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# LEARNING CHARME MODELS WITH (DEEP) NEURAL NETWORKS

JOSÉ G. GÓMEZ-GARCÍA\*, JALAL FADILI\*\*, AND CHRISTOPHE CHESNEAU\*

**ABSTRACT.** In this paper, we consider a model called CHARME (Conditional Heteroscedastic Autoregressive Mixture of Experts), a class of generalized mixture of nonlinear nonparametric AR-ARCH time series. Under certain Lipschitz-type conditions on the autoregressive and volatility functions, we prove that this model is stationary, ergodic and  $\tau$ -weakly dependent. These conditions are much weaker than those presented in the literature that treats this model. Moreover, this result forms the theoretical basis for deriving an asymptotic theory of the underlying (non)parametric estimation, which we present for this model. As an application, from the universal approximation property of neural networks (NN), possibly with deep architectures, we develop a learning theory for the NN-based autoregressive functions of the model, where the strong consistency and asymptotic normality of the considered estimator of the NN weights and biases are guaranteed under weak conditions.

## 1. INTRODUCTION

Statistical models such as AR, ARMA, ARCH, GARCH, ARMA-GARCH, etc. are still popular today in time series analysis (see [41, Part III]). These time series are part of the general class of models called conditional heteroscedastic autoregressive nonparametric (CHARN) process, which takes the form

$$X_t = f(X_{t-1}, \dots, X_{t-p}, \theta^0) + g(X_{t-1}, \dots, X_{t-p}, \lambda^0)\epsilon_t, \quad (1.1)$$

with unknown functions  $f$ ,  $g$  and independent identically distributed zero-mean innovations  $\epsilon_t$ , which provides a flexible class of models for many applications, for example in econometrics or finance, see [22] and [19]. However, in practice, note that it is not always realistic to assume that the observed process has the same trend function  $f$  and the same volatility function  $g$  at each time point (this is for instance the case of EEG signals, see [35]). In particular, if those functions change slowly over time, local stationarity can be assumed (see [6]), in which there is already a good list of appropriate models. Anyway, estimation procedures for those models are mainly based on applying estimators for stationary processes locally in time which do not work well if the structure of the time series generating mechanism changes more or less abruptly. In this paper, we consider a more general class of nonparametric models (called CHARME), which adapt to situations where explosive phases may be included. The basics of this new class are presented below.

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**1.1. The CHARME model.** Let  $(E, \|\cdot\|)$  be a Banach space, and  $E$  endowed with its Borel  $\sigma$ -algebra  $\mathcal{E}$ . The product Banach space  $E^p$  is naturally endowed with its product  $\sigma$ -algebra  $\mathcal{E}^{\otimes p}$ . The conditional heteroscedastic  $p$ -autoregressive mixture of experts CHARME( $p$ ) model, with values in  $E$ , is the random process defined by

$$X_t = \sum_{k=1}^K \xi_t^{(k)} (f_k(X_{t-1}, \dots, X_{t-p}, \theta_k^0) + g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k^0) \epsilon_t) \quad t \in \mathbb{Z}, \quad (1.2)$$

where

- for each  $k \in [K] := \{1, 2, \dots, K\}$ ,  $f_k : E^p \times \Theta_k \rightarrow E$  and  $g_k : E^p \times \Lambda_k \rightarrow \mathbb{R}$  are the so-called autoregressive and volatility functions, with  $\Theta_k$  and  $\Lambda_k$  the spaces of parameters, are respectively  $\mathcal{E}^{\otimes p} \times \mathcal{B}(\Theta_k)$ - and  $\mathcal{E}^{\otimes p} \times \mathcal{B}(\Lambda_k)$ -measurable functions, where  $\mathcal{B}(\Theta_k)$  is the Borel field on  $\Theta_k$  and similarly for  $\Lambda_k$ ;
- $(\epsilon_t)_t$  are  $E$ -valued independent identically distributed (iid) zero-mean innovations;
- $\xi_t^{(k)} = \mathbb{I}_{\{R_t=k\}}$ , with  $\mathbb{I}_{\mathcal{C}}$  the characteristic function of  $\mathcal{C}$  (takes 1 on  $\mathcal{C}$  and 0 otherwise), where  $(R_t)_{t \in \mathbb{Z}}$  is an iid sequence with values in a finite set of states  $[K]$ , which is independent of the innovations  $(\epsilon_t)_{t \in \mathbb{Z}}$ . In the sequel, we will denote  $\pi_k = \mathbb{P}(R_0 = k)$ .

Model (1.2) can be extended to the case where  $p = \infty$ , called CHARME with infinite memory, denoted by CHARME( $\infty$ ) for short. For the related setting, we will define the subset of  $E^{\mathbb{N}}$

$$E^\infty := \{(x_k)_{k>0} \in E^{\mathbb{N}} : x_k = 0 \text{ for } k > N, \text{ for some } N \in \mathbb{N}^*\},$$

which will be considered with its product  $\sigma$ -algebra  $\mathcal{E}^{\otimes \mathbb{N}}$ .

It is obvious that the model (1.2) contains the model (1.1) (corresponding to the case  $K = 1$  in (1.2)). On the other hand, applications of the CHARME model (1.2) have been directly and indirectly seen in various areas, such as financial analysis [47] (for asset management and risk analysis) and [51] (for predictions of daily probability distributions of S&P returns), hydrology [28] (for the detection of structural changes in hydrological data), electroencephalogram (EEG) signals [34] (for the analysis of EEG recordings from human subjects during sleep), among others.

**1.2. Contributions.** The objective of this article is to build an estimation theory for the CHARME and feedforward neural network (NN) based CHARME models. In this regard, we first approach the CHARME model in a general context, showing its  $\tau$ -weak dependence, ergodicity and stationarity under weak conditions. This consequence together with simple conditions allow us to establish strong consistency for the estimators of the parameters  $(\theta_k^0, \lambda_k^0)_{k \in [K]}$  of the model (1.2), which are the minimizers of a general loss function, not necessarily differentiable. Addressing non-differentiable losses and non iid samples is quite challenging and necessitate to invoke intricate arguments from the calculus of variations (in particular on normal integrands and epi-convergence; see Section 4). Such arguments are not that common in the statistical literature and allow us to investigate new cases that have not been considered before. Additionally, under the same weak assumptions to obtain ergodicity and stationarity together with usual regularity conditions on the autoregressive functions, we prove the asymptotic normality of the conditional least-squares estimator of a simpler CHARME model (i.e., (1.2) with  $g_k \equiv 1$ ).

For the NN-based CHARME( $p$ ) model (i.e., the CHARME( $p$ ) model with NN-based autoregressive functions), we specialize the above results that will ensure establish learning guarantees.

Our results are not limited to the case  $p$  finite. Indeed, we will show that the stationary solution of the CHARME( $\infty$ ) model can be approximated by the stationary solution of its associated CHARME( $p$ ) model (see Remark 3.1 and (3.4)), when  $p$  is large enough. Moreover, in Section 6.3, we will argue that CHARME( $p$ ) models can be universally approximated by NN-based, possibly with deep architectures, CHARME( $p$ ) models. Altogether, this will provide us with a provably controlled way to learn infinity memory CHARME models with (deep) neural networks.

**1.3. Relation to prior work.** Stockis *et al.* [45] show geometric ergodicity of CHARME( $p$ ) models, with  $p < \infty$ , under certain conditions, including regularity. Specifically, they demand that the iid random variables  $\epsilon_t$  have a continuous density function, positive everywhere. In contrast, in this paper the innovations are not supposed to be absolutely continuous and our approach can also be applied, for example, to discrete state space processes. Note also that [45] uses this regularity condition in order to obtain some mixing conditions of  $\eta_t = (X_t, \xi_t)_{t \in \mathbb{Z}}$  for deriving asymptotic stability of the model through the results of [37]. However, observe that taking a simple model as the AR(1)-input, solution of the recursion

$$X_t = \frac{1}{2}(X_{t-1} + \epsilon_t), \quad t \in \mathbb{Z}, \quad (1.3)$$

with  $(\epsilon_t)_{t \in \mathbb{Z}}$  iid such that  $\mathbb{P}(\epsilon_0 = 0) = \mathbb{P}(\epsilon_0 = 1) = 1/2$ , we can see that the assumptions in [45] are not satisfied. In fact, this model is not mixing, see [1]. On the other hand, this model is  $\tau$ -weakly dependent and satisfies all our assumptions, see [9].

**1.4. Paper organization.** The paper is organized as follows: In Section 2 we start with the preliminaries such as the definition and most important properties of  $\tau$ -weak dependence which characterize our model, and a summary of neural networks. In Section 3 we study the properties of ergodicity and stationarity of the CHARME model, which will be essential for developing a theory of estimation of the model. In Section 4 we provide estimators of the parameters of the model (1.2) and we prove its strong consistency under very weak conditions. Asymptotic normality of the conditional least-squares estimator is also established in Section 5, but for a simpler CHARME model (the model (1.2) with  $g_k \equiv 1$ ) in order to simplify the presentation. In Section 6 we discuss the previous results in the context of NN-based CHARME models and examine the difference between approximation and exact modeling by NNs. Numerical experiments are included in Section 7 and the proofs in Section 8.

## 2. PRELIMINARIES

Let  $(E, \|\cdot\|)$  be a Banach space and  $h : E \rightarrow \mathbb{R}$ . We define  $\|h\|_\infty = \sup_{x \in E} |h(x)|$  and the Lipschitz constant/modulus of  $h$  as

$$\text{Lip}(h) = \sup_{x \neq y \in E} \frac{|h(x) - h(y)|}{\|x - y\|}.$$

For an  $E$ -valued random variable  $X$  defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and  $m \geq 1$ , we denote by  $\|\cdot\|_m$  the  $\mathbb{L}^m$ -norm, i.e.,  $\|X\|_m = (\mathbb{E}\|X\|^m)^{1/m}$ , where  $\mathbb{E}$  denotes the expectation.

**2.1. Weak dependence.** The appropriate notion of weak dependence for the model (1.2) was introduced in [9]. It is based on the concept of the coefficient  $\tau$  defined below.

*Definition 2.1* ( $\tau$ -dependence). Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space.  $\mathcal{M}$  a  $\sigma$ -sub-algebra of  $\mathcal{A}$  and  $X$  a random variable with values in  $E$  such that  $\|X\|_1 < \infty$ . The coefficient  $\tau$  is defined as

$$\tau(\mathcal{M}, X) = \mathbb{E} \left| \sup \left\{ \left| \int_E h(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int_E h(x) \mathbb{P}_X(dx) \right| : h \text{ s.t. } \text{Lip}(h) \leq 1 \right\} \right|.$$

Note that if  $Y$  is any random variable with the same distribution as  $X$  and independent of  $\mathcal{M}$ , then

$$\tau(\mathcal{M}, X) \leq \|X - Y\|_1.$$

This is a coupling argument that allows us to easily bound the  $\tau$  coefficient. See the examples in [9]. On the other hand, if the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  is rich enough (which we always assume in the sequel), there exists  $X^*$  with the same distribution as  $X$  and independent of  $\mathcal{M}$  such that  $\tau(\mathcal{M}, X) = \|X - X^*\|_1$ .

Using the definition of this  $\tau$  coefficient with the  $\sigma$ -algebra  $\mathcal{M}_p = \sigma(X_t, t \leq p)$  and the norm  $\|x - y\| = \|x_1 - y_1\| + \dots + \|x_k - y_k\|$  on  $E^k$ , we can assess the dependence between the past of the sequence  $(X_t)_{t \in \mathbb{Z}}$  and its future  $k$ -tuples through the coefficients

$$\tau_k(r) = \max_{1 \leq l \leq k} \frac{1}{l} \sup \{ \tau(\mathcal{M}_p, (X_{j_1}, \dots, X_{j_l})) \text{ with } p + r \leq j_1 < \dots < j_l \}.$$

Finally, denoting  $\tau(r) := \tau_\infty(r) = \sup_{k > 0} \tau_k(r)$ , the time series  $(X_t)_{t \in \mathbb{Z}}$  is called  *$\tau$ -weakly dependent* if its coefficients  $\tau(r)$  tend to 0 as  $r$  tends to infinity.

**2.2. Neural networks.** Neural networks produce structured parametric families of functions that have been studied and used for almost 70 years, going back to the late 1940's and the 1950's [24, 42]. An often cited theoretical feature of neural networks, known since the 1980's, is their universal approximation capacity [27], i.e., given any continuous target function  $f$  and a target accuracy  $\epsilon > 0$ , neural networks with enough judiciously chosen parameters give an approximation to the function within an error of size  $\epsilon$ .

Modern machine learning emphasizes the use of deep architectures (as opposed to shallow networks popular in the 1980's-1990's). A few recent works have focused on the advantages of deep versus shallow architectures in neural networks by showing that deep networks can approximate many interesting functions more efficiently, per parameter, than shallow networks (see [23, 49, 52, 53, 8] for a selection of rigorous results).

It appears then natural to use this property when it comes to model the functions  $f_k$  and  $g_k$ ,  $k \in [K]$ , of the process (1.2).

*Definition 2.2.* Let  $d, L \in \mathbb{N}$ . A fully connected feedforward neural network with input dimension  $d$ ,  $L$  layers and activation map  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , is a collection of weight matrices  $(W^{(l)})_{l \in [L]}$  and bias vectors  $(b^{(l)})_{l \in [L]}$ , where  $W^{(l)} \in \mathbb{R}^{N_l \times N_{l-1}}$  and  $b^{(l)} \in \mathbb{R}^{N_l}$ , with  $N_0 = d$ , and  $N_l \in \mathbb{N}$  is the number of neurons for layer  $l \in [L]$ . Let's gather these parameters in the

vector

$$\theta = ((W^{(1)}, b^{(1)}), (W^{(2)}, b^{(2)}), \dots, (W^{(L)}, b^{(L)})) \in \prod_{l=1}^L \mathbb{R}^{N_l \times N_{l-1}} \times \mathbb{R}^{N_l}.$$

Then, a neural network parametrized by<sup>1</sup>  $\theta$  produces a function

$$f : (x, \theta) \in \mathbb{R}^d \times \left( \prod_{l=1}^L \mathbb{R}^{N_l \times N_{l-1}} \times \mathbb{R}^{N_l} \right) \mapsto f(x, \theta) = x^{(L)} \in \mathbb{R}^{N_L},$$

where  $x_L$  results from the recursion:

$$\begin{aligned} x^{(0)} &:= x, \\ x^{(l)} &:= \varphi(W^{(l)}x^{(l-1)} + b^{(l)}), \quad \text{for } l = 1, \dots, L-1, \\ x^{(L)} &:= W^{(L)}x^{(L-1)} + b^{(L)}, \end{aligned}$$

where  $\varphi$  acts componentwise, i.e., for  $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ ,  $\varphi(y) = (\varphi(y_1), \dots, \varphi(y_N))$ .

