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USE OF FORMALISM IN MATHEMATICAL ACTIVITY

CASE STUDY: THE CONCEPT OF CONTINUITY IN HIGHER EDUCATION

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In this paper, we consider logical-mathematical formalism in mathematical statements. We examine formalism regarding the notion of continuity in higher education. The choice of this concept is based on the fact that this concept involves a large number of related variables and that its logical structure nevertheless not too complex because all quantifiers are at the top of the form (Chellougui, 2009).

First we present the didactic transposition of the continuity from knowledge learned to knowledge taught. In a second step, we consider the definition of continuity as presented in various mathematics textbooks for first year university science students.

INTRODUCTION

In Tunisian secondary mathematics education, according to official instructions, logical symbols (logical connectors: \Rightarrow , \Leftrightarrow , ..., and quantifiers: \forall , \exists) are not introduced, and mathematical statements (theorems, definitions) are generally expressed in natural language (Chellougui, 2003). However, from the beginning of the first year of university, scientific formalized statements are used without a specific introduction to symbolism or to the relationships between statements in natural language and formalized statements. This widespread use of formalized language at university is motivated by the supposed superiority in terms of operating statements fully or partially formalized. However, for many students, formalism seems to be an obstacle to mathematical work and therefore to conceptualization (Quine 1970). Thus, in mathematical activity in the first year of university, we identify problems of interpretation of logical-mathematical vocabulary or gap of operating order to students, specifically difficulties in manipulation of complex statements with multiple quantifications. Generally, these issues are not addressed in common textbooks. We adopt the assumption that they reflect ordinary mathematical practice of mathematics teachers (Durand-Guerrier, 2003).

To illustrate this, we chose the concept of continuity of functions which is studied in high school and again in college. For this concept, an understanding of how a definition in natural language can be expressed in form of a formal definition is required. Consider, for example, the definition of continuity of a function at any point proposed by Schwartz (1991):

(For all $a \in \mathbb{R}$) (for all $\varepsilon > 0$) (there exist $\eta > 0$ such that) (for all $x \in \mathbb{R}$ such that $|x - a| \leq \eta$), we have: $|f(x) - f(a)| \leq \varepsilon$. This sentence can be written formally. [...]:
 $(\forall a \in \mathbb{R}) (\forall \varepsilon > 0) (\exists \eta > 0) (\forall x \in \mathbb{R}) [|x - a| \leq \eta \Rightarrow |f(x) - f(a)| \leq \varepsilon]$. (p.20)

Here, there are three universal quantifiers and an existential quantifier in third position. The scope of these quantifiers on implication is between parentheses. This study is concerned with the definition of the term as found in several mathematical textbooks of first year scientific university. Prior to this study, we present some elements of didactic transposition of the concept of continuity from expert knowledge to knowledge to be taught.

SOME ELEMENTS OF DIDACTIC TRANSPOSITION

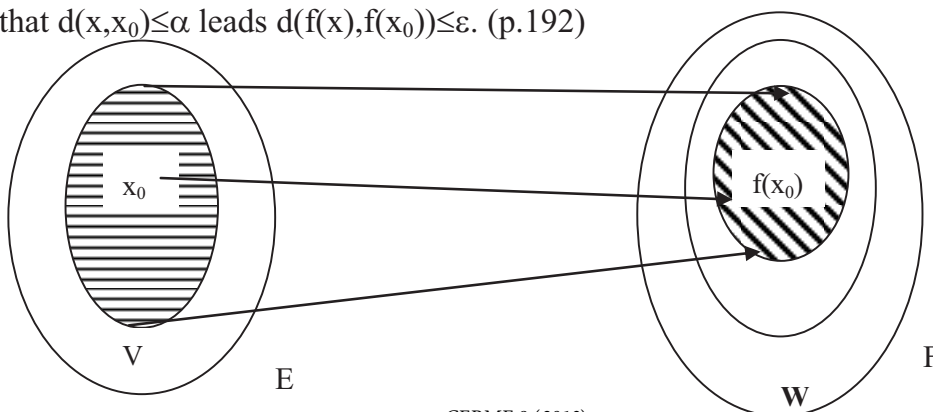
The first comprehensive outline of the Didactic Transposition Theory was developed in Chevallard (1991). This theory aims to produce a scientific analysis of didactic systems and is based on the assumption that the mathematical knowledge set up as a teaching object ('savoir enseigné'), in an institutionalized educational system, normally has a preexistence, which is called "expert knowledge" ('savoir savant').

