

HYDRODYNAMICS FOR SSEP WITH NON-REVERSIBLE SLOW BOUNDARY DYNAMICS: PART I, THE CRITICAL REGIME AND BEYOND

C. ERIGNOUX, P. GONÇALVES, AND G. NAHUM

ABSTRACT. The purpose of this article is to provide a simple proof of the hydrodynamic and hydrostatic behavior of the SSEP in contact with slowed reservoirs which inject and remove particles in a finite size windows at the extremities of the bulk. More precisely, the reservoirs inject/remove particles at/from any point of a window of size K placed at each extremity of the bulk and particles are injected/removed to the first open/occupied position in that window. The hydrodynamic limit is given by the heat equation with non-linear Robin boundary conditions or Neumann boundary conditions, the latter being in the case when the reservoirs are too slow. The proof goes through the entropy method of [16]. We also derive the hydrostatic limit for this model, whose proof is based on the method developed in [18] and [20]. We observe that we do not make use of correlation estimates in none of our results.

1. INTRODUCTION

One of the intriguing questions in Statistical Physics is related to the understanding of how local microscopic perturbations of the dynamics of a particle system, carries through its macroscopic description. In recent years, several articles have been dedicated to the understanding of adding a slow bond, a slow site or a slow boundary to the most classical interacting particle system, namely, *the exclusion process*. For references on this topic, we refer the reader to [13, 14, 1] and references therein, where the hydrodynamic limit for the symmetric simple exclusion process (SSEP) with, respectively, a slow bond, a slow site and a slow boundary was analyzed. Recently, the case of the non-simple symmetric exclusion process with slow boundary has been analyzed in [15, 3, 4] and the asymmetric case in [19]. In the studied cases mentioned above with a slow boundary, the macroscopic PDE, ends up with boundary conditions of the type: Dirichlet, (linear) Robin, or Neumann.

In this article, motivated by deriving other types of boundary conditions, we consider the SSEP in the discrete box $\{1, \dots, N-1\}$ coupled with slow reservoirs, placed at $x=0$ and $x=N$, whose role is to inject and and remove particles in a window of a fixed size $K \geq 1$. A particle may enter to the first free site and leave from the first occupied site in its respective window (*i.e.*, $\{1, \dots, K\}, \{N-K, \dots, N-1\}$). We control the action of the reservoirs by fixing the rates of injection/removal as proportional to $N^{-\theta}$. In this article, we address here the characterization of the hydrodynamic and hydrostatic behavior for the slowed regime $\theta \geq 1$, and we will consider in the second part of this article [9] the case where $\theta \in (0, 1)$, which requires a different set of tools. More precisely, we show that the spatial density of particles is given by a weak solution of the heat equation with non-linear (resp. linear) Robin boundary conditions, if $\theta = 1$ and $K \geq 2$ (resp. $K = 1$, in which we recover the results of [1]), and Neumann boundary conditions, if $\theta > 1$ for any $K \geq 1$. For the case $\theta = 1$, the irreversibility of the boundary dynamics reflects on a non-linear macroscopic boundary evolution for $K \geq 2$ and a simplified version of this model was studied in [6]. The model where particles may enter only through the right and leave only through the left with rates $\frac{1}{2}$ was first introduced by De Masi *et al* in [6], and the reservoirs were termed "current reservoirs", since they do not fix the value of the density at the boundary, but its gradient. The dynamics we consider here is a generalization of the dynamics of [6] since we allow injection and removal from both reservoirs and moreover, the rate is slowed with respect to the bulk

Date: December 20, 2019.

C.E. gratefully acknowledges funding from the European Research Council under the European Unions Horizon 2020 Program, ERC Consolidator GrantUniCoSM (grant agreement no 724939). P.G. thanks FCT/Portugal for support through the project UID/MAT/04459/2013. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovative programme (grant agreement No 715734).

dynamics. In [6], the dynamics was shown to have the Propagation of Chaos property, and that result was obtained by providing sharp estimates on the L^∞ norm of v -functions. As a consequence, the Fick's Law was shown to hold and the hydrostatic limit was proved in [8] and [7], respectively.

When $K = 1$, we are reduced to the SSEP with classical slowed reservoirs, where the hydrodynamic and hydrostatic scenario were both investigated in [1] for $\theta \geq 0$ and for $\theta < 0$, the hydrodynamic behavior was studied in [15]. For $\theta \geq 0$, in [1], Baldasso *et al* showed the hydrodynamic limit by the application of the Entropy method, first presented in [16]. In their case, which corresponds here to the case when $K = 1$, they were able to use an auxiliary measure which is product and given by a suitable profile and for that reason, the entropy production at the boundaries is small enough to enable them to show a replacement lemma at the boundaries. In the present paper, we apply a similar strategy for $\theta \geq 1$, but with an extra difficulty due to the explicit correlation terms at the boundaries, which makes us use another replacement lemma. Unfortunately, that same procedure is not possible for $\theta < 1$ since the comparison measure, namely, a measure close to the stationary state of the system, is quite far from being of product type. Due to the boundary terms of the dynamics, we are not able to control the entropy of the initial measure w.r.t. any product measure and therefore we cannot apply the entropy method except in the case where the boundary is quite slow, that is, when $\theta \geq 1$. For this reason, in the second part of this article [9], we make use of duality estimates already exploited in [10, 11] to derive both the hydrostatic and hydrodynamic limit in the case $\theta \in (0, 1)$.