The rectified linear unit (ReLU) is the activation map of preference in many applications, but other examples of activation maps in the literature include the sigmoid, softplus, ramp or other activations [44, Chapter 20.4].

Let us close this section by insisting on the fact that although a theory that completely quantifies the approximation power of (deep) neural networks is still lacking, there is an active literature that provides increasing evidence that deep neural networks enjoy strong approximation and expressive powers. In particular, the work of [8] has shown that neural networks with sufficient depth and appropriate width, possess greater expressivity and approximation power than traditional methods of nonlinear approximation. They also exhibited large classes of functions which can be exactly or efficiently captured by neural networks whereas classical nonlinear methods fall short of the task.

### 3. ERGODICITY AND STATIONARITY OF CHARME MODELS

In this section we study the properties of ergodicity and stationarity of the model (1.2) for the general case, i.e., for the case  $p = \infty$ , because the case  $p < \infty$  is a straightforward corollary. In turn, these properties will be instrumental in establishing statistical inference guarantees.

**Theorem 3.1.** *Consider the CHARME( $\infty$ ) model, i.e., (1.2) with  $p = \infty$ . Assume that there exist non-negative real sequences  $(a_i^{(k)})_{i \geq 1, k \in [K]}$  and  $(b_i^{(k)})_{i \geq 1, k \in [K]}$ , such that for any  $x, y \in E^\infty$ , and  $\forall k \in [K]$ ,*

$$\|f_k(x, \theta_k^0) - f_k(y, \theta_k^0)\| \leq \sum_{i=1}^{\infty} a_i^{(k)} \|x_i - y_i\|, \quad |g_k(x, \theta_k^0) - g_k(y, \theta_k^0)| \leq \sum_{i=1}^{\infty} b_i^{(k)} \|x_i - y_i\|. \quad (3.1)$$

Denote  $A_k = \sum_{i=1}^{\infty} a_i^{(k)}$ ,  $B_k = \sum_{i=1}^{\infty} b_i^{(k)}$  and  $C(m) = 2^{m-1} \sum_{k=1}^K \pi_k (A_k^m + B_k^m \|\epsilon_0\|_m^m)$ . The following holds:

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<sup>1</sup>We intentionally omit the explicit dependence on  $\varphi$  since the latter is chosen once for all.

(i) if  $c := C(1) < 1$ , then there exists a  $\tau$ -weakly dependent strictly stationary solution  $(X_t)_{t \in \mathbb{Z}}$  of CHARME( $\infty$ ) which belongs to  $\mathbb{L}^1$ , and such that

$$\tau(r) \leq 2 \frac{\mu_1}{1-c} \inf_{1 \leq s \leq r} \left( c^{r/s} + \frac{1}{1-c} \sum_{i=s+1}^{\infty} c_i \right) \xrightarrow{r \rightarrow \infty} 0, \quad (3.2)$$

where

$$\mu_1 = \sum_{k=1}^K \pi_k (\|f_k(0, \theta_k^0)\| + |g_k(0, \theta_k^0)| \|\epsilon_0\|_1) \text{ and } c_i = \sum_{k=1}^K \pi_k \left( a_i^{(k)} + b_i^{(k)} \|\epsilon_0\|_1 \right).$$

(ii) if moreover  $C(m) < 1$  for some  $m > 1$ , then the stationary solution belongs to  $\mathbb{L}^m$ .

*Corollary 3.1.* Consider the CHARME( $p$ ) model (1.2) and suppose that the inequalities (3.1) hold (in this case  $a_i^{(k)} = b_i^{(k)} = 0$  for all  $i > p$  and all  $k \in [K]$ ). Under the notations of Theorem 3.1, if  $c < 1$ , then there exists a  $\tau$ -weakly dependent stationary solution  $(X_t)_{t \in \mathbb{Z}}$  of CHARME( $p$ ) which belongs to  $\mathbb{L}^1$  and such that  $\tau(r) \leq 2\mu_1(1-c)^{-1}c^{r/p}$  for  $r \geq p$ . Moreover, if  $C(m) < 1$  for some  $m > 1$ , then this solution belongs to  $\mathbb{L}^m$ .

*Remark 3.1.*

- (1) Consider the assumptions of Theorem 3.1. The Lipschitz-type assumption (3.1) entails continuity of  $f_k(\cdot, \theta_k^0)$  and  $g_k(\cdot, \theta_k^0)$ , whence we deduce continuity of  $F$  as defined in (8.2). It then follows from [13, Lemma 5.5] and the completeness of  $\mathbb{L}^m$ , that there exists a measurable function  $H$  such that the CHARME( $\infty$ ) process can be written as  $X_t = H(\tilde{\xi}_t, \tilde{\xi}_{t-1}, \dots)$ , where  $\tilde{\xi}_t := (\epsilon_t, \xi_t^{(1)}, \dots, \xi_t^{(K)}) = (\epsilon_t, \xi_t) \in E \times \{e_1, \dots, e_K\}$ , where  $e_1, \dots, e_K$  are the canonical basis vectors for  $\mathbb{R}^K$ . In other words, the CHARME( $\infty$ ) process can be represented as a causal Bernoulli shift. Moreover, under these assumptions,  $(X_t)_{t \in \mathbb{Z}}$  is the unique causal Bernoulli shift solution to (1.2) with  $p = \infty$ . Therefore, the solution  $(X_t)_{t \in \mathbb{Z}}$  is automatically an ergodic process. Finally, the ergodic theorem implies the SLLN for this process. This consequence of Theorem 3.1 will be a key to establish strong consistency when it comes to estimating the autoregressive and volatility functions of the CHARME( $p$ ) model.
- (2) Using the arguments in [13], it can be shown that the stationary solution of CHARME( $\infty$ ) can be approximated by a stationary solution of the CHARME( $p$ ) model (1.2) for some large value of  $p$ . In fact, the bounds of the weak dependence coefficients of [13, Theorem 3.1] come from an approximation with Markov chains of order  $p$  along with its weak dependence and stationarity properties (see [13, Corollary 3.1]). Indeed, let  $X_t$  be the stationary solution of the CHARME( $\infty$ ) model and let  $X_{p,t}$  be the stationary solution of its associated CHARME( $p$ ) model, i.e.,

$$X_{p,t} = F(X_{p,t-1}, \dots, X_{p,t-p}, 0, 0, \dots; \tilde{\xi}_t), \quad (3.3)$$

where  $F$  is defined in (8.2). Then, [13, Lemma 5.5] gives

$$\mathbb{E} \|X_t - X_{p,t}\| \leq \frac{\mu_1}{(1-c)^2} \sum_{i=p+1}^{\infty} c_i. \quad (3.4)$$

- (3) In [45], the authors show that CHARME( $p$ ) is geometrically ergodic for  $p < \infty$  considering the process  $(R_t)_{t \in \mathbb{Z}}$  as a first-order irreducible and aperiodic strictly stationary Markov chain, together with a list of conditions. In particular, they demand that the iid random variables  $\epsilon_t$  have a continuous density function, positive everywhere. In

contrast, in this paper the innovations are not supposed to be absolutely continuous and our approach can also be applied to discrete state space processes. We refer the reader to [15, 16, 17, 18, 12, 11, 10].

Additionally, in [45], the geometric ergodicity of  $\eta_t = (X_t, \xi_t)$ ,  $t \in \mathbb{Z}$ , has been shown in order to obtain some mixing conditions of  $(\eta_t)_{t \in \mathbb{Z}}$  for deriving asymptotic stability of the model and, therefore, for formalizing an asymptotic theory for non-parametric estimation. However, note that, by taking the simple AR(1) model defined in (1.3), we can see that this does not satisfy some the assumptions in [45]. In fact, the AR(1) process (1.3) is not mixing, see [1]. It turns out that the main restrictions of the mixing processes are the regularity conditions required for the noise process  $(\epsilon_t)_{t \in \mathbb{Z}}$ . These regularity conditions, however, are not needed within the framework of  $\tau$ -dependence. For example, the process (1.3) is  $\tau$ -weakly dependent with  $\tau(r) \leq 2^{-r} \sqrt{1/6}$ ; see [9, Application 1].

#### 4. ESTIMATION OF CHARME PARAMETERS: CONSISTENCY

The goal now is to design consistent estimators of the parameters  $(\theta_k, \lambda_k)_{k \in [K]}$  of model (1.2). In the sequel, we will denote the space of parameters as the product spaces  $\Theta := \times_{k=1}^K \Theta_k$  and  $\Lambda := \times_{k=1}^K \Lambda_k$ .

Let  $(X_t)_{-p+1 \leq t \leq n}$ <sup>2</sup> be  $n$  observations of a strictly stationary solution  $(X_t)_{t \in \mathbb{Z}}$  of the model (1.2) (which exists by Theorem 3.1). We assume that the number of states  $K$  is known. In general, a careful analysis of the model usually provides interpretation for the states in terms of physical significance or economical meaning. We also assume that we have access to observations of the hidden iid variables  $(R_t)_{-p+1 \leq t \leq n}$ , or equivalently, the variables  $(\xi_t^{(k)})_{-p+1 \leq t \leq n, k \in [K]}$ . A similar assumption can be found in the literature for special cases of CHARME, see e.g., [47, 45].

Our goal is to study a nonlinear estimator for the parameters

$$(\theta^0, \lambda^0) := (\theta_1^0, \dots, \theta_K^0, \lambda_1^0, \dots, \lambda_K^0)$$

of the CHARME( $p$ ) model (1.2) from observations  $(X_t)_{-p+1 \leq t \leq n}$  and  $(\xi_t^{(k)})_{-p+1 \leq t \leq n, k \in [K]}$ . This will be achieved through solving the minimization problem

$$\begin{aligned} (\widehat{\theta}_n, \widehat{\lambda}_n) &\in \text{Argmin}_{(\theta, \lambda) \in \Theta \times \Lambda} Q_n(\theta, \lambda), \text{ where} \\ Q_n(\theta, \lambda) &:= \frac{1}{n} \sum_{t=1}^n \sum_{k=1}^K \xi_t^{(k)} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)). \end{aligned} \quad (4.1)$$

Here,  $\ell : E \times E \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is some loss function. Typically,  $\ell$  would satisfy  $\ell(u, u, \tau) = 0, \forall \tau$ . Observe that we allow  $\ell$  to be extended-real-valued (i.e., possibly taking value  $+\infty$ ). This will allow to deal equally well with non-classical (and challenging) situations as would be the case if we wanted to include some information/constraints one might have about certain parameters and the relationships between them in the estimation process. Handling

<sup>2</sup>With a slight abuse of notations, we use the same symbol for the observations.



extended-real-valued functions when establishing consistency theorems is very challenging which will necessitate more sophisticated arguments.

It will be convenient to define the processes

$$Y_t = (X_{t-p}, X_{t-p+1}, \dots, X_t) \quad \text{and} \quad \xi_t = (\xi_t^1, \dots, \xi_t^K), \quad t \in \mathbb{Z}.$$

Observations  $(X_t)_{-p+1 \leq t \leq n}$  yield observations  $(Y_t)_{1 \leq t \leq n}$ . Denote  $\{e_1, \dots, e_K\}$  be the set of canonical basis vectors for  $\mathbb{R}^K$ . Let  $(E^{p+1} \times \{e_1, \dots, e_K\}, \mathcal{E}^{\otimes(p+1)} \otimes \Xi, P)$  the (common) probability space on which the random vectors  $Y_t$  and  $\xi_t$  are defined. We use the shorthand notation

$$h(Y_t, \xi_t, \theta, \lambda) := \sum_{k=1}^K \xi_t^{(k)} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)). \quad (4.2)$$

Consistency will be established under the following assumptions. We will denote  $\text{rang}_g := \bigcup_{k \in [K]} g_k(E^p, \Lambda_k) \subset \mathbb{R}$ ; i.e., the union of the ranges of the functions  $g_k$ .

- (A.1)  $\mathcal{E}^{\otimes(p+1)} \otimes \Xi$  is  $P$ -complete, namely, a subset of a null set in  $\mathcal{E}^{\otimes(p+1)} \otimes \Xi$  also belongs to  $\mathcal{E}^{\otimes(p+1)} \otimes \Xi$ .
- (A.2) For each  $k \in [K]$ ,  $\Theta_k$  and  $\Lambda_k$  are Polish spaces, i.e., a complete, separable, metric spaces.
- (A.3) For any  $k \in [K]$ ,  $f_k$  and  $g_k$  are Carathéodory mappings, i.e.,  $f_k(X_1, \dots, X_p, \theta_k)$  (resp.  $g_k(X_1, \dots, X_p, \lambda_k)$ ) is  $\mathcal{E}^{\otimes p}$ -measurable in  $(X_1, \dots, X_p)$  for each fixed  $\theta_k$  (resp.  $\lambda_k$ ) and continuous in  $\theta_k$  (resp.  $\lambda_k$ ) for each fixed  $(X_1, \dots, X_p)$ .
- (A.4)  $\ell$  is  $\mathcal{E} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\text{rang}_g)$ -measurable, and for every  $u \in E$ ,  $(v, \tau) \in E \times \text{rang}_g \mapsto \ell(u, v, \tau)$  is lower semicontinuous (lsc).
- (A.5)  $\inf(\ell) \geq 0$ .
- (A.6) For each  $k \in [K]$  and  $t \in [n]$ , there exists  $\bar{\theta}_k \in \Theta_k$  such that  $f_k(X_{t-1}, \dots, X_{t-p}, \bar{\theta}_k) = 0$ .
- (A.7) There exists non-negative constants  $C$  and  $c$ , and  $\gamma > 0$ , such that for all  $k \in [K]$  and  $t \in [n]$ ,

$$\inf_{\lambda_k \in \Lambda_k} \ell(X_t, 0, g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \leq C \|X_t\|^\gamma + c.$$

Before proceeding, some remarks on these assumptions are in order.

