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► **To cite this version:**

Laurent Boudin, Lara Trussardi. Concentration effects in a kinetic model with wealth and knowledge exchanges. 2020. hal-02470191v1

**HAL Id: hal-02470191**

**<https://hal.science/hal-02470191v1>**

Preprint submitted on 7 Feb 2020 (v1), last revised 5 Jul 2021 (v2)

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# CONCENTRATION EFFECTS IN A KINETIC MODEL WITH WEALTH AND KNOWLEDGE EXCHANGES

LAURENT BOUDIN AND LARA TRUSSARDI

ABSTRACT. Using a kinetic equation of Boltzmann type, we describe the effect of knowledge and wealth exchange in a closed community, where all the interactions inside the population are binary. We model the evolution of the distribution function  $f = f(t, x, v)$  of agents with wealth  $v$  and knowledge  $x$  at time  $t$ . Some numerical experiments demonstrating, among other things, clustering effects, are presented, and some more asymptotic studies are performed.

## 1. INTRODUCTION

The methods and concepts of statistical mechanics appear as successful means to describe and study the collective behaviour of a large number of interacting individuals in social sciences. The behaviour of the population is considered as resulting from a chain of elementary binary mechanisms. The most popular fields explored for complex systems in social sciences are economics and sociology. The idea of using tools from physics to study social phenomena goes back to the 70s and 80s articles [34, 21] for sociology, and to various works around 2000 [28, 3, 26, 17] for economics, opening the lead for both sociophysics and econophysics fields. We refer to various recent review articles and books related to these topics, see, for instance, [8, 9, 10, 20] and the references therein.

Among many other approaches, mainly agent-based, existing in the literature, the kinetic theory of gases, involving Boltzmann-like equations and collision operators, is popular in sociophysics and econophysics. Up to our knowledge, the first kinetic models in social sciences were developed by Helbing [24, 25] to study the social behaviour dynamics of a population, and by Cordier, Pareschi and Toscani [13] to describe a simple market economy. For more details, we refer to [29], in particular to [12, 5], and the very complete book by Pareschi and Toscani [30].

The model we mathematically and numerically investigate here is quite simple. It may lack some realism, because it involves simplified models with respect to [4, 31, 33], to provide a neat mathematical framework. Nevertheless, it allows us to recover clustering effects highlighted in [15, 23], for example. The main idea relies on the same kind of assessment as [6, 16, 27, 31]: the wealth exchanges are also driven by the knowledge/beliefs of each agent in the population. For instance, in [31], the population interacts with a fixed, time-dependent background of common knowledge, which behaves like an information mean field that does not depend on the population itself. This background can then be understood as the media.

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This work was partially funded by the joint Austrian-French Amadeus program headed by A. Jünger and A. Moussa, and the French ANR-13-BS01-0004 project *Kibord* headed by L. Desvillettes. LT acknowledges partial support from the European Union in the FP7-PEOPLE-2012-ITN Program under Grant Agreement Number 304617, and from the Austrian Science Fund (FWF) projects P27052 and F 65. LB and LT acknowledge partial support from the Inria MECI project.

In [27], the population is divided into two groups, the chartists and the fundamentalists, whose interactions allow to steer the price formation of a specific good.

The point of view we choose here is different. We assume that all the exchanges, knowledge or wealth, are of binary kind, inside a homogeneous closed community. The microscopic wealth exchange mechanism between two agents is very similar to the one from [31]: it depends on the knowledge of each agent. In the same way, the microscopic knowledge exchange takes into account the dependence with respect to the agents' wealth, with the quite natural idea that an agent may consider as more trustworthy another agent who owns more than himself. In other words, knowledge plays a fundamental role to improve the social condition, so that we can safely suppose that the more we know, the more we can earn and, at the same time we can imagine that who owns more has a higher knowledge.

For the wealth exchanges, we only take into account the personal saving propensity, and forget, for the time being, the risk perception of the individuals described in [31]. We assume that each agent can use his own personal knowledge to reduce the risk in a trade. A more detailed investigation on kinetic models for wealth distribution and market economy can be found in [2, 18].

This article is divided into four parts and one appendix. Section 2 is dedicated to the presentation of the microscopic and kinetic models of wealth and knowledge exchange processes. In Section 3, we provide numerical experiments for the Boltzmann equation. Section 4 is devoted to the introduction of a quasi-invariant limit for the knowledge and wealth interactions, the derivation of a Fokker-Planck equation and the study of the moments including a numerical study of the quasi-invariant knowledge case. In addition, the appendix gives a proper mathematical framework of the model.

## 2. KINETIC MODEL

We want to foresee the time evolution of a population of agents who are described thanks to two characteristics: their wealth and their knowledge. Using the formalism of kinetic theory, we are led to introduce an unknown distribution function  $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ ,  $(t, x, v) \mapsto f(t, x, v)$ , where  $t \geq 0$  is the time variable,  $x \geq 0$  the knowledge and  $v \geq 0$  the wealth. Actually, we shall see that  $f$  has a compact support when its initial datum has a compact support. The quantity  $f(t, x, v) dx dv$  can be understood as the number of agents of the population inside an elementary volume of the phase space (in both knowledge and wealth variables) centred at  $(x, v)$  and of measure  $dx dv$ .

We first need to describe the microscopic mechanisms between agents for both knowledge and wealth to be able to define the associated mesoscopic collision operators of Boltzmann type, and eventually write the kinetic equation governing the time evolution of  $f$ .

In what follows, for the sake of simplicity, we shall write “agent  $(x, v)$ ” to deal with an agent of knowledge  $x$  and wealth  $v$ .

**2.1. Microscopic exchanges of knowledge and wealth.** An agent in the population can interact with any other one. We here assume that these interactions are of binary type, *i.e.* we suppose that the interactions involving three individuals or more can be seen as the “sum” of binary exchanges between agents. Moreover, the wealth and knowledge exchanges are chosen to be independent. That means that an agent may interact at the same time with two different individuals, one for the exchanging wealth and one for exchanging the knowledge.

Let us first describe the knowledge binary exchange process. The mathematical description of the process of learning is a challenge. There are different works that aim to describe the knowledge growth [7, 14, 22].

Here we consider a very simple model with two agents  $(x, v)$  and  $(y, w)$ . Their knowledges  $x'$  and  $y'$  are updated thanks to the following interaction rule (with the kinetic theory vocabulary)

$$(1) \quad \begin{cases} y' = y + \kappa(v)(x - y) = \kappa(v)x + [1 - \kappa(v)]y, \\ x' = x + \kappa(w)(y - x) = [1 - \kappa(w)]x + \kappa(w)y, \end{cases}$$

where  $\kappa : \mathbb{R}_+ \rightarrow [\alpha, 1/2)$  is a non-decreasing function of the wealth variable, with  $0 < \alpha \ll 1/2$ . Mechanism (1) implies that the knowledge change for an individual depends on the wealth of the other agent involved in the interaction. More precisely,  $\kappa$  can be considered as a *confidence* function, in the sense that the bigger  $v$  is with respect to  $w$ , the more agent  $(y, w)$  trusts agent  $(x, v)$ . Let us point out that this rule is quite similar to the one presented in [33]. As a matter of fact, the post-collisional knowledge  $x'$  is computed from  $x$ , contrary to [4] where the average opinion was used, by adding a quantity involving the relative knowledge  $x - y$  and a coefficient  $\kappa(w)$  depending on the wealth of the other agent. The dependence of this coefficient is a new feature, since it was previously related to  $x$  and not  $w$  as in [33]. We emphasize that the main difference with [31] lies in the fact that we allow interaction between agents for the exchange of information, whereas in [31], the authors used the interaction with a given background of information, with a mean-field point of view. For instance, we can choose  $\kappa$  constant, linear or exponential, e.g.  $\kappa(v) = [1 - (1 - 2\alpha)e^{-2v}]/2$  for any  $v$ .

Since  $x'$  and  $y'$  are convex combinations of  $x$  and  $y$ , the knowledge bounds are preserved at the microscopic level, *i.e.*  $[x', y'] \subset [x, y]$ . There should eventually be a knowledge concentration effect inside the population, if there is no other phenomenon taken into account for the knowledge variable. Note that, nevertheless, there are no reasons for this concentration to go to the initial average knowledge. Indeed, the microscopic post-collisional total knowledge

$$(2) \quad x' + y' = x + y + (\kappa(v) - \kappa(w))(x - y)$$

can be larger than  $x + y$  if we simultaneously have  $x > y$  and  $\kappa(v) > \kappa(w)$ , ensuring that the exchange process is profitable to everyone when a wealthy well-informed agent interacts with the rest of the population.

**Remark 1.** *The microscopic property  $[x', y'] \subset [x, y]$  clearly implies that, if  $f^{\text{in}}$  has a compact support in the knowledge variable, so has  $f$  at any time.*

This whole behaviour remains quite simplistic from the modelling point of view, but it has the mathematical benefit that the collision rule (1) is invertible: both  $x$  and  $y$  can be expressed in terms of  $x'$  and  $y'$ , since the Jacobian  $J_K(v, w)$  of (1), which does not depend on  $x$  and  $y$ , writes  $J_K(v, w) = \kappa(v) + \kappa(w) - 1$ , and clearly remains negative for any  $v, w$ , by assumption on  $\kappa$ .

**Remark 2.** *We can also add to (1) a threshold effect, which is probably more realistic. The model then relies on a bounded-confidence assumption, *i.e.* the knowledge interaction is forbidden between agents  $(x, v)$  and  $(y, w)$  when  $|v - w| \leq \omega$ , where  $\omega > 0$  is given. This assumption is very common in the literature of opinion dynamics, see [15, 23] for instance.*

Let us now focus on the wealth binary exchange process, and consider again two agents  $(x, v)$  and  $(y, w)$ . Their wealth values  $v'$  and  $w'$  after interaction are given by the collision

rule

$$(3) \quad \begin{cases} v' = [1 - \Psi(x)\gamma]v + \Psi(y)\gamma w, \\ w' = \Psi(x)\gamma v + [1 - \Psi(y)\gamma]w, \end{cases}$$

where  $\gamma \in (0, 1)$  is fixed and  $\Psi : \mathbb{R}_+ \rightarrow (0, 1]$  is a non-increasing continuous function of the knowledge variable. The collision rule (3) was first proposed in a simplified version in [13] and then thoroughly in [31]. In this work, we did not take into account the random risk parameter inducing a noise in the interaction rule as in [31]. Indeed, we assume that the saving and risk propensity are directly linked. More precisely, an agent who tends to save money would naturally risk less than another agent who wants to take risks on his investments. Hence, this is why we choose to treat them with the sole quantity  $\gamma\Psi(x)$ , which can then be understood as the saving/risk-taking propensity of agent  $(x, v)$ . The monotonicity of  $\Psi$  implies that the more an agent has knowledge, the less risky the wealth exchange is for him. For instance, we can choose  $\Psi$  to be constant, linear or inverse polynomial, e.g. for any  $x$ ,  $\Psi(x) = (1 + x)^{-\beta}$  with  $\beta > 0$ . Note that, for the mathematical well-posedness result detailed in Appendix A, we need a stronger assumption on the lower bound of  $\Psi$ .

