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# Modeling Error of $\alpha$ -Models of Turbulence on a Two-Dimensional Torus

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## Abstract

This paper is devoted to study the rate of convergence of the weak solutions  $\mathbf{u}_\alpha$  of  $\alpha$ -regularization models, namely the Leray- $\alpha$ , the modified Leray- $\alpha$  and the simplified Bardina models, to the weak solution  $\mathbf{u}$  of the Navier-Stokes equations in the two-dimensional periodic case, as the regularization parameter  $\alpha$  goes to zero.

**Key words :** Rate of convergence,  $\alpha$ -turbulence models, Navier-Stokes Equations.

**2010 MSC:** 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

In this work we study the rate of convergence of weak solutions of several two dimensional  $\alpha$ -models of turbulence to the weak solution of the Navier-Stokes equations (NSE), with periodic boundary conditions. These models belong to the class of Large Eddy Simulation models (LES), used to carry out numerical simulations of turbulence flows, that cannot be performed by the NSE because, according to Kolmogorov laws, it would require  $\mathcal{O}(Re^{d^2/4})$  degrees of freedom where  $d = 2, 3$ , which is inaccessible to modern computers for high Reynolds numbers [7]. Considering the 2D case is appropriate to analyse models that simulate layers of shallow water in stratified flows, such as the ocean or the atmosphere [6, 29].

Let  $L$  denote a given length scale,  $\mathbf{u}(t, \mathbf{x})$  and  $p(t, \mathbf{x})$  for  $t > 0$ ,  $\mathbf{x} \in \mathbb{R}^2/[0, L]^2 = \mathbb{T}_2$ , denote the velocity and the pressure of the fluid, that satisfy the NSE,

$$(1.1) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathbb{T}_2,$$

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{T}_2,$$

$$(1.3) \quad \mathbf{u}|_{t=0} = \mathbf{u}_0,$$

where the constant  $\nu > 0$  denotes the kinematic viscosity,  $\mathbf{u}_0$  and  $\mathbf{f}$  are given as the initial velocity and the external forces. The  $\alpha$ -model aim at regularizing the nonlinear term

$(\mathbf{u} \cdot \nabla)\mathbf{u}$ , and are given by the general abstract form

$$(1.4) \quad \partial_t \mathbf{u}_\alpha + N(\mathbf{u}_\alpha) - \nu \Delta \mathbf{u}_\alpha + \nabla p_\alpha = \mathbf{f}, \quad \text{in } \mathbb{T}_2$$

$$(1.5) \quad \nabla \cdot \mathbf{u}_\alpha = 0, \quad \text{in } \mathbb{T}_2$$

$$(1.6) \quad \mathbf{u}_\alpha|_{t=0} = \mathbf{u}_0,$$

where  $\mathbf{u}_\alpha$  and  $p_\alpha$  are the filtered velocity and pressure at frequencies of order  $1/\alpha$ . The  $\alpha$ -models under study herein are the Leray- $\alpha$ , simplified Bardina and the modified Leray- $\alpha$  models, given by

$$(1.7) \quad N(\mathbf{u}_\alpha) = \begin{cases} (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, & \text{Leray-}\alpha \text{ (L-}\alpha\text{)}, \\ (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, & \text{Bardina model (SB)}, \\ (\mathbf{u}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, & \text{Modified Leray-}\alpha \text{ (ML-}\alpha\text{)}, \end{cases}$$

and the bar operator is given by the Helmholtz equation

$$(1.8) \quad \bar{\mathbf{v}} - \alpha^2 \Delta \bar{\mathbf{v}} = \mathbf{v} \quad \text{in } \mathbb{T}_2.$$

The initial model is due to J. Leray [26] in the whole space  $\mathbb{R}^3$ , where  $\bar{\mathbf{v}} = \mathbf{v} \star \rho_\alpha$ , for a given mollifier  $\rho_\alpha$ ,  $N_\alpha(\mathbf{v}) = (\bar{\mathbf{v}} \cdot \nabla)\mathbf{v}$ . Note that in the whole space there is also a Kernel  $G_\alpha$  such that  $\bar{\mathbf{v}} = \mathbf{v} \star G_\alpha$  for the Helmholtz filter [30]. This class of models has been the subject of many works this last two decades, see for instance [5, 9, 10, 11, 12, 15, 16, 20, 21, 23, 24]. They are known to have regular unique solutions, at least in  $L_t^\infty H_x^1 \cap L_t^2 H_x^2$ , that converge to solutions of the NSE as  $\alpha \rightarrow 0$ , with suitable assumptions about the data, which means that  $\mathbf{u}_\alpha \rightarrow \mathbf{u}$ ,  $p_\alpha \rightarrow p$ , where  $(\mathbf{u}, p)$  is the corresponding weak solution of the NSE.

We aim in this paper to study the rate of convergence, namely the norm of

$$(1.9) \quad \mathbf{e} = \mathbf{u}_\alpha - \mathbf{u}$$

in various space such as  $L_t^\infty L_x^2$ ,  $L_t^2 H_x^1$ ,  $L_t^\infty H_x^1$  and  $L_t^2 H_x^2$ , as  $\alpha \rightarrow 0$ . This study is motivated by the paper of Cao and Titi [4], in which the authors prove that for all 2D  $\alpha$ -models (1.4)-(1.7), one has the following  $L_t^\infty L_x^2$  estimate on a given time interval  $[0, T]$ ,  $\forall \alpha \leq L/2\pi$ :

$$(1.10) \quad \sup_{t \in [0, T]} \|\mathbf{e}(t)\|^2 \leq C\alpha^2 \left( CT \left( 1 + \log \left( \frac{L}{2\pi\alpha} \right) \right) + C \right),$$

where  $C$  is a constant and when no risk of confusion occurs,  $\|\cdot\|$  stands for the  $L^2$  norm. In this results, it is assumed that

$$\mathbf{u}_0 \in \mathcal{D}(-P_\sigma \Delta), \quad \mathbf{f} \in L^2([0, T], \mathcal{P}_\sigma L^2(\mathbb{T}_2)^2),$$

$P_\sigma$  being the Leray projector. The logarithmic factor that appears in (1.10) comes from an inequality initially proved by Brezis-Gallouet in [3].

Cao-Titi's result raises two questions.

- i) Is that possible to improve the  $O(\alpha^2 \log(1/\alpha))$  rate, and in better norms ?
- ii) Can we get an estimate global in time ?

In this paper we positively answer to these questions by showing that when

$$\mathbf{u}_0 \in \mathcal{P}_\sigma H^1(\mathbb{T}_2)^2, \quad \mathbf{f} \in L^2(\mathbb{R}_+, \mathcal{P}_\sigma L^2(\mathbb{T}_2)^2),$$

we get an estimate uniform in time of order  $O(\alpha^3)$  in the  $L_t^\infty L_x^2 \cap L_t^2 H_x^1$  norms, specifically for all  $\alpha$ -models (1.4)-(1.7),

$$(1.11) \quad \|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2 dt \leq C\alpha^3, \quad \forall s \geq 0,$$

where  $C$  is a time-independent constant, see Theorem 4.1 below. We also get a uniform in time estimate in the  $L_t^\infty H_x^1 \cap L_t^2 H_x^2$  norm of order  $O(\alpha^2)$  for the L- $\alpha$  model, and in  $O(\alpha^2 \log(1/\alpha))$  for SB and ML- $\alpha$  model, namely for all  $s \geq 0$ ,

$$(1.12) \quad \|\nabla \mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{e}\|^2 dt \leq \begin{cases} C\alpha^2 & \text{for L-}\alpha, \\ C\alpha^2 \left( C \log \left( \frac{L}{2\pi\alpha} \right) + C \right) & \text{for SB and ML-}\alpha \end{cases}$$

see Theorem 4.2 below. Estimates (1.11) and (1.12) are the main results in the present work.

Thanks to (1.11)-(1.12), we are able to study the rates of convergence of the pressures, by showing (see Theorem 5.1 below)

$$(1.13) \quad \int_0^s \|\nabla q\|^2 dt \leq \begin{cases} C\alpha^{5/2} & \text{for L-}\alpha, \\ C\alpha^2 \left( C \log \left( \frac{L}{2\pi\alpha} \right) + C \right) & \text{for SB models and ML-}\alpha, \end{cases}$$

where  $C$  does not depend on times and  $q = p_\alpha - p$ .

The paper is organized as follows. In Section 2 we set the mathematical framework. In Section 3 we derive from energy balances uniform in time energy estimates weak solutions of the NSE and for all  $\alpha$ -models as well. This is the main step before investigating the rate of convergence in Section 4, where we prove the estimates (1.11)-(1.12). Section 5 is devoted to study the pressure. In Section 6, we make additional remarks about the 3D case for which the situation is quite different and not in the focus of the present paper.

## 2 Mathematical framework

In this section we set the functional spaces we are working with. We show basic properties of the Helmholtz filter, then we carry out the Leray projection of the NSE and Leray- $\alpha$  on free divergence fields spaces. The section ended with a brief state of the art about  $\alpha$ -models.

### 2.1 Functional spaces

Let  $\mathbb{T}_2 = \mathbb{R}^2/[0, L]^2$  be a 2D torus, for  $1 \leq p \leq \infty$ ,  $m \in \mathbb{N}$ , let  $L^p(\mathbb{T}_2)$  and  $H^m(\mathbb{T}_2)$  denote the standard Lebesgue and Sobolev spaces on  $\mathbb{T}_2$ , respectively. The  $L^p(\mathbb{T}_2)$ -norm is denoted by  $\|\cdot\|_p$  for all  $1 \leq p \leq \infty$ , except for the case  $p = 2$  where  $\|\cdot\| \equiv \|\cdot\|_2$ . Bold symbols are used for vectors, matrices, or space of vectors. We denote by  $\Pi$  the set of all trigonometric polynomials on  $\mathbb{T}_2$  with spatial zero mean, *i.e.*,

$$\int_{\mathbb{T}_2} \phi(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \phi \in \Pi.$$

Let us define

$$\Lambda := \{\boldsymbol{\varphi} \in \Pi^2 : \nabla \cdot \boldsymbol{\varphi} = 0\}.$$

As usual when studying the NSE we define the following standard Hilbert functional spaces

$$\begin{aligned}\mathbf{H} &:= \text{the closure of } \mathbf{\Lambda} \text{ in } L^2(\mathbb{T}_2)^2, \\ \mathbf{V} &:= \text{the closure of } \mathbf{\Lambda} \text{ in } H^1(\mathbb{T}_2)^2.\end{aligned}$$

Let  $(\cdot, \cdot)$  and  $\|\cdot\|$  be the standard inner product and norm on  $\mathbf{H}$ , that are

$$(\mathbf{u}, \mathbf{v}) := \int_{\mathbb{T}_2} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} \quad \text{and} \quad \|\mathbf{u}\|^2 := \int_{\mathbb{T}_2} |\mathbf{u}|^2 \, d\mathbf{x}.$$

The inner product  $(\mathbf{u}, \mathbf{v})_{\mathbf{V}}$  and the corresponding norm  $\|\mathbf{u}\|_{\mathbf{V}}$  on  $\mathbf{V}$  are defined as follows

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{V}} := \|\nabla \mathbf{u}\|.$$

In the sequel, we use the notation  $P_\sigma$  for denoting the Helmholtz-Leray orthogonal projection operator of  $\mathbf{L}^2(\mathbb{T}_2)$  onto  $\mathbf{H}$ . We next consider an orthonormal basis  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$  of  $\mathbf{H}$  consisting of eigenfunctions of the the Laplace operator

$$-\Delta : \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V} \longrightarrow \mathbf{H}$$

and for  $m \geq 1$ ,  $\mathbf{H}_m = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ .

