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► **To cite this version:**

Pierre-Antoine Giorgi, Anne Nouri. On a Vlasov-Poisson system in a bounded set with direct reflection boundary conditions. *Annales Henri Lebesgue*, In press. hal-02471632

HAL Id: hal-02471632

<https://hal.science/hal-02471632>

Submitted on 8 Feb 2020

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On a Vlasov-Poisson system in a bounded set with direct reflection boundary conditions

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Abstract. Existence (resp. uniqueness) of global (resp. local) in time continuous solutions to the Vlasov-Poisson system is proven in a one-dimensional bounded domain, with direct reflection boundary conditions. Generalized characteristics are used. Electroneutrality is obtained in the limit.

1 Introduction and main results

We consider the Vlasov-Poisson system

$$\partial_t f + v\partial_x f + E\partial_v f = 0, \quad t > 0, \quad (x, v) \in]0, 1[\times \mathbb{R}, \quad (1.1)$$

$$\partial_t g + v\partial_x g - E\partial_v g = 0, \quad t > 0, \quad (x, v) \in]0, 1[\times \mathbb{R}, \quad (1.2)$$

$$\varepsilon\partial_x E = \int_{\mathbb{R}} f \, dv - \int_{\mathbb{R}} g \, dv, \quad t > 0, \quad x \in]0, 1[, \quad (1.3)$$

$$f(0, x, v) = f^0(x, v), \quad (x, v) \in]0, 1[\times \mathbb{R}, \quad (1.4)$$

$$g(0, x, v) = g^0(x, v), \quad (x, v) \in]0, 1[\times \mathbb{R}. \quad (1.5)$$

This is a well-known model describing the dynamics of particles in a collisionless, electrostatic and non-relativistic plasma composed of ions and electrons. f and g respectively denote the ionic and electronic distribution functions. The electric field $E = -\partial_x \phi$ derives from the potential ϕ . The parameter $\varepsilon > 0$ is equal to the square of the ratio between the Debye and the characteristic observation lengths. The Debye length is a physical length below which charge separation occurs. In many physical situations, ε is small. The distribution functions f and g satisfy direct reflection boundary conditions at the boundary $x \in \{0, 1\}$, and E is given and constant at $x = 0$,

$$f(t, x, v) = f(t, x, -v), \quad t > 0, \quad x \in \{0, 1\}, \quad v \in \mathbb{R}, \quad (1.6)$$

$$g(t, x, v) = g(t, x, -v), \quad t > 0, \quad x \in \{0, 1\}, \quad v \in \mathbb{R}, \quad (1.7)$$

$$E(t, 0) = E_0, \quad t > 0. \quad (1.8)$$

Many works have already been done on the Vlasov-Poisson system. C. Bardos and P. Degond [BAR1985] proved global in time existence and uniqueness of classical solutions to the Cauchy problem related to the Vlasov-Poisson system in the whole three-dimensional space for small initial data. Using characteristics, K. Pfaffelmoser [PFA1992] proved existence of classical solutions to the Vlasov-Poisson system for general initial data. Together with the study of propagation of moments $\int |v|^m f(t, x, v) \, dx \, dv$ with $m > 3$, P.-L. Lions and B. Perthame [LIO1991] proved existence and uniqueness of an $L^1 \cap L^\infty$ weak solution to the Vlasov-Poisson system in the whole space. C. Pallard [PAL2012] proved an analogous result for $m > 2$. Using optimal transport, G. Loeper [LOE2006] proved the uniqueness of weak solutions with bounded mass density. A key point in their proofs is the boundedness of the mass density $\rho(t, x) = \int f(t, x, v) \, dv$. In a domain with boundaries, the treatment of the Vlasov-Poisson system is more complex. Existence of weak solutions to the Vlasov-Poisson system in a bounded domain and given indata were proven by N. Ben Abdallah [BEN1994]. For the Vlasov equation, S. Mischler [MIS1999] extended his result by considering other types of boundary conditions such as specular reflection, proving existence and uniqueness of weak solutions, and studying their traces. On a half-line, H.J. Hwang and J. Schaeffer [HWA2008] proved uniqueness of weak solutions to the one-species Vlasov-Poisson system with specular reflection for the distribution function and given constant electric field at the boundary, pointing inward

of the domain, using an approach with characteristics. Considering the Vlasov-Poisson system for two species as it is done in this paper, the electric field points inward (resp. outward) of the domain for one (resp. the other) species. It is known from a counterexample by Y. Guo [GUO1995] that there is in general no C^1 solution to the Vlasov-Maxwell system with direct reflection boundary conditions on a half-line. This counterexample can be adapted to the Vlasov-Poisson system with direct reflection boundary conditions in a bounded domain. In this paper, we prove global in time existence and local in time uniqueness of continuous (resp. continuous in time and C^1 in space) ionic and electronic distribution functions f and g (resp. electric field E).

If the spatial domain in (1.1)–(1.8) were \mathbb{R} instead of $]0, 1[$, classical characteristics for (1.1) would be defined by $X' = V, V' = E(t, X)$. In the frame of this paper, bounces may occur at the boundary $\{0, 1\}$ of the spatial domain. When a characteristics intersects $x = 0$ (resp. $x = 1$) with a zero velocity at a time $s > 0$, it can be seen that it stays in the domain without any discontinuity. And so, it is still considered as a classical characteristics. Bounces at the boundary of the domain occur when a characteristics intersects the boundary with a non zero velocity. Generalized characteristics involving possible bounces [GUO1995, HWA2008] and continuous solutions to (1.1) and (1.6) are defined as follows.

Definition 1.1

Let $E \in C([0, +\infty[; C^1([0, 1]))$.

The generalized backward characteristics (X, V) from $(t, x, v) \in]0, +\infty[\times]0, 1[\times \mathbb{R}$ related to (1.1) and (1.6) is defined as the union of the classical characteristics which connect (t, x, v) to $(t_1, x_1, v_1), (t_1, x_1, -v_1)$ to $(t_2, x_2, v_2), \dots, (t_n, x_n, -v_n)$ to $(t_{n+1}, x_{n+1}, v_{n+1}), \dots$, where $x_n \in \{0, 1\}, |v_n| > 0$ and $t_n > t_{n+1} \geq 0$.

This gives a set $P \subset \mathbb{N}^*$ counting the number of bounces, and a sequence of bouncing times $(t_n)_{n \in P}$ such that,

$$X(t_n; t, x, v) = 0 \quad \text{and} \quad V(t_n^+; t, x, v) = -V(t_n^-; t, x, v) > 0,$$

or

$$X(t_n; t, x, v) = 1 \quad \text{and} \quad V(t_n^+; t, x, v) = -V(t_n^-; t, x, v) < 0, \quad n \in P.$$

Definition 1.2

Let $E \in C([0, +\infty[; C^1([0, 1]))$. A continuous solution f to (1.1), (1.4) and (1.6) is a function $f \in C([0, +\infty[\times]0, 1[\times \mathbb{R})$ such that

$$f(t, x, v) = f^0(X(0; t, x, v), V(0; t, x, v)), \quad t > 0, \quad (x, v) \in [0, 1] \times \mathbb{R}, \quad (1.9)$$

where $(X(\cdot; t, x, v), V(\cdot; t, x, v))$ is the generalized characteristics from (t, x, v) as in Definition 1.1.

The main results of this paper are the following.

Theorem 1.1

Let $\varepsilon > 0$. Let $f^0, g^0 \in C([0, 1] \times \mathbb{R})$ be nonnegative even functions w.r.t. the v variable, with finite kinetic energy, and such that for any $R > 0$,

$$\sup_{(x,w) \in [0,1] \times \mathbb{R}; |w-v| < R} f^0(x, w) \in L^1(\mathbb{R}_v), \quad \sup_{(x,w) \in [0,1] \times \mathbb{R}; |w-v| < R} g^0(x, w) \in L^1(\mathbb{R}_v). \quad (1.10)$$

Let $E_0 \in \mathbb{R} \setminus \{0\}$ such that

$$E_0 \neq \varepsilon^{-1} \int (g^0 - f^0)(x, v) dx dv. \quad (1.11)$$

There exists a solution $(f, g, E) \in (C([0, +\infty[\times]0, 1[\times \mathbb{R}))^2 \times C([0, +\infty[; C^1([0, 1]))$ of (1.1)–(1.8) in the sense of Definition 1.2. Moreover, f and g are nonnegative.

Remark 1.1 Due to the mass conservation, $E(\cdot, 1)$ is also a constant function. Assumption (1.11) is satisfied if $E_0 \neq 0$ and electroneutrality holds at $t = 0$. It is done for $E(\cdot, 1)$ to be different from zero.

Remark 1.2 As will be seen in Property 3.1, $t \mapsto E(t, 0)$ is assumed to be a constant function in order to ensure the conservation of total energy.

