Qualitative properties of solutions to semilinear elliptic equations from the gravitational Maxwell Gauged O(3) Sigma model

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Abstract

This article is devoted to the study of the following semilinear equation with measure data which originates in the gravitational Maxwell gauged $O(3)$ sigma model,

$$(E) \qquad -\Delta u + A_0(\prod_{j=1}^{k}|x-p_j|^{2n_j})^{-a}\frac{e^u}{(1+e^u)^{1+a}} = 4\pi\sum_{j=1}^{k} n_j\delta_{p_j} - 4\pi\sum_{j=1}^{l} m_j\delta_{q_j} \quad \text{in } \mathbb{R}^2.$$

In this equation the $\{\delta_{p_j}\}_{j=1}^k$ (resp. $\{\delta_{q_j}\}_{j=1}^l$) are Dirac masses concentrated at the points $\{p_j\}_{j=1}^k$, (resp. $\{q_j\}_{j=1}^l$), n_j and m_j are positive integers, and a is a nonnegative real number. We set $N=\sum_{j=1}^k n_j$ and $M=\sum_{j=1}^l m_j$.

In previous works [11, 32], some qualitative properties of solutions of (E) with $a=0$ have been established. Our aim in this article is to study the more general case where $a>0$. The additional difficulties of this case come from the fact that the nonlinearity is no longer monotone and the data are signed measures. As a consequence we cannot anymore construct directly the solutions by the monotonicity method combined with the supersolutions and subsolutions technique. Instead we develop a new and self-contained approach which enables us to emphasize the role played by the gravitation in the gauged $O(3)$ sigma model. Without the gravitational term, i.e. if $a=0$, problem (E) has a layer's structure of solutions $\{u_\beta\}_{\beta\in(-2(N-M),-2]}$, where u_β is the unique non-topological solution such that $u_\beta=\beta\ln|x|+O(1)$ for $-2(N-M)<\beta<-2$ and $u_{-2}=-2\ln|x|-2\ln\ln|x|+O(1)$ at infinity respectively. On the contrary, when $a>0$, the set of solutions to problem (E) has a much richer structure: besides the topological solutions, there exists a sequence of non-topological solutions in type I, i.e. such that u tends to $-\infty$ at infinity, and of non-topological solutions of type II, which tend to ∞ at infinity. The existence of these types of solutions depends on the values of the parameters N, M, β and on the gravitational interaction associated to a.

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Contents

1 Introduction

In this paper our goal is to classify the solutions of the following equation with measure data

$$-\Delta u + A_0(\prod_{j=1}^{k} |x-p_j|^{2n_j})^{-a}\frac{e^u}{(1+e^u)^{1+a}} = 4\pi\sum_{j=1}^{k} n_j\delta_{p_j} - 4\pi\sum_{j=1}^{l} m_j\delta_{q_j} \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $\{\delta_{p_j}\}_{j=1}^k$ (resp. $\{\delta_{q_j}\}_{j=1}^l$) are Dirac masses concentrated at the points $\{p_j\}_{j=1}^k$, (resp. $\{q_j\}_{j=1}^l$), $p_j \neq p_{j'}$ for $j \neq j'$, the related coefficients n_j and m_j are positive integers, $A_0 > 0$ is a given constant, $a = 16\pi G$ with G being the Newton's gravitational constant (or more precisely a dimensionless rescaling factor of the gravitational constant [31]) which is of the order of 10^{-30}. This means that physically speaking the exponent a is very small. Set

$$\mathbf{P}(x) = A_0(\prod_{j=1}^{k} |x-p_j|^{2n_j})^{-a}. \tag{1.2}$$

Since

$$2^{-1-a}\min\{e^u, e^{-au}\} \leq \frac{e^u}{(1+e^u)^{1+a}} \leq \min\{e^u, e^{-au}\}, \tag{1.3}$$

we define the notion of weak solution as follows:

Definition 1.1 *A function $u \in L^1_{loc}(\mathbb{R}^2)$ such that $\mathbf{P}\min\{e^u, e^{-au}\} \in L^1_{loc}(\mathbb{R}^2)$ is called a weak solution of (E), if for any $\xi \in C_c^\infty(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} u(-\Delta)\xi\, dx + \int_{\mathbb{R}^2} \mathbf{P}\frac{e^u}{(1+e^u)^{1+a}}\xi\, dx = 4\pi\sum_{j=1}^{k} n_j\xi(p_j) - 4\pi\sum_{j=1}^{l} m_j\xi(q_j).$$

This definition means that the following equation holds in the sense of distributions in $\mathbb{R}^2$,

$$-\Delta u + \mathbf{P}\frac{e^u}{(1+e^u)^{1+a}} = 4\pi\sum_{j=1}^{k} n_j\delta_{p_j} - 4\pi\sum_{j=1}^{l} m_j\delta_{q_j}. \tag{1.4}$$

We denote by $\Sigma := \{p_1, \cdots, p_k, q_1, \cdots, q_l\}$ the set of the supports of the measures in the right-hand side of (1.1). Since the nonlinearity in (1.4) is locally bounded in $\mathbb{R}^2 \setminus \Sigma$, a weak solution of (1.4) belongs to $C^2(\mathbb{R}^2 \setminus \Sigma)$ and is a strong solution of

$$-\Delta u + \mathbf{P}\frac{e^u}{(1+e^u)^{1+a}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \Sigma. \tag{1.5}$$

The nonlinear term is not monotone, actually the function $u \mapsto \frac{e^u}{(1+e^u)^{1+a}}$ is increasing on $(-\infty, -\ln a)$, and decreasing on $(-\ln a, \infty)$. This makes the structure of solutions of our problem much more complicated than the case where $a = 0$.

1.1 Physical models and related equations

Equation (1.1) comes from the Maxwell gauged $O(3)$ sigma model. When $a = 0$, it governs the self-dual $O(3)$ gauged sigma model developed from Heisenberg ferromagnet, see references [1, 2, 25, 28]. When the sigma model for Heisenberg ferromagnet with magnetic field is two-dimensional, it can be expressed by a local $U(1)$-invariant action density [32, p. 43-49]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{1}{2}(1-\vec{n}\cdot\phi)^2,$$

where $\vec{n} = (0,0,1)$, $\phi = (\phi_1, \phi_2, \phi_3)$ is a spin vector defined over the $(2+1)$-dimensional Minkowski spacetime $\mathbb{R}^{2,1}$, with value in the unit sphere $\mathbb{S}^2$, i.e. $|\phi| = 1$, D_μ are gauge-covariant derivatives on ϕ, defined by

$$D_\mu\phi = \partial_\mu\phi + A_\mu(\vec{n}\times\phi) \quad \text{where } \mu = 0, 1, 2$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic curvature induced from the 3-vector connection A_ν, $\nu = 0, 1, 2$ as detailled in [34, p. 177-189]. When the time gauge A_0 is zero, that is in the static situation, the functional of total energy can be expressed by the following expressions

$$\begin{aligned} E(\phi, A) &= \frac{1}{2}\int_{\mathbb{R}^2}\left((D_1\phi)^2 + (D_2\phi)^2 + (1-\vec{n}\cdot\phi)^2 + F_{12}^2\right)dx \\ &= 4\pi|deg(\phi)| + \frac{1}{2}\int_{\mathbb{R}^2}\left((D_1\phi \pm \phi\times D_2\phi)^2 + (F_{12} \mp (1-\vec{n}\cdot\phi))^2\right)dx, \end{aligned}$$

where $deg(\phi)$ denotes the Brouwer's degree of ϕ. The related Bogomol'nyi equation is obtained by using the stereographic projection $\phi \mapsto \tilde{\phi}$ from the south pole $\mathcal{S} = (0, 0, -1)$ of $\mathbb{S}^2 \setminus \{\mathcal{S}\}$ onto $\mathbb{R}^2$ (see e.g. [7, 34] for details). Then the function $u = \ln |\tilde{\phi}|^2$ satisfies

$$-\Delta u + \frac{4e^u}{1+e^u} = 4\pi \sum_{j=1}^{k} n_j \delta_{p_j} - 4\pi \sum_{j=1}^{l} m_j \delta_{q_j} \quad \text{in } \mathbb{R}^2. \tag{1.6}$$

It is pointed out in [32] that the points p_j $(j = 1, \cdots, k)$, which are the poles of $\tilde{\phi}$ can be viewed as magnetic monopoles and the points q_j $(j = 1, \cdots, l)$, which are the zeros of $\tilde{\phi}$ as antimonopoles (see [34, p. 55]). They are also called magnetic vortices and anti-vortices respectively.

An important quantity for the gauged sigma model is the total magnetic flux. It is customary [27] to identity it to the integral of the curvature as follows:

$$\mathcal{M}(\phi) = \int_{\mathbb{R}^2} F_{12}. \tag{1.7}$$

Using the variable u its value coincides with $\int_{\mathbb{R}^2} \Delta u dx$ (the Laplacian being taken a.e.). Thus, for the sake of simplicity, we identify $\mathcal{M}(\phi)$ and $\mathcal{M}(u)$, an expression which will be called the total flux in the sequel. Here and in what follows, we denote

$$N = \sum_{j=1}^{k} n_j \quad \text{and} \quad M = \sum_{j=1}^{l} m_j.$$

When the gravitation constant G is replaced by zero, a layer's structure of solutions of (1.1) has been determined in the following result:

Theorem 1.1 *[11, 32] (i) If $M = N - 1$, then problem (1.6) has no solution.*
(ii) If $M < N - 1$, then for any $\beta \in [2, 2(N - M))$ problem (1.6) has a unique solution u_β verifying

$$\mathcal{M}(u_\beta) = 2\pi(2(N - M) + \beta),$$

with the following behaviour as $|x| \to \infty$,

$$u_\beta(x) = \begin{cases} -\beta \ln |x| + O(1) & \text{if } \beta \in (2, 2(N - M)), \\ -2 \ln |x| - 2 \ln \ln |x| + O(1) & \text{if } \beta = 2. \end{cases}$$

Furthermore the correspondence $\beta \mapsto u_\beta$ is decreasing.
(iii) If $M < N - 1$ and u is a non-topological solution of (1.6) with finite total magnetic flux, i.e. $\mathcal{M}(u) < \infty$, then there exists a unique $\beta \in [2, 2(N - M))$ such that $u = u_\beta$.

These equations have been studied extensively, motivated by a large range of many applications in physics such as the gauged sigma models with broken symmetry [33], the gravitational Maxwell gauged $O(3)$ sigma model [7, 9, 27, 28], the self-dual Chern-Simons-Higgs model [8, 21], magnetic vortices [19], Toda system [20, 24], Liouville equation [18] and the references therein. It is also motivated by important questions in the theory of nonlinear partial differential equations [5, 29, 30], which has its own features in two dimensional space.

When $a = 16\pi G$, equation (1.1) governs the gravitational Maxwell gauged $O(3)$ sigma model restricted to a plane. Because of the gravitational interaction between particles, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} + \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \frac{1}{2}(1 - \vec{n} \cdot \phi)^2$$

with stress energy tensor

$$T_{\mu\nu} = g^{\mu'\nu'} F_{\mu\nu} F_{\mu'\nu'} + D_\mu \phi D_\mu \phi - g_{\mu\nu} \mathcal{L}.$$

We simplify the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci tensor and R is a scalar tensor of the metric in considering a metric conformal to the $(2 + 1)$-dimensional Minkowski one

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^\eta & 0 \\ 0 & 0 & e^\eta \end{pmatrix}.$$

Then

$$\frac{1}{2} e^{-\eta} \Delta \eta = -8\pi G T_{00},$$

where

$$T_{00} = \frac{1}{2}(e^{-\eta} F_{12} \pm (1 - \vec{n} \cdot \phi))^2 \pm e^{-\eta} F_{12}(1 - \vec{n} \cdot \phi)) \pm e^{-\eta} \phi (D_1 \phi \times D_2 \phi) + \frac{1}{2}(D_1 \phi \pm \phi \times D_2 \phi)^2.$$

The minimum of the energy is achieved if and only if (ϕ, A) satisfies the self-dual equations (the Bogomol'nyi equations)

$$D_1 \phi = \mp \phi \times D_2 \phi, \quad F_{12} = \pm e^\eta (1 - \vec{n} \cdot \phi).$$

Furthermore, a standard analysis yields equation (1.1). In particular, Yang in [34] studied equation (1.1) when there is only one concentrated pole, i.e. $k = 1$ and $l = 0$. For multiple poles, Chae showed in [7] that problem (1.1) has a sequence of non-topological solutions u_β such that

$$u_\beta(x) = \beta \ln |x| + O(1) \quad \text{when } |x| \to \infty$$

for $\beta \in (-\min\{6,\, 2(N - M)\},\, -2)$, when

$$aN < 1 \quad \text{and} \quad N - M \geq 2. \tag{1.8}$$

Under the assumption (1.8), the existence of solutions has been improved up to the range $\beta \in (-2(N - M),\, -2)$ by Song in [28]. However, these existence results do not show the role of the gravitation played in the gauged sigma model and the features of the interaction of the diffusion and the non-monotone nonlinearity of equation (1.1) in the whole two dimensional space.

1.2 Main results

Note that if we take into account the gravitation, the total magnetic flux turns out to be

$$\mathcal{M}(u) = \int_{\mathbb{R}^2} \mathbf{P}(x) \frac{e^u}{(1+e^u)^{1+a}}\, dx, \tag{1.9}$$

which, due to the potential and the decay to zero for $\frac{e^t}{(1+e^t)^{1+a}}$ as $t \to \infty$, allows the existence of solutions with very wild behaviors at infinity. In fact, the following three types of solutions are considered in this paper

$$\begin{cases} \text{a solution } u \text{ of (1.1) is topological} & \text{if} \quad \lim\limits_{|x|\to+\infty} u(x) = \ell \in \mathbb{R}, \\ \text{a solution } u \text{ of (1.1) is non-topological of type I} & \text{if} \quad \lim\limits_{|x|\to+\infty} u(x) = -\infty, \\ \text{a solution } u \text{ of (1.1) is non-topological of type II} & \text{if} \quad \lim\limits_{|x|\to+\infty} u(x) = +\infty. \end{cases}$$

The first result of this paper deals with non-topological solutions of type I for (1.1). For such a task we introduce two important quantities:

$$\beta^{\#} = \max\Big\{-2(N-M),\, \frac{2-2aN}{a}\Big\} \quad \text{and} \quad \beta^* = \min\left\{0,\, 2aN-2, \alpha^* - 2(N-M)\right\}, \tag{1.10}$$

where

$$\alpha^* := \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{P}(x)dx. \tag{1.11}$$

Notice that $\alpha^* = \infty$ if $an_j \geq 1$ for some j or $aN \leq 1$, otherwise α^* is finite, in this case, a free parameter A_0 should be taken into account. If $aN \leq 1$, we have that $\beta^* = 2aN - 2 \leq 0$.

Theorem 1.2 *Let $a = 16\pi G$, $an_j < 1$ for $j = 1, \cdots, k$ and $\mathcal{M}$ be the total magnetic flux given in (1.9).*
(i) If

$$aN \leq 1 \quad \text{and} \quad M < (1+a)N - 1, \tag{1.12}$$

then for any $\beta \in (-2(N-M),\, \beta^*)$, problem (1.1) possesses a minimal solution $u_{\beta,min}$ satisfying

$$u_{\beta,min}(x) = \beta \ln|x| + O(1) \quad \text{as} \quad |x| \to +\infty.$$

Moreover, for some real number C_*,

$$u_{\beta,min}(x) = \beta \ln|x| + C_* + O(|x|^{-\frac{aN-\beta-2}{aN-\beta-1}}) \quad \text{as} \quad |x| \to +\infty, \tag{1.13}$$

and the total magnetic flux of the solution $u_{\beta,min}$ is equal to $2\pi[2(N-M)+\beta]$, i.e.

$$\mathcal{M}(u_{\beta,min}) = 2\pi[2(N-M)+\beta]. \tag{1.14}$$

(ii) If

$$aN > 1 \quad \text{and} \quad M < N, \tag{1.15}$$

then $\beta^{\#} < 0$ and for any $\beta \in \big(\beta^{\#}, 0\big)$, problem (1.1) possesses a sequence of non-topological solutions $u_{\beta,i}$ of type I satisfying

$$u_{\beta,i}(x) = \beta \ln |x| + C_i + O(|x|^{-\frac{2aN-2\beta-2}{2aN-2\beta-1}}) \quad \text{as} \quad |x| \to \infty, \tag{1.16}$$

where

$$C_i < C_{i+1} \to \infty \quad \text{as} \quad i \to +\infty.$$

Moreover, the total magnetic flux of the solutions $\{u_{\beta,i}\}_i$ is equal to $2\pi[2(N-M)+\beta]$.

Note that our assumption (1.12) is much weaker than (1.8) and Theorem 1.2 provides a larger range of β for existence of solutions u_β verifying $u_\beta = \beta \ln |x| + o(1)$ at infinity. Furthermore we obtain a minimal solution and not just a finite energy solution as in [28, Theorem 1.3]. Note also that the assumption $M < (1+a)N - 1$ implies that $\beta^* > -2(N-M)$, and our second interest is to consider this extremal case $\beta = \beta^*$, which is $2aN - 2$ under the assumption (1.12).

