of bars of degree  $p$  and length larger than  $\varepsilon$  is bounded by

$$\begin{aligned} \sum_{i=1}^{N_\varepsilon-1} \#\mathcal{Z}^{(p)}(a_i, a_{i+1}) + \#\mathcal{Z}^{(p+1)}(a_i, a_{i+1}) \\ = \sum_{i=1}^{N_\varepsilon-1} \beta^{(p)}(f^{a_{i+1}}, f^{a_i}) + \beta^{(p+1)}(f^{a_{i+1}}, f^{a_i}) < +\infty. \end{aligned}$$

The conjecture stated in the introduction for a general  $\mathcal{C}^\infty$  function  $f$  has now the following more precise version: For  $\varepsilon > 0$ , the  $\tilde{o}(e^{-\frac{2\varepsilon}{\hbar}})$  eigenvalues of  $\Delta_{f,M,h}^{(p)}$  are given by the  $\lambda_\alpha^{(p)}(h)$  such that  $\alpha$  is of length larger than  $\varepsilon$ ,  $\alpha \in A^{(p)}$  or ( $\alpha \in A^{(p-1)}$  and  $b_\alpha^{(p)} < +\infty$ ), and

$$\lim_{h \rightarrow 0} -h \log(\lambda_\alpha(h)) = 2(b_\alpha - a_\alpha).$$

Our proof, relying on a recurrence on the number of critical values by following their increasing (and decreasing) order, is not adapted to the more general case with an infinite number of critical values. One may think of a different type of induction: Starting from our result for finitely many critical values, one may increase the number of critical values by perturbing the function such that it creates small bars in a given interval  $[a, b]$ , and then try to obtain spectral and resolvent estimates for the spectral parameter  $\lambda \in [0, \tilde{o}(e^{-\frac{2\varepsilon}{\hbar}})]$ , which are uniform with respect to the additional small bars.

**b) What about  $\mathcal{C}^0$ -potentials ?** The stability of bar codes makes sense in the  $\mathcal{C}^0$  topology while a finite bar code can be associated with a continuous function which satisfies Hypothesis B.1. The relation between the exponentially small eigenvalues of  $\Delta_{f,h}$  and the bar code of  $f$  suggests that the bottom of the spectrum of  $\Delta_{f,h}$  makes sense only under Hypothesis B.1. Is there a natural self-adjoint operator “ $\Delta_{f,h}$ ” on  $M$  when  $f$  is only continuous and for which Theorem 1.7 could be extended ?

**c) Applications of the result on  $p$ -forms:** Over decades, the case of functions has received a lot of attention with an easy interpretation in terms of Fokker-Planck equation associated with reversible processes at low temperature and within the modelling e.g. in chemistry as points the title of this text. Here is an attempt to interpret our spectral results for  $p$ -forms. This deserves more precise studies and we hope that relevant applications will be found. Within the stochastic approach, the Witten Laplacian is better written as

$$\mathcal{L}_{f,h} = e^{\frac{f}{\hbar}} \Delta_{f,h} e^{-\frac{f}{\hbar}} = h^2 \Delta_{0,1} + 2h \mathcal{L}_{\nabla f} = d_{0,h} d_{2f,h}^* + d_{2f,h}^* d_{0,h},$$

considered in the  $L^2$ -space associated with the invariant measure  $e^{-\frac{2f}{\hbar}} dx$ ,  $L^2(M, e^{-\frac{2f}{\hbar}} dx; \Lambda T^*M)$ , and where  $\Delta_{0,1} = dd^* + d^*d$  is the usual Hodge Laplacian. There are formulas to express the semigroups associated with Hodge and Witten Laplacians, in terms of expectations values along brownian motion:  $e^{-t\mathcal{L}_{f,h}} v = \mathbb{E}(\xi_t^* v)$  for  $v \in \mathcal{C}^\infty(M; \Lambda T^*M)$ , where  $\xi_t$  is the flow associated with a stochastic differential equation of the type  $dx = X(x_t) \circ dB_t - 2\nabla f(x_t) dt$  where  $B$  is an  $m$ -dimensional brownian motion in  $\mathbb{R}^m$  and  $X : M \times \mathbb{R}^m \rightarrow TM$  is a submersion specified by the metric on  $M$  (see in particular [ELJL, Theorem 1.1.2, formula 1.2.5, and Section 2.4]). Due to the supesymmetric argument, eigenforms of  $\Delta_{f,h}$  (resp.  $\mathcal{L}_{f,h}$ ) can be assumed to solve  $d_{f,h}^* \omega = 0$  (resp.  $d_{2f,h}^* \tilde{\omega} = 0$  with  $\tilde{\omega} = e^{\frac{f}{\hbar}} \omega$ ), because when  $d_{f,h}^* \omega \neq 0$  (resp.  $d_{2f,h}^* \omega \neq 0$ ) then  $d_{f,h}^* \omega$  (resp

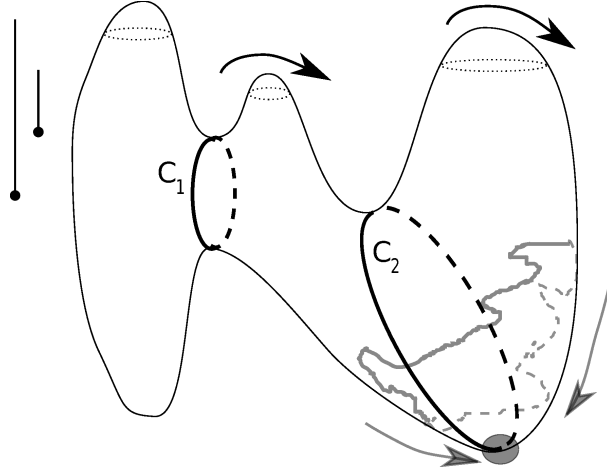
$d_{2f,h}^* \tilde{\omega}$ ) is another eigenform of  $\Delta_{f,h}$  (resp. of  $\mathcal{L}_{f,h}$ ) with degree decreased by 1 and associated with the same eigenvalue. Let  $\tilde{\omega}$  be such an eigenform with  $d^*(e^{\frac{2f}{h}} \tilde{\omega}) = 0$  and  $\mathcal{L}_{f,h} \tilde{\omega}_h = \lambda_h \tilde{\omega}_h$ . By assuming that  $\tilde{\omega}$  is a  $p$ -form and after normalization,  $A_h e^{\frac{2f}{h}} \tilde{\omega}$  may be identified with a  $p$ -cycle via

$$\int_M \eta \wedge (\star e^{\frac{2f}{h}} A_h \tilde{\omega}) = \int_{C_{\tilde{\omega},h}} \eta,$$

where  $\partial C_{\tilde{\omega},h} = 0$  is a consequence of  $d^*(e^{\frac{2f}{h}} \tilde{\omega}) = 0$ . It would be better to think of  $C_{\tilde{\omega},h}$  as a courant but let us forget the regularity issues. When  $f$  is a Morse function with

$$f(x_1, \dots, x_p, x_{p+1}, \dots, x_d) = -\varphi_-(x_1, \dots, x_p) + \varphi_+(x_{p+1}, \dots, x_d)$$

around a critical point of index  $p$  with critical value 0 which is a lower endpoint of a bar of degree  $p$ , the leading term of the WKB-approximation says  $e^{\frac{2f}{h}} \tilde{\omega} = e^{-\frac{2\varphi_+(x_{p+1}, \dots, x_d)}{h}} dx_1 \wedge \dots \wedge dx_p$  and  $C_{\tilde{\omega},h}$  is asymptotically equal to some fixed cycle  $C_{\tilde{\omega},0}$  supported by the unstable manifold of  $-\nabla f$ . We may expect such a behaviour in general. The evolution  $\tilde{\omega}_h(t) = e^{-t\mathcal{L}_{f,h}} \tilde{\omega}_h = e^{-t\lambda_h} \tilde{\omega}_h$  says that this cycle is not changed by the dynamics when  $t \ll \frac{1}{\lambda_h}$  and disappears when  $t \gg \frac{1}{\lambda_h}$ . The reverse eigenvalue  $\frac{1}{\lambda_h}$  appear as the lifetime of the cycle  $C_{\tilde{\omega},h}$  of which an asymptotic form  $C_{\tilde{\omega},0}$  is expected when the normalization factor  $A_h$  is well chosen. Below is a picture for the brownian dynamics of a 1-cycle, which shows the generalization of the metastability picture that we expect.



**Figure 25:** Metastability of cycles: The bars of degree 1 represented on the left-hand side, with lengths  $\ell_1 < \ell_2$ , provide the lifetime  $e^{\frac{2\ell_1}{h}}$  (resp  $e^{\frac{2\ell_2}{h}}$ ) of the cycle  $C_1$  (resp.  $C_2$ ). After a time larger than the lifetime,  $C_1$  is first deformed into  $C_2$  and  $C_2$  is then deformed into the grey cycle which is rapidly retracted to the global minimum.