Theorem 1.2

Let f^0, g^0 be nonnegative Lipschitz functions satisfying (1.10), even w.r.t. the v variable, and such that for some $c_0 > 0$ and $V_0 > 0$,

$$f^0(x, v) = g^0(x, v) = 0, \quad x \in [0, 1], \quad |v| \leq c_0 \quad \text{or} \quad |v| \geq V_0.$$

Let $E_0 \in \mathbb{R} \setminus \{0\}$ satisfying (1.11).

There is a time $T_0 > 0$ such that the continuous solution to (1.1)–(1.8) is unique on $[0, T]$.

The use of generalized characteristics enlightens the solution. The main difficulties to use them are to discard infinitely many bounces accumulating at some boundary point, and to get continuity of the distribution functions f and g from (1.9).

The paper organizes as follows. In Section 2, generalized characteristics are studied, taking into account possible bounces. Theorem 1.1 (resp. Theorem 1.2) is proven in Section 3 (resp. Section 4). The quasineutrality equation is obtained when $\varepsilon \rightarrow 0$ at the end of Section 3.

2 Generalized characteristics

In this section, $T > 0$ and $E \in C([0, T]; C^1([0, 1]))$ are given. We consider the Cauchy problem for the Vlasov equation,

$$\partial_t f + v \partial_x f + E \partial_v f = 0, \quad t \in [0, T], \quad (x, v) \in]0, 1[\times \mathbb{R}, \quad (2.1)$$

$$f(0, x, v) = f^0(x, v), \quad (x, v) \in]0, 1[\times \mathbb{R}, \quad (2.2)$$

$$f(t, x, v) = f(t, x, -v), \quad t \in [0, T], \quad (x, v) \in \{0, 1\} \times \mathbb{R}. \quad (2.3)$$

As recalled above, existence and uniqueness of weak solutions to the problem have been proven by S. Mischler in [MIS1999], using a variety of test functions. Our approach differs from his by considering generalized characteristics and continuous distribution functions. Excluding the case where infinitely many bounces would accumulate at a boundary point, we prove in Propositions 2.1 and 2.2 that the backwards in time generalized characteristics from any $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$,

$$s \mapsto (X(s; t, x, v), V(s; t, x, v)),$$

has a finite number of bounces at the boundary, hence reaches time zero. Example 2.1 exhibits a case where the map $(t, x, v) \mapsto V(0; t, x, v)$ is discontinuous. In Proposition 2.3, the continuity of the map $(x, v) \mapsto (X(0; t, x, v), |V(0; t, x, v)|)$ is proven, for any $t > 0$.

Lemma 2.1 For $t > 0$ and $(x, v) \in [0, 1] \times \mathbb{R}$, the second component V of the generalized characteristics from (t, x, v) satisfies

$$|V(s; t, x, v)| \leq |v| + T \|E\|_\infty, \quad (2.4)$$

for $s \in [0, t]$ if P is finite (resp. $s \in [t_1, t] \cup_{n \in P} [t_{n+1}, t_n]$ if P is not finite).

Proof of Lemma 2.1. The case where P is finite is classical. Assume $P = \mathbb{N}^*$. Denote by $t_0 = t$ and prove by induction on $n \in \mathbb{N}$ that

$$||v| - |V(s; t, x, v)|| \leq (t - s) \|E\|_\infty, \quad s \in [t_{n+1}, t_n]. \quad (2.5)$$

For $n = 0$ and $s \in [t_1, t]$, the equation $\partial_s V(s; t, x, v) = E(s, X(s; t, x, v))$ yields

$$||v| - |V(s; t, x, v)|| \leq \int_s^t |E(r, X(r; t, x, v))| dr \leq (t - s) \|E\|_\infty.$$

Assuming that $||v| - |V(s; t, x, v)|| \leq (t - s) \|E\|_\infty$ for $s \in [t_n, t_{n-1}]$ and $n \geq 1$, it holds

$$\begin{aligned} ||v| - |V(s; t, x, v)|| &\leq ||v| - |V(t_n^+; t, x, v)|| + ||V(t_n^-; t, x, v)| - |V(s; t, x, v)|| \\ &\leq (t - t_n) \|E\|_\infty + \int_s^{t_n} |E(r, X(r; t, x, v))| dr \\ &\leq (t - s) \|E\|_\infty, \quad s \in [t_{n+1}, t_n]. \end{aligned}$$

■

Given $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$, either the backwards in time generalized characteristics from (t, x, v) reaches $\{0\} \times [0, 1] \times \mathbb{R}$ at $(X(0; t, x, v), V(0; t, x, v))$ without any bounce. This is the easy case where $P = \emptyset$. Or bounces occur at $x = 0$ or $x = 1$. We first prove that there is a finite number of them. This is strongly linked with the sign of the electric field at the boundaries. The following analysis distinguishes two cases. We first deal with the case of a negative value of $t \mapsto E(t, 1)$. The case of a positive value of $t \mapsto E(t, 0)$ can be treated analogously.

In the case of a negative value of $E(t, 1)$ on $[0, T]$, let $\delta \in]0, 1[$ be such that

$$E(t, x) < 0, \quad (t, x) \in [0, T] \times [1 - \delta, 1]. \quad (2.6)$$

Let

$$\Delta(v) := \frac{1}{\|E\|_\infty} \left(\sqrt{(|v| + T\|E\|_\infty)^2 + 2\delta\|E\|_\infty} - (|v| + T\|E\|_\infty) \right), \quad v \in \mathbb{R}. \quad (2.7)$$

A bound of the number of possible bounces on the generalized characteristics from $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$ is given in the following proposition.

Proposition 2.1

Assume $t \mapsto E(t, 1)$, $t \in [0, T]$, constant and negative.

The number of bounces occurring at $x = 1$ along the backwards in time generalized characteristics from $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$ is finite and bounded by $\frac{T}{\Delta(v)}$.

Proof of Proposition 2.1. Let (X, V) be a backwards in time generalized characteristics from (t, x, v) with at least two bounces at $x = 1$, occurring at times t_1 and t_2 , with $t_2 < t_1 < t$. In order to prove the result, it is sufficient to bound $t_1 - t_2$ from below.

It holds that $V(t_1^-; t, x, v) > 0$. As $V(\cdot; t, x, v)$ is decreasing when $X(\cdot; t, x, v)$ is in the interval $[1 - \delta, 1]$, X leaves the interval $[1 - \delta, 1]$ at a time $s_1 \in]t_2, t_1[$, and $X(s; t, x, v) \geq 1 - \delta$ for $s \in [s_1, t_1]$. The integration between s and t_1 along the classical characteristics $(X(\cdot; t, x, v), V(\cdot; t, x, v))|_{[t_2^+, t_1^-]}$ yields

$$X(s) - 1 + (t_1 - s)V(t_1^-) \geq -\frac{\|E\|_\infty}{2}(t_1 - s)^2, \quad s \in [s_1, t_1],$$

so that

$$\frac{\|E\|_\infty}{2}(t_1 - s_1)^2 + (t_1 - s_1)V(t_1^-) - \delta \geq 0.$$

This implies that

$$t_1 - t_2 \geq t_1 - s_1 \geq \frac{\sqrt{V(t_1^-)^2 + 2\delta\|E\|_\infty} - V(t_1^-)}{\|E\|_\infty}.$$

And so, by Lemma 2.1,

$$t_1 - t_2 \geq \frac{1}{\|E\|_\infty} \left(\sqrt{(|v| + T\|E\|_\infty)^2 + 2\delta\|E\|_\infty} - (|v| + T\|E\|_\infty) \right).$$

The number of bounces at $x = 1$ on $[0, T]$ is thus smaller than $\frac{T}{\Delta(v)}$. ■

The opposite case where $E(t, 1) > 0$, $t \in [0, T]$, is more complicated. It corresponds to an electric field pointing outward of the domain $]0, 1[$ at the boundary. It cannot be directly expected that the time between two bounces is bounded from below. An infinite number of bounces is a priori not impossible. The distance to the boundary of the X component of the characteristics between two bounces could be arbitrarily small. It is proven in the following proposition that this does not occur. We first prove a preliminary lemma.

Lemma 2.2

For any $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$, the series $\sum_{n \in P} |V(t_n^+; t, x, v)|$ converges.

