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# Analytical theory of motion and new ephemeris of Triton from observations

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## ABSTRACT

Modelling the motion of Triton, the main satellite of Neptune, is specific. Earlier researchers built Triton's ephemeris by numerical integration of the equations of its motion. However, these ephemeris can be accessed only by using online ephemeris server or by borrowing a special calculating program and huge data file from authors of the ephemeris. In addition, the interval of the earlier ephemeris is limited. In this paper, simple and easily programmable formulae are presented for computing Triton's ephemeris for any instant of time. The formulae are based on a new analytical theory of Triton's motion all necessary perturbing factors being taken into consideration. The parameters of the theory are fit to all published observations made from 1847 to 2012 (10 254 observations in total). After the parameters were fit to observations, the root-mean-square residuals were 0.228 arcsec, the weighted average residual being 0.036 arcsec. The new ephemeris of Triton slightly differs from those produced by other authors because of differences in the sets of used observations. The new ephemeris of Triton are put on our online ephemeris server. It is shown that the available observations do not allow to determine reliably the quadratic term in the orbital longitude of Triton. Such a term would be an additional indicator of the accuracy of the theory and observations.

**Key words:** astrometry – ephemerides – planets and satellites: individual: Neptune satellites.

## 1 INTRODUCTION

Theories of motion of the planets and their satellites are necessary for generating the ephemeris of these bodies and carrying out space missions to them. In addition, the models of satellite motions based on observations enable to define satellite orbital deceleration or acceleration caused by the tides from the viscoelastic bodies of the planets or of the satellites themselves (Lainey et al. 2009, 2012; Emelyanov & Nikonchuk 2013).

Observations of a satellite are carried out from the date of its discovery up to current date. The properties of orbital motion are such that the accuracy of the ephemeris essentially depends upon the interval of observations that were used to define the parameters of motion. Hence, to obtain more precise ephemeris, we need observations covering longer time interval.

Considerable dependence of the ephemeris precision on the time interval had been proven in earlier investigations (Desmars et al. 2009; Emelyanov 2010; Jacobson et al. 2012). If the observations made by spacecrafts at very short time intervals, though having

high precision, they could make little contribution to the ephemeris precision.

When solving equations of satellite's motion, it is necessary to take into consideration all relevant perturbations so that the solution would have the accuracy higher than that of observations. To this end, numerical integration of the equations of motion is generally used because, for most natural planetary satellites, construction of analytical theory is difficult enough due to the complexity of the formulae of the perturbation theory. However, there are planetary satellites for which it is quite possible to construct analytical theory with sufficient precision. The Neptunian moon Triton is one of them.

In calculating ephemeris, analytical theories of motion have significant advantages over numerical models. The latter can be used only either via online ephemeris server or with a special calculating program and huge data file provided by the authors of the ephemeris. On the contrary, an analytical theory can be described by a set of rather simple formulae that can be easily programmed for ephemeris calculation. Furthermore, ephemeris built by numerical integration are always obtained for some limited time interval, while analytical theory of motion enables to calculate satellite coordinates for any moment of time. That is why we used the opportunity to construct

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analytical theory of Triton's motion and use it for producing Triton's ephemeris.

Triton is fairly massive Neptunian satellite, other satellites being rather small and exerting negligible influence on its motion. The parameters of the Triton–Neptune system have the following approximate values (Jacobson, Riedel & Taylor 1991). Neptune's equatorial radius is 25 225 km, Triton's radius being 1350 km. The ratio of the satellite's mass to that of the planet is 0.0002089. As will be shown below, Triton moves in almost circular orbit around its planet once every 5.87685421025 d at a mean distance of 354 700 km. Angular distance between Neptune and Triton, as seen from Earth, is 16 arcsec. Inclination of Triton's orbit to the planet's equator is about  $156^{\circ}82'$ . Right ascension and declination of Neptune's north pole is about  $299^{\circ}4'$  and  $43^{\circ}5'$ , respectively. Triton's motion is influenced by Neptune's oblateness and solar attraction.

The model of Triton's motion presented in Jacobson (2009) was elaborated by numerical integration of the equations of motion where influence of the planet's oblateness and perturbations from the Sun and some planets were taken into account. To determine Triton's orbit, Jacobson used all published Earth-based observations of the satellite and observations available to him from private communications. Observations made by the *Voyager* spacecraft were used as well. In total, the observations cover the interval from 1847 to 2008. Triton's ephemeris, available via the HORIZONS ephemeris server (Giorgini et al. 1996), can be obtained for the time interval from 1800 to 2200.

An attempt to build alternative model of Triton's motion based on ground-based observations made from 1975 to 2006 was also undertaken by Zhang (Zhang et al. 2014). However, since the model of Triton's motion used in Jacobson (2009) is based on larger series of observations, it is Jacobson's results that were considered in this research.

In Jacobson (2009) a simplified analytical model of the satellite's motion based on the precessing Keplerian ellipse is also offered. Parameters of such an orbit, called mean elements, were chosen in a way that the satellite's coordinates could be best fit to the results of numerical integration at the 400 yr time interval, from 1800 to 2200. The precessing ellipse model assumes that in a certain coordinate system orbital semimajor axis  $a$ , eccentricity  $e$  and inclination  $I$  remain constant while the longitude of the ascending node  $\Omega$ , the argument of pericentre  $\varpi$  and the mean longitude  $\lambda$  are linear functions of time. In Jacobson (2009) the axis around which the orbital plane is precessing with constant inclination is the line normal to the Laplace plane. The coordinates of the direction of this axis referred to the Earth equator are given. Although using the Laplace plane allows partially to take into consideration solar perturbations on the precessing orbit, the precessing ellipse model turns out to be only approximate because it does not take account of periodic perturbations in the elements of Keplerian orbit.

In this paper, we prove that the precessing ellipse model proposed in Jacobson (2009) does not conform to the properties of Triton's perturbed motion. The perturbed value of the mean anomaly has no secular change, as against the case for the unperturbed Keplerian motion, but exhibits periodic oscillations around zero with an amplitude of several degrees.

We offer simple and easily programmable formulae for calculation of Triton's ephemeris at any instant of time. The formulae are based on a new analytical theory of Triton's motion where all main perturbing forces were taken into account. The accuracy with which the motion of Triton can be presented by using this theory is hundreds times better than that of its observations.

The parameters of the theory were fit to all published observations made between 1847 and 2012. Thus, compared to the model in Jacobson (2009) where the observation time interval ends in 2008, Triton's ephemeris proposed in this paper are based on longer time interval of observations.

## 2 ANALYSIS OF TRITON'S ORBIT FROM NUMERICAL INTEGRATION OF THE EQUATIONS OF MOTION

Applying the methods used in the theory of perturbations to the problem of Triton's motion leads to the expansion of the disturbing function in powers of small parameters, the perturbations being represented as the series. The small parameters of the expansions are the coefficients defining perturbation factors as well as orbital eccentricity of the satellite. For practical purposes, only the most significant terms are left in the series. To find out which perturbations should be taken into consideration in the analytical theory of Triton's motion, we analysed the way in which the elements of the satellite orbit change. The analysis was performed on the basis of numerical integration of the equations of motion.