**d) General statement for subexponential factors:** Specifying the exponential scales of the exponentially small eigenvalues of  $\Delta_{f,M,h}^{(p)}$  associated with the lengths of the bar code of  $f$  was done in Theorem 1.7 and Theorem 7.1, while the spectral version of the stability was given in Corollary 1.8 and Theorem 7.6. Those results are general statements which hold under simple general assumptions like Hypothesis 1.2 or Hypothesis 1.3. The situation is different when we want to specify the subexponential factors. In Section 9, the general construction was used in order to specify the subexponential factors and to show that they were keeping some kind of stability property, possibly within a finite dimensional matricial writing (see Proposition 9.10).

Although the method is clear and heavily relies on Theorem 6.3 and the use of Stokes' theorem like in [LNV], we were not able to take into account all the possible configurations in a uniform and satisfactory presentation. Although the stability of individual subexponential factors cannot hold, a general robust statement or formula for the determination of the subexponential factors would be valuable.

**e) Piecewise affine functions and discretization via triangulation:** In the one dimensional case, a schematic Witten Laplacian for which everything relies on simple linear algebra is provided by a piecewise affine function  $f$ . Eigenforms of degree 0 or 1 are computed by matching exponentials at the discontinuities of the slope of  $f$ . It becomes a fully discrete model, in its coding and in the computation of the eigenforms. The generalization of a piecewise affine function after a triangulation of  $\mathbb{R}^d$  or  $\mathbb{T}^d$  (and for further generalizations, one should consider a Lipschitz triangulated riemannian manifold like in [GMM]) enters in our general assumption Hypothesis 1.3. Away from the singularities of  $f$ , the Witten Laplacian is nothing but a scalar operator  $-\Delta + V(x)$ , where  $V$  is a piecewise constant function, while the Hessian of  $f$  brings a measure potential carried by the singularities of  $f$ . We are led to consider a specific self-adjoint extension of  $-\Delta + V(x)$  on  $\mathcal{C}_0^\infty(\Omega_{reg}; \Lambda T^*M)$ , where  $\Omega_{reg}$  is the open domain where  $f$  is differentiable with a locally constant gradient. Many things have been done on the scalar Laplacian plus simple or double layer potentials, or more general interface conditions (see [AGHKH, BGP]). Here we work with Hodge-type Laplacians and discriminating with respect to the degree will lead to different types of interface conditions and we wonder whether cohomology brings additional restrictions along strata of codimension  $> 1$ . It would be interesting to see if such a finitely coded potential  $f$  leads to a completely solvable linear algebra problem like in dimension 1. It could be an alternative model problem as compared to the case of Morse functions, which could be useful to understand some non trivial boundary or corner problems.

**f) Infinite or large dimensional problems:** After specifying the geometrical problems, especially concerned with the domain issues for the differential, codifferential, and Witten Laplacian, all the analysis is carried out along the real axis of values of  $f$ ,  $\mathbb{R} \supset f(M)$ . In this projective perspective, the dimension of  $M$  does not count until the computation of the subexponential factors, which involve the asymptotics of Laplace integrals. This raises the question of the validity of such an approach for infinite dimensional – or large dimensional – problems, which have applications in statistical physics, and where the asymptotic behaviour of the dimension is related with the small parameter  $h \rightarrow 0^+$  (see e.g. [HelW, DiLe], or the recent [BrDi] where the estimates when  $h \rightarrow 0^+$  are even shown to be uniform in the dimension, and references therein).

**g) Other boundary conditions for Witten Laplacians:** Our results include the case of Witten Laplacians on bounded domains like  $f_a^b$ , provided that one considers Neumann boundary conditions on the upper boundary  $f^{-1}(\{b\})$  and Dirichlet boundary conditions on  $f^{-1}(\{a\})$ . In some applications like in the analysis of quasi-stationary distributions, it is relevant to put Dirichlet boundary conditions everywhere on  $\partial\Omega$  when the manifold  $M$  is replaced at the beginning by some regular domain (see [LeNi, DLLN1, DLLN2, LeNe1, LeNe2]). The cohomology groups  $H^*(f^b; f^a)$  have to be replaced by  $H^*(f^b; f^a \cup \partial\Omega)$ , but additional corner problems at the intersection  $\partial\Omega \cap f^{-1}(\{a, b\})$  have to be analyzed carefully.

**h) Non reversible dynamics and spectral analysis of non self-adjoint related problems:** The analysis of Witten Laplacians enters in the general scope of the semiclassical analysis of self-adjoint Schrödinger-type operators. Within the stochastic analysis, several models, motivated by applications where a non reversible drift is considered, lead to non self-adjoint operators for which a similar analysis can be carried out in the case of functions,  $p = 0$  (see e.g [LeMi]). An interesting non self-adjoint (and non elliptic) operator which has many connections with Witten Laplacian is Bismut's hypoelliptic Laplacian, which is defined in any degree  $0 \leq p \leq 2d$  when we work on  $\mathcal{X} = T^*Q$  with  $\dim Q = d$ . The asymptotic behaviour of exponentially small eigenvalues has been studied so far only when  $p = 0$  and  $Q = \mathbb{R}^d$  in [HHS], where Bismut's hypoelliptic Laplacian is nothing but the Kramers-Fokker-Planck operator of kinetic theory. For studying the case of general  $p$ -forms on a manifold, a better understanding of boundary condi-



tions for Bismut’s hypoelliptic Laplacians (defined in [Nie]) is necessary. When  $f : Q \rightarrow \mathbb{R}$  is the potential, adapting the analysis of this text would lead to “Dirichlet boundary conditions” on  $T_{f^{-1}(\{a\})}^*Q$  and “Neumann boundary conditions” on  $T_{f^{-1}(\{b\})}^*Q$  for the hypoelliptic Laplacian acting in  $\pi^{-1}(f_a^b)$ , where  $\pi : T^*Q \rightarrow Q$  is the fiber projection. Additionally, the non self-adjoint nature of the problem requires different techniques relying on complex deformations in order to handle the exponential decay of resolvents and eigenfunctions.

**i) Remarks about the subanalytic case:** In the subanalytic case and for at least the second time (a previous time was in [GeNi] for the analysis of Mourre estimates for analytically fibered operators), the differentiation along regular strata has been used in order to prove estimates. Instead of considering a non regular solution  $\phi$  to the Hamilton-Jacobi equation  $|\nabla f|^2 = |\nabla \phi|^2$ , we constructed a finite family of regular functions  $\phi_k$ ,  $k = 1, \dots, K$ ,  $|\nabla f|^2 \geq |\nabla \phi_k|^2$ , finally leading to a good enough exponential decay estimate. We were not able to make a direct use of viscosity solutions, which did not allow to absorb all the singular terms in Agmon’s type estimates. In a different context, global subanalytic viscosity solutions to Hamilton-Jacobi with analytic coefficients (which is not the case here) were studied in [Tre]. Is there a better way to introduce viscosity solutions in our problem? In the other way, differentiating along the regular strata could it be used for constructing subsolutions to Hamilton-Jacobi type equations?

**j) Fukaya conjecture and multidimensional persistence:** Determining the homotopy type of a compact manifold  $M$  such that  $\pi_1(M) = 0$  and the  $A_\infty$  structure on harmonic forms induced by the pullback of the wedge product, can be attacked via Witten’s deformation. This was proposed by Fukaya in [Fuk] and more precisely studied via WKB methods a la Helffer-Sjöstrand in [CLM]. It consists in considering several Witten’s deformations of the differential and the Hodge Laplacian,  $d_{f_{ij},h} = e^{-\frac{f_{ij}}{h}}(hd)e^{\frac{f_{ij}}{h}}$ , associated with a sequence  $(f_0, f_1, \dots, f_k)$  such that  $f_{ij} = f_j - f_i$ ,  $0 \leq i < j \leq k$ , are Morse functions. Although it may not bring an additional topological information, replacing Morse functions by more general  $C^\infty$  functions means the understanding of the  $\frac{k(k+1)}{2}$ -dimensional version of persistence diagrams, bars being replaced by multidimensional objects. The multidimensional version of persistence homology, partly motivated by applications in statistical data analysis, is only emerging. We refer again to [KaSc] for a theoretical presentation of multidimensional persistence.