It is clear that the microscopic total wealth is conserved during the exchange process:  $v' + w' = v + w$ . We also emphasize that  $v'$  and  $w'$  are not convex combinations of  $v$  and  $w$ , but satisfy some relevant qualitative properties, such as the following one. Assuming that  $x > y$ , which implies that  $\Psi(x) \leq \Psi(y)$ , we then have

$$v' \geq v + (w - v)\Psi(x)\gamma.$$

This ensures that the interaction of an agent with another one, richer but less informed, is profitable to the first one. Of course, there is an arguable situation when both  $y$  and  $w$  are respectively smaller than  $x$  and  $v$ . In such a case, agent  $(x, v)$  may not want to interact with  $(y, w)$ , since she would have nothing to win in the wealth exchange. Consequently, a wealth threshold effect should also be investigated in the latter case. This will be more discussed in details in the numerical experiments.

Eventually, we must point out a mathematical issue: the collision rule (3) may not be invertible. Indeed, we can check that the Jacobian  $J_W$  of (3), which does not depend on  $v$  and  $w$ , satisfies, for any  $x, y$ ,

$$J_W(x, y) = 1 - \gamma(\Psi(x) + \Psi(y)) \in (1 - 2\gamma, 1).$$

Since  $\Psi$  is continuous, the lower bounding of  $J_W(x, y)$  can be 0 if  $\gamma > 1/2$  for some values of  $x$  and  $y$ . This range of values of  $\gamma$  is realistic at the microscopic level, see [31] for more details. For mathematical reasons, we have to choose  $\gamma \leq 1/2$  to ensure the invertibility of (3). So, thanks to the choice of  $\gamma$  we can relax the hypothesis on  $\Psi$  and we do not need any more the continuity property.

**Remark 3.** *The wealth is conserved only at the whole population level, as we can see from the microscopic property  $v' + w' = v + w$ . So if  $f^{\text{in}}$  has a compact support in the wealth variable  $[0, V]$  where  $V$  is the total wealth of the all society, so has  $f$  at any time.*

**2.2. Collision operators and governing equation.** In order to take into account the microscopic collision rules (1)–(3) in the time evolution of the distribution function  $f$ , we need to write the related collision operators  $Q_K$  and  $Q_W$ . However, the rules may not be diffeomorphisms from  $\mathbb{R}_+^{*2}$  into itself. Thus, to overcome this difficulty, as explained, for instance, in [4, page 511], the natural framework consists in writing the collision operators under weak forms.

A test function  $\phi$  will be *admissible* if  $\phi \in C^0(\mathbb{R}_+^2)$ . We do not need  $\phi$  to be compactly supported. Indeed, thanks to Remarks 1 and 3, we know that, if  $f^{\text{in}}$  has a compact support in  $(x, v)$ , then  $f$  shares the same property a.e.  $t$ . Moreover, realistic initial data can always be considered as compactly supported.

Then, for an admissible test function  $\phi$  of  $(x, v)$ , we write the weak form of the collision operator  $Q_K(f, f)$ , acting on the knowledge variable, as

$$(4) \quad \begin{aligned} \langle Q_K(f, f), \phi \rangle &= \nu_K \iint_{(\mathbb{R}_+^*)^4} f(t, x, v) f(t, y, w) (\phi(x', v) - \phi(x, v)) \, dx \, dy \, dv \, dw \\ &= \frac{\nu_K}{2} \iint_{(\mathbb{R}_+^*)^4} f(t, x, v) f(t, y, w) (\phi(x', v) + \phi(y', w) - \phi(x, v) - \phi(y, w)) \, dx \, dy \, dv \, dw, \end{aligned}$$

where  $\nu_K > 0$  denotes the interaction frequency in the population for the knowledge exchange. Both expressions of  $Q_K(f, f)$  in (4) are equal, thanks to the change of variables  $(x, y, v, w) \mapsto (y, x, w, v)$ , whose Jacobian equals 1. In the same way, for the collision operator  $Q_W(f, f)$ , which acts on the wealth variable, we write, for any admissible test-function  $\phi$ ,

$$(5) \quad \begin{aligned} \langle Q_W(f, f), \phi \rangle &= \nu_W \iint_{(\mathbb{R}_+^*)^4} f(t, x, v) f(t, y, w) (\phi(x, v') - \phi(x, v)) \, dx \, dy \, dv \, dw \\ &= \frac{\nu_W}{2} \iint_{(\mathbb{R}_+^*)^4} f(t, x, v) f(t, y, w) (\phi(x, v') + \phi(y, w') - \phi(x, v) - \phi(y, w)) \, dx \, dy \, dv \, dw, \end{aligned}$$

where  $\nu_W > 0$  denotes the interaction frequency in the population for the wealth exchange.

More mathematical details on the collisional kernels are provided in Appendix A in order to study the well-posedness of the following problem.

Let  $T > 0$ . The previous considerations allow us to eventually formulate the integro-differential equation of Boltzmann type, satisfied, in a weak sense, by the distribution function  $f$ , that is, for any admissible test-function  $\phi$  of  $(x, v)$  and almost every  $t \in [0, T]$ ,

$$(6) \quad \frac{d}{dt} \int_{(\mathbb{R}_+^*)^2} f(t, x, v) \phi(x, v) \, dx \, dv = \langle Q_K(f, f), \phi \rangle + \langle Q_W(f, f), \phi \rangle,$$

with initial condition  $f(0, \cdot, \cdot) = f^{\text{in}}$ , where  $f^{\text{in}} \in L^1((\mathbb{R}_+^*)^2)$  is a given nonnegative function which is compactly supported in both variables. Note that the choice of the admissible test functions allows not to impose any boundary condition on  $f$  on the axes  $[x = 0]$  and  $[v = 0]$ .

The conservation of the total number of agents in the population is a straightforward consequence of the weak formulations (4)–(5), *i.e.*, we have the following

**Proposition 4.** *Let  $f \in L^\infty(0, T; L^1((\mathbb{R}_+^*)^2))$  solving (6) with initial datum  $f^{\text{in}} \in L^1((\mathbb{R}_+^*)^2)$ . Then we have, for a.e.  $t$ ,*

$$\|f(t, \cdot, \cdot)\|_{L^1((\mathbb{R}_+^*)^2)} = \|f^{\text{in}}\|_{L^1((\mathbb{R}_+^*)^2)}.$$

*Proof.* We just have to choose  $\phi \equiv 1$  in (6) and use (4)–(5) for that test-function.  $\square$

This conservation property is crucial for the numerical experiments, and we shall preserve it at the computational level, as we explain in the next section.

Finally, we state below an existence result for weak solutions to (6). The proof is provided in Appendix A.

**Theorem 5.** *Assume that  $\Psi$  is lower-bounded by a constant  $\delta > 0$ . Let  $f^{\text{in}}$  a nonnegative function in  $L^1(\mathbb{R}_+^{*2})$  with compact support in both variables. Then there exists a nonnegative  $f \in L^\infty(0, T; L^1(\mathbb{R}_+^{*2}))$  which weakly solves (6) for almost every  $t$ , with initial datum  $f^{\text{in}}$ .*

The previous result provides a formal framework to our numerical setting which is discussed in the next section.

### 3. NUMERICAL EXPERIMENTS FOR THE BOLTZMANN EQUATION

In this section, we briefly discuss the numerical method and the computational tools for the Boltzmann equation. Then we present some numerical experiments on the model on various situations, in particular, we perform some basic tests and discuss the relevance of thresholds for the collisions.

**3.1. Numerical values, computational strategy.** We must deal with both variables involved, wealth and knowledge, *i.e.* discretize a two-dimensional model. We use a standard particle method [1]. The distribution function  $f$  is approximated as a sum of Dirac masses:

$$f(t, x, v) \simeq \sum_{p=1}^{2N} \delta_{(x_p(t), v_p(t))}(x, v),$$

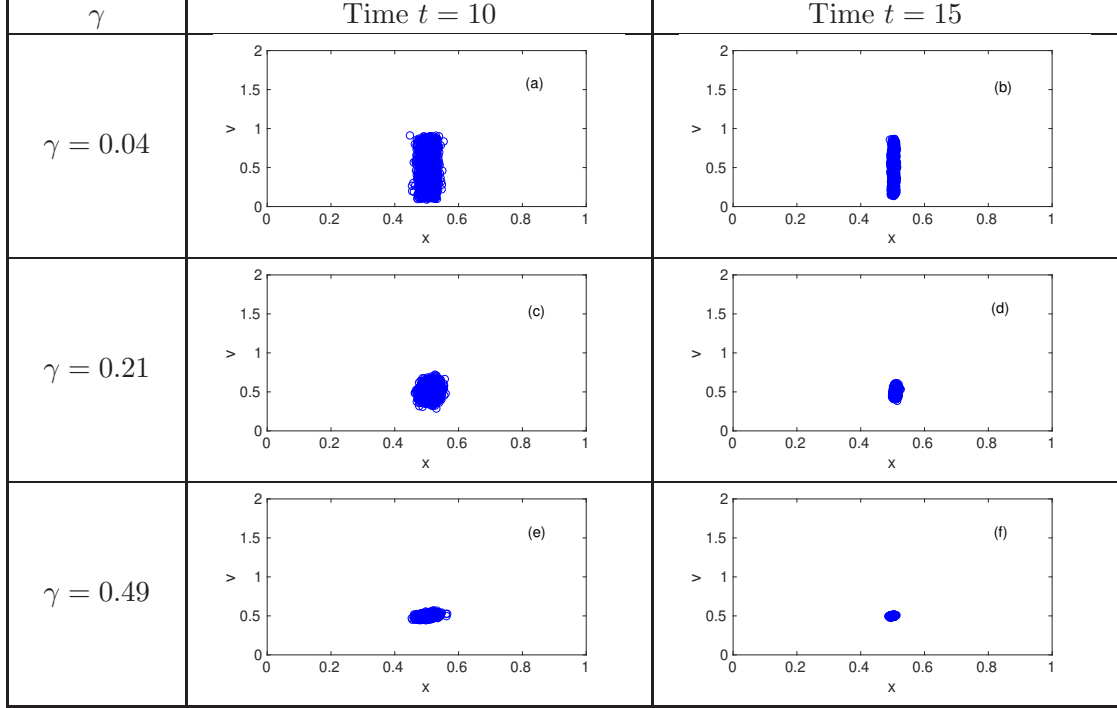
where  $2N$  is the total number of agents in the simulation, and  $x_p(t)$ ,  $v_p(t)$  are the knowledge and wealth at time  $t$  of a numerical agent  $p$ ,  $1 \leq p \leq 2N$ . In what follows, if not specified otherwise,  $N$  is chosen equal to 1000. The usual computational average is performed over 30 different simulations. The final computational time  $T$  will be given for each experiment.