Theorem 1.3 *Assume that $a = 16\pi G$, $an_j < 1$ for $j = 1, \cdots, k$, the magnetic flux $\mathcal{M}$ is given by (1.9) and let (1.12) hold.*
Then problem (1.1) possesses a minimal non-topological solution $u_{\beta^*,min}$ satisfying

$$u_{\beta^*,min}(x) = \beta^* \ln |x| - 2 \ln \ln |x| + O(1) \quad \text{as } |x| \to +\infty, \tag{1.17}$$

and the total magnetic flux of $u_{\beta^*,min}$ is equal to $2\pi[2(N-M)+\beta^*]$.

The existence of non-topological states of type II to (1.1) states as follows.

Theorem 1.4 *Assume that $a = 16\pi G$, $an_j < 1$ for $j = 1, \cdots, k$ and $\beta^{\#}$ is given by (1.10), then* for any $\beta > \beta^{\#}_+ = \max\{0, \beta^{\#}\}$, *problem (1.1) possesses a sequence of non-topological solutions $\{u_{\beta,i}\}_i$ such that*

$$u_{\beta,i}(x) = \beta \ln |x| + C_i + O(|x|^{-\frac{2aN-2\beta-2}{2aN-2\beta-1}}) \quad \text{as } |x| \to +\infty, \tag{1.18}$$

where

$$C_i < C_{i+1} \to +\infty \quad \text{as } i \to +\infty.$$

Moreover, the total magnetic flux of the solutions $\{u_{\beta,i}\}_i$ is equal to $2\pi[2(N-M)+\beta]$.

Concerning topological solutions of (1.1), we have following result,

Theorem 1.5 *Let $a = 16\pi G$, $an_j < 1$ for $j = 1, \cdots, k$ and (1.15) hold true.*
Then problem (1.1) possesses infinitely many topological solutions $u_{0,i}$ satisfying

$$u_{0,i}(x) = C_i + O(|x|^{-\frac{2aN-2}{2aN-1}}) \quad \text{as } |x| \to \infty, \tag{1.19}$$

where

$$C_i < C_{i+1} \to \infty \quad \text{as } i \to \infty.$$

Moreover, the total magnetic flux of the solutions $\{u_{0,i}\}_i$ is equal to $4\pi(N-M)$.

Note that Theorem 1.4 and Theorem 1.5 provide respectively infinitely many non-topological solutions of Type II and topological solutions. Furthermore, there is no upper bound for these solutions, this is due to the failure of the Keller-Osserman condition for the nonlinearity $\frac{4e^u}{(1+e^u)^{1+a}}$, see [17, 23]. More precisely equation (1.1) admits no solution with boundary blow-up in a bounded domain. The existence of these solutions illustrates that the gravitation plays an important role in the Maxwell gauged $O(3)$ sigma model:

(i) the set of solutions is extended to topological and two types of non-topological solutions;

(ii) the uniqueness fails for the solution under the given condition $u_\beta(x) = \beta \ln |x| + O(1)$ at infinity;

(iii) the numbers (counted with multiplicity) of magnetic poles N, M do no longer verify $M < N + 1$. In fact, for the non-topological solution of type I, it becomes $M < (1 + a)N + 1$, but for the non-topological solution of type II, there is no restriction on N and M, if $\beta > 0$ is large enough.

Our existence statements of solutions of (1.1) are summarized in the three tables above.

Table 1: Non-topological solutions of Type I

Assumptions on a, N, M	range of β	solutions	asymptotic behavior at ∞
$aN \leq 1$, $M < (1 + a)N - 1$	$(-2(N - M), \beta^*)$	Minimal	$\beta \ln \|x\| + O(1)$
$aN \leq 1$, $M < (1 + a)N - 1$	$\beta^* = 2(aN - 1)$	Minimal	$\beta^* \ln \|x\| - 2 \ln \ln \|x\| + O(1)$
$N > M$, $aN > 1$	$(\beta^{\#}, 0)$	Multiple	$\beta \ln \|x\| + c_i + o(1)$, $\lim\limits_{i\to\infty} c_i = \infty$

Table 2: Non-topological solutions of Type II

range of β	solutions	asymptotic behavior at ∞
$(\beta^{\#}_+, \infty)$	Multiple	$\beta \ln \|x\| + c_i + o(1)$, $\lim\limits_{i\to\infty} c_i = \infty$

Table 3: Topological solutions

Assumptions on a, N, M	solutions	asymptotic behavior at ∞
$aN > 1$, $M < N$	Multiple	$c_i + o(1)$, $\lim\limits_{i\to\infty} c_i = \infty$

The biggest difference with the case that $a = 0$ is that the nonlinearity is no longer monotone, which makes more difficult to construct super and sub solutions to (1.1). Our main idea is to approximate the solution by monotone iterative schemes for some related equations with an increasing nonlinearity.

Finally, we concentrate on the nonexistence of solutions u_β for (1.1) with the behavior $\beta \ln |x| + O(1)$ at infinity for some β.

Theorem 1.6 *Assume that $a = 16\pi G$ and $an_j < 1$ for $j = 1, \cdots, k$.*

(i) If $aN < 1$ and $\beta^* < \beta < \frac{2-aN}{a}$, then problem (1.1) has no solution u_β with the aymptotic behavior

$$u_\beta(x) = \beta \ln |x| + o(\ln |x|) \quad \text{as} \quad |x| \to \infty.$$

(ii) If $aN = 1$, then problem (1.1) has no topological solution.

The remaining of this paper is organized as follows. In Section 2, we present some decompositions of solutions of (1.1), some important estimates are provided and related forms of equations are considered. We prove that problem 1.1 has a minimal non-topological solution of Type I and minimal solutions in Section 3. Existence of infinitely many non-topological solutions of Type II is obtained in Section 4. Infinitely many topological solutions and minimal topological solution are constructed in Section 5. Finally, Section 6 deals with the classification of general non-topological solutions of (1.1) with infinite total magnetic flux.

2 Preliminary

2.1 Regularity

We begin our analysis by considering the regularity of weak solutions of (1.1). Let ζ be a smooth and increasing function defined in $(0,\infty)$ and such that

$$\zeta(t)=\begin{cases}\ln t & \text{for } \ 0<t\leq 1/2,\\ 0 & \text{for } \ t\geq 1.\end{cases}$$

Set

$$\nu_1(x)=2\sum_{i=1}^{k} n_i\zeta\left(\tfrac{|x-p_i|}{\sigma}\right) \quad \text{and} \quad \nu_2(x)=2\sum_{j=1}^{l} m_j\zeta\left(\tfrac{|x-q_j|}{\sigma}\right), \tag{2.1}$$

where $\sigma\in(0,1)$ is chosen such that any two balls of the set

$$\{B_\sigma(p_i),\, B_\sigma(q_j):\, i=1,\cdots k,\, j=1,\cdots l\}$$

do not intersect. We fix a positive number $r_0\geq e^e$ large enough such that $B_\sigma(p_i), B_\sigma(q_j)\subset B_{r_0}(0)$ for $i=1,\cdots,k$ and $j=1,\cdots,l$, and we denote

$$\Sigma_1=\{p_1,\cdots,p_k\},\quad \Sigma_2=\{q_1,\cdots,q_l\} \quad \text{and} \quad \Sigma=\Sigma_1\cup\Sigma_2.$$

If u is a weak solution of (1.1), we set

$$u=w-\nu_1+\nu_2 \quad \text{in } \ \mathbb{R}^2\setminus\Sigma. \tag{2.2}$$

Then w is a weak solution of

$$-\Delta w+V\frac{e^w}{(e^{\nu_1-\nu_2}+e^w)^{1+a}}=f_1-f_2 \quad \text{in } \ \mathbb{R}^2, \tag{2.3}$$

with

$$V=\mathbf{P}e^{a(\nu_1-\nu_2)},\ f_1=4\pi\sum_{i=1}^{k} n_i\delta_{p_i}-\Delta\nu_1 \ \text{ and } \ f_2=4\pi\sum_{j=1}^{l} m_j\delta_{q_j}-\Delta\nu_2. \tag{2.4}$$

The functions f_1, f_2 are smooth with compact supports in $B_{r_0}(0)$ and they satisfy

$$\int_{\mathbb{R}^2}(f_1-f_2)\,dx=4\pi(N-M). \tag{2.5}$$

Proposition 2.1 *Assume that u is a weak solution of (1.1), then u is a classical solution of*

$$-\Delta u + \mathbf{P}(x)\frac{e^u}{(1+e^u)^{1+a}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \Sigma, \tag{2.6}$$

and $w = u - \nu_1 + \nu_2$ is a classical solution of (2.3) in whole $\mathbb{R}^2$.

Proof. Let u be a weak solution of (1.1). Since $\frac{e^u}{(1+e^u)^{1+a}}$ is uniformly bounded in $\mathbb{R}^2$ and $\mathbf{P}$ is locally bounded and smooth in $\mathbb{R}^2 \setminus \Sigma$, the function u is a classical solution of (2.6) in $\mathbb{R}^2 \setminus \Sigma$. By standard regularity theory it belongs to $C^\infty\left(\mathbb{R}^2 \setminus \Sigma\right)$. Then w is a smooth locally bounded function in $\mathbb{R}^2 \setminus \Sigma$ satisfying (2.3), an equation that we rewrite under the form

$$-\Delta w + h(\cdot, w) = f_1 - f_2 \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{2.7}$$

where the function $h(x,z)$ is defined in $\mathbb{R}^2 \times \mathbb{R}$ by

$$h(x,z) = \begin{cases} V(x)\dfrac{e^z}{(e^{\nu_1-\nu_2}+e^z)^{1+a}} & \text{for } \ x \in \mathbb{R}^2 \setminus \Sigma, \\ 0 & \text{for } \ x \in \Sigma_2, \\ \sigma^{-2an_j}\displaystyle\prod_{i\neq k}|p_j - p_i|^{-2an_i}e^{-az} & \text{for } \ x = p_j \in \Sigma_1. \end{cases}$$

The function h is nonnegative and smooth in $\mathbb{R}^2 \setminus \Sigma$ and continuous in $\mathbb{R}^2 \times \mathbb{R}$. Since w is smooth in $\mathbb{R}^2 \setminus \Sigma$, so is $h(\cdot, w)$. Next we set, with $Z = e^z \geq 0$

$$\phi(Z) = \frac{Ze^{a(\nu_1-\nu_2)}}{\left(e^{a(\nu_1-\nu_2)}+Z\right)^{1+a}} \Longrightarrow \phi'(Z) = \frac{e^{a(\nu_1-\nu_2)}\left(e^{a(\nu_1-\nu_2)}-aZ\right)}{\left(e^{a(\nu_1-\nu_2)}+Z\right)^{2+a}}. \tag{2.8}$$

Then

$$\phi'(Z_0) = 0 \quad \text{with } \ Z_0 = \frac{e^{a(\nu_1-\nu_2)}}{a} \Longrightarrow \phi(Z_0) = \frac{a^a}{(a+1)^{1+a}}e^{(a-a^2)(\nu_1-\nu_2)} = \max\{\phi(Z) : Z > 0\}. \tag{2.9}$$

Hence

$$0 \leq h(x,w) \leq \mathbf{P}(x)\frac{a^a}{(a+1)^{1+a}}e^{(a-a^2)(\nu_1-\nu_2)}. \tag{2.10}$$

Note that $\mathbf{P}$ is locally bounded in $\mathbb{R}^2 \setminus \Sigma_1$, then it follows by standard regularity arguments, (see e.g. [13]) that w belongs to $W^{2,t}_{loc}(\mathbb{R}^2 \setminus \Sigma_1)$ for any $1 < t < \infty$. Hence $h(\cdot, w) \in C^{1,\theta}(\mathbb{R}^2 \setminus \Sigma_1)$ for any $\theta \in (0,1)$, and finally $w \in C^{3,\theta}(\mathbb{R}^2 \setminus \Sigma_1)$ is a strong solution in $\mathbb{R}^2 \setminus \Sigma_1$. In a neighborhood of Σ_1 we write h under the form

$$h(x,z) = \mathbf{P}(x)e^{a\nu_1}\frac{e^{z-a\nu_2}}{(e^{\nu_1-\nu_2}+e^z)^{1+a}}.$$

Since h is nonnegative, then w satisfies the inequality

$$-\Delta w \leq f_1 - f_2 \quad \text{in } \mathcal{D}'(\mathbb{R}^2),$$

and as $f_1 - f_2$ is bounded with compact support, it follows that w is locally bounded from above in $\mathbb{R}^2$. Furthermore, there exist an open set $\mathcal{O}$ such that $\Sigma_1 \subset \mathcal{O}$ and $\overline{\mathcal{O}} \cap \Sigma_2 = \emptyset$ and a function $\zeta_1 \in C(\overline{\mathcal{O}})$ such that

$$h(x,z) = \zeta_1 \frac{e^z}{(e^{\nu_1-\nu_2}+e^z)^{1+a}} \quad \text{for all } \ (x,z) \in \overline{\mathcal{O}} \times \mathbb{R}.$$

For a given $p_j \in \Sigma_1$, we set $r_j = \sup\{w(x) : x \in \overline{B}_\sigma(p_j)\}$ and $v_j = r_j - w$. Then $v_j \geq 0$ in $\overline{B}_\sigma(p_j)$ and

$$-\Delta v_j = f_2 - f_1 + \zeta_1 \frac{e^{r_j - v_j}}{(e^{\nu_1-\nu_2}+e^{r_j-v_j})^{1+a}} = f_2 - f_1 + \zeta_1 e^{-ar_j} \frac{e^{-v_j}}{(e^{\nu_1-\nu_2-r_j}+e^{-v_j})^{1+a}}.$$

Since f_1, f_2 are smooth, hence $\zeta_1 e^{-ar_j} \frac{e^{-v_j}}{(e^{\nu_1-\nu_2-r_j}+e^{-v_j})^{1+a}} \in L^1(B_\sigma(p_j))$ by [4]. If $0 < \sigma' \leq \sigma$, we denote by $\phi_j^{B_{\sigma'}}$ the harmonic lifting of $v_j \lfloor_{\partial B_{\sigma'}}$ in $B_{\sigma'}(p_j)$ and put $\tilde{v}_{\sigma'} = v_j - \phi_j^{B_{\sigma'}}$. Then for $\sigma' \leq \sigma$,

$$\begin{cases} -\Delta \tilde{v}_{\sigma'} = f_2 - f_1 + \zeta_1 e^{-ar_j} \frac{e^{-v_j}}{(e^{\nu_1-\nu_2-r_j}+e^{-v_j})^{1+a}} := F_j & \text{in } \ B_{\sigma'}(p_j), \\ \tilde{v}_{\sigma'} = 0 & \text{on } \ \partial B_{\sigma'}(p_j). \end{cases}$$

Let $M^2(B_{\sigma'}(p_j))$ denote the Marcikiewicz space also known as the Lorentz space $L^{2,\infty}(B_{\sigma'}(p_j))$. Then there holds

$$\|\nabla \tilde{v}_{\sigma'}\|_{M^2(B_{\sigma'}(p_j))} \leq c_0 \|F_j\|_{L^1(B_{\sigma'}(p_j))} \tag{2.11}$$

and the constant c_0 is independent of σ'. We recall below John-Nirenberg's theorem [13, Theorem 7.21]: *Let $u \in W^{1,1}(G)$ where $G \subset \Omega$ is convex and suppose there is a constant K such that*

$$\int_{G \cap B_r} |\nabla u| dx \leq Kr \quad \textit{for any ball } \ B_r(0). \tag{2.12}$$

Then there exist positive constants μ_0 and c_1 such that

$$\int_G \exp\Big(\frac{\mu}{K}|u - u^G|\Big)\, dx \leq c_1 (diam(G))^2, \tag{2.13}$$

where $\mu = \mu_0 |G| (diam(G))^{-2}$ and u^G is the average of u on G.
From (2.11) with $G = B_{\sigma'}(p_j)$,

$$\int_{B \cap B_r} |\nabla \tilde{v}_{\sigma'}| dx \leq c_3 r \|F_j\|_{L^1(B_{\sigma'}(p_j))} := K(\sigma')r, \tag{2.14}$$

and since $|G|(diam(G))^{-2} = \pi$, we obtain

$$\int_{B_{\sigma'}(p_j)} \exp\left(\frac{\pi\mu_0}{K(\sigma')}|\tilde{v}_{\sigma'} - \tilde{v}_{\sigma'}^{B_{\sigma'}}|\right) dx \leq c_1 \sigma'^2. \tag{2.15}$$

Hence, for any $\kappa > 0$ there exists $\sigma' \in (0, \sigma]$ such that

$$\int_{B_{\sigma'}(p_j)} \exp\left(\kappa|\tilde{v}_{\sigma'} - \tilde{v}_{\sigma'}^{B_{\sigma'}}|\right) dx \le c_1\sigma'^2 \Longrightarrow \int_{B_{\sigma'}(p_j)} \exp(\kappa\tilde{v}_{\sigma'})dx) \le c_1\sigma'^2 \exp(\kappa\tilde{v}_{\sigma'}^{B_{\sigma'}}). \quad (2.16)$$