Some objects of mathematical expert knowledge are defined as direct teaching objects and constructed in the didactic system (by definition or construction), i.e. mathematical notions, such as for example addition, the circle, or second order differential equations with constant coefficients. However, there are other knowledge objects, termed para-mathematical notions, useful in mathematical activities but often not set up as teaching objects per se but pre-constructed, such as the notions of parameter, equation, or proof (Klisinska, 2009).

We try to analyze the question of the use of para-mathematical logical symbolism in mathematical activity.

We start with the general definition of continuity of a function at a point made in a Dictionary of Mathematics (Bouvier and al., 1979):

Application continuous at a point. – An application f of a topological space E into a topological space F is continuous at $x_0 \in E$ if for all neighbourhoods W of $f(x_0)$ in F , there exists neighbourhood V of x_0 in E whose image $f(V)$ is contained in W . This is the mathematical expression of the sentence "f(x) tends to f(x₀) as x tends to x₀". In the case where E and F are metric spaces, f is continuous at $x_0 \in E$ if any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d(x, x_0) \leq \alpha$ leads $d(f(x), f(x_0)) \leq \varepsilon$. (p.192)



(p.192)

This definition is given in topological spaces and neighbourhoods; it is then translated into metric spaces. It is formulated in a mixed language, an association of natural language and mathematical symbols. The authors do not use any logical symbol but illustrate this definition by charts. They were inspired by the approach of Bourbaki (1971), which itself offers the following definition:

Definition 1. - We say that f of a topological space X into a topological space X' is continuous at a point $x_0 \in X$ if any neighbourhood V' of $f(x_0)$ in X' , there exists a neighbourhood V of x_0 in X such that $x \in V$ implies $f(x) \in V'$. (P.I.8)

In comparison, Durand-Guerrier and Arsac (2003) use logical symbolism and even suggest a fully formal definition to define a uniformly continuous application:

Application uniformly continuous. – An application f of a metric space E with values in a metric space F is uniformly continuous if for all $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $d(x, y) \leq \eta(\varepsilon)$ implies $d(f(x), f(y)) \leq \varepsilon$. This can be symbolized by:

$$(\forall \varepsilon > 0) (\exists \eta(\varepsilon) > 0) (\forall (x, y) \in E^2) (d(x, y) \leq \eta(\varepsilon) \Rightarrow d(f(x), f(y)) \leq \varepsilon). \text{ (p.193)}$$

In this definition the quantification is complete and dependency relationship between η and ε appears.

To give a definition of continuity in \mathbb{R} , it is possible to get a definition equivalent with intervals since the intervals form a basis of neighbourhood in set \mathbb{R} .

Haug (2000) suggests the following definition:

Definition 6.a Let E be a set of real numbers. Let f be an application of E into \mathbb{R} , a and b are real numbers.

We say that b is a *limit* of f if: any open interval J centred on b , there exists an open interval I centred on a such that $f(E \cap I) \subset J$. (p.107)

He then notes:

Show that if we replace the above equation by the following equation we obtain an equivalent definition.

$$\forall_{\mathbb{R}^*} \varepsilon, \exists_{\mathbb{R}^*} \eta, \forall_E t, |a - t| \leq \eta \Rightarrow |b - f(t)| \leq \varepsilon. \text{ (p.107)}$$

Here, we note that there are no parentheses to express the scope of quantifiers on implication, which is a fairly common practice among authors of textbooks and mathematicians. Some students may not be aware of the difficulties related scope of quantifiers and use of parentheses. It is indeed important to understand the effects on the meaning of a statement and interpret in a mathematical theory, when changing the order of quantifiers (Dubinsky & Yiparaki, 2000).