Let  $A = -P_\sigma \Delta$  be the Stokes operator, the domain of which is given by  $\mathcal{D}(A) := \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}$ . Then we know [4, 16]:

$$(2.1) \quad A\mathbf{u} = -P_\sigma \Delta \mathbf{u} = -\Delta \mathbf{u} \quad \forall \mathbf{u} \in \mathcal{D}(A).$$

Let  $\lambda_1 > 0$  be the first eigenvalue of  $A$ , i.e.,  $A\varphi_1 = \lambda_1 \varphi_1$ , and the above setting leads to  $\lambda_1 = (2\pi/L)^2$ . By the virtue of the Poincaré inequality we have

$$(2.2) \quad \lambda_1 \|\mathbf{u}\|^2 \leq \|\nabla \mathbf{u}\|^2 \quad \forall \mathbf{u} \in V,$$

$$(2.3) \quad \lambda_1 \|\nabla \mathbf{u}\|^2 \leq \|A\mathbf{u}\|^2 = \|\Delta \mathbf{u}\|^2 \quad \forall \mathbf{u} \in \mathcal{D}(A).$$

Then it follows by (2.2)-(2.3) that there exist positive dimensionless constants  $c_1, c_2$  such that

$$(2.4) \quad c_1 \|A\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{H}^2(\mathbb{T}_2)} \leq c_2 \|A\mathbf{u}\| \quad \forall \mathbf{u} \in \mathcal{D}(A).$$

In the following, we will make an intensive use of the following 2D-Ladyzhenskaya inequality [22]:

$$(2.5) \quad \forall \mathbf{u} \in \mathbf{V}, \quad \|\mathbf{u}\|_4 \leq C \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2},$$

where  $C$  is a non-negative dimensionless constant.

## 2.2 Helmholtz filter

The filter operator used to construct the turbulence models is the Helmholtz filter, see Germano [18], or [2, 14, 25]. Given a cut length  $\alpha > 0$  (which will be called the filter radius), for each  $\mathbf{u} \in \mathbf{H}$ ,  $\bar{\mathbf{u}} \in \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}$  is the unique solution of the following Helmholtz equation:

$$(2.6) \quad \bar{\mathbf{u}} - \alpha^2 \Delta \bar{\mathbf{u}} = \mathbf{u} \quad \text{in } \mathbb{T}_2,$$

We deduce from (2.6),

$$\|\mathbf{u} - \bar{\mathbf{u}}\| = \alpha^2 \|\Delta \bar{\mathbf{u}}\| \quad \forall \mathbf{u} \in \mathbf{H}.$$

We already know that the filter satisfies the inequality, see [13]:

$$(2.7) \quad \|\bar{\mathbf{u}}\| + \alpha \|\nabla \bar{\mathbf{u}}\| + \alpha^2 \|\Delta \bar{\mathbf{u}}\| \leq C \|\mathbf{u}\| \quad \forall \mathbf{u} \in \mathbf{H},$$

where  $C$  is a Sobolev constant. It follows that

$$(2.8) \quad \|\nabla \mathbf{u} - \nabla \bar{\mathbf{u}}\| = \alpha^2 \|\nabla \Delta \bar{\mathbf{u}}\| \leq C\alpha \|\Delta \mathbf{u}\| \quad \forall \mathbf{u} \in \mathcal{D}(A).$$

## 2.3 Leray Projector

Troughout the rest of the paper we assume

$$(2.9) \quad \mathbf{u}_0 \in \mathbf{V}, \quad \mathbf{f} \in L^2(\mathbb{R}_+, \mathbf{H}).$$

In order to eliminate the pressure, we apply the Helmholtz-Leray orthogonal projection  $\mathcal{P}_\sigma$  on free divergence fields to both the NSE and  $\alpha$ -models. We get the differential systems:

$$(2.10) \quad \begin{aligned} \frac{d\mathbf{u}}{dt} + P_\sigma[(\mathbf{u} \cdot \nabla)\mathbf{u}] - \nu \Delta \mathbf{u} &= \mathbf{f}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, \end{aligned}$$

as well as

$$(2.11) \quad \begin{aligned} \frac{d\mathbf{u}_\alpha}{dt} + P_\sigma[N(\mathbf{u}_\alpha)] - \nu \Delta \mathbf{u}_\alpha &= \mathbf{f}, \\ \mathbf{u}_\alpha|_{t=0} &= \mathbf{u}_0, \end{aligned}$$

where one has used the facts that  $P_\sigma \mathbf{f} \equiv \mathbf{f}$  since  $\mathbf{f} \in \mathbf{H}$ ,  $P_\sigma \Delta \mathbf{u} = \Delta \mathbf{u}$  and  $P_\sigma(\nabla p) = P_\sigma(\nabla p_\alpha) = 0$ . Once the velocity is calculated, the pressure is  $p$  and  $p_\alpha$  are solutions of the equations

$$(2.12) \quad -\Delta p = \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}), \quad -\Delta p_\alpha = \nabla \cdot (N(\mathbf{u}_\alpha)).$$

From now we will only consider the velocity speaking as a solution to the NSE and the  $\alpha$ -models.

**Remark 2.1.** *Thanks to the Leray-Helmholtz decomposition and for simplicity we assume that  $\mathbf{f}$  is divergence free. Otherwise, the gradient part of  $\mathbf{f}$  can be added to the modified pressure and  $P_\sigma \mathbf{f}$  is replaced by  $\mathbf{f}$ .*

**Remark 2.2.** *A common property of all  $\alpha$ -models which are considered in the present paper is that these models reduce to the NSE when  $\alpha = 0$ . It can be seen directly from the equality (2.6).*

## 2.4 Brief State of the art

In the 2D case, there exists a unique solution of the NSE, global in time, without any singularity, see Temam [36, 37].

The proof of the existence and uniqueness of solution of the  $\alpha$ -models given by (1.7) can be established by using the standard Galerkin method. The L- $\alpha$  model was implemented computationally by Cheskidov-Holm-Olson-Titi [12].

Ilyin-Lunasin-Titi introduced and studied the ML- $\alpha$  model in the 3D periodic case, see [21]. It was tested numerically in [19]. However, the global existence and uniqueness for 2D can be proved in a similar way.

The Bardina closure model of turbulence was firstly introduced by Bardina-Ferziger-Reynolds in [1] to perform simulations of the atmosphere. A simplified version of the Bardina's model, was modeled and studied by Layton-Lewandowski in [23, 24], then studied by Berselli-Lewandowski [30] in the whole space. This model is designed by  $N(\mathbf{u}_\alpha) = \nabla \cdot (\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha)$ . Cao-Lunasin-Titi proposed a variant of this model [5], which is the one we consider in this paper and that we still call "Simplified Bardina model" (SB).

### 3 A priori estimates

#### 3.1 General Orientation

As (2.9) holds We know that both the NSE (1.1)-(1.3) and the  $\alpha$ -model (1.5)-(1.6), for any  $N$  given in (1.7), admit a unique solution  $\mathbf{u}$  such that

$$(3.1) \quad \mathbf{u} \in L^\infty(\mathbb{R}_+, \mathbf{V}) \cap L^2(\mathbb{R}_+, \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V}).$$

For simplicity, we note in the following

$$(3.2) \quad \mathcal{F} := \|\mathbf{f}\|_{\mathbf{L}^2(\mathbb{R}_+; \mathbf{H})}^2$$

In this section, we detail the  $L^2(\mathbb{R}_+, \mathbf{H}^2(\mathbb{T}_2) \cap \mathbf{V})$  estimates to get the right constants, each model one after another. The analysis is based on 2D energy inequalities, using the Ladyzhenskaya inequality (2.5). About the  $\alpha$ -models, we start by estimating  $\bar{\mathbf{u}}_\alpha$ . This analysis is based on the following identities.

$$(3.3) \quad (\mathcal{P}_\sigma((\mathbf{u} \cdot \nabla)\mathbf{u}), \mathbf{u}) = (\mathcal{P}_\sigma((\mathbf{u} \cdot \nabla)\mathbf{u}), \Delta \mathbf{u}) = 0,$$

for the L- $\alpha$  and the SB,

$$(3.4) \quad (\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \mathbf{u}_\alpha) = (\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \Delta \bar{\mathbf{u}}_\alpha) = 0.$$

However the ML- $\alpha$  is more tricky, since we only have

$$(3.5) \quad (\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \bar{\mathbf{u}}_\alpha) = 0.$$

#### 3.2 NSE Estimates

**Lemma 3.1 (NSE).** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then the unique weak solution  $\mathbf{u}$  of the NSE satisfies*

$$(3.6) \quad \|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2 dt \leq \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu \lambda_1} =: C_{NSE1} \quad \forall s \geq 0,$$

and

$$(3.7) \quad \|\nabla \mathbf{u}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}\|^2 dt \leq \|\nabla \mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu} =: C_{NSE2} \quad \forall s \geq 0.$$

**Remark 3.1.** *Estimate (3.6) in the previous theorem can be obtained more generally when  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{V}')$ . We use the condition  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$  for both estimates (3.6) and (3.7) for conciseness.*

*Proof.* We argue step by step, first proving (3.6).

*Step 1.  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate.* Take the scalar product of the NSE (2.10) with  $\mathbf{u}$  and use the identity  $(\mathcal{P}_\sigma[(\mathbf{u} \cdot \nabla)\mathbf{u}], \mathbf{u}) = 0$ , which lead to the following estimate

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + \nu \|\nabla \mathbf{u}\|^2 \leq \|\mathbf{f}\| \|\mathbf{u}\|.$$

Using Poincaré and Young inequalities on the r.h.s (right-hand side) of (3.8) yields:

$$(3.9) \quad \frac{d}{dt} \|\mathbf{u}\|^2 + \nu \|\nabla \mathbf{u}\|^2 \leq \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2.$$

Integrating (3.9) on  $[0, s]$  for  $s \geq 0$ , one has

$$(3.10) \quad \|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2 dt \leq \|\mathbf{u}_0\|^2 + \frac{1}{\nu \lambda_1} \int_0^s \|\mathbf{f}\|^2 dt.$$

Finally, the estimate (3.6) follows by (3.10) since  $s$  can be chosen arbitrary.