Now we observe that there holds in $B_{\sigma'}(p_j)$,

$$|F_j| \le |f_2 - f_1| + \zeta_1 e^{-ar_j} e^{av_j} \le |f_2 - f_1| + \zeta_1 e^{a \sup |v_j \lfloor_{\partial B_{\sigma'}}|} e^{a\tilde{v}_{\sigma'}}.$$

For $\kappa > a$,

$$\int_{B_{\sigma'}(p_j)} |F_j(x)|^{\frac{\kappa}{a}} dx \le 2^{\frac{\kappa}{a}-1} \int_{B_{\sigma'}(p_j)} \left(|f_2 - f_1|^{\frac{\kappa}{a}} + \left(\zeta_1 e^{a \sup |v_j \lfloor_{\partial B_{\sigma'}}|}\right)^{\frac{\kappa}{a}} e^{\kappa\tilde{v}_{\sigma'}}\right) dx.$$

By (2.15) the right-hand side of the above inequality is bounded, hence $F_j \in L^{\frac{\kappa}{a}}(B_{\sigma'}(p_j))$. Since $\tilde{v}_{\sigma'}$ vanishes on $\partial B_{\sigma'}(p_j)$, it follows by L^p regularity theory that $\tilde{v}_{\sigma'} \in W^{2,\frac{\kappa}{a}}(B_{\sigma'}(p_j)) \cap W_0^{1,\frac{\kappa}{a}}(B_{\sigma'}(p_j))$. By Sobolev embedding theorem, $\tilde{v}_{\sigma'} \in L^\infty(B_{\sigma'}(p_j))$. Hence $F_j \in L^\infty(B_{\sigma'}(p_j))$ and again $\tilde{v}_{\sigma'} \in W^{2,q}(B_{\sigma'}(p_j))$ for any $q \in [1, \infty)$ and thus $\tilde{v}_{\sigma'} \in C^{1,\theta}(\overline{B}_{\sigma'}(p_j))$ for any $\theta \in (0, 1)$. Therefore v_j remains bounded in $C^{1,\theta}(\overline{B}_{\sigma''}(p_j))$ for any $\sigma'' < \sigma'$. In a neighborhood of p_j, $x \mapsto |x - p_j|^{-2an_j} e^{2a\zeta(\frac{|x-p_j|}{\sigma})}$ is Hölder continuous (of order $2an_j$ if $2an_j < 1$), and so is $x \mapsto \mathbf{P}(x)e^{a(\nu_1-\nu_2)(x)}$. For the same reason, $x \mapsto \frac{e^{w(x)}}{\left(e^{(\nu_1-\nu_2)(x)+e^{w(x)}}\right)^{1+a}}$ is Hölder continuous (with the same exponent) near p_j. Finally we infer that there exists $\theta \in (0, 1)$ such that $v_j \in C^{2,\theta}(\overline{B}_{\sigma''}(p_j))$, which implies that $w \in C^{2,\theta}(\mathbb{R}^2)$ is a strong solution of (2.3) in $\mathbb{R}^2$. □

Remark. Since ϑ_1 and ν_2 have compact support, we note that a weak solution u_β with the asymptotic behavior $\beta \ln|x| + O(1)$ at infinity can be decomposed

$$u_\beta = w_\beta - \nu_1 + \nu_2, \quad (2.17)$$

where w_β is a classical solution of (2.3) with the same asymptotic behavior $\beta \ln|x| + O(1)$ at infinity. In fact, we shall continue to take out the singular source of the solution w_β at infinity in our derivation of non-topological solutions of (1.1).

2.2 Basic estimates

The following estimates play an important role in our construction of solutions to (1.1).

Lemma 2.1 *Let Γ be the fundamental solution of $-\Delta$ in $\mathbb{R}^2$, $F \in L^p_{loc}(\mathbb{R}^2)$, $p > 1$, with the support in $\overline{B_R(0)}$ for some $R > 0$ such that*

$$\int_{\mathbb{R}^2} F(x)dx = 0. \quad (2.18)$$

Then there holds

$$|\Gamma * F(x)| \le \frac{R}{|x|}\|F\|_{L^1(\mathbb{R}^2)} \quad \text{for } |x| > 4R, \quad (2.19)$$

and for some $c_2 > 0$ depending on p and R,

$$\|\Gamma * F\|_{L^\infty(\mathbb{R}^2)} \le c_2\|F\|_{L^p(\mathbb{R}^2)}. \quad (2.20)$$

Proof. As $\mathrm{supp}(F) \subset \overline{B_R(0)}$ and $F \in L^p_{loc}(\mathbb{R}^2)$, $F \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ by Hölder's inequality. Since (2.18) holds, we have for $|x| > 4R$,

$$\begin{aligned}
|\Gamma * F(x)| &= \frac{1}{2\pi}\Big| \int_{B_R(0)} \ln|x-y|\, F(y)dy - \int_{B_R(0)} \ln|x|\, F(y)dy\Big| \\
&= \frac{|x|^2}{2\pi}\Big| \int_{B_{\frac{R}{|x|}}(0)} \ln|e_x - z|\, F(|x|z)dz\Big| \\
&\le \frac{|x|^2}{\pi} \int_{B_{\frac{R}{|x|}}(0)} |z||F(|x|z)|dz \\
&\le \frac{2R}{\pi|x|} \int_{B_R(0)} |F(y)|dy \\
&< \frac{R}{|x|}\|F\|_{L^1(\mathbb{R}^2)},
\end{aligned}$$

where $e_x = \frac{x}{|x|}$. Besides (2.18) we have used the fact that

$$|\ln|e_x - z|| \le 2|z| \le 2\frac{R}{|x|} \quad \text{for any } \ z \in B_{R/|x|}(0) \subset B_{1/4}(0).$$

Therefore, (2.19) is proved. On the other hand, for $|x| \le 4R$, we have that

$$\begin{aligned}
|\Gamma * F(x)| &= \frac{1}{2\pi}\Big| \int_{B_R(0)} F(y) \ln|x-y|dy\Big| \\
&\le \Big(\int_{B_R(0)} |F(y)|^p dx\Big)^{\frac{1}{p}} \Big(\int_{B_R(0)} |\ln|x-y||^{p'} dx\Big)^{\frac{1}{p'}} \\
&\le c_p\|F\|_{L^p(B_R(0))},
\end{aligned}$$

where $p' = \frac{p}{p-1}$ and $c_p = \max\limits_{|x|\le 4R}(\int_{B_R(0)} |\ln|x-y||^{p'} dx)^{\frac{1}{p'}}$. Thus, (2.20) follows and the proof is complete. □

For functions with non-compact supports, we have the following estimates.

Lemma 2.2 *Let $F \in L^p_{loc}(\mathbb{R}^2)$ with $p > 1$ satisfy that*

$$\int_{\mathbb{R}^2} F(x)\, dx = 0 \tag{2.21}$$

and

$$|F(x)| \le c_3|x|^{-\tau} \quad \text{for } |x| \ge r \tag{2.22}$$

for some $\tau > 2$, $c_3 > 0$ and $r > 0$. Then for some $c_4 > 0$

$$\|\Gamma * F\|_{L^\infty(\mathbb{R}^2)} \le c_4,$$

and there exist $c_5 > 0$ and $r_0 > r$ such that for $|x| \ge r_0$

$$|x||\nabla\Gamma * F(x)| + |\Gamma * F(x)| \le \frac{c_5}{(\tau-2)^2}|x|^{-\frac{\tau-2}{\tau-1}}. \tag{2.23}$$

Proof. If $F \in L^p_{loc}(\mathbb{R}^2)$ satisfies (2.22), then $F \in L^1(\mathbb{R}^2)$. Let $\eta_r : \mathbb{R}^2 \to [0,1]$ be a smooth and radially symmetric function such that

$$\eta_r = 1 \text{ in } B_r(0), \quad \eta_r = 0 \text{ in } B_{r+1}(0),$$

and denote

$$F_1 = F\eta_r - \left(\int_{\mathbb{R}^2} F\eta_r dx\right) \frac{\eta_r}{\|\eta_r\|_{L^1(\mathbb{R}^2)}}, \quad F_2 = F - F_1.$$

By (2.21), we have that

$$\int_{\mathbb{R}^2} F_1\, dx = \int_{\mathbb{R}^2} F_2\, dx = 0.$$

Since $F_1 \in L^p_{loc}(\mathbb{R}^2)$ has compact support, it follows by Lemma 2.1, $\Gamma * F_1$ is bounded and satisfies (2.19). Concerning F_2, we have

$$F_2 = F - F_1 = \frac{\int_{\mathbb{R}^2} F\eta_r dx}{\|\eta_r\|_{L^1(\mathbb{R}^2)}} \quad \text{on } B_r,$$

and F_2 satisfies (2.22) on B_r^c, with may be another constant. It is locally bounded hence $\Gamma * F_2$ is also locally bounded in $\mathbb{R}^2$. If (2.22) holds true, we have to prove that F_2 verifies (2.23). Since $\int_{\mathbb{R}^2} F_2\, dx = 0$, then for all $|x| > 4r$ and $R \in (r, \frac{|x|}{4})$ which will be chosen latter on,

$$\begin{aligned}
2\pi(\Gamma * F_2)(x) &= |x|^2 \int_{\mathbb{R}^2} \ln|e_x - z| F_2(|x|z)dz + |x|^2 \ln|x| \int_{\mathbb{R}^2} F_2(|x|z)dz \\
&= |x|^2 \int_{B_{R/|x|}(0)} \ln|e_x - z| F_2(|x|z)dz + |x|^2 \int_{B_{1/2}(e_x)} \ln|e_x - z| F(|x|z)dz \\
&\quad + |x|^2 \int_{\mathbb{R}^2 \setminus (B_{R/|x|}(0) \cup B_{1/2}(e_x))} \ln|e_x - z|\, F_2(|x|z)dz \\
&=: I_1(x) + I_2(x) + I_3(x),
\end{aligned}$$

using the fact that $B_{R/|x|}(0) \cap B_{1/2}(e_x)) = \emptyset$. By a direct computation, we have that

$$\begin{aligned}
|I_1(x)| &\le |x|^2 \int_{B_{R/|x|}(0)} |z||F_2(|x|z)|dz \\
&= 2\frac{R}{|x|} \int_{B_R(0)} |F_2(y)|dy \\
&\le 2\frac{R}{|x|} \|F_2\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

For $z \in B_{1/2}(e_x)$, there holds $|x||z| \ge \frac{1}{2}|x| > 2r$, then $|F(|x|z)| \le c_3|x|^{-\tau}|z|^{-\tau}$ and

$$\begin{aligned}
|I_2(x)| &\le c_3|x|^{2-\tau} \int_{B_{1/2}(e_x)} (-\ln|e_x - z|)|z|^{-\tau}dz \\
&\le 2^\tau c_3|x|^{2-\tau} \int_{B_{1/2}(e_x)} (-\ln|e_x - z|)dz \\
&\le c_6 R^{2-\tau},
\end{aligned}$$

where $c_6 = 2^{2(N-M)}c_3\big(\int_{B_{1/2}(0)}(-\ln|z|)dz\big)$ can be chosen independently of τ in $(2, 2(N-M))$. Next, if $z \in \mathbb{R}^2\backslash(B_{R/|x|}(0)\cup B_{1/2}(e_x))$, then $\big|\ln|e_x - z|\big| \le \ln(1+|z|)$ and $|F(|x|z)| \le c_7|x|^{-\tau}|z|^{-\tau}$, since $|z| \ge \frac{R}{|x|} > \frac{r}{|x|}$. By the integration by parts we get

$$\begin{aligned}|I_3(x)| &\le c_8|x|^{2-\tau}\int_{\mathbb{R}^2\backslash B_{R/|x|}(0)} \ln(1+|z|)\,|z|^{-\tau}dz\\ &\le \frac{2\pi c_8}{\tau-2}R^{2-\tau}\ln\left(1+\frac{R}{|x|}\right) + \frac{2\pi c_5}{(\tau-2)^2}R^{2-\tau}\\ &\le \frac{2\pi c_8}{(\tau-2)^2}\Big((\tau-2)\ln 2 + 1\Big)R^{2-\tau}.\end{aligned}$$

Thus, taking $R = |x|^{\frac{1}{\tau-1}}$ and $|x|$ sufficiently large (certainly $R \in (r, \frac{|x|}{4})$ is satisfied), we have

$$\begin{aligned}|\Gamma * F_2(x)| &\le \frac{R}{\pi|x|}\|F\|_{L^1(\mathbb{R}^2)} + \frac{c_8}{2\pi}R^{2-\tau} + \frac{c_8}{(\tau-2)^2}\Big(2(N-M-1)\ln(e+1)+1\Big)R^{2-\tau}\\ &\le \frac{c_9}{(\tau-2)^2}|x|^{-\frac{\tau-2}{\tau-1}},\end{aligned}$$

where $c_9 > 0$ can be chosen independently of τ. In order to prove the gradient estimate, we denote by (r,θ) the polar coordinates in $\mathbb{R}^2$, set $t = \ln r$ and

$$\omega(t,\theta) = \tilde{\omega}(r,\theta) = r^{-\frac{\tau-2}{\tau-1}}\Gamma * F(r,\theta) \quad \text{and} \quad \phi(t,\theta) = r^{\tau}F(r,\theta).$$

Then ω and ϕ are bounded on $[\ln r_1, \infty)\times S^1$ where there holds

$$\mathcal{L}\omega := \frac{\partial^2\omega}{\partial t^2} - 2\frac{\tau-2}{\tau-1}\frac{\partial\omega}{\partial t} + \left(\frac{\tau-2}{\tau-1}\right)^2\omega + \frac{\partial^2\omega}{\partial\theta^2} = \tau^{-\frac{(\tau-2)^2}{\tau-1}}\phi.$$

Since the operator $\mathcal{L}$ is uniformly elliptic on $[\ln r_1,\infty)\times S^1$, for any $T > \ln r_1 + 2$ there holds by standard elliptic equations regularity estimates [13],

$$\sup_{[T-1,T+1]\times S^1}\left(\left|\frac{\partial\omega}{\partial\theta}\right| + \left|\frac{\partial\omega}{\partial t}\right|\right) \le c_{10}\sup_{[T-2,T+2]\times S^1}\left(|\omega| + \left|\tau^{-\left(\frac{\tau-2}{\tau-1}\right)^2}\phi\right|\right) \le c_{11},$$

and c_{11} does not depend on T. As $|x||\nabla\tilde{\omega}(x)| = \left(\left|\frac{\partial\omega}{\partial\theta}\right|^2 + \left|\frac{\partial\omega}{\partial t}\right|^2\right)^{\frac{1}{2}}$, this implies the claim. □

The decay estimate on the gradient at infinity does not use the identity (2.21). It is actually more general.

Corollary 2.1 *Let $F \in L^p_{loc}(\mathbb{R}^2)$ with $p > 1$ satisfy (2.22) with $\tau > 2$ and w be a solution of*

$$-\Delta w = F \quad \text{in } \mathbb{R}^2. \tag{2.24}$$

(i) If $\lim_{|x|\to\infty}|x|^{\frac{\tau-2}{\tau-1}}|w(x)| < \infty$, then

$$|\nabla w(x)| \le c_{12}|x|^{-\frac{2\tau-3}{\tau-1}} \quad \text{for } |x| \ge r_0. \tag{2.25}$$

(ii) If there exists a constant C such that $w(x) = C + O(|x|^{-\frac{\tau-2}{\tau-1}})$ when $|x|\to\infty$, then estimate (2.25) holds.

Proof. The assertion (i) is clear since the starting point of the gradient estimate in the previous lemma is

$$|w(x)| \leq c_{13}|x|^{-\frac{\tau-2}{\tau-1}} \quad \text{for } |x| \text{ large enough.}$$

For assertion (ii), we set $w(x) = C + \tilde{w}(x)$ where $|\tilde{w}(x)| = O(|x|^{-\frac{\tau-2}{\tau-1}})$. Then $-\Delta\tilde{w} = -\Delta w$ and $\nabla w = \nabla\tilde{w}$. We conclude by (i). □

When $\tau = 2$, Lemma 2.2 is no longer valid, however the following limit case is available.

Lemma 2.3 *Let $F \in L^p_{loc}(\mathbb{R}^2)$ with $p > 1$ satisfy (2.21) and*

$$|F(x)| \leq c_{14}|x|^{-2}(\ln|x|)^{-\nu} \quad \text{for} \quad |x| \geq r, \tag{2.26}$$

for some $\nu > 2$, $c_{14} > 0$ and $r > 0$. Then

$$\|\Gamma * F\|_{L^\infty(\mathbb{R}^2)} \leq c_{15},$$

and

$$|x||\nabla\Gamma * F(x)| + |\Gamma * F(x)| \leq c_{16}(\ln|x|)^{-\nu} \quad \textit{for} \quad |x| \geq r_1, \tag{2.27}$$

where c_{15}, $c_{16} > 0$ and $r_1 > r$ is large enough.