**k) Comparison with the instantonic picture:** The instantonic picture makes sense within Thom-Smale transversality condition, which ensures that any critical point of index  $p + 1$  is related to some critical points of index  $p$  by a finite number of regular integral curves of  $-\nabla f$ . This gives rise to the standard Thom-Smale complex structure. More recently, it has received an accurate analysis in terms of the analysis of the dynamical system of  $-2\nabla f$  perturbed by a brownian motion in [DaRi] by applying Faure-Sjöstrand theory of weighted Sobolev spaces. We already mentioned that our approach is orthogonal to the instantonic point of view: Instead of exploring the geometry of the potential landscape  $M \ni x \mapsto f(x) \in \mathbb{R}$ , we considered globally the sublevel sets  $f^\lambda$  and their homological properties. We can parallel this with the comparison between Riemann’s and Lebesgue’s integration theory. This global approach avoids to consider possibly complicated cancellation phenomena in the general method of tunnel effect computations described in [HeSj2, HeSj3]. It is a question whether such a global and topological approach makes sense for other spectral problems related with dynamical systems.

## A Abstract Hodge theory

The abstract version of Hodge theory provides spectral results, like (155) or Corollary A.2 below, which hold in general with weak regularity assumptions. For a proof, we refer for example to [GMM, Section 2] (see in particular Propositions 2.3 and 2.4, Corollary 2.5, and Theorem 2.8 there).

**Proposition A.1.** *Let  $(H, \|\cdot\|_H)$  be a Hilbert space and let  $T : D(T) \subset H \rightarrow H$  be a closed*



densely defined unbounded linear operator such that

$$\text{Ran } T \subset \ker T \quad \text{and} \quad D(T) \cap D(T^*) \quad \text{embeds compactly into } H,$$

where  $D(T) \cap D(T^*)$  is equipped with the graph norm

$$\|u\|_{D(T) \cap D(T^*)} := \sqrt{\|u\|_H^2 + \|Tu\|_H^2 + \|T^*u\|_H^2}.$$

We then have the following properties:

i) The operator  $(T + T^*, D(T) \cap D(T^*))$  is self-adjoint with a compact resolvent and satisfies

$$\ker(T + T^*) = \ker T \cap \ker T^*.$$

In particular, the linear space  $D(T) \cap D(T^*)$  is dense in  $H$  and  $T + T^*$  is a self-adjoint Fredholm operator with index 0, that is more precisely

$$\ker T \cap \ker T^* \quad \text{has finite dimension} \quad \text{and} \quad \text{Ran}(T + T^*) = (\ker T \cap \ker T^*)^\perp.$$

ii) The operator  $\Delta := TT^* + T^*T$  with domain

$$D(\Delta) := \{u \in D(T) \cap D(T^*) \text{ s.t. } Tu \in D(T^*) \text{ and } T^*u \in D(T)\}$$

is a nonnegative self-adjoint operator with kernel

$$\ker \Delta = \ker T \cap \ker T^* = \ker(T + T^*).$$

In particular,  $\Delta$  has a compact resolvent (since  $D(\Delta)$  with its graph norm embeds continuously into  $D(T) \cap D(T^*)$ ) and is the Friedrichs extension associated with the closed nonnegative quadratic form  $Q$  on  $D(T) \cap D(T^*)$  defined by

$$Q(u, v) := \langle Tu, Tv \rangle_H + \langle T^*u, T^*v \rangle_H.$$

Let us also note the following consequences of Proposition A.1 underlining the supersymmetric structure of the operator  $\Delta$  defined there : when  $T$  is as in the statement of Proposition A.1, the resolvent satisfies for every  $z \in \mathbb{C} \setminus \sigma(\Delta)$ ,  $u \in D(T)$ , and  $u' \in D(T^*)$ ,

$$(z - \Delta)^{-1}Tu = T(z - \Delta)^{-1}u \quad \text{and} \quad (z - \Delta)^{-1}T^*u' = T^*(z - \Delta)^{-1}u'. \quad (155)$$

Let us prove the first relation, the proof of the second one being similar. Let us then consider  $u \in D(T)$  and let us define  $v = (z - \Delta)^{-1}u$  for some  $z \in \mathbb{C} \setminus \sigma(\Delta)$ . Then  $v \in D(\Delta)$  and  $(z - \Delta)v = u \in D(T)$ , which implies  $\Delta v = T^*Tv + TT^*v \in D(T)$  and hence, since  $\text{Ran } T \subset \ker T$ ,  $T^*Tv \in D(T)$ . In particular, one has  $Tv \in D(TT^*)$ , and hence  $Tv \in D(\Delta)$ , and

$$(z - \Delta)Tv = zTv - TT^*Tv = T(z - \Delta)v = Tu \quad \text{and then} \quad Tv = (z - \Delta)^{-1}Tu,$$

that is precisely the first relation in (155).

An easy consequence of (155) is the following: for any eigenvalue  $\lambda$  of  $\Delta$  and associated eigenvector  $u \in D(\Delta)$ , we have  $Tu \in D(\Delta)$  and  $T^*u \in D(\Delta)$ , with

$$T\Delta u = \Delta Tu = \lambda Tu \quad \text{and} \quad T^*\Delta u = \Delta T^*u = \lambda T^*u \quad (156)$$

Note that if in addition  $\lambda \neq 0$ , one element among  $Tu, T^*u$  is nonzero (since in this case  $u \notin \ker \Delta = \ker T \cap \ker T^*$ ).

**Corollary A.2.** *Assume the hypotheses of Proposition A.1 and define  $\Delta := TT^* + T^*T$  as there. The following orthogonal decompositions then hold:*

$$H = \text{Ran}T \oplus^\perp \text{Ran}T^* \oplus^\perp \ker \Delta \quad \text{and, for } \mathbf{T} = T \text{ or } \mathbf{T} = T^*, \quad \ker \mathbf{T} = \text{Ran}\mathbf{T} \oplus^\perp \ker \Delta.$$

In particular, the operators  $T$  and  $T^*$  have closed ranges and

$$\ker T / \text{Ran}T \simeq \ker T^* / \text{Ran}T^* \simeq \ker \Delta.$$

*Proof.* This result is the statement of [GMM, Proposition 2.9] and is an easy consequence of Proposition A.1. First, since  $\text{Ran}(T + T^*) = (\ker T \cap \ker T^*)^\perp = (\ker \Delta)^\perp$  according to Proposition A.1, we deduce the inclusions (since  $T$  and  $T^*$  are closed),

$$\text{Ran}T + \text{Ran}T^* \supset (\ker \Delta)^\perp = \overline{\text{Ran}T + \text{Ran}T^*} \supset \text{Ran}T + \text{Ran}T^*.$$

The linear space  $\text{Ran}T + \text{Ran}T^*$  is then closed in  $H$  and, owing to  $T^2 = 0$ , this sum is moreover orthogonal. The spaces  $\text{Ran}T$  and  $\text{Ran}T^*$  are consequently closed and

$$H = (\ker \Delta)^\perp \oplus^\perp \ker \Delta = \text{Ran}T \oplus^\perp \text{Ran}T^* \oplus^\perp \ker \Delta.$$

Furthermore, the inclusion  $\ker T \supset \text{Ran}T \oplus^\perp \ker \Delta$  is clear, owing again to  $T^2 = 0$ . To prove the reverse inclusion, just notice that any  $v \in \ker T$  writes as the sum  $v = u_0 + Tu_1 + T^*u_2$ , where  $u_0 \in \ker \Delta$ ,  $u_1 \in D(T)$ , and  $u_2 \in D(T^*)$ . It follows that  $T^*u_2 \in D(T)$  and  $TT^*u_2 = 0$ , which implies  $T^*u_2 = 0$  (by taking the scalar product with  $u_2$ ) and then  $v = u_0 + Tu_1 \in \text{Ran}T \oplus^\perp \ker \Delta$ .

Lastly, the relation  $\ker T^* = \text{Ran}T^* \oplus^\perp \ker \Delta$  follows by applying the relation  $\ker T = \text{Ran}T \oplus^\perp \ker \Delta$  with  $T$  replaced by  $T^*$ , which satisfies  $\text{Ran}T^* \subset \ker T^*$  and  $T^{**} = T$ .  $\square$

## B Persistent cohomology and bar codes

### B.1 A sheaf theoretic presentation

Let  $f$  be a  $\mathcal{C}^\infty$  function on the compact manifold  $M$  having finitely many critical values (but we do not assume  $f$  is a Morse function). We shall define its bar code following the sheaf theoretic presentation of [KaSc].