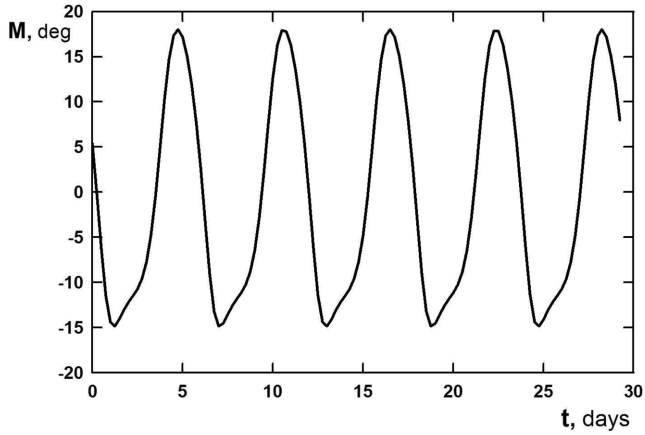
We introduce the following notations for the elements of the Keplerian orbit:  $a$  is the orbital semimajor axis,  $e$  the eccentricity and  $I$  the inclination of the orbit to a certain reference plane. Other elements are:  $M$  the mean anomaly,  $\omega$  the angular distance of pericentre to the ascending node of the orbit (argument of pericentre) and  $\Omega$  the longitude of the ascending node. The orbital elements are referred to a non-rotating planetocentric coordinate system. As mentioned above, in Jacobson (2009) the coordinate system was referred to the Laplace plane.

The elements of the Keplerian osculating orbit were computed from the results of numerical integration of the equations of motion with all relevant perturbing factors taken into consideration. The results of numerical integration were obtained by using the HORIZONS online ephemeris service (Jet Propulsion Laboratory, California Institute of Technology, USA). We used the telnet interface of the server.

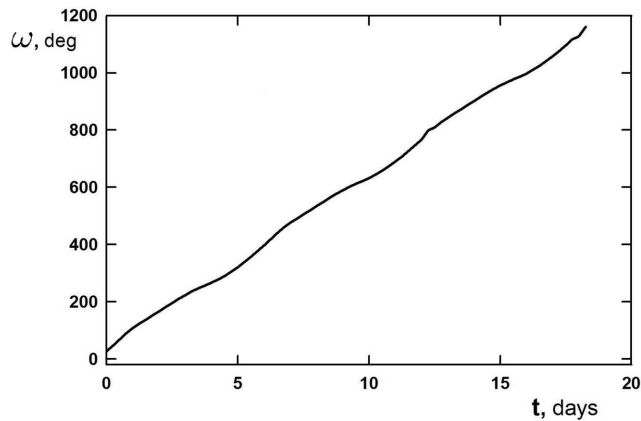
Triton's planetocentric coordinates and velocities referred to the Earth's equator were obtained for a series of time instants with a stepsize of 0.25 d at a 400 yr time interval between 1800 and 2200. After that, the coordinates and the velocities were transformed in the coordinate system referred to the Laplacian plane (Jacobson 2009). The resulting values of coordinates and velocities were used to compute at each moment of time the elements  $a$ ,  $e$ ,  $I$ ,  $M$ ,  $\omega$ , and  $\Omega$  of the Keplerian osculating orbit.

The changes in the element  $M$  over a time-scale of five revolutions of the satellite are shown in Fig. 1. It is seen that the perturbed value of the mean anomaly  $M$  oscillates near zero with an amplitude of about  $17^{\circ}$ . Note that the mean anomaly was calculated directly from coordinates and velocities of the satellite and no secular changes were excluded from obtained values. The question arises: how in the world is it possible for this satellite to revolve around the planet? The explanation can be found in Fig. 2 which shows the change of the argument of pericentre  $\omega$  over a time-scale of about three satellite revolutions. It is seen that the line of apses is rotating with an angular velocity which is approximately equal to that of the satellite's rotation around the planet. Thus, the motion of the satellite along its orbit is determined by the argument of latitude  $u = M + \omega$  (its mean rate of change we denote thereafter as  $\dot{u}$ ).

As applied here to the Triton motion, this effect is generally known. The possibility of perturbed circular orbital motion of a satellite around the oblate planet with non-zero eccentricity of



**Figure 1.** The change of Triton's mean anomaly  $M$  over a time-scale of five revolutions in its orbit.



**Figure 2.** The change of Triton's argument of pericentre  $\omega$  over a time-scale of three revolutions in its orbit.

Keplerian osculating orbit is mentioned by Beletskii (1963). The properties of motion with low eccentricity is discussed also by Greenberg (1981).

These results prove that the precessing ellipse model proposed in Jacobson (2009) does not correspond to the properties of Triton's Keplerian osculating orbit because the changes in  $M$  and  $\omega$  have quite different character.

Note that Jacobson took the eccentricity of Triton's orbit to be 0.00001.

Analysis of changes in the osculating Keplerian elements obtained from numerical integration of the equations of motion revealed that the amplitudes of the short-period perturbations in the elements  $I$ ,  $u$  and  $\Omega$  do not exceed 0.00001 rad, whereas those in  $a$  are not greater than 3.5 km which makes 0.00001 from the value of the semimajor axis (354 700 km). Taking into consideration the mean distance between the Earth and Neptune, we conclude that the 1 arcsec error in the satellite's topocentric angular coordinates would correspond to 22 000 km error in orbital position.

When determining the necessary precision of the theory of Triton's motion, the starting point was the accuracy of the best Earth-based observations used in Jacobson (2009). If expressed in topocentric angular coordinates, this accuracy is 0.03 arcsec, which corresponds to 660 km in Triton's orbital position.

It follows from the analysis given above that the short-period perturbations and oscillations in the satellite's distance to the planet's

centre are approximately 200 times less than the accuracy of observations. That is why in our model of Triton's motion the short-period perturbations are neglected and the orbit is assumed to be circular. Moreover, it follows from Lagrange equations (Murray & Dermott 2010) that there are no long-period perturbations of the first order in the semimajor axis  $a$ . Hence, we assume  $a$  to be constant.

At this stage only the changes in the elements over a time-scale of several revolutions of the satellite were examined. Obviously, when examining Triton's motion at the 400 yr interval, it is necessary to take into account long-period perturbations whose amplitudes are much greater than those of the short-period perturbations.

### 3 THE ANALYTIC THEORY OF TRITON'S MOTION

Having taken into consideration the above-given analysis of Triton's orbit, we have come to the construction of the analytical theory with the view of using it to generate Triton's ephemeris.

In our theory, we take into account secular perturbations caused by Neptune's non-sphericity as well as secular and long-period perturbations resulting from solar attraction.

Neptune's non-sphericity is mainly described by the second harmonic coefficient  $J_2$  in the expansion of the potential function of Neptune's gravitational field. This term does not give any long-period perturbations of the first order. As for the long-period perturbations resulting from fourth harmonic coefficient as well as second-order perturbations, we esteem that, with the accuracy assumed above, they can be neglected.

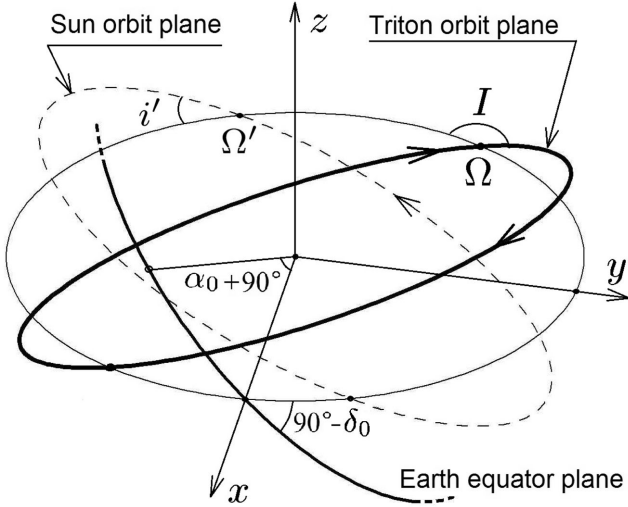
All secular perturbations in the elements  $u$  and  $\Omega$  are taken into account by including linear terms into their perturbations. The corresponding coefficients  $\dot{u}$  and  $\dot{\Omega}$  will be afterwards fit to observations. In this way, all secular perturbations in these elements will be taken into consideration, including those from the Sun.