We investigate different compactly supported initial data. Concerning the knowledge variable, we assume that  $x$  lies in  $[0, X]$  with  $X = 1$  without loss of generality. For the wealth variable, we proceed in the same way, noticing that the wealth is only conserved at the whole-population level. For some of the first time steps, an individual may reach a post-interaction value  $v'$  larger than the pre-interaction value  $v$ . But the expected maximal value cannot be larger than the total wealth of the society given by  $V = \sum_{p=1}^{2N} v_p(0)$ . Nevertheless, if we choose the initial datum in  $v$  so that any wealth value lies in  $[0, W]$  with  $W = 2$ , we can observe that the distribution after a few time steps satisfy  $(x, v) \in [0, X] \times [0, W]$  thanks to Remarks 1 and 3.

The collisions in  $x$  and  $v$  are independent. Hence, the wealth and knowledge interactions can simultaneously happen, but may involve different agents, which are randomly chosen. Moreover, since we are interested in the case when there is no predominance of one kind of collision, we take  $\nu_K = \nu_W = 1$ , for instance. The time step is then chosen as  $1/\nu_K = 1/\nu_W$ . This choice means that, during each time step, one knowledge collision and one wealth collision happens for each individual.

The whole algorithm is embedded in a numerical code written in C. Note that it exactly conserves the total number of agents, as suggested by the conservation property in Proposition 4.

**3.2. First numerical experiments.** In the following section, we investigate the influence of the various functions and parameters appearing in the collision rules (1)–(3). We show the steady states which we obtain when we have only one kind of collisions and introduce some thresholds in order to get some cluster formations. Let us first start with an initial datum which is uniformly distributed with respect to  $(x, v)$  on  $[0, 1] \times [0, 1]$ . We take  $\Psi(x) = (1+x)^{-\beta}$


 TABLE 1. Influence of  $\gamma$  on the concentration effect with  $\kappa = 0.34$ .

with  $\beta = 1$  in the collision rule (3) and a constant confidence function  $\kappa$ . We consider different values for  $\gamma$  (Table 1), in particular equal to 0.04, 0.21 and 0.49 for  $\kappa = 0.34$  and then in Table 2, taking  $\gamma = 0.21$ , we choose different values of  $\kappa$ , in particular equal to 0.03, 0.12 and 0.49.

The collision operators have a concentration effect and the convergence rate towards the Dirac mass does not depend on the initial datum but on the choice we make for the functions in (1) and (3).

In Tables 1–2, we can see the different screenshots of the function  $f$  at different times. The evolution respectively depends on the choice of  $\gamma$  and  $\kappa$ . Note that a bigger  $\gamma$  leads to a faster convergence to the mean value for  $v$ , while a bigger  $\kappa$  leads to a faster convergence to the mean value for  $x$ . That implies that, if one of both values is small, then we first observe the concentration effect for the other variable.

In any case, the distribution functions converge to a Dirac mass centred at  $(c_K, c_W)$  with  $c_K = c_W = 0.5$ , which is exactly the mean knowledge and wealth, *i.e.*

$$c_K = \int_{\mathbb{R}_+^2} x f^{\text{in}} dx dv \quad c_W = \int_{\mathbb{R}_+^2} v f^{\text{in}} dx dv.$$

Now we consider a non trivial confidence function  $\kappa$ . If  $\kappa$  is not constant, e.g.  $\kappa(v) = [1 - (1 - 2\alpha)e^{-2v}]/2$  with  $\alpha = 0.05$ , the concentration effect also happens, but not necessarily at the average value of  $x$ , since the collision rule (1) for  $x$  then does not conserve the total knowledge as explained in (2). Consequently, with this choice for  $\kappa$ , we do not know the value of  $c_K$  a priori. This leads us to study the moments of  $f$  which are investigated in Section 4.2.



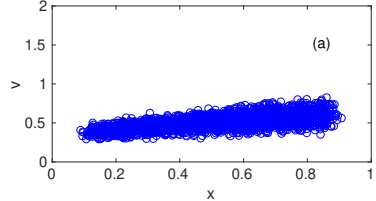
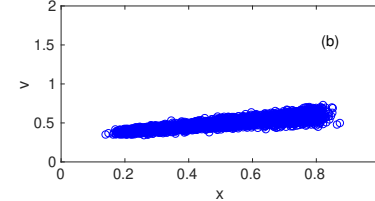
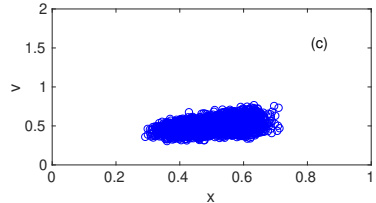
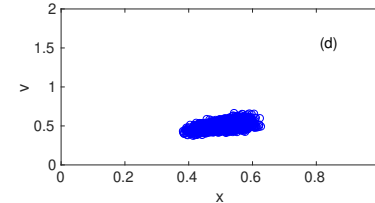
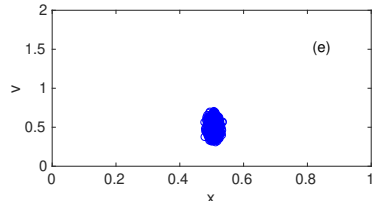
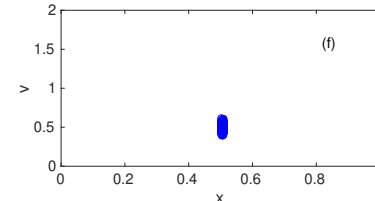
Value of the parameter $\kappa$	Time $t = 10$	Time $t = 15$
$\kappa = 0.03$		
$\kappa = 0.12$		
$\kappa = 0.49$		

TABLE 2. Influence of  $\kappa$  on the concentration effect, choosing  $\gamma = 0.21$ .

We can observe that  $f(t, \cdot, \cdot)$  converges again to a Dirac mass when  $t$  goes to  $+\infty$  (more details on the convergence are presented in Section 4.4). Nevertheless, this large-time result, both for constant and non constant  $\kappa$ , is not realistic from the modelling viewpoint: a society where all the individuals exactly share the same wealth and knowledge is utopian.

**3.2.1. One-type collision.** If one type of collision is dominant, we can observe the effect of the corresponding collision operator in the numerical simulations. For instance, we take the same initial datum as before, i.e. uniformly distributed with respect to  $(x, v)$  on  $[0, 1] \times [0, 1]$ . The distribution at final time demonstrates a concentration effect, but on a straight line (since the other variable has no influence). This behaviour has also been observed in Tables 1–2 for very small values of these parameters.

The knowledge collision rules (1) induce concentrating the agents at the average knowledge value (which, for this initial datum, is 0.5) on Figure 1a, with no effect on the wealth distribution. The situation with the wealth collision rule (3) is different, as shown on Figure 1b. When time grows, all the agents are on the same straight line, but they do not have the same wealth, because the less informed agents become poorer, and the more informed richer.

The equations for the straight lines are obtained considering the case when the post-interaction value is equal to the pre-interaction value, i.e.  $x' = x$  and  $v' = v$ . See Section 4.4 for more details.

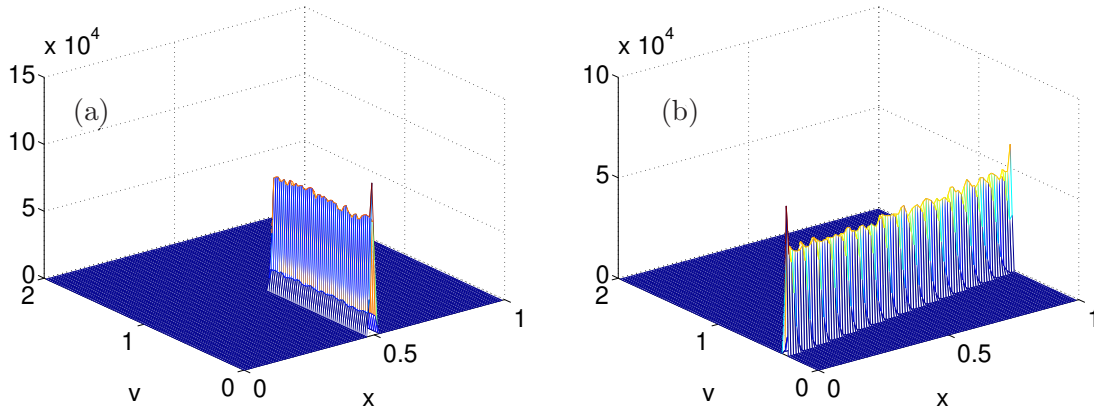


FIGURE 1. Profile of  $f$  at final time  $T = 70$ : (a) averaged result with only information exchanges; (b) averaged result with only wealth exchanges.

**3.3. Thresholds and clusters.** Let us now consider some more realistic situations. As we already explained, interaction thresholds are often used in both knowledge and wealth exchanges, see [9, 15, 23] for instance. Those thresholds usually induce formation of clusters.

Let us choose, as an initial datum for  $f$ , a truncated inverse-Gaussian distribution with mean  $\mu$  and shape parameter  $\lambda$  both for  $x$  and  $v$ . The truncation happens so that the support of  $f^{\text{in}}$  lies in  $[0, 1] \times [0, 2]$ . The distribution characteristics are the following:  $\mu = 0.25$  and  $\lambda = 0.5$  for knowledge,  $\mu = 0.4$  and  $\lambda = 0.2$  for wealth. This distribution implies that a lot of individuals have low values of knowledge and wealth, and a few agents have higher values of knowledge and/or wealth, as shown in Figure 2.

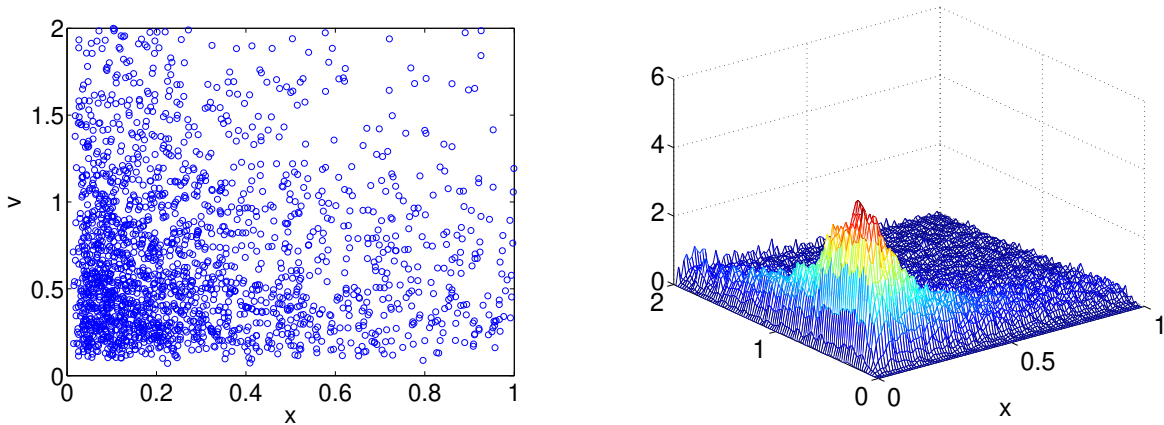


FIGURE 2. A more realistic initial datum for the distribution function.

We can introduce a threshold for the wealth collisions and/or the knowledge collision.