As a consequence of the hydrodynamic limit we derive Fick's Law. More precisely, we consider two currents related to the system, the conservative and the non-conservative. The former counts the net number of particles going through a bond, while the later counts the number of particles injected minus the number of particles removed from the system through a site. Then, we associate the corresponding fields and we show their convergence. This is the content of Theorem 2.8 whose proof is given in Section 4.

Having the hydrodynamic limit proved, it is simple to obtain the hydrostatic limit, by showing that the stationary correlations of the system vanish as the system size grows to infinity. When $K = 1$ that is exactly the strategy pursued in [1]. In our case, when $K \geq 2$ we do not have any information about the stationary correlations of the system and for that reason we have to do it in a different way. Therefore, here the hydrostatic behavior is investigated through the methods developed in [20] and [18]. In particular, we will follow essentially [20], where the hydrostatic limit was shown for $K = 1$. The proof presented in [18] is robust enough for the hydrostatic limit to follow directly from the hydrodynamic limit when $\theta = 1$, thus we will focus on the case $\theta > 1$ and refer the interested reader to [18] and references therein. Our main interest is when $\theta > 1$, where the macroscopic evolution is governed by a Neumann Laplacian on $[0, 1]$. In contrast to the arguments in [1], where the hydrostatic limit was shown through estimates on the density and correlation fields, the method in [20] is based on the study of the system's evolution at a subdiffusive time scale. This allows us to show replacement lemmas that, under a different time scale, do not hold. In this sense, our results regarding the hydrostatic limit also extend the ones obtained in [1] for $\theta \geq 1$ by the application of a simpler method and when correlation estimates are not easy to obtain.

Regarding the results of the present paper, as already mentioned, the model expresses a macroscopic phase transition from non-linear Robin to Neumann boundary conditions. In particular, we derive the following hydrodynamic equation when $\theta = 1$

$$\left\{ \begin{array}{l} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), \quad (t, u) \in [0, T] \times (0, 1), \\ \partial_u \rho_t(0) = -D_{\alpha, \gamma} \rho_t(0), \quad t \in (0, T], \\ \partial_u \rho_t(1) = D_{\beta, \delta} \rho_t(1), \quad t \in (0, T], \\ \rho(0, \cdot) = f_0(\cdot), \end{array} \right. \quad (1)$$

where $\alpha = (\alpha_1, \dots, \alpha_K)$, $\beta = (\beta_1, \dots, \beta_K)$, $\delta = (\delta_1, \dots, \delta_K)$, $\gamma = (\gamma_1, \dots, \gamma_K)$ are parameters of the boundary dynamics and the operator $D_{\lambda, \sigma}$ is defined for any vectors $\lambda = (\lambda_1, \dots, \lambda_K)$, $\sigma = (\sigma_1, \dots, \sigma_K)$ and f :

$[0, 1] \rightarrow \mathbb{R}$ as

$$(D_{\lambda, \sigma} f)(u) = \sum_{x=1}^K \{\lambda_x (1-f(u)) f^{x-1}(u) - \sigma_x f(u) (1-f(u))^{x-1}\}. \quad (2)$$

In the case $\theta > 1$, the non linear Robin boundary conditions are replaced with Neumann boundary conditions

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ \partial_u \rho_t(0) = \partial_u \rho_t(1) = 0, & t \in (0, T], \\ \rho(0, \cdot) = f_0(\cdot) \end{cases} \quad (3)$$

We also prove uniqueness of the weak solution of (1) in the case where the parameters satisfy suitable conditions (cf. (H0) below).

Our results generalize those of [6] and our proof is much simpler and does not require any knowledge on the decay of ν -functions. Our proof relies on good estimates between the Dirichlet form and the carré du champ operator and a few replacement lemmas which allow to control the boundary terms. Throughout the paper we will state the results for $K \geq 2$, but in some cases present the proofs in detail for $K = 2$ only, since for $K > 2$ the techniques are exactly the same and the biggest change is in the notation. Nevertheless, whenever required, we will state some appropriate remarks regarding the general case $K > 2$. For $\beta_x = \gamma_x = 1$ and $\delta_x = \alpha_x = 0$ for all $x \in \{1, \dots, K\}$, the uniqueness for the Cauchy problem (1) was shown in [8]. For $K = 2$ with $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$ the proof reduces to the case of linear Robin boundary conditions, whose uniqueness problem was studied in [1].

Since we treat in [9] the case $\theta \in (0, 1)$, the main issue left open is related to the fluctuations around the hydrodynamic limit, for which we need to obtain very sharp estimates on the space-time correlations of the system. Large deviations from the stationary state is also another challenge to look at in the near future. Note that in order to get exact information about the stationary state of the system, we cannot make use of the preliminary work on the matrix product ansatz of Derrida [5], since it does not straightforwardly apply to this dynamics in general, and encompasses the case $K = 1$ only.