The planet's rotation axis precesses synchronously with the motion of Triton's orbital plane around the vector of summary angular momentum of both planet and satellite. The constant angle between these axes is about 0:506 (Jacobson 2009). Orbital inclinations referred to both the planet's rotation axis and the vector of summary angular momentum also remain constant. As demonstrated in Jacobson (2009), such model of motions of the planet and the satellite is the most realistic. Only solar perturbations can produce small deviations from it.

We use coordinate system referred to the fixed vector of summary angular momentum of both planet and satellite. The  $z$ -axis of the coordinate system is directed along this vector, the  $x$ -axis pointing to the node of the  $xy$ -plane over the Earth equator so that the  $y$ -axis forms a sharp angle with the Earth equator. Hereafter, we shall refer to this system as the *orbital coordinate system*. Since the orbital inclination and the longitude of the ascending node are determined from observations, practically the orbital coordinate system is defined by the condition that the inclination is constant, provided that long-period solar perturbations are excluded.

Let  $x_g$ ,  $y_g$  and  $z_g$  be the axes of planetocentric coordinate system referred to the Earth equator. The coordinates in these two coordinate systems are interrelated via the right ascension  $\alpha_0$  and declination  $\delta_0$  of the  $z$ -axis. The coordinates  $x_g$ ,  $y_g$  and  $z_g$  are calculated by the formulae:

$$\begin{aligned} x_g &= -\sin \alpha_0 x - \cos \alpha_0 \sin \delta_0 y + \cos \alpha_0 \cos \delta_0 z, \\ y_g &= \cos \alpha_0 x - \sin \alpha_0 \sin \delta_0 y + \sin \alpha_0 \cos \delta_0 z, \\ z_g &= \cos \delta_0 y + \sin \delta_0 z. \end{aligned} \quad (1)$$



**Figure 3.** Coordinate systems centred at Neptune. The main planes are shown with the lines of intersection with a unit sphere.

The parameters  $\alpha_0$  and  $\delta_0$  are not known in advance, they can be determined only from observations of the satellite.

The elements of the Keplerian osculating orbit are not affected by long-period perturbations of first order caused by the second zonal harmonic with coefficient  $J_2$  in the expansion of the planet's gravitational potential. Thus, if we have neglected the second-order perturbations and the solar perturbations, the satellite would move in plane orbit which is precessing with constant angular velocity and constant inclination to the main reference plane.

In our model, the elements  $a, I, u$  and  $\Omega$  give the satellite's position in the orbital coordinate system relative to the planet's centre. Three of them are functions of time:

$$\begin{aligned} I &= I_0 + \delta I(t), \\ u &= u_0 + \dot{u}(t - t_0) + \delta u(t), \\ \Omega &= \Omega_0 + \dot{\Omega}(t - t_0) + \delta \Omega(t), \end{aligned} \quad (2)$$

where  $I_0, u_0, \dot{u}, \Omega_0, \dot{\Omega}$  are constants,  $t_0$  is a given initial epoch and  $\delta I(t), \delta u(t), \delta \Omega(t)$  are long-period perturbations of the corresponding elements.

Due to assumed simplifications, the formulae of Keplerian motion converted into the following simple relationships:

$$\begin{aligned} x &= a (\cos u \cos \Omega - \sin u \sin \Omega \cos I), \\ y &= a (\cos u \sin \Omega + \sin u \cos \Omega \cos I), \\ z &= a \sin u \sin I. \end{aligned} \quad (3)$$

A scheme describing the geometry of the problem is presented in Fig. 3.

To complete the model of Triton's motion, with the assumptions made above, we need to determine the long-period perturbations  $\delta I(t), \delta u(t)$  and  $\delta \Omega(t)$  caused by the Sun's attraction.

The expansion of the disturbing function  $R'$  has been taken from Murray & Dermott (2010). It is the series in powers of ratio  $a/a'$ , where  $a'$  is the semimajor axis of the Sun's orbit around the planet. The series begins with the term containing  $(a/a')^2$ .

Let us make some additional justified simplifications. Since the semimajor axis  $a$  of Triton is 354 700 km and the mean distance from Neptune to the Sun is 4504 449 760 km, we obtain that  $a/a' = 0.000078757455$ . This means that we can neglect higher degrees

**Table 1.** The parameters of the model of solar motion.

Parameter	Value from the mean elements	Value from the INPOP10 ephemeris
$a'$ (km)	4504449760	4499478064
$i'$ ( $^\circ$ )	27.923658	27.923678
$\Omega'$ ( $^\circ$ )	200.788305	200.788181
$u'_0$ ( $^\circ$ )	258.329018	258.727508
$\dot{u}'$ (deg d $^{-1}$ )	0.00598182615	0.00598084154

of this ratio, leaving only terms with  $(a/a')^2$ . We also assume that the Sun moves in a circular orbit in the invariable plane. Let  $i'$  and  $\Omega'$  be the inclination and the longitude of the ascending node of the Sun's orbit in the orbital coordinate system. We assume that  $i'$  and  $\Omega'$  are known constant values, while the Sun's argument of latitude  $u'$  is a known linear function of time  $u' = u'_0 + \dot{u}'(t - t_S)$ , where  $t_S$  is a given epoch. The values of  $i', \Omega', u'_0$  and  $\dot{u}'$  can be determined from the mean elements of Neptune's orbit (Simon et al. 1994). After simplifications, the disturbing function is as follows:

$$\begin{aligned} R' &= \frac{Gm'a^2}{a'^3} \sum_{k=0}^2 (2 - \delta_{0,k}) \frac{(2-k)!}{(2+k)!} \sum_{p'=0}^{2'} F_{2k1}(I) F_{2kp'}(i') \\ &\times \cos[(2 - 2p')u' + k(\Omega' - \Omega)], \end{aligned}$$

where  $m'$  is the mass of the Sun,  $\delta_{0,k} = 1$  if  $k = 0$ ,  $\delta_{0,k} = 0$  if  $k \neq 0$ , and the prime of the second sum indicates that the term with the indices  $k = 0$  and  $p' = 1$ , i.e. the secular term, is omitted.  $F_{2k1}(I)$  are the inclination functions. The resulting expression has eight terms, two of which are equal because  $F_{200}(i') = F_{202}(i')$ . The necessary expressions for the inclination functions have been taken from Kaula (1966).

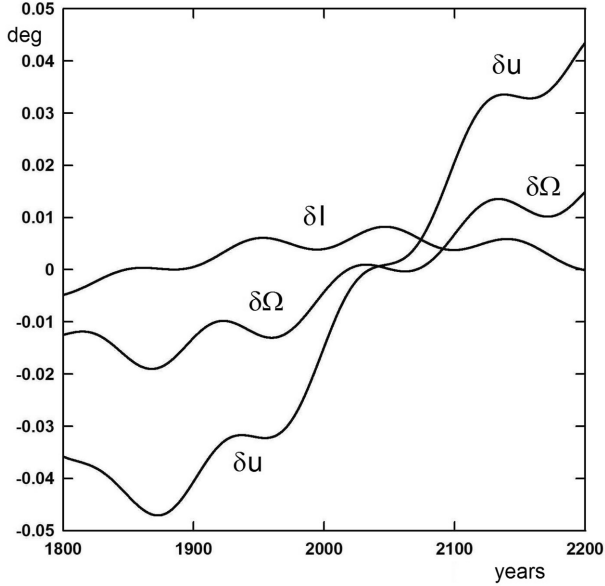
The Lagrange equations for the elements  $\delta I(t), \delta u(t)$  and  $\delta \Omega(t)$  were solved by using the small parameter method. The first-order perturbations were defined with respect to the small parameter  $(m'/m)(a/a')^3$ . In addition, it was taken into account that secular perturbations from the second zonal harmonics with the coefficient  $J_2$  in the elements  $u$  and  $\Omega$  depend on  $I(t)$  which undergoes long-period perturbations. Thus, perturbations proportional to  $(m'/m)(a/a')^3 J_2$  were also taken into consideration.

