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Interval observer design for unknown input estimation of linear time-invariant discrete-time systems

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Abstract: In this paper, the problem of joint state and unknown input estimation for linear time-invariant (LTI) discrete-time systems using interval observer is addressed. This problem has already been studied in the context of continuous-time systems. To the best of our knowledge, unknown input interval-based estimation for discrete-time systems has not been considered in the litterature. Assuming that the measurement noise and disturbances are bounded, lower and upper bounds are first computed for the unmeasured state and then for the unknown inputs. The results obtained with a numerical example highlight the efficiency of the method.

Keywords: Interval estimation, discrete-time systems, unknown input observer, state transformation, cooperative systems.

1. INTRODUCTION

Consider the following LTI discrete-time system:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Dd(k) + \omega(k) \\ y(k) &= Cx(k) + \delta(k) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, the input and the measurement vectors; $d \in \mathbb{R}^q$ is the unknown input vector which does not affect the outputs. A, B, C and D are contant matrices of appropriate dimensions. Finally, $\omega \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^p$ are the state and measurement noises which are assumed to be bounded with a priori known bounds $|\omega| \leq \overline{\omega}$ and $|\delta| \leq \overline{\delta}$ where $\overline{\omega} \in \mathbb{R}^n$ and $\overline{\delta} \in \mathbb{R}^p$ are constant component-wise positive vectors and $|\cdot|$ is the component-wise absolute value for vectors. Moreover, it is assumed that $n \geq q$ and $p \geq q$.

Systems can be subject to disturbances that affect the inputs and/or the outputs, and when these disturbances cannot be measured, they are referred to as unknown inputs. In order to solve the problem of state and unknown input estimation, unknown input observers (UIO) have been developed since 1970's (Meditch and Hostetter (1974); Hostetter (1973); Wang et al. (1975)), and are mostly used in the field of fault detection (Frank and Ding (1997); Chen et al. (1996)). However, in presence of measurement noise or uncertain parameters, classical Luenberger-based observers face some limitations. This is why interval observers have been proposed, to cope with uncertainties by evaluating the set of admissible values of the state at each time instant and computing the lower and upper bounds.

The problem of state estimation without unknown inputs using interval observers has been widely studied. In Gouzé et al. (2000), it was shown that in the particular case of asymptotically stable and cooperative systems (*i.e* systems where the Jacobian matrix of the state vector field has non-negative off-diagonal elements), interval observers can be designed directly. This assumption of cooperativity is the main limitation of interval observers as most of the systems are not cooperative. However, in the case of LTI systems this hypothesis can be relaxed by using a time-varying change of coordinates (Mazenc and Bernard, 2011) or by time-invariant ones (Raïssi et al., 2012).

In the set-membership framework, joint state and unknown input estimation has been considered but only for continuous-time systems (Gucik-Derigny et al., 2016). Standard discrete-time UIOs have already been used for state estimation in the presence of unknown inputs (Valcher (1999); Darouach (2004)), and also for unknown input estimation (Maquin et al., 1994). Based on these works, the goal of this paper is to establish a discrete-time interval observer to jointly estimate the state and the unknown input of systems described by (1).

The idea is to use the results from Maquin et al. (1994) to obtain an unknown input-free subsystem. A second state transformation using a time-invariant change of coordinates is performed in order to ensure the cooperativity property of the observation error in the new coordinates. Then, an interval observer is designed in the new coordinates and allows to deduce lower and upper bounds for the state in the original basis. Finally, the bounds of the unknown input are computed.

The paper is organized as follows. In Section 2, some useful properties and notations for the comprehension of the paper are given. Section 3 introduces the problem statement and the assumptions required in this paper. In Section 4, the methodology used to compute the state and

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the unknown input bounds is explained. Finally, Section 5 presents numerical results to illustrate the efficiency of the proposed technique.

2. PRELIMINARIES

2.1 Notations and definitions

- Given a matrix $M \in \mathbb{R}^{n\times m}$, define $M^+ = \max(0, M)$, $M^- = M - M^+$, $M^* = [M^+ \quad M^-]$ and $\overline{\overline{M}} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$.
- For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$, the operators $\leq, <, >$ are understood componentwise.
- For $\underline{x}, \overline{x} \in \mathbb{R}^n$ with $\underline{x} \leq \overline{x}$, define $\overline{X}^T = [\overline{x}\ \underline{x}]$ and $\underline{X}^T = [\underline{x}\ \overline{x}]$.
- A matrix $M \in \mathbb{R}^{n\times n}$ is called Schur stable if its spectral radius is less than one.
- A matrix $M \in \mathbb{R}^{n\times n}$ is called nonnegative if all its elements are nonnegative.