We first suppose that two agents  $(x, v)$  and  $(y, w)$  only have a threshold for the wealth interactions and so, they exchange their wealth if and only if  $|v - w| \leq \omega$ , where  $\omega > 0$  is given.

The threshold value  $\omega$  has an impact on the cluster formation: if we choose  $\omega$  too small, we observe that the society will not have very rich individuals, while if  $\omega$  is too big, there is no cluster formation. This is obvious if we consider that without a threshold we have the convergence to the mean wealth. So, a big  $\omega$  is similar to this generic case.

Hence, for this first experiment, with the same initial datum as in Figure 2, we provide a threshold effect on the wealth variable. Two agents  $(x, v)$  and  $(y, w)$  can only exchange information when their respective wealth values are close, for instance  $|v - w| \leq \omega$ , with  $\omega = 0.1$ . The distribution function at final time is shown in Figure 3.

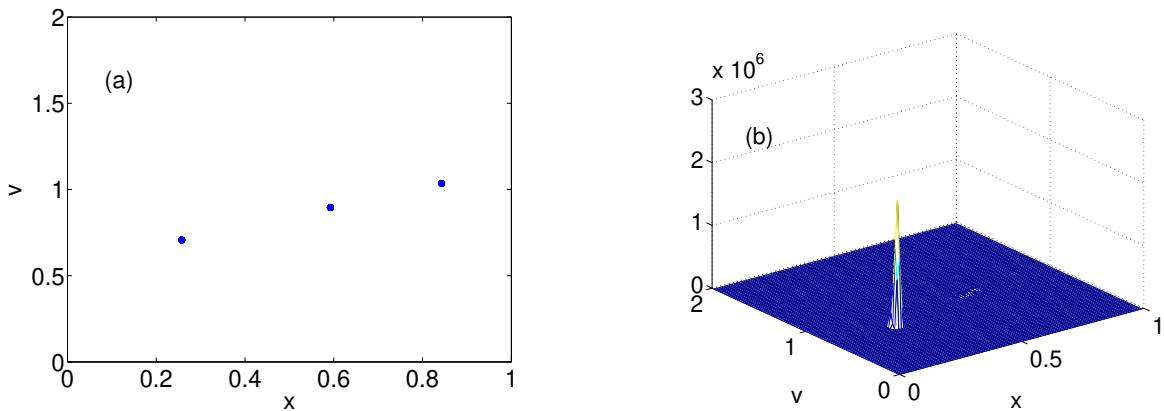


FIGURE 3. Experiment 2 (with thresholds) and final time  $T = 70$ : distribution at final time for one simulation (on the left), and averaged on 30 computations (on the right).

We observe that these concentration points are aligned on a straight line which corresponds to the one obtained in Figure 1b.

In the second experiment, we introduce a threshold also for the knowledge collisions. We assume that a wealth exchange between agents  $(x, v)$  and  $(y, w)$  can only occur when  $|v - w| \leq 0.5$ , *i.e.*  $\omega = 0.5$ , while the information exchange only occurs if both agents have the same level of knowledge and wealth. For being more clear, two agents  $(x, v)$  and  $(y, w)$  exchange their knowledge if and only if they have the same “general background”: we divide the domain into four parts and we let the knowledge interaction (1) happens only between agents in the same area. This four areas represent the following cases: rich individuals with big knowledge ( $v, w \geq 1$  and  $x, y \geq 0.5$ ); rich individuals with small knowledge ( $v, w \geq 1$  and  $x, y \leq 0.5$ ); poor individuals with big knowledge ( $v, w \leq 1$  and  $x, y \geq 0.5$ ); poor individuals with small knowledge ( $v, w \leq 1$  and  $x, y \leq 0.5$ ).

This choice reflects the fact that people usually interact with other agents which have more or less the same level of wealth and/or knowledge, for socio-professional networking reasons (*i.e.* “bounded confidence” and “bounded exchange”).

The restriction for the knowledge collisions has been introduced because the interaction takes place between people who have more or less the same cultural level, and the restriction for the wealth collisions takes into account the fact that someone who owns a lot often does not want to share what he knows. The distribution obtained at final time with those thresholds is shown in Figure 4.

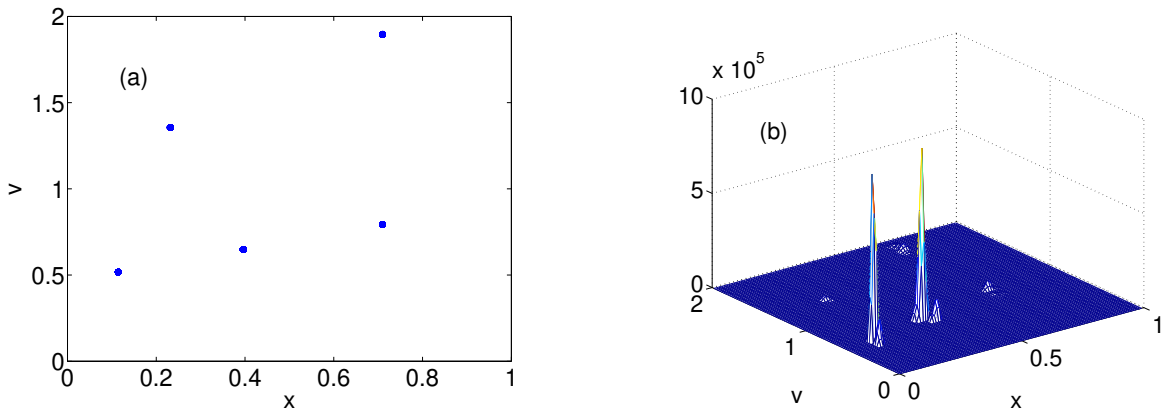


FIGURE 4. Experiment 1 (with threshold) and final time  $T = 450$ : distribution at final time for one simulation (on the left), and averaged on 30 computations (on the right).

Due to the kind of thresholds we impose, we can expect to have at least one Dirac mass in each “region of interaction” and, knowing that the collisions bring the individuals towards the mean value of the knowledge, it is not a surprise that some clusters are concentrated at the value of  $x$  corresponding to the average value of the “region of interaction”. Observing the position of the clusters, we can see that they lie on the straight line of equation  $v = D(1 + x)$ ,  $D > 0$ , which we obtained in Figure 1b. We can then deduce that we have two concentration effects at the beginning: on the curve  $v = D/\Psi(x)$  and one on the line  $x = c_K$ ,  $c_K \in (0, 1)$  which eventually goes to the Dirac mass at  $(c_K, \frac{D}{\Psi(c_K)})$ . Observe that we can compute a priori the value of  $c_K$  and  $D$  since they correspond respectively to the average of the knowledge values and the average of the wealth values of the region of interaction (see Section 4.4 for more details).

#### 4. QUASI-INVARIANT LIMIT

We want to describe the asymptotic behaviour of the Boltzmann equation (6). We focus on the case when only small interactions for the knowledge and the wealth are allowed, and are of the same order of magnitude. Let  $0 < \varepsilon \ll 1$  be such that  $\kappa = \varepsilon \hat{\kappa}$  and  $\gamma = \varepsilon \hat{\gamma}$ . We rescale the interaction rule (1) for the knowledge and the interaction rule (3) obtaining respectively

$$(7) \quad \begin{cases} y' = y + \varepsilon \hat{\kappa}(v)(x - y) \\ x' = x + \varepsilon \hat{\kappa}(w)(y - x) \end{cases}$$

and

$$(8) \quad \begin{cases} v' = (1 - \Psi(x)\varepsilon\hat{\gamma})v + \Psi(y)\varepsilon\hat{\gamma}w \\ w' = (1 - \Psi(y)\varepsilon\hat{\gamma})w + \Psi(x)\varepsilon\hat{\gamma}v. \end{cases}$$

Let us consider a test function  $\phi$  and write the following Taylor expansions up to the second order. For any  $(x, x')$  and any  $(v, v')$ , there exist  $\theta_1, \theta_2 \in [0, 1]$  such that

$$\begin{aligned} \phi(x', v) - \phi(x, v) &= (x' - x)\partial_x\phi(x, v) + \frac{(x' - x)^2}{2}\partial_{xx}^2\phi(\theta_1x' + (1 - \theta_1)x, v) \\ \phi(x, v') - \phi(x, v) &= (v' - v)\partial_v\phi(x, v) + \frac{(v' - v)^2}{2}\partial_{vv}^2\phi(x, \theta_2v' + (1 - \theta_2)v). \end{aligned}$$

Plugging (7) and (8) in the previous equalities, we can write:

$$\begin{aligned} \phi(x', v) - \phi(x, v) &= (x' - x)\partial_x\phi(x, v) + O(\varepsilon^2) \\ \phi(x, v') - \phi(x, v) &= (v' - v)\partial_v\phi(x, v) + O(\varepsilon^2). \end{aligned}$$

We recall that we consider the case with  $\nu_K = \nu_W = 1$ . Then we denote by  $f_\varepsilon$  the solution to equation (6) where the collision rules are now given by (7) and (8). The weak form (4)–(5) of  $Q_K$  and  $Q_W$  then become, for any admissible test function  $\phi$ ,

$$(9) \quad \langle Q_K(f_\varepsilon, f_\varepsilon), \phi \rangle = \varepsilon \iint_{\mathbb{R}_+^4} f_\varepsilon(t, x, v) f_\varepsilon(t, y, w) [\hat{\kappa}(w)(y - x)\partial_x\phi(x, v)] dx dy dv dw + O(\varepsilon^2),$$

$$(10) \quad \begin{aligned} \langle Q_W(f_\varepsilon, f_\varepsilon), \phi \rangle \\ = \hat{\gamma} \iint_{\mathbb{R}_+^4} f_\varepsilon(t, x, v) f_\varepsilon(t, y, w) [\Psi(y)w - \Psi(x)v]\partial_v\phi(x, v) dx dy dv dw + O(\varepsilon^2). \end{aligned}$$

We are now in a position to derive a Fokker-Planck equation.

