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# Interval observers design for continuous-time linear switched systems

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**Abstract:** This paper is devoted to investigate interval observers design for linear switched systems. The considered systems are subject to disturbances which are assumed to be unknown but bounded. First, observer gains are computed to ensure the stability of the estimation error. Then, under some changes of coordinates an interval observer is designed. Efficiency of the proposed method is demonstrated through a numerical example.



Keywords: Interval observer, stability, continuous-time switched systems, hybrid systems, cooperativity

## 1. INTRODUCTION

Switched systems are one of the most important classes of Hybrid Dynamical Systems (HDS). They consist of continuous or discrete-time subsystems where only one subsystem is active at each time. A switching rule orchestrating among them. Several works treated switched systems such as (Djemai and Defoort (2015); Liberzon (2012); Branicky (1998)). Among these studies, control and state estimation were crucial and fundamental problems.

In order to study the stability of switched systems, specific results have been developed. For example, in (Hu et al. (2002)) a common Lyapunov function yields sufficient conditions for the global asymptotic stability. However, it may not always be possible to get this common function. Therefore, multiple Lyapunov functions were proposed for instance in (Liberzon and Morse (1999)). Approaches based on average dwell time have also been studied to ensure the stability of switched systems (see Hetel (2007); Serres et al. (2011) and the references therein).

As far as the stability problem is widely concerned, it is worth pointing out that the state is not always directly measured but may be estimated from the input and the output of the process. State estimation for switched systems remains a challenging problem by reason of the combined discrete and continuous states (see Hocine et al. (2005); Tian et al. (2009); Birouche et al. (2006); Arichi et al. (2015)).

In practice, measurements are usually subject to noises. To compute robust estimates, interval observers assume that noises and disturbances are bounded without any stochastic assumptions. This approach assumes that the measurement and process errors are unknown but bounded.

In the last decade, several works have investigated estimation problems in an unknown but bounded error (UBBE) framework for different classes of continuous systems such as Linear Time-Invariant (LTI), Linear Time-Varying (LTV) and Linear Parameter-Varying (LPV) systems (see Raïssi et al. (2012); Mazenc and Bernard (2011); Videau et al.; Raïssi et al. (2010); Mazenc and Bernard (2010); Gouzé et al. (2000); Lamouchi et al. (2016); Yousfi et al. (2016b,a) and the references therein).

The main restriction of interval observer theory is the cooperativity feature. Considering that most of systems are not cooperative, it has been shown in the literature that any linear invariant-time system could be changed into a cooperative form. The technique proposed in (Mazenc and Bernard (2011)) considers that it is possible to design an interval observer via a time-varying change of coordinates based on the Jordan representation. In (Raïssi et al. (2012)) an invariant change of coordinates is used to ensure the cooperativity property.

The extension of interval observers to switched systems has not been fully considered in the literature. To the best of the authors knowledge, only a preliminary work has been developed in (He and Xie (2016, 2015)) with the strong assumption that there exist gains $L_q$ such that $(A_q - L_q C_q)$, $q \in \overline{1, N}$, are Metzler (the off-diagonal elements are nonnegative). Unfortunately, this assumption is rarely verified.

In this paper, the main contribution is to design an interval observer for switched linear systems subject to disturbances. The measurement noise and the state dis-


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turbance are unknown but bounded with known bounds. State estimation conditions are given in terms of Linear Matrix Inequalities (LMIs). The suggested methodology allows one to overcome the strong limitations in (He and Xie (2016, 2015)). It will be shown that the constructive methodology can be applied for a large class of linear switched systems.

This paper is structured as follows. Some preliminaries are described in Section 2. Interval state estimation for switched linear systems is stated in Section 3. Section 4 is dedicated to show efficiency of the proposed method via a numerical example. Section 5 concludes the paper.

## 2. PRELIMINARIES

The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$ respectively. $|x|$ is the elementwise absolute value of a vector $x \in \mathbb{R}^n$. The sequence of integers $1, ..., N$ is denoted by $\overline{1, N}$. $E_p$ is a $(p \times 1)$ vector whose elements are equal to 1. $I$ is the identity matrix of proper dimension. For a matrix $P = P^T$, $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^n$ is negative (positive) definite. Lower and upper bounds $\underline{x}$ and $\overline{x}$ of $x$ satisfy $\underline{x} \leq x \leq \overline{x}$, where the relation $\leq$ is interpreted elementwise for vectors and matrices. For a matrix $A \in \mathbb{R}^{m\times n}$, let $A^+ = \max\{0, A\}$ and $A^- = A^+ - A$.

*Lemma 1.* (Chebotarev et al. (2015)) Let $x \in \mathbb{R}^n$ be a vector satisfying $\underline{x} \leq x \leq \overline{x}$ and $A \in \mathbb{R}^{m\times n}$ be a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

A matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ is called Metzler if all its off-diagonal elements are nonnegative, i.e. $a_{ij} \geq 0, \ \forall i \neq j$.