Proof of Lemma 2.2. Assume $P = \mathbb{N}^*$, and first prove that the number of consecutive bounces between $x = 0$ and $x = 1$ is finite. Integration along the classical characteristics on $[t_{k+1}, t_k]$, with $(k, k+1) \in P^2$, yields

$$X(t_k) - X(t_{k+1}) = (t_k - t_{k+1})V(t_{k+1}^+) + \int_{t_{k+1}}^{t_k} (t_k - r)E(r, X(r))dr.$$

It holds that $|X(t_k) - X(t_{k+1})| = 1$. Hence

$$1 \leq (t_k - t_{k+1})|V(t_{k+1}^+)| + \frac{\|E\|_\infty}{2}(t_k - t_{k+1})^2,$$

so that

$$t_k - t_{k+1} \geq \frac{1}{\|E\|_\infty} \left(\sqrt{V(t_{k+1}^+)^2 + 2\|E\|_\infty} - |V(t_{k+1}^+)| \right).$$

It follows from Lemma 2.1 that

$$t_k - t_{k+1} \geq \frac{1}{\|E\|_\infty} \left(\sqrt{(|v| + T\|E\|_\infty)^2 + 2\|E\|_\infty} - (|v| + T\|E\|_\infty) \right).$$

This implies a finite number of consecutive bounces between $x = 0$ and $x = 1$ on $[0, T]$. Consequently we can assume $X(t_k) = 0$ for $k \geq n_0$ for some $n_0 \in \mathbb{N}^*$, the other case $X(t_k) = 1$ for $k \geq n_0$ being analogous. It holds that

$$\begin{aligned} 2 \sum_{k=1}^n |V(t_k^+)| &= 2 \left(\sum_{k=1}^{n_0-1} |V(t_k^+)| \right) + V(t_{n_0}^+) + V(t_n^+) + \sum_{k=n_0}^{n-1} \underbrace{(|V(t_{k+1}^+)| + |V(t_k^+)|)}_{=V(t_{k+1}^+) - V(t_k^-)} \\ &= 2 \left(\sum_{k=1}^{n_0-1} |V(t_k^+)| \right) + V(t_{n_0}^+) + V(t_n^+) - \int_{t_n}^{t_{n_0}} E(s, X(s)) ds, \quad n \geq n_0 + 1. \end{aligned}$$

By (2.4) applied to $s = t_n^+$,

$$\sum_{k=1}^n |V(t_k^+)| \leq \sum_{k=1}^{n_0-1} |V(t_k^+)| + \frac{V(t_{n_0}^+) + |v|}{2} + T\|E\|_\infty, \quad n \geq n_0.$$

■

Proposition 2.2

When $E(t, 1)$ takes a constant value $E_1 > 0$ on $[0, T]$, the number of bounces occurring at $x = 1$ along the backwards in time generalized characteristics from $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$ is finite.

Proof of Proposition 2.2. It is a proof by contradiction. Assume infinitely many bounces at $(t_n, 1)_{n \in \mathbb{N}}$ along the backwards in time generalized characteristics from $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$. $(t_n)_{n \in \mathbb{N}}$ being a decreasing sequence in $[0, T]$, converges to a time $t^* \geq 0$ when $n \rightarrow +\infty$. Denote by

$$V_n = V(t_n^-) > 0, \quad s_n \in]t_{n+1}, t_n[\quad \text{such that } s_n = \min_{s \in [t_{n+1}, t_n]} X(s), \quad y_n = 1 - X(s_n).$$

It holds that $\lim_{n \rightarrow +\infty} s_n = t^*$, $\lim_{n \rightarrow +\infty} V_n = 0$ by Lemma 2.2, and $\lim_{n \rightarrow +\infty} y_n = 0$. Indeed, if for a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ of (y_n) , $\lim_{k \rightarrow +\infty} y_{n_k} = y^* > 0$, then

$$\begin{aligned} \frac{\|E\|_\infty}{2}(t_{n_k} - s_{n_k})^2 + (t_{n_k} - s_{n_k}) - \frac{y^*}{2} &\geq \int_{s_{n_k}}^{t_{n_k}} (t_{n_k} - r)E(r, X(r))dr + V(s_{n_k})(t_{n_k} - s_{n_k}) - \frac{y^*}{2} \\ &= X(t_{n_k}) - X(s_{n_k}) - \frac{y^*}{2} = y_{n_k} - \frac{y^*}{2} \geq 0, \end{aligned}$$

for k large enough. Consequently,

$$t_{n_k} - s_{n_k} \geq \frac{\sqrt{1 + y^* \|E\|_\infty} - 1}{\|E\|_\infty},$$

for k large enough. This would contradict the infinite number of bounces at $x = 1$.

By the continuity of E , there is $\delta_1 > 0$ such that

$$E(t, x) \geq \frac{E_1}{2}, \quad (t, x) \in [0, T] \times [1 - \delta_1, 1].$$

It follows from

$$0 = \int_{t_{n+1}}^{t_n} V(s)ds = (t_n - t_{n+1})V_n - \int_{t_{n+1}}^{t_n} (r - t_{n+1})E(r, X(r))dr,$$

that

$$t_n - t_{n+1} \leq \frac{4}{E_1}V_n, \quad (2.8)$$

for n large enough. Moreover,

$$\begin{aligned} y_n &= 1 - X(s_n) = \int_{s_n}^{t_n} V(s)ds \\ &= (t_n - s_n)V_n - \int_{s_n}^{t_n} (r - s_n)E(r, X(r))dr \\ &\leq (t_n - s_n)V_n, \end{aligned}$$

for n large enough. And so,

$$y_n \leq \frac{4}{E_1}V_n^2. \quad (2.9)$$

In a neighborhood of $(t^*, 1)$, $E(s, X(s))$ expresses as

$$E(s, X(s)) = E_1 + (X(s) - 1)\partial_x E(t^*, 1) + (X(s) - 1)\varepsilon(s), \quad (2.10)$$

with

$$\lim_{s \rightarrow t^*} \varepsilon(s) = 0. \quad (2.11)$$

Indeed, the function ε introduced in (2.10) satisfies

$$\varepsilon(s) = \frac{1}{X(s) - 1} \int_1^{X(s)} (\partial_x E(s, y) - \partial_x E(t^*, 1))dy,$$

which tends to zero when s tends to t^* , by the uniform continuity of $\partial_x E$ on $[0, T] \times [0, 1]$. The case $\partial_x E(t^*, 1) < 0$, i.e. $\partial_x E(t^*, 1) = -\alpha^2$ with $\alpha > 0$, is treated here. α is taken as 1 for the sake of simplicity. By definition of the characteristics (X, V) and (2.9)–(2.11), $s \mapsto (X(s; t, x, v), V(s; t, x, v))$ satisfies

$$\begin{aligned} X''(s) + X(s) &= E_1 + 1 + g(s), \quad s \in [t_{n+1}, t_n], \\ V(s) &= X'(s), \quad s \in]t_{n+1}, t_n[, \\ X(t_n) &= 1, \quad X'(t_n^-) = V_n, \end{aligned}$$

where

$$g(s) := (X(s) - 1)\varepsilon(s) = o(V_n^2), \quad s \in [t_{n+1}, t_n].$$

Here, $o(V_n^2)$ means that $\lim_{n \rightarrow +\infty} \frac{o(V_n^2)}{V_n^2} = 0$. Hence,

$$\begin{aligned} X(s) &= E_1(1 - \cos(s - t_n)) + V_n \sin(s - t_n) + 1 + o(V_n^3), \quad s \in [t_{n+1}, t_n], \\ V(s) &= E_1 \sin(s - t_n) + V_n \cos(s - t_n) + o(V_n^2), \quad s \in]t_{n+1}, t_n[, \end{aligned}$$

or

$$X(s) = \frac{E_1}{2}(s - t_n)^2 + V_n(s - t_n) + 1 + o(V_n^3), \quad s \in [t_{n+1}, t_n], \quad (2.12)$$

$$V(s) = E_1(s - t_n) + V_n + o(V_n^2), \quad s \in]t_{n+1}, t_n[. \quad (2.13)$$

Since t_{n+1} is a solution to $X(t_{n+1}) = 1$, $t_n - t_{n+1}$ satisfies

$$E_1(t_n - t_{n+1})^2 - 2V_n(t_n - t_{n+1}) + o(V_n^3) = 0,$$

i.e.

$$E_1(t_n - t_{n+1}) = V_n \pm \sqrt{V_n^2 + o(V_n^3)}.$$

By definition of V_{n+1} ,

$$V_{n+1} = -V(t_{n+1}^+) = E_1(t_n - t_{n+1}) - V_n + o(V_n^2).$$

Given the positive sign of V_n and V_{n+1} , it results

$$E_1(t_n - t_{n+1}) = V_n + \sqrt{V_n^2 + o(V_n^3)} = 2V_n + o(V_n^2) \quad \text{and} \quad V_{n+1} = V_n + o(V_n^2).$$

Thus, for some n_0 large enough,

$$V_{n+1} \geq V_n(1 - V_n), \quad n \geq n_0.$$

Denote by $h : x \mapsto x - x^2$ and $h^p = \underbrace{h \circ \dots \circ h}_p$, $p \in \mathbb{N}^*$. Let $n_1 \geq n_0$ be such that $V_n \leq \frac{1}{2}$, $n \geq n_1$. It holds that

$$V_{n_1+p} \geq h^p(V_{n_1}), \quad p \in \mathbb{N}. \quad (2.14)$$

Moreover, it can easily be proven by induction that

$$h^p(x) \geq \frac{x}{p+1}, \quad p \geq 1, \quad x \in \left[0, \frac{1}{2}\right]. \quad (2.15)$$

It results from (2.14)–(2.15) that

$$\sum_{p=n_1+1}^n V_p \geq \sum_{p=1}^{n-n_1} h^p(V_{n_1}) \geq V_{n_1} \sum_{p=1}^{n-n_1} \frac{1}{p}, \quad n \geq n_1 + 1,$$

which contradicts the statement of Lemma 2.2. Hence the number of bounces at $x = 1$ is finite. The case $\partial_x E(t^*, 1) > 0$ (resp. $\partial_x E(t^*, 1) = 0$) is similar and also leads to (2.12)–(2.13) (see [GIO2019]). ■

Proposition 2.2 can similarly be extended to the case where $t \mapsto E(t, 1)$ is not a constant function.