Further, we read:

Definition 6.b– Let E be a set of real, let a be an element of E , let f an application from E to \mathbb{R} .

We say that f is *continuous* at a if f has a limit at a . (p.111)

The definition of a limit of a function is followed by a geometric representation based primarily on intervals of \mathbb{R} , which can enlighten the definition; there is also an explanation of the passage from the “neighbourhood” point of view to the “distance” point of view, which is not very common in other textbooks studied. Use of intervals favours the didactic transposition of the definition with neighbourhoods; it reduces the number of quantifiers and may in some cases be easier to handle (Chellougui, 2009).

STUDY OF SOME TEXTBOOKS: CONCEPT OF CONTINUITY IN THE KNOWLEDGE TO BE TAUGHT

Below we present a study of certain textbooks for students in their first year of university. We chose these textbooks because they were used by students and teachers of the Faculty of Sciences of Bizerte. The textbooks we have examined are: Chambadal and Ovaert, *Mathematics*, 1966; Arnaudies and Fraysse, 1988; Schwartz, 1991; Guégand and Gavini, 1995.

Our focus is on the different types of language used. Three phenomena emerged from this study which will be analyzed below.

Implication versus bounded quantification

In a mathematics textbook (Chambadal & Ovaert, 1966), we find the definition of a limit of a function at a point followed by that of continuity.

Definition 19. - Limit of a function at a point. - Let f be a function defined on a part A of $\overline{\mathbb{R}}$ and x_0 an accumulation point of A . We say that f has a limit at x_0 if it has a limit when x tends to x_0 remaining in $P=A-\{x_0\}$. We also say more briefly that f has a limit when x tends to x_0 . (pp.394-395)¹

This definition, first given in natural language, is then made more explicit and formalized:

-If x_0 is finite, so that f tends towards l when x tends to x_0 , it is necessary and sufficient that:

$$\forall \varepsilon \in \mathbb{R}_+^*, \exists \eta \in \mathbb{R}_+^* : \forall x \in A \cap ([x_0 - \eta, x_0 + \eta] - \{x_0\}), |f(x) - l| \leq \varepsilon \quad (1)^2$$

what writes:

$$\forall \varepsilon \in \mathbb{R}_+^*, \exists \eta \in \mathbb{R}_+^* : \forall x \in A, |x - x_0| \leq \eta \text{ and } x \neq x_0 \Rightarrow |f(x) - l| \leq \varepsilon. \quad (2) \text{ (pp.395-396)}$$

A beginning reader might ask where the implication that appears in (2) comes.

In another analysis textbook of first year science (Guégand & Gavini, 1995), we can learn:

2.1 Definition: Let I be an interval of \mathbb{R} , $a \in I$ and $f : I \rightarrow \mathbb{R}$

¹ Our translation

² To facilitate the study of different official statements, we have numbered (1) to (12) in the illustrations taken from different books or in our own analysis

We say that f is **continuous at a** if and only if f admits a limit in a equal to $f(a)$.

Otherwise we say that f is discontinuous at a .

Let us clarify this definition:

f is continuous at a

$$\Leftrightarrow \forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I, |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon \quad (3)$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I, |x - a| \leq \alpha \Rightarrow |f(x) - f(a)| \leq \varepsilon \quad (4)$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I \cap [a - \alpha, a + \alpha], |f(x) - f(a)| \leq \varepsilon. \quad (5) \text{ (p.108)}^3$$

Here the beginning reader might wonder why the implication has disappeared in (5). This game appearance/disappearance of implication is related to mathematical practice of bounded quantification. This type of quantification is present in mathematics, but absent in the predicate calculus. For example, the mathematical writing:

$\forall x \in A F(x)$ is reflected in the predicate calculus by: $\forall x(x \in A \Rightarrow F(x))$.