*Step 2.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate.* In order to prove the other estimate (3.7), instead of  $\mathbf{u}$ , we take  $-\Delta \mathbf{u}$  as a test in the NSE (2.10). As we already have said, in the 2D case periodic the nonlinear term vanishes, see [35, 36], i.e.,

$$(3.11) \quad (P_\sigma[(\mathbf{u} \cdot \nabla) \mathbf{u}], -\Delta \mathbf{u}) = 0.$$

By the Young inequality the term corresponding to the body forces can be estimated by

$$(3.12) \quad (\mathbf{f}, -\Delta \mathbf{u}) \leq \frac{\nu}{2} \|\mathbf{f}\|^2 + \frac{\nu}{2} \|\Delta \mathbf{u}\|^2.$$

Therefore, the rest of the proof follows as the same as what have been done above. Thus, the proof is complete.  $\square$

### 3.3 Leray- $\alpha$ estimates

**Lemma 3.2 (L- $\alpha$ ).** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then the unique weak solution  $\mathbf{u}_\alpha$  of the L- $\alpha$  satisfies  $\forall s \geq 0$*

$$(3.13) \quad \|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \leq \frac{CC_{L1}^2}{\nu^4} \left( \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu \lambda_1} \right) + \frac{2\mathcal{F}}{\nu} =: C_L,$$

where  $C_{L1}$  is given in (3.18).

*Proof.* For the L- $\alpha$  model, the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha \quad \text{where} \quad \bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha = \mathbf{u}_\alpha.$$

We argue in 3 steps, with an intermediate step to estimate  $\bar{\mathbf{u}}_\alpha$ .

*Step 1.  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate of  $\mathbf{u}_\alpha$ .* Taking  $\mathbf{u}_\alpha$  as a test in the L- $\alpha$  model (2.11) gives

$$(3.14) \quad \frac{d}{dt} \|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 \leq \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2.$$

Here

$$(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha], \mathbf{u}_\alpha) = 0$$

has been used, see in [4]. It leads to for all  $s \geq 0$

$$(3.15) \quad \|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 dt \leq \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu \lambda_1}.$$

*Step 2.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate of  $\bar{\mathbf{u}}_\alpha$ .* Testing (2.11) by  $-\Delta \bar{\mathbf{u}}_\alpha$  and replacing  $\mathbf{u}_\alpha$  by  $\bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha$  yield

$$(3.16) \quad \frac{d}{dt} (\|\nabla \bar{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\Delta \bar{\mathbf{u}}_\alpha\|^2) + \nu \|\Delta \bar{\mathbf{u}}_\alpha\|^2 + \nu \alpha^2 \|\nabla \Delta \bar{\mathbf{u}}_\alpha\|^2 \leq \frac{\|\mathbf{f}\|^2}{\nu}.$$

Here the vanishing of the nonlinear term has been used, i.e.,

$$(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha], -\Delta \bar{\mathbf{u}}_\alpha) = ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha), -\Delta \bar{\mathbf{u}}_\alpha) = 0.$$



Therefore, by (3.16) for all  $s \geq 0$

$$(3.17) \quad \|\nabla \bar{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2 \|\Delta \bar{\mathbf{u}}_\alpha(s)\|^2 + \nu \int_0^s (\|\Delta \bar{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\nabla \Delta \bar{\mathbf{u}}_\alpha\|^2) dt \leq C_{L1},$$

where  $C_{L1}$  is given by

$$(3.18) \quad \|\nabla \bar{\mathbf{u}}_0\|^2 + \alpha^2 \|\Delta \bar{\mathbf{u}}_0\|^2 + \frac{\mathcal{F}^2}{\nu} \leq (1 + \lambda_1) \|\nabla \mathbf{u}_0\|^2 + \frac{\mathcal{F}^2}{\nu} =: C_{L1},$$

here the facts

$$\|\nabla \bar{\mathbf{u}}_0\| \leq \|\nabla \mathbf{u}_0\|, \quad \alpha^2 \|\Delta \bar{\mathbf{u}}_0\|^2 \leq \|\mathbf{u}_0\|^2,$$

given by (2.7) and the Poincaré inequality have been applied.

*Step 3.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate of  $\mathbf{u}_\alpha$ .* We test (2.11) again by  $-\Delta \mathbf{u}_\alpha$  which leads to

$$(3.19) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 = (P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha], \Delta \mathbf{u}_\alpha) + (\mathbf{f}, -\Delta \mathbf{u}_\alpha).$$

The first term on the r.h.s of (3.19) can be estimated by:

$$(3.20) \quad \begin{aligned} (P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha], \Delta \mathbf{u}_\alpha) &\leq C \|\bar{\mathbf{u}}_\alpha\|_4 \|\nabla \mathbf{u}_\alpha\|_4 \|\Delta \mathbf{u}_\alpha\| \\ &\leq C \|\nabla \bar{\mathbf{u}}_\alpha\| \|\nabla \mathbf{u}_\alpha\|^{1/2} \|\Delta \mathbf{u}_\alpha\|^{3/2} \\ &\leq \frac{C}{\nu^3} \|\nabla \bar{\mathbf{u}}_\alpha\|^4 \|\nabla \mathbf{u}_\alpha\|^2 + \frac{\nu}{4} \|\Delta \mathbf{u}_\alpha\|^2. \end{aligned}$$

Here one has used the Hölder, 2D-Ladyzhenskaya (2.5), Sobolev and Young inequalities, respectively. From (3.19)-(3.20) one obtains

$$(3.21) \quad \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 \leq \frac{2}{\nu} \|\mathbf{f}\|^2 + \frac{C}{\nu^3} \|\nabla \bar{\mathbf{u}}_\alpha\|^4 \|\nabla \mathbf{u}_\alpha\|^2.$$

The previous estimate yields for all  $s \geq 0$

$$(3.22) \quad \|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu^3} \int_0^s \|\nabla \bar{\mathbf{u}}_\alpha\|^4 \|\nabla \mathbf{u}_\alpha\|^2 dt.$$

Finally, both estimates (3.15) and (3.17) are applied in (3.22) to get (3.13), which ends the proof.  $\square$

### 3.4 Simplified Bardina estimates

**Lemma 3.3 (SB).** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then the unique weak solution  $\mathbf{u}_\alpha$  of the SB model satisfies*

$$(3.23) \quad \|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \leq \frac{CC_S^2}{\nu^2 \lambda_1} + \frac{2\mathcal{F}}{\nu} =: C_{SB} \quad \forall s \geq 0,$$

where  $C$  is a positive constant and  $C_S$  is given by (3.25).

*Proof.* For this model, the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha.$$

We make it in two step, starting with  $\bar{\mathbf{u}}_\alpha$ .

Step 3.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate of  $\bar{\mathbf{u}}_\alpha$ . Taking  $-\Delta \bar{\mathbf{u}}_\alpha$  as a test in (2.11) and using the fact  $\mathbf{u}_\alpha = \bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha$  give us

$$(3.24) \quad \frac{d}{dt} (|\nabla \bar{\mathbf{u}}_\alpha|^2 + \alpha^2 |\Delta \bar{\mathbf{u}}_\alpha|^2) + \nu |\Delta \bar{\mathbf{u}}_\alpha|^2 + \nu \alpha^2 |\nabla \Delta \bar{\mathbf{u}}_\alpha|^2 \leq \frac{1}{\nu} \|\mathbf{f}\|^2,$$

where the identity

$$(P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], -\Delta \bar{\mathbf{u}}_\alpha) = 0$$

has been used. Thus, for all  $s \geq 0$

$$(3.25) \quad \|\nabla \bar{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2 \|\Delta \bar{\mathbf{u}}_\alpha(s)\|^2 + \nu \int_0^s (|\Delta \bar{\mathbf{u}}_\alpha|^2 + \alpha^2 |\nabla \Delta \bar{\mathbf{u}}_\alpha|^2) dt \leq C_S,$$

where  $C_S := C_{L1}$  as given in (3.18).

Step 2.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate of  $\mathbf{u}_\alpha$ . One takes  $-\Delta \mathbf{u}_\alpha$  as a test in (2.11) to obtain

$$(3.26) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 = (P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) - (\mathbf{f}, \Delta \mathbf{u}_\alpha).$$

The nonlinear term on the r.h.s of (3.26) is estimated by:

$$(3.27) \quad \begin{aligned} (P_\sigma[(\bar{\mathbf{u}}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) &\leq C \|\bar{\mathbf{u}}_\alpha\|_4 \|\nabla \bar{\mathbf{u}}_\alpha\|_4 \|\Delta \mathbf{u}_\alpha\| \\ &\leq C \|\bar{\mathbf{u}}_\alpha\|^{1/2} \|\nabla \bar{\mathbf{u}}_\alpha\| \|\Delta \bar{\mathbf{u}}_\alpha\|^{1/2} \|\Delta \mathbf{u}_\alpha\| \\ &\leq \frac{C}{\nu} \|\bar{\mathbf{u}}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \|\Delta \bar{\mathbf{u}}_\alpha\| + \frac{\nu}{4} \|\Delta \mathbf{u}_\alpha\|^2. \end{aligned}$$

In the above inequalities the Hölder, 2D-Ladyzhenskaya and Young inequalities have been applied, respectively. The estimates (3.26)-(3.27) lead to

$$(3.28) \quad \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 \leq \frac{2}{\nu} \|\mathbf{f}\|^2 + \frac{C}{\nu} \|\bar{\mathbf{u}}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \|\Delta \bar{\mathbf{u}}_\alpha\|.$$

and by (3.25) for all  $s \geq 0$

$$(3.29) \quad \begin{aligned} \|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt &\leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu} \int_0^s \|\bar{\mathbf{u}}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \|\Delta \bar{\mathbf{u}}_\alpha\| dt \\ &\leq \frac{2\mathcal{F}}{\nu} + \frac{CC_{SB}}{\nu \lambda_1} \int_0^s \|\Delta \bar{\mathbf{u}}_\alpha\|^2 dt \\ &\leq \frac{2\mathcal{F}}{\nu} + \frac{CC_{SB}^2}{\nu^2 \lambda_1}. \end{aligned}$$

Therefore, the proof is complete. □

### 3.5 Modified Leray- $\alpha$

**Lemma 3.4 (ML- $\alpha$ ).** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then the unique weak solution  $\mathbf{u}_\alpha$  of the ML- $\alpha$  model satisfies*

$$(3.30) \quad \|\nabla \mathbf{u}_\alpha(t)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \leq \frac{C_{ML4}}{\nu^4} + \frac{2\mathcal{F}}{\nu} =: C_{ML\alpha} \quad \forall s \geq 0,$$

where  $C_{ML4} = CC_{ML1}C_{ML2}C_{ML3}$  with  $C$  is a positive constant and for  $i = 1, 2, 3$ ,  $C_{MLi}$  are given by (3.33), (3.37) and (3.42), respectively.

*Proof.* The nonlinear term of this model is given by

$$N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha.$$

This case is more tricky than the others. We make it into 3 steps, starting with a  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate of  $\bar{\mathbf{u}}_\alpha$ , then a  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate of  $\mathbf{u}_\alpha$ , to finally get the conclusion.