The following assumption on  $f$  which is weaker than Hypothesis 1.2 allows us to use this construction in a low regularity setting. We keep the notation of Definition 1.1

$$f^t = \{x \in M, \quad f(x) < t\} \quad \text{and} \quad f^{\leq t} = \{x \in M, \quad f(x) \leq t\}.$$

**Hypothesis B.1.** *The function  $f : M \rightarrow \mathbb{R}$  is continuous and there exist finitely many values  $\min f = c_1 < \dots < c_{N_f} = \max f$  with the following property: For any  $n \in \{1, \dots, N_f - 1\}$  and all  $a < b \in ]c_n, c_{n+1}[$ ,  $f^{\leq a}$  is a deformation retract of  $f^{\leq b}$ . The values  $c_1, \dots, c_{N_f}$  are called “critical values” of  $f$ .*

We shall need the following

**Lemma B.2.** *With the assumptions of Hypothesis B.1, we the space  $H^*(f^b, f^a)$  is finite dimensional.*

*Proof.* It is enough to prove that if  $t$  is in some  $]c_j, c_{j+1}[$ , then  $H^*(f^{\leq t})$  is finite dimensional. The general case follows by applying the long exact sequence of the pair  $(f^{\leq b}, f^{\leq a})$ . Now let  $\varepsilon$  be small enough,  $g$  a smooth function such that  $\|g - f\| \leq \varepsilon$ . Then the inclusions

$$f^{\leq t} \subset g^{\leq t+\varepsilon} \subset f^{\leq t+2\varepsilon}$$

hold true and for  $\varepsilon$  small enough,

$$f^{\leq t-2\varepsilon} \subset f^{\leq t} \subset f^{\leq t+2\varepsilon}$$

are homotopy equivalences. Notice that  $g$  being smooth and  $t + \varepsilon$  being a regular value for  $\varepsilon$  generic, the cohomologies  $H^*(g^{t+\varepsilon})$  are finite dimensional, and we have maps

$$H^*(f^{\leq t+2\varepsilon}) \longrightarrow H^*(g^{\leq t+\varepsilon}) \longrightarrow H^*(f^{\leq t})$$

but the composition of the above two arrows must be an isomorphism, and it factors through a finite dimensional space, therefore  $H^*(f^{\leq t})$  is finite dimensional and we have a uniform bound for  $t$  in  $]c_j, c_{j+1}[$ .  $\square$

By using the deformation along the gradient flow away from the “critical values”  $c_1, \dots, c_{N_f}$ , Hypothesis B.1 is obviously true when  $f$  satisfies Hypothesis 1.2. It is also true for a general Lipschitz function satisfying Hypothesis 1.6 as mentioned in Subsection 8.3.1. It implies that for any  $a, b \notin \{c_1, \dots, c_{N_f}\}$ ,  $a < b$ , the relative homology groups ( $\mathbb{K}$ -vector spaces)  $H^*(f^{\leq b}, f^{\leq a}; \mathbb{K})$  are finite dimensional and change only when  $a$  or  $b$  passes a “critical value”,  $c_1, \dots, c_{N_f}$ .

For the introduction of a persistent sheaf on  $\mathbb{R}$ , we need to consider all the sublevel sets, and only at the end, do we restrict our attention to the relative cohomology groups  $H^*(f^b, f^a; \mathbb{K})$  with  $a < b$ ,  $a, b \notin \{c_1, \dots, c_{N_f}\}$ .

In order to use standard results of sheaf theory it is better to work with the closed sublevel set  $f^{\leq t}$  for a general  $t \in \mathbb{R}$  which may be a “critical value”.

For a field  $\mathbb{K}$ ,  $\mathbb{K}_M$  denotes the locally constant sheaf on  $M$  and we consider a  $c$ -soft injective resolution

$$0 \longrightarrow \mathbb{K}_M \longrightarrow \mathcal{L}^0 \longrightarrow \mathcal{L}^1 \longrightarrow \dots,$$

$c$ -soft meaning that the restriction morphism  $\Gamma(M; \mathcal{L}^q) \rightarrow \Gamma(K; \mathcal{L}^q)$  is surjective for any compact subset  $K \subset M$  and any  $q \in \mathbb{N}$ . A bounded  $c$ -soft resolution ending with  $\mathcal{L}^{\dim M} \rightarrow 0$  exists because  $M$  is a compact manifold.

Such a resolution can be obtained by introducing the canonical injective resolution or the sheaf of  $\mathbb{K}$ -valued Alexander-Spanier cochains on  $M$ . When  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  we can use the de Rham resolution

$$0 \longrightarrow \mathbb{K}_M \longrightarrow \mathcal{C}^\infty(M; \mathbb{K}) \xrightarrow{d} \mathcal{C}^\infty(M; \Lambda^1 T^* M) \xrightarrow{d} \dots$$

showing that  $\mathbb{K}_M$  is quasi-isomorphic to the de Rham complex

$$0 \longrightarrow \mathcal{C}^\infty(M; \mathbb{K}) \xrightarrow{d} \mathcal{C}^\infty(M; \Lambda^1 T^* M) \xrightarrow{d} \dots$$

and the homology groups of  $\mathbb{K}_M$ , denoted  $H^i(M; \mathbb{K})$ , are obtained by computing the homology of the complex  $\mathcal{L}^\bullet$ .

For any locally closed subset (i.e. the intersection of a closed and an open set)  $A$ ,  $\mathcal{L}_A$  is  $c$ -soft. When  $A$  and  $B$  are closed,  $A \subset B$ , the short exact sequence

$$0 \longrightarrow \mathcal{L}_{B \setminus A}^\bullet \longrightarrow \mathcal{L}_B^\bullet \longrightarrow \mathcal{L}_A^\bullet \longrightarrow 0$$

leads to the long exact sequence

$$\dots \longrightarrow H_c^*(B \setminus A, \mathcal{L}^\bullet) \longrightarrow H^*(B, \mathcal{L}^\bullet) \longrightarrow H^*(A, \mathcal{L}^\bullet) \longrightarrow H_c^{*+1}(B \setminus A, \mathcal{L}^\bullet) \longrightarrow \dots$$

With our choice of  $\mathcal{L}^\bullet$ , this says

$$\dots \longrightarrow H_c^*(B \setminus A, \mathbb{K}) \longrightarrow H^*(B, \mathbb{K}) \longrightarrow H^*(A, \mathbb{K}) \longrightarrow H_c^{*+1}(B \setminus A, \mathbb{K}) \longrightarrow \dots \quad (157)$$

when  $A$  is a closed subspace of  $M$ . We have just summarized Godement's arguments for Theorem 4.10.1 of [God] defining the long exact sequence associated to a closed subset. For general values  $a < b$  in  $\mathbb{R}$ , the relative cohomology groups  $H^*(f^{\leq b}, f^{\leq a}; \mathbb{K})$  can be understood in terms of the cohomology with compact support in  $\{x \in M, a < f(x) \leq b\}$ . Under Hypothesis B.1,  $H^*(f^{\leq a-\varepsilon'}, \mathbb{K}) \sim H^*(f^{\leq a-\varepsilon}, \mathbb{K})$  for any  $\varepsilon, \varepsilon' > 0$  small enough, the Mittag-Leffler condition (see [KaScBook]-chap I) is satisfied and the cohomology groups of open sublevel sets are given by the projective limits  $H^*(f^a; \mathbb{K}) = \varprojlim_{\varepsilon \rightarrow 0^+} H^*(f^{\leq a-\varepsilon}; \mathbb{K}) \sim H^*(f^{\leq a-\varepsilon_0}, \mathbb{K})$  for  $\varepsilon_0 > 0$  small enough.

Persistent cohomology is introduced in this way in [KaSc] (we refer the reader to [CaZo][EdHa][LSV] for other presentations) via the direct image functor  $Rp_*$ , in the derived category, applied to the locally constant sheaf  $\mathbb{K}_{\Gamma_f^+}$  on  $\Gamma_f^+ = \{(x, t) \in M \times \mathbb{R}, f(x) \leq t\}$  where  $p : M \times \mathbb{R} \rightarrow (\mathbb{R}, \gamma)$  is given by  $p(x, t) = t$ . The notation  $(\mathbb{R}, \gamma)$  means that  $\mathbb{R}$  is endowed with the non-Hausdorff  $\gamma$ -topology for which open (resp. closed) sets are  $]-\infty, t[$  (resp.  $[t, +\infty[$ ),  $t \in \mathbb{R}$ . Note that here we do not need to consider the values  $\pm\infty$  because  $M$  is compact. So we set  $\mathcal{P} = Rp_*\mathbb{K}_{\Gamma_f^+}$ . For a  $\gamma$ -open set  $]-\infty, t[$  the set of sections  $\Gamma(]-\infty, t[; \mathcal{P})$  is quasi-isomorphic to the de Rham complex

$$0 \longrightarrow \mathcal{C}^\infty(f^t; \mathbb{K}) \xrightarrow{d} \mathcal{C}^\infty(f^t; \Lambda^1 T^* M) \xrightarrow{d} \dots, \text{ when } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C},$$

while the stalk at  $t \in \mathbb{R}$ ,  $\mathcal{P}_t = \varinjlim_{t' > t} \Gamma(]-\infty, t'[; \mathcal{P})$  is quasi-isomorphic to the de Rham complex of  $f^{\leq t}$ . With the  $\gamma$ -topology on  $\mathbb{R}$  an example of a locally constant sheaf is  $\mathbb{K}_{[a, b]}$ ,  $-\infty < a < b \leq +\infty$  with

$$\text{Hom}(\mathbb{K}_{[a, b]}; \mathbb{K}_{[c, d]}) = \begin{cases} \mathbb{K} & \text{if } a \leq c < b \leq d \\ 0 & \text{else} \end{cases}$$