Proof. The assumption (2.26) jointly with $F \in L^p_{loc}(\mathbb{R}^2)$ implies $F \in L^1(\mathbb{R}^2)$. We write $F = F_1 + F_2$ in the same way as in Lemma 2.2. Clearly $\Gamma * F_1$ is uniformly bounded and satisfies (2.19) and (2.20). Then for $|x| > 4r > 4$ and $R \in (r, \frac{|x|}{4}]$,

$$2\pi(\Gamma * F_2)(x) = I_1(x) + I_2(x) + I_3(x),$$

where I_1, I_2 and I_3 are defined in the proof of Lemma 2.2 and where $|I_1(x)| \leq 2\frac{R}{|x|}\|F_2\|_{L^1(\mathbb{R}^2)}$.

When $z \in B_{\frac{1}{2}}(e_x)$ we have $|x||z| \geq \frac{1}{2}|x| > 2r$, hence

$$|F_2(|x|z)| \leq c_{14}|x|^{-2}|z|^{-2}\left(\ln|x| + \ln|z|\right)^{-\nu} \leq c_{14}|x|^{-2}|z|^{-2}\left|\ln|x| - \ln 2\right|^{-\nu}, \tag{2.28}$$

and

$$I_2(x) \leq -c_{14}\left|\ln|x| - \ln 2\right|^{-\nu}\int_{B_{\frac{1}{2}}(e_x)} \ln|z - e_x||z|^{-2}dz \leq c_{17}\left(\ln|x|\right)^{-\nu}.$$

Finally, if $z \in \mathbb{R}^2 \setminus (B_{R/|x|}(0) \cup B_{1/2}(e_x))$, then $|\ln|e_x - z|| \leq \ln(1 + |z|)$ and

$$|F_2(|x|z)| \leq c_{17}(1 + |x||z|)^{-2}\left(\ln(1 + |x||z|)\right)^{-\nu} \leq c_{17}|x|^{-2}|z|^{-2}\left(\ln(1 + |x||z|)\right)^{-\nu}.$$

Since $|z| \geq \frac{R}{|x|} > \frac{r}{|x|}$, we have

$$I_3(x) \leq c_{18}\int_{\frac{R}{|x|}}^{\infty} \ln(1+t)(\ln(1 + t|x|)^{-\nu}\frac{dt}{t} \leq c_{18}\int_R^\infty (\ln(1+s))^{1-\nu}\frac{ds}{s}.$$

Since $R > r > 1$,

$$I_3(x) \leq c_{19}\int_R^\infty (\ln(1+s))^{1-\nu}\frac{ds}{1+s} = \frac{c_{19}}{\nu(\ln(1+R))^\nu}.$$

If we choose $R = \frac{|x|}{4}$, we obtain that $(\ln(1+|x|))^\nu(\Gamma * F)(x)$ remains uniformly bounded on $\mathbb{R}^2$.

Next we prove the gradient estimate. Set $t = \ln r$, $\omega(t,\theta) = (\Gamma * F)t, \theta)$ and $\phi(t,\theta) = F(r,\theta)$, then

$$\mathcal{L}\omega := \frac{\partial^2\omega}{\partial t^2} + \frac{\partial^2\omega}{\partial\theta^2} = \phi$$

and $|\omega(t,\theta)| \le c_{20}t^{-\nu}$ and $|\phi(t,\theta)| \le c_{20}t^{-\nu}$ for $t \ge t_1$. Since the operator $\mathcal{L} = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\theta^2}$ is uniformly elliptic, then we have for $T \ge \max\{4, t_1\}$,

$$\sup_{[T-1,T+1]\times S^1}\left(\left|\frac{\partial\omega}{\partial\theta}\right| + \left|\frac{\partial\omega}{\partial t}\right|\right) \le c_{10}\sup_{[T-2,T+2]\times S^1}(|\omega| + |\phi|) \le 2c_{10}c_{20}(T-2)^{-\nu} \le c_{21}T^{-\nu}.$$

Returning to the variable x, we infer (2.27). □

Similarly as in Corollary 2.1, the following extension of (2.27) holds.

Corollary 2.2 *Let $F \in L^p_{loc}(\mathbb{R}^2)$ with $p > 1$ satisfy (2.26) with $\nu > 2$ and w satisfiesf (2.24).*
(i) If $\lim_{|x|\to\infty}(\ln|x|)^\nu|w(x)| < \infty$, then there exist $c_{22} > 0$ and $r_2 > 1$ such that

$$|\nabla w(x)| \le c_{22}|x|^{-1}(\ln|x|)^{-\nu} \quad \text{for } |x| \ge r_2. \tag{2.29}$$

(ii) If there exists a constant C such that $w(x) = C + O((\ln|x|)^{-\nu})$ when $|x| \to +\infty$, then estimate (2.29) holds.

2.3 Related problems with increasing nonlinearity

In order to remove the condition $\beta\ln|x| + O(1)$ as $|x| \to \infty$ satisfied by the solutions of (1.1), we introduce two functions λ and Λ, which are positive smooth functions such that

$$\lambda(x) = |x|, \quad \Lambda(x) = \ln|x| \quad \text{for } |x| \ge e^e. \tag{2.30}$$

Since $\Delta\Lambda = 0$ in $B^c_{e^e}(0)$,

$$\Delta\ln\Lambda = \frac{\Delta\Lambda}{\Lambda} - \frac{|\nabla\Lambda|^2}{\Lambda^2} = -\frac{1}{|x|^2(\ln|x|)^2} \quad \text{in } B^c_{e^e}(0), \tag{2.31}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}(\Delta\ln\lambda)dx = 1, \tag{2.32}$$

it implies

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}^2}(\Delta\ln\Lambda)dx &= \lim_{r\to+\infty}\frac{1}{2\pi}\int_{\partial B_r(0)}\frac{\nabla\Lambda(x)}{\Lambda(x)}\cdot\frac{x}{|x|}d\omega(x)\\ &= \lim_{r\to+\infty}\frac{1}{r\ln r} = 0.\end{aligned}$$

In what follows we classify the solutions of the following equations

$$-\Delta u + WF_i(\lambda^\beta e^u) = g_\beta \quad \text{in } \mathbb{R}^2, \tag{2.33}$$

where $i = 1, 2$, $F_1(s) = s$, $F_2(\cdot, s) = \frac{s}{e^{(\nu_1-\nu_2)(\cdot)}+s}$,

$$g_\beta = f_1 - f_2 + \beta\Delta \ln \lambda, \tag{2.34}$$

and where W satisfies the following assumption:

($\mathcal{W}_0$) *The function W is positive and locally Hölder continuous in $\mathbb{R}^2 \setminus \Sigma_1$ and*

$$W(x) \le c_{23}|x - p_j|^{-\tau_{p_j}} \quad in \quad B_\sigma(p_j) \quad \text{and} \quad \limsup_{|x|\to+\infty} W(x)|x|^{\gamma_\infty} < +\infty,$$

where $c_{23} > 0$, $\tau_{p_j} \in [0, 2)$ and $\gamma_\infty > 0$.

It is important to note that from (2.5), (2.32) and (2.34), there holds

$$\int_{\mathbb{R}^2} g_\beta \, dx = 2\pi[2(N - M) + \beta]. \tag{2.35}$$

Theorem 2.1 *Assume that $F_1(s) = s$, $F_2(x, s) = \frac{s}{e^{(\nu_1-\nu_2)(x)}+s}$ and g_β is a Hölder continuous function with compact support in $B_{r_0}(0)$ satisfying the relation (2.35) for some nonnegative integers N and M. Let W verify ($\mathcal{W}_0$) with*

$$\gamma_\infty > \beta + 2. \tag{2.36}$$

(i) Then problem (2.33) with $i = 1$ has a unique bounded solution v verifying, for some $C_\beta \in \mathbb{R}$,

$$v(x) = C_\beta + O(|x|^{-\frac{\gamma_\infty-\beta-2}{\gamma_\infty-\beta-1}}) \ \text{ and } \ |\nabla v(x)| = O(|x|^{-1-\frac{\gamma_\infty-\beta-2}{\gamma_\infty-\beta-1}}) \quad as \ |x| \to +\infty. \tag{2.37}$$

(ii) Assume additionally that $2(N - M) + \beta > 0$ and

$$2\pi[2(N - M) + \beta] < \int_{\mathbb{R}^2} W dx \le +\infty. \tag{2.38}$$

Then problem (2.33) with $i = 2$ has a unique bounded solution verifying

$$v(x) = C_\beta + O(|x|^{-\frac{\gamma_\infty+\beta_--2}{\gamma_\infty+\beta_--1}}) \ \text{ and } \ |\nabla v(x)| = O(|x|^{-1-\frac{\gamma_\infty+\beta_--2}{\gamma_\infty+\beta_--1}}) \quad as \ |x| \to +\infty. \tag{2.39}$$

Proof. *Step 1.* Since $0 \le \tau_{p_j} < 2$, $W \in L^1_{loc}(\mathbb{R}^N)$. For $t \in \mathbb{R}$, we set

$$h_{i,t}(x) = W(x)F_i(x, \lambda^\beta(x)e^t) \qquad \forall\, x \in \mathbb{R}^2 \setminus \Sigma_1.$$

Notice that $h_{2,t}$ is defined on Σ_2 by

$$h_{2,t}(q_j) = 0 = \lim_{x\to q_j} h_{2,t}(x) \quad \text{for all } \ q_j \in \Sigma_2.$$

The function $h_{i,t}$ is Hölder continuous in $\mathbb{R}^2 \setminus \Sigma_1$, $t \mapsto h_{i,t}$ is increasing in $\mathbb{R}^2 \setminus \Sigma_1$, and there holds

$$h_{1,t} \to \infty \quad \text{locally in } \mathbb{R}^2 \setminus \Sigma_1 \quad \text{as } t \to +\infty,$$
$$h_{2,t} \to W \quad \text{locally in } \mathbb{R}^2 \setminus \Sigma_1 \quad \text{as } t \to +\infty.$$

Furthermore,

$$h_{i,t} \to 0 \quad \text{locally in } \mathbb{R}^2 \setminus \Sigma_1 \quad \text{as } t \to -\infty, \quad i = 1, 2.$$

Using assumption $(\mathcal{W}_0)$, we obtain that

$$h_{1,t}(x) \leq c_{24} e^t |x|^{-\gamma_\infty + \beta} \quad \text{for } |x| \geq r_3 \tag{2.40}$$

for some $r_3 > 0$. Since $-\gamma_\infty + \beta < -2$, we have that

$$\lim_{t \to -\infty} \int_{\mathbb{R}^2} h_{1,t}(x) dx = 0.$$

Concerning h_2, we have $(\nu_1 - \nu_2)(x)(x) = 0$ if $\operatorname{dist}(x, \Sigma) \geq \sigma$, and there holds

$$F_2(x, \lambda(x)^\beta e^t) = \frac{\lambda(x)^\beta e^t}{1 + \lambda(x)^\beta e^t} = \frac{|x|^\beta e^t}{1 + |x|^\beta e^t} \quad \text{for } |x| \geq r_3,$$

then

$$h_{2,t}(x) \leq c_{25} e^t |x|^{-\gamma_\infty - \beta_-} \quad \text{for } |x| \geq r_3, \tag{2.41}$$

which implies

$$\lim_{t \to -\infty} \int_{\mathbb{R}^2} h_{2,t}(x) dx = 0.$$

We claim that there exists $t_i \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} h_{i,t_i}(x) dx = \int_{\mathbb{R}^2} g_\beta(x) dx = 2\pi[2(N - M) + \beta]. \tag{2.42}$$

From the definition of F_i, (2.38) and the assumption on g_β,

$$\lim_{t \to +\infty} \int_{\mathbb{R}^2} h_{1,t}(x) dx = \infty \quad \text{and} \quad \lim_{t \to +\infty} \int_{\mathbb{R}^2} h_{2,t}(x) dx = \int_{\mathbb{R}^2} W dx > \int_{\mathbb{R}^2} g_\beta \, dx.$$

Since $t \mapsto \int_{\mathbb{R}^2} h_{i,t}(x) dx$ is continuous and increasing, it follows by the mean value theorem that there exists $t_i \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} h_{i,t_i}(x) dx = \int_{\mathbb{R}^2} g_\beta(x) dx.$$

Step 2. We use Lemma 2.2 to obtain some basic estimates on $w_{0,i} = \Gamma * (g_\beta - h_{i,t_i})$, taking into account the fact that $\int_{\mathbb{R}^2} (g_\beta - h_{i,t_i}) \, dx = 0$ and

$$-\Delta w_{0,i} = g_\beta - h_{i,t_i} \quad \text{in} \quad \mathbb{R}^2.$$

The function g_β is smooth with compact support, the functions h_{i,t_i} are locally integrable in $\mathbb{R}^2$ and satisfy

$$|h_{1,t_1}(x)| \leq c_{26} |x|^{-\gamma_\infty - \beta} \quad \text{and} \quad |h_{2,t_2}(x)| \leq c_{26} |x|^{-\gamma_\infty - \beta_-} \quad \text{for } |x| \text{ large enough.}$$

Since (2.36) holds, then, by Lemma 2.2, the functions $w_{0,i}$ are uniformly bounded in $\mathbb{R}^2$,

$$|w_{0,i}(x)| \leq c_{27}|x|^{\varrho_i} \quad \text{and} \quad |\nabla w_{0,i}(x)| \leq c_{28}|x|^{\varrho_i - 1} \quad \text{for } |x| \text{ large enough,} \tag{2.43}$$

where

$$\varrho_1 = -\frac{\gamma_\infty + \beta - 2}{\gamma_\infty + \beta - 1} \quad \text{and} \quad \varrho_2 = -\frac{\gamma_\infty + \beta_- - 2}{\gamma_\infty + \beta_- - 1}. \tag{2.44}$$

Step 3. In order to apply the classical iterative method we have to construct suitable supersolutions and subsolutions for equation (2.33).

Construction of the supersolution. Set $\overline{v}_i = (t_i)_+ + w_{0,i} + \|w_{0,i}\|_{L^\infty(\mathbb{R}^2)}$, then

$$\begin{aligned} -\Delta \overline{v}_i + WF_i(\lambda^\beta e^{\overline{v}_i}) &= g_\beta - WF_i(\lambda^\beta e^{t_i}) + WF_i(\lambda^\beta e^{\overline{v}_i}) \\ &\geq g_\beta, \end{aligned}$$

since $F_i(\lambda^\beta e^{\overline{v}_i}) \geq F_i(\lambda^\beta e^{t_i})$ as $\overline{v}_i \geq t_i$. Hence $\overline{v}_i$ is a super solution of (2.33) for $i = 1, 2$.

Construction of the subsolution. Set $\underline{v}_i = -(t_i)_- + w_{0,i} - \|w_{0,i}\|_{L^\infty(\mathbb{R}^2)}$, then

$$-\Delta \underline{v}_i + WF_i(\lambda^\beta e^{\underline{v}_i}) = g_\beta - WF_i(\lambda^\beta e^{t_i}) + WF_i(\lambda^\beta e^{\underline{v}_i}) \leq g_\beta,$$

since $F_i(\lambda^\beta e^{\underline{v}_i}) \leq F_i(\lambda^\beta e^{t_i})$ as $\underline{v}_i \leq t_i$. Hence $\underline{v}_i$ is a subsolution of (2.33) for $i = 1, 2$. As $\overline{v}_i > \underline{v}_i$ in $\mathbb{R}^2$, by a standard iterating process, see [31, Section 2.4.4], there exists a solution v_i of (2.33) such that

$$\underline{v}_i \leq v_i \leq \overline{v}_i \quad \text{in } \mathbb{R}^2.$$

Note that v_i belongs to $C^2(\mathbb{R}^2 \setminus \Sigma_1) \cap C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Uniqueness: Let $\tilde{v}_i$ be another solution of (2.33) and $w_i = \tilde{v}_i - v_i$, then

$$\begin{aligned} \Delta(w_i^2) &= 2w_i \Delta w_i + 2|\nabla w_i|^2 \\ &\geq 2w_i \Delta w_i \\ &= 2w_i \left(F_i(\lambda^\beta e^{\tilde{v}_i}) - F_i(\lambda^\beta e^{v_i}) \right) \geq 0, \end{aligned}$$

hence w_i^2 is bounded and subharmonic in $\mathbb{R}^2$. Thus w_i^2 is a constant by Liouville's theorem, that is $\tilde{v}_i = v_i + C$. Then $F_i(\lambda^\beta e^{v_i}) = F_i(\lambda^\beta e^{v_i + C})$. Thus $C = 0$ and uniqueness follows. We denote by $v_{\beta,i}$ this unique solution.