*Remark 4.1.*

1. The completeness assumption (A.1) is harmless and for technical convenience. Standard techniques can be used to eliminate it.
2. Functions verifying Assumption (A.4) are known as *random lsc* or *normal integrands*. The concept of a random lsc function is due to [39], who introduced it in the context of the calculus of variations under the name of normal integrand. Properties of random lsc functions are studied [40, Chapter 14]. The proof of our consistency theorem will rely on stability properties of the family of random lsc functions under various operations, and on their powerful ergodic properties set forth in the series of papers [31, 30, 32]. Unlike other works on the Law of Large Numbers for random lsc functions [4, 2, 26], which postulate iid sampling, only stationarity is needed in our

context.

3. Lower-semicontinuity wrt the parameters is a much weaker assumption than those found in the literature. In addition to allowing to handle constraints on the parameters easily (see the discussion after Theorem 4.1), it will also allow for non-smooth activations maps in DNN-based learning such as the very popular ReLU. In fact, even continuity is not needed in our context whereas differentiability is an important assumption in existing works; see e.g., [45, 47].
4. Assumption (A.5) can be weakened to lower-boundedness by a negative combination of powers (with appropriate exponents) of the norm. We leave the details to the interested reader.
5. Assumption (A.6) is quite natural and is verified in most applications we have in mind (e.g., neural networks).
6. Our proof technique does not really need  $p$  to be finite. Thus our result can be extended equally well to the CHARME( $\infty$ ) model by considering the process  $Y_t$  as valued in  $E^\infty$  and assume  $\mathcal{E}^{\otimes \mathbb{N}}$ -measurability in our assumptions.

*Example 4.1.* A prominent example in applications is where the loss function  $\ell$  takes the form

$$\ell(u, v, \tau) = \frac{\|u - v\|^\gamma}{|\tau|^\gamma}, \quad \gamma > 0.$$

In view of the role played by  $\tau$ , it is natural to impose the following assumption on  $g_k$ :

$$(A_g) \quad \exists \delta > 0 \text{ such that } \forall k \in [K], \inf_{x_1, \dots, x_p, \lambda_k} |g_k(x_1, \dots, x_p, \lambda_k)| \geq \delta.$$

Let us show that  $\ell$  comply which assumptions (A.4), (A.5) and (A.7). (A.5) is obviously verified. As for (A.7), we have from (A<sub>g</sub>) that

$$\ell(X_t, 0, g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \leq \delta^{-\gamma} \|X_t\|^\gamma,$$

whence assumption (A.7) holds with  $C = \delta^{-\gamma}$  and  $c = 0$ . It remains to check (A.4). Since (A<sub>g</sub>) implies that  $0 \notin \text{ran}_g = \mathbb{R} \setminus ]-\delta, \delta[$ , continuity of the norm and (A<sub>g</sub>) entail that  $\ell$ , which is the the ratio of continuous functions on Borel spaces, is continuous, hence a Borel function.

We are now in position to state our consistency theorem.

**Theorem 4.1.** *Let  $(X_t)_{t \in \mathbb{Z}}$  be a strictly stationary ergodic solution of (1.2), which exists under the assumptions of Theorem 3.1 with  $C(m) < 1$  for some  $m \geq 1$ . Let  $(\widehat{\theta}_n, \widehat{\lambda}_n)$  the estimator defined by (4.1), and assume that (A.1)-(A.7) are verified with  $\gamma = m$ . Then, the following holds:*

- (i) *each cluster point of  $(\widehat{\theta}_n, \widehat{\lambda}_n)_{n \in \mathbb{N}}$  belongs to  $\text{Argmin}_{(\theta, \lambda) \in \Theta \times \Lambda} \mathbb{E}h(Y, \xi, \theta, \lambda)$  a.s.*
- (ii) *if moreover the sequence  $(Q_n)_{n \in \mathbb{N}}$  is equi-coercive, and  $\text{Argmin}_{(\theta, \lambda) \in \Theta \times \Lambda} \mathbb{E}h(Y, \xi, \theta, \lambda) = \{\theta^0, \lambda^0\}$ , then*

$$(\widehat{\theta}_n, \widehat{\lambda}_n) \rightarrow (\theta^0, \lambda^0) \quad \text{and} \quad Q_n(\widehat{\theta}_n, \widehat{\lambda}_n) \rightarrow \mathbb{E}h(Y, \xi, \theta^0, \lambda^0) \quad \text{a.s.}$$

Recall that a sequence of functions  $(\phi_n)_{n \in \mathbb{N}}$  is equi-coercive if there exists a lsc coercive function  $\psi$  such that  $\phi_n \geq \psi$ ,  $\forall n \in \mathbb{N}$ , see [7, Definition 7.6 and Proposition 7.7]. This entails in particular that the sublevel sets of the functions  $\phi_n$  are compact<sup>3</sup> uniformly in  $n$ .

For instance, a sufficient condition to ensure equi-coerciveness in our context is that, for each  $k \in [K]$ , there exists a  $\mathcal{B}(\Theta_k) \otimes \mathcal{B}(\Lambda_k)$ -measurable compact subset  $\mathcal{C}_k \subset \Theta_k \times \Lambda_k$  such that<sup>4</sup>

$$\text{dom}(\ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \cdot), g_k(X_{t-1}, \dots, X_{t-p}, \cdot))) \subset \mathcal{C}_k, \quad \forall t \in [n].$$

Indeed, it is immediate to see that such a condition entails

$$\text{dom}(Q_n) \subset \bigtimes_{k=1}^K \mathcal{C}_k,$$

which is a compact set.

The sets  $\mathcal{C}_k$  can be used to impose some prior constraints on the parameters  $(\theta_k, \lambda_k)$  which might follow from certain physical, economic or mathematical considerations. For instance, these parameters can be constrained to comply with the strict stationarity assumption in Theorem 3.1. Other constraints can be also used to remove some underdeterminacies as in the case of neural networks (see Section 6 for further discussion). In general, to account for constraints, one sets  $\ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) = +\infty$  for the unfeasible cases, or more precisely,

$$\begin{aligned} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) = \\ \tilde{\ell}(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) + \iota_{\mathcal{C}_k}(\theta_k, \lambda_k), \quad \forall k \in [K], \end{aligned}$$

where  $\tilde{\ell}$  is a full-domain loss verifying (A.4), (A.5) and (A.7), and  $\iota_{\mathcal{C}_k}$  is the indicator function of  $\mathcal{C}_k$ , taking 0 on  $\mathcal{C}_k$  and  $+\infty$  otherwise. By assumptions on  $\mathcal{C}_k$ ,  $\iota_{\mathcal{C}_k}$  is  $\mathcal{B}(\Theta_k) \otimes \mathcal{B}(\Lambda_k)$ -measurable and lsc, and thus  $\ell$  inherits (A.4) from  $\tilde{\ell}$ . (A.5) is trivially verified, and for (A.6) to hold, it is necessary and sufficient that for each  $k \in [K]$  and  $t \in [n]$ ,  $f_k(X_{t-1}, \dots, X_{t-p}, \cdot)^{-1}(0) \times \Lambda_k \cap \mathcal{C}_k \neq \emptyset$ .

We finally stress that the constraints above need not be separable, as soon as one takes  $h(Y_t, \xi_t, \theta, \lambda)$  as

$$h(Y_t, \xi_t, \theta, \lambda) = \sum_{k=1}^K \xi_t^{(k)} \tilde{\ell}(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) + \iota_{\mathcal{C}}(\theta, \lambda),$$

where  $\mathcal{C} \subset \Theta \times \Lambda$  is a  $\mathcal{B}(\Theta) \otimes \mathcal{B}(\Lambda)$ -measurable compact set. Thus, depending on the application at hand, our reasoning above can be extended to more complicated situations.

<sup>3</sup>We here specialized [7, Definition 7.6] to metric spaces (see (A.2)) where compactness implies closeness and countable compactness.

<sup>4</sup>Observe that accounting for this constraint does not compromise assumption (A.4) thanks to compactness of  $\mathcal{C}_k$ .

## 5. ESTIMATION OF CHARME PARAMETERS: ASYMPTOTIC NORMALITY

To establish asymptotic normality, we need to restrict ourselves to a finite-dimensional framework where  $E = \mathbb{R}^d$  and  $\Theta_k = \mathbb{R}^{d_k}$ . Throughout this section,  $\|\cdot\|$  denotes the standard Euclidean norm and the corresponding (Euclidean) space is to be understood from the context.

In this section, we consider the following constant-volatility special case of the model in (1.2):

$$X_t = \sum_{k=1}^K \xi_t^{(k)} (f_k(X_{t-1}, \dots, X_{t-p}, \theta_k^0) + \epsilon_{k,t}) \quad t \in \mathbb{Z}. \quad (5.1)$$

We then specialize the estimator in (4.1) to (5.1) and the quadratic loss, which now reads

$$\hat{\theta}_n \in \operatorname{Argmin}_{\theta \in \Theta} \left\{ Q_n(\theta) := \frac{1}{n} \sum_{t=1}^n \sum_{k=1}^K \xi_t^{(k)} \|X_t - f_k(X_{t-1}, \dots, X_{t-p}, \theta_k)\|^2 \right\}. \quad (5.2)$$

This corresponds to the conditional least-squares method. We focus on this simple loss although our results hereafter can be extended easily, through tedious calculations, to any loss  $\ell$  which is three-times continuously differentiable wrt its second argument.

For a three-times continuously differentiable mapping  $h : \nu \in \mathbb{R}^{d_k} \mapsto h(\nu) \in \mathbb{R}^d$ , we will denote  $\partial h / \partial \nu_i(\mu) \in \mathbb{R}^d$  the derivative of  $h$  wrt to the  $i$ -th entry of  $\nu$  evaluated at  $\mu \in \mathbb{R}^{d_k}$ , and  $J[h](\mu) = (\partial h / \partial \nu_1(\mu) \dots \partial h / \partial \nu_{d_k}(\mu))$  the Jacobian of  $h$ . Similarly the second and third order (mixed) derivatives are denoted respectively  $\partial^2 h / (\partial \nu_i \partial \nu_j)(\mu)$  and  $\partial^3 h / (\partial \nu_i \partial \nu_j \partial \nu_l)(\mu)$ . For a differentiable scalar-valued function on an Euclidean space,  $\nabla$  will denote its gradient operator (the vector of its partial derivatives).

From Example 4.1, Theorem 4.1 applies, hence showing consistency of the estimator (5.2). On the other hand, to establish asymptotic normality of this estimator, we will invoke [48, Theorem 3.2.23 or 3.2.24] (which are in turn due to [29]). This requires to impose the more stringent regularity assumptions:

- (B.1) For each  $k \in [K]$ , the function  $\theta_k \in \Theta_k \mapsto f_k(X_p, \dots, X_1, \theta_k)$  is three-times continuously differentiable almost everywhere in an open neighborhood  $\mathcal{V}$  of  $\theta^0 = (\theta_1^0, \dots, \theta_K^0)$ .
- (B.2) For all  $k \in [K]$  and all  $i, j \in [d_k]$ ,

$$\mathbb{E} \left\| \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,i}} \right\|^2 < \infty \quad \text{and} \quad \mathbb{E} \left\| \frac{\partial^2 f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,j} \partial \theta_{k,i}} \right\|^2 < \infty.$$

- (B.3) The vectors  $\{\partial f_k(X_p, \dots, X_1, \theta_k^0) / \partial \theta_{k,i}\}_{i \in [d_k], k \in [K]}$ , are linearly independent in the sense that if  $(a_{k,i})_{i \in [d_k], k \in [K]}$  are arbitrary real numbers such that

$$\mathbb{E} \left\| \sum_{k=1}^K \sum_{i=1}^{d_k} a_{k,i} \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,i}} \right\|^2 = 0,$$

then  $a_{k,i} = 0$  for all  $i \in [d_k]$  and all  $k \in [K]$ .

(B.4) For  $k \in [K]$  and  $i, j, r \in [d_k]$

$$G_k^{ijr} := \mathbb{E} \left| \left( \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,i}} \right)^\top \frac{\partial^2 f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,j} \partial \theta_{k,r}} \right| < \infty$$

and

$$H_k^{ijr} := \mathbb{E} \left| (X_{p+1} - f_k(X_p, \dots, X_1, \theta_k^0))^\top \frac{\partial^3 f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,i} \partial \theta_{k,j} \partial \theta_{k,r}} \right| < \infty.$$

(B.5) For all  $k \in [K]$  and all  $i, j \in [d_k]$ ,  $|W_{k,ij}| < \infty$ , where

$$W_{k,ij} = \mathbb{E} \left[ \xi_t^{(k)} (X_{p+1} - f_k(X_p, \dots, X_1, \theta_k^0))^\top \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,i}} \cdot (X_{p+1} - f_k(X_p, \dots, X_1, \theta_k^0))^\top \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial \theta_{k,j}} \right].$$

Let us denote by  $W = (W_{kl})_{1 \leq k, l \leq K}$  the block-diagonal matrix defined by the sub-matrices

$$W_{kl} = \begin{cases} 0_{d_k \times d_l} & \text{if } k \neq l \\ (W_{k,ij})_{1 \leq i, j \leq d_k} & \text{if } k = l. \end{cases} \quad (5.3)$$

**Theorem 5.1.** *Let  $(X_t)_{t \in \mathbb{Z}}$  be a strictly stationary ergodic solution of (5.1) with  $\mathbb{E} \|X_t\|^2 < \infty$ , which exists under the assumptions of Theorem 3.1 with  $C(m) < 1$  for  $m = 2$ . Suppose that (B.1)-(B.5) hold. Then there exists a sequence of estimators  $\hat{\theta}_n$  such that*

$$\hat{\theta}_n \rightarrow \theta^0 \quad \text{a.s.}$$

and for any  $\varepsilon > 0$ , there exists  $N$  large enough and an event with probability at least  $1 - \varepsilon$  on which, for all  $n > N$ ,  $\nabla Q_n(\hat{\theta}_n) = 0$ , and  $Q_n$  attains a relative minimum at  $\hat{\theta}_n$ . Furthermore,

$$\sqrt{n} (\hat{\theta}_n - \theta^0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V^{-1} W V^{-1}),$$

as  $n \rightarrow \infty$ , where  $V = (V_{kl})_{1 \leq k, l \leq K}$  is the block-diagonal matrix defined by the sub-matrices

$$V_{kl} = \begin{cases} 0_{d_k \times d_l} & \text{if } k \neq l \\ \pi_k \mathbb{E} \left[ (J[f_k(X_p, \dots, X_1, \cdot)](\theta_k^0))^\top J[f_k(X_p, \dots, X_1, \cdot)](\theta_k^0) \right] & \text{if } k = l. \end{cases} \quad (5.4)$$

Observe that the covariance matrix  $V^{-1} W V^{-1}$  is also block-diagonal with diagonal blocks  $V_{kk}^{-1} W_{kk} V_{kk}^{-1}$ .