*Lemma 2.* (Gouzé et al. (2000)) Let

$$\dot{x}(t) = Ax + u, \quad x(0) = x_0$$

where $A$ is a Metzler matrix and $u \geq 0$. If $x_0 \geq 0$, then

$$x(t) \geq 0, \quad \forall t \geq 0. \tag{2}$$

A continuous-time linear system $\dot{x}(t) = Ax$ is said to be cooperative if A is a Metzler matrix.

*Lemma 3.* (Jiang et al. (2002)) Let $\delta > 0$ be a scalar and $P \in \mathbb{R}^n$ be a symmetric positive definite matrix, then

$$2x^T y \leq \frac{1}{\delta}x^T P x + \delta y^T P^{-1} y \quad x, y \in \mathbb{R}^n. \tag{3}$$

Consider the system described by:

$$\begin{cases} \dot{x} = Ax + \phi(t) \\ y = Cx \end{cases} \tag{4}$$

where $\phi$ is a continuous function and assume that there exist two known functions $\overline{\phi}$ and $\underline{\phi} : \mathbb{R} \to \mathbb{R}^n$ Lipschitz continuous such that $\underline{\phi}(t) \leq \phi(t) \leq \overline{\phi}(t)$ for all $t \geq 0$.

*Theorem 1.* (Gouzé et al. (2000)) If there exists a gain K such that $(A - KC)$ is Metzler and if $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, then the system

$$\begin{cases} \dot{\overline{x}} = A\overline{x} + \overline{\phi} + K(y - C\overline{x}) \\ \dot{\underline{x}} = A\underline{x} + \underline{\phi} + K(y - C\underline{x}) \end{cases} \tag{5}$$

is a framer for the system (4) such that

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \quad \forall t \geq 0$$

*Theorem 2.* (Gouzé et al. (2000)) The system (5) is called an interval observer for system (4) if the lower $(x - \underline{x})$ and upper $(\overline{x} - x)$ estimation errors are asymptotically stable.

## 3. INTERVAL STATE ESTIMATION FOR SWITCHED LINEAR SYSTEMS

Consider a Switched Linear System (SLS) described by:

$$\begin{cases} \dot{x}(t) = A_q x(t) + B_q u(t) + w(t) \\ y_m(t) = C_q x(t) + v(t) \end{cases}, q \in \overline{1, N}, N \in \mathbb{N} \tag{6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y_m \in \mathbb{R}^p$, $w \in \mathbb{R}^n$, $v \in \mathbb{R}^p$ are respectively the state vector, the input, the output, the disturbance and the measurement noise. $A_q$, $B_q$ and $C_q$ are constant matrices of proper dimensions. $q$ is the index of the active subsystem and $N$ is the number of subsystems. The measurement noise and the state disturbance are assumed to be unknown but bounded with *a priori* known bounds such that

$$|w(t)| \leq \overline{w}, \ \ |v(t)| \leq \overline{V}E_p, \quad \forall t \geq 0 \tag{7}$$

where $\overline{w} \in \mathbb{R}^n$ and $\overline{V}$ is a scalar.

The aim is to derive two trajectories $\underline{x}(t)$ and $\overline{x}(t)$ where $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, $\forall t \geq 0$, despite the disturbances, starting from the initial condition $x_0$ which is assumed to be bounded by two known bounds $\underline{x}_0 \leq x_0 \leq \overline{x}_0$.

To design an interval observer for (6), a necessary condition is given in the following assumption.

*Assumption 1.* There exist gains $L_q$ such that the matrices $(A_q - L_q C_q)$ are Metzler for $q \in \overline{1, N}$.

The matrices $L_q$ $(q \in \overline{1, N})$ denote the observer gains associated with each subsystem $q$.

A candidate interval observer structure for the estimation of $\overline{x}$, $\underline{x}$ is described by:

$$\begin{cases} \dot{\overline{x}} = (A_q - L_q C_q)\overline{x} + B_q u + \overline{w} + L_q y_m + |L_q|\overline{V}E_p \\ \dot{\underline{x}} = (A_q - L_q C_q)\underline{x} + B_q u - \overline{w} + L_q y_m - |L_q|\overline{V}E_p \end{cases} \tag{8}$$

Similarly to *Theorem* 1, the following theorem gives the conditions for achieving partially the desired design goal.

*Theorem 3.* Consider the system described by (6). Let *Assumption 1* be satisfied. For any initial condition $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, if there exist observer gains $L_q$ such that

$$(A_q - L_q C_q) \text{ are Metzler}, \forall q \in \overline{1, N} \tag{9}$$

then the system (8) is a framer for the system (6) with

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \quad \forall t \geq 0. \tag{10}$$

**Proof.** Let $\overline{e}(t) = \overline{x} - x$ be the upper observation error and $\underline{e}(t) = x - \underline{x}$ be the lower observation error. Let us show that $\overline{e}$ and $\underline{e}$ are positive.