Under Hypothesis B.1, the cohomology  $H^*(f^{< t}; \mathbb{K})$  is finite dimensional and locally constant on  $\mathbb{R} \setminus \{c_1, \dots, c_{N_f}\}$ . Therefore the sheaf  $\mathcal{P}$  is an  $\mathbb{R}$ -constructible sheaf of  $\mathbb{K}$ -vector spaces. By applying results of Crawley-Boevey in [Cra] (see also Guillermou in [Gui]), Kashiwara and Schapira show in [KaSc] that

$$\mathcal{P} \sim \bigoplus_{p=0}^{\dim M} \bigoplus_{\alpha \in A^{(p)}} \mathbb{K}_{[a_\alpha^{(p)}, b_\alpha^{(p+1)}]}[p], \quad -\infty < a_\alpha^{(p)} < b_\alpha^{(p+1)} \leq +\infty.$$

As pointed out in [KaSc] this equivalence has to be understood as an equivalence of the objects in the bounded derived category, for  $\text{Ext}^1(\mathbb{R}_{[0, +\infty[}, \mathbb{R}_{]-\infty, 0]}) = \mathbb{R}$ . This subtlety has no consequence as long as we focus on those objects which are the  $H^j(f^{< t}, \mathbb{R})$ . Because the sheaf is locally constant in  $\mathbb{R} \setminus \{c_1, \dots, c_{N_f}\}$ , the endpoints  $a_\alpha$  belong to  $\{c_1, \dots, c_{N_f}\}$  and the endpoints  $b_\alpha$  to  $\{c_2, \dots, c_{N_f}, +\infty\}$ . The reason why we put the exponent  $(p+1)$  for  $b_\alpha$  will become clear below. When we allow  $a_\alpha = b_\alpha$  the finite cardinal of  $A$  can be augmented arbitrarily by adding  $[a_\alpha, b_\alpha[ = \emptyset, \mathbb{K}_\emptyset = 0$ , with  $b_\alpha = a_\alpha$ .

Remember that when  $F$  is a sheaf on the topological space  $X$  and  $Z$  is locally closed,  $F_Z$  is the sheaf on  $X$  characterized by

$$\begin{cases} F_Z|_Z = F|_Z \\ F_Z|_{X \setminus Z} = 0 \end{cases}$$

and when  $Z$  is closed one has the exact sequence

$$0 \longrightarrow F_{X \setminus Z} \longrightarrow F \longrightarrow F_Z \longrightarrow 0.$$

Applied to  $X = (\mathbb{R}, \gamma)$  and  $F = \mathcal{P} \sim \bigoplus_{\alpha \in A} \mathbb{K}_{[a_\alpha, b_\alpha[}$  we obtain

$$\begin{aligned} \mathcal{P}_{[t_0, +\infty[} &\sim \bigoplus_{\alpha \in A, t_0 < b_\alpha} \mathbb{K}_{[\max(a_\alpha, t_0), b_\alpha[}, \\ \mathcal{P}_{]-\infty, t_0[} &\sim \bigoplus_{\alpha \in A, b_\alpha \leq t_0} \mathbb{K}_{[a_\alpha, b_\alpha[}, \\ \mathcal{P}_{[a, b[} &\sim \bigoplus_{\alpha \in A, a < b_\alpha \leq b} \mathbb{K}_{[\max(a, a_\alpha), b_\alpha[}. \end{aligned}$$

and the obvious graded analogous result holds. From the long exact sequence (157) written

$$\dots \longrightarrow H^{(p-1)}(f^{\leq t}) \longrightarrow H^{(p-1)}(f^{\leq a}) \longrightarrow H_c^{(p)}(f^{\leq t} \setminus f^{\leq a}) \longrightarrow H^{(p)}(f^{\leq t}) \longrightarrow H^{(p)}(f^{\leq a}) \longrightarrow \dots$$

and because we are working with  $\mathbb{K}$ -vector spaces we obtain

$$\mathcal{P}(a)^{(p)}|_t \sim \ker[H^{(p)}(f^{\leq t}; \mathbb{K}) \rightarrow H^{(p)}(f^{\leq a}; \mathbb{K})] \oplus \operatorname{coker}[H^{(p-1)}(f^{\leq t}; \mathbb{K}) \rightarrow H^{(p-1)}(f^{\leq a}; \mathbb{K})],$$

or

$$\mathcal{P}(a)^{(p)} \sim \ker(\mathcal{P}_{[a, +\infty[}^{(p)} \rightarrow \mathcal{P}_a^{(p)}) \oplus \operatorname{coker}(\mathcal{P}_{[a, +\infty[}^{(p-1)} \rightarrow \mathcal{P}_a^{(p-1)}).$$

Using  $\mathcal{P}_{[a, +\infty[} \sim \bigoplus_{\alpha \in A, a < b_\alpha} \mathbb{K}_{[\max(a_\alpha, a), b_\alpha[}$ , we deduce

$$\begin{aligned} \ker(\mathcal{P}_{[a, +\infty[}^{(p)} \rightarrow \mathcal{P}_a^{(p)}) &\sim \bigoplus_{\alpha \in A^{(p)}, a < a_\alpha^{(p)}} \mathbb{K}_{[a_\alpha^{(p)}, b_\alpha^{(p+1)}[} \\ \operatorname{coker}(\mathcal{P}_{[a, +\infty[}^{(p-1)} \rightarrow \mathcal{P}_a^{(p-1)}) &\sim \bigoplus_{\alpha \in A^{(p-1)}, a_\alpha^{(p-1)} \leq a < b_\alpha^{(p)} < +\infty} \mathbb{K}_{[b_\alpha^{(p)}, +\infty[}. \end{aligned}$$

We obtain

$$H_c^{(p)}(f^{\leq b} \setminus f^{\leq a}; \mathbb{K}) \sim \left( \bigoplus_{\alpha \in A^{(p)}, a < a_\alpha^{(p)} \leq b < b_\alpha^{(p+1)}} \mathbb{K} \right) \bigoplus \left( \bigoplus_{\alpha \in A^{(p-1)}, a_\alpha^{(p-1)} \leq a < b_\alpha^{(p)} \leq b} \mathbb{K} \right).$$

When  $a, b$  do not belong to  $\{c_1, \dots, c_{N_f}\}$ , the inequalities in the sums can be replaced by strict inequalities and

$$H^{(p)}(f^b, f^a; \mathbb{K}) \sim \left( \bigoplus_{\alpha \in A^{(p)}, a < a_\alpha^{(p)} < b < b_\alpha^{(p+1)}} \mathbb{K} \right) \bigoplus \left( \bigoplus_{\alpha \in A^{(p-1)}, a_\alpha^{(p-1)} < a < b_\alpha^{(p)} < b} \mathbb{K} \right).$$

## B.2 Trivialized complex

We now establish the relationship with the bar codes used in [LNV] which was inspired by Barannikov's presentation of Morse theory in [Bar] (see also [LSV]).

With the definitions of [LNV], the equality  $\partial_B b = a$  holds true for two critical values  $a, b$  if and only the map

$$H^p(f^{b+\varepsilon}, f^{a-\varepsilon}) \longrightarrow H^p(f^{\leq a+\varepsilon}, f^{a-\varepsilon})$$

vanishes, while

$$H^p(f^{b-\varepsilon}, f^{a-\varepsilon}) \longrightarrow H^p(f^{\leq a+\varepsilon}, f^{a-\varepsilon})$$

is non-zero. But we have  $H^*(f^y, f^x) = H^*([x, y[, \mathcal{P})$ , where  $\mathcal{P} = Rp_* \mathbb{K}_{\Gamma_f^+}$  and by assumption

$$\mathcal{P} = \bigoplus_{p=0}^{\dim M} \bigoplus_{\alpha \in A^{(p)}} \mathbb{K}_{[a_\alpha^{(p)}, b_\alpha^{(p+1)}[ [p], \quad -\infty < a_\alpha^{(p)} < b_\alpha^{(p+1)} \leq +\infty.$$

so that

$$H^*([x, y[, \mathcal{P}) = \bigoplus_{p=0}^{\dim M} \bigoplus_{\alpha \in A^{(p)}} H^*([x, y[, \mathbb{K}_{[a_\alpha^{(p)}, b_\alpha^{(p+1)}[ [p]), \quad -\infty < a_\alpha^{(p)} < b_\alpha^{(p+1)} \leq +\infty.$$

so it is enough to consider the case of  $\mathcal{P} = \mathbb{K}_{[a_\alpha^{(p)}, b_\alpha^{(p+1)}][p]}$  and then it is obvious that  $\partial_B b_\alpha^{(p+1)} = a_\alpha^{(p)}$ . We thus proved

**Proposition B.3.** *If*

$$\mathcal{P} = Rp_*\mathbb{K}_{\Gamma_f^+}$$

and

$$\partial_B b_\alpha^{(p+1)} = a_\alpha^p, \partial_B a_\alpha^{(p)} = 0$$

With the persistent cohomology described above, we are now able to extend it under the general Hypothesis B.1 and we fix the corresponding notations.