In fact bounded quantification hides the implication the domain of quantification is limited to the elements that satisfy the antecedent of the conditional statement, which removes the implication. The practice of bounded quantification is present in textbooks, several authors provide a formulation of the definition of continuity without the conditional, often without providing explanations allowing students to be able to restore the conditional by changing the domain of quantification (e.g. Chambadal & Ovaert 1966, Guégand and Gavini 1995). Durand-Guerrier (2003) considers that being able to restore or remove correctly the conditional according with the domain of quantification in such cases contribute to the understanding of implication.

Implication versus conjunction

In another textbook of mathematics (Arnaudies & Fraysse, 1988), the authors begin defining continuity at a point with the neighbourhoods in a metric space:

Definition III.4.1 - Let A be a part of \mathbb{R} and $f: A \rightarrow \mathbb{R}$ a function. We say that f is continuous at $a \in A$ if and only if for every neighbourhood W of $f(a)$ there exists a neighbourhood V of a such that $f(V \cap A) \subset W$.

We say that f is discontinuous at point $a \in A$ if and only if it is not continuous at that point.

The function f is continuous if and only if it is continuous at every point of A . (p.108)⁴

After they present the classic definition using a mixed language:

[...] we obtain in particular the following definitions of continuity of f at a equivalent to the definition III.4.1:

(I) For all real $\varepsilon > 0$, there exists a real $\eta > 0$ such that

$$(x \in A \text{ and } |x - a| \leq \eta) \Rightarrow (|f(x) - f(a)| \leq \varepsilon)$$

(II) For all real $\varepsilon > 0$, there exists a real $\eta > 0$ such that

$$(x \in A \text{ and } |x - a| < \eta) \Rightarrow (|f(x) - f(a)| < \varepsilon). \text{ (p.108)}^5$$

³ Our translation

⁴ Our translation

We can ask: why is there "and" in the antecedent of the implication?

We rather expect the following entry:

For all real $\varepsilon > 0$ there exists a real $\eta > 0$ such that for all $x \in A$ $(|x-a| < \eta) \Rightarrow (|f(x)-f(a)| < \varepsilon)$

or:

$$\forall \varepsilon > 0 \exists \eta > 0 \forall x \in A (|x-a| < \eta) \Rightarrow (|f(x)-f(a)| < \varepsilon) \quad (6)$$

Another formulation of the statement (II) presented in the textbook, using logical symbols for the quantification of each variable ε and η gives:

$$\forall \varepsilon > 0 \exists \eta > 0 (x \in A \text{ and } |x-a| < \eta) \Rightarrow (|f(x)-f(a)| < \varepsilon) \quad (7)$$

Are statements (6) and (7) equivalent?

As noted above, removing the bounded quantification on the variable x to the statement (6), we obtain:

$$\forall \varepsilon > 0 \exists \eta > 0 \forall x [x \in A \Rightarrow (|x-a| < \eta \Rightarrow |f(x)-f(a)| < \varepsilon)].$$

Our question refers back to logical equivalence between:

$$(x \in A \wedge |x-a| < \eta) \Rightarrow (|f(x)-f(a)| < \varepsilon) \quad (8)$$

$$\text{and } [x \in A \Rightarrow (|x-a| < \eta \Rightarrow |f(x)-f(a)| < \varepsilon)] \quad (9)$$

Considering only variable x , statements (8) and (9) are respectively of the form:

$$(p(x) \wedge q(x)) \Rightarrow r(x) \quad \text{and} \quad [p(x) \Rightarrow (q(x) \Rightarrow r(x))].$$

It is known that in the propositional calculus, the following equivalence:

$$[p \Rightarrow (q \Rightarrow r)] \Leftrightarrow [(p \wedge q) \Rightarrow r] \quad \text{is a tautology.}$$

By extension, in predicate calculus, the two following equivalences are universally valid:

$$[p(x) \Rightarrow (q(x) \Rightarrow r(x))] \Leftrightarrow [(p(x) \wedge q(x)) \Rightarrow r(x)]$$

$$\forall x [p(x) \Rightarrow (q(x) \Rightarrow r(x))] \Leftrightarrow [(p(x) \wedge q(x)) \Rightarrow r(x)].$$