Lemma 1. (Hirsch and Smith, 2005); Consider the linear system

$$x(k+1) = Ax(k) + \omega(k) \tag{2}$$

where $\omega \in \mathbb{R}^n_+$ and A is a nonnegative matrix. Then, $\forall k > 0$, $x(k) \geq 0$ provided that $x(0) \geq 0$. Such dynamical systems are called cooperative.

The following lemma was stated and proven in the continuous-time case in Gouzé et al. (2000) and is derived here in the discrete-time context.

Lemma 2. Consider the system described by (2) and suppose that the following assumptions are fulfilled:

- The matrix A is Schur stable and nonnegative;
- ω is bounded by a fixed positive vector $\overline{\Omega}$;
- $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.

Then, the state vector x is asymptotically lower than the positive vector

$$\rho_x = (I - A)^{-1}\overline{\Omega} \tag{3}$$

Proof. Starting from (2), it is shown that

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^i \omega(k-1-i) \tag{4}$$

Then, as $\omega \leq \overline{\Omega}$, we can deduce that $\forall k \in \mathbb{Z}_+$:

$$x(k) \leq A^k x(0) + \sum_{i=0}^{k-1} A^i \overline{\Omega} \tag{5}$$

A is Schur stable, therefore the term $A^k x(0)$ converges to 0. Moreover, using classical results on the convergence of geometric series, the fact that A is Schur stable allows to deduce the convergence of $\sum_{i=0}^{k-1} A^i$ to

$$\sum_{i=0}^{\infty} A^i = (I - A)^{-1} \tag{6}$$

Finally, we can conclude that $\forall k \in \mathbb{Z}_+$:

$$x(k) \leq (I - A)^{-1}\overline{\Omega}. \tag{7}$$

2.2 Interval observers and cooperative discrete-time linear systems

Consider the following system without unknown inputs:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{8}$$

The aim of an interval observer is to compute two trajectories $\underline{x}(k)$ and $\overline{x}(k)$ such that $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$ for all $k \in \mathbb{Z}_+$ with an initial condition verifying $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. The upper and lower bounds could be obtained with Luenberger-based observers defined by:

$$\begin{cases} \underline{x}(k+1) = (A - LC)\underline{x}(k) + Bu(k) + Ly(k) \\ \overline{x}(k+1) = (A - LC)\overline{x}(k) + Bu(k) + Ly(k) \end{cases} \tag{9}$$

The dynamics of the errors $\underline{e}(k) = x(k) - \underline{x}(k)$ and $\overline{e}(k) = \overline{x}(k) - x(k)$ are given by:

$$\begin{cases} \underline{e}(k+1) = (A - LC)\underline{e}(k) \\ \overline{e}(k+1) = (A - LC)\overline{e}(k) \end{cases} \tag{10}$$

Based on Lemma 1, the observation errors (10) are always positive and bounded if and only if the matrix $(A-LC)$ is Schur stable and nonnegative (Smith (1995);Efimov et al. (2013b)).

In the following sections, interval observers whose structures are similar to (9) will be designed to compute lower and upper bounds for the state vector even in the presence of noise and disturbances. It will also be shown that the assumption on the non-negativity of $(A - LC)$ can be relaxed by using some changes of coordinates. Finally, an original approach will be presented to estimate the bounds of the unknown inputs.

3. PROBLEM STATEMENT

The methodology proposed to jointly estimate the state and the unknown inputs of a LTI discrete-time system is splitted in two steps. First, two bounds $\underline{x}_k$,$\overline{x}_k \in \mathbb{R}^n$ for the state are estimated. Then, a technique to build the lower and upper bounds $\underline{d}_k$,$\overline{d}_k \in \mathbb{R}^q$ for the unknown input is described.

The unknown input interval observer developed in this paper is based on the UIO proposed in Maquin et al. (1994) whose methodology is extended to systems with bounded disturbances and noise. Following a change of coordinates, the state is divided into two subsytems, one affected by the unknown input and the second one is unknown input-free. This allows to design an interval observer in the new coordinate basis to estimate the upper and lower bounds of the state. Then, by returning into the initial coordinates, the upper and lower bounds for the unknown input can be computed. First of all, the following assumption is required.

Assumption 1. C is a full row rank matrix and D is a full column rank matrix.

Under Assumption 1, there exists an orthogonal matrix $H \in \mathbb{R}^{n\times n}$ and matrices $R_0 \in \mathbb{R}^{q\times q}$ and $K \in \mathbb{R}^{q\times q}$ such that:

$$D = H \begin{bmatrix} R_0 \\ 0 \end{bmatrix} K^T \tag{11}$$

This leads to the transformation of the system (1) into an equivalent one:

$$\begin{cases} z(k+1) &= \tilde{A}z(k) + \tilde{B}u(k) + \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tilde{d}(k) + \tilde{\omega}(k) \\ y(k) &= \tilde{C}z(k) + \delta(k) \end{cases} \quad (12)$$

where:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \tilde{A} = H^T A H = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

$$\tilde{B} = H^T B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{C} = CH = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$

$$z(k) = H^T x(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \quad \tilde{d}(k) = K^T d(k)$$

$$\tilde{\omega}(k) = H^T \omega = \begin{bmatrix} \tilde{\omega}_1(k) \\ \tilde{\omega}_2(k) \end{bmatrix}$$