From (6) and (8) the dynamics of the interval estimation errors are given by:

$$\dot{\overline{e}}(t) = \dot{\overline{x}} - \dot{x} = (A_q - L_q C_q)\overline{e}(t) + \overline{\Gamma}_q \tag{11}$$

$$\text{and} \quad \dot{\underline{e}}(t) = \dot{x} - \dot{\underline{x}} = (A_q - L_q C_q)\underline{e}(t) + \underline{\Gamma}_q \tag{12}$$

where

$$\overline{\Gamma}_q = \overline{w} - w + L_q v + |L_q|\overline{V}E_p \tag{13}$$

$$\underline{\Gamma}_q = w + \overline{w} - L_q v + |L_q|\overline{V}E_p \tag{14}$$

As $|v(t)| \leq \overline{V}E_p$, $\forall t \geq 0$ and by construction $(\overline{w} - w)$ and $(w + \overline{w})$ are positive then $\overline{\Gamma}_q$ and $\underline{\Gamma}_q$ are positive for all $q \in \overline{1,N}$. In addition, since $L_q$ are computed to verify (9) and $\underline{x}_0$ and $\overline{x}_0$ are chosen such that

$$\begin{cases} \overline{e}(0) = \overline{x}_0 - x_0 \geq 0 \\ \underline{e}(0) = x_0 - \underline{x}_0 \geq 0 \end{cases}$$

then, according to *Lemma* 2, $\overline{e}(t)$ and $\underline{e}(t)$ are positive $\forall t \geq 0$. Thus, $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

■

*Remark 1.* The framer (8) is initialized with the initial conditions $\underline{x}_0$ and $\overline{x}_0$ for the active subsystem $(q = 1)$. At the switching time instant, the output of the previous active subsystem $(q = i)$ is used to initialize (8) with the subsystem $(q = i + 1)$.
In addition, *Theorem* 3 ensures only the inclusion relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$. However, the errors $(x - \underline{x})$ and $(\overline{x} - x)$ are not guaranteed to be bounded.

For the stability analysis of (11) and (12), let us introduce the following lemma.

*Lemma 4.* (Liberzon and Morse (1999)) Let

$$\dot{x}(t) = A_q x(t), \quad q \in \overline{1,N} \tag{15}$$

The switched system (15) is globally asymptotically stable if there exists a matrix $S = S^T > 0$ such that

$$\dot{V}(x) = x^T \left(A_q{}^T S + S A_q\right) x < 0, \quad q \in \overline{1,N} \tag{16}$$

where $V(x)$ is the common Lyapunov function given by:

$$V(x) = x^T S x. \tag{17}$$

*Theorem 4.* Let *Assumption 1* be satisfied. Given scalars $\delta_q > 0$, if there exists a symmetric positive definite matrix $S \in \mathbb{R}^n$ for all $q \in \overline{1,N}$ such that

$$A_q{}^T S + S A_q - C_q{}^T W_q{}^T - W_q C_q + \alpha_q S \prec 0 \tag{18}$$

where $\alpha_q = \frac{2}{\delta_q}$ and $W_q = S L_q$, then the framer (8) is an asymptotically stable interval observer for (6).

**Proof.** As mentioned in *Theorem* 2, the global asymptotic stability of the interval observer is guaranteed by applying a common Lyapunov function to the estimation errors. Consider the following Lyapunov function:

$$V(\overline{e}) = \overline{e}(t)^T S \overline{e}(t), \quad S \succ 0. \tag{19}$$

$$\begin{aligned} \dot{V}(\overline{e}) &= \dot{\overline{e}}^T S \overline{e} + \overline{e}^T S \dot{\overline{e}} \\ &= \overline{e}^T \left[ (A_q - L_q C_q)^T S + S(A_q - L_q C_q) \right] \overline{e} - 2\overline{e}^T S w \\ &\quad + 2\overline{e}^T S L_q v + 2\overline{e}^T S \overline{w} + 2\overline{e}^T S |L_q| \overline{V} E_p \end{aligned} \tag{20}$$

From *Lemma* 3, it is clear that

$$\dot{V}(\overline{e}) \leq \overline{e}^T \mathcal{B}_1 \overline{e} + \mathcal{C}_1 \tag{21}$$

where

$$\mathcal{B}_1 = A_q{}^T S + S A_q - C_q{}^T W_q{}^T - W_q C_q + \frac{2}{\delta_q} S \tag{22}$$

$$\begin{aligned} \mathcal{C}_1 &= w^T \left[-\delta_q S\right] w + \overline{w}^T \left[\delta_q S\right] \overline{w} + v^T \left[\delta_q L_q^T S L_q\right] v \\ &\quad + E_p{}^T \left[\delta_q \overline{V} L_q^T S L_q \overline{V}\right] E_p \end{aligned} \tag{23}$$

From (18), it is assumed that $\mathcal{B}_1 \prec 0$. In addition, the noises and disturbances are bounded it follows that $\mathcal{C}_1$ is bounded. Therefore the error $\overline{e}$ is bounded. The same arguments show that the error $\underline{e}$ is also bounded.