Step 3: asymptotic expansion. Now we shall employ Lemma 2.2 with $\Phi_i = WF_i(\lambda^\beta e^t) - g_\beta$, where g_β has compact support and (2.36) holds, thus

$$\limsup_{|x| \to +\infty} \left(|\Phi_1(x)||x|^{\gamma_\infty + \beta} + |\Phi_2(x)||x|^{\gamma_\infty + \beta_-} \right) < +\infty.$$

Therefore we have that

$$\limsup_{|x| \to +\infty} |\Gamma * \Phi_i(x)]|x|^{-\varrho_i} < +\infty. \tag{2.45}$$

The function $w = v_{\beta,i} - \Gamma * \Phi_i$ is harmonic and bounded, hence it is constant by Louville theorem. Denote this constant by $C_{\beta,i}$, we deduce that

$$v_{\beta,i} = C_{\beta,i} + O(|x|^{\varrho_i}) \ \text{ as } \ |x| \to +\infty.$$

The gradient estimates in (2.37) are the consequences of Corollary 2.1, which ends the proof. □

Corollary 2.3 *Under the assumptions of Theorem 2.1 the unique solutions $v_{\beta,i}$ of problem (2.33) with $i = 1, 2$ respectively, satisfy the flux identity*

$$\int_{\mathbb{R}^2} WF_i(\lambda^\beta e^{v_{\beta,i}})dx = 2\pi(2(N-M)+\beta). \tag{2.46}$$

Proof. For any $R > 0$, there holds

$$-\int_{|x|=R} \frac{\partial v_{\beta,i}}{\partial r} dS + \int_{B_R} WF_i(\lambda e^{v_{\beta,i}})dx = \int_{B_R} g_\beta dx.$$

By (2.37),

$$\left| \int_{|x|=R} \frac{\partial v_{\beta,i}}{\partial r} dS \right| = O(|x|^{\rho_i}) \quad \text{as} \ \ |x| \to +\infty,$$

where ρ_i is defined in (2.44). The result follows from (2.35). □

In the critical case $\beta = \beta^* := 2aN - 2$ where $a > 0$ and $0 < aN \leq 1$, the problem related to (1.1) is the following

$$-\Delta u + WF_2(\lambda^{\beta^*}\Lambda^{-2}e^u) = g_{\beta^*} \quad \text{in} \ \ \mathbb{R}^2, \tag{2.47}$$

where g_{β^*} expressed by

$$g_{\beta^*} = f_1 - f_2 + \beta^*\Delta \ln \lambda - 2\Delta \ln \Lambda, \tag{2.48}$$

is subject to the condition

$$\int_{\mathbb{R}^2} g_{\beta^*}\, dx = 2\pi[2(N-M)+\beta^*],$$

and W satisfies that

($\mathcal{W}_1$) *The function W is positive, locally Hölder continuous in $\mathbb{R}^2 \setminus \Sigma_1$ and satisfies*

$$W(x) \leq c_{29}|x-p_j|^{-2n_j a} \quad \text{in} \quad B_\sigma(p_j) \quad \text{and} \quad \limsup_{|x|\to\infty} \left(\left| |x|^{2aN} W(x) - 2 \right| |x| \right) < +\infty,$$

where $c_{29} > 0$, $n_j a < 1$ with $j = 1, \cdots, k$.

Theorem 2.2 *Let $F_2(s) = \frac{s}{e^{\nu_1-\nu_2+s}}$, g_{β^*} be defined in (2.48) with $\beta^* = 2(aN-1) \leq 0$ and W satisfies ($\mathcal{W}_1$). Assume furthermore that $M < (1+a)N - 1$ and set $\theta^* = \min\{3, 2-\beta^*\} \geq 2$. Then problem (2.47) has a unique bounded solution v and there exists $C_* \in \mathbb{R}$ such that*

$$\begin{aligned} v_*(x) &= C_* + O(|x|^{-\frac{\theta^*-2}{\theta^*-1}}) \qquad \text{as } |x| \to +\infty \\ |\nabla v(x)| &= O(|x|^{-1-\frac{\theta^*-2}{\theta^*-1}}) \qquad \text{as } |x| \to +\infty, \end{aligned} \tag{2.49}$$

if $aN < 1$, or

$$\begin{aligned} v_*(x) &= C_* + O\left((\ln|x|)^{-4}\right) \qquad \text{as } |x| \to +\infty \\ |\nabla v(x)| &= O\left(|x|^{-1}(\ln|x|)^{-4}\right) \qquad \text{as } |x| \to +\infty, \end{aligned} \tag{2.50}$$

if $aN = 1$.

Proof. Notice that the assumptions $aN \leq 1$ and $M < (1+a)N - 1$ imply $N - M > 0$. Set

$$\Lambda_0(x) = \frac{1}{1+|x|^2} \quad \text{for any } x \in \mathbb{R}^2, \tag{2.51}$$

and for $t \in \mathbb{R}$,

$$h_t(x) = \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t\Lambda_0(x)}}{e^{\nu_1(x)-\nu_2(x)} + \lambda^{\beta^*}\Lambda^{-2}e^{t\Lambda_0(x)}} \quad \text{for any } x \in \mathbb{R}^2 \setminus \Sigma,$$

with $h_t(x) = 0$ for $x \in \Sigma_2$. The function $h_t(\cdot)$ is continuous in $\mathbb{R}^2 \setminus \Sigma_1$ and $t \mapsto h_t(x)$ is increasing for all $x \in \mathbb{R}^2 \setminus \Sigma$. Direct computation implies the following properties:

$$h_t(x) \to W \quad \text{locally in } \mathbb{R}^2 \setminus \Sigma_2 \quad \text{as } t \to +\infty,$$

and

$$h_t(x) \to 0 \quad \text{locally in } \mathbb{R}^2 \setminus \Sigma_1 \quad \text{as } t \to -\infty.$$

Since $2aN \leq 2$, there holds

$$\int_{\mathbb{R}^2} W(x)dx = \infty.$$

Furthermore, there exist $\tau \in \mathbb{R}$ and $r_* > 0$ such that for any $t \leq \tau$ and $|x| \geq r_*$,

$$h_t(x) \leq c_{30}\frac{|x|^{-2}(\ln(|x|+1))^{-2}}{1+|x|^{2aN-2}(\ln(|x|+1))^{-2}},$$

where $c_{30} > 0$ depends on τ. Since $2aN - 2 \leq 0$, it follows that for $|x| \geq r_*$,

$$h_t(x) \leq c_{31}|x|^{-2}(\ln(|x|+1))^{-2}. \tag{2.52}$$

Hence, by the dominated convergence theorem,

$$\lim_{t\to-\infty}\int_{\mathbb{R}^2} h_t(x)dx = 0.$$

Using the fact that $t \mapsto \int_{\mathbb{R}^2} h_t(x)dx$ is increasing, there exists $t_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} h_{t_0}(x)dx = 2\pi[2(N-M)+\beta^*] = \int_{\mathbb{R}^2} g_{\beta^*}(x)dx. \tag{2.53}$$

We claim that for some $c_{32} > 0$,

$$|w_0(x)| \leq c_{32}|x|^{-\frac{\theta^*-2}{\theta^*-1}} \quad \text{for } |x| \text{ large enough}, \tag{2.54}$$

and if this holds true it will follow that $\|w_0\|_{L^\infty} < \infty$, where $w_0 = \Gamma * (g_{\beta^*} - h_{t_0})$.
Using (2.48),

$$g_{\beta^*}(x) = \frac{2}{|x|^2\Lambda^2(x)} \quad \text{for } |x| \geq r_1,$$

and

$$h_{t_0}(x) = \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}}{1+\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}} = \frac{2|x|^{-2}e^{t_0\Lambda_0}(1+O(|x|^{-1})}{\Lambda^2+|x|^{\beta^*}e^{t_0\Lambda_0}} \quad \text{as } |x| \to +\infty.$$

Therefore, we obtain that

$$\begin{aligned} g_{\beta^*} - h_{t_0} &= \frac{2e^{t_0\Lambda_0}}{|x|^2}\left(\frac{\Lambda^2 e^{t_0\Lambda_0}(1+O(|x|^{-1})) - \Lambda^2 - |x|^{\beta^*}e^{t_0\Lambda_0}}{\Lambda^2 e^{t_0\Lambda_0}(\Lambda^2+|x|^{\beta^*}e^{t_0\Lambda_0})}\right) \\ &= \frac{2e^{t_0\Lambda_0}\left(\Lambda^2(e^{t_0\Lambda_0}-1) - e^{t_0\Lambda_0}(|x|^{\beta^*} - \Lambda^2 O(|x|^{-1}))\right)}{|x|^2\Lambda^2 e^{t_0\Lambda_0}(\Lambda^2+|x|^{\beta^*}e^{t_0\Lambda_0})}. \end{aligned} \tag{2.55}$$

Since $\Lambda_0(x)$ is defined by (2.51), $e^{t_0\Lambda_0}-1 = O(|x|^{-2})$ at infinity. Noticing that $\beta^* = 0$ if $aN = 1$, we conclude that

$$|g_{\beta^*} - h_{t_0}| \leq c_{33}\max\left\{|x|^{-3}\Lambda^{-2}, |x|^{-2+\beta^*}\Lambda^{-4}\right\} \leq c_{33}\begin{cases} |x|^{-\theta^*} & \text{if } \ aN<1 \\ |x|^{-2}(\ln|x|)^{-4} & \text{if } \ aN=1. \end{cases} \tag{2.56}$$

Additionally, $\int_{\mathbb{R}^2} w_0 dx = 0$. Therefore, from Lemmas 2.2 and 2.3, we have that w_0 remains bounded on $\mathbb{R}^2$ and there exists $c_{34} > 0$ such that

$$\big|w_0(x)\big| \leq c_{34}(1+|x|)^{-\frac{\theta^*-2}{\theta^*-1}} \qquad \text{for all } \ x\in\mathbb{R}^2 \quad \text{if } aN<1 \tag{2.57}$$

and

$$\big|w_0(x)\big| \leq c_{34}\big(\ln(2+|x|)\big)^{-4} \qquad \text{for all } \ x\in\mathbb{R}^2 \quad \text{if } aN=1. \tag{2.58}$$

Existence. We first construct a supersolution. Set

$$\overline{v} = (t_0)_+ + w_0 + \|w_0\|_{L^\infty(\mathbb{R}^2)} \quad \text{in} \quad \mathbb{R}^2.$$

Since $\Lambda_0 : \mathbb{R}^2 \to (0,1]$, then $\overline{v} \geq t_0\Lambda_0$ in $\mathbb{R}^2$. The function $t \mapsto \frac{\Lambda^{-2}e^t}{e^{\nu_1-\nu_2}+\Lambda^{-2}e^t}$ is increasing, therefore,

$$\frac{W\lambda^{\beta^*}\Lambda^{-2}e^{\overline{v}}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{\overline{v}}} \geq \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}},$$

which implies,

$$-\Delta\overline{v} + \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{\overline{v}}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{\overline{v}}} - g_{\beta^*} \geq g_{\beta^*} - h_{t_0} + \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}} - g_{\beta^*} = 0,$$

then $\overline{v}$ is a super solution of (2.33).

Similarly we construct a subsolution by setting $\underline{v} = (t_0)_- + w_0 - \|w_0\|_{L^\infty(\mathbb{R}^2)}$. Using $\underline{v} \leq t_0\Lambda_0$ in $\mathbb{R}^2$ and by monotonicity, we have that

$$\frac{W\lambda^{\beta^*}\Lambda^{-2}e^{\underline{v}}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{\underline{v}}} \leq \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}},$$

thus,

$$-\Delta\underline{v} + \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{\underline{v}}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{\underline{v}}} - g_{\beta^*} \leq g_{\beta^*} - h_{t_0} + \frac{W\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2}e^{t_0\Lambda_0}} - g_{\beta^*} = 0,$$

thus $\underline{v}$ is a subsolution.

Since $\overline{v} > \underline{v}$, the standard iterative process, yields the existence of a solution v_* of (2.33) such that

$$\underline{v} \leq v_* \leq \overline{v} \quad \text{in} \quad \mathbb{R}^2.$$

As in the proof of Theorem 2.1 the solutions are unique in the class of bounded solutions, a class to which v_* belongs. Put $\Phi_* = g_{\beta^*} - WF_2(\lambda^*\Lambda^{-2}e^{v_*})$, then $w_* = v_* - \Gamma * \Phi_*$ is harmonic and bounded, hence it is a constant, say C_*. Since Φ_* satisfies the same estimate (2.57), with possibly another constant, we deduce from Lemma 2.2 that for all $x \in \mathbb{R}^2$

$$| \Gamma * \Phi_* | \leq c_{35} \begin{cases} (1+|x|)^{-\frac{\theta^*-2}{\theta^*-1}} & \text{if } \ aN < 1 \\ (1+|x|)^{-2}(\ln(2+|x|))^{-4} & \text{if } \ aN = 1. \end{cases} \tag{2.59}$$

This implies inequalities (2.49) and (2.50) by Lemma 2.2 and Lemma 2.3 and Corollary 2.1 and Corollary 2.2. □

Similarly as Corollary 2.3, there holds,

Corollary 2.4 *Under the assumptions of Theorem 2.2 the solutions v_{β^*} satisfy*

$$\int_{\mathbb{R}^2} WF_2(\lambda^{\beta^*}\Lambda^{-2}e^{v_{\beta^*}})dx = 2\pi(2(N-M)+\beta^*). \tag{2.60}$$

From the existence and uniqueness of solutions of (2.33) and (2.47), it is easy to prove the following statements.

Corollary 2.5 *Under the assumptions of Theorem 2.1, if $\overline{w}_i$ and $\underline{w}_i$ are respectively a bounded supersolution and a bounded subsolution of (2.33) such that $\underline{w}_i \leq \overline{w}_i$, then the standard iterative process will converge to the unique bounded solution v_i of (2.33), and $\underline{w}_i \leq v_i \leq \overline{w}_i$. A similar result holds concerning equation (2.47) under the assumption of Theorem 2.1.*

Corollary 2.6 *Under the assumptions of Theorem 2.2, the function $w_* := \lambda^{\beta^*}\Lambda^{-2} + v_*$ where v_* is the unique bounded solution of (2.47) satisfies*

$$-\Delta w_* + WF_2(e^{w_*}) = f_1 - f_2 \quad \text{in } \mathbb{R}^2. \tag{2.61}$$

3 Minimal solution

In order to consider solutions w of (2.3) with asymptotic behavior $\beta \ln|x| + O(1)$, we look for w under the form $w = \beta \ln|x| + v$ where v is a bounded function satisfying some related equation. In particular, we look for non-topological solution u_β of problem (1.1) under the form

$$u_\beta = -\nu_1 + \nu_2 + \beta \ln\lambda + v_\beta \quad \textit{or} \quad w_\beta = v_\beta + \beta \ln\lambda,$$

where λ is given by (2.30) and v_β is a bounded classical solution of

$$-\Delta v + \frac{V\lambda^\beta e^v}{(e^{\nu_1-\nu_2} + \lambda^\beta e^v)^{1+a}} = g_\beta \quad \text{in } \mathbb{R}^2, \tag{3.1}$$

with V being defined in (2.4) and where g_β is defined in (2.34). *Here and in what follows, we always assume that $a = 16\pi G$, $an_j < 1$ for $j = 1, \cdots, k$ and $\mathcal{M}$ is the total magnetic flux given in (1.9).*

We first consider the non-topological solutions of type I for problem (1.1) in the subcritical case, which are solutions verifying $u(x) = \beta \ln |x| + O(1)$ as $|x| \to \infty$ with $\beta < 0$. It is equivalent to look for classical solutions of (3.1) with $\beta < 0$.