H^T is supposed to be bounded, therefore $|\tilde{\omega}| \leq \overline{\tilde{\omega}}$ where $\overline{\tilde{\omega}}$ is a constant positive vector. The system (12) is decomposed into an unknown input depending subsystem and an unknown input-free subsytem described by:

$$\begin{cases} z_1(k+1) &= \tilde{A}_{11}z_1(k) + \tilde{A}_{12}z_2(k) + \tilde{B}_1 u(k) + R_0\tilde{d}(k) \\ & \quad +\tilde{\omega}_1(k) \\ z_2(k+1) &= \tilde{A}_{21}z_1(k) + \tilde{A}_{22}z_2(k) + \tilde{B}_2 u(k) + \tilde{\omega}_2(k) \\ y(k) &= \tilde{C}_1 z_1(k) + \tilde{C}_2 z_2(k) + \delta(k) \end{cases} \quad (13)$$

where $\tilde{C}_1 \in \mathbb{R}^{p\times q}$ and $\tilde{C}_2 \in \mathbb{R}^{p\times(n-q)}$.

$\tilde{C}_1$ is supposed to be a full column rank matrix (Hou and Muller, 1992) and can be decomposed as:

$$\tilde{C}_1 = H_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix} K_1^T \quad (14)$$

with $H_1 = [H_{011}\ H_{012}]$ ($H_{011} \in \mathbb{R}^{p\times q}$ and $H_{012} \in \mathbb{R}^{p\times(p-q)}$) and $\tilde{y}(k) = H_1^T y(k)$; the measurements equation can be decomposed as

$$\begin{cases} \tilde{y}_1(k) &= R_1 K_1^T z_1(k) + H_{011}^T \tilde{C}_2 z_2(k) + H_{011}^T \delta(k) \\ \tilde{y}_2(k) &= H_{012}^T \tilde{C}_2 z_2(k) + H_{012}^T \delta(k) = C_2 z_2(k) + H_{012}^T \delta(k) \end{cases} \quad (15)$$

As $\tilde{y}_1(k) = G_s^T \tilde{y}(k)$ with $G_s^T = \begin{bmatrix} I_q & O_{q\times(p-q)} \end{bmatrix}$, the expression of z_1 is extracted from (15):

$$z_1(k) = E(y(k) - \tilde{C}_2 z_2(k) - \delta(k)) \quad (16)$$

with $E = K_1 R_1^{-1} G_s^T H_1^T$.

By replacing this expression of $z_1(k)$ in the second equation of (13) we obtain:

$$\begin{cases} z_2(k+1) &= \tilde{A}_{21}E[y(k) - \tilde{C}_2 z_2(k) - \delta(k)] + \tilde{A}_{22}z_2(k) \\ & \quad +\tilde{B}_2 u(k) + \tilde{\omega}_2(k) \end{cases} \quad (17)$$

Finally we have the following dynamical system:

$$\begin{cases} z_2(k+1) &= A_2 z_2(k) + B_2 u(k) + D_2 y(k) - D_2\delta(k) \\ & \quad +\tilde{\omega}_2(k) \\ \tilde{y}_2(k) &= C_2 z_{2(k)} + H_{012}^T \delta(k) \end{cases} \quad (18)$$

where $A_2 = \tilde{A}_{22} - \tilde{A}_{21}E\tilde{C}_2$, $B_2 = \tilde{B}_2$, $C_2 = H_{012}^T \tilde{C}_2$ and $D_2 = \tilde{A}_{21}E$.

In order to be able to design an interval observer for the discrete-time system (18), the following assumption which is standard in the field of observer design is required (Hou and Muller, 1992).

Assumption 2. The pair (A_2, C_2) is detectable.

Based on Assumption 2 the following lemma allows to transform (18) into a suitable form for interval observer design (Efimov et al., 2013a).

Lemma 3. There exists a gain $L \in \mathbb{R}^{(n-q)\times(p-q)}$ and a transformation matrix P of appropriate dimensions such that $(A_2 - LC_2)$ is Schur stable and $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

Such a transformation always exists, and in the case where the eigenvalues of $(A_2 - LC_2)$ are real, R can be chosen as diagonal or as Jordan form of $A_2 - LC_2$ (Efimov et al., 2013a). After the change of coordinates $r_2 = Pz_2$, the system (18) is described in the new coordinates by:

$$\begin{cases} r_2(k+1) &= Rr_2(k) + PB_2 u(k) + My(k) - M\delta(k) \\ & \quad +P\tilde{\omega}_2(k) \\ \tilde{y}_2(k) &= C_2 P^{-1} r_{2(k)} + H_{012}^T \delta(k) \end{cases} \quad (19)$$

where $M = P(D_2 + LH_{012}^T)$.