**4.1. Derivation of the Fokker-Planck equation.** We first introduce some other notations:

$$\begin{aligned} \varepsilon\hat{D}(x, y, w) &= \varepsilon\hat{\kappa}(w)(y - x) = x' - x, \\ \varepsilon\hat{E}(x, y, v, w) &= \varepsilon\hat{\gamma}(\Psi(y)w - \Psi(x)v) = v' - v. \end{aligned}$$

The Taylor expansions of  $\phi$  then read:

$$\begin{aligned} \phi(x', v) - \phi(x, v) &= \varepsilon\hat{D}(x, y, w)\partial_x\phi(x, v) + O(\varepsilon^2), \\ \phi(x, v') - \phi(x, v) &= \varepsilon\hat{E}(x, y, v, w)\partial_v\phi(x, v) + O(\varepsilon^2). \end{aligned}$$

We consider the scaling  $\tau = \varepsilon t$  and let  $f(t, x, v) = f_\varepsilon(\tau, x, v)$ . Putting together (9) and (10) we obtain that  $f_\varepsilon$  satisfies

$$(11) \quad \begin{aligned} \varepsilon \frac{d}{d\tau} \iint_{\mathbb{R}_+^2} \phi(x, v) f_\varepsilon(\tau, x, v) dx dv &= \frac{d}{dt} \iint_{\mathbb{R}_+^2} \phi(x, v) f(t, x, v) dx dv \\ &= \varepsilon \iint_{\mathbb{R}_+^4} f_\varepsilon(\tau, x, v) f_\varepsilon(\tau, y, w) \hat{D}(x, y, w) \partial_x \phi(x, v) dx dy dv dw \\ &+ \varepsilon \iint_{\mathbb{R}_+^4} f_\varepsilon(\tau, x, v) f_\varepsilon(\tau, y, w) \hat{E}(x, y, v, w) \partial_v \phi(x, v) dx dy dv dw, \end{aligned}$$

which implies

$$(12) \quad \frac{d}{d\tau} \iint_{\mathbb{R}_+^2} \phi(x, v) f_\varepsilon(\tau, x, v) dx dv = \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau, x, v) \left( \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau, y, w) \left[ \hat{E}(x, y, v, w) \partial_v \phi(x, v) + \hat{D}(x, y, w) \partial_x \phi(x, v) \right] dw dy \right) dv dx.$$

Let us now define

$$(13) \quad \mathcal{E}(\tau, x, v) = \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau, y, w) \hat{E}(x, y, v, w) dw dy,$$

$$(14) \quad \mathcal{D}(\tau, x) = \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau, y, w) \hat{D}(x, y, w) dw dy.$$

Letting  $\varepsilon$  go to 0 and denoting by  $h$  the formal limit of  $f_\varepsilon$ , we get:

$$\frac{d}{d\tau} \iint_{\mathbb{R}_+^2} \phi(x, v) h(\tau, x, v) dx dv = \iint_{\mathbb{R}_+^2} \left[ \mathcal{E}(\tau, x, v) \partial_v \phi(x, v) + \mathcal{D}(\tau, x) \partial_x \phi(x, v) \right] h(\tau, x, v) dv dx.$$

This eventually gives the following Fokker-Planck equation

$$(15) \quad \frac{\partial h}{\partial \tau} = \gamma \frac{\partial}{\partial v} (\mathcal{M}_W h) + \frac{\partial}{\partial x} (\mathcal{M}_K h),$$

where we set

$$(16) \quad \mathcal{M}_W(\tau, x, v) = \iint_{\mathbb{R}_+^2} [\Psi(x)v - \Psi(y)w] h(\tau, y, w) dy dw,$$

$$(17) \quad \mathcal{M}_K(\tau, x) = \iint_{\mathbb{R}_+^2} \kappa(w)(x - y) h(\tau, y, w) dy dw$$

and with initial datum  $h(x, v, 0) = h_0(x, v)$ .

We observe that  $\mathcal{M}_W$  and  $\mathcal{M}_K$  allow to define the fluxes in the Fokker-Planck equation. As we already pointed out, the Boltzmann equation (6) does not require any boundary condition on  $f$ . On the contrary, for the Fokker-Planck equation (15), which is written as a conservation law with fluxes, it is mandatory to impose boundary condition on  $h$ . In order to take into account the fact that  $h$  must remain compactly supported (as  $f$ ), it is natural to choose

$$\begin{aligned} \mathcal{M}_K(\tau, x) h(\tau, x, v) &= 0 \text{ on } x = 0 \text{ and } x = X, \forall v, \forall \tau \\ \mathcal{M}_W(\tau, x, v) h(\tau, x, v) &= 0 \text{ on } v = 0 \text{ and } v = V, \forall x, \forall \tau. \end{aligned}$$

The values  $X$  and  $V$  of course depend on the initial datum  $f^{\text{in}}$  of  $h$ .

For the hypothesis on the function  $\kappa$ , the boundary condition for the term  $\mathcal{M}_K$  reduce to a Dirichlet boundary condition, *i.e.*  $h = 0$  on  $x = 0$  and  $x = X$  and we observe that this choice is consistent to Remark 1. The choice of the boundary conditions for the term  $\mathcal{M}_W$  is a bit more delicate since the support of the function is conserved only at the whole population level. In order to prevent the variable  $v$  to leave the domain  $[0, V]$  we need to choose a quite big value for  $V$  and we only consider initial data such that  $h(x, 0) = 0$  at the initial time  $t = 0$ .

In the case with a constant value for  $\Psi$ , then we can also reduce to homogeneous Dirichlet boundary condition, *i.e.*  $h = 0$  on  $v = 0$  and  $v = V$  and on  $x = 0$  and  $x = X$ .

**4.2. Study of the moments.** In this subsection, we make some specific assumptions on the functions  $\kappa$  and  $\Psi$  to be able to investigate the behaviour of the zero-th and first-order moment of  $h$ . They both have a physical meaning: the zero-th-order moment corresponds to the number of individuals we consider; the first-order moment with respect to  $x$  corresponds to the mean knowledge and the first-order moment with respect to  $v$  to the mean wealth. The study of the moments is relevant to help understanding the steady states.

Let us define the moment of order  $i$  in  $x$  and  $j$  in  $v$ :

$$M_{ij}(\tau) = \iint_{\mathbb{R}_+^2} x^i v^j f(\tau/\varepsilon, x, v) dx dv.$$

In the literature these crossed moments are usually called bivariate moments and they are especially used for numerical resolution methods as in [11]. Another example is [32] where the bivariate moments are used for studying the deformation of the shape of particles (and they also allow to have rational exponents  $i$  and  $j$ ).

Note that the moment  $M_{00}$  is always conserved since it corresponds to the initial mass.

First, let us consider constant parameter functions, for which the following result holds.

**Proposition 6.** *Set  $\kappa = k_1$  and  $\Psi = k_2$ , where  $k_1, k_2$  are positive constants. Then  $M_{00} = m_{00}$ ,  $M_{01} = m_{01}$  and  $M_{10} = m_{10}$  are constants. Moreover*

$$(18) \quad M_{11}(\tau) = \frac{m_{10}m_{01}}{m_{00}} + \left( M_{11}(0) - \frac{m_{10}m_{01}}{m_{00}} \right) \exp[-m_{00}(k_1 + \gamma k_2)\tau].$$

The moment  $M_{11}$  can be seen as the correlation between the two variables  $x$  and  $v$ . Given two random variables  $X$  and  $Y$ , we say that they are not correlated (or independent) if  $E(XY) = E(X)E(Y)$ . This is exactly what happens when taking constant parameter functions  $\kappa$  and  $\Psi$ , since we obtain that  $M_{11} = \frac{m_{10}m_{01}}{m_{00}}$  when  $t$  goes to  $+\infty$ , and the independence is easily understood just looking at collision rules (1) and (3).

*Proof.* We can rewrite (9), for any admissible test function  $\phi$ , as

$$\begin{aligned} \langle Q_K(f_\varepsilon, f_\varepsilon), \phi \rangle &= k_1 \left( \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, x, v) \partial_x \phi(x, v) dx dv \iint_{\mathbb{R}_+^2} y f_\varepsilon(\tau/\varepsilon, y, w) dy dw \right. \\ &\quad \left. - \iint_{\mathbb{R}_+^2} x f_\varepsilon(\tau/\varepsilon, x, v) \partial_x \phi(x, v) dx dv \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, y, w) dy dw \right) \\ &= k_1 \left( M_{10} \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, x, v) \partial_x \phi(x, v) dx dv - m_{00} \iint_{\mathbb{R}_+^2} x f_\varepsilon(\tau/\varepsilon, x, v) \partial_x \phi(x, v) dx dv \right), \end{aligned}$$

and (10) as

$$\begin{aligned} \langle Q_W(f_\varepsilon, f_\varepsilon), \phi \rangle &= \gamma k_2 \left( \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, x, v) \partial_v \phi(x, v) dx dv \iint_{\mathbb{R}_+^2} w f_\varepsilon(\tau/\varepsilon, y, w) dy dw \right. \\ &\quad \left. - \iint_{\mathbb{R}_+^2} v f_\varepsilon(\tau/\varepsilon, x, v) \partial_v \phi(x, v) dx dv \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, y, w) dy dw \right) \\ &= \gamma k_2 \left( M_{01} \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, x, v) \partial_v \phi(x, v) dx dv - m_{00} \iint_{\mathbb{R}_+^2} v f_\varepsilon(\tau/\varepsilon, x, v) \partial_v \phi(x, v) dx dv \right). \end{aligned}$$

In the previous equalities, the quantity

$$m_{00} = \iint_{\mathbb{R}_+^2} f_\varepsilon(\tau/\varepsilon, x, v) dx dv$$

is constant.

By letting  $\varepsilon \rightarrow 0^+$  and for  $\tau \geq 0$ , we get:

$$\begin{aligned} \frac{d}{d\tau} \iint_{\mathbb{R}_+^2} h(\tau, x, v) \phi(x, v) dx dv &= \\ &= k_1 M_{10} \iint_{\mathbb{R}_+^2} h(\tau, x, v) \partial_x \phi(x, v) dx dv + \gamma k_2 M_{01} \iint_{\mathbb{R}_+^2} h(\tau, x, v) \partial_v \phi(x, v) dx dv \\ &\quad - m_{00} \iint_{\mathbb{R}_+^2} h(\tau, x, v) (k_1 x \partial_x \phi(x, v) + \gamma k_2 v \partial_v \phi(x, v)) dx dv. \end{aligned}$$

Of course  $h$  still satisfies the mass conservation property. The first moment with respect both to  $x$  and  $v$  are constants, *i.e.*  $M_{10}(\tau) = M_{10}(0) = m_{10}$  and  $M_{01}(\tau) = M_{01}(0) = m_{01}$ .

Then, choosing  $\phi(x, v) = xv$ , we get

$$\frac{dM_{11}}{d\tau} = (k_1 + \gamma k_2)(M_{10}M_{01} - m_{00}M_{11}) = (k_1 + \gamma k_2)(m_{10}m_{01} - m_{00}M_{11})$$

which allows to obtain an explicit expression (18) for  $M_{11}(\tau)$  for any  $\tau$ . This moment somehow represents the average wealth weighted by the knowledge.  $\square$

From a numerical viewpoint, let us perform a test with an initial datum with support in  $(x, v) \in [0.7, 0.9] \times [0.2, 0.5] \cup [0.1, 0.3] \times [1.5, 1.8]$ . An average on 30 simulations gives

$$M_{11}(0) = \iint_{\mathbb{R}_+^2} xv f^{\text{in}}(t, x, v) dx dv \simeq 0.308.$$

We can also compute the asymptotical value

$$\lim_{\tau \rightarrow \infty} M_{11}(\tau) = \frac{m_{10}m_{01}}{m_{00}} \simeq 0.505.$$

In Figure 5 we numerically show the evolution of the moment  $M_{11}$  computed from (18) (in red) and obtained for the Boltzmann equation (in blue) with respect to time, with  $k_1 = 0.25$ ,  $k_2 = 0.6$ ,  $\gamma = 0.5$ ,  $\varepsilon = 0.1$  and 10000 agents. The value of  $M_{11}(\tau)$  (line in blue) is obtained from an average on 30 simulations.