To calculate the long-period perturbations resulting from the Sun's attraction, the above described approximate model of the Sun's circular motion around Neptune was used. This model's parameters  $a', i', \Omega', u'_0$  and  $\dot{u}'$  were taken in two versions. In the first one, the parameters were calculated by using Neptune's mean orbital elements taken from Simon et al. (1994). The second version of the parameters of solar motion was calculated by the least-squares method from heliocentric coordinates of the Sun computed with the INPOP10 ephemeris (Fienga et al. 2011) for a series of time instants with a stepsize 10 d for the time interval from 1800 to 2200. Obtained parameters are given in Table 1. The date JD = 2451545.0 (TT) was taken as an initial epoch  $t_S$  for the parameter  $u'_0$ . In these versions, the values do not differ significantly. For the final variant of the theory, we took the parameters of the Sun's planar circular orbit computed with the INPOP10 ephemeris.

At first, the long-period perturbations from the Sun were obtained in analytical form. After this, the above-given parameters of solar motion as well as the parameters of Triton's motion found from observations were substituted into the formulae. As a result, for any

**Table 2.** The coefficients appearing in the formulae for the long-period perturbations resulting from the Sun's attraction.

$i$	$K_l^{(i)}$ (°)	$K_u^{(i)}$ (°)	$K_\Omega^{(i)}$ (°)	$k_1^{(i)}$	$k_2^{(i)}$
1	0.0	-0.00012327	0.00063339	2	0
2	0.00096486	-0.00279453	-0.00178908	2	1
3	0.00664662	-0.04335625	-0.01560110	0	1
4	0.00004687	-0.00017215	-0.00009186	-2	1
5	0.00095975	-0.00233686	-0.00218071	2	2
6	-0.00037627	0.00170605	0.00096231	0	2
7	-0.00000225	0.00000730	0.00000536	-2	2



**Figure 4.** The long-period perturbations from the Sun in the elements of Triton's orbit.

given time instant, the perturbations can be calculated by using the following simple formulae:

$$\begin{aligned} \delta I(t) &= \sum_{i=1}^7 K_l^{(i)} \cos \left[ k_1^{(i)} u' + k_2^{(i)} (\Omega' - \bar{\Omega}) \right], \\ \delta u(t) &= \sum_{i=1}^7 K_u^{(i)} \sin \left[ k_1^{(i)} u' + k_2^{(i)} (\Omega' - \bar{\Omega}) \right], \\ \delta \Omega(t) &= \sum_{i=1}^7 K_\Omega^{(i)} \sin \left[ k_1^{(i)} u' + k_2^{(i)} (\Omega' - \bar{\Omega}) \right], \end{aligned} \quad (4)$$

where

$$u' = u'_0 + \dot{u}'(t - t_0), \quad \bar{\Omega} = \Omega_0 + \dot{\Omega}(t - t_0). \quad (5)$$

The coefficients involved in these formulae are given in Table 2.

The graphs illustrating the long-period perturbations from the Sun in Triton's elements  $I$ ,  $u$  and  $\Omega$  at the time interval from 1800 to 2200 are given in Fig. 4.

Now, we need to determine from observations the eight parameters  $a$ ,  $I_0$ ,  $u_0$ ,  $\dot{u}$ ,  $\Omega_0$ ,  $\dot{\Omega}$ ,  $\alpha_0$  and  $\delta_0$ , two of which,  $u_0$  and  $\Omega_0$ , should be attributed to the given initial epoch  $t_0$ .

We have made an attempt to find from observations the coefficient of the empiric quadratic term in the perturbations of the argument of latitude  $u$ . The corresponding expression for the argument of latitude can be presented in the following way

$$u = u_0 + \dot{u}(t - t_0) + \hat{k}(t - t_0)^2 + \delta u(t), \quad (6)$$

**Table 3.** Parameters of the model of Triton's motion obtained from numerical integration of the equations of motion.

Parameter	Value
$a$ (km)	354758.98
$I$ (°)	156.86561883
$u_0$ (°)	32.66861530
$\dot{u}$ (deg d <sup>-1</sup> )	61.2586972029
$\Omega_0$ (°)	72.89882654
$\dot{\Omega}$ (deg d <sup>-1</sup> )	0.001433819551
$\alpha_0$ (°)	299.46088779
$\delta_0$ (°)	43.40655561

where the coefficient  $\hat{k}$  is also to be determined from observations. The quadratic term in the argument of latitude is translated to the orbital longitude of the satellite. The reasons why this quadratic term may appear is discussed in the Section 6.

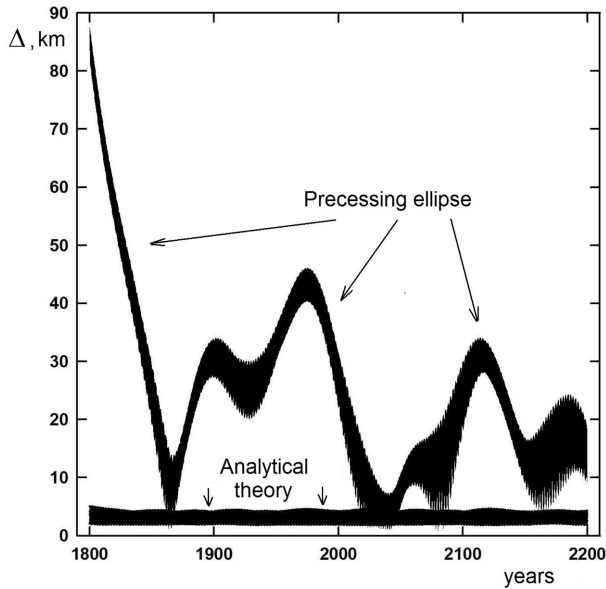
After the parameters are determined, Triton's planetocentric coordinates referred to the Earth's equator can be computed by using the following chain of formulae: (5), (4), (6), (2), (3) and (1).

#### 4 ANALYSIS OF THE PRECISION OF TRITON'S ANALYTICAL THEORY OF MOTION

When constructing the analytical theory of Triton's motion, the above described simplifications were applied. Before using the theory for producing ephemeris, it is necessary to evaluate how accurately it can represent the motion of the satellite. To this end, the parameters of the orbit should be refined on the basis of some precise reference model of motion. As such reference model, Triton's ephemeris were used accessible at the HORIZONS ephemeris server (Giorgini et al. 1996) and produced from the results obtained in Jacobson (2009). We used the telnet interface of the ephemeris server which can produce the satellite's rectangular planetocentric coordinates with an accuracy of 0.001 mm as well as velocities referenced to the Earth's equator. The coordinates and velocities were taken for 584 161 time instants with the stepsize of 0.25 d at a the time interval between 1800 and 2200. Using the least-squares method, the coordinates were used to refine the eight parameters of the theory, that is  $a$ ,  $I_0$ ,  $u_0$ ,  $\dot{u}$ ,  $\Omega_0$ ,  $\dot{\Omega}$ ,  $\alpha_0$  and  $\delta_0$ . The values of the parameters that were obtained are given in Table 3. As an initial epoch  $t_0$  of the parameters  $u_0$  and  $\Omega_0$ , the date JD= 2378520.5 (in TT scale) was taken.