■

The methodology described above, although simple, is not always constructive. Indeed, it is not always possible to find gains $L_q$ such that *Assumption 1* is satisfied. Hence, the key point of the idea is to find a change of coordinates that transforms the observation errors into cooperative forms. The changes of coordinates proposed for instance in (Raïssi et al. (2012); Mazenc and Bernard (2011)) for continuous systems can be used to transform the matrices $(A_q - L_q C_q)$ into a Metzler form.

Let us assume that there exists a non singular transformation matrix $P$ such that, with the new coordinates $z = Px$, the system (6) is transformed into the form

$$\begin{cases} \dot{z} = P A_q P^{-1} z + P B_q u + P w \\ y = C_q P^{-1} z + v \end{cases} \tag{24}$$

A Luenberger based candidate observer for the system (24) can be written in the new coordinates $z$ as:

$$\begin{cases} \dot{\overline{z}} = P(A_q - L_q C_q) P^{-1} \overline{z} + P B_q u + P^+ \overline{w} - P^- \underline{w} \\ \qquad + P L_q y_m + |P L_q| \overline{V} E_p \\ \dot{\underline{z}} = P(A_q - L_q C_q) P^{-1} \underline{z} + P B_q u + P^+ \underline{w} - P^- \overline{w} \\ \qquad + P L_q y_m - |P L_q| \overline{V} E_p \end{cases} \tag{25}$$

$$\forall q \in \overline{1,N}$$

where

$$\overline{z}(0) = P^+ \overline{x}_0 - P^- \underline{x}_0 \tag{26}$$

$$\underline{z}(0) = P^+ \underline{x}_0 - P^- \overline{x}_0 \tag{27}$$

$P$ is the solution of the Sylvester equation given by

$$P A_q - R_q P = Q_q C_q, \quad Q_q = P L_q \tag{28}$$

and

$$R_q = P(A_q - L_q C_q) P^{-1}$$

Let $\overline{e_z}(t) = \overline{z} - z$ be the upper observation error and $\underline{e}_z(t) = z - \underline{z}$ be the lower one.

From systems (24) and (25), the dynamics of the interval estimation errors are given by:

$$\begin{aligned} \dot{\overline{e}}_z(t) &= \dot{\overline{z}} - \dot{z} = \left[\left(P^+ \overline{w} - P^- \underline{w}\right) - P w\right] + |P L_q| \overline{V} E_p \\ &\quad + P(A_q - L_q C_q) P^{-1} \overline{e_z} + P L_q v = R_q \overline{e_z} + \overline{\Upsilon}_q \end{aligned} \tag{29}$$

$$\begin{aligned} \dot{\underline{e}}_z(t) &= \dot{z} - \dot{\underline{z}} = \left[P w - \left(P^+ \underline{w} - P^- \overline{w}\right)\right] + |P L_q| \overline{V} E_p \\ &\quad + P(A_q - L_q C_q) P^{-1} \underline{e}_z - P L_q v = R_q \underline{e}_z + \underline{\Upsilon}_q \end{aligned} \tag{30}$$

where

$$\overline{\Upsilon}_q = \left[\left(P^+ \overline{w} - P^- \underline{w}\right) - P w\right] + |P L_q| \overline{V} E_p + P L_q v \tag{31}$$

$$\underline{\Upsilon}_q = \left[P w - \left(P^+ \underline{w} - P^- \overline{w}\right)\right] + |P L_q| \overline{V} E_p - P L_q v \tag{32}$$

Similarly to the proof of *Theorem* 4, the asymptotic stability of the observer (25) is ensured by applying a common Lyapunov function to the observation errors as follows:

$$V(\overline{e}_z) = \overline{e}_z{}^T M \overline{e}_z, \; V(\underline{e}_z) = \underline{e}_z{}^T M \underline{e}_z \tag{33}$$

In the following, only the dynamics of the upper observation error are considered. The derivative of $V(\overline{e}_z)$ is:

$$\dot{V}(\overline{e}_z) = \dot{\overline{e}}_z^T M\overline{e}_z + \overline{e}_z{}^T M\dot{\overline{e}}_z \leq \overline{e}_z{}^T \mathcal{B}_2 \overline{e}_z + \mathcal{C}_3 \quad (34)$$

where

$$\begin{aligned}\mathcal{C}_3 = {} & \overline{w}^T \left[\delta_q {P^+}^T M P^+\right] \overline{w} - \underline{w}^T \left[\delta_q {P^-}^T M P^-\right] \underline{w} \\ & - w^T \left[\delta_q P^T M P\right] w + v^T \left[\delta_q L_q^T P^T M P L_q\right] v \\ & + {E_p}^T \left[\delta_q \overline{V} |P L_q|^T M |P L_q| \overline{V}\right] E_p \end{aligned} \quad (35)$$

and

$$\begin{aligned}\mathcal{B}_2 = {} & P(A_q - L_q C_q) {P^{-1}}^T M + M\left[P(A_q - L_q C_q) P^{-1}\right]^T \\ & + \frac{1}{\delta_q} M = {P^{-1}}^T {A_q}^T P^T M + M P A_q P^{-1} + \frac{1}{\delta_q} M \\ & - {P^{-1}}^T {C_q}^T {L_q}^T P^T M - M P L_q C_q P^{-1} \prec 0 \end{aligned} \quad (36)$$