*Step 1.*  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate of  $\bar{\mathbf{u}}_\alpha$ . Taking  $\bar{\mathbf{u}}_\alpha$  as a test in (2.11) and replacing  $\mathbf{u}_\alpha$  by  $\bar{\mathbf{u}}_\alpha - \alpha^2 \Delta \bar{\mathbf{u}}_\alpha$  to obtain

$$(3.31) \quad \frac{d}{dt} (|\bar{\mathbf{u}}_\alpha|^2 + \alpha^2 |\nabla \bar{\mathbf{u}}_\alpha|^2) + \nu |\nabla \bar{\mathbf{u}}_\alpha|^2 + \nu \alpha^2 |\Delta \bar{\mathbf{u}}_\alpha|^2 \leq \frac{1}{\nu \lambda_1} \|\mathbf{f}\|^2.$$

Here the fact  $(P_\sigma[(\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], \bar{\mathbf{u}}_\alpha) = 0$  and the Poincaré inequality have been used on the r.h.s. Then One gets from (3.31) for all  $s \geq 0$

$$(3.32) \quad |\bar{\mathbf{u}}_\alpha(s)|^2 + \alpha^2 |\nabla \bar{\mathbf{u}}_\alpha(s)|^2 + \nu \int_0^s (|\nabla \bar{\mathbf{u}}_\alpha|^2 + \alpha^2 |\Delta \bar{\mathbf{u}}_\alpha|^2) ds \leq C_{ML1},$$

where as in (3.18) above  $C_{ML1}$  is given by

$$(3.33) \quad |\bar{\mathbf{u}}_0|^2 + \alpha^2 |\nabla \bar{\mathbf{u}}_0|^2 + \frac{\mathcal{F}^2}{\nu \lambda_1} \leq (1 + \lambda_1) \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}^2}{\nu \lambda_1} =: C_{ML1}.$$

*Step 2.*  $L^2 H_x^1 \cap L_t^\infty L_x^2$  estimate of  $\mathbf{u}_\alpha$ . Taking  $\mathbf{u}_\alpha$  as a test in (2.11) yields

$$(3.34) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 = -((\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) + (\mathbf{f}, \mathbf{u}_\alpha).$$

The nonlinear term on the r.h.s of (3.34) can be estimated by

$$\begin{aligned} ((\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) &\leq C \|\mathbf{u}_\alpha\|_4^2 \|\nabla \bar{\mathbf{u}}_\alpha\| \\ &\leq C \|\mathbf{u}_\alpha\| \|\nabla \mathbf{u}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\| \\ &\leq \frac{C}{\nu} \|\mathbf{u}_\alpha\|^2 \|\nabla \bar{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4} \|\nabla \mathbf{u}_\alpha\|^2. \end{aligned}$$

Here we have used the Hölder, 2D-Ladyzhenskaya and Young inequalities, respectively. Using the Young inequality for the other term on the r.h.s of (3.34) gives

$$(3.35) \quad \frac{d}{dt} \|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 \leq \frac{2}{\lambda_1 \nu} \|\mathbf{f}\|^2 + \frac{C}{\nu} \|\mathbf{u}_\alpha\|^2 \|\nabla \bar{\mathbf{u}}_\alpha\|^2.$$

Using the estimate (3.32) for all  $s \geq 0$  leads to

$$(3.36) \quad \begin{aligned} \int_0^s \|\mathbf{u}_\alpha\|^2 \|\nabla \bar{\mathbf{u}}_\alpha\|^2 dt &= \int_0^s (|\bar{\mathbf{u}}_\alpha|^2 + 2\alpha^2 |\nabla \bar{\mathbf{u}}_\alpha|^2 + \alpha^4 |\Delta \bar{\mathbf{u}}_\alpha|^2) \|\nabla \bar{\mathbf{u}}_\alpha\|^2 dt \\ &\leq \frac{4C_{ML1}^2}{\nu}. \end{aligned}$$

Here one has used the following identity

$$\|\mathbf{u}_\alpha\|^2 = |\bar{\mathbf{u}}_\alpha|^2 + 2\alpha^2 |\nabla \bar{\mathbf{u}}_\alpha|^2 + \alpha^4 |\Delta \bar{\mathbf{u}}_\alpha|^2.$$

Therefore, by (3.35)-(3.36) for all  $s \geq 0$

$$(3.37) \quad \|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 dt \leq \frac{2\mathcal{F}}{\nu \lambda_1} + \frac{4CC_{ML1}^2}{\nu^2} =: C_{ML2}.$$

Step 3.  $L^2 H_x^2 \cap L_t^\infty H_x^1$  estimate of  $\mathbf{u}_\alpha$ . We take  $-\Delta \mathbf{u}_\alpha$  as a test in (2.11) to obtain

$$(3.38) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 = (P_\sigma[(\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) - (\mathbf{f}, \Delta \mathbf{u}_\alpha).$$

The nonlinear integral can be estimated by

$$(3.39) \quad \begin{aligned} (P_\sigma[(\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) &\leq C \|\mathbf{u}_\alpha\|_4 \|\nabla \bar{\mathbf{u}}_\alpha\|_4 \|\Delta \mathbf{u}_\alpha\|_2 \\ &\leq C \|\mathbf{u}_\alpha\|^{1/2} \|\nabla \mathbf{u}_\alpha\|^{1/2} \|\nabla \bar{\mathbf{u}}_\alpha\|^{1/2} \|\Delta \mathbf{u}_\alpha\|^{3/2} \\ &\leq \frac{C}{\nu^3} \|\mathbf{u}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 \|\nabla \bar{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4} \|\Delta \mathbf{u}_\alpha\|^2. \end{aligned}$$

Here one has used the Hölder, 2D-Ladyzhenskaya, Sobolev and Young inequalities, respectively. From (3.38)-(3.39) we obtain:

$$(3.40) \quad \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 + \nu \|\Delta \mathbf{u}_\alpha\|^2 \leq \frac{C}{\nu^3} \|\mathbf{u}_\alpha\|^2 \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 + \frac{2}{\nu} \|\mathbf{f}\|^2$$

and in particular

$$(3.41) \quad \frac{d}{dt} \|\nabla \mathbf{u}_\alpha\|^2 \leq \frac{CC_{ML2}}{\nu^3} \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \|\nabla \mathbf{u}_\alpha\|^2 + \frac{2}{\nu} \|\mathbf{f}\|^2.$$

Hence, by (3.41) for all  $s \geq 0$

$$(3.42) \quad \|\nabla \mathbf{u}_\alpha(s)\|^2 \leq \left( \|\nabla \mathbf{u}_0\|^2 + \frac{2\mathcal{F}}{\nu} \right) \exp \left\{ \frac{CC_{ML2}^2}{\nu^4} \right\} =: C_{ML3}.$$

Together (3.40) and (3.42) one obtains (3.30). Thus, the proof is complete for this model.  $\square$

## 4 The rate of convergence of $\mathbf{u}_\alpha$ to $\mathbf{u}$

In this section, we study the rate of convergence of the weak solutions  $\mathbf{u}_\alpha$  of the four  $\alpha$ -models to the weak solution  $\mathbf{u}$  of the NSE in some suitable norms in terms of  $\alpha$  as  $\alpha$  tends to zero. For simplicity, throughout this section  $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$  denotes the error between  $\mathbf{u}$  and  $\mathbf{u}_\alpha$  which are the weak solutions of the NSE (2.10) and  $\alpha$ -models (2.11), respectively.

### 4.1 Error estimate in $L^2 H_x^1 \cap L_t^\infty L_x^2$

The first main result in this section is given by the following theorem:

**Theorem 4.1.** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then*

$$(4.1) \quad \|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2 dt \leq C_r \alpha^3 \quad \forall s \geq 0,$$

where  $C_r$  is given by

$$\begin{cases} (4.9) & \text{for the } L\text{-}\alpha \text{ model,} \\ (4.11) & \text{for the SB model,} \\ (4.12) & \text{for the ML-}\alpha \text{ model,} \end{cases}$$

*Proof.* As the models share common features, in a first step we consider these common features, and in a second step we separate the models to distinguish their differences.

*Step 1. Common features.* We subtract (2.11) from (2.10) and by multiplying  $\mathbf{e}$  and integrating the result reads

$$(4.2) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{e}\|^2 + \nu \|\nabla \mathbf{e}\|^2 = (-P_\sigma[(\mathbf{u} \cdot \nabla) \mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}).$$

We add and subtract on the r.h.s of (4.2) the term  $((\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, \mathbf{e})$  and then rewrite it in the following form:

$$(4.3) \quad \begin{aligned} RHS &= (-P_\sigma[(\mathbf{u} \cdot \nabla) \mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}) \\ &= (-\mathbf{u} \cdot \nabla \mathbf{u} + N(\mathbf{u}_\alpha), P_\sigma \mathbf{e}) \\ &= (-\mathbf{u} \cdot \nabla \mathbf{u} + N(\mathbf{u}_\alpha), \mathbf{e}) \\ &= (-\mathbf{u} \cdot \nabla \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, \mathbf{e}) + (-\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e}). \end{aligned}$$

We will deal with the two terms on the r.h.s of (4.3) separately. Replacing  $\mathbf{u}_\alpha$  by  $\mathbf{u} - \mathbf{e}$  the first term in (4.3) is rewritten as follows:

$$\begin{aligned} (-\mathbf{u} \cdot \nabla \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, \mathbf{e}) &= (-\mathbf{u} \cdot \nabla \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla)(\mathbf{u} - \mathbf{e}), \mathbf{e}) \\ &= (-\mathbf{u} \cdot \nabla \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}, \mathbf{e}) \\ &= ((-\mathbf{e} \cdot \nabla) \mathbf{u}, \mathbf{e}) \\ &= ((\mathbf{e} \cdot \nabla) \mathbf{e}, \mathbf{u}), \end{aligned}$$

where  $(\mathbf{u}_\alpha \cdot \nabla) \mathbf{e}, \mathbf{e}) = 0$  has been used and the result is continuous estimated by

$$(4.4) \quad \begin{aligned} ((\mathbf{e} \cdot \nabla) \mathbf{e}, \mathbf{u}) &\leq C \|\mathbf{e}\|_4 \|\nabla \mathbf{e}\| \|\mathbf{u}\|_4 \\ &\leq C \|\mathbf{e}\|^{1/2} \|\nabla \mathbf{e}\|^{3/2} \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \\ &\leq \frac{C}{\nu^3} \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2 \|\mathbf{e}\|^2 + \frac{\nu}{4} \|\nabla \mathbf{e}\|^2. \end{aligned}$$

The first inequality from above is due to the Hölder inequality with the pairing  $(1/4, 1/2, 1/4)$ , the second one is obtained by applying the 2D-Ladyzhenskaya inequality and the last one comes from using the Young inequality with the pairing  $(1/4, 3/4)$ . The residual term will be estimated for each model separately, one after the others.