We now consider the continuity of the function f defined by (1.9). Despite the continuity of f^0 , f may be discontinuous. Actually, issues arise when $X(0; t, x, v)$ is exactly zero (or one). This is illustrated in the following example.

Example 2.1

Let the field E be a positive constant. Let $t \in]0, T]$ and $x > \frac{Et^2}{2}$.

The map $v \mapsto V(0; t, x, v)$ is discontinuous at $v = \frac{x}{t} + \frac{E}{2}t$ and continuous elsewhere.

Proof of Example 2.1. The backward in time characteristics from (t, x, v) before any bounce at $x = 0$ is given by

$$X(s) = x - v(t - s) + \frac{E}{2}(t - s)^2, \quad V(s) = v - E(t - s), \quad s \in [0, t].$$

For $v = \frac{x}{t} + \frac{E}{2}t$, there is no bounce on $]0, t]$, $X(0; t, x, v) = 0$ and $V(0; t, x, v) > 0$. For $\tilde{v} < v$, the backward in time characteristics from (t, x, \tilde{v}) has no bounce and

$$V(0; t, x, \tilde{v}) = \tilde{v} - tE.$$

For $\tilde{v} > v$, the backward in time characteristics from (t, x, \tilde{v}) encounters a bounce at time

$$t_1 = t - \frac{\tilde{v} - \sqrt{\tilde{v}^2 - 2xE}}{E} > 0, \quad \text{and} \quad V(0; t, x, \tilde{v}) = \tilde{v} - tE - 2\sqrt{\tilde{v}^2 - 2xE}.$$

Thus,

$$\lim_{\tilde{v} \rightarrow v^-} V(0; t, x, \tilde{v}) = v - tE, \quad \lim_{\tilde{v} \rightarrow v^+} V(0; t, x, \tilde{v}) = -(v - tE).$$

Hence the map $v \mapsto V(0; t, x, v)$ is discontinuous at $v = \frac{x}{t} + \frac{E}{2}t$.

Proposition 2.3

The map $(t, x, v) \mapsto (X(0; t, x, v), |V(0; t, x, v)|)$ is continuous on $[0, T] \times [0, 1] \times \mathbb{R}$.

Proof of Proposition 2.3. Let $(\tilde{t}, \tilde{x}, \tilde{v})$ be given. If the backwards characteristics from $(\tilde{t}, \tilde{x}, \tilde{v})$ reaches $t = 0$ at $X(0; \tilde{t}, \tilde{x}, \tilde{v}) \in]0, 1[$, then analogous arguments as for classical characteristics imply that

$$\lim_{(t, x, v) \rightarrow (\tilde{t}, \tilde{x}, \tilde{v})} (X(0; t, x, v), V(0; t, x, v)) = (X(0; \tilde{t}, \tilde{x}, \tilde{v}), V(0; \tilde{t}, \tilde{x}, \tilde{v})).$$

What remains to be proven is the continuity of $(t, x, v) \mapsto (X(0; t, x, v), |V(0; t, x, v)|)$ at $(\tilde{t}, \tilde{x}, \tilde{v})$ such that its backwards characteristics reaches $t = 0$ at $X(0; \tilde{t}, \tilde{x}, \tilde{v}) \in \{0, 1\}$. Assume $X(0; \tilde{t}, \tilde{x}, \tilde{v}) = 0$. Consider (t, x, v) such that the backwards characteristics $(X, V)(\cdot; t, x, v)$ has an earliest bounce at time $t_1(t, x, v) > 0$. For (t, x, v) close enough to $(\tilde{t}, \tilde{x}, \tilde{v})$, there is no bounce of the backwards characteristics from $(\tilde{t}, \tilde{x}, \tilde{v})$ on the interval $[0, t_1(t, x, v)]$. Define the extended electric field E^e on $[0, T] \times \mathbb{R}$ by

$$\begin{aligned} E^e(t, x) &= E(t, x), & t \geq 0, x \in [0, 1], \\ E^e(t, x) &= E(t, 0), & t \geq 0, x < 0, & E^e(t, x) = E(t, 1), & t \geq 0, x > 1, \end{aligned}$$

and the extended classical characteristics (X^e, V^e) from $(t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}$ by

$$\partial_s X^e = V^e, \quad X^e(t; t, x, v) = x, \quad \partial_s V^e = E^e(s, X^e), \quad V^e(t; t, x, v) = v.$$

First consider the case where $E_0 < 0$. There is no restriction to consider $\tilde{v} > 0$, and (t, x) (resp. (\tilde{t}, \tilde{x})) in the strip close to $x = 0$ where $E < 0$.

Let $\varepsilon > 0$ be given, and $v \in]\frac{\tilde{v}}{2}, \frac{3\tilde{v}}{2}[$. For (t, x, v) in an appropriate neighborhood of $(\tilde{t}, \tilde{x}, \tilde{v})$, it holds that

$$|X^e(s; t, x, v) - X^e(s; \tilde{t}, \tilde{x}, \tilde{v})| + |V^e(s; t, x, v) - V^e(s; \tilde{t}, \tilde{x}, \tilde{v})| \leq \varepsilon, \quad s \in [0, \tilde{t}]. \quad (2.16)$$

It follows from

$$\begin{aligned} V^e(t_1; t, x, v) &\geq v, & X^e(t_1; t, x, v) &= 0, \\ X^e(s; t, x, v) &= V^e(t_1; t, x, v)(s - t_1) + \int_s^{t_1} (r - s) E^e(r, X^e(r; t, x, v)) dr, \end{aligned}$$

that

$$X^e(s; t, x, v) \leq -2\varepsilon, \quad s \leq t_1 - \frac{4\varepsilon}{\tilde{v}}.$$

Together with (2.16) and $X(0, \tilde{t}, \tilde{x}, \tilde{v}) = 0$, this implies that $t_1 < \frac{4\varepsilon}{\tilde{v}}$. Consequently,

$$|V(t_1^-; t, x, v) - V(0; t, x, v)| + |V(t_1; \tilde{t}, \tilde{x}, \tilde{v}) - V(0; \tilde{t}, \tilde{x}, \tilde{v})| \leq 2\|E\|_\infty t_1 \leq c\varepsilon,$$

and

$$\begin{aligned} &|V(0; t, x, v) + V(0; \tilde{t}, \tilde{x}, \tilde{v})| \\ &\leq |V(0; t, x, v) - V(t_1^-; t, x, v)| + |V(0; \tilde{t}, \tilde{x}, \tilde{v}) - V(t_1; \tilde{t}, \tilde{x}, \tilde{v})| + |V(t_1; \tilde{t}, \tilde{x}, \tilde{v}) - V(t_1^+; t, x, v)| \\ &\leq c\varepsilon. \end{aligned}$$

The inequality

$$|X(0; t, x, v) - X(0; \tilde{t}, \tilde{x}, \tilde{v})| \leq c\varepsilon,$$

can be proven by bounding $|X(0; t, x, v) - X(0; \tilde{t}, \tilde{x}, \tilde{v})|$ from above by

$$|X(0; t, x, v) - X(t_1; t, x, v)| + |X(0; \tilde{t}, \tilde{x}, \tilde{v}) - X(t_1; \tilde{t}, \tilde{x}, \tilde{v})| + |X(t_1; \tilde{t}, \tilde{x}, \tilde{v}) - X(t_1; t, x, v)|.$$

Consider the case where $E_0 > 0$ and (t, x, v) (resp. $(\tilde{t}, \tilde{x}, \tilde{v})$) such that the backwards characteristics $(X, V)(\cdot; t, x, v)$ (resp. $(X, V)(\cdot; \tilde{t}, \tilde{x}, \tilde{v})$) has an earliest bounce at time $t_1(t, x, v) > 0$ (resp. has no bounce on $[0, t_1(t, x, v)]$) and is such that $X(0; \tilde{t}, \tilde{x}, \tilde{v}) = 0$. There is no restriction to consider $\tilde{v} > 0$, and

(t, x) (resp. (\tilde{t}, \tilde{x})) in the strip close to $x = 0$ where $E > \frac{E_0}{2}$. Let $\varepsilon > 0$ be given. It holds that for (t, x, v) in an appropriate neighborhood of $(\tilde{t}, \tilde{x}, \tilde{v})$,

$$|X^e(s; t, x, v) - X^e(s; \tilde{t}, \tilde{x}, \tilde{v})| + |V^e(s; t, x, v) - V^e(s; \tilde{t}, \tilde{x}, \tilde{v})| \leq \varepsilon, \quad s \in [0, \tilde{t}].$$

Consider the extreme case where $V(t_1(t, x, v); t, x, v) = 0$. Then $0 \leq V(s; \tilde{t}, \tilde{x}, \tilde{v}) \leq \varepsilon$, for $s \in [0, t_1(t, x, v)]$. And so,

$$\frac{E_0}{2} t_1 \leq \int_0^{t_1} E(r, X(r; \tilde{t}, \tilde{x}, \tilde{v})) dr = V(t_1; \tilde{t}, \tilde{x}, \tilde{v}) - V(0; \tilde{t}, \tilde{x}, \tilde{v}) \leq c\varepsilon.$$

From here the proof is analogous to the case where $E_0 < 0$. ■

Consequently, taking f^0 continuous and even w.r.t. the v variable and defining

$$f(t, x, v) = f^0(X(0; t, x, v), V(0; t, x, v)), \quad (t, x, v) \in [0, +\infty[\times [0, 1] \times \mathbb{R},$$

as in (1.9), makes f continuous.