4. MAIN RESULTS

In this section, an interval observer is designed for state estimation and then for unknown input estimation.

4.1 State estimation

The state estimation is first performed in the coordinates r_2. In the sequel, we define $\overline{\Delta}^T = \begin{bmatrix} \overline{\delta} & -\overline{\delta} \end{bmatrix}$, $\underline{\Delta}^T = \begin{bmatrix} -\overline{\delta} & \overline{\delta} \end{bmatrix}$ and $\overline{\Omega}^T = [\overline{\omega}\ -\overline{\omega}]$, $\underline{\Omega}^T = [-\overline{\omega}\ \overline{\omega}]$.

The following theorem allows to carry out an interval state estimation in the coordinates r_2.

Theorem 1. Assume that $\underline{r}_2(0) \leq r_2(0) \leq \overline{r}_2(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{r}_2(k)$ and $\overline{r}_2(k)$ given by

$$\begin{cases} \overline{r}_2(k+1) &= R\overline{r}_2(k) + PB_2 u(k) + My(k) + (-M)^* \overline{\Delta} \\ & \quad +P^* \tilde{\overline{\Omega}}_2 \\ \underline{r}_2(k+1) &= R\underline{r}_2(k) + PB_2 u(k) + My(k) + (-M)^* \underline{\Delta} \\ & \quad +P^* \tilde{\underline{\Omega}}_2 \end{cases} \quad (20)$$

are bounded and verify

$$\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k) \quad (21)$$

In addition, if the gain L is chosen such that $(A_2 - LC_2)$ is Schur stable, then $\overline{r}_2$ and $\underline{r}_2$ are bounded.

Proof. There are two results to prove:

(1) $\forall k \in \mathbb{Z}_+$, $\underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k)$ *i.e* the upper and lower observation errors are positive.
(2) Stability of the interval observer.

Step 1: Positivity of the observation errors.

The upper and lower observation errors are defined as

$$\begin{cases} \overline{e}_{r_2}(k) = \overline{r}_2(k) - r_2(k) \\ \underline{e}_{r_2}(k) = r_2(k) - \underline{r}_2(k) \end{cases}$$

The dynamics of the errors are given by

$$\begin{cases}\overline{e}_{r_2}(k+1) = R\overline{e}_{r_2}(k) + V_1(k)\\ \underline{e}_{r_2}(k+1) = R\underline{e}_{r_2}(k) + V_2(k)\end{cases}$$

where

$$\begin{aligned}V_1(k) &= (-M)^+(\overline{\delta} - \delta(k)) - (-M)^-(\overline{\delta} + \delta(k))\\ &\quad + P^+(\overline{\tilde{\omega}}_2 - \tilde{\omega}_2(k)) - P^-(\overline{\tilde{\omega}}_2 + \tilde{\omega}_2(k))\end{aligned}$$

and

$$\begin{aligned}V_2(k) &= (-M)^+(\overline{\delta} + \delta(k)) - (-M)^-(\overline{\delta} - \delta(k))\\ &\quad + P^+(\overline{\tilde{\omega}}_2 + \tilde{\omega}_2(k)) - P^-(\overline{\tilde{\omega}}_2 - \tilde{\omega}_2(k))\end{aligned}$$

According to Lemma 3, the matrix $R = P(A_2 - LC_2)P^{-1}$ is nonnegative.

For a given matrix $M \in \mathbb{R}^{n\times m}$ we have defined in Section 2 that $M = M^+ + M^-$, therefore M^+ and $-M^-$ are nonnegative matrices. Moreover, $|\delta| \leq \overline{\delta}$ and $|\tilde{\omega}_2| \leq \overline{\tilde{\omega}}_2$. Therefore, as V_1 and V_2 are the sums of positive terms, we can deduce that $\forall k \in \mathbb{Z}_+,\ V_1(k) \geq 0$ and $V_2(k) \geq 0$.

Adding to that, as $\underline{r}_2(0) \leq r_2(0) \leq \overline{r}_2(0)$, we have $\overline{e}_{r_2}(0) \geq 0$ and $\underline{e}_{r_2}(0) \geq 0$. Therefore Lemma 1 allows to conclude that the upper and lower observation errors are always positive.

Step 2: Stability and convergence of the interval observer.

Defining $E_{r_2}^T = \begin{bmatrix}\overline{e}_{r_2} & \underline{e}_{r_2}\end{bmatrix}$ leads to the following dynamics:

$$E_{r_2}(k+1) = \overline{\overline{R}}E_{r_2}(k) + V(k) \tag{22}$$

where R is defined in Section 3 as $R = P(A_2 - LC_2)P^{-1}$ and $V^T = [V_1\ V_2]$.