Proposition 3.1 *Let N, M be positive integers verifying (1.12) and $aN \leq 1$, then for any $\beta \in (-2(N-M),\, \beta^*)$, problem (3.1) has a minimal bounded solution $v_{\beta,min}$ such that*

$$\int_{\mathbb{R}^2} V \frac{\lambda^\beta e^{v_{\beta,min}}}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_{\beta,min}})^{1+a}} dx = 2\pi(2(N-M)+\beta). \tag{3.2}$$

Proof. *Step 1: construction of an approximating scheme.* We recall that

$$\mathbf{P} = \frac{V}{e^{a(\nu_1-\nu_2)}} \quad \text{in } \mathbb{R}^2 \setminus \Sigma,$$

then

$$\lim_{x\to p_j} \mathbf{P}(x)|x-p_j|^{2n_j a} = A_0 \prod_{i\neq j}^{k} |p_i - p_j|^{-2an_i}, \qquad \lim_{|x-q_j|\to 0^+} \mathbf{P}(x) = 0,$$

and

$$\lim_{|x|\to\infty} \mathbf{P}(x)|x|^{2aN} = A_0. \tag{3.3}$$

Since $aN \leq 1$, there holds

$$\int_{\mathbb{R}^2} \mathbf{P}(x)dx = \infty,$$

then that $\mathbf{P}$ verifies the assumption $(\mathcal{W}_0)$ with $\tau_{p_j} = 2n_j a < 2$ and $\tau_\infty = 2aN > 2 - 2(N-M)$. Theorem 2.1-$(ii)$ implies that for any $\beta \in (-2(N-M),\, 2aN-2\})$, the nonlinear elliptic problem

$$-\Delta v + \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{\lambda^\beta e^v}{e^{\nu_1-\nu_2} + \lambda^\beta e^v} = g_\beta \quad \text{in } \mathbb{R}^2, \tag{3.4}$$

has a unique bounded solution v_0, which is continuous in $\mathbb{R}^2$, smooth in $\mathbb{R}^2 \setminus \Sigma$ and

$$\int_{\mathbb{R}^2} \left(\frac{V}{e^{a(\nu_1-\nu_2)}} \frac{\lambda^\beta e^{v_0}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_0}} - g_\beta \right) dx = 0$$

by the same argument as in Theorem 2.1-(ii); then there exists a constant $C_{0,\beta}$ such that

$$\lim_{|x|\to+\infty} v_0(x) = C_{0,\beta} \quad \text{and} \quad v_0(x) - C_{0,\beta} = O\left(|x|^{-\frac{2aN-\beta-2}{2aN-\beta-1}}\right) \quad \text{as } |x| \to +\infty. \tag{3.5}$$

We set

$$W_0 = \mathbf{P} \text{ and } W_1 = \frac{V}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_0})^a} = \frac{e^{a(\nu_1-\nu_2)}}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_0})^a} W_0 \quad \text{in } \mathbb{R}^2 \setminus \Sigma.$$

The function W_1 is positive and Hölder continuous in $\mathbb{R}^2 \setminus \Sigma_1$, and since

$$0 < W_1(x) \leq W_0(x) \quad \forall x \in \mathbb{R}^2, \tag{3.6}$$

it satisfies $\mathcal{W}_0$. Furthermore, as $N - M > 0$, $v_0(x) \to 0$ as $|x| \to \infty$ and $\beta < 0$ and therefore $W_1(x) = W_0(x)(1 + o(1))$ as $|x| \to \infty$. Applying Theorem 2.1-(ii), with $\gamma_\infty = 2aN$, we see that there exists a unique bounded function v_1 satisfying

$$-\Delta v_1 + W_1 \frac{\lambda^\beta e^{v_1}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1}} = g_\beta \quad \text{in } \mathbb{R}^2. \tag{3.7}$$

Furthermore, $v_1(x)$ converges to some constant $C_{1,\beta}$ when $x \to +\infty$ and

$$v_1(x) = C_{1,\beta} + O\left(|x|^{-\frac{2aN-\beta-2}{2aN-\beta-1}}\right) \quad \text{as } |x| \to +\infty. \tag{3.8}$$

Set $z = v_0 - v_1$. Since the function $t \mapsto \frac{\lambda^\beta e^t}{e^{\nu_1-\nu_2}+\lambda^\beta e^t}$ is nondecreasing, it follows that

$$\begin{aligned} -\Delta z_+^2 &= 2z_+(W_1 - W_0)\frac{\lambda^\beta e^{v_1}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1}} - 2W_0 z_+ \left(\frac{\lambda^\beta e^{v_0}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_0}} - \frac{\lambda^\beta e^{v_1}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1}}\right) - 2|\nabla z_+|^2 \\ &\leq 0. \end{aligned}$$

Hence z_+^2 is subharmonic and bounded, it is therefore constant. Hence $(v_0 - v_1)_+ = C \geq 0$. If $C > 0$ then $\sup\{v_0 - v_1, 0\} = C$, which implies that $v_0 - v_1 = C$. Replacing v_0 by $v_1 + C$ we deduce from (3.4), (3.7)

$$\frac{V}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1+c})^a}\frac{\lambda^\beta e^{v_1}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1}} = \frac{V}{e^{a(\nu_1-\nu_2)}}\frac{\lambda^\beta e^{v_1+c}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1+C}},$$

which yields

$$e^C\left(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1}\right) = e^{a(\nu_1-\nu_2)}\left(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_1+c}\right)^{1-a} \quad \text{in } \mathbb{R}^2 \setminus \Sigma.$$

Since $\beta < 0$, we obtain $e^C = 1$ by letting $|x| \to \infty$. Hence $C = 0$ which implies $v_0 \leq v_1$ in $\mathbb{R}^2$ and $C_{1,\beta} \geq C_{0,\beta}$.

By induction, we suppose that for $n \geq 2$ we have constructed the sequence $\{v_k\}_{k<n}$ of bounded solutions to

$$-\Delta v_k + W_k \frac{\lambda^\beta e^{v_k}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_k}} = g_\beta \quad \text{in } \mathbb{R}^2, \tag{3.9}$$

where

$$W_k = \frac{e^{a(\nu_1-\nu_2)}}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_{k-1}})^a} W_0.$$

Then $0 < W_k \leq W_{k-1} \leq ... \leq W_0$ and therefore $v_0 \leq ... \leq v_{k-1} \leq v_k$, and furthermore

$$v_k(x) = C_{k,\beta} + O\left(|x|^{-\frac{2aN-\beta-2}{2aN-\beta-1}}\right) \quad \text{as } |x| \to +\infty. \tag{3.10}$$

Then v_n is the unique bounded solutions of

$$-\Delta v_n + W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2}+\lambda^\beta e^{v_n}} = g_\beta \quad \text{in } \mathbb{R}^2, \tag{3.11}$$

where

$$W_n = \frac{e^{a(\nu_1-\nu_2)}}{(e^{\nu_1-\nu_2}+\lambda^\beta e^{v_{n-1}})^a} W_0 \leq W_{n-1} = \frac{e^{a(\nu_1-\nu_2)}}{(e^{\nu_1-\nu_2}+\lambda^\beta e^{v_{n-2}})^a} W_0,$$

since $v_{n-2} \leq v_{n-1}$ by induction. Furthermore, by Lemma 2.2 (since $\beta < 0$ and $N - M > 0$),

$$v_n(x) = C_{n,\beta} + O\left(|x|^{-\frac{2aN-\beta-2}{2aN-\beta-1}}\right) \quad \text{as } |x| \to +\infty. \tag{3.12}$$

As above the function $(v_{n-1} - v_n)_+^2$ is subharmonic and bounded, hence it is constant, which implies $v_{n-1} = v_n + C$, $C \geq 0$. Then from the equations satisfied by v_n and v_{n-1},

$$1 \geq \frac{W_n}{W_{n-1}} = e^C \frac{e^{\nu_1-\nu_2}+\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2}+\lambda^\beta e^C e^{v_n}} \geq 1.$$

Hence $e^C = 1$, then $C = 0$ and $v_{n-1} \leq v_n$. Consequently $n \mapsto C_{n,\beta}$ is increasing.

Let $R > 1$ be such that $\mathrm{supp}(g_\beta) \subset B_R$ and $\Theta(x) := \Gamma * |g_\beta|(x) = \frac{1}{2\pi}\int_{B_R} |g_\beta(x)| \ln|x-y| dy$. For $|x| \geq R+1$, one has $1 \leq |x-y| \leq |x| + R$, hence

$$0 \leq \ln|x-y| \leq \ln(|x|+R) \leq \ln|x| + \frac{R}{|x|} \leq \ln|x| + 1,$$

therefore

$$0 \leq \Theta(x) \leq \frac{R^2\|g_\beta\|_{L^\infty}}{2}(\ln|x|+1) \quad \text{if } |x| \geq R+1.$$

Since $|\Theta|$ is bounded from above on B_{R+1} by some constant c_{36}, we deduce

$$|\Theta(x)| \leq c_{36} + \frac{R^2\|g_\beta\|_{L^\infty}}{2}(\ln_+|x|+1) \quad \text{for all } x \in \mathbb{R}^2. \tag{3.13}$$

Set $z = v_n - \Theta$, then

$$-\Delta z_+^2 \leq -2z_+\Delta z = -2z_+ W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2}+\lambda^\beta e^{v_n}} \leq 0.$$

The function z_+ has compact support because of (3.13). It is subharmonic, nonnegative and bounded, hence it is constant with zero value necessarily, hence, for any $n \in \mathbb{N}$,

$$v_0(x) \leq v_n(x) \leq c_{36} + \frac{R^2\|g_\beta\|_{L^\infty}}{2}(\ln_+|x|+1) \quad \text{for all } x \in \mathbb{R}^2. \tag{3.14}$$

For $\epsilon > 0$ set

$$w_\epsilon(x) = \epsilon \ln|x| + c_{36} + \frac{R^2\|g_\beta\|_{L^\infty}}{2}(\ln R + 1).$$

Then w_ϵ is harmonic in $\overline{B_R}^c$. It is larger than v_n for $|x| = R$ and also at infinity, since v_n is bounded. If we set $Z = v_n - w_\epsilon$, then as above the function Z_+^2 is subharmonic, nonnegative and bounded in B_R^c. Since it vanishes for $|x| = R$, its extension ζ by 0 in $\overline{B_R}$ is still subharmonic nonnegative and bounded. It is therefore constant. Since it vanishes at infinity, it is identically 0. Hence $v_n - w_\epsilon \leq 0$. Letting $\epsilon \to 0$ we obtain

$$v_0(x) \leq v_n(x) \leq c_{36} + \frac{R^2 \|g_\beta\|_{L^\infty}}{2} (\ln R + 1) \quad \text{for all } x \in \mathbb{R}^2. \tag{3.15}$$

Combining Lemma 2.2 with (3.15) for $n = 1$ we infer

$$| \, v_n(x) - C_{n,\beta} \, | \leq c_{37}(1 + |x|)^{-\frac{2aN-\beta-2}{2aN-\beta-1}} \quad \text{for all } \ x \in \mathbb{R}^2, \tag{3.16}$$

where $c_{37} > 0$ is independent of n. By Lemma 2.2

$$| \, \nabla v_n(x) \, | \leq c_{38}|x|^{-1-\frac{2aN-\beta-2}{2aN-\beta-1}} \quad \text{for } |x| \text{ large enough}, \tag{3.17}$$

then,

$$-\int_{|x|=R} \frac{\partial v_n}{\partial r} dS + \int_{B_R} W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_n}} dx = \int_{B_R} g_\beta dx.$$

By (3.17), the first integral tends to 0 when $R \to +\infty$, therefore

$$\int_{\mathbb{R}^2} W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_n}} dx = \int_{\mathbb{R}^2} g_\beta dx = 2\pi(2(N-M) + \beta). \tag{3.18}$$

Set $v_{\beta,min} = \lim\limits_{n\to\infty} v_n$ and $C_\beta = \lim\limits_{n\to\infty} C_{n,\beta}$, then

$$W_n \to W_\infty := \frac{e^{a(\nu_1-\nu_2)}}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_{\beta,min}})^a} W_0$$

and

$$| \, v_{\beta,min}(x) - C_\beta \, | \leq c_{39}(1 + |x|)^{-\frac{2aN+2(N-M)-\beta-2}{2aN+2(N-M)-\beta-1}} \quad \text{for all } \ x \in \mathbb{R}^2. \tag{3.19}$$

Furthermore

$$0 \leq W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_n}} \leq W_0 \frac{\lambda^\beta e^{v_{\beta,min}}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_{\beta,min}}}.$$

The right-hand side of the above inequality is an integrable function, therefore

$$W_n \frac{\lambda^\beta e^{v_n}}{e^{\nu_1-\nu_2} + \lambda^\beta e^{v_n}} \to W_0 \frac{e^{a(\nu_1-\nu_2)} \lambda^\beta e^{v_{\beta,min}}}{(e^{\nu_1-\nu_2} + \lambda^\beta e^{v_{\beta,min}})^{1+a}} \quad \text{in } \ L^1(\mathbb{R}^2) \quad \text{as } \ n \to +\infty.$$

This implies that $v_{\beta,min}$ is a weak solution of (1.1) and relation (3.2) holds.

Step 2: $v_{\beta,min}$ is minimal among the bounded solutions. Let $\tilde{v}$ be any bounded solution. Then $\frac{V}{(e^{\nu_1-\nu_2+\lambda^\beta e^{\tilde{v}}})^a} \leq \frac{V}{e^{a(\nu_1-\nu_2)}}$, and by uniqueness, it implies $v_0 \leq \tilde{v}$. Hence $\frac{V}{(e^{\nu_1-\nu_2+\lambda^\beta e^{\tilde{v}}})^a} \leq \frac{V}{(e^{\nu_1-\nu_2+\lambda^\beta e^{v_0}})^a}$ and therefore $v_1 \leq \tilde{v}$. By induction we obtain $v_n \leq \tilde{v}$ and finally $v_{\beta,min} \leq \tilde{v}$.

Step 3: asymptotic behaviour. Put

$$F = g_\beta - \frac{V\lambda^\beta e^{v_{\beta,min}}}{(e^{\nu_1-\nu_2}+\lambda^\beta e^{v_{\beta,min}})^{1+a}}.$$

Then $\int_{\mathbb{R}^2} F\,dx = 0$ and $|F(x)| \le c_{40}|x|^{-(2aN-\beta)}$ for $|x| \ge r_0$. So we have (3.2), and applying Lemma 2.2 yields the estimate

$$|\Gamma * F| \le c_{41}(1+|x|)^{-\frac{2aN-\beta-2}{2aN-\beta-1}} \quad \text{for all } x \in \mathbb{R}^2.$$

Therefore $w = v_{\beta,min} - \Gamma * F$ is harmonic and bounded in $\mathbb{R}^2$. It is therefore constant. This implies

$$v_{\beta,min} = C_\beta + O(|x|^{-\frac{2aN-\beta-2}{2aN-\beta-1}}) \quad \text{as } |x| \to +\infty. \tag{3.20}$$

This ends the proof. □

Proof of Theorem 1.2 part (i)**.** Let

$$u_{\beta,min} = -\nu_1 + \nu_2 + \beta \ln \lambda + v_{\beta,min},$$

where $v_{\beta,min}$ is the minimal bounded solution of (3.1) obtained in Proposition 3.1. Then $u_{\beta,min}$ is the minimal non-topological solution of type I of (1.1) in the sense that

$$u_{\beta,min}(x) - \beta \ln|x| = O(1) \quad \text{as} \quad |x| \to +\infty.$$

Moreover, $u_{\beta,min}$ verifies (1.13) and its total magnetic flux is $2\pi[2(N-M)+\beta]$ by (3.2). □

4 Critical-minimal solutions

4.1 Non-topological solutions

If $N > M$ and $aN < 1$, we recall that by Theorem 2.1, for any $\beta \in (-2(N-M), \beta^*)$, with $\beta^* = 2(aN-1) < 0$, there exists a unique bounded solution v_β to equation

$$-\Delta v + \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{\lambda^\beta e^v}{e^{\nu_1-\nu_2}+\lambda^\beta e^v} = g_\beta \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

and by Theorem 2.2, there exists a unique bounded solution v_{β^*} to

$$-\Delta v + \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{\lambda^{\beta^*}\Lambda^{-2} e^v}{e^{\nu_1-\nu_2}+\lambda^{\beta^*}\Lambda^{-2} e^v} = g_{\beta^*} \quad \text{in } \mathbb{R}^2. \tag{4.2}$$

For $\beta \in (-2(N-M),\ \beta^*)$, we first set

$$w_\beta = v_\beta + \beta \ln \lambda. \tag{4.3}$$

Then w_β is a the unique solution of

$$-\Delta w + \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{e^w}{e^{\nu_1-\nu_2}+e^w} = f_1 - f_2 \quad \text{in } \mathbb{R}^2 \tag{4.4}$$

such that $w - \beta \ln \lambda$ is bounded in $\mathbb{R}^2$.
When $\beta = \beta^*$, we set

$$w_{\beta^*} = v_{\beta^*} + \beta^* \ln \lambda - 2 \ln \Lambda, \tag{4.5}$$

thus w_{β^*} is the unique solution of (4.4) such that $w - \beta^* \ln \lambda - 2 \ln \Lambda$ remains bounded in $\mathbb{R}^2$.