*Step 2. Analysis of the models one after eachother*

**L- $\alpha$  model.** For this model the nonlinear term is given by  $N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha$ . The residual term is written as

$$R = (-\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha, \mathbf{e}) = -(((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla) \mathbf{u}_\alpha, \mathbf{e}).$$

The Hölder, 2D-Ladyzhenskaya, (2.6), (2.8), Sobolev, Poincaré and Young inequalities are

going to apply to get the following estimates:

$$\begin{aligned}
R &\leq C\|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\|_4\|\nabla\mathbf{u}_\alpha\|\|\mathbf{e}\|_4 \\
&\leq C\|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha - \nabla\bar{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha\|\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2} \\
&\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}\|\Delta\bar{\mathbf{u}}_\alpha\|^{1/2}\|\Delta\mathbf{u}_\alpha\|^{1/2}\|\nabla\mathbf{e}\| \\
&\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}\|\Delta\mathbf{u}_\alpha\|\|\nabla\mathbf{e}\| \\
(4.5) \quad &\leq \frac{CC_L\alpha^3}{\nu\lambda_1}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}\|^2.
\end{aligned}$$

Notice that  $\|\nabla\mathbf{u}_\alpha(t)\|$  in the above estimate is uniformly bounded by  $C_L^{1/2}$  where  $C_L$  given by Lemma 3.2. Putting (4.4) and (4.5) into (4.2) gives us

$$(4.6) \quad \frac{d}{dt}\|\mathbf{e}\|^2 + \nu\|\nabla\mathbf{e}\|^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\mathbf{e}\|^2.$$

Here we are going to apply the Gronwall's lemma for (4.6). Although the argument is standard we still provide the details for this model and for the other models the details will be shipped. Let

$$A(s) := -\frac{C}{\nu^3} \int_0^s \|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 dt \quad \forall s \geq 0,$$

where  $C$  is given in (4.6). Multiplying both sides of (4.6) by  $A(t)$  yields for all  $s \geq 0$

$$(4.7) \quad \|\mathbf{e}(s)\|^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1} \exp\{-A(s)\} \int_0^s \|\Delta\mathbf{u}_\alpha\|^2 dt,$$

where one has used the fact  $\mathbf{e}_0 = 0$ . Thus, combine (4.7) with Lemmas ?? and 3.2 to obtain

$$(4.8) \quad \|\mathbf{e}(s)\|^2 \leq \frac{CC_L^2\alpha^3}{\nu^2\lambda_1} \exp\left\{\frac{C_{NSE1}^2}{\nu^4}\right\} =: E_L\alpha^3 \quad \forall s \geq 0,$$

where  $C_L$  and  $C_{NSE1}$  are given by Lemmas 3.2 and 3.1, respectively. Finally, we combine (4.6) and (4.8) to get (4.1) with  $C_r$  given by

$$(4.9) \quad C_{rL} = C \left( \frac{C_L}{\nu^2} + \frac{C_{NSE1}^2 E_L}{\nu^4} \right).$$

**SB model.** In this case the residual term is given by

$$\begin{aligned}
R &= -(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, \mathbf{e} \\
&= -(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, \mathbf{e} \\
&= -((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e} - ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha), \mathbf{e}) \\
(4.10) \quad &= R_1 + R_2.
\end{aligned}$$

The term  $R_1$  on the r.h.s of (4.10) can be handled as (4.5) in the case of L- $\alpha$  model. The second term  $R_2$  can be estimated as in the ML- $\alpha$  case. Therefore, the constant  $C_r$  in this case has the following form

$$(4.11) \quad C_{rSB} = C_{rL} + C_{rML}.$$

□

**ML- $\alpha$  model.** In this case the residual term is rewritten as

$$\begin{aligned} R &= (-\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha, \mathbf{e} \\ &= ((\mathbf{u}_\alpha \cdot \nabla) \mathbf{e}, \mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha), \end{aligned}$$

and is handled precisely as in the L- $\alpha$  case. Then the proof for this case follows by that of the L- $\alpha$  model with  $C_r$  is given by

$$(4.12) \quad C_{r_{ML_a}} = C \left( \frac{C_{ML_a}}{\nu^2} + \frac{C_{NSE1}^2 E_{ML_a}}{\nu^4} \right),$$

where  $C_{ML_a}$  is given by Lemma 3.4 and

$$E_{ML_a} = \frac{C C_{ML_a}^2}{\nu^2 \lambda_1} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} \right\}.$$

From Theorem 4.1 we have immediately the following results:

**Corollary 4.1.** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then for all  $s \geq 0$*

$$(4.13) \quad \|\bar{\mathbf{e}}(s)\|^2 + \nu \int_0^s \|\nabla \bar{\mathbf{e}}\|^2 dt \leq 3C_r \alpha^3,$$

where  $\bar{\mathbf{e}} = \bar{\mathbf{u}} - \bar{\mathbf{u}}_\alpha$  and  $C_r$  is given by Theorem 4.1 for each  $\alpha$ -model.

**Corollary 4.2.** *Let  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ . Then for all  $s \geq 0$*

$$(4.14) \quad \|(\mathbf{u} - \bar{\mathbf{u}}_\alpha)(s)\|^2 + \nu \int_0^s \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\|^2 dt \leq C_{cor}(\alpha^2 + \alpha^3),$$

where  $C_{cor}$  is given by (4.17).

*Proof.* The triangle inequality, Theorem 4.1, Lemma 3.1, relation (2.7) and Poincaré inequality yield for all  $s \geq 0$

$$\begin{aligned} \|\mathbf{u} - \bar{\mathbf{u}}_\alpha(s)\|^2 &\leq 2 \left( \|(\mathbf{u} - \mathbf{u}_\alpha)(s)\|^2 + \|(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)(s)\|^2 \right) \\ &\leq 2C_r \alpha^3 + 2\alpha^4 \|\Delta \bar{\mathbf{u}}_\alpha(s)\|^2 \\ &\leq 2C_r \alpha^3 + 2C\alpha^2 \|\mathbf{u}_\alpha(s)\|^2 \\ (4.15) \quad &\leq 2C_r \alpha^3 + 2C \frac{C_E}{\lambda_1} \alpha^2. \end{aligned}$$

Here for each  $\alpha$ -model  $C_E$  is given by  $C_L, C_{NS_a}, C_{SB}$  or  $C_{ML_a}$  in Lemmas 3.2, ??, 3.3 and 3.4, respectively. Moreover,  $C_r$  is given by Theorem 4.1. Similarity, we have for all  $s \geq 0$

$$\begin{aligned} \nu \int_0^s \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\|^2 dt &\leq 2\nu \left( \int_0^s \|\nabla(\mathbf{u} - \mathbf{u}_\alpha)\|^2 dt + \int_0^s \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\|^2 dt \right) \\ &\leq 2C_r \alpha^3 + 2C\alpha^2 \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \\ (4.16) \quad &\leq 2C_r \alpha^3 + 2CC_E \alpha^2. \end{aligned}$$

Thus, (4.14) follows by (4.15) and (4.16) with the constant  $C$  given by

$$(4.17) \quad C_{cor} = 2 \max\{C_r, CC_E, CC_E/\lambda_1\}.$$

□

## 4.2 Error estimate in $L^2H_x^2 \cap L_t^\infty H_x^1$

Throughout the rest of the paper, we assume  $\alpha < L/2\pi$ . Before going on to state the results, we start with a technical result that follows from a useful result due to H. Brézis and T. Gallouët [3].

**Lemma 4.1.** *Let  $0 \leq \alpha < \lambda_1^{-1/2} = L/2\pi$ ,  $\mathbf{u}_\alpha$  be the weak solutions of any  $\alpha$ -model. Then there exist  $K_1$  and  $K_2$  such that for all  $t \geq 0$*

$$(4.18) \quad \|\bar{\mathbf{u}}_\alpha(t)\|_\infty^2 \leq K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2$$

*Proof.* Assume that  $0 \leq \alpha < \lambda_1^{-1/2} = L/2\pi$ . Thanks to the Brezis-Gallouët inequality [3], there exists a constant  $C = C(\mathbb{T}_2) > 0$  such that

$$\begin{aligned} \|\bar{\mathbf{u}}_\alpha\|_\infty^2 &\leq C \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \left[ 1 + \log\left(\frac{L}{2\pi} \frac{\|\Delta \bar{\mathbf{u}}_\alpha\|}{\|\nabla \bar{\mathbf{u}}_\alpha\|}\right) \right] \\ &= C \left[ \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \left( 1 + \log\left(\frac{L}{2\pi\alpha}\right) \right) + \|\nabla \bar{\mathbf{u}}_\alpha\|^2 \log\left(\frac{\alpha \|\Delta \bar{\mathbf{u}}_\alpha\|}{\|\nabla \bar{\mathbf{u}}_\alpha\|}\right) \right] \\ &\leq C \left[ C_E \left( 1 + \log\left(\frac{L}{2\pi\alpha}\right) \right) + \frac{C_1^2}{\lambda_1^{1/2}} \frac{\|\nabla \bar{\mathbf{u}}_\alpha\|}{C_1} \log\left(\frac{C_1}{\|\nabla \bar{\mathbf{u}}_\alpha\|}\right) \right] \\ &\leq C \left[ C_E \left( 1 + \log\left(\frac{L}{2\pi\alpha}\right) \right) + \frac{C_1^2}{e\lambda_1^{1/2}} \right]. \end{aligned}$$

Here one has used the following estimate in the third inequality above

$$(4.19) \quad \alpha^2 \|\Delta \bar{\mathbf{u}}_\alpha\|^2 \leq C \|\mathbf{u}_\alpha\|^2 \leq C \frac{\|\nabla \mathbf{u}_\alpha\|^2}{\lambda_1} \leq C \frac{C_E}{\lambda_1} =: C_1^2, \quad (C_1 > 0)$$

and

$$(4.20) \quad \|\nabla \bar{\mathbf{u}}_\alpha\| \leq C \|\nabla \mathbf{u}_\alpha\| \leq C C_E^{1/2} \leq \frac{C C_1}{\lambda_1^{1/2}},$$

where the two first estimates in (4.19)-(4.20) are given by (2.6) and the constant  $C_E$  is given as in Corollary 4.1. Thus, in this case the proof follows by choosing  $K_1 = C C_E$  and  $K_2 = K_1 + C_1^2 / (e\lambda_1^{1/2})$ .  $\square$

We are now in order to state the next main result in this section.

**Theorem 4.2.** *Let  $\alpha < L/2\pi$ ,  $\mathbf{u}_0 \in \mathbf{V}$ ,  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$  and*

$$(4.21) \quad D(s) := \|\nabla \mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{e}\|^2 dt \quad \forall s \geq 0.$$

*Then*

1. *For the L- $\alpha$  model*

$$(4.22) \quad D(s) \leq C_R \alpha^2,$$

*where  $C_R$  is given by (4.36).*

2. *For the SB model*

$$(4.23) \quad D(s) \leq C_R \alpha^2 (K_1 \log(L/2\pi\alpha) + K_2 + C_{SB}),$$



where  $C_R$  given by (4.38).

3. For the ML- $\alpha$  model

$$(4.24) \quad D(s) \leq C_R \alpha^2 (K_1 \log(L/2\pi\alpha) + K_2 + C_{ML\alpha})$$

where  $C_R$  given by (4.44). Here  $C_{ML\alpha}$ ,  $C_{SB}$  and  $K_1, K_2$  are given by Lemmas 3.4, 3.3 and 4.1, respectively.