The existence of a common transformation matrix $P$ such that $P(A_q - L_q C_q) P^{-1}$ for all $q \in \overline{1, N}$ are Metzler is difficult since (36) is a nonlinear inequality. Therefore, the stability of the observer (25) can not be easily ensured.

However, it is rare, even impossible, to determine a non singular transformation matrix $P$ to transform the system (6) into a cooperative form such that $P(A_q - L_q C_q) P^{-1}$ $(q \in \overline{1, N})$ are Metzler. As a solution to this problem, a second method is proposed. The main idea consists in redesigning two conventional observers in the original base "$x$". Then, stability conditions will be given in terms of LMIs by applying a common Lyapunov function to the estimation errors. Consider the SLS (6) and two point observers described by

$$\begin{cases} \dot{\hat{x}}^+ = (A_q - L_q C_q)\hat{x}^+ + B_q u + P_q^{-1}\left(P_q^+ \overline{w} + P_q^- \overline{w}\right) \\ \qquad\qquad + L_q y_m + P_q^{-1} |P_q L_q| \overline{V} E_p \\ \dot{\hat{x}}^- = (A_q - L_q C_q)\hat{x}^- + B_q u + P_q^{-1}\left(-P_q^+ \overline{w} - P_q^- \overline{w}\right) \\ \qquad\qquad + L_q y_m - P_q^{-1} |P_q L_q| \overline{V} E_p \end{cases} \quad (37)$$

with $P_q$, $q \in \mathcal{I}$, are chosen as in the following theorem and

$$\begin{cases} \hat{x}_0^+ = Q_q\left(P_q^+ \overline{x}_0 - P_q^- \underline{x}_0\right) \\ \hat{x}_0^- = Q_q\left(P_q^+ \underline{x}_0 - P_q^- \overline{x}_0\right) \end{cases} \quad (38)$$

with

$$Q_q = {P_q}^{-1}$$

The observer (37) is not an interval observer for (6) and its structure is similar to the one proposed in Dinh et al. (2014) for the case of non switched systems. However, the estimates computed by (37) are used in *Theorem 5* to deduce an interval estimation.

*Theorem 5.* Consider matrices $P_q$ $(q \in \mathcal{I})$ such that $F_q = P_q(A_q - L_q C_q) P_q^{-1}$ are Metzler. If the initial condition $x_0$ verifies $\underline{x}_0(t) \leq x_0(t) \leq \overline{x}_0(t)$, then an interval estimation for (6) is given by:

$$\begin{cases} \underline{x} = {Q_q}^+ P_q \hat{x}^- - {Q_q}^- P_q \hat{x}^+ \\ \overline{x} = {Q_q}^+ P_q \hat{x}^+ - {Q_q}^- P_q \hat{x}^- \end{cases} \quad (39)$$

satisfying

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \quad (40)$$

In addition, if there exists a symmetric definite positive matrix $M$ such that

$${A_q}^T M + M A_q - {C_q}^T {W_q}^T - W_q C_q + \sigma_q M \prec 0 \quad (41)$$

where

$$\sigma_q = \frac{3}{\delta_q} \text{ and } W_q = M L_q$$

then (37) is asymptotically stable and $\overline{x}$, $\underline{x}$ are bounded.

**Proof.** Consider the errors $E_q^+ = P_q \hat{x}^+ - P_q x$ and $E_q^- = P_q x - P_q \hat{x}^-$.

Let us show that $\overline{x} - x \geq 0$ and $x - \underline{x} \geq 0$ where $\overline{x}$ and $\underline{x}$ are computed by (39).

From (6) and (37) the dynamics of the errors $E_q^+$ and $E_q^-$ are given by:

$$\begin{aligned}\dot{E}_q^+ = {} & P_q \dot{\hat{x}}^+ - P_q \dot{x} = P_q(A_q - L_q C_q)\hat{x}^+ + |P_q L_q| \overline{V} E_p \\ & - P_q(A_q - L_q C_q) x + \left[\left(P_q^+ \overline{w} + P_q^- \overline{w}\right) - P_q w\right] + P_q L_q v \\ = {} & P_q(A_q - L_q C_q) P_q^{-1}\left(P_q \hat{x}^+ - P_q x\right) + \gamma_q^+ \\ = {} & F_q E_q^+ + \gamma_q^+ \end{aligned} \quad (42)$$

where

$$\gamma_q^+ = \left[\left(P_q^+ \overline{w} + P_q^- \overline{w}\right) - P_q w\right] + P_q L_q v + |P_q L_q| \overline{V} E_p$$