Proposition 4.1 *Under the assumptions of Theorem 2.2, the mapping $\beta \mapsto w_\beta$ is increasing for $\beta \in (-2(N-M),\ \beta^*)$ and*

$$w_{\beta^*} = \sup \left\{ w_\beta \text{ in } \mathbb{R}^2 :\ \beta \in (-2(N-M),\ \beta^*) \right\}.$$

Proof. If $\beta^* > \beta > \beta' > -2(N-M)$, the function $z = w_{\beta'} - w_\beta$ is negative in B_R^c for some $R > 0$. Hence

$$-\Delta z_+^2 = -2 z_+ \Delta z - 2|\nabla z_+|^2 \leq -\frac{2V}{e^{a(\nu_1-\nu_2)}} \left(\frac{e^w_{\beta'}}{e^{\nu_1-\nu_2} + e^{w_{\beta'}}} - \frac{e^w_\beta}{e^{\nu_1-\nu_2} + e^{w_\beta}} \right) (w_{\beta'} - w_\beta) \leq 0.$$

Hence z_+^2 is a nonnegative and bounded subharmonic function in $\mathbb{R}^2$, it is therefore constant. Since it vanishes in B_R^c, it is identically 0, which yields $w_{\beta'} \leq w_\beta$. Actually the inequality is strict since it is the case at infinity and there cannot exist $x_0 \in \mathbb{R}^2$ such that $w_{\beta'}(x_0) = w_\beta(x_0)$, because of the strong maximum principle. Similarly, if $\beta < \beta^*$, there holds by (4.3) and (4.5),

$$(w_\beta - w_{\beta^*})(x) = (\beta - \beta^*) \ln |x| + 2 \ln(\ln |x|) + O(1) \quad \text{as } |x| \to +\infty.$$

Hence $z_+^2 = (w_\beta - w_{\beta^*})_+^2$ is subharmonic nonnegative and bounded, hence it is constant and necessarily with value zero. Therefore $w_\beta \leq w_{\beta^*}$, and actually $w_\beta < w_{\beta^*}$ by the strong maximum principle. We set

$$\tilde{w}_{\beta^*} := \sup\{w_\beta : \beta \in (-2(N-M), \beta^*)\} = \lim_{\beta \uparrow \beta^*} w_\beta.$$

Then $\tilde{w}_{\beta^*} \leq w_{\beta^*}$ and w_{β^*} is a solution of (4.4). By the strong maximum principle, either $\tilde{w}_{\beta^*} < w_{\beta^*}$ or $\tilde{w}_{\beta^*} = w_{\beta^*}$. In order to identify w_{β^*}, we use the flux identities obtained in Corollaries 2.3 and 2.4, replacing $v_{\beta,2}$ and v_{β^*} by their respective expressions from (4.3) and (4.5):

$$\mathcal{M}(w_\beta) = \int_{\mathbb{R}^2} \frac{V}{e^{\nu_1-\nu_2}} \frac{e^{w_\beta}}{e^{\nu_1-\nu_2} + e^{w_\beta}} = 2\pi(2(N-M) + \beta),$$

and

$$\mathcal{M}(w_{\beta^*}) = \int_{\mathbb{R}^2} \frac{V}{e^{\nu_1-\nu_2}} \frac{e^{w_{\beta^*}}}{e^{\nu_1-\nu_2} + e^{w_{\beta^*}}} dx = 2\pi(2(N-M) + \beta^*).$$

Since the mapping $\beta \mapsto \frac{e^{w_\beta}}{1+e^{w_\beta}}$ is increasing, there holds by the monotone convergence theorem,

$$\begin{aligned} \mathcal{M}(\tilde{w}_{\beta^*}) &= \int_{\mathbb{R}^2} \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{e^{\tilde{w}_{\beta^*}}}{e^{\nu_1-\nu_2} + e^{\tilde{w}_{\beta^*}}} dx \\ &= \lim_{\beta \uparrow \beta^*} \int_{\mathbb{R}^2} \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{e^{w_\beta}}{e^{\nu_1-\nu_2} + e^{w_\beta}} dx \\ &= 2\pi(2(N-M) + \beta^*) \\ &= \int_{\mathbb{R}^2} \frac{V}{e^{a(\nu_1-\nu_2)}} \frac{e^{w_{\beta^*}}}{e^{\nu_1-\nu_2} + e^{\tilde{w}_{\beta^*}}} dx, \end{aligned} \tag{4.6}$$

where $\frac{e^{\tilde{w}_{\beta^*}}}{e^{\nu_1-\nu_2}+e^{\tilde{w}_{\beta^*}}} \leq \frac{e^{w_{\beta^*}}}{e^{\nu_1-\nu_2}+e^{w_{\beta^*}}}$. This implies that $\tilde{w}_{\beta^*} = w_{\beta^*}$ almost everywhere and actually everywhere by continuity. □

4.2 Proof of Theorem 1.3

If $v_{\beta,min}$ is the minimal bounded solution of (3.1) obtained in Proposition 3.1, we set

$$w_{\beta,min} = v_{\beta,min} + \beta \ln \lambda \quad \text{in} \quad \mathbb{R}^2. \tag{4.7}$$

Then $w_{\beta,min}$ is a solution of

$$\begin{cases} -\Delta w + \dfrac{Ve^w}{(e^{\nu_1-\nu_2}+e^w)^{1+a}} = f_1 - f_2 \quad \text{in } \mathbb{R}^2, \\ w = \beta \ln \lambda + O(1) \qquad \text{as } |x| \to +\infty. \end{cases} \tag{4.8}$$

Since $v_{\beta,min}$ is the minimal bounded solution of (3.1), $w_{\beta,min}$ is the minimal solution of (4.8).

Furthermore, $v_{\beta,min}$ is the limit of the increasing sequence of the bounded solutions $\{v_n\}$ of (3.11), therefore $w_{\beta,min}$ is the limit of the increasing sequence $\{w_{\beta,n}\} := \{v_n + \beta \ln \lambda\}$ of the solutions of

$$\begin{cases} -\Delta w_{\beta,n} + \dfrac{V}{(e^{\nu_1-\nu_2}+e^{w_{\beta,n-1}})^a} \dfrac{e^{w_{\beta,n}}}{e^{\nu_1-\nu_2}+e^{w_{\beta,n}}} = f_1 - f_2 \quad \text{in } \mathbb{R}^2, \\ w_{\beta,n} = \beta \ln \lambda + O(1) \qquad \text{as } |x| \to \infty. \end{cases} \tag{4.9}$$

By the comparison principle, the mapping $\beta \in (-2(N-M),\, \beta^*) \mapsto w_{\beta,n}$ is increasing for any n, and this is also true for $\beta \mapsto w_{\beta,min}$. By (3.15) there holds for any $n \in \mathbb{N}$,

$$w_0(x) \leq w_{\beta,n}(x) \leq w_{\beta,min}(x) \leq c_{42} + \frac{R^2 \|g_\beta\|_{L^\infty}}{2}(\ln R + 1) + \beta \ln \lambda(x) \qquad \text{in } \mathbb{R}^2. \tag{4.10}$$

Uniformly upper bound for $\{w_{\beta,min}\}_\beta$. Let $\overline{v}_2 = \Gamma * g_{\beta^*}$, then $\overline{v}_2 = \Gamma * (f_1 - f_2) + \beta^* \ln \lambda - 2 \ln \Lambda$ and

$$\lim_{|x|\to\infty} \frac{\overline{v}_2(x)}{\ln |x|} = 2(N-M) + \beta^* > 0.$$

Since $\overline{v}_2$ is a super solution of (4.9), we have by comparison

$$w_{\beta,n} \leq \overline{v}_2 \quad \text{in} \quad \mathbb{R}^2,$$

which implies that for any $\beta \in (-2(N-M),\, \beta^*)$

$$w_{\beta,min} \leq \overline{v}_2 \quad \text{in} \quad \mathbb{R}^2.$$

Hence there exists $w_{\beta^*,min} = \lim\limits_{\beta \uparrow \beta^*} w_{\beta,min}$ and

$$w_{\beta^*,min} \leq \overline{v}_2 \quad \text{in} \quad \mathbb{R}^2,$$

and therefore

$$w_{\beta^*,min}(x) \leq \beta^* \ln|x| - 2\ln\ln|x| + C \tag{4.11}$$

for some $C \in \mathbb{R}$.

Lower bound for $w_{\beta^*,min}$. From Proposition 4.1, the equation

$$-\Delta w + \frac{V}{e^{a(\nu_1-\nu_2)}}\frac{e^w}{e^{\nu_1-\nu_2}+e^w} = f_1 - f_2 \quad \text{in } \mathbb{R}^2 \tag{4.12}$$

has a unique solution w_{β^*}, with the following asymptotic behavior

$$w_{\beta^*}(x) = \beta^* \ln|x| - 2\ln\ln|x| + O(1) \quad \text{as} \quad |x| \to +\infty,$$

and w_{β^*} is the limit of the solutions w_β of (4.4) for $\beta \in (-2(N-M), \beta^*)$ satisfying

$$w_\beta(x) = \beta \ln|x| + O(1) \quad \text{as } |x| \to +\infty.$$

Since w_β is a subsolution for (4.8) it is bounded from above by $w_{\beta,min}$ by the same comparison method as the ones used previously. Therefore $w_{\beta^*} \leq w_{\beta^*,min}$. Combining (4.11) with the expression of w_{β^*} given in (4.11), we infer that

$$w_{\beta^*,m}(x) = \beta^* \ln|x| - 2\ln\ln|x| + O(1) \quad \text{as } |x| \to +\infty.$$

Clearly the flux identity holds as in the previous theorem, which ends the proof. □

5 Multiple solutions

5.1 Non-topological solutions

Let $\beta \neq 0$ and u_β be a solution of problem (1.1) with the asymptotic behavior

$$u_\beta(x) = \beta \ln|x| + O(1) \quad \text{as } |x| \to +\infty.$$

Then u_β can written under the form

$$u_\beta = -\nu_1 + \nu_2 + \beta \ln\lambda + v_\beta,$$

where v_β is a bounded solution of the following equation equivalent to (3.1)

$$-\Delta v + W_\beta \frac{e^v}{(e^{\nu_1-\nu_2}\lambda^{-\beta}+e^v)^{1+a}} = g_\beta \quad \text{in } \mathbb{R}^2 \tag{5.1}$$

with $W_\beta = V\lambda^{-a\beta}$, and where g_β is expressed by

$$g_\beta = f_1 - f_2 + \beta\Delta\ln\lambda.$$

Note that it is a smooth function with compact support in $B_{r_0}(0)$ and it verifies

$$\int_{\mathbb{R}^2} g_\beta \, dx = 2\pi[2(N-M)+\beta].$$

As for W_β it satisfies

$$\lim_{x\to p_j} W_\beta(x) = A_0(\prod_{i\neq j}|p_j-p_i|^{2n_i})^{-a}\,, \quad \lim_{x\to q_j} W_\beta(x)=0 \ \text{ and } \ \lim_{|x|\to\infty} W_\beta(x)|x|^{2aN+a\beta}=A_0.$$

The existence of multiple solutions states as follows:

Proposition 5.1 *Let N, M be positive integers and $\beta^\sharp$ be given in (1.10). Then for any $\beta>\beta^\sharp$ problem (5.1) possesses a sequence of solutions $v_{\beta,i}$ such that*

$$\int_{\mathbb{R}^2} W_\beta \frac{e^{v_{\beta,i}}}{(e^{\nu_1-\nu_2}\lambda^{-\beta}+e^{v_{\beta,i}})^{1+a}}dx = 2\pi[2(N-M)+\beta], \tag{5.2}$$

and

$$v_{\beta,i}(x) = C_{\beta,i} + O(|x|^{-\frac{a\beta+2aN-2}{a\beta+2aN-1}}) \quad as \ \ |x|\to+\infty$$

with

$$C_{\beta,i}\to\infty \ \text{ as } \ i\to+\infty.$$

Proof. By Theorem 2.1, for any $A>0$ the equation

$$-\Delta w + e^{-A(1+a)}W_\beta e^w = g_\beta \quad \text{in } \mathbb{R}^2 \tag{5.3}$$

has a unique bounded solution w_A. We note that

$$w_A = w_0 + A(1+a),$$

where w_0 is the bounded solution of (5.3) with $A=0$. Note that for any $A\geq A^*=a^{-1}\|w_0\|_{L^\infty(\mathbb{R}^2)}$,

$$w_A \geq A \quad \text{in} \quad \mathbb{R}^2.$$

Step 1: construction of an approximating sequence. We set $v_0:=w_A$ and define $H_t(.)$ by

$$H_0(t,\cdot) = \begin{cases} A_0 & \text{in } \Sigma_1, \\ 0 & \text{in } \Sigma_2, \\ W_\beta \dfrac{e^t}{(e^{\nu_1-\nu_2}\lambda^{-\beta}+e^{v_0})^{1+a}} & \text{in } \mathbb{R}^2\setminus\Sigma. \end{cases}$$

Under the assumptions, $H_0(t,\cdot)\in L^\delta(\mathbb{R}^2)$ for some $\delta>1$ and there exists a unique (and explicit) real number t_1 such that

$$\int_{\mathbb{R}^2} H_0(t_1,x)dx = 2\pi(2(N-M)+\beta).$$

We construct first a bounded solution v_1 of

$$-\Delta v + W_\beta \frac{e^v}{(e^{\nu_1-\nu_2}\lambda^{-\beta}+e^{v_0})^{1+a}} = g_\beta \quad \text{in } \mathbb{R}^2. \tag{5.4}$$

We set

$$w_1 = \Phi * (g_\beta - H_0(t_1, \cdot)).$$

By Lemma 2.2, w_1 is bounded. Put $\overline{v} = \|v_0\|_{L^\infty} + w_1 + \|w_1\|_{L^\infty} + |t_1|$. Then

$$\frac{e^{\overline{v}}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}} \geq \frac{e^{t_1}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}},$$

therefore

$$-\Delta\overline{v} + W_\beta \frac{e^{\overline{v}}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}} - g_\beta \geq g_\beta - H(t_1, \cdot) + W_\beta \frac{e^{t_1}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}} - g_\beta \geq 0.$$

Hence $\overline{v}$ is a supersolution of (5.4). Since

$$-\Delta v_0 + W_\beta \frac{e^{v_0}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}} - g_\beta \leq W_\beta e^{v_0} \left(\frac{1}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_0})^{1+a}} - \frac{1}{e^{(1+a)A}} \right) \leq 0,$$

v_0 is a subsolution of (5.4) dominated by $\overline{v}$. Hence there exists a solution $v = v_1$ of (5.4) satisfying

$$v_0 \leq v_1 \leq \overline{v}.$$

Since $a\beta + 2aN > 2$, we have from Lemma 2.2

$$v_1(x) = C_{1,\beta} + O(|x|^{-\frac{a\beta+2aN-2}{a\beta+2aN-1}}) \quad \text{as} \quad |x| \to \infty.$$

We define a sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_0 = w_A$ and $v = v_n$ is the bounded solution of

$$-\Delta v + W_\beta \frac{e^{v}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{n-1}})^{1+a}} = g_\beta \quad \text{in } \mathbb{R}^2. \tag{5.5}$$

Assume that we have proved the existence and boundedness of the functions v_k for $k < n$ and that there holds $v_0 \leq v_1 \leq ... \leq v_{n-1}$. We define $H_{n-1}(t, .)$ by

$$H_{n-1}(t, .) = \begin{cases} A_0 & \text{in } \Sigma_1 \\ 0 & \text{in } \Sigma_2 \\ W_\beta \dfrac{e^{t}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{n-1}})^{1+a}} & \text{in } \mathbb{R}^2 \setminus \Sigma, \end{cases}$$

and denote by t_n the unique real number such that

$$\int_{\mathbb{R}^2} H_{n-1}(t, x)dx = 2\pi(2(N - M) + \beta).$$

Since $v_0 \leq v_1 \leq ... \leq v_{n-1}$, there holds $t_0 < t_1 < ... < t_n$. If we set $w_n = \Gamma * (g_\beta - H_{n-1}(t_n, .))$, clearly $\overline{v}_n := \|v_{n-1}\|_{L^\infty} + w_n + \|w_n\|_{L^\infty} + |t_n|$ is a supersolution. Furthermore

$$-\Delta v_{n-1} + W_\beta \frac{e^{v_{n-1}}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{n-1}})^{1+a}} - g_\beta$$
$$\leq W_\beta e^{v_{n-1}} \left(\frac{1}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{n-1}})^{1+a}} - \frac{1}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{n-2}})^{1+a}} \right) \leq 0.$$

Hence v_{n-1} is a subsolution. A solution $v = v_n$ of (5.5) satisfying $v_{n-1} \leq v_n \leq \overline{v}_n$ exists. It is bounded and satisfies

$$v_n(x) = C_{n,\beta} + O(|x|^{-\frac{a\beta+2aN-2}{a\beta+2aN-1}}) \quad \text{as} \quad |x| \to \infty \tag{5.6}$$

for some $C_{n,\beta}$. Furthermore the sequence $\{C_{n,\beta}\}$ is nondecreasing, and by Corollary 2.1

$$|\nabla v_n(x)| = O(|x|^{-1-\frac{a\beta+2aN-2}{a\beta+2aN-1}}) \quad \text{as} \quad |x| \to \infty. \tag{5.7}$$

Uniformly upper bound for $\{v_n\}_n$. Let $\overline{v}_\beta = \Gamma * g_\beta$, then it is a supersolution of (5.5) for any $n \in \mathbb{N}$ and satisfies

$$\lim_{|x|\to\infty} \frac{\overline{v}_\beta(x)}{\ln|x|} = 2(N-M) + \beta.$$

This implies that for any for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\overline{v}_\beta(x) \leq (2(N-M) + \beta + \epsilon)\ln(|x|+1) + C_\epsilon \quad \text{in} \quad \mathbb{R}^2. \tag{5.8}$$

Note that $\overline{v}_\beta$ is a super solution of (5.5) and by the comparison principle

$$v_n \leq \overline{v}_\beta \quad \text{in} \quad \mathbb{R}^2.$$

Therefore the limit of the sequence $\{v_n\}$ as $n \to \infty$ exists. As it depends also on A, we denote it by $v_{\beta,A}$ and there holds

$$v_{\beta,A} \leq \overline{v}_\beta \quad \text{in} \quad \mathbb{R}^2.$$

Furthermore $v_{\beta,A}$ is a locally bounded solution of (5.1) which satisfies

$$A \leq v_{\beta,A}(x) \leq (2(N-M) + \beta + \epsilon)\ln(|x|+1) + C_\epsilon \quad \text{in} \quad \mathbb{R}^2. \tag{5.9}$$