*Proof.* Subtracting (2.11) from (2.10) and taking  $-\Delta \mathbf{e}$  as a test yield:

$$(4.25) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{e}\|^2 + \nu \|\Delta \mathbf{e}\|^2 = (-P_\sigma[(\mathbf{u} \cdot \nabla) \mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], -\Delta \mathbf{e}).$$

Adding and subtracting the term  $((\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, -\Delta \mathbf{e})$  to the r.h.s of (4.25):

$$(4.26) \quad RHS = (-\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, -\Delta \mathbf{e} + (-\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha + N(\mathbf{u}_\alpha), -\Delta \mathbf{e}.$$

By using  $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$  the first term on the r.h.s of (4.26) can be estimated by

$$(4.27) \quad \begin{aligned} I_1 &= (-\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha, -\Delta \mathbf{e} \\ &= (-\mathbf{u} \cdot \nabla) \mathbf{u} + ((\mathbf{u} - \mathbf{e}) \cdot \nabla) (\mathbf{u} - \mathbf{e}), -\Delta \mathbf{e} \\ &= (-\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{e} - (\mathbf{e} \cdot \nabla) \mathbf{u} + (\mathbf{e} \cdot \nabla) \mathbf{e}, -\Delta \mathbf{e} \\ &= ((\mathbf{u} \cdot \nabla) \mathbf{e}, \Delta \mathbf{e}) + (\mathbf{e} \cdot \nabla) \mathbf{u}, \Delta \mathbf{e} = I_{11} + I_{12}, \end{aligned}$$

where the vanishing of the term  $((\mathbf{e} \cdot \nabla) \mathbf{e}, -\Delta \mathbf{e})$  has been used. The first term on the r.h.s of (4.27) is bounded by

$$(4.28) \quad \begin{aligned} I_{11} &= ((\mathbf{u} \cdot \nabla) \mathbf{e}, \Delta \mathbf{e}) \leq C \|\mathbf{u}\|_4 \|\nabla \mathbf{e}\|_4 \|\Delta \mathbf{e}\| \\ &\leq C \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\nabla \mathbf{e}\|^{1/2} \|\Delta \mathbf{e}\|^{3/2} \\ &\leq \frac{C}{\nu^3} \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2 \|\nabla \mathbf{e}\|^2 + \frac{\nu}{6} \|\Delta \mathbf{e}\|^2. \end{aligned}$$

In (4.28), the Hölder, 2D-Ladyzhenskaya and Young inequalities have been applied. Similarly, the other term on the r.h.s of (4.27) can be handled by:

$$(4.29) \quad \begin{aligned} I_{12} &= ((\mathbf{e} \cdot \nabla) \mathbf{u}, \Delta \mathbf{e}) \leq C \|\mathbf{e}\|_4 \|\nabla \mathbf{u}\|_4 \|\Delta \mathbf{e}\| \\ &\leq C \|\mathbf{e}\|^{1/2} \|\nabla \mathbf{e}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\Delta \mathbf{u}\|^{1/2} \|\Delta \mathbf{e}\| \\ &\leq \frac{C}{\lambda_1^{1/2}} \|\nabla \mathbf{e}\| \|\nabla \mathbf{u}\|^{1/2} \|\Delta \mathbf{u}\|^{1/2} \|\Delta \mathbf{e}\| \\ &\leq \frac{C}{\nu \lambda_1} \|\nabla \mathbf{e}\|^2 \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| + \frac{\nu}{6} \|\Delta \mathbf{e}\|^2. \end{aligned}$$

Using (4.28)-(4.29) the quantity  $I_1$  in (4.27) can be bounded by

$$(4.30) \quad I_1 \leq \left( \frac{C}{\nu^3} \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2 + \frac{C}{\nu \lambda_1} \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| \right) \|\nabla \mathbf{e}\|^2 + \frac{\nu}{3} \|\Delta \mathbf{e}\|^2.$$

In the following parts, we will estimate the second term on the r.h.s of (4.26) for each  $\alpha$ -model. We start with the first one as follows.

**L- $\alpha$  model.** The nonlinear term is given by  $N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ . Therefore, the residual term can be estimated by

$$\begin{aligned}
I_2 &= (-\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e} \\
&= ((\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e} \\
&\leq C\|\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|_4\|\nabla\mathbf{u}_\alpha\|_4\|\Delta\mathbf{e}\| \\
&\leq C\|\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|^{1/2}\|\nabla\bar{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha\|^{1/2}\|\Delta\mathbf{u}_\alpha\|^{1/2}\|\Delta\mathbf{e}\| \\
&\leq C\alpha\|\nabla\mathbf{u}_\alpha\|\|\Delta\mathbf{u}_\alpha\|\|\Delta\mathbf{e}\| \\
&\leq \frac{C}{\nu}\alpha^2\|\nabla\mathbf{u}_\alpha\|^2\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2 \\
(4.31) \quad &\leq \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2.
\end{aligned}$$

Here the Hölder, 2D-Ladyzhenskaya, (2.6)-(2.7) and Young inequalities have been applied. Moreover,  $C_L$  is given by Lemma 3.2. Using estimates (4.25)-(4.31) leads to

$$\begin{aligned}
\frac{d}{dt}\|\nabla\mathbf{e}\|^2 + \nu\|\Delta\mathbf{e}\|^2 &\leq \left( \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| \right) \|\nabla\mathbf{e}\|^2 \\
(4.32) \quad &\quad + \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2
\end{aligned}$$

and in particular

$$(4.33) \quad y'(t) - g(t)y(t) \leq h(t) \quad \forall t \geq 0,$$

where for all  $t \geq 0$

$$\begin{cases}
y(t) &= \|\nabla\mathbf{e}(t)\|^2, \\
g(t) &= \frac{C}{\nu^3}\|\mathbf{u}(t)\|^2\|\nabla\mathbf{u}(t)\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}(t)\|\|\Delta\mathbf{u}(t)\|, \\
h(t) &= \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha(t)\|^2.
\end{cases}$$

Therefore, the Gronwall's lemma gives for all  $s \geq 0$

$$(4.34) \quad \|\nabla\mathbf{e}(s)\|^2 \leq \frac{CC_L}{\nu^2} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1} \right\} \alpha^2 =: R_L\alpha^2.$$

Here we use the fact that  $\nabla\mathbf{e}(0) = 0$ . Finally, combine (4.32) and (4.34) yield

$$(4.35) \quad \|\nabla\mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta\mathbf{e}\|^2 dt \leq C_R \alpha^2 \quad \forall s \geq 0,$$

where

$$(4.36) \quad C_R = \left( \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2}C_{NSE2}^{1/2}}{\nu^2\lambda_1} \right) R_L + \frac{CC_L^2}{\nu^2}.$$

Thus, the proof is complete.

**SB model.** The nonlinear term is given by  $N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha$ . Adding and subtracting the term  $(\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$  give

$$\begin{aligned}
I_2 &= (-\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, -\Delta\mathbf{e} \\
&= (-\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, -\Delta\mathbf{e} \\
(4.37) \quad &= I_{21} + I_{22}.
\end{aligned}$$

Here, the first term on the r.h.s of (4.37) can be handled by

$$\begin{aligned}
I_{21} &= (-\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e} \\
&= ((\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \Delta\mathbf{e},
\end{aligned}$$

which is similar to (4.31) in the L- $\alpha$  model. The other term is rewritten as

$$\begin{aligned}
I_{22} &= (-\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, -\Delta\mathbf{e} \\
&= ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}),
\end{aligned}$$

which is similar to (4.41) in the ML- $\alpha$  model. Therefore, the constant  $C_R$  in this case is similar as in the ML- $\alpha$  model and has the form

$$(4.38) \quad C_R = \left( \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{SB} + \frac{CC_{SB}}{\nu^2}$$

Here  $C_{SB}$  is given by Lemma 3.3 and

$$R_{SB} := \frac{CC_{SB}}{\nu^2} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right\}.$$

Thus, the proof is complete.

**ML- $\alpha$  model.** The nonlinear term is given by  $N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha$ . Hence, the residual term can be rewritten by

$$\begin{aligned}
I_2 &= (-\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha, -\Delta\mathbf{e} \\
&= ((\mathbf{u}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
&= (((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) + ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
(4.39) \quad &= I_{21} + I_{22}.
\end{aligned}$$

The first term on the r.h.s of (4.39) can be estimated by

$$\begin{aligned}
I_{21} &= (((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\
&\leq C \|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\|_4 \|\nabla\bar{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|_4 \|\Delta\mathbf{e}\|_2 \\
&\leq C \|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\|^{1/2} \|\nabla\bar{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\| \|\Delta\bar{\mathbf{u}}_\alpha - \Delta\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{e}\| \\
&\leq C\alpha \|\Delta\mathbf{u}_\alpha\| \|\nabla\mathbf{u}_\alpha\| \|\Delta\mathbf{e}\| \\
&\leq \frac{C}{\nu} \alpha^2 \|\nabla\mathbf{u}_\alpha\|^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12} \|\Delta\mathbf{e}\|^2 \\
(4.40) \quad &\leq \frac{CC_{ML\alpha}}{\nu} \alpha^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12} \|\Delta\mathbf{e}\|^2,
\end{aligned}$$

where  $C_{ML_a}$  is given by Lemma 3.4. Next, we bound the second term on the r.h.s of (4.39) as follows:

$$\begin{aligned}
(4.41) \quad I_{22} &= ((\bar{\mathbf{u}}_\alpha \cdot \nabla)(\bar{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&\leq C \|\bar{\mathbf{u}}_\alpha\|_\infty \|\nabla \bar{\mathbf{u}}_\alpha - \nabla \mathbf{u}_\alpha\| \|\Delta \mathbf{e}\| \\
&\leq C\alpha \|\bar{\mathbf{u}}_\alpha\|_\infty \|\Delta \mathbf{u}_\alpha\| \|\Delta \mathbf{e}\| \\
&\leq \frac{C\alpha^2}{\nu} \|\bar{\mathbf{u}}_\alpha\|_\infty^2 \|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12} \|\Delta \mathbf{e}\|^2 \\
&\leq \frac{C\alpha^2}{\nu} \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 \right) \|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12} \|\Delta \mathbf{e}\|^2,
\end{aligned}$$

in the case  $\alpha < L/2\pi$ . Here  $K_1$  and  $K_2$  are given in Lemma 4.1 in the Appendix. Otherwise,

$$(4.42) \quad I_{22} \leq \frac{CK_2\alpha^2}{\nu} \|\Delta \mathbf{u}_\alpha\|^2 + \frac{\nu}{12} \|\Delta \mathbf{e}\|^2 \quad \text{for } \alpha \geq L/2\pi.$$

Here in (4.40)-(4.41), we have used the inequalities Hölder, 2D-Ladyzhenskaya, Young and formula (4.18) in Lemma 4.1. Putting (4.30) and (4.40)-(4.41) into the r.h.s of (4.25), we obtain

$$\begin{aligned}
(4.43) \quad \frac{d}{dt} \|\nabla \mathbf{e}\|^2 + \nu \|\Delta \mathbf{e}\|^2 &\leq \left( \frac{C}{\nu^3} \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2 + \frac{C}{\nu\lambda_1} \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| \right) \|\nabla \mathbf{e}\|^2 \\
&\quad + \frac{C\alpha^2}{\nu} \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{ML_a} \right) \|\Delta \mathbf{u}_\alpha\|^2,
\end{aligned}$$

Since both (4.43) share a similar structure than (4.32) then the rest of the proof follows by that of the L- $\alpha$  model. The constant  $C_R$  in this case is given by

$$(4.44) \quad C_R = \left( \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{ML_a} + \frac{C_{ML_a}}{\nu^2}$$

Here

$$R_{ML_a} := \frac{C_{ML_a}}{\nu^2} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right\}.$$