Having determined the parameters, the differences  $\Delta$  between the satellite's planetocentric position calculated from the analytical theory and that obtained from the initial coordinates were calculated for each time instant. The root-mean-square value of these differences over all time instants turned out to be 3.3 km. This quantity can serve as an evaluation of the theory's precision. Thus, with the obtained value of orbital radius 354700.30 km, relative error is about 0.00001. Corresponding error in Triton's topocentric position is 0.00015 arcsec which is 200 times less than the errors of the best on-ground observations of Triton.

We made estimates of the precision of Triton's analytical theory of motion and the precessing ellipse model as well. The differences  $\Delta$  between the satellite's positions based on the precessing ellipse model and those based on the initial coordinates computed from numerical integration were also calculated for a series of time instants. In fact, these residuals are the errors of tested model. Fig. 5 demonstrates the dependences of the errors  $\Delta$  on time for both the



**Figure 5.** Residuals between Triton’s positions calculated from both the analytical theory and the precessing ellipse model and the positions obtained from numerical integration.

precessing ellipse model used in Jacobson (2009) and the analytical theory proposed in this paper. It is seen from the graphs that the errors of our theory do not exceed 4 km over the whole time interval between 1800 and 2200. The errors of the precessing ellipse model proposed in Jacobson (2009) exceed 20 km for the greater part of the time interval, the root-mean-square value being equal to 30 km. Note that the precessing orbit model is most probably designed for giving a somewhat rough idea of the orbital properties.

Our analysis proves that the constructed analytical theory of Triton’s motion has an accuracy that is satisfactory for producing ephemeris at the 400 yr time interval.

## 5 OBSERVATIONS USED TO DETERMINE THE ORBIT

Our main goal was to get the time interval of observations used to define the parameters of the model of Triton’s motion to be as long as possible. The main sources of data were publications in scientific journals. In addition, small portions of observations were sent by observers directly to the Natural Satellite Data Center (Arlot & Emelyanov 2009). In the latter case, the bibliographic reference contain only observer’s name, the year and the indication: ‘Communication to the Natural Satellite Data Center’.

Triton was discovered in 1846 and, since 1847, a lot of micrometric observations of the satellite have been made. This type of observations gives us the value of the apparent angular distance  $s$  between the planet and the satellite as well as the corresponding position angle  $P$  measured from the direction to the true north of the epoch of the date. In most cases, these two angles were measured at different moments of time. Generally, using the observations published during the 19th and at the beginning of the 20th centuries posed many problems. For instance, it was not possible sometimes to identify from publication the time-scale; sometimes it was not clear whether reductions were made when data were prepared for publication.

At the beginning of the 20th century photographic observations began to be used, and at the end of this century observations made

with CCD cameras appeared. Such observations can be relative or absolute. Relative observations provide us with Triton’s coordinates relative to Neptune while absolute observations give us satellite positions relative to stars. Absolute observations always have a systematic error resulting from inaccuracy in star coordinates taken from star catalogues.

We have also used the observations made by the *Voyager 2* spacecraft in 1988–1989 (the results have been published in Jacobson 1991). It is reported in Jacobson (2009) that the spacecraft trajectory was reconstructed, i.e. refined and transformed into the International Celestial Reference Frame. The corresponding computer file `vgr2-nep081.bsp` available at the Jet Propulsion Laboratory (JPL; Acton et al. 2015) site <http://naif.jpl.nasa.gov/pub/naif/VOYAGER/kernels/spk/> allows to obtain refined coordinates of the *Voyager 2* spacecraft. Earlier, we used these coordinates in Emelyanov & Arlot (2011) to fit the orbit of the Neptunian moon Nereid to observations. The old *Voyager 2* position published in Jacobson (1991) were replaced by the positions calculated with the SPICE software of JPL using the new kernel file. For the first 54 observations published in Jacobson (1991), the kernel file `vgr2-nep081.bsp` does not allow to calculate the *Voyager 2* positions. As result, the only 359 observations of Triton made with *Voyager 2* were used in our fit of the theory to the observations.

The type of observations is designated by two identifiers. The first one describes photodetector: ‘microm’ is micrometric, ‘phot’ photographic, ‘CCD’ CCD camera, ‘*Voyager 2*’ the *Voyager 2* spacecraft and ‘mer’ meridian circle. The second identifier describes the way the satellite’s coordinates were measured: ‘abs’ means absolute measurements and ‘rel’ the relative.

All observations were distributed into groups, so that each group comprises observations of the same type made at the same observatory or obtained from the same source. The main principle was the hypothesis that observations in a group have the same accuracy, which was necessary to provide weights for conditional equations used to refine the orbital parameters by the least-squares method. As a result, we have obtained 57 groups of observations including those made by *Voyager 2*.

When refining the orbital parameters, the residuals  $\Delta$  were calculated for each observation, so that one measurement of any coordinate gave one residual. The way the residuals were calculated depended on the type of the observation.

Let  $\Delta\alpha$  and  $\Delta\delta$  be the differences between observed and calculated values of right ascension and declination, respectively. These differences can be determined for both relative and absolute observations made by photographic camera, CCD receiver or spacecraft. Then, the residual  $\Delta$  in right ascension is given by the formula

$$\Delta = \Delta\alpha \cos \delta,$$

where  $\delta$  is the planet’s declination calculated from the existing theory of its motion. As for declination, it was assumed that  $\Delta = \Delta\delta$ .

For micrometric observations, when the angular distance between the planet and the satellite is measured, the residual is the difference between observed distance and that calculated from the theory, that is  $\Delta = \Delta s$ . If position angle is measured, the residual is

$$\Delta = s \Delta P,$$

where  $\Delta P$  is the difference between observed position angle and that obtained from the theory, the value of  $s$  being taken from the theory.

As to the *Voyager 2* observations, the Triton’s coordinates were measured at various distances ranging from 2.4 to 685 million

kilometres. Thus, the same errors in angular coordinates gave essentially different errors in the satellite's orbital positions. In order to be able to compare the accuracy of the Earth-based observations with those made by the spacecraft, the residual  $\Delta$  was normalized by the mean geocentric distance of Triton. Normalized residuals are given below.

When fitting the parameters of Triton's motion, we calculated the root-mean-square residuals  $\sigma_k$  for each group of observations ( $k$  is the group index) as well as the root-mean-square residual  $\sigma$  for all observations. Table 4 gives data for all groups of observations as well as the root-mean-square residuals obtained after parameters were fitted.

In total, to fit the parameters of Triton's orbit, 10 254 observations were used made at the 165 yr interval, since 1847 to 2012. These observations gave 16 096 measurements of the satellite's coordinates.

## 6 DETERMINATION OF THE ORBIT OF TRITON AND EVALUATION OF THE ACCURACY OF OBSERVATIONS

Determination of the parameters of the analytic theory of Triton's motion was made by computing corrections to the parameters by the least-squares method. For each measured coordinate, a conditional equation was built.

Initial values of the parameters were taken from the results given in Jacobson (2009). After the first refinement was made, the iterations continued until corrections, diminishing with each iteration, became 100 times less than their errors. In result, no more than six iterations were necessary.