## 6. LEARNING CHARME MODELS WITH NEURAL NETWORKS

In this section, we apply our results to the case where  $E = \mathbb{R}^d$  and each of the functions  $f_k$  and  $g_k$  in the CHARME( $p$ ) model (1.2) is exactly modeled by a feedforward neural network (see Section 2.2). More precisely, given an activation map  $\varphi$ , and for each  $k \in [K]$ ,  $f_k$  and  $g_k$  are feedforward neural networks according to Definition 2.2, parameterized by weights and biases given respectively by  $\theta_k = \left( (W_k^{(1)}, b_k^{(1)}), \dots, (W_k^{(L_k)}, b_k^{(L_k)}) \right)$  and  $\lambda_k = \left( (\bar{W}_k^{(1)}, \bar{b}_k^{(1)}), \dots, (\bar{W}_k^{(L_k)}, \bar{b}_k^{(L_k)}) \right)$ . For each  $k \in [K]$ , we have:

- for each layer  $l \in [L_k]$  of the  $k$ -th NN modeling  $f_k$ ,  $W_k^{(l)} = (w_{k,ij}^{(l)})_{(i,j) \in [N_{k,l}] \times [N_{k,l-1}]}$  and  $b_k^{(l)} = (\beta_{k,1}^{(l)}, \dots, \beta_{k,N_{k,l}}^{(l)})^\top$  are respectively the matrix of weights and vector of biases;
- for each layer  $l \in [\bar{L}_k]$  of the  $k$ -th NN modeling  $g_k$ ,  $\bar{W}_k^{(l)} = (\bar{w}_{k,ij}^{(l)})_{(i,j) \in [\bar{N}_{k,l}] \times [\bar{N}_{k,l-1}]}$  and  $\bar{b}_k^{(l)} = (\bar{\beta}_{k,1}^{(l)}, \dots, \bar{\beta}_{k,\bar{N}_{k,l}}^{(l)})^\top$  are respectively the matrix of weights and vector of biases;
- $N_{k,0} = \bar{N}_{k,0} = d \cdot p$ ,  $N_{k,L} = d$  and  $\bar{N}_{k,L} = 1$ .

We throughout make the standard assumption that the activation map  $\varphi$  is Lipschitz continuous<sup>5</sup>.

**6.1. Ergodicity and stationarity.** Considering the notations of Theorem 3.1, let  $x^\top = (x_1, \dots, x_p) \in \mathbb{R}^{dp}$  and  $y^\top = (y_1, \dots, y_p) \in \mathbb{R}^{dp}$ . Split the matrix into  $p$  column blocks  $W_{k,i}^{(1)} \in \mathbb{R}^{N_{k,1} \times d}$  such that  $W_k^{(1)} = \begin{pmatrix} W_{k,1}^{(1)} & W_{k,2}^{(1)} & \dots & W_{k,p}^{(1)} \end{pmatrix}$ . It is easy to see that

$$\begin{aligned} \|f_k(x, \theta_k) - f_k(y, \theta_k)\| &\leq (\text{Lip}(\varphi))^{L_k-1} \prod_{l=2}^{L_k} \|W_k^{(l)}\| \|W_k^{(1)}(x - y)\| \\ &= (\text{Lip}(\varphi))^{L_k-1} \prod_{l=2}^{L_k} \|W_k^{(l)}\| \left\| \sum_{i=1}^p W_{k,i}^{(1)}(x_i - y_i) \right\| \\ &\leq (\text{Lip}(\varphi))^{L_k-1} \prod_{l=2}^{L_k} \|W_k^{(l)}\| \sum_{i=1}^p \|W_{k,i}^{(1)}\| \|x_i - y_i\|, \end{aligned}$$

where  $\|W_k^{(l)}\|$  is the spectral norm of the corresponding matrix. Similarly, we have

$$|g_k(x, \lambda_k) - g_k(y, \lambda_k)| \leq (\text{Lip}(\varphi))^{\bar{L}_k-1} \prod_{l=2}^{\bar{L}_k} \|\bar{W}_k^{(l)}\| \sum_{i=1}^p \|\bar{W}_{k,i}^{(1)}\| \|x_i - y_i\|.$$

Identifying with (3.1), we recognize

$$A_k = (\text{Lip}(\varphi))^{L_k-1} \prod_{l=2}^{L_k} \|W_k^{(l)}\| \sum_{i=1}^p \|W_{k,i}^{(1)}\| \quad \text{and} \quad B_k = (\text{Lip}(\varphi))^{\bar{L}_k-1} \prod_{l=2}^{\bar{L}_k} \|\bar{W}_k^{(l)}\| \sum_{i=1}^p \|\bar{W}_{k,i}^{(1)}\|. \quad (6.1)$$

Therefore, if  $C(m) = 2^{m-1} \sum_{k=1}^K \pi_k (A_k^m + B_k^m \|\epsilon_0\|_m^m) < 1$  for some  $m \geq 1$ , there exists a stationary solution of the NN-based CHARME( $p$ ) model such that the coefficient  $\tau(r) \leq M(C(1))^{r/p}$  for  $r > p$  and some  $M > 0$ .

## 6.2. Learning guarantees.

**6.2.1. Consistency.** To invoke the consistency result of Theorem 4.1, we need to check that  $f_k$  and  $g_k$  verify the corresponding assumptions. Obviously, the Euclidean spaces of parameters  $\Theta_k$  and  $\Lambda_k$  obey (A.2). As for (A.3), it is also fulfilled thanks to obvious continuity properties of NN functions, defined as composition of affine and Lipschitz continuous mappings. (A.3) is obviously verified, for instance, by zeroing both the weight matrix and bias vector at any same layer (a fortiori, this is true for  $\theta_k = 0$ ). When the volatility functions  $g_k$  are not (non-zero) constant, we need to ensure that (A<sub>g</sub>) is verified, which will in turn guarantee

<sup>5</sup>Actually the Lipschitz constant is even 1 in general, e.g., ReLU.

that (A.7) holds when the loss is as in Example 4.1. For this, if  $\varphi$  is positive-valued (as for the ReLU), then it would be sufficient to impose that for any  $k \in [K]$ , the weights  $\bar{W}_k^{(\bar{L}_k)}$  of the last layer are non-negative and the bias  $\bar{b}_k^{(\bar{L}_k)} \geq \delta$  for some  $\delta > 0$ .

Thus, since there exists a stationary solution of the NN-based CHARME( $p$ ) under the condition of the previous section, the statement of Theorem 4.1(i) applies to the estimator (4.1) of the NN parameters.

To be able to apply Theorem 4.1(ii), we need some equi-coerciveness and uniqueness of the true parameters  $(\theta^0, \lambda^0)$ . First, it is important to note that neural networks are often non-identifiable models, which means that different parameters can represent the same function, or equivalently,  $f_k(\cdot, \theta_k) = f_k(\cdot, \theta'_k) \not\Rightarrow \theta_k = \theta'_k$ . In fact there are invariances in the NN parametrization which induce ambiguities in the solutions of the estimation problem (4.1). Clearly, (4.1) is a non-convex problem which may not have a global minimizer, not to mention uniqueness of the latter, even with the population risk  $\mathbb{E}h(Y, \xi, \cdot, \cdot)$  if the weights and biases are allowed to vary freely over the parameters space<sup>6</sup>. Clearly, there is a need of regularization to reduce the space of candidate weights and biases by appropriately constraining them to remove ambiguities and prevent overfitting.

While there is empirical evidence that suggests that when the size of the network is large enough and ReLU non-linearities are used all local minima could be global, there is currently no complete rigorous theory that provides a precise mathematical explanation for these observed phenomena. This is the subject of intense research activity which goes beyond the scope of this paper; see the review paper [50]. A few sufficient deterministic conditions for the existence of global minimizers of (4.1)<sup>7</sup> can be found in [21, 54]. In [21], it is shown that for certain network architectures with positively homogeneous activations and regularizations, any sparse local minimizer is a global one. The work in [54] deals with general architectures but with smooth activations but no regularization, and delivers conditions under which any critical point is a global minimizer. The authors in [38] showed that imposing certain norm regularizations on the weights parameter space of deep networks with ReLU is an effective way of controlling the sample complexity and generalization error. Their group-norm type regularizer takes the form<sup>8</sup>

$$\left( \sum_{l=1}^{L_k} \sum_{i=1}^{N_{k,l}} \left( \sum_{j=1}^{N_{k,l-1}} |w_{k,ij}^{(l)}|^s \right)^{q/s} \right)^{1/q}, \quad 1 \leq s, q \leq +\infty.$$

In particular, for  $q = s$ , the regularizer measures the overall  $s$ -norm of the weights. If moreover  $s \in [1, 2]$ , then it follows from the arithmetic-geometric mean inequality and standard

<sup>6</sup>This is the case for rescaling when the activation is positively homogeneous, in which case multiplying one layer of a global minimizer by a positive constant and dividing another layer by the same constant produces a pair of different global minimizers

<sup>7</sup>More precisely, in all the works cited here, their framework amounts to considering  $g_k$  as a constant and  $\ell$  as quadratic in our setting.

<sup>8</sup>We only give derivations for  $f_k$  since the same reasoning holds for  $g_k$ .

matrix norm inequalities that

$$\left( \sum_{l=1}^{L_k} \sum_{i=1}^{N_{k,l}} \sum_{j=1}^{N_{k,l-1}} |w_{k,ij}^{(l)}|^s \right)^{1/s} \geq L_k^{1/s} \left( \prod_{l=1}^{L_k} \|W_k^{(l)}\| \right)^{1/L_k},$$

and from (6.1), we get the upper-bound (assuming  $p > 1$ )

$$A_k \leq p L_k^{-L_k} \text{Lip}(\varphi)^{L_k-1} \left( \sum_{l=1}^{L_k} \sum_{i=1}^{N_{k,l}} \sum_{j=1}^{N_{k,l-1}} |w_{k,ij}^{(l)}|^s \right)^{L_k/s} \leq \left( \sum_{l=1}^{L_k} \sum_{i=1}^{N_{k,l}} \sum_{j=1}^{N_{k,l-1}} |w_{k,ij}^{(l)}|^s \right)^{\log p} \quad (6.2)$$

for  $L_k \geq \max(1, s \log p)$ . While  $A_k$  is scale-invariant, its upper-bound (6.2) allows to define a compact constraint, whose radius can be chosen such that it satisfies  $C(m) < 1$  for  $m \geq 1$  known.

To summarize, if (4.1) is solved with  $\ell$  and constraint sets  $\mathcal{C}_k$  involving the upper-bound (6.2), or more generally any lsc coercive regularizers, see the discussion after Theorem 4.1, then equi-coerciveness holds true. If uniqueness is assumed (see discussion above), then Theorem 4.1(ii) yields that the estimator (4.1) of the NN parameters is (strongly) consistent.

**6.2.2. Asymptotic normality.** We now turn to asymptotic normality of the estimator (5.2) for the CHARME( $p$ ) model (5.1), where  $f_k$  is neutral network. We need to check the assumptions of Theorem 5.1. For this, we assume in this section that the activation map of the NN is three-times continuously differentiable with bounded derivatives (this is the case for softplus, smoothed ReLU, sigmoid, etc.). In turn, this will entail that  $\varphi$  is Lipschitz continuous, and that for all  $k \in [K]$ ,  $\theta_k \mapsto f_k(X_p, \dots, X_1, \theta_k)$  is almost surely three-times continuously differentiable at any  $\theta_k \in \Theta_k$ , i.e., (B.2) holds.

Let us now check our assumptions. In view of the derivatives of  $f_k$  in (A.1) (see Section A), boundedness of the derivatives of  $\varphi$  and stationarity, it is not difficult to check that

$$\begin{aligned} \mathbb{E} \left\| \frac{\partial f_k(X_p, \dots, X_1, \theta_k^0)}{\partial w_{k,ij}^{(l)}} \right\|^2 &= O(\mathbb{E} \|X_t\|^2), \quad \mathbb{E} \left\| \frac{\partial^2 f_k(X_p, \dots, X_1, \theta_k^0)}{\partial w_{k,ij}^{(l)} \partial w_{k,i'j'}^{(l)}} \right\|^2 = O(\max_{s \in \{2,4\}} (\mathbb{E} \|X_t\|^s)), \\ \mathbb{E} \left\| \frac{\partial^3 f_k(X_p, \dots, X_1, \theta_k^0)}{\partial w_{k,ij}^{(l)} \partial w_{k,i'j'}^{(l)} \partial w_{k,i''j''}^{(l)}} \right\|^2 &= O(\max_{s \in \{2,4,6\}} (\mathbb{E} \|X_t\|^s)). \end{aligned}$$

The derivatives wrt biases  $\beta_{k,i}^{(l)}$  as given in (A.2) are bounded in view of boundedness of the derivative of  $\varphi$ . Thus, if Theorem 3.1 holds with  $C(m) = 2^{m-1} \sum_{k=1}^K \pi_k A_k^m < 1$ , for  $m \in \{2, 4, 6\}$ , then  $\max_{s \in \{2,4,6\}} (\mathbb{E} \|X_t\|^s) < \infty$ , whence conditions (B.2), (B.4), and (B.5) hold. As far as assumption (B.3) is concerned, it captures the fact that  $\theta^0$  is a strict local minimizer of (5.2), which is in turn closely related to our discussion on uniqueness in the previous section. Assuming that it holds, we are in position to invoke Theorem 5.1 to prove asymptotic normality of the estimator (5.2) of the NN-parameters of the CHARME( $p$ ) model (5.1).



**6.3. Approximation vs exact modeling by neural networks.** Until now, we have assumed that the autoregressive and volatility functions  $f_k$  are  $g_k$  are *exactly* modeled by feedforward NNs with finitely many neurons. A natural question we ask is: what are the consequences if the NN architecture (depth and width) is such that it provides only  $\varepsilon$ -approximations to  $f_k$  and  $g_k$  ?