The second moments with respect to  $x$  and  $v$ , *i.e.*  $M_{20}$  and  $M_{02}$  can also be computed and they are both exponentially decreasing. More precisely, we have

$$\frac{d}{d\tau} M_{02} = C_1 m_{01}^2 - C_2 m_{00} M_{02}$$

with  $C_1 = 2k_2\gamma \geq 0$  and  $C_2 = 2k_2\gamma \geq 0$  and

$$\frac{d}{d\tau} M_{20} = C_3 m_{10}^2 - C_4 m_{00} M_{20}$$

with  $C_3 = 2k_1 \geq 0$  and  $C_4 = 2k_1 \geq 0$ , which again allows to obtain explicit values for  $M_{02}$  and  $M_{20}$ .



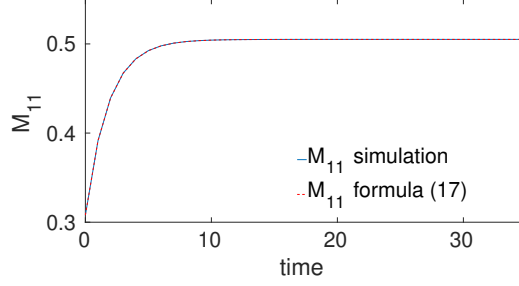


FIGURE 5. Behaviour of the moment  $M_{11}$  for the Boltzmann equation (in blue) and computed from the formula (18) (in red).  $(x, v) \in [0.7, 0.9] \times [0.2, 0.5] \cup [0.1, 0.3] \times [1.5, 1.8]$ .

In this framework, we can find an explicit expression of  $h$  thanks to the characteristics method. Of course, this kind of situation is quite simple, but it allows to analytically prove convergence of the distribution function towards a Dirac mass in large time.

Let consider (15). With the choice of constant functions  $\kappa = k_1$  and  $\Psi = k_2$ , we get

$$(19) \quad \partial_t h = (\gamma k_2 + k_1) m_{00} h + k_2 (v m_{00} - m_{01}) \partial_v h + k_1 (x m_{00} - m_{10}) \partial_x h.$$

where  $m_{00}$ ,  $m_{10}$  and  $m_{01}$  are respectively the mass, the first moment with respect to  $x$  and the first moment with respect to  $v$  and they are constants.

Equation (19) leads to an explicit expression of  $h$ , which reads

$$h(\tau, x, v) = e^{(\gamma k_2 + k_1) m_{00} \tau} f^{\text{in}}(\xi, \zeta)$$

where

$$\xi = (x - m_{10}/m_{00}) e^{k_1 m_{00} \tau} + m_{10}/m_{00}, \quad \zeta = (v - m_{01}/m_{00}) e^{\gamma k_2 m_{00} \tau} + m_{01}/m_{00}.$$

Note that  $\xi$  and  $\zeta$  are respectively convex combinations of  $x$  and  $m_{10}/m_{00}$ , and  $v$  and  $m_{01}/m_{00}$ , which imply that they belong to  $\mathbb{R}_+^*$  if  $x$  and  $v$  do too, and clearly conversely.

We can also prove the convergence of  $h$  towards the Dirac mass centred in  $(\bar{x}, \bar{v})$  with  $\bar{x} = m_{10}/m_{00}$  and  $\bar{v} = m_{01}/m_{00}$  in large time. Indeed, let us study the limit, when  $\tau$  goes to  $+\infty$ , of

$$\iint_{(\mathbb{R}_+)^2} h(\tau, x, v) \varphi(x, v) dx dv - \varphi(\bar{x}, \bar{v})$$

for any  $C^\infty$  function  $\varphi$  with compact support in  $(\mathbb{R}_+^*)^2$ . With the change of variable  $(x, v) \mapsto (\xi, \zeta)$ , the previous integral becomes

$$\iint_{(\mathbb{R}_+)^2} \left[ \varphi \left( \left( \xi - \frac{m_{10}}{m_{00}} \right) e^{-k_1 m_{00} \tau} + \frac{m_{10}}{m_{00}}, \left( \zeta - \frac{m_{01}}{m_{00}} \right) e^{-\gamma k_2 \tau} + \frac{m_{01}}{m_{00}} \right) - \varphi(\bar{x}, \bar{v}) \right] f^{\text{in}}(\xi, \zeta) d\xi d\zeta.$$

Letting  $\tau$  go to  $+\infty$ , by dominated convergence, we eventually obtain the following value of the limit, for any  $\varphi$ :

$$\iint_{(\mathbb{R}_+)^2} f^{\text{in}}(\xi, \zeta) \left[ \varphi \left( \frac{m_{10}}{m_{00}}, \frac{m_{01}}{m_{00}} \right) - \varphi(\bar{x}, \bar{v}) \right] d\xi d\zeta = 0.$$

Now let us consider a linear expression for the functions  $\kappa$  and  $\Psi$ . In that case,  $M_{00} = m_{00}$  and  $M_{01} = m_{01}$  are still constants, but  $M_{01}$  depends on moments of higher order:

$$\frac{d}{d\tau} M_{11}(\tau) = \gamma c_2 (M_{10} M_{11} - m_{01} M_{21}).$$

Hence, the only moments we can exactly compute are  $m_{00}$  and  $m_{01}$ . But, performing some numerical simulations, we observe the same behaviour as in Figure 5 for  $M_{11}$ . In particular, we highlight that the moments do not depend on the choice of  $\kappa$  and  $\Psi$ , but only on the initial distribution of the individuals. Indeed the expression to compute the moments does not depend on the choice of the functions. Consequently, if we suppose, as suggested from the numerical simulations, that

$$\lim_{\tau \rightarrow +\infty} M_{11} = m_{11},$$

then we can also obtain an expression for  $M_{21} = \frac{M_{10} m_{11}}{m_{01}}$  for  $\tau$  going to  $+\infty$  where we can write an explicit expression for  $M_{10}$ .

For more results on the derivation of Fokker-Planck equations and the large-time behaviour, the reader can refer to [19].

**4.3. Numerical comparisons between the Boltzmann and Fokker-Planck equations.** Since we want to compare the Boltzmann equation with the Fokker-Planck equation, we need then to introduce a discretization for the second one.

As we did for the Boltzmann equation, also here we assume that  $x$  lies in  $[0, X]$  with  $X = 1$  without loss of generality. For the wealth variable, we recall that it is conserved at the whole-population level and we defined  $V = \sum_{p=1}^{2N} v_p(0)$  the total wealth of the society.

For the implementation of the Fokker-Planck equation (15), we use an explicit Euler finite-volume scheme. Again we have a two-dimensional model and we want the mass to remain conserved. Equation (15) is posed in the phase space domain  $(x, v) \in [0, 1] \times [0, V]$ . We choose uniform subdivisions  $(x_0, \dots, x_l)$  for the variable  $x$  and  $(v_0, \dots, v_m)$  for the variable  $v$ . Since the Boltzmann simulations has been done for  $v \in [0, 2]$ , we can take  $V = 2$ , and we choose  $\ell = m = 100$  in the simulations. When  $x \in (x_i, x_{i+1})$ ,  $v \in (v_j, v_{j+1})$  at time  $t_n = n\Delta t$ , the value  $h(t, x, v_n)$  is approximated by

$$h_{ij}^n = \frac{1}{\Delta x \Delta v} \iint_{C_i \times H_j} h(t, x, v^n) dx dv$$

with  $C_i \times H_j = (x_i, x_{i+1}) \times (v_j, v_{j+1})$ .

We consider the evolution of the support when we only have one collision operator, *i.e.* only the changes with respect to the knowledge or only the changes with respect to the wealth. This test has been done for the Boltzmann equation and for the Fokker-Planck equation by rescaling the parameters.

In Figure 6, we show the evolution of the support of the distribution function for the Fokker-Planck equation (blue) and the Boltzmann equation (red) when we only have knowledge collisions ( $\Psi \equiv 0$ ). We present the results starting from two different uniform initial data. In particular we have chosen  $\kappa = 0.25$ ,  $\varepsilon = 0.1$  and 10000 agents for the particle code. On the left, evolution of the support starting from a uniform initial data for  $x \in [0.2, 0.875]$  and  $v \in [0, 2]$ ; on the right, evolution of the support starting from an uniform initial data for  $x \in [0.235, 0.485]$  and  $v \in [0, 2]$ .

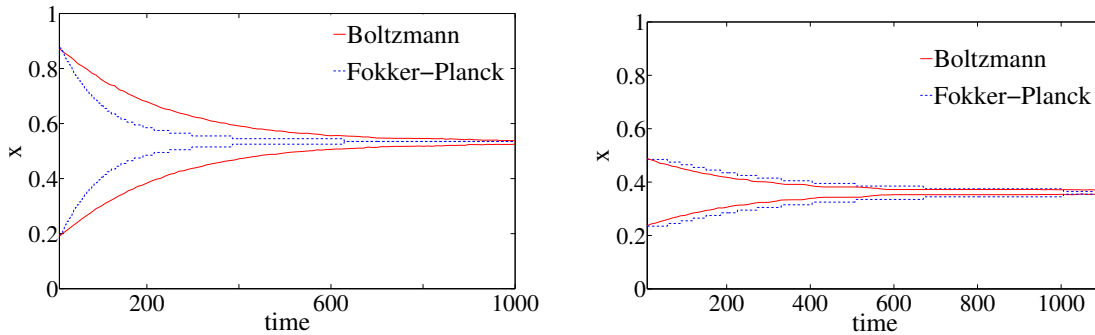


FIGURE 6. Evolution of the support for  $h$  (in blue) and for  $f$  (in red) where we only have knowledge interactions starting from two different initial data.

In Figure 7, we show the evolution of the support for the Fokker-Planck equation (blue) and the Boltzmann equation (red) when we only have wealth collisions (so by taking  $\kappa \equiv 0$ ). We took  $\Psi = 0.6$ ,  $\gamma = 0.5$ ,  $\varepsilon = 0.1$  and 10000 agents for the kinetic code. The initial datum was given for  $v \in [0.4, 1.38]$  and  $x \in [0, 1]$ .

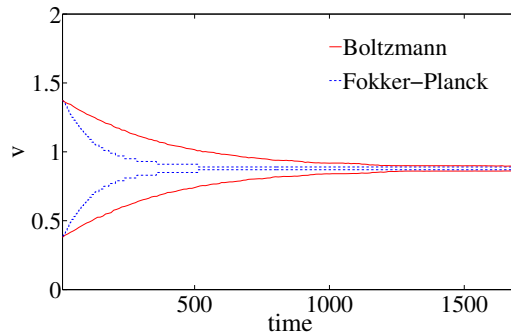


FIGURE 7. Evolution of the support for  $h$  (in blue) and for  $f$  (in red) where we only have wealth interactions.

**4.4. Quasi-invariant knowledge.** The knowledge exchange seems to be the key mechanism of the whole process, in the sense that there is no way to improve its wealth without information. In this subsection, we consider the quasi-invariant knowledge case, *i.e.* we assume that the function  $\kappa$  which appears in the knowledge collision rule (1) is of order  $\varepsilon$  with  $0 < \varepsilon \ll 1$ . The collisions are then only given by (7). From the modelling viewpoint, that means that there are few exchanges of information inside the population.