So using logical arguments, we prove the equivalence between the two statements: (6) and (7). We summarize this equivalence in the following table with a justification in logical syntax by translating writing mathematics into predicate calculus (Kouki, 2008):

Writing mathematical	Predicates calculus
$\forall x \in A (F(x) \Rightarrow G(x))$	$\forall x [x \in A \Rightarrow (F(x) \Rightarrow G(x))] \equiv$
	$\forall x [(x \in A \wedge F(x)) \Rightarrow G(x)]$
$\forall x [(x \in A \text{ and } F(x)) \Rightarrow G(x)]$	

⁵ Out translation

Negation of formalized statements

a- It is generally recognized that formal logic is to facilitate the transition to negation (e.g. Guégand and Gavini 1995).

Those quantified expressions can be easily denied. So to formulate that f does not tend to L (real) at a (real), we have (negation of the definition)

$$\exists \varepsilon > 0, \forall \eta > 0, \exists x \in U, |x-a| < \eta \quad |f(x)-L| \geq \varepsilon \quad (10) \text{ (p.108)}^6$$

In the textbook *those quantified expressions* mean:

$$(1') \quad \forall \varepsilon > 0, \exists \eta > 0, \forall x \in U, |x-a| < \eta \Rightarrow |f(x)-L| < \varepsilon$$

$$(2') \quad \forall \varepsilon > 0, \exists \eta > 0, \forall x \in U, |x-a| \leq \eta \Rightarrow |f(x)-L| \leq \varepsilon$$

$$(3') \quad \forall \varepsilon > 0, \exists \alpha > 0, \forall x \in U \cap [a-\alpha, a+\alpha], |f(x)-L| \leq \varepsilon. \text{ (p.107)}$$

In statement (10), we find a blank between the two inequalities:

$(|x-a| < \eta)$ and $(|f(x)-L| \geq \varepsilon)$. How to fill this blank? Is it the negation of (1'), (2') or (3')?

The negation of (1') is: $\exists \varepsilon > 0, \forall \eta > 0, \exists x \in U, |x-a| < \eta \wedge |f(x)-L| \geq \varepsilon$

The negation of (2') is: $\exists \varepsilon > 0, \forall \eta > 0, \exists x \in U, |x-a| \leq \eta \wedge |f(x)-L| > \varepsilon$

The negation of (3') is: $\exists \varepsilon > 0, \forall \eta > 0, \exists x \in U \cap [a-\alpha, a+\alpha] |f(x)-L| > \varepsilon$

Expression (10) doesn't correspond to any of the previous negations. We hypothesize that the authors aimed to negate statement (1) and did not want to use logical symbolism for the conjunction "and". If they did not put the word "and" in this blank to keep all words in formal language and to avoid using a mixed language. This is based on the fact that in mathematics we very rarely use the logical symbol " \wedge " which represents conjunction. This is reflected in the textbooks studied. Indeed, the logical symbol of the conjunction is identified just once among these textbooks: in the first paragraph of the part entitled: *Set Theory*, of Laurent Schwartz (1991). The author, in this section, uses the definition of continuity of a real function to illustrate the rules for handling negation: inversion of two types of quantification, negation of implication. For example:

For express now that the function is continuous at every point, we write:

[...]

$$"(\forall a \in \mathbb{R}) (\forall \varepsilon > 0) (\exists \eta > 0) (\forall x \in \mathbb{R}) [(|x-a| \leq \eta) \Rightarrow |f(x)-f(a)| \leq \varepsilon] \text{ (p.20)}$$

Further, we read:

For example, the property for a function f of a real variable not is everywhere continuous, that is to say to be discontinuous at least one point is expressed by the single line:

$$(\exists a \in \mathbb{R}) (\exists \varepsilon > 0) (\forall \eta > 0) (\exists x \in \mathbb{R}) [(|x-a| \leq \eta) \wedge (|f(x)-f(a)| > \varepsilon)] \text{ (p.21)}$$