To settle this question, let  $X_t$  the CHARME process as given in (1.2), and  $\tilde{X}_t$  is be the CHARME process defined by the same innovations and hidden process  $(R_t)_{t \in \mathbb{Z}}$  but with functions  $\tilde{f}_k$  and  $\tilde{g}_k$ , i.e.,

$$\tilde{X}_t = \sum_{k=1}^K \xi_t^{(k)} \left( \tilde{f}_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \tilde{\theta}_k) + \tilde{g}_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \tilde{\lambda}_k) \epsilon_t \right), \quad t \in \mathbb{Z}. \quad (6.3)$$

The functions  $\tilde{f}_k$  and  $\tilde{g}_k$  are supposed to be two neural networks providing approximations to  $f_k$  and  $g_k$ . Denote the approximation accuracy as

$$\varepsilon_k := \sup_{(x_1, \dots, x_p) \in E^p} \left( \left\| \tilde{f}_k(x_1, \dots, x_p, \tilde{\theta}_k) - f_k(x_1, \dots, x_p, \theta_k^0) \right\|, \right. \\ \left. \left\| \tilde{g}_k(x_1, \dots, x_p, \tilde{\lambda}_k) - g_k(x_1, \dots, x_p, \lambda_k^0) \right\| \right). \quad (6.4)$$

To compare the two processes, it is natural to assume that the functions  $(\tilde{f}_k, \tilde{g}_k)_{k \in \mathbb{N}}$  verify the assumptions of Theorem 3.1 so that  $(\tilde{X}_t)_{t \in \mathbb{Z}}$  is a strictly stationary solution of (6.3). Thus,  $\forall t \in \mathbb{Z}$

$$\begin{aligned} \|\tilde{X}_t - X_t\| &= \left\| \sum_{k=1}^K \xi_t^{(k)} \left( \left( \tilde{f}_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \tilde{\theta}_k) - f_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \theta_k^0) \right) \right. \right. \\ &\quad + \left( \tilde{g}_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \tilde{\lambda}_k) - g_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \theta_k^0) \right) \epsilon_t \\ &\quad + \left( f_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \theta_k^0) - f_k(X_{t-1}, \dots, X_{t-p}, \theta_k^0) \right) \\ &\quad \left. \left. + \left( g_k(\tilde{X}_{t-1}, \dots, \tilde{X}_{t-p}, \theta_k^0) - g_k(X_{t-1}, \dots, X_{t-p}, \theta_k^0) \right) \epsilon_t \right) \right\| \\ &\leq \sum_{k=1}^K \xi_t^{(k)} \left( \varepsilon_k (1 + \|\epsilon_t\|) + \sum_{i=1}^p \left( a_i^{(k)} + b_i^{(k)} \|\epsilon_t\| \right) \|\tilde{X}_{t-i} - X_{t-i}\| \right). \end{aligned}$$

Taking expectations in both sides and thanks to stationarity of both processes, and by assumptions on  $\epsilon_t$  and  $\xi_t^{(k)}$ , we get

$$\begin{aligned}
\mathbb{E}\|\tilde{X}_t - X_t\| &\leq (1 + \mathbb{E}\|\epsilon_0\|) \sum_{k=1}^K \pi_k \epsilon_k + \mathbb{E}\|\tilde{X}_t - X_t\| \left( \sum_{k=1}^K \pi_k (A_k + B_k \mathbb{E}\|\epsilon_0\|) \right) \\
&\leq (1 + \mathbb{E}\|\epsilon_0\|) \sum_{k=1}^K \pi_k \epsilon_k + \mathbb{E}\|\tilde{X}_t - X_t\| \left( \sum_{k=1}^K \pi_k (A_k + B_k \mathbb{E}\|\epsilon_0\|)^m \right)^{1/m} \\
&\leq (1 + \mathbb{E}\|\epsilon_0\|) \sum_{k=1}^K \pi_k \epsilon_k + \mathbb{E}\|\tilde{X}_t - X_t\| \left( \sum_{k=1}^K \pi_k 2^{m-1} (A_k^m + B_k^m \mathbb{E}\|\epsilon_0\|^m) \right)^{1/m} \\
&\leq (1 + \mathbb{E}\|\epsilon_0\|) \sum_{k=1}^K \pi_k \epsilon_k + \mathbb{E}\|\tilde{X}_t - X_t\| C(m)^{1/m},
\end{aligned}$$

where we have used that  $m \geq 1$  in the second line and Jensen's inequality in the third. Since by assumption  $C(m) < 1$  for some  $m \geq 1$ , see Theorem 3.1, we get that

$$\mathbb{E}\|\tilde{X}_t - X_t\| \leq \frac{(1 + \mathbb{E}\|\epsilon_0\|) \sum_{k=1}^K \pi_k \epsilon_k}{1 - C(m)^{1/m}}. \quad (6.5)$$

In a nutshell, this inequality highlights the fact that, as expected, the mean error between  $\tilde{X}_t$  and the true process  $X_t$  is within a factor of the average approximation accuracy of  $f_k$  and  $g_k$ . This bound also shed light on the role of  $C(m)$ , and the smaller, the better.

Notice also that if  $\bar{X}_t$  is the stationary solution of the CHARME( $\infty$ ) model ((1.2), for  $p = \infty$ ) and  $X_t$  is the stationary solution of its associated CHARME( $p$ ) model (defined in (3.3)), we can then approximate this solution by  $\tilde{X}_t$ , for some large integer value of  $p$  and  $\epsilon_k$  small enough for all  $k \in [K]$ . Precisely, we would get that

$$\mathbb{E}\|\bar{X}_t - \tilde{X}_t\| \leq E\|\bar{X}_t - X_t\| + \mathbb{E}\|X_t - \tilde{X}_t\| \leq (3.4) + (6.5) \longrightarrow 0,$$

as  $\epsilon_k \rightarrow 0$  for all  $k \in [K]$  and  $p \rightarrow \infty$ . This justifies that one could learn infinity memory CHARME models with deep neural networks, by approximating them by a CHARME( $p$ ) for  $p$  finite but sufficiently large. Of course, strictly speaking, learning a CHARME( $\infty$ ) would necessitate infinitely many observations.

## 7. NUMERICAL EXPERIMENTS

In order to assess numerically the performance (consistency and asymptotic normality) of our estimator and support our theoretical predictions, we here report some numerical experiments. The CHARME( $p$ ) models in (5.1) were generated in two scenarios: when the autoregressive functions  $f_k$  are generated by feedforward NNs (i.e.,  $f_k$  are exactly modeled by neural networks) and when they are not (i.e., a neural network may provide only an  $\epsilon_k$ -approximations to  $f_k$ ). In all cases, we parametrize the functions  $f_k$  with feedforward NNs, and we train the NNs by minimizing (5.2) to estimate the corresponding weights and biases  $\theta_k$ . The estimation/training step is accomplished using stochastic (sub)gradient descent (SGD). For smooth activation maps, the gradient is computed via the chain rule through reverse mode automatic differentiation (i.e., backpropagation algorithm); see [20].

For non-smooth activations such as the ReLU, we invoke the theory of conservative fields and definability proposed recently in [5] to justify our use of the non-smooth chain rule and automatic differentiation.

All experiments were conducted under R with an interface to Keras 2.2.5 [14]. R notebooks that allow to reproduce our experiments are publicly available for download at [https://github.com/jose3g/Learning\\_CHARME\\_models\\_with\\_DNN.git](https://github.com/jose3g/Learning_CHARME_models_with_DNN.git).

*Experiment 1* (Learning NNs from NN-based CHARME data). We simulate a NN-based CHARME( $p$ ) model as in (5.1) with  $K = 3$  and  $p = 30$ , where  $f_k = f_k(\cdot, \theta_k^0)$ ,  $k = 1, 2, 3$ , are NNs with  $\#\text{neu}(\theta_1^0) = (N_{1,0}, \dots, N_{1,5}) = (30, 50, 60, 40, 20, 1)$ ,  $\#\text{neu}(\theta_2^0) = (N_{2,0}, \dots, N_{2,3}) = (30, 20, 5, 1)$  and  $\#\text{neu}(\theta_3^0) = (N_{3,0}, \dots, N_{3,3}) = (30, 25, 30, 1)$ , all with a ReLU activation function. We have taken the weights  $w_{k,ij}^{(l)}$  arbitrarily (randomly uniform over a small interval  $[-\delta, \delta]$ ) and  $(\pi_1, \pi_2, \pi_3) = (0.1, 0.4, 0.5)$  such that  $C(1) < 1$  (the explicit expression is provided in (6.1)) in order to guarantee the stationarity of the model. Precisely,  $C(1) = 0.9882313$  for this model. The biases  $b_k^{(l)} = (\beta_{k,1}^{(l)}, \dots, \beta_{k,N_{k,l}}^{(l)})^\top$  are also taken arbitrarily but in  $\mathbb{R}$  and we have set particularly  $(b_1^{(5)}, b_2^{(3)}, b_3^{(3)}) = (1, 0, -1)$ . Then, from this model and with innovations  $\epsilon_t \sim \mathcal{N}(0, 1)$ , we have generated a dataset of  $n = 10^5$  observations.

Let us turn to the estimation/training step. For this, we consider the quadratic loss function defined in (5.2) with the same configurations of the model that generates the data, that is, with  $K = 3$ ,  $(\pi_1, \pi_2, \pi_3) = (0.1, 0.4, 0.5)$  and  $f_k$  such that  $\#\text{neu}(\theta_1) = (30, 50, 60, 40, 20, 1)$ ,  $\#\text{neu}(\theta_2) = (30, 20, 5, 1)$  and  $\#\text{neu}(\theta_3) = (30, 25, 30, 1)$ , and the ReLU activation function. We run  $10^3$  iterations of the SGD algorithm with learning rate/step-size 0.001 which decays at the rate of 0.5. Let  $\hat{\theta}_n^* = (\hat{\theta}_{n,1}^*, \hat{\theta}_{n,2}^*, \hat{\theta}_{n,3}^*)$  be the parameters obtained in the last iteration.

On the left side in Figure 1, we show the histogram of the errors  $\hat{\epsilon}_t = X_t - \hat{X}_t$ , where

$$\hat{X}_t = \sum_{k=1}^K \xi_t^{(k)} f_k(X_{t-1}, \dots, X_{t-p}, \hat{\theta}_{n,k}^*). \quad (7.1)$$

The Gaussian density with mean and variance equal to the empirical mean and variance of  $\hat{\epsilon}_t$  is also displayed in a blue solid line.

*Experiment 2* (Learning NNs from non NN-based CHARME data). In this experiment we simulate a CHARME(5) model as follows:

$$X_t = \epsilon_t + \begin{cases} X_{t-1} + 3, & \text{if } R_t = 1 \\ \sqrt{0.2X_{t-1}^2 + 0.1X_{t-2}^2 + 0.25X_{t-3}^2 + 0.2X_{t-4}^2 + 0.05X_{t-5}^2} - 3, & \text{if } R_t = 2 \\ 0.05X_{t-1} + 0.2X_{t-2} + 0.15X_{t-3} + 0.03X_{t-4} + 0.01X_{t-5} + 0.1, & \text{if } R_t = 3 \end{cases}$$

with  $(\pi_1, \pi_2, \pi_3) = (0.15, 0.35, 0.5)$ . Note that the first autoregressive process  $X_t^{(1)} = X_{t-1} + 3 + \epsilon_t$  is not stationary, although the entire process is stationary (because  $C(1) < 1$ ). By taking  $\epsilon_t \sim \mathcal{N}(0, 1)$ , we generate again a dataset of  $n = 10^5$ .

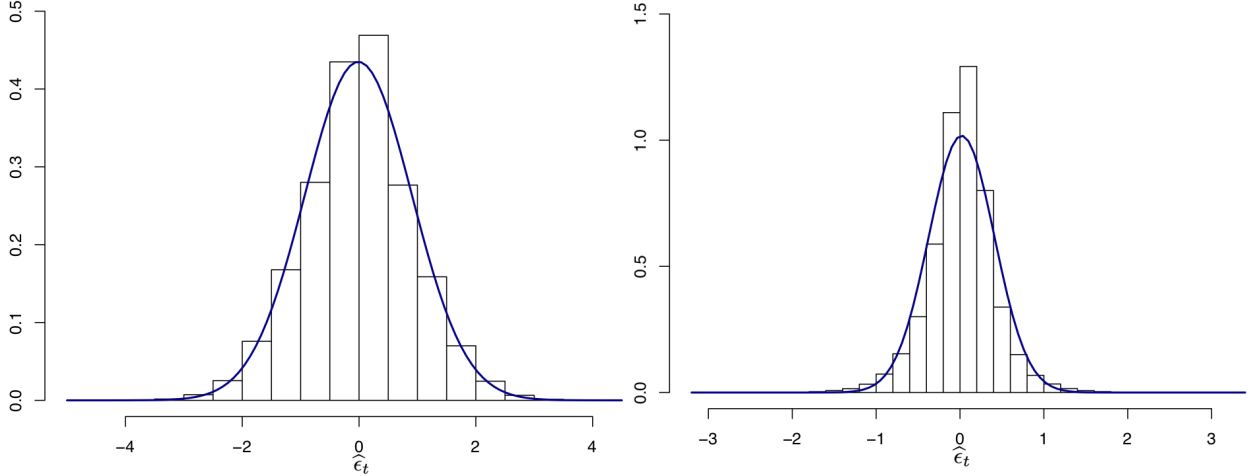


FIGURE 1. Histograms for the estimated errors  $\hat{\epsilon}_t$  of Experiment 1 (left) and Experiment 2 (right).

For the estimation/training procedure, we consider also the quadratic loss function (5.2) with three NNs  $f_k(\cdot, \theta_k)$ ,  $k = 1, 2, 3$ , such that  $\#\text{neu}(\theta_1) = (5, 300, 400, 200, 1)$ ,  $\#\text{neu}(\theta_2) = (5, 500, 600, 400, 1)$  and  $\#\text{neu}(\theta_3) = (5, 300, 400, 200, 1)$ , all with a ReLU activation map. We run 2000 iterations of the SGD algorithm with learning rate/step-size 0.01 which decays at the rate  $10^{-6}$ . Let  $\hat{\theta}_n^* = (\hat{\theta}_{n,1}^*, \hat{\theta}_{n,2}^*, \hat{\theta}_{n,3}^*)$  be the parameters obtained in the last iteration.

Similarly to Experiment 1, we show on the right side of Figure 1 the histogram of the errors  $\hat{\epsilon}_t = X_t - \hat{X}_t$ , where  $\hat{X}_t$  is as given in (7.1). The Gaussian density with mean and variance equal to the empirical mean and variance of  $\hat{\epsilon}_t$  is also displayed in a blue solid line.