As $|\delta| \leq \overline{\delta}$ and $|\tilde{\omega}_2| \leq \overline{\tilde{\omega}}_2$, it is straightforward to prove that $V_1 \leq \overline{V}$ and $V_2 \leq \overline{V}$ with:

$$\overline{V} = 2((-M)^+ - (-M)^-)\overline{\delta} + 2(P^+ - P^-)\overline{\tilde{\omega}}_2$$

Moreover, Lemma 3 states that $\overline{\overline{R}}$ is Schur stable and nonnegative, therefore we can ensure the stability of the observation error dynamics (22).

Finally, with Lemma 2, we can deduce that E_{r_2} is asymptotically lower than the non-negative vector

$$\rho_{r_2} = (I - \overline{\overline{R}})^{-1}\overline{V}. \tag{23}$$

Furthermore, since $r_2 = Pz_2$, the bounds of $z_2(k)$ are given by the following corollary.

Corollary 1. Under the conditions of Theorem 1, we have $\underline{z}_2(k) \leq z_2(k) \leq \overline{z}_2(k)$ with

$$\begin{cases}\overline{z}_2(k) = (P^{-1})^+\overline{r}_2(k) + (P^{-1})^-\underline{r}_2(k)\\ \underline{z}_2(k) = (P^{-1})^+\underline{r}_2(k) + (P^{-1})^-\overline{r}_2(k)\end{cases} \tag{24}$$

Proof. We have $P^{-1}z_2(k) = ((P^{-1})^+ + (P^{-1})^-)z_2(k)$. If $\forall k \in \mathbb{Z}_+,\ \underline{r}_2(k) \leq r_2(k) \leq \overline{r}_2(k)$, then

$$\begin{aligned}((P^{-1})^+\underline{r}_2(k) + (P^{-1})^-\overline{r}_2(k) &\leq (P^{-1})r_2(k)\\ &\leq (P^{-1})^+\overline{r}_2(k) + (P^{-1})^-\underline{r}_2(k)\end{aligned}$$

which is equivalent to $\underline{z}_2(k) \leq z_2(k) \leq \overline{z}_2(k)$. Finally, as $\underline{r}_2(k)$ and $\overline{r}_2(k)$ are bounded, $\underline{z}_2(k)$ and $\overline{z}_2(k)$ are bounded as well.

The last step consists in computing the bounds for the whole state in the original coordinates $\underline{x}^T = [\underline{x}_1 \& \underline{x}_2]$ and $\overline{x}^T = [\overline{x}_1\ \overline{x}_2]$. Based on Theorem 1 and Corollary 1, the following theorem ensures the interval estimation of the state x.

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{x}(k)$ and $\overline{x}(k)$ given by

$$\begin{cases}\overline{x}_1(k) &= H_{11}Ey + (H_{12})^*\overline{Z}_2(k) + (-E_1)^*\overline{Z}_2(k)\\ &\quad +(-H_{11}E)^*\overline{\Delta}\\ \underline{x}_1(k) &= H_{11}Ey + (H_{12})^*\underline{Z}_2(k) + (-E_1)^*\underline{Z}_2(k)\\ &\quad +(-H_{11}E)^*\underline{\Delta}\\ \overline{x}_2(k) &= H_{21}Ey + (H_{22})^*\overline{Z}_2(k) + (-E_2)^*\overline{Z}_2(k)\\ &\quad +(-H_{21}E)^*\overline{\Delta}\\ \underline{x}_2(k) &= H_{21}Ey + (H_{22})^*\underline{Z}_2(k) + (-E_2)^*\underline{Z}_2(k)\\ &\quad +(-H_{21}E)^*\underline{\Delta}\end{cases} \tag{25}$$

are bounded and verify

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k) \tag{26}$$

with $E_1 = H_{11}E\tilde{C}_2$ and $E_2 = H_{21}E\tilde{C}_2$.

Proof. We have $x = Hz$:

$$\begin{aligned}\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}E(y(k) - \tilde{C}_2z_2(k) - \delta(k))\\ z_2(k)\end{bmatrix}\\ &= \begin{bmatrix}H_{11}Ey(k) + H_{12}z_2(k) - E_1z_2(k) - H_{11}E\delta(k)\\ H_{21}Ey(k) + H_{22}z_2(k) - E_2z_2(k) - H_{21}E\delta(k)\end{bmatrix}\end{aligned}$$