The errors of all observations are considered to be uncorrelated. However, the observations that were used cannot be regarded as having the same accuracy. That is why, when composing the conditional equations, it is necessary to assign weighting factors to each of them. As a priori, we have no knowledge of the accuracy of observations, assigning the weighting factors to the conditional equations was made in the following way. At the first iteration, all observations were regarded as having the same accuracy. For each group of observations, the root-mean-square residuals of observations  $\sigma_k$  were determined as explained in the previous section. At subsequent iterations, the weighting factors were assigned to the equations by dividing the left- and right-hand sides of each equation by  $\sigma_k$  obtained at the previous iteration. That is, for each measurement number  $i$  a weighting factor  $w_i = 1/\sigma_k^2$  was assigned where  $k$  is the number of the group to which belongs the observation.

We have also calculated the weighted root-mean-square value of residuals for all observations  $\sigma_w$  by the formula

$$\sigma_w = \sqrt{\left(\sum_{i=1}^N w_i \Delta_i^2\right) \left(\sum_{i=1}^N w_i\right)^{-1}},$$

where  $N$  is the number of measurements. The least-squares method provides us also with the evaluations of precision of the parameters being determined.

When calculating the theoretical values of the measured coordinates, the using of the planetary motion model is inevitable. If the satellite's right ascension and declination are measured, the errors in calculated planetary coordinates are included into theoretical values of the satellite coordinates. That is why it was important to choose appropriate planetary theory among those available. The last version INPOP13C (Fienga et al. 2015) of the INPOP theory (Fienga et al. 2011) was chosen for our calculations.

We made an attempt to detect a quadratic term in the longitude of Triton by the fit to the observations using the constructed analytical theory. This term may appear for three reasons only. First, it can be an error in the theory. Some term of very long period being omitted in the theory may appear as a quadratic term in the errors of the mean longitude of Triton. Secondly, it may be a perturbation force of dissipative properties that was neglected in the equations of satellite motion. Thirdly, the errors of observations can be haphazardly distributed in time so that they produce a quadratic term in the mean longitude of Triton. To exclude the first possible reason, we performed a fit of all orbital parameters including the quadratic term in the mean longitude to the model of motion generated with the JPL ephemeris of Triton on the period of 200 yr. The JPL ephemeris of Triton are based on a model of satellite motion where the perturbing forces of gravitational nature only are taken into account. There is no energy dissipation in this model. A quadratic terms being appeared in the mean longitude errors could be explained as the errors in the tested theory. In result of this experiment the value of the quadratic term coefficient  $\hat{k}$  was found to be  $(-7.1 \pm 4.5) 10^{-9} \text{ deg yr}^{-2}$ . This value is negligible compared to the value of this coefficient obtained from observations. This means that the quadratic term in the mean longitude of Triton cannot be explained by the errors in the constructed analytical theory. The question about the nature of the quadratic term appeared from observations we reserve for following investigations.

There were various ways of fitting the parameters. We considered both the variant where the coefficient of the quadratic term in the satellite's longitude  $\hat{k}$  is refined and that where this coefficient is put to zero. Obviously, the accuracy with which the parameter  $\hat{k}$  can be determined significantly depends upon the time interval of the observations. In particular, the time interval can turn out to be not sufficiently large for reliable determination of the parameter  $\hat{k}$ . We also considered variants both without observations made by the *Voyager 2* spacecraft and with them. The algorithm of fitting the parameters allows to throw away those observations that have residuals greater than some given value.

The differences in the measured and calculated values of the satellite's right ascension and declination contain systematic errors of observations as well as errors of the planetary theory. The mean values of errors at some time intervals can be significantly greater than random errors. These systematic biases have nothing to do with the model of the satellite motion around the planet, and that is why it is desirable to find them and exclude from observations. We have made this for several groups of observations, for some of which the biases were determined separately for various time intervals. Table 5 gives those systematic biases that were excluded from some groups of observations.

Initially, all available observations (10 254 in total) were used, which gave 16 096 conditional equations. For these observations, residuals did not exceed 2.5 arcsec.

The root-mean-square residuals  $\sigma$  and its weighted average values  $\sigma_w$ , after the orbital parameters were fit to observations (for various sets of the parameters and observations), are given in Table 6. It can be concluded from these results that the addition of quadratic term in orbital longitude to the theory does not improve the agreement between theory and observations.

We have also examined the case where only those observations are left that have residuals not greater than 1 arcsec. In this case, the total number of observations decreased to 10 164, that is only less than 1 per cent of observations were thrown away. For this set of observations, we have obtained  $\sigma = 0.199 \text{ arcsec}$  and  $\sigma_w = 0.036 \text{ arcsec}$ . It is seen that, compared with the case of full set of



**Table 4.** Composition of observations and description of groups of observations. Here,  $k$  is the group's index,  $N_k$  the number of observations in the group,  $\sigma_k$  the root-mean-square residual in the group,  $\Delta T_k$  the interval of observations. The column 'Type' gives one of the types of observations as described in the text. The names of the observatories are preceded by their IAU observatory codes (if they are known).