3 Proof of the existence Theorem 1.1

Up to Property 3.1 in this section, the parameter ε in (1.3) is taken as one for the sake of simplicity. Let $T > 0$ be given. Theorem 1.1 is proven with a fixed point argument for the map S defined on

$$K := \left\{ a \in C([0, T] \times [0, 1]); \int_0^1 a(t, x) dx = \|f^0\|_{L^1} - \|g^0\|_{L^1}, \quad t \in [0, T] \right\},$$

by $S = S_3 \circ S_2 \circ S_1$. Here,

$$\begin{aligned} S_1(a)(t, x) &= E_0 + \int_0^x a(t, y) dy, \quad (t, x) \in [0, T] \times [0, 1], \\ S_2(E) &:= (f, g), \end{aligned}$$

where f (resp. g) is the solution to the linear Vlasov equation with force field E (resp. $-E$), initial datum f^0 (resp. g^0) and direct reflection boundary conditions, and

$$S_3(f, g) := \int (f - g)(\cdot, \cdot, v) dv.$$

In Lemma 3.1 we prove that S maps K into K . Lemma 3.2 proves the compactness of S in $C([0, T] \times [0, 1])$. Lemma 3.3 proves its continuity. We then conclude that there is a fixed point a of S , such that $(f, g) = S_2 \circ S_1(a)$ is a solution to the Cauchy problem (1.1)–(1.8).

Lemma 3.1 *The map S is well defined and maps K into K .*

Proof of Lemma 3.1.

$S_1(a) := E$ is continuous on $[0, T] \times [0, 1]$ like a , and globally Lipschitz with respect to x since

$$|E(t, x') - E(t, x)| \leq \|a\|_{\infty} |x' - x|, \quad t \in [0, T], \quad (x, x') \in [0, 1]^2. \quad (3.1)$$

Moreover, $E(t, 0) = E_0$ and $E(t, 1) = E_0 + \|f^0\|_{L^1} - \|g^0\|_{L^1}$ are constants different from zero by (1.11). The analysis from Section 2 and the evenness of (f^0, g^0) with respect to v allow to define $S_2(E) = (f, g)$, where f (resp. g) is the solution to the linear Vlasov equation with force field E (resp. $-E$), initial datum f^0 (resp. g^0) and direct reflection boundary conditions. Recall that

$$\begin{aligned} f(t, x, v) &= f^0(X(0; t, x, v), |V(0; t, x, v)|), \\ (\text{resp. } g(t, x, v) &= g^0(Y(0; t, x, v), |W(0; t, x, v)|)), \quad (t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}, \end{aligned}$$

where (X, V) (resp. (Y, W)) are the generalized characteristics associated to E (resp. $-E$), such that

$$(t, x, v) \rightarrow (X(0; t, x, v), |V(0; t, x, v)|, Y(0; t, x, v), |W(0; t, x, v)|) \quad (3.2)$$

is continuous. Consequently, f (resp. g) is continuous, and nonnegative like f^0 (resp. g^0). $S(a) = \int (f - g)(\cdot, \cdot, v) dv$ belongs to $C([0, T] \times [0, 1])$. Indeed, let

$$R = (|E_0| + \|a\|_\infty) T.$$

By (1.10), there is $U > 0$ such that

$$\int_{|v|>U} \sup_{x \in [0, 1], |w-v|<R} f^0(x, w) dv \quad \text{and} \quad \int_{|v|>U} \sup_{x \in [0, 1], |w-v|<R} g^0(x, w) dv$$

are arbitrarily small. It follows from the continuity of (3.2) and (f^0, g^0) , that the map

$$\begin{aligned} (t, x) &\mapsto \int_{|v|<U} (f - g)(t, x, v) dv \\ &= \int_{|v|<U} (f^0(X(0; t, x, v), |V(0; t, x, v)|) - g^0(Y(0; t, x, v), |W(0; t, x, v)|)) dv \end{aligned}$$

is continuous on $[0, T] \times [0, 1]$. Finally, the mass conservation of f (resp. g) implies that

$$\int_0^1 \int (f - g)(t, x, v) dv dx = \|f^0\|_{L^1} - \|g^0\|_{L^1}, \quad t \in [0, T].$$

Consequently $S(a)$ belongs to K . ■

Lemma 3.2 *S is compact in $C([0, T] \times [0, 1])$.*

Proof of Lemma 3.2. Let $(a_n)_{n \in \mathbb{N}}$ with $a_n \in C([0, T] \times [0, 1])$ bounded by M . Denote by

$$E_n = S_1(a_n) \quad \text{and} \quad (f_n, g_n) = (S_2 \circ S_1)(a_n).$$

By (1.10), $(S(a_n))_{n \in \mathbb{N}}$ is bounded in $C([0, T] \times [0, 1])$ by

$$\int \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} f^0(x, w) dv + \int \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} g^0(x, w) dv.$$

Prove its uniform equicontinuity. Let $\eta > 0$ be given. By (1.10), there is $U > 0$ such that,

$$\int_{|v|>U} \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} f^0(x, w) dv + \int_{|v|>U} \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} g^0(x, w) dv < \frac{\eta}{2}.$$

Let (X_n, V_n) (resp. (Y_n, W_n)) be the generalized characteristics associated to the field $S_1(a_n)$ (resp. $-S_1(a_n)$). The existence of $h_0 > 0$ such that

$$\sup_{(t, x) \in [0, T] \times [0, 1]} \int_{|v|<U} |f^0(X_n(0; t+h, x+k, v), |V_n(0; t+h, x+k, v)|) - f^0(X_n(0; t, x, v), |V_n(0; t, x, v)|)| dv < \frac{\eta}{4},$$

(resp.

$$\sup_{(t, x) \in [0, T] \times [0, 1]} \int_{|v|<U} |g^0(Y_n(0; t+h, x+k, v), |W_n(0; t+h, x+k, v)|) - g^0(Y_n(0; t, x, v), |W_n(0; t, x, v)|)| dv < \frac{\eta}{4},$$

$$n \in \mathbb{N}, \quad |h| + |k| < h_0,$$

follows from the uniform continuity on $[0, T] \times [0, 1] \times [-U, U]$ of the map

$$(t, x, v) \mapsto (X_n(0; t, x, v), |V_n(0; t, x, v)|, Y_n(0; t, x, v), |W_n(0; t, x, v)|),$$

and its continuous dependence with respect to the fields. The Ascoli theorem applies, which ends the proof of the lemma. ■

Lemma 3.3 *S is continuous in $C([0, T] \times [0, 1])$.*

Proof of Lemma 3.3. Let $(a_n)_{n \in \mathbb{N}}$ converging to a in $C([0, T] \times [0, 1])$. Denote by

$$\begin{aligned} E_n &= S_1(a_n), \quad (f_n, g_n) = (S_2 \circ S_1)(a_n), \quad S(a_n) = \int_{\mathbb{R}} (f_n - g_n) dv, \\ E &= S_1(a), \quad (f, g) = (S_2 \circ S_1)(a), \quad S(a) = \int_{\mathbb{R}} (f - g) dv. \end{aligned}$$

The sequence $(E_n)_{n \in \mathbb{N}}$ converges to E in $C([0, T] \times [0, 1])$, because

$$\max_{[0, T] \times [0, 1]} |E_n - E| \leq \max_{[0, T] \times [0, 1]} |a_n - a|.$$

Let $\eta > 0$ be given. By (1.10), there is $U > 0$ such that,

$$\int_{|v| > U} \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} f^0(x, w) dv + \int_{|v| > U} \sup_{x \in [0, 1], |w-v| < (|E_0| + M)T} g^0(x, w) dv < \eta.$$