The observation errors relative to the state x are given by:

$$\begin{cases}\overline{e}_{x_1}(k) = \overline{x}_1(k) - x_1(k)\\ \underline{e}_{x_1}(k) = x_1(k) - \underline{x}_1(k)\\ \overline{e}_{x_2}(k) = \overline{x}_2(k) - x_2(k)\\ \underline{e}_{x_2}(k) = x_2(k) - \underline{x}_2(k)\end{cases}$$

and in a developed form we get:

$$\begin{cases}\overline{e}_{x_1}(k) &= (H_{12}^+ + (-E_1)^+)\overline{e}_{z_2}(k)\\ &\quad -(H_{12}^- + (-E_1)^-)\underline{e}_{z_2}(k)\\ &\quad +(-H_{11}E)^+(\overline{\delta} - \delta(k)) - (-H_{11}E)^-(\overline{\delta} + \delta(k))\\ \underline{e}_{x_1}(k) &= (H_{12}^+ + (-E_1)^+)\underline{e}_{z_2}(k)\\ &\quad -(H_{12}^- + (-E_1)^-)\overline{e}_{z_2}(k)\\ &\quad +(-H_{11}E)^+(\overline{\delta} + \delta(k)) - (-H_{11}E)^-(\overline{\delta} - \delta(k))\\ \overline{e}_{x_2}(k) &= (H_{22}^+ + (-E_2)^+)\overline{e}_{z_2}(k)\\ &\quad -(H_{22}^- + (-E_2)^-)\underline{e}_{z_2}(k)\\ &\quad +(-H_{21}E)^+(\overline{\delta} - \delta(k)) - (-H_{21}E)^-(\overline{\delta} + \delta(k))\\ \underline{e}_{x_2}(k) &= (H_{22}^+ + (-E_2)^+)\underline{e}_{z_2}(k)\\ &\quad -(H_{22}^- + (-E_2)^-)\overline{e}_{z_2}(k)\\ &\quad +(-H_{21}E)^+(\overline{\delta} + \delta(k)) - (-H_{21}E)^-(\overline{\delta} - \delta(k))\end{cases} \tag{27}$$

With the same reasoning as in the proof of Theorem 1, it can be deduced from (27) that the state observation errors are positive. Therefore, $\underline{x}(k) \leq x(k) \leq \overline{x}(k)\ ,\ \forall k \geq k_0$.

By defining the compact error E_x as

$$E_x = \begin{bmatrix}\overline{e}_{x_1}^T & \underline{e}_{x_1}^T & \overline{e}_{x_2}^T & \underline{e}_{x_2}^T\end{bmatrix}^T$$

and the matrices

$$F = \begin{bmatrix}H_{12}^+ + (-E_1)^+ & H_{12}^- + (-E_1)^-\\ H_{12}^- + (-E_1)^- & H_{12}^+ + (-E_1)^+\\ H_{22}^+ + (-E_2)^+ & H_{22}^- + (-E_2)^-\\ H_{22}^- + (-E_2)^- & H_{22}^+ + (-E_2)^+\end{bmatrix}$$

$$J = \begin{bmatrix}(-H_{11}E)^+ - (-H_{11}E)^-\\ -(-H_{11}E)^- + (-H_{11}E)^+\\ (-H_{21}E)^+ - (-H_{21}E)^-\\ -(-H_{21}E)^- + (-H_{21}E)^+\end{bmatrix}$$

it can be deduced that

$$E_x \leq FE_{z_2} + 2J\overline{\delta}.$$

Moreover, from Corollary 1, we have $E_{z_2} = \mathcal{R}E_{r_2}$ with $\mathcal{R} = \begin{bmatrix} (P^{-1})^+ & -(P^{-1})^- \\ -(P^{-1})^- & (P^{-1})^+ \end{bmatrix}$.

It follows that $E_x \leq F\mathcal{R}E_{r_2} + 2J\overline{\delta}$.

Finally, it is shown that E_x is asymptotically element-wise lower than the non-negative vector $\rho_x = F\mathcal{R}\rho_{r_2} + 2J\overline{\delta}$.

4.2 Unknown input estimation

In this subsection, the upper and lower bounds of the unknown input d will be estimated. The expression of d is expressed from the first equation of (13):

$$\begin{aligned} d(k) = KR_0^{-1}[z_1(k+1) - \tilde{A}_{11}z_1(k) - \tilde{A}_{12}z_2(k) \\ - \tilde{B}_1u(k) - \tilde{\omega}_1(k)] \end{aligned} \quad (28)$$

By replacing z_1 with its expression in (16), equation (28) becomes:

$$\begin{aligned} d(k) = KR_0^{-1}[Ey(k+1) - E\tilde{C}_2z_2(k+1) - E\delta(k+1) \\ - \tilde{A}_{11}(Ey(k) - E\tilde{C}_2z_2(k) - E\delta(k)) - \tilde{A}_{12}z_2(k) \\ - \tilde{B}_1u(k) - \tilde{\omega}_1(k)] \end{aligned} \quad (29)$$

The following theorem ensures the interval estimation of the unknown input d.