Similarly to $E_q^+$, the dynamics of $E_q^-$ are given by:

$$\dot{E}_q^- = P_q \dot{x} - P_q \dot{\hat{x}}^- = R_q E_q^- + \gamma_q^- \quad (43)$$

where

$$\gamma_q^- = \left[P_q w - \left(-P_q^+ \overline{w} - P_q^- \overline{w}\right)\right] + |P_q L_q| \overline{V} E_p - P_q L_q v$$

According to *Lemma* 1 we have

$$-P_q^+ \overline{w} - P_q^- \overline{w} \leq P_q w \leq P_q^+ \overline{w} + P_q^- \overline{w}$$

Since $P_q(A_q - L_q C_q) P_q^{-1}$ are assumed to be Metzler, and by construction $\gamma_q^+$ and $\gamma_q^-$ are positive for all $t \geq 0$. Then, if $\overline{x}_0$ and $\underline{x}_0$ are chosen such that $E_q^+(0)$ and $E_q^-(0)$ are positive, then the errors $E_q^+(t)$ and $E_q^-(t)$ stay positive $\forall t \geq 0$ such that

$$P_q \hat{x}^- \leq P_q x \leq P_q \hat{x}^+$$

As $Q_q = {P_q}^{-1}$, then $\underline{x} \leq x \leq \overline{x}$ where

$$\begin{cases} \underline{x} = {Q_q}^+ P_q \hat{x}^- - {Q_q}^- P_q \hat{x}^+ \\ \overline{x} = {Q_q}^+ P_q \hat{x}^+ - {Q_q}^- P_q \hat{x}^- \end{cases}$$

For the stability analysis, Let us now show that $E_q^+$ and $E_q^-$ are asymptotically stable or simply show that $(\hat{x}^+ - x)$ and $(x - \hat{x}^-)$ are asymptotically stable.

Let $e^+ = (\hat{x}^+ - x)$ and $e^- = (x - \hat{x}^-)$ be the observation errors and consider the following Lyapunov function:

$$V\left(e^+\right) = {e^+}^T M e^+ \quad (44)$$

where $M$ is a symmetric positive definite matrix. As in proof of *Theorem* 4, the derivative of the Lyapunov function can be given as follows:

$$\begin{aligned}\dot{V}\left(e^+\right) = {} & \dot{e}^{+T} M e^+ + {e^+}^T M \dot{e}^+ \\ = {} & {e^+}^T\left[(A_q - L_q C_q)^T M + M(A_q - L_q C_q)\right] e^+ \\ & + 2{e^+}^T M {P_q}^{-1}\left(P_q^+ \overline{w} + P_q^- \overline{w}\right) - 2{e^+}^T M w \\ & + 2{e^+}^T M L_q v + 2{e^+}^T M |L_q| \overline{V} E_p \end{aligned} \quad (45)$$

According to *Lemma* 3, we have

$$\dot{V}\left(e^+\right) \leq {e^+}^T \mathcal{B}_3 e^+ + \mathcal{C}_5$$

where

$$\mathcal{B}_3 = (A_q - L_q C_q)^T M + M(A_q - L_q C_q) + \frac{3}{\delta_q} M$$
$$= {A_q}^T M + M A_q - {C_q}^T {W_q}^T - W_q C_q + \frac{3}{\delta_q} M \quad (46)$$

$$\mathcal{C}_5 = \overline{w}^T \left[\delta_q {P_q^+}^T {P_q^{-1}}^T M P_q^{-1} P_q^+\right] \overline{w} + v^T \left[\delta_q {L_q}^T M L_q\right] v$$
$$+ \overline{w}^T \left[\delta_q {P_q^-}^T {P_q^{-1}}^T M P_q^{-1} P_q^-\right] \overline{w} - w^T [\delta_q M] w$$
$$+ {E_p}^T \left[\delta_q \overline{V} |L_q|^T M |L_q| \overline{V}\right] E_p \quad (47)$$

The noise $v$ and disturbance $w$ are bounded, it follows that $\mathcal{C}_5$ is bounded. Therefore, if $\mathcal{B}_3 \prec 0$, the observation error $e^+$ is bounded. The same arguments allow one to show that the observation error $e^-$ is also bounded. In addition, since $P_q$ and $Q_q$ are bounded for all $(q \in \overline{1,N})$ then $E_q^+$ and $E_q^-$ are bounded.

■

In the next section, the performance of the suggested method is shown through a numerical example.