And so,

$$\begin{aligned} & \int_{|v| > U} (f^0(X_n(0; t, x, v), |V_n(0; t, x, v)|) + g^0(Y_n(0; t, x, v), |W_n(0; t, x, v)|)) dv \\ & + \int_{|v| > U} (f^0(X(0; t, x, v), |V(0; t, x, v)|) + g^0(Y(0; t, x, v), |W(0; t, x, v)|)) dv \leq 4\eta, \quad n \in \mathbb{N}^*. \end{aligned}$$

The convergence of

$$\int_{|v| < U} (f^0(X_n(0; t, x, v), |V_n(0; t, x, v)|) - g^0(Y_n(0; t, x, v), |W_n(0; t, x, v)|)) dv$$

to

$$\int_{|v| < U} (f^0(X(0; t, x, v), |V(0; t, x, v)|) - g^0(Y(0; t, x, v), |W(0; t, x, v)|)) dv$$

in $C([0, T] \times [0, 1])$ when $n \rightarrow +\infty$ follows from the continuous dependence of

$$[0, T] \times [0, 1] \times [-U, U] \ni (t, x, v) \mapsto (X(0; t, x, v), |V(0; t, x, v)|, Y(0; t, x, v), |W(0; t, x, v)|)$$

with respect to the fields. ■

End of the proof of Theorem 1.1. The existence of a solution (f, g, E) to the Cauchy problem (1.1)–(1.8) follows from Lemmas 3.2–3.3 and the Schauder fixed-point theorem for the map S in K . ■

The obtention of quasineutrality from the Vlasov-Poisson system, i.e. the passage to the limit when $\varepsilon \rightarrow 0$ in (1.1)–(1.8) is a difficult problem. The formal limit does not hold for unstable profiles, as proven by D. Han-Kwan and M. Hauray in [HAN2015]. In [HAN2016], D. Han-Kwan and F. Rousset justified the quasineutral limit of a Vlasov-Poisson system with adiabatic electrons for small times in Sobolev spaces, and for initial data satisfying a Penrose stability condition.

In the following property, we prove that the electroneutrality equation (3.6) holds at the limit $\varepsilon \rightarrow 0$.

Property 3.1

For every $\varepsilon > 0$ let $(f_\varepsilon, g_\varepsilon, E_\varepsilon)$ be a solution to the Vlasov-Poisson system (1.1)–(1.8) with initial datum $(f_\varepsilon^0, g_\varepsilon^0, E_{\varepsilon,0})$ satisfying the assumptions of Theorem 1.1, and

$$\int_{[0, 1] \times \mathbb{R}} v^2 (f_\varepsilon^0 + g_\varepsilon^0) dx dv \leq C, \quad \varepsilon > 0, \quad (\text{finite initial kinetic energy}), \quad (3.3)$$

$$\sqrt{\varepsilon} E_{\varepsilon,0} \leq C, \quad \varepsilon > 0, \quad (3.4)$$

$$\int_{\mathbb{R}} f_\varepsilon^0(x, v) dv = \int_{\mathbb{R}} g_\varepsilon^0(x, v) dv, \quad \text{a.a. } (t, x) \in [0, T] \times [0, 1], \quad \varepsilon > 0, \quad (\text{initial electroneutrality}), \quad (3.5)$$

$$\|f_\varepsilon^0\|_{L^2} + \|g_\varepsilon^0\|_{L^2} \leq C, \quad \varepsilon > 0,$$

for some $C > 0$. There exist a subsequence $(f_{\varepsilon_n}, g_{\varepsilon_n})$ of $(f_\varepsilon, g_\varepsilon)$, a subsequence (E_{ε_n}) of (E_ε) , and functions (f, g) such that $(f_{\varepsilon_n}, g_{\varepsilon_n})$, (resp. $(\sqrt{\varepsilon_n} E_{\varepsilon_n})$) weakly converges to (f, g) in $L^2([0, T] \times [0, 1] \times \mathbb{R})$ (resp. $L^2([0, T] \times [0, 1])$). Moreover, $(\int_{\mathbb{R}} f_{\varepsilon_n} dv)$ (resp. $(\int_{\mathbb{R}} g_{\varepsilon_n} dv)$) weakly converges in $L^2([0, T] \times [0, 1])$ to $\int_{\mathbb{R}} f dv$ (resp. $\int_{\mathbb{R}} g dv$), and

$$\int_{\mathbb{R}} f(t, x, v) dv = \int_{\mathbb{R}} g(t, x, v) dv, \quad \text{a.a. } (t, x) \in [0, T] \times [0, 1]. \quad (3.6)$$

Proof of Property 3.1. The energy associated to the Vlasov-Poisson system (1.1)–(1.8) is the sum of the kinetic and potential energies,

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2} \int_{[0,1] \times \mathbb{R}} v^2 (f_\varepsilon + g_\varepsilon)(t, x, v) dx dv + \frac{\varepsilon}{2} \int_{[0,1]} E_\varepsilon(t, x)^2 dx. \quad (3.7)$$

The boundary conditions have been chosen in order to ensure the conservation of the energy. This classically follows from the multiplication by v^2 of the Vlasov equations (1.1)–(1.2), their integration with respect to (x, v) , and the use of the continuity equations,

$$\partial_t \left(\int_{\mathbb{R}} (f_\varepsilon - g_\varepsilon) dv \right) + \partial_x \left(\int_{\mathbb{R}} v (f_\varepsilon - g_\varepsilon) dv \right) = 0.$$

More details can be found in [GIO2019]. And so,

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2} \int_{[0,1] \times \mathbb{R}} v^2 (f_\varepsilon^0 + g_\varepsilon^0)(x, v) dx dv + \frac{\varepsilon}{2} E_{\varepsilon,0}^2, \quad t \in [0, T].$$

The family $(\sqrt{\varepsilon} E_\varepsilon)_{\varepsilon > 0}$ being uniformly bounded in $L^2([0, T] \times [0, 1])$, there is a sequence (ε_n) tending to zero when $n \rightarrow +\infty$ such that $(\sqrt{\varepsilon_n} E_{\varepsilon_n})$ weakly converges in $L^2([0, T] \times [0, 1])$. Hence $(\varepsilon_n E_{\varepsilon_n})$ weakly converges to zero in $L^2([0, T] \times [0, 1])$ when $n \rightarrow +\infty$. The family (f_ε) and (g_ε) being uniformly bounded in $L^2([0, T] \times [0, 1] \times \mathbb{R})$, there is a subsequence of (ε_n) , still denoted by (ε_n) for the sake of simplicity, and functions f and g in $L^2([0, T] \times [0, 1] \times \mathbb{R})$, such that (f_{ε_n}) (resp. (g_{ε_n})) weakly converges in $L^2([0, T] \times [0, 1] \times \mathbb{R})$ to f (resp. g), and

$$\int v^2 (f + g)(t, x, v) dx dv < +\infty.$$

Moreover, $(\int f_{\varepsilon_n}(t, x, v) dv)_{n \in \mathbb{N}}$ (resp. $(\int g_{\varepsilon_n}(t, x, v) dv)_{n \in \mathbb{N}}$) weakly converges in $L^2([0, T] \times [0, 1])$ to $\int f(t, x, v) dv$ (resp. $\int g(t, x, v) dv$). Indeed, for any function $\alpha \in L^2([0, T] \times [0, 1])$,

$$\begin{aligned} & \left| \int \alpha(t, x) \int (f_{\varepsilon_n} - f)(t, x, v) dv dx dt \right| \\ & \leq \frac{\|\alpha\|_\infty}{K^2} \int v^2 (f_{\varepsilon_n} + f)(t, x, v) dv dx dt + \left| \int \alpha(t, x) \chi_K(v) (f_{\varepsilon_n} - f)(t, x, v) dv dx dt \right|, \end{aligned} \quad (3.8)$$

where χ_K denotes the characteristic function of $] -K, K[$. The first term in the r.h.s. of (3.8) tends to zero when $K \rightarrow +\infty$ uniformly with respect to n . The second term in the r.h.s. of (3.8) tends to zero for any fixed K , given that the map $\alpha(t, x) \chi_K(v)$ belongs to $L^2([0, T] \times [0, 1] \times \mathbb{R})$. The passage to the limit when $n \rightarrow +\infty$ in (1.3) leads to the electroneutrality equation (3.6).

4 Proof of the uniqueness Theorem 1.2

This section splits into two lemmas. Under the assumptions of Theorem 1.2 and locally in time, Lemma 4.1 provides a bound on the number of possible bounces at the boundary of the domain of the generalized characteristics associated to a solution to (1.1)–(1.8). Lemma 4.2 proves the local in time uniqueness result of Theorem 1.2.