Theorem 3. Assume that the conditions of Theorem 1 are satisfied. Then, for all $k \in \mathbb{Z}_+$ the estimates $\underline{d}(k)$ and $\overline{d}(k)$ given by

$$\begin{cases} \overline{d}(k) & = Qy(k+1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1u(k) \\ & +G_1^*\overline{Z}_2(k+1) + G_2^*\overline{Z}_2(k) + G_3^*\overline{\Delta} + G_4^*\overline{\Delta} \\ & +G_5^*\overline{\tilde{\Omega}}_1 \\ \underline{d}(k) & = QEy(k+1) - Q\tilde{A}_{11}Ey(k) - Q\tilde{B}_1u(k) \\ & +G_1^*\underline{Z}_2(k+1) + G_2^*\underline{Z}_2(k) + G_3^*\underline{\Delta} + G_4^*\underline{\Delta} \\ & +G_5^*\underline{\tilde{\Omega}}_1 \end{cases} \quad (30)$$

are bounded and verify

$$\underline{d}(k) \leq d(k) \leq \overline{d}(k) \quad (31)$$

With $Q = KR_0^{-1}$, $G_1 = -QE\tilde{C}_2$, $G_2 = Q(\tilde{A}_{11}E\tilde{C}_2 - \tilde{A}_{12})$, $G_3 = -QE$, $G_4 = Q\tilde{A}_{11}E$ and $G_5 = -Q$.

Proof. The lower and upper observation errors of the unknown input d are:

$$\begin{cases} \overline{e}_d(k) = \overline{d}(k) - d(k) \\ \underline{e}_d(k) = d(k) - \underline{d}(k) \end{cases}$$

By developing these expressions we obtain:

$$\begin{cases} \overline{e}_d(k) & = G_1^+\overline{e}_{z_2}(k+1) - G_1^-\underline{e}_{z_2}(k+1) \\ & +G_2^+\overline{e}_{z_2}(k) - G_2^-\underline{e}_{z_2}(k) \\ & +G_3^+(\overline{\delta} - \delta(k+1)) - G_3^-(\overline{\delta} + \delta(k+1)) \\ & +G_4^+(\overline{\delta} - \delta(k)) - G_4^-(\overline{\delta} + \delta(k)) \\ & +G_5^+(\overline{\tilde{\omega}}_1 - \tilde{\omega}_1(k)) - G_5^-(\overline{\tilde{\omega}}_1 + \tilde{\omega}_1(k)) \\ \underline{e}_d(k) & = G_1^+\underline{e}_{z_2}(k+1) - G_1^-\overline{e}_{z_2}(k+1) \\ & +G_2^+\underline{e}_{z_2}(k) - G_2^-\overline{e}_{z_2}(k) \\ & +G_3^+(\overline{\delta} + \delta(k+1)) - G_3^-(\overline{\delta} - \delta(k+1)) \\ & +G_4^+(\overline{\delta} + \delta(k)) - G_4^-(\overline{\delta} - \delta(k)) \\ & +G_5^+(\overline{\tilde{\omega}}_1 + \tilde{\omega}_1(k)) - G_5^-(\overline{\tilde{\omega}}_1 - \tilde{\omega}_1(k)) \end{cases} \quad (32)$$

With the same reasoning as in the proof of Theorem 1, it is deduced from (32) that the unknown input observation errors are positive.

As $\overline{e}_{z_2}$, $\underline{e}_{z_2}$, δ and $\tilde{\omega}_1$ are bounded, then $\overline{d}$ and $\underline{d}$ are bounded as well.

If we define $E_d^T = [\overline{e}_d \ \underline{e}_d]$ and:

$$N_1 = \begin{bmatrix} G_1^+ & -G_1^- \\ -G_1^- & G_1^+ \end{bmatrix}, \quad N_2 = \begin{bmatrix} G_2^+ & -G_2^- \\ -G_2^- & G_2^+ \end{bmatrix},$$
$$N_3 = G_3^+ - G_3^-, \quad N_4 = G_4^+ - G_4^-, \quad N_5 = G_5^+ - G_5^-,$$

then it can be deduced that

$$E_d \leq (N_1 + N_2)E_{z_2} + 2(\overline{\overline{N_3}} + \overline{\overline{N_4}}) \begin{bmatrix} \overline{\delta} \\ \overline{\delta} \end{bmatrix} + 2\overline{\overline{N_5}} \begin{bmatrix} \overline{\tilde{\omega}}_1 \\ \overline{\tilde{\omega}}_1 \end{bmatrix}.$$

As $E_{z_2} = \mathcal{R}E_{r_2}$, we obtain

$$E_d \leq (N_1 + N_2)\mathcal{R}E_{r_2} + +2(\overline{\overline{N_3}} + \overline{\overline{N_4}}) \begin{bmatrix} \overline{\delta} \\ \overline{\delta} \end{bmatrix} + 2\overline{\overline{N_5}} \begin{bmatrix} \overline{\tilde{\omega}}_1 \\ \overline{\tilde{\omega}}_1 \end{bmatrix}.$$