The bar code  $\mathcal{B}(f) = ([a_\alpha, b_\alpha]_{\alpha \in A})$  associated with  $f$  with  $a_\alpha < b_\alpha$ ,  $a_\alpha \in \{c_1, \dots, c_{N_f}\}$ ,  $b_\alpha \in \{c_2, \dots, c_{N_f}, +\infty\}$  and graded according to  $\mathcal{B}^{(p)}(f) = ([a_\alpha^{(p)}, b_\alpha^{(p+1)}]_{\alpha \in A^{(p)}})$  is the one introduced in the previous paragraph. We use the superscript  $*$  when we do not want to specify  $(p)$ . When  $a < b$  are not “critical values” we write

$$\begin{aligned} A^*(a, b) &= \{\alpha \in A^*, [a_\alpha^*, b_\alpha^{*+1}[\cap]a, b[\not\subseteq \{\emptyset, ]a, b[\}\}, \\ A_c^*(a, b) &= \{\alpha \in A^*(a, b), [a_\alpha^*, b_\alpha^{*+1}[\cap]a, b[\text{relatively compact in } ]a, b[\}, \\ &\alpha \in A^*(a, b) \Leftrightarrow a < a_\alpha^* < b \text{ or } a < b_\alpha^{*+1} < b, \\ &\alpha \in A_c^*(a, b) \Leftrightarrow a < a_\alpha^* < b_\alpha^{*+1} < b. \end{aligned}$$

In order to keep track of the possible multiplicities of the values  $a_\alpha$  and  $b_\alpha$ , we set

$$\begin{aligned} \mathcal{X}^*(a, b) &= \{(\alpha, a_\alpha^*), \alpha \in A_c^*(a, b)\} \\ \mathcal{Y}^*(a, b) &= \{(\alpha, b_\alpha^*), \alpha \in A_c^{*-1}(a, b)\} \\ \mathcal{Z}^*(a, b) &= \{(\alpha, a_\alpha^*), \alpha \in A^*(a, b) \setminus A_c^*(a, b), a < a_\alpha < b\} \\ &\quad \sqcup \{(\alpha, b_\alpha^*), \alpha \in A^{*-1}(a, b) \setminus A_c^{*-1}(a, b), a < b_\alpha^* < b\}, \\ \mathcal{J}^*(a, b) &= \mathcal{X}^*(a, b) \sqcup \mathcal{Y}^*(a, b) \sqcup \mathcal{Z}^*(a, b). \end{aligned}$$

We now consider the complex defined on

$$\mathbb{K}^{\mathcal{J}^*(a, b)} = \bigoplus_{p=0}^{\dim M} \mathbb{K}^{\mathcal{J}^{(p)}(a, b)} \sim \mathbb{K}^{2\#A_c(a, b) + \#(A(a, b) \setminus A_c(a, b))}$$

with natural basis  $(x \in \mathcal{X}(a, b), y \in \mathcal{Y}(a, b), z \in \mathcal{Z}(a, b))$  and with the differential  $\mathbf{d}_{\mathcal{B}}$  defined by

$$\begin{aligned} \mathbf{d}_{\mathcal{B}}x^{(p)} &= y^{(p+1)} \quad \text{if } x^{(p)} \in \mathcal{X}^{(p)}(a, b), y^{(p+1)} \in \mathcal{Y}^{(p+1)}(a, b), p_1(x) = \alpha = p_1(y), \\ \mathbf{d}_{\mathcal{B}}y^{(p)} &= 0 \quad \text{if } y^{(p)} \in \mathcal{Y}^{(p)}(a, b), \\ \mathbf{d}_{\mathcal{B}}z^{(p)} &= 0 \quad \text{if } z^{(p)} \in \mathcal{Z}^{(p)}(a, b). \end{aligned}$$

By construction, when  $-\infty < a < b < +\infty$  are not “critical values” of  $f$ ,

$$H^p(\mathbb{K}^{\mathcal{J}^*(a, b)}, \mathbf{d}_{\mathcal{B}}) = \bigoplus_{z \in \mathcal{Z}^{(p)}(a, b)} \mathbb{K}z \sim \left( \bigoplus_{\alpha \in A^{(p)}, a < a_\alpha^{(p)} < b < b_\alpha^{(p+1)}} \mathbb{K} \right) \oplus \left( \bigoplus_{\alpha \in A^{(p-1)}, a_\alpha^{(p-1)} < a < b_\alpha^{(p)} < b} \mathbb{K} \right),$$

and the complex  $(\mathbb{K}^{\mathcal{J}^*(a, b)}, \mathbf{d}_{\mathcal{B}})$  computes all the relative cohomology groups  $H^*(f^b, f^a; \mathbb{K})$ . The sets  $\mathcal{X}(a, b)$ ,  $\mathcal{Y}(a, b)$ ,  $A_c(a, b)$ , play a role when we compute the positive exponentially small eigenvalues of Witten Laplacians with Dirichlet boundary conditions on  $f^{-1}(\{a\})$  and Neumann boundary conditions on  $f^{-1}(\{b\})$ .

### B.3 Stability theorem

The bar code associated with  $f$  is given by a family  $\mathcal{B}(f) = ([a_\alpha, b_\alpha])_{\alpha \in A}$ , now containing possibly empty sets when  $a_\alpha = b_\alpha$ , with the equivalence  $([a_\alpha, b_\alpha])_A \sim ([c_\beta, d_\beta])_{\beta \in B}$  if there is a bijection between  $j : \{\alpha \in A, a_\alpha < b_\alpha\} \rightarrow \{\beta \in B, c_\beta < d_\beta\}$  such that  $c_{j(\alpha)} = a_\alpha$  and  $d_{j(\alpha)} = b_\alpha$ . Following [CEH] they can be represented as a family of points  $((a_\alpha, b_\alpha))_{\alpha \in A}$  in  $\{(x, y) \in \mathbb{R} \times (\mathbb{R} \cup \{+\infty\}), x \leq y\}$ , appearing with multiplicities, and the bottleneck distance between two general bar codes  $\mathcal{B}_A = ([a_\alpha, b_\alpha])_{\alpha \in A}$  and  $\mathcal{B}_B = ([c_\beta, d_\beta])_{\beta \in B}$ , where  $A$  and  $B$  can be assumed with the same cardinal when we authorize  $a_\alpha = b_\alpha$ ,  $c_\beta = d_\beta$ , is given by

$$d_{bot}(\mathcal{B}_A, \mathcal{B}_B) = \inf_{j:A \rightarrow B} \max_{\alpha \in A} \max(|a_\alpha - c_{j(\alpha)}|, |b_\alpha - d_{j(\alpha)}|),$$

with the convention  $|(+\infty) - (+\infty)| = 0$ . The stability theorem says that for two different functions  $f, g$  on  $M$  which satisfy Hypothesis B.1, the bottleneck distance between the bar codes  $\mathcal{B}(f)$  and  $\mathcal{B}(g)$  associated with  $f$  and  $g$  satisfies

$$d_{bot}(\mathcal{B}(f), \mathcal{B}(g)) \leq \|f - g\|_{C^0}.$$

It is proved in [KaSc] by using the convolution of sheaves. In the one-dimensional case and for  $\varepsilon \geq 0$  we have  $\mathbb{K}_{[-\varepsilon, \varepsilon]} * \mathbb{K}_{[a, b[} = \mathbb{K}_{[a-\varepsilon, b-\varepsilon[}$  (in terms of constructible functions according to [Sch], simply use  $1_{[a, b[} = 1_{[a, +\infty[} - 1_{[b, +\infty[}$  and  $1_{[-\varepsilon, \varepsilon]} * 1_{[a, +\infty[} = 1_{[a-\varepsilon, +\infty[}$ ) and this convolution is nothing but a translation by  $-\varepsilon$  on the real axis. Two  $\mathbb{R}$ -constructible sheaves on  $(\mathbb{R}, \gamma)$ ,  $F, G$  are said  $\varepsilon$ -isomorphic,  $F \stackrel{\varepsilon}{\sim} G$ , if there are morphisms  $i : \mathbb{K}_{[-\varepsilon, \varepsilon]} * F \rightarrow G$  and  $j : \mathbb{K}_{[-\varepsilon, \varepsilon]} * G \rightarrow F$  such that natural morphisms  $\mathbb{K}_{[-2\varepsilon, 2\varepsilon]} * F \rightarrow F$  and  $\mathbb{K}_{[-2\varepsilon, 2\varepsilon]} * G \rightarrow G$  are factored via

$$\begin{aligned} \mathbb{K}_{[-2\varepsilon, 2\varepsilon]} * F &\xrightarrow{\mathbb{K}_{[-\varepsilon, \varepsilon]} * i} \mathbb{K}_{[-\varepsilon, \varepsilon]} * G \xrightarrow{j} F \\ \mathbb{K}_{[-2\varepsilon, 2\varepsilon]} * G &\xrightarrow{\mathbb{K}_{[-\varepsilon, \varepsilon]} * j} \mathbb{K}_{[-\varepsilon, \varepsilon]} * F \xrightarrow{i} G. \end{aligned}$$