Thus, the proof is complete.  $\square$

**Corollary 4.3.** *Let  $\mathbf{u}_0 \in \mathbf{V}$ ,  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$  and*

$$(4.45) \quad E(s) = \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)(s)\|^2 + \nu \int_0^s \|\Delta(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\|^2 dt \quad \forall s \geq 0.$$

*Then*

$$(4.46) \quad E(s) \leq 2C_R h(\alpha) + 2CC_E \alpha^2 \quad \forall s \geq 0.$$

*Proof.* The proof shares the same idea with Corollary 4.2. We start with

$$\begin{aligned}
(4.47) \quad \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)(s)\|^2 &\leq 2(\|\nabla(\mathbf{u} - \mathbf{u}_\alpha)(s)\|^2 + \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)(s)\|^2) \\
&\leq 2C_R h(\alpha) + 2\alpha^4 \|\nabla \Delta \bar{\mathbf{u}}_\alpha(s)\|^2 \\
&\leq 2C_R h(\alpha) + 2C\alpha^2 \|\nabla \mathbf{u}_\alpha(s)\|^2 \\
&\leq 2C_R h(\alpha) + 2CC_E \alpha^2 \quad \forall s \geq 0,
\end{aligned}$$

where (2.7) has been used in the third inequality. As previous parts  $C_E$  is given as in Corollary 4.2. Similarity, for all  $s \geq 0$

$$\begin{aligned}
(4.48) \quad I &= \nu \int_0^s \|\Delta \mathbf{u} - \Delta \bar{\mathbf{u}}_\alpha\|^2 dt \\
&\leq 2\nu \int_0^s \|\Delta \mathbf{u} - \Delta \mathbf{u}_\alpha\|^2 dt + 2\nu \int_0^s \|\Delta \mathbf{u}_\alpha - \Delta \bar{\mathbf{u}}_\alpha\|^2 dt \\
&\leq 2C_R h(\alpha) + 2\nu \alpha^4 \int_0^s \|\Delta \Delta \bar{\mathbf{u}}_\alpha\|^2 dt \\
&\leq 2C_R h(\alpha) + 2\alpha^2 \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 dt \\
&\leq 2C_R h(\alpha) + 2CC_E \alpha^2.
\end{aligned}$$

Therefore, (4.46) follows by combining (4.47) and (4.48).  $\square$

## 5 The rate of convergence of $p_\alpha$ to $p$

In this section we focus on the error of the pressure by using the results from the previous sections. Let  $p$  and  $p_\alpha$  are the pressures associated to the weak solutions  $\mathbf{u}$  and  $\mathbf{u}_\alpha$  of the NSE (1.1)-(1.3) and all  $\alpha$ -models (1.4)-(1.6), respectively. It will be shown that the difference  $q = p - p_\alpha$  is bounded in terms of the parameter  $\alpha$  uniformly in time in a suitable norm.

**Theorem 5.1.** *Let  $\mathbf{u}_0 \in \mathbf{V}$ ,  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$  and*

$$(5.1) \quad I(s) := \int_0^s \|\nabla q\|^2 dt \quad \forall s \geq 0.$$

*Then*

1. *For for the L- $\alpha$  and model*

$$I(s) \leq C\alpha^{5/2} + C\alpha^3,$$

*where  $C$  given by (5.9).*

2. *For for the SB model*

$$I(s) \leq C\alpha^3 + C\alpha^{5/2}(\log(L/2\pi\alpha) + 1)^{1/2} + C\alpha^2(\log(L/2\pi\alpha) + 1),$$

*where  $C$  is given by (5.14).*

3. *For for the ML- $\alpha$  model*

$$I(s) \leq C\alpha^4 + C\alpha^3 + C(\alpha^{5/2} + \alpha^2)(\log(L/2\pi\alpha) + 1),$$

*where  $C$  given by (5.19).*

*Proof.* It follows from the NSE (1.1)-(1.3) and  $\alpha$ -models (1.4)-(1.6) that

$$(5.2) \quad -\Delta q = \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)] =: \nabla \cdot \mathbf{g}.$$

Assumes that  $p$  and  $p_\alpha$  are periodic and zero averages. The vanishing of the mean values of  $p$  and  $p_\alpha$  ensure their uniqueness determined. Multiplying (5.2) by  $q$  and integrating on  $\mathbb{T}_2$  with using Cauchy-Schwarz inequality yield

$$(5.3) \quad \|\nabla q\|^2 \leq \|\mathbf{g}\|^2 = \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 dx.$$

In order to estimate the error of the pressure we are led to bound the r.h.s of (5.3). Replacing  $\mathbf{e}$  by  $\mathbf{u} - \mathbf{u}_\alpha$ , adding and subtracting the term  $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha$  give

$$\begin{aligned}
\|\mathbf{g}\|^2 &= \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 d\mathbf{x} \\
&= \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 d\mathbf{x} \\
&= \int_{\mathbb{T}_2} |-(\mathbf{e} \cdot \nabla)\mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{e} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 d\mathbf{x} \\
(5.4) \quad &= C \int_{\mathbb{T}_2} (|(\mathbf{e} \cdot \nabla)\mathbf{u}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2) d\mathbf{x}.
\end{aligned}$$

By (5.4) one has for all  $t \geq 0$ :

$$(5.5) \quad I(t) = \int_0^t \|\mathbf{g}\|^2 ds \leq C(I_1 + I_2 + I_3).$$

The proof is given for each  $\alpha$ -model separately below one after the others.

**L- $\alpha$  model.** We have  $N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$  in this case. Each term on the r.h.s of (5.5) will be estimated below. Firstly,

$$\begin{aligned}
I_1 &= \int_0^t \int_{\mathbb{T}_2} |(\mathbf{e} \cdot \nabla)\mathbf{u}|^2 d\mathbf{x} ds \\
&\leq \int_0^t \|\mathbf{e}\|_4^2 \|\nabla\mathbf{u}\|_4^2 ds \\
&\leq \int_0^t \|\mathbf{e}\| \|\nabla\mathbf{e}\| \|\nabla\mathbf{u}\| \|\Delta\mathbf{u}\| ds \\
&\leq C_r^{1/2} C_{NSE2}^{1/2} \alpha^{3/2} \left( \int_0^t \|\nabla\mathbf{e}\|^2 ds \right)^{1/2} \left( \int_0^t \|\Delta\mathbf{u}\|^2 ds \right)^{1/2} \\
(5.6) \quad &\leq \frac{C_r C_{NSE}}{\nu} \alpha^3 \quad \forall t \geq 0,
\end{aligned}$$

here we have used the Hölder and 2D-Ladyzhenskaya inequalities, Lemma 3.1, 4.1 and 4.2, respectively. Secondly,

$$\begin{aligned}
I_2 &= \int_0^t \int_{\mathbb{T}_2} |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}|^2 d\mathbf{x} ds \\
&\leq \int_0^t \|\mathbf{u}_\alpha\| \|\nabla\mathbf{u}_\alpha\| \|\nabla\mathbf{e}\| \|\Delta\mathbf{e}\| ds \\
&\leq \frac{C_L}{\lambda^{1/2}} \left( \int_0^t \|\nabla\mathbf{e}\|^2 ds \right)^{1/2} \left( \int_0^t \|\Delta\mathbf{e}\|^2 ds \right)^{1/2} \\
(5.7) \quad &\leq \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \quad \forall t \geq 0,
\end{aligned}$$

here we have used the Hölder and 2D-Ladyzhenskaya inequalities, Lemma 3.2, Theorems

4.1 and 4.2, respectively. Thirdly,

$$\begin{aligned}
I_3 &= \int_0^t \int_{\mathbb{T}_2} |((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla) \mathbf{u}_\alpha|^2 dx ds \\
&\leq \int_0^t \|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\| \|\nabla \mathbf{u}_\alpha\| \|\Delta \mathbf{u}_\alpha\| ds \\
&\leq 2CC_L \alpha^3 \int_0^t \|\Delta \bar{\mathbf{u}}_\alpha\| \|\Delta \mathbf{u}_\alpha\| ds \\
(5.8) \quad &\leq \frac{2CC_L^{3/2}}{\nu} \alpha^3 \quad \forall t \geq 0,
\end{aligned}$$

here in addition as above we have used (2.6), (4.47) and Lemma 3.2. Thus the proof for this model follows by (5.5)-(5.8)

$$(5.9) \quad I(s) \leq \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} + \left( \frac{C_r C_{NSE}}{\nu} + \frac{2CC_L^{3/2}}{\nu} \right) \alpha^3.$$

**SB model.** For this model we have for all  $t \geq 0$

$$\begin{aligned}
I(t) &= \int_0^t \int_{\mathbb{T}_2} |(\mathbf{u} \cdot \nabla) \mathbf{u} - (\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{u}_\alpha - (\bar{\mathbf{u}}_\alpha \cdot \nabla) \bar{\mathbf{u}}_\alpha|^2 dx ds \\
(5.10) \quad &\leq 4(I_1 + I_2 + I_3).
\end{aligned}$$

One has used the fact that  $\mathbf{u}_\alpha = \mathbf{u} - \mathbf{e}$  in the second term inside the integral. Similarity, by Corollary 4.1 and Lemma 3.3

$$\begin{aligned}
I_1 &= \int_0^t \int_{\mathbb{T}_2} |((\mathbf{u} - \bar{\mathbf{u}}_\alpha) \cdot \nabla) \mathbf{u}|^2 dx ds \\
&\leq \int_0^t \|\mathbf{u} - \bar{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\| \|\nabla \mathbf{u}\| \|\Delta \mathbf{u}\| \\
&\leq \frac{C_{NSE2}^{1/2}}{\nu} C_{cor}^{1/2} (\alpha^2 + \alpha^3)^{1/2} \left( \nu \int_0^t \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\|^2 ds \right)^{1/2} \left( \nu \int_0^t \|\Delta \mathbf{u}\|^2 ds \right)^{1/2} \\
(5.11) \quad &\leq \frac{C_{NSE2}}{\nu} C_{cor} (\alpha^2 + \alpha^3) \quad \forall t \geq 0.
\end{aligned}$$

We deal with the second integral by using Lemma 3.3 and Theorems 4.1, 4.2

$$\begin{aligned}
I_2 &= \int_0^t \int_{\mathbb{T}_2} |(\bar{\mathbf{u}}_\alpha \cdot \nabla) \mathbf{e}|^2 dx ds \\
&\leq \int_0^t \|\bar{\mathbf{u}}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\| \|\nabla \mathbf{e}\| \|\Delta \mathbf{e}\| dx ds \\
&\leq \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left( \nu \int_0^t \|\nabla \mathbf{e}\|^2 ds \right)^{1/2} \left( \nu \int_0^t \|\Delta \mathbf{e}\|^2 ds \right)^{1/2} \\
(5.12) \quad &\leq \begin{cases} \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right)^{1/2} & \text{if } \alpha < L/2\pi, \\ \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} & \text{if } \alpha < L/2\pi. \end{cases}
\end{aligned}$$

Similarity, the last term can be estimated for all  $t \geq 0$  by

$$\begin{aligned}
I_3 &= \int_0^t \int_{\mathbb{T}_2} |(\bar{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u} - \bar{\mathbf{u}}_\alpha)|^2 d\mathbf{x} ds \\
&\leq \int_0^t \|\bar{\mathbf{u}}_\alpha\| \|\nabla \bar{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\| \|\Delta(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\| ds \\
&\leq \frac{C_{SB}^{1/2}}{\nu \lambda_1^{1/2}} (2C_R h(\alpha) + 2CC_{SB}\alpha^2)^{1/2} \left( \nu \int_0^t \|\nabla \mathbf{u}_\alpha\|^2 ds \right)^{1/2} \left( \nu \int_0^t \|\Delta(\mathbf{u} - \bar{\mathbf{u}}_\alpha)\|^2 ds \right)^{1/2} \\
&\leq \frac{C_{SB}}{\nu \lambda_1^{1/2}} (2C_R h(\alpha) + 2CC_{SB}\alpha^2) \\
(5.13) \quad &= \begin{cases} \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[ C_R \alpha^2 \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right) + C_{SB}\alpha^2 \right] & \text{if } \alpha < L/2\pi, \\ \frac{CC_{SB}}{\nu \lambda_1^{1/2}} (C_R + C_{SB})\alpha^2 & \text{if } \alpha < L/2\pi. \end{cases}
\end{aligned}$$

Therefore, by (5.11)-(5.13)

$$\begin{aligned}
I(s) &\leq \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right)^{1/2} \\
(5.14) \quad &+ \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[ C_R \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{SB} \right) + C_{SB} \right] \alpha^2
\end{aligned}$$

Thus the proof for this model is completed.