Using the upper bound ρ_{r_2} of the observation error E_{r_2}, we can deduce that E_d is asymptotically lower than the non-negative vector:

$$\rho_d = (N_1 + N_2)\mathcal{R}\rho_{r_2} + 2(\overline{\overline{N_3}} + \overline{\overline{N_4}}) \begin{bmatrix} \overline{\delta} \\ \overline{\delta} \end{bmatrix} + 2\overline{\overline{N_5}} \begin{bmatrix} \overline{\tilde{\omega}}_1 \\ \overline{\tilde{\omega}}_1 \end{bmatrix}.$$

5. NUMERICAL EXAMPLE

In this section, a numerical example of a LTI discrete-time system from Maquin et al. (1994) is given to illustrate the overall proposed methodology.

5.1 Model description

Let us consider the discrete-time system described by (1) with:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad (33)$$

The initial state is chosen as $x_0 = [1 \ 1 \ 1]^T$.

The state and measurement noises are assumed to be uniformly distributed and bounded with $\overline{\delta} = 10^{-2}[1 \ 1]^T$ and $\overline{\omega} = 10^{-2}[1 \ 1 \ 1]^T$. The unknown input d is simulated by $d(k) = \cos(0.5k)$.

5.2 Interval observer design

State estimation

Assumption 1 is verified since D is a full column rank matrix and C is a full row rank matrix. Therefore, we obtain the following matrices:

$$H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K = 1 \qquad R = 1$$

In addition, Assumption 2 holds as the pair (A_2, C_2) is observable. The gain L is chosen with a Schur stable pole assignment $\{0.5\ , 0.6\}$ and is given by $L^T = [-0.3 \ -2.1]$.

Finally, Lemma 1 is satisfied with a transformation matrix P given by:

$$P = \begin{bmatrix} 10 & -5 \\ -10 & 6 \end{bmatrix} \quad (34)$$

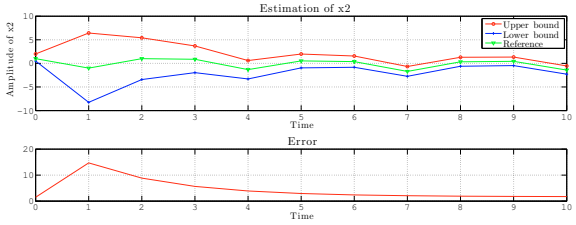

Fig. 1. Lower and upper bounds for the state x_2

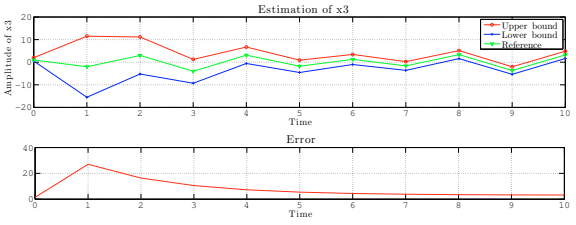

Fig. 2. Lower and upper bounds for the state x_3

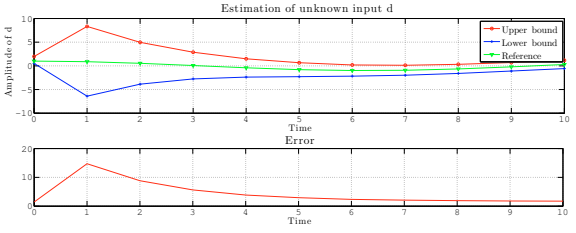

Fig. 3. Lower and upper bounds for unknown input d.

allowing to obtain a Schur stable and nonnegative matrix $R = P(A_2 - LC_2)P^{-1}$. The bounds of the initial state are chosen as $\overline{x}_0 = [2\ \ 2\ \ 2]^T$ and $\underline{x}_0 = [0.5\ \ 0.5\ \ 0.5]^T$.

The simulation results for state estimation and the errors are depicted in Fig. 1 and Fig. 2.

The results show that the proposed methodology is well suited for state estimation in discrete-time with the presence of unknown inputs. In the following, the results obtained for the computation of the lower and upper bounds of the unknown input are presented.

Unknown input estimation
The estimation of the bounds of the unknown input may be required in many applications. These bounds are computed using (30) and the results are shown in Fig. 3. The proposed interval observer for LTI discrete-time systems converges asymptotically and gives satisfying estimation of the state and unknown input bounds despite the presence of measurement noise and disturbances.

6. CONCLUSION

In this paper, the joint estimation of state and unknown input for LTI discrete-time systems has been studied using interval observers. The results have been obtained using a change of coordinates that allows to decouple a part of the state from the unknown input. The interval observer is first used to estimate the bounds of the unknown input-free part of the state vector, then for the reconstruction of the unknown input. In future works, this methodology will be investigated for nonlinear discrete-time systems.

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