Because of the above lower estimate, the functions $x \mapsto \frac{e^{v_n(x)}}{(e^{\nu_1-\nu_2}\lambda^{-\beta}+e^{v_n(x)})^{1+a}}$ are upper bounded on $\mathbb{R}^2$ by some constant depending on A and β but independent of n, and this estimate holds true if v_n is replaced by $v_{\beta,A}$. Hence for any $R > 0$,

$$-\int_{|x|=R} \frac{\partial v_n}{\partial r} dS + \int_{B_R} W_\beta \frac{e^{v_n}}{e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_n}} dx = \int_{B_R} g_\beta dx.$$

By (5.7) the integral term on $|x| = R$ tends to 0 when $R \to \infty$, therefore

$$\int_{\mathbb{R}^2} W_\beta \frac{e^{v_n}}{e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_n}} dx = \int_{\mathbb{R}^2} g_\beta dx = 2\pi(2(N-M)+\beta). \tag{5.10}$$

Since $\frac{e^{v_n}}{e^{\nu_1-\nu_2}\lambda^{-\beta}+e^{v_n}}$ is bounded independently of n, it follows by the dominated convergence theorem that

$$\int_{\mathbb{R}^2} W_\beta \frac{e^{v_{\beta,A}}}{e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{\beta,A}}} dx = \int_{\mathbb{R}^2} g_\beta dx. \tag{5.11}$$

Combining this identity with the estimate

$$\left| g_\beta(x) - W_\beta(x) \frac{e^{v_{\beta,A}(x)}}{(e^{\nu_1-\nu_2}\lambda^{-\beta} + e^{v_{\beta,A}(x)})^{1+a}} \right| \leq c_{43}(1+|x|)^{-2aN-a\beta},$$

and using Lemma 2.2, we infer that $v_{\beta,A}$ is uniformly bounded in $\mathbb{R}^2$ and that there exists $C_{\beta,A} > A$ such that

$$v_{\beta,A}(x) = C_{\beta,A} + O(|x|^{-\frac{a\beta+2aN-2}{a\beta+2aN-1}}) \quad \text{as} \quad |x| \to +\infty. \tag{5.12}$$

In order to construct the sequence of solutions, we start with $A = A_0 = 1$, then take $A = A_1 = \inf\{k \in \mathbb{N} : k > C_{\beta,A}\}$ and we iterate this process, defining by induction A_{i+1} by $A_{i+1} = \inf\{k \in \mathbb{N} : k > C_{\beta,A_i}\}$.

□

Proof of Theorem 1.2 part (ii) **and Theorem 1.4.** *Multiple solutions.* Let

$$u_{\beta,i} = -\nu_1 + \nu_2 + \beta \ln \lambda + v_{\beta,i},$$

where $\{v_{\beta,i}\}_i$ is a sequence solutions of (5.1) which exist by Proposition 5.1. Then $\{u_{\beta,i}\}_i$ is a sequence of non-topological solutions in type II of (1.1) verifying (1.13) and with total magnetic flux $2\pi[2(N-M)+\beta]$. The proof is now complete. □

5.2 Topological solution

Proof of Theorem 1.5. *Multiple Topological solutions.* Let u be a topological solution of problem (1.1). We can write it as $u = -\nu_1 + \nu_2 + v$ where v is a bounded regular solution of

$$-\Delta v + V\frac{e^v}{(e^{\nu_1-\nu_2}+e^v)^{1+a}} = g_0 \quad \text{in } \mathbb{R}^2 \tag{5.13}$$

with

$$g_0 = f_1 - f_2,$$

and where the functions f_1 and f_2 have been defined in (2.4). They are smooth, have compact support in $B_{r_0}(0)$ and the flux identity (2.5) is satisfied.

Claim: Problem (5.13) possesses a sequence of bounded solutions $\{v_i\}_i$ such that

$$\int_{\mathbb{R}^2} V\frac{e^{v_i}}{(e^{\nu_1-\nu_2}+e^{v_i})^{1+a}}dx = 4\pi(N-M), \tag{5.14}$$

and

$$v_i(x) = C_i + O(|x|^{-\frac{2aN-2}{2aN-1}}) \quad \text{as} \quad |x| \to +\infty, \tag{5.15}$$

with $C_i \to +\infty$ as $i \to +\infty$.

This can be proved as follows: given $A > 0$, let w_A be the bounded solution of

$$-\Delta w + e^{-A(1+a)}Ve^w = g_0 \quad \text{in } \mathbb{R}^2. \tag{5.16}$$

We note that

$$w_A = w_0 + A(1+a),$$

where w_0 is a bounded solution of (5.3) with $A = 0$. Note also that if $A \geq A^* = a^{-1}\|w_0\|_{L^\infty(\mathbb{R}^2)}$, then

$$w_A \geq A \quad \text{in} \quad \mathbb{R}^2.$$

We set $\mu_0 = w_A$, and define μ_n ($n \in \mathbb{N}$) to be the solution of

$$-\Delta\mu_n + V\frac{e^{\mu_n}}{(e^{\nu_1-\nu_2}+e^{\mu_{n-1}})^{1+a}} = g_0 \quad \text{in } \mathbb{R}^2. \tag{5.17}$$

As in the proof of Proposition 5.1, the mapping $n \mapsto \mu_n$ is increasing and μ_n is uniformly upper bounded. It converges to some function v_A as $n \to +\infty$, and v_A is a weak solution of (5.13). Since $V(x) \le c_{22}|x|^{-2aN}$ when $|x| \to +\infty$, and $2aN > 2$, there holds

$$\mu_n(x) = C_{n,A} + O\left(|x|^{-\frac{2aN-2}{2aN-1}}\right) \quad \text{and} \quad |\nabla\mu_n(x)| \le c_{44}|x|^{-1-\frac{2aN-2}{2aN-1}} \quad \text{as } |x| \to +\infty.$$

Integrating (5.17) on B_R and letting $R \to \infty$ yields

$$\int_{\mathbb{R}^2} V\frac{e^{\mu_n}}{(e^{\nu_1-\nu_2}+e^{\mu_{n-1}})^{1+a}}dx = \int_{\mathbb{R}^2}(f_1-f_2)dx = 4\pi(N-M).$$

Because $\frac{e^{\mu_n}}{(e^{\nu_1-\nu_2}+e^{\mu_n-1})^{1+a}}$ is uniformly bounded and $V \in L^1(\mathbb{R}^2)$ we obtain by the dominated convergence theorem

$$\int_{\mathbb{R}^2} V\frac{e^{v_A}}{(e^{\nu_1-\nu_2}+e^{v_A})^{1+a}}dx = 4\pi(N-M). \tag{5.18}$$

Therefore,

$$v_A(x) = C_A + O(|x|^{-\frac{2aN-2}{2aN-1}}) \quad \text{as } |x| \to +\infty, \tag{5.19}$$

and the end of the proof is similar as the one of Proposition 5.1.

□

6 Nonexistence

Lemma 6.1 *Let $aN < 1$. Then*

(i) Problem (1.1) has no solution u_β verifying

$$u_\beta(x) - \beta\ln|x| = o(\ln|x|) \quad \text{as } |x| \to +\infty \tag{6.1}$$

for $\beta^* < \beta \le 0$.

(ii) Problem (1.1) has no solution u_β verifying (6.1) if $0 \le \beta < \frac{2-aN}{a}$.

(iii) Problem (1.1) has no topological solution.

Proof. We recall that a solution verifying (6.1) with $\beta < 0$ (resp. $\beta > 0$) is called non-topological of type II (resp. type I). Given a function, we denote by $\overline{w}$ the circular average of w, i.e.

$$\overline{w}(r) = \frac{1}{2\pi r}\int_{\partial B_r(0)} w(\xi)d\theta(\xi) = \frac{1}{2\pi}\int_0^{2\pi} w(r,\theta)d\theta.$$

For $|x| \ge r_0$, there exists $c_{45} > 0$ such that

$$\mathbf{P}(x) \ge c_{45}|x|^{-aN},$$

and we set, for all $x \in \mathbb{R}^2$

$$h_u(x) = \mathbf{P}(x)\frac{e^{u(x)}}{(1+e^{u(x)})^{1+a}}. \tag{6.2}$$

Part (i). If u is a non-topological solution of Type I, it satisfies $u(x) \leq c_{46}$ for $|x| \geq r_0$ and $c_{46} > 0$. By Jensen inequality there exists positive constants c_{47} and c_{48} such that

$$\overline{h}_u(r) \geq \frac{c_{47}A_0 r^{-aN}}{(1+e^{c_{24}})^{1+a}}\overline{e^{u(x)}} \geq c_{48}e^{\overline{u}(r)} \quad \text{for} \quad r > r_0, \tag{6.3}$$

and from (6.1), there exist $\epsilon_0 \in (0,\, 1)$ and $c_{49} > 0$ such that for $r > r_0$,

$$\overline{h}_u(r) \geq \frac{c_{49}}{r^{2-\epsilon_0}}. \tag{6.4}$$

Then (1.1) implies that

$$(r\overline{u}_r)_r \geq \frac{c_{49}}{r^{1-\epsilon_0}} \quad \text{for} \quad r \geq r_0,$$

thus, integrating the above inequalities, we obtain

$$r\overline{u}_r(r) - r_0\overline{u}_r(r_0) \geq c_{50}(r^{\epsilon_1} - r_0^{\epsilon_1}),$$

where $\overline{u}_r = \frac{d\overline{u}}{dr}$ and c_{49}, $c_{50} > 0$. Hence there holds

$$\overline{u}(r) \geq \overline{u}(r_0) + (r_0\overline{u}_r(r_0) - c_{50}r_0^{\epsilon_1})\ln r + \frac{c_{50}}{\epsilon_1}r^{\epsilon_1} \quad \text{for } r > r_0.$$

As a consequence,

$$\overline{u}(r) \to +\infty \quad \text{as} \quad r \to +\infty, \tag{6.5}$$

which contradicts the fact that u is bounded from above.

Part (ii). If u is a non-topological solution of Type II and $0 \leq \beta < \frac{2-aN}{a}$, then

$$h_u(x) \quad \geq \quad \mathbf{P}(x)\frac{1}{(1+e^u)^a} \geq \frac{c_{51}}{|x|^{2-\epsilon_1}}$$

for some $\epsilon_1 > 0$ and $c_{51} > 0$. Then (1.1) implies that

$$(r\overline{w}_r)_r \geq \frac{c_{52}}{r^{1-\epsilon_1}} \quad \text{for} \quad r \geq r_0.$$

Hence there holds

$$\overline{w}(r) \geq \overline{w}(r_0) + (r_0\overline{w}_r(r_0) - c_{52}r_0^{\epsilon_1})\ln r + \frac{c_{52}}{\epsilon_1}r^{\epsilon_1} \quad \text{for } r > r_0,$$

which contradicts (6.1).

Part (iii). The proof is the same as above. □

Proof of Theorem 1.6. If $aN < 1$, Lemma 6.1 implies that then problem (1.1) has no solution u_β for $\beta^* < \beta < \frac{2-aN}{a}$ verifying $u_\beta(x) = \beta \ln|x| + O(1)$.

Next we assume that $aN = 1$, and u is a topological solution (1.1). Hence u is bounded at infinity and

$$h_u(x) \geq \mathbf{P}(x)\frac{e^u}{(1+e^u)^a} \geq \frac{c_{53}}{|x|^{-2}}.$$

Then (1.1) implies that

$$(r\overline{u}_r)_r \geq \frac{c_{54}}{r} \quad \text{for } r \geq r_0.$$

By integrating this inequality we encounter a contradiction with the fact that u is bounded at infinity. □

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References

[1] R. Beeker. *Electromagnetic Fields and Interactions*, Dover, New York (1982).

[2] A. Belavin and A. Polyakov. Metastable states of two-dimensional isotropic ferromagnets, *JETP Lett. 22,* 245-247 (1975).

[3] P. Bénilan and H. Brezis. Nonlinear problems related to the Thomas-Fermi equation, in Nonlinear Evolution Equations and Related Topics, *J. Evolution Eq. 3*, 673-770 (2004).

[4] H. Brezis and P. L. Lions. A note on isolated singularities for linear elliptic equations, *Math. Anal. and Appl. 7A*, 263-266 (1981).

[5] H. Brezis and F. Merle. Uniform estimates and blow-up behavior for solutions of $\Delta u = V(x)e^u$ in two dimensions, *Comm. Part. Diff. Eq. 16*, 1223-1253 (1991).

[6] M. Cantor. Elliptic operators and the decomposition of tensor fields, *Bull. Amr. Math. Soc. 5*, 235-262 (1981).

[7] M. Chae. Existence of multi-string solutions of the gauged harmonic map model, *Lett. Math. Phys. 59(2)*, 173-188 (2002).

[8] H. Chan, C. Fu and C. Lin. Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation, *Comm. Math. Phys. 231*, 189-221 (2002).

[9] J. Chern and Z. Yang. Evaluating solutions on an elliptic problem in a gravitational gauge field theory, *J. Funct. Anal. 265(7)*, 1240-1263 (2013).

[10] H. Chen and F. Zhou. On governing equation of Gauged Sigma model for Heisenberg ferromagnet, *Nonlinear Analysis 196,* 111788, 16 pp (2020).

[11] H. Chen and H. Hajaiej, Classification of non-topological solutions of a self-dual Gauged Sigma model, *Preprint* (2018).

[12] N. Choi and J. Han, Classification of solutions of elliptic equations arising from a gravitational $O(3)$ gauge field model, *J. Diff. Eq. 264(8)*, 4944-4988 (2018).

[13] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin/New York (1983).

[14] J. Han and H. Huh. Existence of topological solutions in the Maxwell gauged $O(3)$ sigma models, *J. Math. Anal. Appl. 386*, 61-74 (2012).

[15] W. Hayman. Slowly growing integral and subharmonic functions, *Comment. Math. Helv. 34*, 75-84 (1960).

[16] A. Jaffe and C. Taubes. *Vortices and Monoples*, Birkhäuser, Boston (1980).

[17] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math. 10*, 503-510 (1957).

[18] J. Jost and G. Wang. Analytic aspects of the Toda system: I. A Moser-Trudinger inequality, *Comm. Pure Appl. Math. 54(11)*, 1289-1319 (2001).

[19] F. Lin and Y. Yang. Gauged harmonic maps, Born-Infeld electromagnetism, and magnetic vortices, *Comm. Pure Appl. Math. 56(11)*, 1631-1665 (2003).

[20] C. Lin, J. Wei. and D. Ye. Classification and nondegeneracy of $SU(n+1)$ Toda system with singular sources, *Invent. Math. 190(1)*, 169-207 (2012).

[21] C. Lin, A. Ponce, Y. Yang. A system of elliptic equations arising in Chern-Simons field theory, *J. Funct. Anal. 247(2)*, 289-350 (2007).

[22] R. McOwen. The behavior of the Laplacian on weighted Sobolev spaces, *Comm. Pure Appl. Math. 32*, 783-795 (1979).

[23] R. Osserman. On the inequality $\Delta u = f(u)$, *Pacific J. Math. 7*, 1641-1647 (1957).

[24] A. Poliakovsky and G. Tarantello. On non-topological solutions for planar Liouville Systems of Toda-type, *Comm. Math. Phys. 347*, 223-270 (2016).

[25] R. Rajaraman. *Solitons and Instantons*, North Holland, Amsterdam (1982).

[26] Y. Richard and L. Véron. Isotropic inequalities of solutions of nonlinear elliptic inequalities, *Ann. Inst. H. Poincaré, Anal. Non Linéaire 6*, 37-72 (1989).

[27] B. Schroers. Bogomol'nyi solitons in a gauged $O(3)$ sigma model, *Phys. Lett. B. 356,* 291-296 (1995).

[28] K. Song. Improved existence results of solutions to the gravitational Maxwell gauged $O(3)$ sigma model, *Proc. Amer. Math. Soc. 144*, 3499-3505 (2016).

[29] J. Vázquez. On a semilinear equation in $\mathbb{R}^2$ involving bounded measures, *Proc. Roy. Soc. Edinburgh 95*, 181-202 (1983).

[30] L. Véron. Elliptic equations involving Measures, Stationary Partial Differential equations, *Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam* (2004).

[31] Y. Yang. *Solitons in Field Theory and Nonlinear Analysis*, Springer Science & Business Media (2013).

[32] Y. Yang. A necessary and sufficient conditions for the existence of multisolitons in a self-dual gauged sigma model, *Comm. Math. Phys. 181*, 485-506 (1996).

[33] Y. Yang. The Existence of Solitons in Gauged Sigma Models with Broken Symmetry: Some Remarks, *Lett. Math. Phys. 40*, 177-189 (1997).

[34] Y. Yang. *Advances in Nonlinear Partial Differential Equations and Related Areas*, World Scientific, Singapore, (1998).