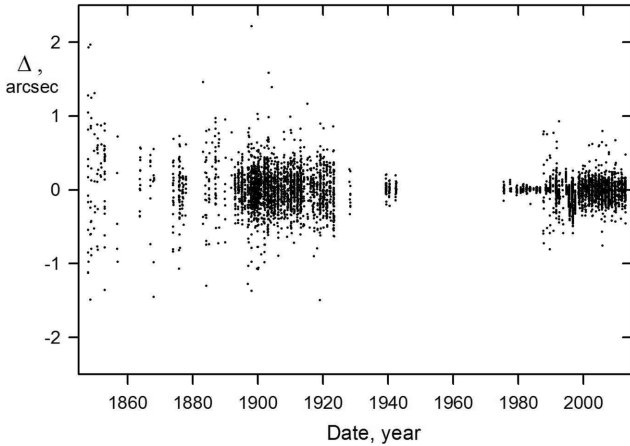
$k$	Type	Publication	Place of observations	$\Delta T_k$	$N_k$	$\sigma_k$
1	microm, rel	Lassell (1849a)	802 Cambridge	1847–1847	4	1.254
2	microm, rel	Hall (1900)	802 Cambridge	1847–1847	24	0.859
3	microm, rel	Lassell (1849a)	992 Liverpool	1848–1848	4	0.985
4	microm, rel	Lassell (1849c)	992 Liverpool	1849–1849	5	0.858
5	microm, rel	Lassell (1849b, 1850, 1851, 1852a,b, 1857)	992 Liverpool	1849–1856	91	0.816
6	microm, rel	Lassell (1864)	992 Liverpool	1863–1863	20	0.365
7	microm, rel	Winlock & Pickering (1888)	802 Cambridge	1866–1868	18	0.706
8	microm, rel	Davis (1874); USNO (1875)	787 Washington	1873–1874	94	0.350
9	microm, rel	USNO (1881); Hall (1876, 1877)	787 Washington	1875–1877	151	0.353
10	microm, rel	Henry, Boinot & Sy (1886)	007 Paris	1883–1884	31	0.873
11	microm, rel	Young (1888)	H80 Halsted observatory	1883–1888	68	0.523
12	microm, rel	Henry (1884a,b)	007 Paris	1884–1884	21	0.722
13	microm, rel	Lohse (1887)	968 Haverhill	1885–1886	3	0.775
14	microm, rel	Perrotin (1887)	020 Nice	1886–1887	27	0.266
15	microm, rel	Parrish & Stone (1888)	780 Leander McCormick	1889–1889	18	0.655
16	microm, rel	Barnard (1893, 1894, 1895)	662 Mount Hamilton	1892–1895	320	0.309
17	microm, rel	Schaeberle (1895, 1897, 1898)	662 Mount Hamilton	1894–1897	133	0.148
18	phot, rel	USNO (1911)	786 Washington	1894–1906	388	0.294
19	microm, rel	Drew (1897, 1899)	690 Lowell Observatory	1896–1898	121	0.362
20	microm, rel	Greenwich, Royal Observatory (1900)	000 Greenwich	1896–1898	28	0.778
21	microm, rel	Barnard (1898, 1899)	754 Williams Bay	1897–1899	326	0.243
22	microm, rel	Aitken (1899, 1904); Hussey (1899, 1902)	662 Mount Hamilton	1898–1902	91	0.264
23	phot, rel	Greenwich, Royal Observatory (1899)	000 Greenwich	1899–1899	12	0.297
24	microm, rel	Barnard (1901)	754 Williams Bay	1899–1901	361	0.235
25	phot, rel	Kostinsky (1902)	084 Pulkovo	1899–1901	21	0.324
26	microm, rel	See (1900); Hammond & Rice (1905); Hammond (1906, 1908); Dinwiddie (1903)	786 Washington	1899–1908	269	0.325
27	phot, rel	Balanovskii (1923)	084 Pulkovo	1899–1920	101	0.374
28	microm, rel	Barnard (1903)	754 Williams Bay	1901–1903	223	0.290
29	phot, rel	Perrine (1903)	662 Mount Hamilton	1902–1903	51	0.284
30	microm, rel	Wirtz (1905)	522 Strasbourg	1903–1905	13	0.670
31	phot, rel	Greenwich, Royal Observatory (1903, 1904, 1905, 1906, 1907, 1908)	000 Greenwich	1903–1908	294	0.234
32	phot, rel	Barnard (1906a, 1907, 1909, 1910, 1912, 1913, 1915, 1916, 1917, 1919, 1927)	754 Williams Bay	1903–1922	986	0.291
33	microm, rel	Barnard (1906b)	754 Williams Bay	1905–1906	32	0.331
34	phot, rel	Albrecht & Smitil (1909)	662 Mount Hamilton	1905–1906	6	0.345
35	microm, rel	Hall (1911); Hall & Burton (1913)	786 Washington	1908–1911	136	0.311
36	phot, rel	Greenwich, Royal Observatory (1913)	000 Greenwich	1909–1910	59	0.169
37	no inf, rel	Burton (1913)	786 Washington	1911–1912	85	0.193
38	microm, rel	Hall & Burton (1919)	786 Washington	1912–1919	282	0.301
39	microm, rel	Hall (1920); Bower & Hall (1923)	786 Washington	1919–1923	84	0.302
40	microm, rel	Hall (1922)	786 Washington	1920–1920	46	0.231
41	microm, rel	Neuimin & Pokrovskii (1926)	084 Pulkovo	1923–1923	118	0.362
42	microm, rel	Crawford (1928)	662 Mount Hamilton	1928–1928	15	0.225
43	microm, rel	Alden (1940, 1943)	077 Johannesburg	1939–1942	73	0.088
44	phot, rel	Walker & Harrington (1988); Walker, Christy & Harrington (1978); Harrington & Walker (1984)	689 Flagstaff	1975–1986	198	0.059
45	phot, abs	Chanturia & Kisseleva (2006)	119 Abastumani	1986–1993	54	0.412
46	<i>Voyager 2</i>	Jacobson (1991)	<i>Voyager 2</i>	1988–1989	359	0.008
47	CCD, rel	Veiga & Vieira Martins (1996)	874 Itajuba	1989–1994	423	0.120
48	phot, abs	Kulyk, Izakevich & Shatkhina (1990)	188 Majdanak	1990–1990	5	0.241
49	CCD, rel	Veiga & Vieira Martins (1998)	874 Itajuba	1995–1997	759	0.186
50	mer, abs	Arlot, Dourneau & Le Campion (2008)	999 Bordeaux	1995–2007	95	0.229
51	CCD, abs	Qiao et al. (2007)	337 Sheshan	1996–2006	943	0.049
52	CCD, abs	Stone (2001)	689 Flagstaff	1998–2000	124	0.118
53	CCD, abs	Owen (1999)	673 Table Mountain	1999–1999	3	0.215
54	CCD, abs	Vieira Martins et al. (2004)	874 Itajuba	2000–2002	66	0.205
55	CCD, abs	Stone & Harris (2000); Stone (2000, 2001)	689 Flagstaff	2000–2012	874	0.138
56	CCD, abs	Owen (2001)	673 Table Mountain	2001–2001	3	0.042
57	CCD, abs	Qiao et al. (2014)	327 Pekin, 337 Sheshan	2007–2009	1095	0.065

**Table 5.** Systematic biases that were determined and excluded from the results of some groups of observations:  $\Delta\alpha$  for right ascension,  $\Delta\delta$  for declination.

Group index	Time interval (yr)	$\Delta\alpha$ (arcsec)	$\Delta\delta$ (arcsec)
45	1986–1993	−0.075	−0.330
50	1995–2007	−0.005	−0.202
51	1996–2006	0.087	−0.052
55	2000–2012	−0.009	−0.067
57	2007–2009	0.027	−0.075

**Table 6.** The root-mean-square residuals of observations after the parameters were fit.

Observations used	$\hat{k} = 0$		with $\hat{k}$ determined	
	$\sigma$ (arcsec)	$\sigma_w$ (arcsec)	$\sigma$ (arcsec)	$\sigma_w$ (arcsec)
All observations	0.228	0.036	0.228	0.036
Without <i>Voyager 2</i> observations	0.234	0.094	0.234	0.094



**Figure 6.** The residuals of the observations of the first type ( $\Delta\alpha \cos \delta$ ,  $s\Delta P$ ).

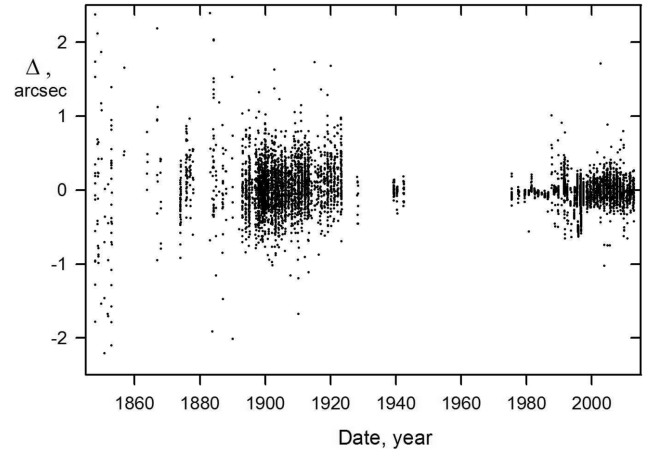
observations, the weighted average residual did not change at all. The fitted parameters  $a$ ,  $I_0$ ,  $u_0$ ,  $\dot{u}$ ,  $\Omega_0$ ,  $\dot{\Omega}$ ,  $\alpha_0$  and  $\delta_0$  obtained for both sets of observations differ at values and these differences are less than their errors calculated by the least-squares method. It is to note also that rough observations were taken into consideration with the small weights assigned to them. For these reasons, we have chosen the variant of the theory where all observations were used.

To see the residuals  $\Delta$  for different dates of observations, we have drawn the graphs of the residuals for two types of the measured values. The first one is residuals in the right ascension and position angle, that is the values

$$\Delta\alpha \cos \delta, \quad s\Delta P.$$

The residuals of the second type are the values  $\Delta\delta$  and  $\Delta s$ . The graphs are given in Figs 6 and 7.