## 4. NUMERICAL EXAMPLE

Consider the linear switched system subject to disturbances described by:

$$\begin{cases} \dot{x}(t) = A_q x(t) + B_q u(t) + w(t) \\ y(t) = C_q x(t) + v(t) \end{cases}, \forall q \in \overline{1,3} \quad (48)$$

where

$$\mathrm{A}_1 = \begin{bmatrix} -1.5 & 0.262 \\ 0 & -1 \end{bmatrix},\ \mathrm{A}_2 = \begin{bmatrix} -0.5 & 2 \\ 0 & -1 \end{bmatrix}$$
$$\mathrm{A}_3 = \begin{bmatrix} -0.6 & 1.5 \\ 0 & -1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C_1 = [1\ 0],\ C_2 = [1\ 1],\ C_3 = [1\ 1.5]$$

$w(t)$ and $v(t)$ are uniformly distributed bounded signals such that $-\overline{w} \le w(t) \le \overline{w}$ with $\overline{w} = [0.03\ 0.03]^T$ and $-\overline{V} E_p \le v(t) \le \overline{V} E_p$ with $\overline{V} = 0.3$.

To verify the cooperativity property, a transformation of coordinates must be used such that $P(A_q - L_q C_q) P^{-1}$ are Metzler. However, it is not possible to compute this common transformation matrix $P$. Hence, non singular transformation matrices $P_q$ are calculated. Consequently, a conventional observer (37) is constructed for the system (48).

Now, using the Matlab LMI toolbox, one can solve the LMI defined by (41). One feasible solution is given by:

$$L_1 = \begin{bmatrix} -0.6555 \\ -0.1011 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.9397 \\ -0.2497 \end{bmatrix},\ L_3 = \begin{bmatrix} 0.5734 \\ -0.2512 \end{bmatrix}$$
$$M = \begin{bmatrix} 81.6804 & 21.7133 \\ 21.7133 & 55.5904 \end{bmatrix},\ \delta_q = 84.5569,\ \forall q \in \overline{1,3}$$

Note that $P_q$ are computed to verify that $P_q (A_q - L_q C_q) {P_q}^{-1}$ are Metzler for all $q \in \overline{1,3}$ and given by:

$$P_1 = \begin{bmatrix} 0.2845 & -0.7262 \\ 0.4770 & 0.4841 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.7776 & -0.8577 \\ 0.1483 & 0.5718 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.7829 & -0.6580 \\ 0.2171 & 0.6580 \end{bmatrix}$$

All conditions of *Theorem 5* are satisfied; it follows that the system (37) is asymptotically stable verifying

$$\underline{x}(t) \le x(t) \le \overline{x}(t)\ \ \forall t \ge 0,\ \forall q \in \overline{1,3}.$$

with

$$\underline{x} = {Q_q}^+ P_q \hat{x}^- - {Q_q}^- P_q \hat{x}^+$$

and

$$\overline{x} = {Q_q}^+ P_q \hat{x}^+ - {Q_q}^- P_q \hat{x}^-$$

where $Q_q = P_q^{-1}$ are given by:

$$Q_1 = \begin{bmatrix} 1.0000 & 1.5000 \\ -0.9853 & 0.5876 \end{bmatrix},\ Q_2 = \begin{bmatrix} 1.0000 & 1.5000 \\ -0.2593 & 1.3599 \end{bmatrix}$$
$$Q_3 = \begin{bmatrix} 1.0000 & 1.0000 \\ -0.3299 & 1.1899 \end{bmatrix}$$

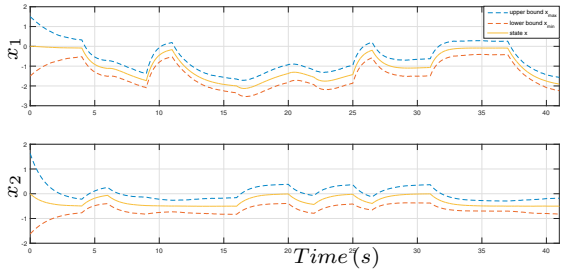
The results of simulation of the obtained observer are depicted in Fig. 1(a) for both coordinates where solid lines present the state and dashed lines present the estimated bounds.

The switching between the three subsystems is governed by the switching signal plotted in Fig. 1(b).

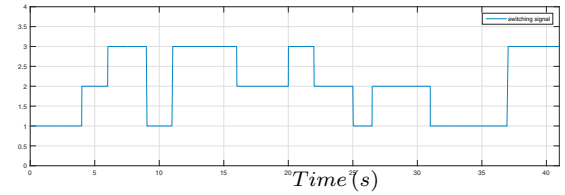
The initial state $x_0$ is assumed to be bounded such that

$$\underline{x}_0 \le x_0 \le \overline{x}_0$$

where $\overline{x}_0 = [1.5\ 1.5]^T$, $\underline{x}_0 = [-1.5\ -1.5]^T$




(a) State and estimated bounds




(b) Switching signal

Fig. 1. Interval state estimation for the switched system with disturbances.

The results show that, despite the disturbances, the state is always inside the upper and the lower trajectories. The interval observer has exhibited approved stability properties. The inclusion

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad \text{for all } t \ge t_0$$

is always verified.

As shown in Fig. 1(a), the interval is quite large at the beginning, although its width decreases despite the uncertainties on the measurements. Finally, the interval observer remains stable despite the switching instants.