*Experiment 3* (Asymptotic normality of trained NNs parameters). We set a CHARME( $p$ ) model as in (5.1) with  $K = 3$  and  $p = 16$ , where  $f_k = f_k(\cdot, \theta_k^0)$ ,  $k = 1, 2, 3$ , are DNNs with  $\#\text{neu}(\theta_1^0) = (16, 32, 64, 32, 1)$ ,  $\#\text{neu}(\theta_2^0) = (16, 64, 32, 1)$ ,  $\#\text{neu}(\theta_3^0) = (16, 32, 64, 1)$ , all with sigmoid activation function (this is because for the CLT result of Theorem 5.1 to apply, the activation function must be three-times continuously differentiable). Of course, the weights generated satisfy the condition  $C(1) < 1$ . In particular,  $C(1) = 0.9743731$  for the weights generated in this model.

We now perform the following steps  $N = 125$  times:

- (i) By taking normal standard innovations with the aforementioned model, we generate a dataset of  $n = 2 \cdot 10^4$ ,
- (ii) By considering the quadratic loss function (5.2) with the same configurations of the model that generates the data and the sigmoid activation function, we run 2000 iterations of the SGD algorithm with learning rate/step-size 0.01 and decay rate  $10^{-6}$ , in order to obtain an approximation  $\hat{\theta}_n^*$  of  $\theta_n^*$ .

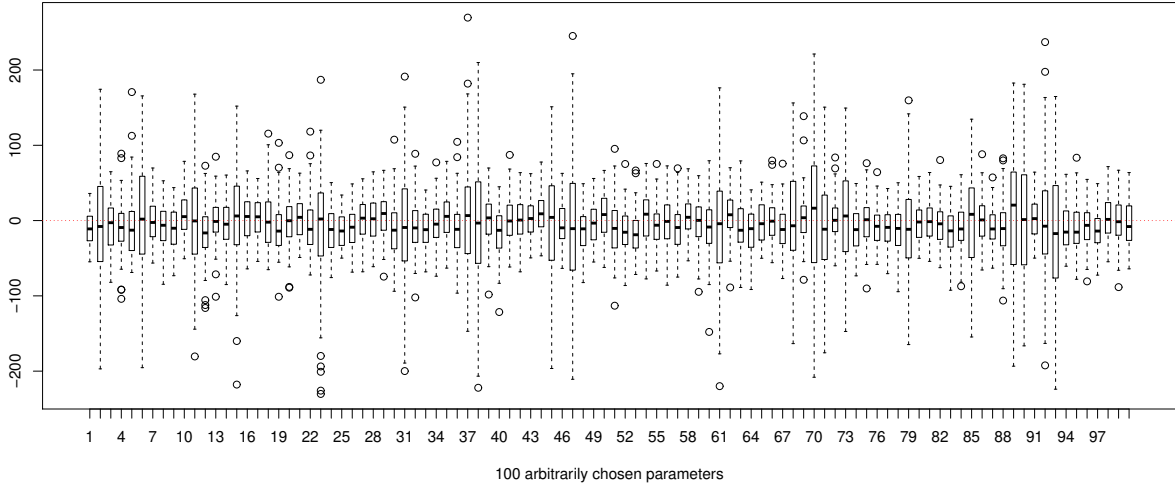


FIGURE 2. Boxplots of 100 coordinates of  $\eta_n$ .

TABLE 1. Multivariate normality test results.

Test	Test Statistic	$p$ -value
Mardia		
Skewness	687.5626	0.4120106
Kurtosis	-0.4461589	0.6554825
Henze-Zirkler	0.9931136	0.7983517
Royston	22.35297	0.0987472

Let  $\hat{\theta}_n^*(t)$ ,  $t = 1, \dots, 125$ , be the estimates<sup>9</sup> obtained in each step of the Monte Carlo simulation and let  $\eta_n(t) := \sqrt{n} \left( \hat{\theta}_n^*(t) - \theta^0 \right)$ ,  $t = 1, \dots, 125$ . One can easily check that the number of parameters to learn is 10691, i.e.,  $\theta^0 \in \mathbb{R}^{10691}$ , and in turn each  $\eta_n(t)$  is a vector in  $\mathbb{R}^{10691}$ .

Figure 2 shows the box-plots of the coordinates of  $\eta_n$ . For the sake of readability, we only show 100 coordinates selected arbitrarily.

To test normality of  $\eta_n$ , as predicted by Theorem 5.1, we apply three multivariate normality tests: Mardia, Henze-Zirkler and Royston test (for the details of these tests, see [25, 36, 43, 33]). Given that the dimension of  $\eta_n(t)$  is quite large (anyway larger than  $N = 125$ ), to avoid numerical instabilities due to matrix inversion, these tests were not applied to the entire set of coordinates of  $\eta_n(t)$ , but to an arbitrary subset of 15 parameters (i.e., 15 arbitrary coordinates of  $\eta_n$  that we will call  $\eta_n|_B$ , where  $B \subset [10691]$ ), which yield the results shown in Table 1.

<sup>9</sup>These are really the SGD-approximations of the conditional least-squares estimates.

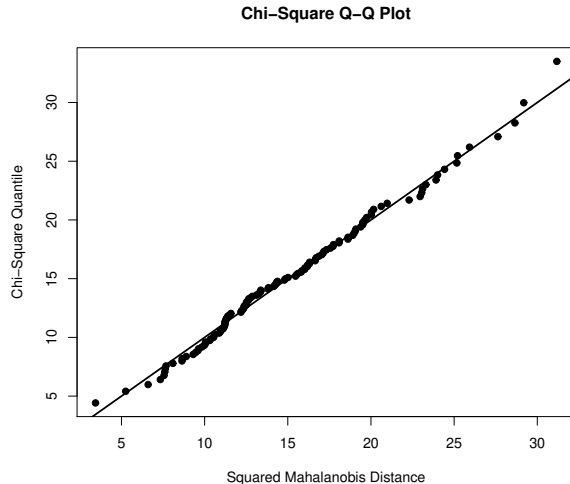


FIGURE 3. Chi-Square Q-Q plot: empirical quantiles of squared Mahalanobis distance from  $\eta_n|_B$  to  $\vec{0}$  vs chi-square quantiles.

We also report the Chi-Square Q-Q plot for Squared Mahalanobis Distance from  $\eta_n|_B$  to  $0$  on Figure 3. We can see that the Q-Q plots are, in fact, almost all on the straight line. Therefore, observing this behavior and the  $p$ -values obtained in the three tests of normality on Table 1, we can conclude that the vector  $\eta_n|_B$  has indeed the predicted Gaussian behavior.

## 8. PROOFS

### 8.1. Proof of Theorem 3.1.

- (i) Note that the CHARME( $\infty$ ) model defined in (1.2) with  $p = \infty$ , can be written as a Markov process:

$$X_t = F(X_{t-1}, X_{t-2}, \dots; \tilde{\xi}_t), \quad t \in \mathbb{Z}, \quad (8.1)$$

by taking the function

$$F(x; (\xi^{(0)}, \dots, \xi^{(K)})) = \sum_{k=1}^K \xi^{(k)} (f_k(x, \theta_k^0) + g_k(x, \lambda_k^0) \xi^{(0)}) \quad (8.2)$$

with innovations  $\tilde{\xi}_t := (\epsilon_t, \xi_t^{(1)}, \dots, \xi_t^{(K)}) = (\epsilon_t, \xi_t) \in E \times \{e_1, \dots, e_K\}$ . Therefore, verifying [13, conditions (3.1) and (3.3)], we will obtain the result by [13, Theorem 3.1]. Note that Condition (3.2) of that paper is already assumed.

Indeed, since the sequences  $(\epsilon_t)_{t \in \mathbb{Z}}$  and  $(Q_t)_{t \in \mathbb{Z}}$  are independent and  $\xi_0 \in \{e_1, \dots, e_K\}$ , denoting  $\mathbb{E}_\epsilon$  the expectation with respect to the distribution of  $\epsilon$ , we obtain for

$x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ , that

$$\begin{aligned}
\|F(x; \tilde{\xi}_0) - F(y; \tilde{\xi}_0)\|_1 &= \mathbb{E} \left[ \|F(x; \tilde{\xi}_0) - F(y; \tilde{\xi}_0)\| \right] \\
&= \mathbb{E} \left[ \left\| \sum_{k=1}^K \xi_0^{(k)} (f_k(x, \theta_k^0) - f_k(y, \theta_k^0) + (g_k(x, \lambda_k^0) - g_k(y, \lambda_k^0))\epsilon_0) \right\| \right] \\
&= \mathbb{E}_\epsilon \left[ \sum_{j=1}^K \left\| \sum_{k=1}^K e_j^{(k)} (f_k(x, \theta_k^0) - f_k(y, \theta_k^0) + (g_k(x, \lambda_k^0) - g_k(y, \lambda_k^0))\epsilon_0) \right\| \mathbb{P}(\xi_0 = e_j) \right] \\
&= \mathbb{E}_\epsilon \left[ \sum_{k=1}^K \pi_k \left\| (f_k(x, \theta_k^0) - f_k(y, \theta_k^0) + (g_k(x, \lambda_k^0) - g_k(y, \lambda_k^0))\epsilon_0) \right\| \right] \\
&= \sum_{k=1}^K \pi_k \mathbb{E}_\epsilon \left\| (f_k(x, \theta_k^0) - f_k(y, \theta_k^0) + (g_k(x, \lambda_k^0) - g_k(y, \lambda_k^0))\epsilon_0) \right\| \\
&\leq \sum_{i=1}^\infty \left( \sum_{k=1}^K \pi_k (a_i^{(k)} + b_i^{(k)} \|\epsilon_0\|_1) \right) \|x_i - y_i\|,
\end{aligned}$$

by the Minkowski inequality and the Lipschitz-type assumptions (3.1) on  $f_k$  and  $g_k$ . So, this verifies (3.1) of [13].

On the other hand, using the same arguments as above, we can verify that

$$\tilde{\mu}_1 = \|F(0; \tilde{\xi}_0)\|_1 \leq \sum_{k=1}^K \pi_k (\|f_k(0, \theta_k^0)\| + |g_k(0, \lambda_k^0)| \|\epsilon_0\|_1),$$

which is finite because  $\epsilon_0 \in \mathbb{L}^1$ . The first part of the theorem is proven.

- (ii) Suppose now that  $C(m) < 1$  for some  $m \in \mathbb{N} \cap (1, \infty)$ . Let  $x = (x_1, \dots)$  and rewrite  $f_k(x, \theta_k^0) = f_k(x, \theta_k^0) - f_k(0, \theta_k^0) + f_k(0, \theta_k^0)$ . Then, from (3.1) and Minkowski inequality, we have that

$$\|f_k(x, \theta_k^0)\| \leq \sum_{i=1}^\infty a_i^{(k)} \|x_i\| + o_k = w_k(x) + o_k, \tag{8.3}$$

where  $o_k = \|f_k(0, \theta_k^0)\|$  and  $w_k(x) = \sum_{i=1}^\infty a_i^{(k)} \|x_i\|$ . Thus,

$$\|f_k(x, \theta_k^0)\|^m \leq \sum_{j=0}^{m-1} \binom{m}{j} w_k^j(x) o_k^{m-j} + w_k^m(x). \tag{8.4}$$

Taking the probability weights  $\lambda_i = a_i^{(k)}/A_k$  (recall that  $A_k = \sum_{i=1}^\infty a_i^{(k)}$ ), we can apply Jensen's inequality for any  $s \geq 1$  as follows:

$$w_k^s(x) = A_k^s \left( \sum_{i=1}^\infty \frac{a_i^{(k)}}{A_k} \|x_i\| \right)^s \leq A_k^{s-1} \sum_{i=1}^\infty a_i^{(k)} \|x_i\|^s. \tag{8.5}$$

Let us denote  $Y_{t-1} = (X_{t-1}, X_{t-2}, \dots)$ . From stationarity of  $(X_t)_{t \in \mathbb{Z}}$ , for  $s \geq 1$ , we obtain

$$\mathbb{E}[w_k^s(Y_{t-1})] \leq A_k^{s-1} \sum_{i=1}^{\infty} a_i^{(k)} \mathbb{E}\|X_{t-i}\|^s = A_k^s \mathbb{E}\|X_0\|^s \quad (8.6)$$

and therefore

$$\mathbb{E}\|f_k(Y_{t-1}, \theta_k^0)\|^m \leq A_k^m \mathbb{E}\|X_0\|^m + \mathbb{E}[R_{k,m}(\|X_0\|)], \quad (8.7)$$

where  $R_{k,s}(x) := \sum_{j=0}^{s-1} \binom{s-1}{j} A_k^j o_k^{s-j} x^j$ .

Similarly, with the same steps, we can prove that

$$\mathbb{E}|g_k(Y_{t-1}, \lambda_k^0)|^m \leq B_k^m \mathbb{E}\|X_0\|^m + \mathbb{E}[\bar{R}_{k,m}(\|X_0\|)], \quad (8.8)$$

where  $\bar{R}_{k,s}(x) := \sum_{j=0}^{s-1} \binom{s-1}{j} B_k^j O_k^{s-j} x^j$ , with  $B_k = \sum_{i=1}^{\infty} b_i^{(k)}$  and  $O_k = |g_k(0, \lambda_k^0)|$ .