Let us first perform an experiment with a piecewise constant initial datum  $f^{\text{in}}$ . We choose  $\gamma = 0.21$ ,  $\alpha = 0.05$  in the expression of  $\tilde{\kappa}$ , and  $\varepsilon = 0.1$ . We already saw in Table 1 and 2 the role of  $\gamma$  and  $\kappa$  on the speed of convergence and we observed that, for example, this choice for  $\gamma$  is not too weak and not too strong. Figure 8 shows the evolution of the the distribution function  $f$  for one numerical simulation.

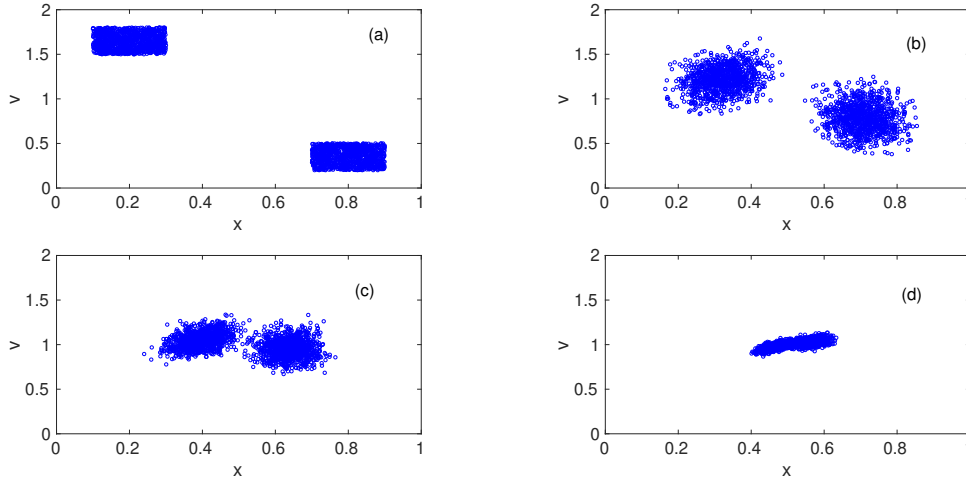


FIGURE 8. Quasi-invariant knowledge with  $\Psi(x) = 1/(1+x)$ : time evolution of  $f$  at times (a)  $t = 0$ , (b)  $t = 5$ , (c)  $t = 10$ , (d)  $t = 20$ .

The average on 50 simulations for the Boltzmann equation (6) at time  $t = 20$  is presented in Figure 9. We observe that also in this case we get a similar behaviour for  $f$  as in Figure 1b: depending on the value of  $\varepsilon$ , before the convergence to a delta, the support of  $f$  firstly concentrate along the quasi-steady-state for the wealth, where the richest have more knowledge, and the poorest have less knowledge. This behaviour, as we saw in Section 3.2, is also due to the choice of the different parameters.

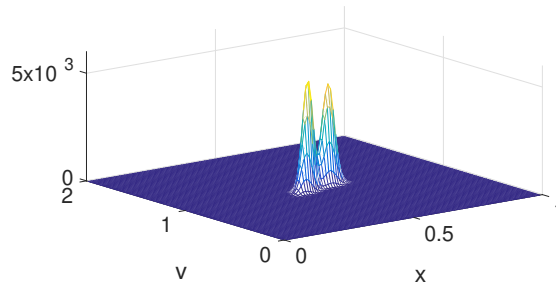


FIGURE 9. Quasi-invariant knowledge, with  $\Psi(x) = 1/(1+x)$ : distribution function at time  $t = 20$  averaged on 50 simulations.

Let us consider two agents  $(x, v)$  and  $(y, w)$ . They exchange their wealth according to the rule (3). Setting  $\Psi(x) = (1+x)^{-\beta}$  for any  $x$ , we can rewrite the post-interaction value for the first agent  $v'$  as

$$v' = v + \gamma \left[ \frac{w}{(1+y)^\beta} - \frac{v}{(1+x)^\beta} \right].$$

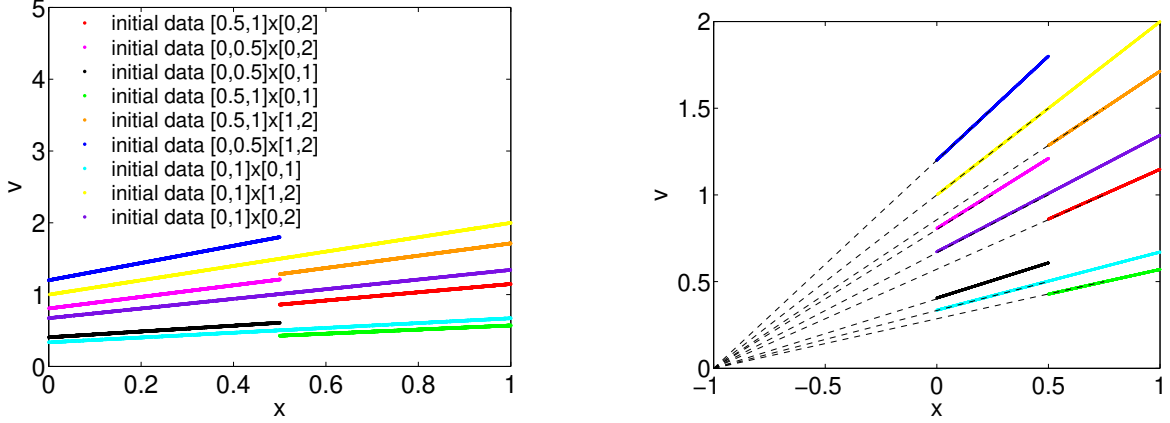


FIGURE 10. Distribution functions obtained with different initial data at large time, with  $\Psi(x) = 1/(1+x)$ .

The stationary state is reached when the post-interaction value  $v'$  corresponds to the pre-interaction value  $v$ , *i.e.*  $v = v'$ . In this case we obtain

$$\frac{v}{(1+x)^\beta} = \frac{w}{(1+y)^\beta}$$

which is a curve  $\Gamma_D$  of equation  $v = D(x+1)^\beta$ , where  $D$  is the average value, *i.e.*

$$D = \frac{1}{XW} \int_0^X \int_0^W \frac{w}{(1+y)^\beta} dy dw$$

is a constant positive value depending on the initial distribution of the individuals. If two agents  $(x, v)$  and  $(y, w)$  are located on that curve, then it is easy to notice that

$$v' = v + \gamma \left[ \frac{D(y+1)^\beta}{(1+y)^\beta} - \frac{D(x+1)^\beta}{(1+x)^\beta} \right] = v, \quad w' = w$$

*i.e.* the curve  $\Gamma_D$  is the steady state for the wealth collision, thus invariant. In particular we have an invariant curve for any admissible function  $\Psi$  which has the expression  $v = D/\Psi(x)$ .

In Figure 10 and 11 are presented the invariant curves  $\Gamma_D$  of equation  $v = D/\Psi(x)$  for different values of  $\beta$ .

We highlight that in Figures 10–11, the curves obtained are steady states when we consider only wealth collisions. While, in the full quasi-invariant case, they are intermediate states and we already observed that for the final time the density converges to a Dirac mass. This is due to the fact that the quasi-invariant knowledge case corresponds somehow to the case when we have only wealth collisions for a vanishing  $\varepsilon$ . This reflects the fact that if the exchange of knowledge is negligible, we obtain a society where people who know a lot are rich and the people with less knowledge are poor, *i.e.* a society with inequalities.

A natural step is to check the convergence rate to this steady state. We know the equation for the invariant curve from the initial distribution of the individuals so we can numerically compute the average distance of all the individuals from  $\Gamma_D$  at each time step. The idea is the following: from the initial distribution we compute the equation of  $\Gamma_D$  and, for each time step and each value  $v$ , we compute the distance for each individual from  $\Gamma_D$ .

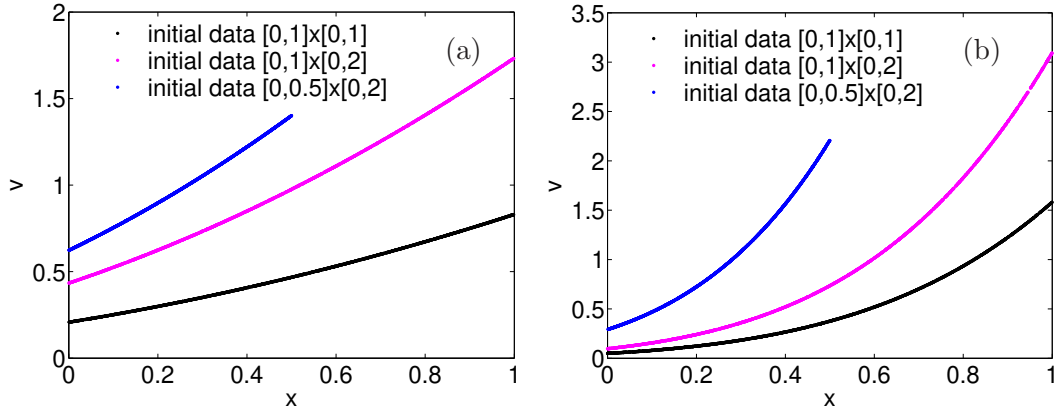


FIGURE 11. Distribution functions obtained with different initial data with large time,  $\kappa = 0.34$  and (a)  $\beta = 2$ , (b)  $\beta = 5$ .

We define a *distance* function which is computed in the following way

$$d(t) = \sum_{p=1}^{2N} \left| v_p(t) - \frac{D}{\Psi(x_p(t))} \right|.$$

The idea is to compute the distance between the wealth value  $v_p$  of each individual and the wealth value  $v$  corresponding to the knowledge of the given individual  $x_p$  on  $\Gamma_D$  and to sum together all over the individuals.

Figure 12, allows to highlight an exponential convergence of  $d(t)$  to 0 when  $t \rightarrow \infty$ .

The evolution with respect to time of the function distance has been obtained starting with initial data  $(x, v) \in [0, 1] \times [0, 2]$  and  $\kappa = 0.52$ ,  $\Psi = \frac{1}{1+x}$ ,  $\varepsilon = 0.1$ ,  $\gamma = 0.21$ . The picture is an average obtained from 50 simulations.

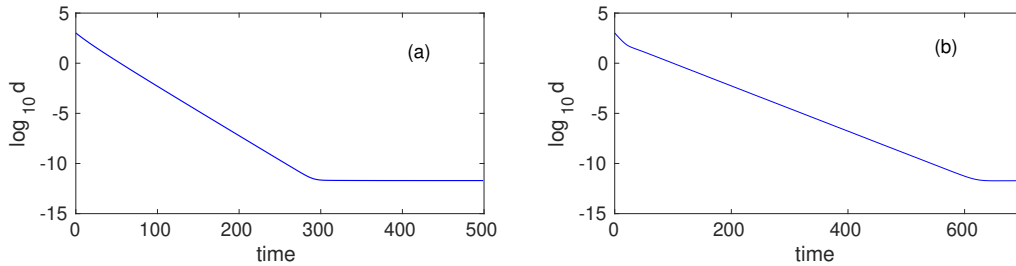


FIGURE 12. Evolution with respect to time of the distance function: (a) convergence only with wealth interaction; (b) convergence for the quasi-invariant knowledge case.