The bottleneck distance is then equal to

$$d_{bot}(F, G) = \inf \left\{ \varepsilon \geq 0, F \stackrel{\varepsilon}{\sim} G \right\},$$

and coincides with  $d_{bot}(\mathcal{B}_A, \mathcal{B}_B)$  after writing  $F \sim \bigoplus_{\alpha \in A} \mathbb{K}_{[a_\alpha, b_\alpha[}$  and  $G \sim \bigoplus_{\beta \in B} \mathbb{K}_{[c_\beta, d_\beta[}$ .

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## References

- [AGHKH] S. Alberverio, F. Gestezy, R. Hoegh-Krohn, H. Holden. *Solvable models in quantum mechanics*. Texts and monographs in physics, Springer (1988).
- [AGV] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko. *Singularities of differentiable maps. Volume 2. Monodromy and asymptotics of integrals*. Reprint of the 1988 translation. Modern Birkhäuser Classics. Birkhäuser/Springer, 2012.
- [APS] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry. Bull. Lond. Math. Soc. 5, 229-234 (1973).
- [Arr] S. Arrhenius. *Z. Phys. Chem.* 4, pp. 96–116 and 226–248 (1889).

- [Bar] S. A. Barannikov. The framed Morse complex and its invariants. Singularities and bifurcations, Adv. Soviet Math., 21, Amer. Math. Soc. (1994) , pp. 93–115.
- [Ber] N. Berglund. Kramer’s law: validity, derivation and generalisation. Markov Process. Related Fields Vol. 19 no. 3 (2013) pp 459–490.
- [BeGe] N. Berglund, B. Gentz. The Eyring-Kramers law for potentials with non quadratic saddles. Markov Process. Related Fields Vol. 16 no. 3 (2010) pp 549–598.
- [BeDu] N. Berglund, S. Dutercq. The Eyring-Kramers law for Markovian jump processes with symmetries. J. Theoret. Probab. Vol. 29 no. 4 (2016) pp 1240–1279.
- [BEGK] A. Bovier, M. Eckhoff, V. Gaynard, M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. JEMS 6 (4), pp. 399–424 (2004).
- [BGK] A. Bovier, V. Gaynard, M. Klein Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. JEMS 7 (1), pp. 69–99 (2004).
- [BiMi] E. Bierstone, P.D. Milman Semianalytic and subanalytic sets. Publications mathématiques IHES, tome 67 (1988), pp 5–42.
- [Bis] J.M. Bismut. The Witten complex and the degenerate Morse inequalities. J. Differ. Geom. 23, pp. 207-240 (1986).
- [BiZh] J.M. Bismut, W. Zhang. Milnor and Ray-Singer Metrics of the Equivariant Determinant of flat bundle. Geom. Funct. Anal., Vol. 4 No. 2, pp. 136–212 (1994).
- [BDLS] J. Bolte, A. Daniilidis, A. Lewis, M. Shiota. Clarke critical values of subanalytic Lipschitz continuous functions. Ann. Polon. Math. 87 (2005), pp. 13-25.
- [BGP] J. Brüning, V. Geyler, K. Pankrashkin. Spectra of self-adjoint extensions and applications to solvable Schrödinger operators. Rev. Math. Phys. Vol. 20, no (2008) pp 1–70.
- [BrDi] M. Brooks and G. Di Gesù, Sharp Tunneling Estimates for a Double-Well Model in Infinite Dimension. arxiv:1911.03187 (2019).
- [CaZo] G. Carlsson, A. Zomorodian. Computing persistent homology. Discrete Comput. Geom. Vol. 33, no 2 (2005) pp 249–274.
- [CLM] K. Chan, N.C. Leung, Z.N. Ma. Witten deformation of product structures on de Rham complex. arXiv: 1401.5876.
- [ChLi] K. C. Chang and J. Liu. A cohomology complex for manifolds with boundary. Topological Methods in Non Linear Analysis, Vol. 5 (1995) pp. 325–340 .
- [ChPi] A. Chazarain, A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*. Gauthiers-Villars (1981)
- [Cla] F.H. Clarke. *Optimization and nonsmooth analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, (1983).
- [CEH] D. Cohen-Steiner, H. Edelsbrunner, J. Harer. Stability of persistence diagrams. Discrete Comput. Geom. Vol. 37 (2007) pp 103–120.
- [CdVPY] Y. Colin de Verdière, Y. Pan, B. Ycart. Singular limits of Schrödinger operators and Markov processes. J. Operator Theory Vol. 41 no. 1 (1999) 151–173.
- [CoMcI] M. Costabel, A. McIntosh. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Math. Z. Vol. 265 no. 23 (2010) pp 265–297.
- [Cra] W. Crawley-Boevey. Decomposition of pointwise persistence finite-dimensional persistence modules. J. Algebra Appl. Vol. 14 (2014).



- [CFKS] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Text and Monographs in Physics, Springer-Verlag (1987).
- [CzRi] M.O. Czarnecki, L. Rifford. Approximation and regularization of Lipschitz functions: convergence of the gradients. *Trans. Amer. Math. Soc* 358 no. 10 (2006), pp. 4467-4520.
- [DaRi] Ñ.V. Dang, G. Rivière. Spectral analysis of Morse-Smale gradient flows. <https://arxiv.org/abs/1605.05516>
- [DeLe] J.M. Delort, G. Lebeau. Microfonctions  $I$ -Lagrangiennes. *J. Math. pures et appl.*, Vol. 67 (1988) pp. 39-84.
- [DiLe] G. Di Gesù, D. Le Peutrec. Small noise spectral gap asymptotics for a large system of nonlinear diffusions. *J. Spectral Theory* Vol 7 No 4 (2017) pp 939–984.
- [DLLN1] G. Di Gesù, T. Lelièvre, D. Le Peutrec, and B. Nectoux. Sharp asymptotics of the first exit point density. *Ann. PDE*, Vol. 5, No. 1 (2019).
- [DLLN2] G. Di Gesù, T. Lelièvre, D. Le Peutrec, and B. Nectoux. The exit from a metastable state: concentration of the exit point distribution on the low energy saddle points, part 1. *J. Math. pures et appl.*, in press: <https://doi.org/10.1016/j.matpur.2019.06.003> (2019).
- [DiSj] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the semi-classical limit*. London Mathematical Society, Lecture Note Series 268, Cambridge University Press (1999).
- [EdHa] H. Edelsbrunner, J. Harer. Persistent homology—a survey. *Surveys in discrete and computational geometry*. *Contemp. Math. Amer. Math. Soc.*, Vol. 453 (2008).
- [ELJL] K.D. Elworthy, Y. Le Jan, X.M. Li. *On the Geometry of Diffusion Operators and Stochastic Flows*. *Lect. Notes in Math.* 1720, Springer (1999).
- [Eyr] H. Eyring. The activated complex in chemical reactions. *J. Chem. Phys.* 3 (2), pp. 107–115, (1935).
- [FrWe] M.I. Freidlin, A.D. Wentzell. *Random perturbations of dynamical systems*. second ed., Springer-Verlag (1998).
- [Fuk] K. Fukaya. Multivalued Morse theory, asymptotic analysis and mirror symmetry. *Graphs and patterns in mathematics and theoretical physics*. *Proc. Sympos. Pure Math.* Vol 73, Amer. Math. Soc. (2005) pp 205–278.
- [GeNi] C. Gérard, F. Nier. Mourre theory for analytically fibered operators. *J. Funct. Anal.* Vol. 152 no. 1 (1998) pp 202–219.
- [GMM] V. Gold’shtein, I. Mitrea, I. Mitrea. Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds. *Journal of Mathematical Sciences*, Vol. 172, no. 2 (2011).
- [God] R. Godement. *Topologie algébrique et théorie des faisceaux*. Hermann (1960).
- [Gui] S. Guillermou. The three cusps conjecture, [arXiv:1603.07876](https://arxiv.org/abs/1603.07876).
- [Hardt] R.M. Hardt. Stratification of real analytic Mappings and Images *Inventiones Mathematicae* 28 (1975) pp 193–208.
- [Hel] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications*. *Lecture Notes in Mathematics* 1336, Springer-Verlag (1988).
- [HelW] B. Helffer. *Semiclassical analysis, Witten Laplacians and statistical mechanics*. *Series in Partial Differential Equations and Applications* 1 World Scientific Publishing (2002).
- [HKN] B. Helffer, M. Klein, F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. *Matematica Contemporanea*, 26, pp. 41–85 (2004).