This second statement is obtained by recursively applying transformation rules:

$$\neg(\forall x Fx) \equiv \exists x \neg Fx \quad (11)$$

$$\neg(\exists x Fx) \equiv \forall x \neg Fx \quad (12)$$

⁶ Our translation

These rules allow change progressively the quantifiers and finally to focus on the negation of the open sentence into brackets. The application of the general rule: $\neg(p(x)\Rightarrow q(x)) \equiv p(x)\wedge\neg q(x)$ to this open sentence provides the conjunction of an atomic formula and of the negation of another atomic formula. So, finally we need to focus only on the negation of the atomic formula: $|f(x)-f(a)|\leq\varepsilon$. Then, the logical negation symbol disappears by the equivalence between the relation ">" and the negation of the relation " \leq ".

Note that these transformation process of quantifiers only apply if they are heading the formula. Indeed, a quantifier in the antecedent of an implication is not modified by the negation:

$$\neg[(\forall x Fx) \Rightarrow G] \equiv (\forall x Fx) \wedge (\neg G)$$

b-In another textbook (Arnaudies and Fraysse (1988) mentioned above), following the definition of continuity, the authors define the discontinuity noting:

The discontinuity of f at a point $a\in A$ means:

(III) There exist $\varepsilon>0$ such that for all $\eta>0$, $(x\in A \text{ and } |x-a|\leq\eta) \not\Rightarrow (|f(x)-f(a)|\leq\varepsilon)$

i.e

(IV) There exists $\varepsilon>0$ such that for all $\eta>0$ we can find at least one x in A such that $|x-a|\leq\eta$ and $|f(x)-f(a)|>\varepsilon$. (p.109)⁷

In statement (III), the authors use the symbol $\not\Rightarrow$ that does not conform to the syntax of logic. It is \Rightarrow . It leads to the question: what is negated? Especially since the universal quantification remains implicit: $[(x\in A \text{ and } |x-a|\leq\eta) \Rightarrow (|f(x)-f(a)|\leq\varepsilon)]$

The authors answer in statement (IV) using a given vocabulary in a language of action, where we would expect more usage of the logic symbol of the existential quantifier. This shows that what is negated is of course the implicitly universally quantified statement. This point could not be obvious for some students; indeed, some beginners could consider the following statement: $\forall x (x\in A \text{ and } |x-a|\leq\eta) \Rightarrow (|f(x)-f(a)|\leq\varepsilon)$ which is interpreted by: "None x satisfies the implication", which is not the negation of the definition.

The following negation: $\neg [(x\in A \wedge |x-a|\leq\eta) \Rightarrow (|f(x)-f(a)|\leq\varepsilon)]$ does not give rise to the appearance of an existential quantifier.

The negation is obviously on the universal $\neg[(\forall x (x\in A \wedge |x-a|\leq\eta) \Rightarrow (|f(x)-f(a)|\leq\varepsilon))]$

CONCLUSION

Logical-mathematical formalizations in definitions of continuity, limits and discontinuity are different from one textbook to another. One might think that they reflect the everyday practices of mathematicians. Anyway, the authors of the textbooks

⁷ Our translation

we have studied do not provide to students means to overcome the linguistic difficulties raised by the use of formalism, in particular concerning its relationships with natural language, so that it seems that there is an “illusion of transparency of mathematical language”.

In some expressions, the presence of bounded quantification is indicated; practice creates a phenomenon of appearance/disappearance of involvement and quantification product entries do not conform to the syntax and logic that generates ambiguity, then the transition formalism is supposed ambiguities of ordinary language (Kouki, 2006). In addition, the use of automatic syntactic rules, not problematized, to construct recursively the negation of a sentence obscures many fundamental questions for operative use of formalism (Durand-Guerrier and al., 2012).

The introduction of logical-mathematical formalism in the learned knowledge aims to introduce a certain level of mathematical rigor in mathematical discourse in order to get rid of ambiguities, implicit assumptions and call to evidence. In the knowledge to be taught, the study showed a wide variety of formulations as well in formal language as in natural or mixed language, for which we have identified and analyzed syntactic difficulties, which are likely to affect student work.

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