Since  $(\xi_t^{(1)}, \dots, \xi_t^{(K)}) \in \{e_1, \dots, e_K\}$ , for  $m \in \mathbb{N}^*$ ,

$$\begin{aligned} \|X_t\|^m &= \sum_{k=1}^K \xi_t^{(k)} \|f_k(Y_{t-1}, \theta_k^0) + g_k(Y_{t-1}, \lambda_k^0) \epsilon_t\|^m \\ &\leq 2^{m-1} \sum_{k=1}^K \xi_t^{(k)} (\|f_k(Y_{t-1}, \theta_k^0)\|^m + |g_k(Y_{t-1}, \lambda_k^0)|^m \|\epsilon_t\|^m), \end{aligned} \quad (8.9)$$

where the last line is due to Jensen's inequality. On the other hand, as  $R_t$  is independent of the random vector  $(\epsilon_t, Y_{t-1})$  and  $\epsilon_t$  is independent of  $Y_{t-1}$ , then, under the invariant measure (the existence of this measure is from the stationarity of  $(X_t)_{t \in \mathbb{Z}}$ ), we obtain that

$$\begin{aligned} \mathbb{E}\|X_0\|^m &= \mathbb{E}\|X_t\|^m \leq 2^{m-1} \sum_{k=1}^K \pi_k (\mathbb{E}\|f_k(Y_{t-1}, \theta_k^0)\|^m + \|\epsilon_0\|_m^m \mathbb{E}|g_k(Y_{t-1}, \lambda_k^0)|^m) \\ &\leq 2^{m-1} \sum_{k=1}^K \pi_k (A_k^m + B_k^m \|\epsilon_0\|_m^m) \mathbb{E}\|X_0\|^m + C, \end{aligned} \quad (8.10)$$

where  $C = 2^{m-1} \sum_{k=1}^K \pi_k (\mathbb{E}[R_{k,m}(\|X_0\|)] + \|\epsilon_0\|_m^m \mathbb{E}[\bar{R}_{k,m}(\|X_0\|)]) < \infty$  since from recursion  $\mathbb{E}\|X_0\|^{m-1} < \infty$ . Therefore, by taking

$$D = \sum_{k=1}^K \pi_k (A_k^m + B_k^m \|\epsilon_0\|_m^m) < \frac{1}{2^{m-1}}, \quad (8.11)$$

we conclude that

$$\mathbb{E}\|X_0\|^m < \frac{C}{1 - 2^{m-1}D} < \infty.$$



For the case  $m \in (1, \infty) \setminus \mathbb{N}$ , we write  $m = n + \delta$ , where  $n = \lfloor m \rfloor$  and  $\delta \in (0, 1)$ . Then, by using the expression (8.3), we have that

$$\begin{aligned} \|f_k(x, \theta_k^0)\|^m &= \|f_k(x, \theta_k^0)\|^\delta \|f_k(x, \theta_k^0)\|^n \\ &\leq (w_k(x) + o_k)^\delta \sum_{j=0}^n \binom{n}{j} w_k^j(x) o_k^{n-j} \\ &\leq \sum_{j=0}^n \binom{n}{j} w_k^{j+\delta}(x) o_k^{n-j} + \sum_{j=0}^n \binom{n}{j} w_k^j(x) o_k^{n+\delta-j} \\ &= w_k^m(x) + o_k^\delta w_k^n(x) + \sum_{j=0}^{n-1} \binom{n}{j} w_k^{j+\delta}(x) o_k^{n-j} + \sum_{j=0}^{n-1} \binom{n}{j} w_k^j(x) o_k^{n+\delta-j}. \end{aligned}$$

As in the previous case, using (8.5) and (8.6), we get that

$$\mathbb{E}\|f_k(Y_{t-1}, \theta_k^0)\|^m \leq A_k^m \mathbb{E}\|X_0\|^m + o_k^\delta A_k^n \mathbb{E}\|X_0\|^n + \mathbb{E}[R_{k,m}^*(\|X_0\|)], \quad (8.12)$$

where  $R_{k,s}^*(x) := \sum_{j=0}^{\lfloor s \rfloor - 1} \binom{\lfloor s \rfloor}{j} A_k^{j+s-\lfloor s \rfloor} o_k^{\lfloor s \rfloor - j} x^{j+s-\lfloor s \rfloor} + \sum_{j=0}^{\lfloor s \rfloor - 1} \binom{\lfloor s \rfloor}{j} A_k^j o_k^{s-j} x^j$ .

Similarly, with the same steps, we can prove that

$$\mathbb{E}|g_k(Y_{t-1}, \lambda_k^0)|^m \leq B_k^m \mathbb{E}\|X_0\|^m + O_k^\delta B_k^n \mathbb{E}\|X_0\|^n + \mathbb{E}[\bar{R}_{k,m}^*(\|X_0\|)], \quad (8.13)$$

where  $\bar{R}_{k,s}^*(x) := \sum_{j=0}^{\lfloor s \rfloor - 1} \binom{\lfloor s \rfloor}{j} B_k^{j+s-\lfloor s \rfloor} O_k^{\lfloor s \rfloor - j} x^{j+s-\lfloor s \rfloor} + \sum_{j=0}^{\lfloor s \rfloor - 1} \binom{\lfloor s \rfloor}{j} B_k^j O_k^{s-j} x^j$ , with  $B_k = \sum_{i=1}^{\infty} b_i^{(k)}$  and  $O_k = |g_k(0, \lambda_k^0)|$ .

Using the same arguments to prove (8.10), we proved that

$$\begin{aligned} \mathbb{E}\|X_0\|^m &= \mathbb{E}\|X_t\|^m \leq 2^{m-1} \sum_{k=1}^K \pi_k (\mathbb{E}\|f_k(Y_{t-1}, \theta_k^0)\|^m + \|\epsilon_0\|_m^m \mathbb{E}|g_k(Y_{t-1}, \lambda_k^0)|^m) \\ &\leq 2^{m-1} \sum_{k=1}^K \pi_k (A_k^m + B_k^m \|\epsilon_0\|_m^m) \mathbb{E}\|X_0\|^m + C^*, \end{aligned}$$

$$\begin{aligned} \text{where } C^* &= 2^{m-1} \sum_{k=1}^K \pi_k ((o_k^\delta A_k^n + O_k^\delta B_k^n \|\epsilon_0\|_m^m) \mathbb{E}\|X_0\|^n \\ &\quad + \mathbb{E}[R_{k,m}^*(\|X_0\|) + \|\epsilon_0\|_m^m \bar{R}_{k,m}^*(\|X_0\|)]) \end{aligned} \quad (8.14)$$

which is finite by recursion, because  $\mathbb{E}\|X_0\|^{m-1} < (\mathbb{E}\|X_0\|^n)^{\frac{m-1}{n}} < \infty$ .

Therefore,

$$E\|X_0\|^m < \frac{C^*}{1 - 2^{m-1}D} < \infty.$$

□

8.2. **Proof of Theorem 4.1.** The proof consists in showing that all conditions of [32, Theorem 1.1] are in force under our assumptions, and to combine this with epi-convergence arguments; see [40, 3, 7] for more about epi-convergence theory and applications.

By virtue of (A.3) and (A.4), it follows from the composition rule in [40, Proposition 14.45(a)] that

$$(Y_t, (\lambda_k, \theta_k)) \mapsto \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k))$$

is random lsc. This entails that  $\xi_t^{(k)} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k))$  is also random lsc thanks to [40, Corollary 14.46]. In turn,  $h$  (see (4.2)), which is the sum of such  $K$  random lsc, is also random lsc in view of [40, Proposition 14.44(c)].

It remains to show that  $\inf_{\Theta \times \Lambda} h(Y_t, \xi_t, \cdot, \cdot) \in \mathbb{L}^1$ . We have

$$\begin{aligned} 0 &\stackrel{\text{(A.5)}}{\leq} \mathbb{E} \left[ \inf_{\theta, \lambda} h(Y_t, \xi_t, \theta, \lambda) \right] \\ &= \mathbb{E} \left[ \inf_{\theta, \lambda} \sum_{k=1}^K \xi_t^{(k)} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \right] \\ &\stackrel{\text{Separability}}{=} \mathbb{E} \left[ \sum_{k=1}^K \xi_t^{(k)} \inf_{\theta_k, \lambda_k} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \theta_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \right] \\ &\stackrel{\text{Optimality}}{\leq} \mathbb{E} \left[ \sum_{k=1}^K \xi_t^{(k)} \inf_{\lambda_k} \ell(X_t, f_k(X_{t-1}, \dots, X_{t-p}, \bar{\theta}_k), g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \right] \\ &\stackrel{\text{(A.6)}}{=} \mathbb{E} \left[ \sum_{k=1}^K \xi_t^{(k)} \inf_{\lambda_k} \ell(X_t, 0, g_k(X_{t-1}, \dots, X_{t-p}, \lambda_k)) \right] \\ &\stackrel{\text{(A.7)}}{\leq} \left( \sum_{k=1}^K \pi_k \right) (C \mathbb{E} \|X_t\|^\gamma + c) = C \mathbb{E} \|X_t\|^\gamma + c. \end{aligned}$$

Using the fact that  $\gamma = m$  and  $\mathbb{E} \|X_t\|^m < +\infty$  by Theorem 3.1, we deduce that  $\inf_{\Theta \times \Lambda} h(Y_t, \xi_t, \cdot, \cdot) \in \mathbb{L}^1$ .

Now, by (A.2),  $\Theta \times \Lambda$ , as a product space of Polish spaces is also Polish. Thus combining this with (A.1), that  $h$  is random lsc, and the summability property we have just shown, as well as the stationarity and ergodicity of  $Y_t$  which are inherited from those of  $X_t$ , it follows from [32, Theorem 1.1] that  $Q_n$  epi-converges to  $\mathbb{E}h(Y, \xi, \cdot, \cdot)$  a.s. It remains now to invoke standard epi-convergence arguments that entail the convergence of the minimizers of  $Q_n$  to those of  $\mathbb{E}h(Y, \xi, \cdot, \cdot)$ .

- (i) Apply [7, Corollary 7.20].
- (ii) Apply [7, Corollary 7.24].

□

8.3. **Proof of Theorem 5.1.** The proof consists in showing that the conditions (A1)-(A4) of [48, Theorem 3.2.23] are fulfilled.

Indeed, let us denote  $Y_{t-1} = (X_{t-1}, \dots, X_{t-p})$ . Then, from strict stationarity and ergodicity, the ergodic theorem and (B.2) it follows that

$$\begin{aligned} \frac{1}{n} \frac{\partial Q_n(\theta^0)}{\partial \theta_{k,i}} &= -\frac{2}{n} \sum_{t=1}^n \xi_t^{(k)} (X_t - f_k(Y_{t-1}, \theta_k^0))^\top \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i}} \\ &\xrightarrow{a.s.} -2\pi_k \mathbb{E} \left[ (X_{p+1} - f_k(Y_p, \theta_k^0))^\top \frac{\partial f_k(Y_p, \theta_k^0)}{\partial \theta_{k,i}} \right] = 0, \end{aligned}$$

for all  $k \in [K]$  and all  $i \in [d_k]$ . Hence, condition (A1) of [48, Theorem 3.2.23] is satisfied.

Similarly, using again (B.2) and the ergodic theorem we have that

$$\begin{aligned} \frac{1}{n} \frac{\partial^2 Q_n(\theta^0)}{\partial \theta_{l,j} \partial \theta_{k,i}} &= \frac{2}{n} \sum_{t=1}^n \xi_t^{(k)} \left[ \left( \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,j}} \right)^\top \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i}} \right. \\ &\quad \left. - (X_t - f_k(Y_{t-1}, \theta_k^0))^\top \frac{\partial^2 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,j} \partial \theta_{k,i}} \right] \mathbb{I}_{\{l=k\}} \\ &\xrightarrow{a.s.} 2\pi_k \mathbb{E} \left[ \left( \frac{\partial f_k(Y_p, \theta_k^0)}{\partial \theta_{k,j}} \right)^\top \frac{\partial f_k(Y_p, \theta_k^0)}{\partial \theta_{k,i}} \right] \mathbb{I}_{\{l=k\}} = 2(V_{kl})_{ij}, \quad (8.15) \end{aligned}$$

for all  $k, l \in [K]$  and all  $(i, j) \in [d_k] \times [d_l]$ , because

$$n^{-1} \sum_{t=1}^n \xi_t^{(k)} (X_t - f_k(Y_{t-1}, \theta_k^0))^\top \frac{\partial^2 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,j} \partial \theta_{k,i}} \mathbb{I}_{\{l=k\}} \xrightarrow{a.s.} 0,$$

for all  $k, l \in [K]$  and all  $(i, j) \in [d_k] \times [d_l]$ ; see [46]. In the expression (8.15),  $(V_{kl})_{ij}$  denotes the  $(i, j)$ -th entry of the matrix  $V_{kl}$  defined in (5.4). From (B.3), the Gram matrix of each Jacobian  $J[f_k(X_p, \dots, X_1, \cdot)](\theta_k^0)$  is invertible for any  $k \in [K]$ , whence we deduce that  $V$  is positive definite since it is block-diagonal whose diagonal blocks are those Gram matrices (up to multiplication by  $\pi_k > 0$ ). Thus, assumption (A2) of [48, Theorem 3.2.23] is also satisfied.

Let  $\theta \in \mathcal{V}$ , and  $\delta > 0$  such that the ball  $\|\theta - \theta^0\| < \delta$  is contained in  $\mathcal{V}$  ( $\delta$  can be chosen arbitrarily small for this to hold). Let the closed segment  $[\theta_0, \theta] = \{\rho\theta + (1-\rho)\theta_0 : \rho \in [0, 1]\}$  and the open segment  $] \theta_0, \theta[ = \{\rho\theta + (1-\rho)\theta_0 : \rho \in ]0, 1[ \}$ . Then, for  $\bar{\theta} \in ] \theta_0, \theta[$ , and any  $k, l \in [K]$  and  $(i, j) \in [d_k] \times [d_l]$ , we have from the mean value theorem that

$$\begin{aligned} (T_n(\bar{\theta}))_{kl,ij} &:= \begin{cases} \frac{\partial^2 Q_n(\bar{\theta})}{\partial \theta_{k,j} \partial \theta_{k,i}} - \frac{\partial^2 Q_n(\theta^0)}{\partial \theta_{k,j} \partial \theta_{k,i}} & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \\ &= (\bar{\theta} - \theta^0)^\top \nabla \left( \frac{\partial^2 Q_n(\bar{\theta})}{\partial \theta_{k,j} \partial \theta_{k,i}} \right) \mathbb{I}_{\{l=k\}}, \quad \text{for some } \bar{\theta} \in ] \theta^0, \bar{\theta} [. \end{aligned}$$

Since by definition  $\|\bar{\theta} - \theta^0\| < \delta$ , we have  $\|\bar{\theta} - \theta^0\| < \delta$  and thus  $\bar{\theta} \in \mathcal{V}$ . Hence from continuity of the norm and that of the derivatives of  $Q_n$  up to third-order on  $\mathcal{V}$ , we get, upon using