## APPENDIX A. WELL-POSEDNESS OF THE PROBLEM

This appendix is dedicated to the statement and proof of the existence of solutions to the Boltzmann equation (6).

Let us set, for any  $w > 0$ ,

$$D_K(w) = \{(x, x') \in \mathbb{R}^2 \mid 0 < (1 - \kappa(w))x < x'\} \subset \mathbb{R}_+^{*2}$$

and, for any  $x > 0$ ,

$$D_W(x) = \{(v, v') \in \mathbb{R}^2 \mid 0 < (1 - \gamma\Psi(x))v < v'\} \subset \mathbb{R}_+^{*2}.$$

It is then easy to check that the transformations  $(x, y) \mapsto (x, x')$  for a fixed  $w > 0$  and  $(v, w) \mapsto (v, v')$  for a fixed  $x > 0$  are bijections, respectively  $D_K(w) \rightarrow \mathbb{R}_+^{*2}$  and  $D_W(x) \rightarrow \mathbb{R}_+^{*2}$ . Both weak forms (4)–(5) can be written as the difference between the weak form of gain terms  $Q_K^+(f, f)$ ,  $Q_W^+(f, f)$ , and loss terms  $Q_K^-(f, f)$ ,  $Q_W^-(f, f)$ , which do not use the post-collisional variables at all. More precisely, we have, for any test-function  $\phi$ ,

$$(20) \quad \langle Q_K^+(f, f), \phi \rangle = \iint_{\mathbb{R}_+^{*4}} \frac{\nu_K \mathbb{1}_{D_K(w)}(x', x)}{\kappa(w)} f(t, x', v) f\left(t, \frac{x - (1 - \kappa(w))x'}{\kappa(w)}, w\right) \phi(x, v) dx dx' dw dv,$$

$$(21) \quad \langle Q_W^+(f, f), \phi \rangle = \iint_{\mathbb{R}_+^{*4}} \frac{\nu_W \mathbb{1}_{D_W(x)}(v', v)}{\gamma\Psi(y)} f(t, x, v') f\left(t, y, \frac{v - ((1 - \gamma\Psi(x))v')}{\gamma\Psi(y)}\right) \phi(x, v) dv dv' dy dx,$$

$$(22) \quad \langle Q_K^-(f, f), \phi \rangle = \nu_K \iint_{\mathbb{R}_+^{*4}} f(t, x, v) f(t, y, w) \phi(x, v) dx dy dv dw,$$

$$(23) \quad \langle Q_W^-(f, f), \phi \rangle = \nu_W \iint_{\mathbb{R}_+^{*4}} f(t, x, v) f(t, y, w) \phi(x, v) dx dy dv dw,$$

where  $\mathbb{1}_E$  denotes the characteristic function of any subset  $E$  of  $\mathbb{R}_+^{*2}$ . The gain terms quantify the exchanges of knowledge/wealth between individuals which produce, after the interaction with another agent, an agent  $(x, v)$ . The loss terms take into account the exchanges of knowledge/wealth where an agent  $(x, v)$  is involved before the collisional process.

Note that the collision rules and operators about the knowledge variable do not imply the possibility of a time delay in the learning process. The way how the agents gather knowledge is an intricate process and modelling it remains difficult.

We first need a priori estimates on the collision operators.

**Lemma 7.** *Assume that  $\Psi$  is lower-bounded by a constant  $\delta > 0$ . Let  $g \in L^1(\mathbb{R}_+^{*2})$ . Then  $Q_K^\pm(g, g)$  and  $Q_W^\pm(g, g)$  also lie in  $L^1(\mathbb{R}_+^{*2})$ , and the following estimates hold:*

$$\begin{aligned} \|Q_K^+(g, g)\|_{L^1(\mathbb{R}_+^{*2})} &\leq \frac{\nu_K}{\alpha} \|g\|_{L^1(\mathbb{R}_+^{*2})}^2, & \|Q_K^-(g, g)\|_{L^1(\mathbb{R}_+^{*2})} &\leq \nu_K \|g\|_{L^1(\mathbb{R}_+^{*2})}^2, \\ \|Q_W^+(g, g)\|_{L^1(\mathbb{R}_+^{*2})} &\leq \frac{\nu_W}{\gamma\delta} \|g\|_{L^1(\mathbb{R}_+^{*2})}^2, & \|Q_W^-(g, g)\|_{L^1(\mathbb{R}_+^{*2})} &\leq \nu_W \|g\|_{L^1(\mathbb{R}_+^{*2})}^2. \end{aligned}$$

*Proof.* This lemma is a straightforward consequence of (20)–(23) with  $\phi \equiv 1$  and Prop. 4.  $\square$

**Remark 8.** *The additional hypothesis on  $\Psi$  in Lemma 7 is not that limiting. It is enough, for instance, to replace it by an assumption stating that  $g$  is compactly supported in  $x$ , see Remark 1. That implies that  $\Psi$  is straightforwardly lower bounded by  $\delta$  on that compact support, since  $\Psi$  is continuous.*

Let us now conclude the proof of Theorem 5. We follow the same kind of strategy as in [4], but without any diffusion term. Note that the proof arguments are not exactly the same as in [4]. The idea of the proof is the following. We introduce a sequence of functions  $(f^n)$ , defined by induction, for which we prove existence, nonnegativity and monotonicity. The existence of each  $f^n$  is straightforward. In order to prove the nonnegativity, we multiply the equation satisfied by  $f^n$  by a relevant time exponential term. This allows to prove nonnegativity of the sequence. For the non-decreasing property, we use an induction argument. Then these properties ensure the existence of a function  $f$  such that the sequence  $(f^n)$  converges towards  $f$  almost everywhere. The last part of the proof exactly follows [4].

Let us now proceed with the proof itself. Set

$$\rho = \int_{\mathbb{R}_+^{*2}} f^{\text{in}}(x, v) \, dx \, dv.$$

and define  $(f^n)_{n \in \mathbb{N}}$  by induction with  $f^0 \equiv 0$ , solving, for any  $t \in [0, T]$ ,

$$(24) \quad \partial_t f^{n+1} + \sigma f^{n+1} = Q_K^+(f^n, f^n) + Q_W^+(f^n, f^n),$$

with the initial condition  $f^{n+1}(0, \cdot, \cdot) = f^{\text{in}}$ , where we set  $\sigma = \rho(\nu_K + \nu_W) > 0$ . Existence of solutions to (24) is straightforward, since  $f^{n+1}$  is not involved in the right-hand side of (24) (which we know is in  $L^1(\mathbb{R}_+^{*2})$ ), and we also have  $f^{n+1} \in C^0([0, T]; L^1(\mathbb{R}_+^{*2}))$ .

Let us first prove by induction that  $f^n$  is nonnegative for any  $n$ . It is clear for  $n = 0$ . Assume now that  $f^n \geq 0$ . We want to prove that  $f^{n+1} \geq 0$ . From (24), we immediately get

$$\int_{\mathbb{R}_+^{*2}} \partial_t f^{n+1} \phi \, dx \, dv + \sigma \int_{\mathbb{R}_+^{*2}} f^{n+1} \phi \, dx \, dv \geq 0$$

for any nonnegative test-function  $\phi$ . Multiplying by  $e^{\sigma t}$  allows to prove that, for any nonnegative test-function  $\phi$ ,

$$\int_{\mathbb{R}_+^{*2}} f^{n+1} \phi \, dx \, dv \geq e^{-\sigma t} \int_{\mathbb{R}_+^{*2}} f^{\text{in}} \phi \, dx \, dv \geq 0,$$

which leads to the non-negativity of  $f^{n+1}$ .

In the same way, we can prove that  $(f^n)$  is non-decreasing. We of course have  $f^1 \geq f^0 \equiv 0$ . Suppose that  $f^n \geq f^{n-1}$ , for a given  $n \geq 0$ . The difference  $f^{n+1} - f^n$  satisfies the following equation, for any  $\phi$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+^{*2}} \partial_t (f^{n+1} - f^n) \phi \, dx \, dv + \sigma \int_{\mathbb{R}_+^{*2}} (f^{n+1} - f^n) \phi \, dx \, dv \\ &= \langle Q_K^+(f^n, f^n), \phi \rangle - \langle Q_K^+(f^{n-1}, f^{n-1}), \phi \rangle + \langle Q_W^+(f^n, f^n), \phi \rangle - \langle Q_W^+(f^{n-1}, f^{n-1}), \phi \rangle. \end{aligned}$$

The right-hand side of the previous equality is nonnegative because  $f^n \geq f^{n-1}$ . Consequently, we can write, for any nonnegative  $\phi$ ,

$$\frac{d}{dt} \left( e^{\sigma t} \int_{\mathbb{R}_+^{*2}} (f^{n+1} - f^n) \phi \, dx \, dv \right) \geq 0.$$



Noticing that the initial datum for  $f^{n+1} - f^n$  is zero, that allows to conclude that  $(f^n)$  is non-decreasing. In particular, that ensures that

$$(25) \quad \int_{\mathbb{R}_+^{*2}} f^n \, dx \, dv \leq \int_{\mathbb{R}_+^{*2}} f^{n+1} \, dx \, dv.$$

We can then prove, by induction, that

$$(26) \quad \int_{\mathbb{R}_+^{*2}} f^n \, dx \, dv \leq \rho.$$

We can write, from (24), thanks to the invariance properties of  $Q_K(f^n, f^n)$  and  $Q_W(f^n, f^n)$ ,

$$\frac{d}{dt} \left( \int_{\mathbb{R}_+^{*2}} f^{n+1} \, dx \, dv \right) = (\nu_K + \nu_W) \left[ \left( \int_{\mathbb{R}_+^{*2}} f^n \, dx \, dv \right)^2 - \rho \int_{\mathbb{R}_+^{*2}} f^{n+1} \, dx \, dv \right].$$

Using (25) and the inductive hypothesis, we observe that the right-hand side of the previous equality is non-positive, which allows to recover (26) for  $f^{n+1}$ .

Because of the monotonicity and the uniform bound (26) of  $(f^n)$ , the monotone convergence theorem ensures the existence of  $f \in L^\infty(0, T; L^1(\mathbb{R}_+^{*2}))$  such that  $(f^n)$  converges towards  $f$  almost everywhere, and in  $L^1(\mathbb{R}_+^{*2})$ , for almost every  $t$ .

We conclude that  $f$  solves (6) in the distributional sense in time and in a weak sense in  $L^1(\mathbb{R}_+^{*2})$ , exactly in the same way as in [4].

#### ACKNOWLEDGEMENTS

LT wants to thank the colleagues from the Research Centre INRIA in Paris (France) for all the scientific discussions and for the great atmosphere during her staying in Paris.

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