Lemma 4.1 *Assume that for some $c_0 > 0$ and $V_0 > c_0$,*

$$f^0(x, v) = 0, \quad x \in [0, 1], \quad |v| \leq c_0 \quad \text{or} \quad |v| \geq V_0. \quad (4.1)$$

Let $T \in \left]0, \frac{c_0}{2c_i}\right]$, where

$$c_i = |E_0| + \frac{1}{\varepsilon} \int_0^1 \int (f^0 + g^0)(x, v) dv dx. \quad (4.2)$$

Then the number of possible bounces of the generalized characteristics of a solution to (1.1)–(1.8) at the boundaries $x = 0$ and $x = 1$ of the domain is bounded by

$$\frac{Tc_i}{\sqrt{V_0^2 + 2c_i} - 2V_0}.$$

Proof of Lemma 4.1. By the mass conservations,

$$\int f(t, x, v) dx dv = \int f^0(x, v) dx dv, \quad \int g(t, x, v) dx dv = \int g^0(x, v) dx dv, \quad t \in [0, T].$$

Hence, c_i is a bound from above of $\|E\|_\infty$. Let $x_0 \in [0, 1]$ and $c_0 < |v_0| < V_0$. By (2.5) and (4.2), the velocities $V(s; 0, x_0, v_0)$ along the generalized characteristics starting at $(0, x_0, v_0)$ satisfy

$$\frac{c_0}{2} \leq |V(s; 0, x_0, v_0)| \leq 2V_0, \quad s \in [0, T]. \quad (4.3)$$

Assume a first bounce on the generalized characteristics starting at $(0, x_0, v_0)$ occurs at s_1 on the $x = 1$ boundary. It follows from the positive bound from below of $|V(s; 0, x_0, v_0)|$ in (4.3) that the next possible bounce will occur at $x = 0$. Denote by s_2 the time of such a bounce. Since

$$X(s_2; 0, x_0, v_0) = X(s_2; s_1, 1, V(s_1^+; 0, x_0, v_0)),$$

it holds that

$$1 + V(s_1^+; 0, x_0, v_0)(s_2 - s_1) + \int_{s_1}^{s_2} (s_2 - r)E(r, X(r; 0, x_0, v_0))dr = 0.$$

Taking into account that $|V(s_1^+; 0, x_0, v_0)|$ (resp. $E(r, X(r; 0, x_0, v_0))$) is bounded from above (resp. from below) by $2V_0$ (resp. $-c_i$) implies that

$$c_i(s_2 - s_1)^2 + 2V_0(s_2 - s_1) - 2 \geq 0.$$

The result of the lemma follows. ■

Lemma 4.2

Assume f^0 Lipschitz with respect to the (x, v) variable, and even with respect to the variable v . For $T > 0$ small enough, there is a unique solution to the Cauchy problem (1.1)–(1.8).

Proof of Lemma 4.2.

Let (f, g) and (\tilde{f}, \tilde{g}) be two solutions to the problem. Denote by $(X(\cdot; t, x, v), V(\cdot; t, x, v))$ (resp. $(Y(\cdot; t, x, v), W(\cdot; t, x, v))$, resp. $(\tilde{X}(\cdot; t, x, v), \tilde{V}(\cdot; t, x, v))$, resp. $(\tilde{Y}(\cdot; t, x, v), \tilde{W}(\cdot; t, x, v))$) the generalized characteristics associated to f (resp. g , resp. \tilde{f} , resp. \tilde{g}) starting at (t, x, v) . Let us prove that

$$\begin{aligned} & (X(0; t, x, v), |V(0; t, x, v)|, Y(0; t, x, v), |W(0; t, x, v)|) \\ &= (\tilde{X}(0; t, x, v), |\tilde{V}(0; t, x, v)|, \tilde{Y}(0; t, x, v), |\tilde{W}(0; t, x, v)|), \quad (t, x, v) \in [0, T] \times [0, 1] \times \mathbb{R}. \end{aligned}$$

By Lemma 4.1, the number of bounces of the generalized characteristics starting at $(0, x_0, v_0)$ is uniformly bounded. Let us first consider the case where no bounce (resp. at most one bounce) occurs at $x = 0$ (resp. at $x = 1$). Denote by A_0 (resp. A_2) the set of $(r, y, u) \in [0, T] \times [0, 1] \times \mathbb{R}$ such that the (X, V) and (\tilde{X}, \tilde{V}) characteristics respectively associated to f and \tilde{f} , passing at (r, y, u) , both have no bounce (resp. one bounce) at $x = 1$ on $[0, T]$. Denote by A_1 the set of $(r, y, u) \in [0, T] \times [0, 1] \times \mathbb{R}$ such that the (X, V) (resp. the (\tilde{X}, \tilde{V})) characteristics passing at (r, y, u) has one bounce (resp. no bounce) at

$x = 1$ on $[0, T]$. Denote by $(B_i)_{0 \leq i \leq 2}$ analogous sets relative to the (Y, W) and (\tilde{Y}, \tilde{W}) characteristics respectively associated to g and \tilde{g} . Denote by

$$\begin{aligned} \alpha &= \sup_{(r,y,u) \in A_0 \cup A_2} \left(|(X - \tilde{X})(0; r, y, u)| + |(V - \tilde{V})(0; r, y, u)| \right) \\ &+ \sup_{(r,y,u) \in A_1} \left(|(X - \tilde{X})(0; r, y, u)| + |(V + \tilde{V})(0; r, y, u)| \right) \\ &+ \sup_{(r,y,u) \in B_0 \cup B_2} \left(|(Y - \tilde{Y})(0; r, y, u)| + |(W - \tilde{W})(0; r, y, u)| \right) \\ &+ \sup_{(r,y,u) \in B_1} \left(|(Y - \tilde{Y})(0; r, y, u)| + |(W + \tilde{W})(0; r, y, u)| \right). \end{aligned}$$

Using the Lipschitz assumption on f_0 and g_0 , and their evenness when considering A_1 and B_1 , notice that

$$\begin{aligned} |E - \tilde{E}|(r, z) &= \left| \int_0^z \int \left[f^0(X(0; r, y, u), V(0; r, y, u)) - f^0(\tilde{X}(0; r, y, u), \tilde{V}(0; r, y, u)) \right] dudy \right. \\ &\quad \left. - \int_0^z \int \left[g^0(Y(0; r, y, u), W(0; r, y, u)) - g^0(\tilde{Y}(0; r, y, u), \tilde{W}(0; r, y, u)) \right] dudy \right| \\ &\leq c\alpha, \quad (r, z) \in [0, T] \times [0, 1]. \end{aligned}$$

Let $(t, x, v) \in A_0$. Both generalized characteristics (X, V) and (\tilde{X}, \tilde{V}) without backward bounce, starting at (t, x, v) , are given by

$$\begin{aligned} X(s; t, x, v) &= x + v(s - t) + \int_s^t (r - s)E(r, X(r; t, x, v))dr, \quad V(s; t, x, v) = v - \int_s^t E(r, X(r; t, x, v))dr, \\ \tilde{X}(s; t, x, v) &= x + v(s - t) + \int_s^t (r - s)\tilde{E}(r, \tilde{X}(r; t, x, v))dr, \quad \tilde{V}(s; t, x, v) = v - \int_s^t \tilde{E}(r, \tilde{X}(r; t, x, v))dr. \end{aligned}$$

Hence

$$\begin{aligned} & \left(|X - \tilde{X}| + |V - \tilde{V}| \right) (s; t, x, v) \\ & \leq c \left(\int_s^t |E - \tilde{E}|(r, \tilde{X}(r; t, x, v))dr + \int_s^t \left| E(r, X(r; t, x, v)) - E(r, \tilde{X}(r; t, x, v)) \right| dr \right) \\ & \leq c \left(\int_s^t |E - \tilde{E}|(r, \tilde{X}(r; t, x, v))dr + \int_s^t |X - \tilde{X}|(r; t, x, v)dr \right) \\ & \leq c \left(T\alpha + \int_s^t |X - \tilde{X}|(r; t, x, v)dr \right). \end{aligned}$$

Consequently,

$$\left(|X - \tilde{X}| + |V - \tilde{V}| \right) (s; t, x, v) \leq cT\alpha, \quad s \in [0, T], \quad (t, x, v) \in A_0.$$

In particular,

$$\left(|X - \tilde{X}| + |V - \tilde{V}| \right) (0; t, x, v) \leq cT\alpha, \quad (t, x, v) \in A_0. \quad (4.4)$$

Let $(t, x, v) \in A_2$. Both generalized characteristics with a backward bounce at $(t_1(t, x, v), 1)$, starting at (t, x, v) , are given by

$$\begin{aligned} X(s; t, x, v) &= x + v(s - t) + \int_s^t (r - s)E(r, X(r; t, x, v))dr, \\ V(s; t, x, v) &= v - \int_s^t E(r, X(r; t, x, v))dr, \quad s \in [t_1(t, x, v), t], \\ X(s; t, x, v) &= 1 + (t_1 - s) \left(v - \int_{t_1}^t E(r, X(r; t, x, v))dr \right) + \int_s^{t_1} (r - s)E(r, X(r; t, x, v))dr, \\ V(s; t, x, v) &= -v + \int_{t_1}^t E(r, X(r; t, x, v))dr - \int_s^{t_1} E(r, X(r; t, x, v))dr, \quad s \in [0, t_1(t, x, v)], \end{aligned}$$

and similar equations for (\tilde{X}, \tilde{V}) with a backward bounce at $(\tilde{t}_1(t, x, v), 1)$. Assume $t_1(t, x, v) \leq \tilde{t}_1(t, x, v)$. It holds that

$$\begin{aligned}(X - \tilde{X})(s) &= \int_s^t (r - s) \left[E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right] dr, \\ (V - \tilde{V})(s) &= \int_s^t \left[E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right] dr, \quad s \in [\tilde{t}_1(t, x, v), t].\end{aligned}$$