## 5. CONCLUSION

This paper investigates state estimation for switched linear systems subject to disturbances. An interval observer is designed under some transformations where two conventional observers are reformulated in the base "$x$". The stability and the cooperativity conditions are represented in terms of LMIs. Effectiveness of the proposed methodology is illustrated through a numerical example. In this work, the switching instants are assumed to be known nonetheless it is not the case for the most of switched systems. This concern will be the subject of further contributions.

## REFERENCES


Fayssal Arichi, Mohamed Djemai, Brahim Cherki, and Noureddine Manamanni. Continuous and discrete state estimation for a class of nonlinear switched systems. *IEEE Trans. On Circuits and Systems II*, 62(7):691–695, 2015.

Abderazik Birouche, Jamal Daafouz, and Claude Iung. Observer design for a class of discrete time piecewise-linear systems. *IFAC Proceedings Volumes*, 39(5):12–17, 2006.

Michael S Branicky. Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on automatic control*, 43(4):475–482, 1998.

Stanislav Chebotarev, Denis Efimov, Tarek Raïssi, and Ali Zolghadri. Interval observers for continuous-time lpv systems with l1/l2 performance. *Automatica*, 58:82–89, 2015.

T. N. Dinh, F. Mazenc, and S. I. Niculescu. Interval observer composed of observers for nonlinear systems. In *European Control Conference*, pages 660–665, 2014.

Mohamed Djemai and Michael Defoort. *Hybrid Dynamical Systems*. Springer, 2015.

Jean-Luc Gouzé, A Rapaport, and Mohamed Zakaria Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological modelling*, 133(1):45–56, 2000.

Zhongwei He and Wei Xie. Interval state observer for nonlinear switched systems with average dwell time. In *34th IEEE Chinese Control Conference (CCC)*, pages 2285–2288, 2015.

Zhongwei He and Wei Xie. Control of non-linear switched systems with average dwell time: interval observer-based framework. *IET Control Theory & Applications*, 10(1):10–16, 2016.

Laurentiu Hetel. *Robust stability and control of switched linear systems*. PhD thesis, Nancy University, 2007.

Abdelfettah Hocine, Didier Maquin, and José Ragot. Finite memory observer for switching systems: Application to diagnosis. *IFAC Proceedings Volumes*, 38(1):406–411, 2005.

Bo Hu, Guisheng Zhai, and Anthony N Michel. Common quadratic lyapunov-like functions with associated switching regions for two unstable second-order lti systems. *International Journal of Control*, 75(14):1127–1135, 2002.

Bin Jiang, Jian Liang Wang, and Yeng Chai Soh. An adaptive technique for robust diagnosis of faults with independent effects on system outputs. *International Journal of Control*, 75(11):792–802, 2002.

Rihab Lamouchi, Messaoud Amairi, Tarek Raïssi, and Mohamed Aoun. Interval observer design for linear parameter-varying systems subject to component faults. In *24th IEEE Mediterranean Conference on Control and Automation (MED)*, pages 707–712, 2016.

Daniel Liberzon. *Switching in systems and control*. Springer Science & Business Media, 2012.

Daniel Liberzon and A Stephen Morse. Basic problems in stability and design of switched systems. *IEEE Control systems*, 19(5):59–70, 1999.

Frédéric Mazenc and Olivier Bernard. Asymptotically stable interval observers for planar systems with complex poles. *IEEE Transactions on Automatic Control*, 55(2):523–527, 2010.

Frédéric Mazenc and Olivier Bernard. Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1):140–147, 2011.

Tarek Raïssi, Gaétan Videau, and Ali Zolghadri. Interval observer design for consistency checks of nonlinear continuous-time systems. *Automatica*, 46(3):518–527, 2010.

Tarek Raïssi, Denis Efimov, and Ali Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):260–265, 2012.

Ulysse Serres, Jean-Claude Vivalda, and Pierre Riedinger. On the convergence of linear switched systems. *IEEE Transactions on Automatic Control*, 56(2):320–332, 2011.

Yang Tian, Thierry Floquet, Lotfi Belkoura, and Wilfrid Perruquetti. Switching time estimation for linear switched systems: an algebraic approach. In *Proceedings of the 48th IEEE Conference on Decision and Control, held jointly with the 28th Chinese Control Conference. CDC/CCC 2009.*, pages 3909–3913, 2009.

Gaétan Videau, Tarek Raïssi, and Ali Zolghadri. Guaranteed state estimation for nonlinear continuous-time systems based on qlpv approximations. In *IEEE European Control Conference (ECC), 2009*, pages 904–909.

Basma Yousfi, Tarek Raïssi, Messaoud Amairi, and Mohamed Aoun. Robust state estimation for singularly perturbed systems. *International Journal of Control*, pages 1–14, 2016a.

Basma Yousfi, Tarek Raïssi, Messaoud Amairi, and Mohamed Aoun. Set-membership methodology for model-based prognosis. *ISA Transactions*, 2016b.