Cauchy-Schwartz inequality, that

$$\begin{aligned}
& \sup_{\delta \rightarrow 0} \frac{1}{n\delta} \left| (T_n(\bar{\theta}))_{kl,ij} \right| \\
& \leq \liminf_{\delta \rightarrow 0} \frac{1}{n} \left\| \nabla \left( \frac{\partial^2 Q_n(\bar{\theta})}{\partial \theta_{k,j} \partial \theta_{k,i}} \right) \mathbb{I}_{\{l=k\}} \right\| = \lim_{\delta \rightarrow 0} \frac{1}{n} \left\| \nabla \left( \frac{\partial^2 Q_n(\bar{\theta})}{\partial \theta_{k,j} \partial \theta_{k,i}} \right) \mathbb{I}_{\{l=k\}} \right\| \\
& \leq \lim_{\delta \rightarrow 0} \frac{2}{n} \mathbb{I}_{\{l=k\}} \sum_{r=1}^{d_k} \sum_{t=1}^n \xi_t^{(k)} \left( \left| \left( \frac{\partial f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,i}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,j} \partial \theta_{k,r}} \right| \right. \\
& + \left| \left( \frac{\partial f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,j}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,i} \partial \theta_{k,r}} \right| + \left| \left( \frac{\partial f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,r}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,i} \partial \theta_{k,j}} \right| \\
& \quad \left. + \left| \left( X_t - f_k(Y_{t-1}, \bar{\theta}_k) \right)^\top \frac{\partial^3 f_k(Y_{t-1}, \bar{\theta}_k)}{\partial \theta_{k,i} \partial \theta_{k,j} \partial \theta_{k,r}} \right| \right) \\
& = \frac{2}{n} \mathbb{I}_{\{l=k\}} \sum_{r=1}^{d_k} \sum_{t=1}^n \xi_t^{(k)} \left( \left| \left( \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,j} \partial \theta_{k,r}} \right| \right. \\
& + \left| \left( \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,j}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i} \partial \theta_{k,r}} \right| + \left| \left( \frac{\partial f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,r}} \right)^\top \frac{\partial^2 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i} \partial \theta_{k,j}} \right| \\
& \quad \left. + \left| \left( X_t - f_k(Y_{t-1}, \theta_k^0) \right)^\top \frac{\partial^3 f_k(Y_{t-1}, \theta_k^0)}{\partial \theta_{k,i} \partial \theta_{k,j} \partial \theta_{k,r}} \right| \right).
\end{aligned}$$

From strict stationarity, ergodicity and the condition **(B.4)**, by using the ergodic theorem again, it follows that

$$\lim_{n \rightarrow \infty} \sup_{\delta \rightarrow 0} \frac{1}{n\delta} \left| (T_n(\bar{\theta}))_{kl,ij} \right| \leq 2\pi_k \mathbb{I}_{\{l=k\}} \sum_{r=1}^{d_k} (G_k^{ijr} + G_k^{jir} + G_k^{rij} + H_k^{ijr}) < \infty.$$

With this we have shown that condition (A3) of [48, Theorem 3.2.23] holds.

Finally, by using [48, Theorem 1.3.3], the vector process  $(Z_t)_{t \in \mathbb{Z}}$  defined by

$$\begin{aligned}
Z_t = -2 \left( \xi_t^{(1)} (X_t - f_1(Y_{t-1}, \theta_1^0))^\top \frac{\partial f_1(Y_{t-1}, \theta_1^0)}{\partial \theta_{1,1}}, \dots \right. \\
\left. \dots, \xi_t^{(K)} (X_t - f_1(Y_{t-1}, \theta_1^0))^\top \frac{\partial f_K(Y_{t-1}, \theta_K^0)}{\partial \theta_{K,d_K}} \right)
\end{aligned}$$

is strictly stationary and ergodic. Therefore, condition (A4) of [48, Theorem 3.2.23] follows by combining **(B.5)** and [48, Theorem A.2.14]. This completes the proof.  $\square$

## APPENDIX A. DERIVATIVES WITH RESPECT TO NN PARAMETERS

Let  $\theta = ((W^{(1)}, \beta^{(1)}), \dots, (W^{(L)}, b^{(L)}))$  be an architecture of a NN  $f : (x, \theta) \in \mathbb{R}^d \rightarrow \mathbb{R}^{N_L}$  and denote  $W^{(l)} = (w_{j_i j_{i-1}}^{(l)})_{(j_i, j_{i-1}) \in [N_i] \times [N_{i-1}]}$  and  $\beta^{(l)} = (\beta_{j_i}^{(l)})_{j_i \in [N_i]}$ , with  $l \in [L]$ . We denote  $D[f]_{W^{(l)}}(x, \theta)$  the Fréchet derivatives of  $f$  wrt to  $W^{(l)}$  evaluated at  $(x, \theta)$ . Recalling the recursion in Definition 2.2, and by the standard chain rule,  $D[f]_{W^{(l)}}$  acting in the direction  $H^{(l)} \in \mathbb{R}^{N_i \times N_{i-1}}$  reads, for  $l \in [L]$ ,

$$D[f]_{W^{(l)}}(x, \theta)(H^{(l)}) = \left( \prod_{i=L-1}^l W^{(i+1)} J[\varphi] (W^{(i)} x^{(i-1)} + b^{(i)}) \right) H^{(l)} x^{(l-1)}. \quad (\text{A.1})$$

Similarly, we have

$$J[f]_{b^{(l)}}(x, \theta) = \prod_{i=L-1}^l W^{(i+1)} J[\varphi] (W^{(i)} x^{(i-1)} + b^{(i)}). \quad (\text{A.2})$$

As usual, the partial derivatives  $\frac{\partial f(x, \theta)}{\partial w_{ij}^{(l)}}(x, \theta)$  (resp.  $\frac{\partial f(x, \theta)}{\partial \beta_i^{(l)}}(\theta)$ ) is nothing but (A.1) (resp. (A.2)) evaluated in the direction  $H^{(l)}$  (resp.  $i$ -th standard basis vector of  $\mathbb{R}^{L_i}$ ) such that  $H_{ij}^{(l)} = 1$  and 0 otherwise.

A similar calculation can be carried out to get the second- and third-order derivatives that we leave to the reader.

## REFERENCES

- [1] D.W.K. Andrews. Non strong mixing autoregressive processes. *J. Appl. Prob.*, 21:930–934, 1984.
- [2] Z. Artstein and R. J-B. Wets. Consistency of minimizers and the SLLN for stochastic programs. *J. Convex Anal.*, 2:1–17, 1995.
- [3] H. Attouch. *Variational convergence for functions and operators*. Applicable mathematics series. Pitman Advanced Publishing Program, 1984.
- [4] H. Attouch and R. J.-B. Wets. Epigraphical processes: laws of large numbers for random lsc functions. *Sém. Anal. Convexe, Montpellier*, 13:1–29, 1990.
- [5] C. Castera, C. Bolte, C. Févotte, and E. Pauwels. An inertial newton algorithm for deep learning. <https://arxiv.org/pdf/1905.12278.pdf>, 2019.
- [6] R. Dahlhaus. A likelihood approximation for locally stationary processes. *Annals of Statistics*, 28:1762–1794, 2000.
- [7] G. Dal Maso. *An introduction to  $\Gamma$ -convergence*, volume 8. Springer Science & Business Media, 2012.
- [8] I. Daubechies, R. DeVore, S. Foucart, B. Hanin, and G. Petrova. Nonlinear approximation and (deep) ReLU networks. arxiv preprint arxiv:1905.02199, 2019.
- [9] J. Dedecker and C. Prieur. Coupling for  $\tau$ -dependent sequences and applications. *J. Theoret. Probab.*, 17(4):861–885, 2004.
- [10] R. Douc, K. Fokianos, and E. Moulines. Asymptotic properties of quasi-maximum likelihood estimators in observation-driven time series models. *Electronic Journal of Statistic*, 11:2707–2740, 2017.

- [11] P. Doukhan, K. Fokianos, and X. Li. On weak dependence conditions: The case of discrete valued processes. *Statistics and Probability Letters*, 82:1941–1948, 2012.
- [12] P. Doukhan, K. Fokianos, and D. Tjøstheim. On weak dependence conditions for Poisson autoregressions. *Statistics and Probability Letters*, 82:942–948, 2012.
- [13] P. Doukhan and O. Wintenberger. Weakly dependent chains with infinite memory. *Stochastic Processes and their Applications*, 118:1997–2013, 2008.
- [14] D. Falbel, JJ. Allaire, F. Chollet, RStudio, Google, Y. Tang, W. Van Der Bijl, M. Studer, and S. Keydana. Package ‘keras’. <https://cran.r-project.org/web/packages/keras/keras.pdf>, 2019.
- [15] R. Ferland, A. Latour, and D. Oraichi. Integer-valued GARCH processes. *Journal of Time Series Analysis*, 27:923–942, 2006.
- [16] K. Fokianos and R. Fied. Interventions in INGARCH processes. *Journal of Time Series Analysis*, 31:210–225, 2010.
- [17] K. Fokianos, A. Rahbek, and D. Tjøstheim. Poisson autoregression. *Journal of the American Statistical Association*, 104:1430–1439, 2009.
- [18] K. Fokianos and D. Tjøstheim. Nonlinear poisson autoregression. *Annals of the Institute of Statistical Mathematics*, 64:1205–1225, 2012.
- [19] J. Franke, W. Hardle, and C. Hafner. *Statistics of Financial Markets: An introduction*. Springer, 5 edition, 2019.
- [20] A. Griewank and A. Walther. *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation, Second Edition*. SIAM, 2008.
- [21] B. Haeffele and R. Vidal. Global optimality in neural network training. In *CVPR*, 2017.
- [22] C. Hafner. *Nonlinear Time Series Analysis with Applications to Foreign Exchange Rate Volatility*. Contributions to Economics. Springer-Verlag Berlin Heidelberg GmbH, 1998.
- [23] B. Hanin and M. Sellke. Approximating continuous functions by ReLU nets of minimal width. arxiv preprint arXiv:1710.11278, 2017.
- [24] D. Hebb. *The organization of behavior: A neuropsychological theory*. Wiley, 1949.
- [25] N. Henze and B. Zirkler. A class of invariant consistent tests for multivariate normality. *Commun Stat Theory Methods*, 19(10):3595–3617, 1990.
- [26] C. Hess. Epi-convergence of sequences of normal integrands and strong consistency of the maximum likelihood estimator. *Annals of Statistics*, 24(3):1298–1315, 1996.
- [27] K. Hornik, M. Stinchcombe, and H. White. Multilayer feedforward networks are universal approximators. *Neural networks*, 2(5):359–366, 1989.
- [28] C. Kirch and Ta. Testing for parameter stability in nonlinear autoregressive models. *Journal of Time*, 33(3):365–385, 2012.
- [29] L.A. Klimko and P.I. Nelson. On conditional least squares estimation for stochastic processes. *The Annals of Statistics*, 6(3):629–642, 1978.
- [30] L. A. Korf and R. J.-B. Wets. An ergodic theorem for stochastic programming problems. In V.H. Nguyen, J.J. Strodiot, and P. Tossings, editors, *Proceedings of the 9th Belgian-French-German Conference on Optimization*, volume 481 of *Lecture Notes in Economics and Mathematical Sciences*, pages 203–217. Springer, 2000.
- [31] L. A. Korf and R. J.-B. Wets. Random lsc functions: An ergodic theorem. In *Stochastic Programming E-print Series (SPEPS)*. Humboldt-Universität, 2000.
- [32] L. A. Korf and R. J.-B. Wets. Random lsc functions: An ergodic theorem. *Mathematics of Operations Research*, 26(2):421–445, May 2001.

- [33] S. Korkmaz, D. Goksuluk, and G. Zarasiz. An r package for assessing multivariate normality. *R Journal*, 6(2):151–162, 2014.
- [34] S. Liehr, K. Pawelzik, J. Kohlmorgen, and K.R. Moler. Hidden markov mixtures of experts with an application to eeg recordings from sleep. *Theory of Biosciences*, 118:246–260, 1999.
- [35] M.T. Lo, P.H. Tsai, P.F. Lin, C. Lin, and Y.L. Hsin. The nonlinear and nonstationary properties in eeg signals: probing the complex fluctuations by hilbert-huang transform. *Advances in Adaptive Data Analysis*, 1(3):461–482, 2009.
- [36] KV. Mardia. Measures of multivariate skewness and kurtosis with applications. *Biometrika*, 57(3):519–530, 1970.
- [37] S.P. Meyn and R.L. Tweedie. *Markov Chain and Stochastic Stability*. Springer-Verlag, 1993.
- [38] B. Neyshabur, Y. Wu, R. Salakhutdinov, and N. Srebro. Path-normalized optimization of recurrent neural networks with relu activations. In *NIPS*, 2016.
- [39] R. T. Rockafellar. Integral functionals, normal integrands and measurable selections. In J. Gossez and L. Waelbroeck, editors, *Nonlinear Operators and the Calculus of Variations*, number 543 in Lecture Notes in Mathematics, pages 157–207. Springer, 1976.
- [40] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer, 1998.
- [41] I. Rojas, H. Pomares, and O. Valenzuela, editors. *Advances in Time Series Analysis and Forecasting: Selected Contributions from ITISE 2016*. Contributions to Statistics. Springer International Publishing, 2017.
- [42] F. Rosenblatt. The perceptron: a probabilistic model for information storage and organization in the brain. *Psychological review*, 65(6):386, 1958.
- [43] P. Royston. Approximating the shapiro-will w test for non-normality. *Statistics and Computing*, 2(3):117–119, 1992.
- [44] S Shalev-Shwartz and S Ben-David. *Understanding machine learning: from theory to algorithms*. Cambridge University Press, 2014.
- [45] J-P. Stockis, J. Franke, and J. Tadjuidje Kamgaing. On geometric ergodicity of charme models. *Journal of Time Series Analysis*, 31:141–152, 2010.
- [46] W.F. Stout. *Almost Sure Convergence*. Academic Press, New York, 1974.
- [47] J. Tadjuidje Kamgaing. *Competing neural networks as model for nonstationary financial time series*. PhD thesis, University of Kaiserslautern, 2005.
- [48] M. Taniguchi and Y. Kakizawa. *Asymptotic Theory of Statistical Inference for Time Series*. Springer-Verlag, New York, 2000.
- [49] M. Telgarsky. Representation benefits of deep feedforward networks. arXiv preprint arXiv:1509.08101, 2015.
- [50] R. Vidal, J. Bruna, R. Giryes, and S. Soatto. Mathematics of deep learning. In *IEEE CDC*, 2017.
- [51] A.S. Weigend and S. Shi. Predicting daily probability distributions of s&p500 returns. *Journal of Forecasting*, 19(4):375–392, 2000.
- [52] D. Yarotsky. Error bounds for approximations with deep relu networks. *Neural Networks*, 94:103–114, 2017.
- [53] D. Yarotsky. Quantified advantage of discontinuous weight selection in approximations with deep neural networks. arXiv preprint arXiv:1705.01365, 2017.

- [54] C. Yun, S. Sra, and A. Jadbabaie. Global optimality conditions for deep neural networks. In *ICLR*, 2018.

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