Hence,

$$\left(|X - \tilde{X}| + |V - \tilde{V}| \right)(s; t, x, v) \leq cT\alpha, \quad s \in [\tilde{t}_1(t, x, v), t], \quad (t, x, v) \in A_2, \quad (4.5)$$

as in the A_0 case. Moreover,

$$\begin{aligned}(X - \tilde{X})(s) &= 2(\tilde{t}_1 - s) \left(-v + \int_{\tilde{t}_1}^t \tilde{E}(r, \tilde{X}(r)) dr \right) + \int_s^{\tilde{t}_1} (r - s) (E(r, X(r)) - \tilde{E}(r, \tilde{X}(r))) dr \\ &\quad + \int_{\tilde{t}_1}^t (r - s) (E(r, X(r)) - \tilde{E}(r, \tilde{X}(r))) dr, \quad s \in [t_1(t, x, v), \tilde{t}_1(t, x, v)].\end{aligned} \quad (4.6)$$

The distance from t_1 to \tilde{t}_1 can be controlled in the following way. The definition of $(t_1(t, x, v), V(t_1^+))$,

$$x - 1 + v(t_1 - t) + \int_{t_1}^t (r - t_1) E(r, X(r)) dr = 0, \quad v - \int_{t_1}^t E(r, X(r)) dr = V(t_1^+),$$

implies that

$$V(t_1^-)(\tilde{t}_1 - t_1) = \int_{\tilde{t}_1}^{t_1} (r - \tilde{t}_1) E(r, X(r)) dr + \int_{\tilde{t}_1}^t (r - \tilde{t}_1) \left(\tilde{E}(r, \tilde{X}(r)) - E(r, X(r)) \right) dr.$$

Hence, for T small enough,

$$\begin{aligned}0 \leq \tilde{t}_1 - t_1 &\leq cT \left(\sup_{r \in [\tilde{t}_1, t]} |X - \tilde{X}|(r) + \alpha \right) \\ &\leq cT\alpha,\end{aligned} \quad (4.7)$$

by (4.5). Consequently,

$$|X - \tilde{X}|(s) \leq c|\tilde{t}_1 - t_1| \leq cT\alpha, \quad s \in [t_1(t, x, v), \tilde{t}_1(t, x, v)]. \quad (4.8)$$

Finally,

$$\begin{aligned}(X - \tilde{X})(s) &= v(t_1 - \tilde{t}_1) - (t_1 - s) \int_{t_1}^t E(r, X(r)) dr + (\tilde{t}_1 - s) \int_{\tilde{t}_1}^t \tilde{E}(r, \tilde{X}(r)) dr \\ &\quad + \int_s^{t_1} (r - s) E(r, X(r)) dr - \int_s^{\tilde{t}_1} (r - s) \tilde{E}(r, \tilde{X}(r)) dr \\ &= (\tilde{t}_1 - t_1) \left(-v + \int_{t_1}^t E(r, X(r)) dr \right) - (\tilde{t}_1 - s) \int_{\tilde{t}_1}^t \left(E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right) dr \\ &\quad - (\tilde{t}_1 - s) \int_{t_1}^{\tilde{t}_1} E(r, X(r)) dr + \int_s^{\tilde{t}_1} (r - s) \left(E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right) dr \\ &\quad + \int_{\tilde{t}_1}^{t_1} (r - s) E(r, X(r)) dr, \\ (V - \tilde{V})(s) &= 2 \int_{t_1}^{\tilde{t}_1} E(r, X(r)) dr + \int_{\tilde{t}_1}^t \left(E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right) dr \\ &\quad - \int_s^{\tilde{t}_1} \left(E(r, X(r)) - \tilde{E}(r, \tilde{X}(r)) \right) dr, \quad s \in [0, t_1].\end{aligned}$$

Hence,

$$\begin{aligned}
|X - \tilde{X}|(s) &\leq c|\tilde{t}_1 - t_1| + cT\alpha + cT \sup_{r \in [t_1, t]} |X - \tilde{X}|(r) + c \int_s^{t_1} |X - \tilde{X}|(r) dr \\
&\leq cT\alpha + c \int_s^{t_1} |X - \tilde{X}|(r) dr, \quad \text{by (4.7), (4.5) and (4.8),} \\
&\leq cT\alpha, \quad s \in [0, t_1[, \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
|V - \tilde{V}|(s) &\leq c|\tilde{t}_1 - t_1| + cT\alpha + cT \sup_{r \in [0, t]} |X - \tilde{X}|(r) \\
&\leq cT\alpha, \quad s \in [0, t_1[, \quad \text{by (4.5), (4.8) and (4.9).} \tag{4.10}
\end{aligned}$$

It follows from (4.9)–(4.10) taken at $s = 0$ that

$$(|X - \tilde{X}| + |V - \tilde{V}|)(0; t, x, v) \leq cT\alpha, \quad (t, x, v) \in A_2. \tag{4.11}$$

Let us consider the last case, where $(t, x, v) \in A_1$. Again,

$$(|X - \tilde{X}| + |V - \tilde{V}|)(s; t, x, v) \leq cT\alpha, \quad s \in [t_1(t, x, v), t]. \tag{4.12}$$

Moreover,

$$\begin{aligned}
X(s) &= X(t_1) + (t_1 - s)V(t_1^+) + \int_s^{t_1} (r - s)E(r, X(r))dr, \\
V(s) &= -V(t_1^+) - \int_s^{t_1} E(r, X(r))dr, \\
\tilde{X}(s) &= \tilde{X}(t_1) - (t_1 - s)\tilde{V}(t_1) + \int_s^{t_1} (r - s)\tilde{E}(r, \tilde{X}(r))dr, \\
\tilde{V}(s) &= \tilde{V}(t_1) - \int_s^{t_1} \tilde{E}(r, \tilde{X}(r))dr, \quad s \in [0, t_1(t, x, v)[.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(|X - \tilde{X}| + |V + \tilde{V}|)(0; t, x, v) &\leq (|X - \tilde{X}| + |V - \tilde{V}|)(t_1^+(t, x, v); t, x, v) + ct_1 \\
&\leq cT\alpha, \quad (t, x, v) \in A_1, \tag{4.13}
\end{aligned}$$

since, using that \tilde{X} is non-increasing on $[0, t]$ because the sign of \tilde{V} cannot change,

$$0 \leq \tilde{X}(0) - \tilde{X}(t_1) \leq 1 - \tilde{X}(t_1) = (X - \tilde{X})(t_1) \leq cT\alpha.$$

Here, the last inequality results from (4.12). Moreover, $\tilde{X}(0) - \tilde{X}(t_1) = t_1|\tilde{V}(\tau)|$ for some $\tau \in [0, t_1]$. This implies that

$$t_1 \leq cT\alpha.$$

It follows from (4.4), (4.11) and (4.13) that

$$\sup_{(t, x, v) \in A_0 \cup A_2} (|X - \tilde{X}| + |V - \tilde{V}|)(0; t, x, v) + \sup_{(t, x, v) \in A_1} (|X - \tilde{X}| + |V + \tilde{V}|)(0; t, x, v) \leq cT\alpha.$$

It similarly holds that

$$\sup_{(t, x, v) \in B_0 \cup B_2} (|Y - \tilde{Y}| + |W - \tilde{W}|)(0; t, x, v) + \sup_{(t, x, v) \in B_1} (|Y - \tilde{Y}| + |W + \tilde{W}|)(0; t, x, v) \leq cT\alpha.$$

The case where more than zero (resp. one) bounce occurs at $x = 0$ (resp. at $x = 1$) can be analogously treated by splitting the characteristics between those bouncing an even (resp. odd) number of times. And so, $\alpha \leq cT\alpha$. Hence $\alpha = 0$ for T small enough. Consequently,

$$\begin{aligned}
(X, Y)(0; t, x, v) &= (\tilde{X}, \tilde{Y})(0; t, x, v), \\
(V, W)(0; t, x, v) &= \pm(\tilde{V}, \tilde{W})(0; t, x, v), \quad (t, x) \in [0, T] \times [0, 1], \quad \frac{c_0}{2} \leq |v| \leq 2V_0,
\end{aligned}$$

so that $(f, g) = (\tilde{f}, \tilde{g})$. ■

Acknowledgements

The project leading to this publication has received funding from the Excellence Initiative of Aix-Marseille University - A*MIDEX, a French 'Investissements d'Avenir' program.

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