**ML- $\alpha$  model.** For this model  $I_1$  is estimated as above. We start with  $I_2$  by

$$\begin{aligned}
I_2 &\leq \int_0^t \|\mathbf{u}_\alpha\| \|\nabla \mathbf{u}_\alpha\| \|\nabla \mathbf{e}\| \|\Delta \mathbf{e}\| ds \\
&\leq \frac{C_{MLa}}{\lambda_1^{1/2}} \left( \int_0^t \|\nabla \mathbf{e}\|^2 ds \right)^{1/2} \left( \int_0^t \|\Delta \mathbf{e}\|^2 ds \right)^{1/2} \\
(5.15) \quad &\leq \frac{C_{MLa} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left( K_1 \log \left( \frac{L}{2\pi\alpha} \right) + K_2 + C_{MLa} \right)^{1/2},
\end{aligned}$$

for all  $t \geq 0$ . One has used the results Lemma 3.4, Theorems 4.1 and 4.2. The term  $I_3$  is bounded by

$$(5.16) \quad I_3 = \int_0^t \int_{\mathbb{T}_2} |(\mathbf{u}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)|^2 d\mathbf{x} ds \leq 2(I_{31} + I_{32}).$$

By (4.47) and Lemma 3.4 yield

$$\begin{aligned}
I_{31} &= \int_0^t \int_{\mathbb{T}_2} |((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha) \cdot \nabla)(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)|^2 d\mathbf{x} ds \\
&\leq \int_0^t \|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\|_4^2 \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\|_4^2 ds \\
&\leq \int_0^t \|\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\|^2 \|\Delta(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\| ds \\
(5.17) \quad &\leq \frac{CC_{MLa}^2}{\nu} \alpha^4 \quad \forall t \geq 0.
\end{aligned}$$



The other term can be estimated for all  $t \geq 0$  by

$$\begin{aligned}
I_{32} &= \int_0^t \int_{\mathbb{T}_2} |(\bar{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)|^2 d\mathbf{x} ds \\
&\leq \int_0^t \|\bar{\mathbf{u}}_\alpha\|_\infty^2 \|\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\|^2 ds \\
&\leq \left( K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2 \int_0^t \|\Delta \mathbf{u}_\alpha\|^2 ds \\
(5.18) \quad &\leq \frac{C_{MLa}}{\nu} \left( K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2
\end{aligned}$$

here Lemma 4.1 and (2.8) have been applied. There for the proof for this model is finished by (5.15)-(5.18)

$$\begin{aligned}
I(s) &\leq \frac{C_r C_{NSE}}{\nu} \alpha^3 + \frac{C_{MLa} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left( K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{MLa} \right)^{1/2} \\
(5.19) \quad &+ \frac{C C_{MLa}^2}{\nu} \alpha^4 + \frac{C_{MLa}}{\nu} \left( K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2,
\end{aligned}$$

which concludes the proof.  $\square$

## 6 The 3D case : additional remarks

The rate of convergence has been estimated in the 3D case by Chen-Guenther-Kim-Thomann-Waymire[8]. They show the following estimate.

$$(6.1) \quad \int_0^T \|\mathbf{e}\| dt \leq C(T)\alpha.$$

Their analysis is carried out in the 3D periodic setting and assumes a small data condition (in which the existence and uniqueness of weak solutions  $\mathbf{u}$  of the 3D NSE is ensured). Here  $\mathbf{u}$  and  $\mathbf{u}_\alpha$  are the weak solutions of the NSE and Navier-Stokes- $\alpha$ , respectively, with periodic boundary conditions.

Another result concerning the convergence rate of  $\alpha$ -models of turbulence has been obtained in [13] (both for 2D and 3D) where it's proved that

$$(6.2) \quad \sup_{t \in [0, T]} \|\mathbf{e}(t)\|^2 + \int_0^T \|\nabla \mathbf{e}\|^2 dt \leq C(T)\alpha^2.$$

The result is obtained with  $\mathbf{u}_\alpha(0, \cdot) = \mathbf{u}_0 \in \mathbf{V}$ ,  $\mathbf{f} \in \mathbf{L}^2(0, T; \mathbf{H})$  and under an extra assumption that the weak solution of the 3D NSE  $\mathbf{u} \in \mathbf{L}^4(0, T; \mathbf{H}^1(\mathbb{T}_3))$ . The latter condition ensures that the existence and uniqueness of weak solutions are established, see Section 6 for more details. The logarithmic term in (1.10) is removed in his results for both 2D and 3D periodic cases.

This section is devoted to find out the rate of convergence of weak solutions of the  $\alpha$ -models to that of the NSE in the 3D case. If  $\mathbf{u} \in \mathbf{L}^4([0, T]; \mathbf{H}^1(\mathbb{T}_3))$  the standard Sobolev embedding implies that  $\mathbf{u} \in \mathbf{L}^4([0, T]; \mathbf{L}^6(\mathbb{T}_2))$  which is a special case of the well-known

Leray-Serrin-Prodi (LSP) 3D uniqueness assumption, where  $r = 4$  and  $s = 6$ , see formula (6.3) below, see Leray [27], Prodi [31] and Serrin [34]. More specifically, that is

$$(6.3) \quad \mathbf{u} \in \mathbf{L}^r([0, T]; \mathbf{L}^s(\mathbb{T}_2)) \quad \text{where} \quad \frac{3}{s} + \frac{2}{r} = 1, \quad s \geq 3.$$

It is also known that, see for example Galdi [17, Definition 2.1, Theorem 4.2], weak solutions satisfy the LSP condition are unique and regular in the set of all Leray-Hopf weak solutions. Recently, under the conditions  $\mathbf{f} \in \mathbf{L}^2([0, T]; \mathbf{H})$ ,  $\mathbf{u}_0 \in \mathbf{H}^1(\mathbb{T}_2)$  and an extra condition  $\mathbf{u} \in \mathbf{L}^4([0, T]; \mathbf{H}^1(\mathbb{T}_3))$ , the author of [13] showed that the rate of convergence of weak solutions  $\mathbf{u}_\alpha$  of the four  $\alpha$ -models to  $\mathbf{u}$  is  $\mathcal{O}(\alpha)$  for some suitable norms. More precisely, that is

$$(6.4) \quad \sup_{t \in [0, T]} \|\mathbf{e}(t)\|^2 + \nu \int_0^T \|\nabla \mathbf{e}\|^2 dt \leq C(T)\alpha^2,$$

where  $C$  is the Sobolev constant and  $C_T$  is given by

$$(6.5) \quad C(T) = C_1 \exp \left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4 ds \right\}.$$

here  $C_1 = C_1(\mathbf{u}_0, \mathbf{f}, \nu)$ . On one hand, it follows that in the case  $\mathbf{u} \in \mathbf{L}^4(\mathbb{R}_+; \mathbf{H}^1(\mathbb{T}_2))$ , which satisfies (6.3), we get the error is uniformly bounded in time, i.e.,

$$(6.6) \quad \sup_{t \geq 0} \|\mathbf{e}(t)\|^2 + \nu \int_0^\infty \|\nabla \mathbf{e}\|^2 dt \leq C_\infty \alpha^2,$$

where

$$(6.7) \quad C_\infty = C_1 \exp \left\{ \frac{C}{\nu^3} \int_0^\infty \|\nabla \mathbf{u}\|^4 ds \right\}.$$

On the other hand, if a weak solution  $\mathbf{u}$  of the NSE regular up to a limit time  $T_* < \infty$  or we say that  $\mathbf{u}$  becomes irregular at the time  $T_*$ . Assume that  $T_*$  is the first time that  $\mathbf{u}$  becomes irregular, see Definition 6.1 in Galdi [17], then it is proved that the  $\mathbf{H}^1(\mathbb{T}_3)$ -norm of  $\mathbf{u}$ ,  $\|\nabla \mathbf{u}(t)\|^2$  will blow-up as  $t$  closes to  $T_*$  from below, see for instance [17, Theorem 6.4], Leray [27] and Scheffer [33]. That is given in the following form: there exists  $\epsilon = \epsilon_{T_*} > 0$  small enough such that

$$(6.8) \quad \|\nabla \mathbf{u}(t)\| \geq \frac{C\nu^{3/4}}{(T_* - t)^{1/4}} \quad \forall t \in (T_* - \epsilon, T_*),$$

where  $C > 0$  only depending on  $\mathbb{T}_2$ . In that case, by (6.8), we consider  $C(T)$  in (6.5) with  $T_* - \epsilon < T < T_*$ , which will also blow-up as in the following way

$$(6.9) \quad \begin{aligned} C(T) &= C_1 \exp \left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4 ds \right\} \\ &\geq C_1 \exp \left\{ \frac{C}{\nu^3} \int_{T_* - \epsilon}^T \|\nabla \mathbf{u}\|^4 ds \right\} \\ &\geq C_1 \exp \left\{ C \int_{T_* - \epsilon}^T \frac{1}{T_* - s} ds \right\} \\ &= C_1 \frac{\epsilon^C}{(T_* - T)^C}. \end{aligned}$$

## 7 Conclusions

In this work, assuming  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+, \mathbf{H})$ , we provide the rate of convergence of  $\mathbf{u}_\alpha$  to  $\mathbf{u}$  as well as  $p_\alpha$  to  $p$ . In addition our argument is tied up to the periodic case mostly because special properties of the Stokes operator  $A$ . The extension of the results to other boundary conditions such as the Dirichlet boundary conditions or to the Euler equations are left as future works. In the 3D case extra assumptions should be assumed for the uniqueness of solution of the NSE before studying the rate of convergence.

**Remark 7.1.** *It seems to be the case that all results herein can be established when the periodic domain  $\mathbb{T}_2 = \mathbb{R}^2/[0, L]^2$  is replaced by the whole space  $\mathbb{R}^2$ . However, the existence and uniqueness of weak solutions of all  $\alpha$ -models herein need to be studying carefully first. That will be investigated for the forthcoming works.*

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