The parameters of Triton's motion obtained from observations without taking into consideration the quadratic term in the orbital longitude are given in Table 7. The case when all the parameters were fit, including  $\hat{k}$ , is illustrated in Table 8. Evaluations of the errors in the parameters given in the tables were obtained by the least-squares method ( $1\sigma$ ). Note that extra digits given in the tables



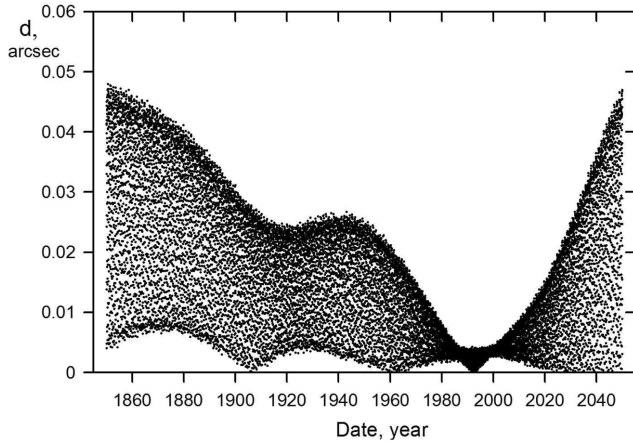
**Figure 7.** The residuals of the observations of the second type ( $\Delta\delta$ ,  $\Delta s$ ).

**Table 7.** The parameters of Triton's motion obtained from observations in the case when  $\hat{k} = 0$ .

Parameter	Without the <i>Voyager 2</i> observations	With all observations
$a$ (km)	354703.78 $\pm 31.01$	354696.76 $\pm 11.84$
$I$ ( $^\circ$ )	158.029057 $\pm 0.210026$	157.268439 $\pm 0.140387$
$u_0$ ( $^\circ$ )	28.456422 $\pm 0.910969$	31.791760 $\pm 0.559837$
$\dot{u}$ (deg d $^{-1}$ )	61.25877585 $\pm 0.00001557$	61.25871809 $\pm 0.00000925$
$\Omega_0$ ( $^\circ$ )	69.657290 $\pm 0.770769$	73.395781 $\pm 0.483272$
$\dot{\Omega}$ (deg d $^{-1}$ )	0.001505781 $\pm 0.000014488$	0.001452458 $\pm 0.000008674$
$\alpha_0$ ( $^\circ$ )	298.586 $\pm 0.126$	299.090 $\pm 0.081$
$\delta_0$ ( $^\circ$ )	42.237 $\pm 0.212$	43.019 $\pm 0.136$

**Table 8.** The parameters of Triton's motion, including  $\hat{k}$ , obtained from observations.

Parameter	Without the <i>Voyager 2</i> observations	With all observations
$a$ (km)	354704.98 $\pm 31.13$	354697.66 $\pm 11.84$
$I$ ( $^\circ$ )	158.030878 $\pm 0.210121$	157.362584 $\pm 0.143993$
$u_0$ ( $^\circ$ )	28.379690 $\pm 0.928935$	31.179424 $\pm 0.610095$
$\dot{u}$ (deg d $^{-1}$ )	61.25877878 $\pm 0.00001705$	61.25873359 $\pm 0.00001111$
$\Omega_0$ ( $^\circ$ )	69.657839 $\pm 0.771176$	72.127604 $\pm 0.495714$
$\dot{\Omega}$ (deg d $^{-1}$ )	0.001505879 $\pm 0.000014496$	0.001458494 $\pm 0.000008997$
$\hat{k}$ (deg yr $^{-2}$ )	−0.0000032 $\pm 0.0000077$	−0.0000098 $\pm 0.0000039$
$\alpha_0$ ( $^\circ$ )	298.583 $\pm 0.126$	299.017 $\pm 0.085$
$\delta_0$ ( $^\circ$ )	42.237 $\pm 0.213$	42.930 $\pm 0.140$



**Figure 8.** The angular distances between Triton's positions calculated from both our theory and the JPL ephemeris (Jacobson 2009).

will be necessary for calculations of the Triton's coordinates since the values of these parameters should be strictly agreed with each other.

We performed calculations of the coefficient  $\hat{k}$  in two versions including the *Voyager 2* observations and without them. The difference of the results turned out to be small in comparison with the evaluation made with the least-squares method ( $1\sigma$ ). The values of  $\hat{k}$  in two versions are comparables with their errors. The analysis of the results of determination of  $\hat{k}$  from observations proves that the accuracy of this parameter is not high.

In result, for calculating the ephemeris of Triton, we recommend to use the parameters obtained with  $\hat{k} = 0$ , taking into consideration all observations, including those made by the *Voyager 2* spacecraft, that is the values in the third column (with all observations) of Table 7.

We have compared the ephemeris of Triton produced with our analytical theory with the ephemeris described in the paper (Jacobson 2009) and available at the JPL HORIZONS ephemeris server (Giorgini et al. 1996). For this, the satellite's geocentric angular coordinates were calculated for a series of time instants using the recommended set of the parameters ( $\hat{k} = 0$ , *Voyager 2* observations included). As the JPL model uses the DE431 planet and lunar ephemeris, in this case we also used the DE431 ephemeris while calculating the ephemerides. For each moment of time, topocentric angular distance  $d$  between the satellite position obtained from both our model and that described in Jacobson (2009) was obtained. In fact, the value of  $d$  was calculated by the formula

$$d = \sqrt{(\Delta\alpha \cos \delta)^2 + (\Delta\delta)^2},$$

where  $\Delta\alpha$  and  $\Delta\delta$  are the differences in the right ascensions and declinations computed with both ephemeris. The graphs for  $d$  at the time interval from 1850 to 2050 are given in Fig. 8. The graphs demonstrate that until 2050 the differences between the satellite positions computed with two different ephemerides will not exceed 0.047 arcsec. These differences are caused by the fact that the ephemeris in comparison are based on differing sets of observations.

## 7 CONCLUSION

The main result presented in this paper is the new analytical model of motion of Neptune's satellite Triton based on the observations. The theory is used to generate the new ephemeris of Triton. The newly produced model and ephemeris have the following advan-

tages compared with earlier results of other researchers. First, the ephemeris can be computed for any time instants by using simple formulae presented in this paper. Secondly, the model of motion is based on the observations of Triton carried out at the time interval between 1847 and 2012 which is 4 yr longer than that used in Jacobson (2009).

To fit the parameters of Triton's motion, 16 096 measurements of the satellite coordinates were used obtained from 10 254 observations. The root-mean-square residual of observations in topocentric angular coordinates is 0.228 arcsec. The corresponding weighted average of the residuals turned out to be 0.036 arcsec.

The new ephemeris of Triton have been compared with those obtained with the JPL ephemeris server and described in Jacobson (2009). As the ephemeris are based on differing sets of observations, the results should differ. At the time interval between 1850 and 2050, the differences between Triton's topocentric positions computed with these different ephemeris do not exceed 0.047 arcsec.

An attempt has also been made to determine the coefficient  $\hat{k}$  of the quadratic term in the satellite's orbital longitude. The quadratic term may be caused by a perturbation force of dissipative properties that was neglected in the equations of satellite motion or by the errors of observations. We performed calculations of the coefficient  $\hat{k}$  in two versions including the *Voyager 2* observations and without them. The difference of the results turned out to be small in comparison with the evaluation made with the least-squares method ( $1\sigma$ ). The values of  $\hat{k}$  in two versions are comparables with their errors. It is clear that the available observations do not allow to determine the quadratic term in orbital longitude reliably.

The new ephemeris of Triton are added to the Natural Satellites Ephemeride Server MULTI-SAT (Emel'yanov & Arlot 2008).

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