- [HeNi] B. Helffer, F. Nier Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary Bernard Helffer - Francis Nier Mémoires de la SMF 105 (2006), vi+89 pages
- [HeSj2] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique II - Interaction moléculaire - Symétries - Perturbations. *Ann. Inst. H. Poincaré Phys. Théor.* 42 (2), pp. 127–212 (1985).
- [HeSj3] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit III. *Math. Nachrichte* 124, pp. 263–313 (1985).
- [HeSj4] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique IV - Étude du complexe de Witten -. *Comm. Partial Differential Equations* 10 (3) (1985), pp. 245–340.
- [HeSj6] B. Helffer and J. Sjöstrand. Puits multiples en mécanique semi-classique VI. Cas des puits sous-variétés. *Ann. Inst. H. Poincaré Phys. Théor.* 46 (4) (1987), pp. 353–372.
- [HHS] F. Hérau, M. Hitrik, and J. Sjöstrand. Tunnel effect and symmetries for Kramers-Fokker-Planck type operators. *J. Inst. Math. Jussieu* 10 (3), pp. 567–634 (2011).
- [Hirs] M.W. Hirsch. *Differential topology* Graduate Text in Mathematics 33, Springer (1976).
- [HKS] R. A. Holley, S. Kusuoka, and D. Stroock. Asymptotics of the spectral gap with applications to the theory of simulated annealing. *Journal of functional analysis* 83 (2), pp. 333–347 (1989).
- [Hiro] H. Hironaka. Subanalytic sets. *Number Theory, Algebraic Geometry and Commutative Algebra in honor of Y. Akizuki*, Kimokuniya Publications (1973) pp 453–493.
- [JMM] T. Jakab, I. Mitrea, and M. Mitrea. On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains. *Indiana Univ. Math. J.* 58, no. 5, pp. 2043–2071 (2009).
- [KaSc] M. Kashiwara, P. Schapira Persistent homology and microlocal sheaf theory arXiv:1705.00955v5
- [KaScBook] M. Kashiwara, P. Schapira *Sheaves on Manifolds* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag (1990).
- [Kra] H. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica* 7 (4), pp. 284–304 (1940).
- [Lan] S. Lang. *Fundamentals of Differential Geometry* Graduate Texts in Mathematics 191, Springer (1999).
- [LauB] F. Laudenbach. *Transversalité, Courants et Théorie de Morse: Un cours de topologie algébrique* Editions de l’Ecole Polytechnique (2012).
- [Lau] F. Laudenbach. A Morse complex on manifolds with boundary. *Geometrica Dedicata* 153-1, pp. 47–57 (2011).
- [Lee] J.M. Lee. *Riemannian Manifolds. An Introduction to Curvature* Graduate Text in Mathematics 176, Springer (1997).
- [LeMi] D. Le Peutrec, L. Michel. Sharp spectral asymptotics for non-reversible metastable diffusion processes. arXiv:1907.09166 (2019).
- [LeNi] T. Lelièvre, F. Nier. Low temperature asymptotics for Quasi-Stationary Distributions in a bounded domain. *Analysis & PDE*, 8-3, pp. 561–628 (2015).
- [LeNe1] D. Le Peutrec, B. Nectoux. Repartition of the quasi-stationary distribution and first exit point density for a double-well potential. arXiv:1902.06304 (2019).
- [LeNe2] D. Le Peutrec, B. Nectoux. Small eigenvalues of the Witten Laplacian with Dirichlet boundary conditions: the case with critical points on the boundary. arXiv:1907.07517 (2019).

- [Lep1] D. Le Peutrec. Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian. *Ann. Fac. Sci. Toulouse Math.* (6), Vol. 19, no 3–4, pp. 735–809 (2010).
- [Lep2] D. Le Peutrec. *Small Eigenvalues of the Witten Laplacian Acting on  $p$ -Forms on a Surface*. *Asymptotic Analysis*, Vol. 73, no. 4, pp. 187–201 (2011).
- [LNV] D. Le Peutrec, F. Nier, C. Viterbo. Precise Arrhenius law for  $p$ -forms: The Witten Laplacian and Morse-Barannikov complex *Ann. Henri Poincaré* Vol. 14, No 3, pp 567–610 (2013)
- [LSV] F. Le Roux, S. Seyfaddini, C. Viterbo. Barcodes and area-preserving homeomorphisms. arXiv: 1810.03139.
- [Loja] S. Lojasiewicz. On semi-analytic and subanalytic geometry. *Panoramas of Mathematics*, Banach Center Publications 34 (1995) pp 89–104.
- [Mic] L. Michel. About small eigenvalues of the Witten Laplacian. *Pure Appl. Anal.* Vol. 1 no. 2 (2019) pp. 149–206.
- [Micl] L. Miclo. Comportement de spectres d’opérateurs de Schrödinger à basse température. *Bulletin des sciences mathématiques* 119 (6), pp. 529–554 (1995).
- [Mit] M. Mitrea. Sharp Hodge decompositions, Maxwell’s equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds. *Duke Math. J.* Vol. 125 no. 3 (2004) 467–547.
- [MiMo] M. Mitrea, S. Monniaux. The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains. *J. Funct. Anal.* Vol. 254 no. 6 (2008) 1522–1574.
- [MMMT] D. Mitrea, I. Mitrea, M. Mitrea, M. Taylor. *The Hodge Laplacian. Boundary value problems on Riemannian manifolds*. De Gruyter Studies in Mathematics Vol. 64 (2016).
- [MTV] M. Mitrea, M. Taylor, A. Vasy Lipschitz domains, domains with corners and the Hodge Laplacian, *Comm. Partial Diff. Eq.* Vol. 30, No 10-12 (2005) pp 1445–1462.
- [Nie] F. Nier. *Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries*. *Mem. Amer. Math. Soc.* 252 no. 1200 (2018).
- [PoSh] L. Polterovich, E. Shelukhin. Autonomous Hamiltonian flows, Hofer’s geometry and persistence modules. *Selecta Math.* (N.S.) Vol. 22 no. 1 (2016) pp 227–296.
- [RoWe] R.T. Rockafellar, R.J-B. Wets *Variational analysis*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 317. Springer-Verlag, Berlin, (1998).
- [Scha] P. Schapira. Operations on constructible functions. *Journal of Pure and Applied Algebra*, Vol 72 (1991) pp 83–93.
- [Sch] G. Schwarz. *Hodge decomposition. A method for Solving Boundary Value Problems*. *Lecture Notes in Mathematics* 1607, Springer-Verlag (1995).
- [Tay] M.E. Taylor. *Partial Differential Equations 1, Basic Theory*. *Applied Mathematical Sciences* 115, Springer-Verlag (1997).
- [Tre] E. Trelat. Global subanalytic solutions of Hamilton-Jacobi type equations. *Annales Henri Poincaré* Vol. 23 no. 3 (2006) pp 363–387.
- [UsZh] M. Usher, J. Zhang. Persistent homology and Floer-Novikov theory. *Geom. Topol.* Vol. 20 no. 6 (2016) pp 3333-3430.
- [VeFr1] A.D. Ventcel, M.I. Freidlin. Small random perturbations of a dynamical system with stable equilibrium position. *Dokl. Akad. Nauk. SSSR*, Vol. 187 (1969) pp. 506–509.

- [VeFr2] A.D. Ventcel, M.I. Freidlin. Small random perturbations of dynamical systems. *Uspehi Mat. Nauk.* Vol. 25 no. 1(151) (1970) pp. 3–55.
- [Ver] G. Verchota. Layer Potentials and Regularity for the Dirichlet Problem for Laplace's Equation in Lipschitz Domains. *J. Funct. Analysis* 59 (1984) pp 572–611.
- [Wit] E. Witten. Supersymmetry and Morse inequalities. *J. Diff. Geom.* 17, no. 4, pp. 661–692 (1982).
- [Zha] W. Zhang. *Lectures on Chern-Weil theory and Witten deformations.* Nankai Tracts in Mathematics, 4, World Scientific Publishing Co. (2001).