The article is divided as follows. In section 2 we present the model, the notation, the weak formulation for the solution of the Cauchy problem and the main results, namely, the hydrodynamic limit (Theorem 2.7), a law of large numbers for the current (Theorem 2.8), and the hydrostatic limit (Theorem 2.9). In section 3 we show the hydrodynamic limit: we start presenting an heuristic proof for finding the notion of weak solution of the PDEs, we identify the main difficulties in the proof and we present the tools to solve them. Then we proceed with the entropy method: in Proposition 3.2 we show tightness of the sequence of empirical measures, which shows that there exists convergent subsequences. With the assumption on the uniqueness of the solution of (1), we proceed with the characterization of limit points. In particular, in Proposition 3.3 we show that the spatial density of particles converges to the solution of (1). Section 4 is devoted to the proof of the law of large numbers for the current fields associated to the system. Section 5 is devoted to the proof of the hydrostatic limit. In the Robin case ($\theta = 1$), we require the existence of a unique stationary solution, to which the hydrodynamic solution converges. These two elements are obtained in Sections 5.2 and 5.3, respectively. In the Neumann case ($\theta > 1$), however, any constant profile is stationary, so that we need one further argument. We therefore show in Section 5.1 that the total mass of the system evolves in the subdiffusive time scale $N^{1+\theta}$, and on this time scale it converges to a unique constant which determines the stationary profile. In the appendix, we prove some technical results required throughout the proofs, namely the replacement lemmas (Appendix A), an energy estimate (Appendix B), and the uniqueness of the weak solution to (1) (Appendix C).

2. MODEL AND RESULTS

2.1. The microscopic model. Denote by N a scaling parameter, which will be taken to infinity later on. For $N \geq 2$ we call bulk the discrete set of points $\Lambda_N := \{1, \dots, N-1\}$. The exclusion process in contact with stochastic reservoirs is a Markov process, that we denote by $\{\eta_t : t \geq 0\}$, whose state

space is $\Omega_N := \{0, 1\}^{\Lambda_N}$. The configurations of the state space Ω_N are denoted by η , so that for $x \in \Lambda_N$, $\eta(x) = 0$ means that the site x is vacant while $\eta(x) = 1$ means that the site x is occupied. For any fixed $K \in \mathbb{N}^+$, we define $I_-^K := \{1, \dots, K\}$, $I_+^K := \{N-K, \dots, N-1\}$. We introduce the infinitesimal generator

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \frac{1}{N^\theta} \mathcal{L}_{N,b} \quad (4)$$

acting on functions $f : \Omega_N \rightarrow \mathbb{R}$ by

$$(\mathcal{L}_{N,0}f)(\eta) = \sum_{x=1}^{N-2} (f(\eta^{x,x+1}) - f(\eta)) \quad \text{and} \quad (\mathcal{L}_{N,b}f)(\eta) = (\mathcal{L}_{N,-}f)(\eta) + (\mathcal{L}_{N,+}f)(\eta)$$

where

$$(\mathcal{L}_{N,\pm}f)(\eta) = \sum_{x \in I_\pm^K} c_x^\pm(\eta) (f(\eta^x) - f(\eta)) \quad (5)$$

and for $x \in I_\pm^K \setminus \{1, N-1\}$

$$\begin{aligned} c_x^-(\eta) &= \alpha_x \eta(1) \cdots \eta(x-1) (1 - \eta(x)) + \gamma_x (1 - \eta(1)) \cdots (1 - \eta(x-1)) \eta(x), \\ c_x^+(\eta) &= \beta_{N-x} (1 - \eta(x)) \eta(x+1) \cdots \eta(N-1) + \delta_{N-x} \eta(x) (1 - \eta(x+1)) \cdots (1 - \eta(N-1)) \end{aligned} \quad (6)$$

and $c_1^-(\eta) = \alpha_1 (1 - \eta(1)) + \gamma_1 \eta(1)$ and $c_{N-1}^+(\eta) = \beta_{N-1} (1 - \eta(N-1)) + \delta_{N-1} \eta(N-1)$. To simplify notation, we will identify $\beta_x \equiv \beta_{N-x}$, $\delta_x \equiv \delta_{N-x}$. In the formulae above, we shortened

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & z \neq x, y \\ \eta(y), & z = x \\ \eta(x), & z = y \end{cases}, \quad \eta^x(z) = \begin{cases} \eta(z), & z \neq x, \\ 1 - \eta(x), & z = x \end{cases}, \quad (7)$$

and the $\alpha_i, \gamma_i, \beta_i, \delta_i$, for $i = 1, \dots, K$ are fixed non-negative constants. The size K of the boundary is considered to be a fixed constant as well. In other words, as illustrated in Figure 1, we consider a stirring dynamics in the bulk, and at the two boundary sets I_\pm^K , particles get created (resp. removed) at the empty (resp. occupied) site closest to the boundary. The role of the parameter θ appearing in (4) is to slow down ($\theta \geq 0$) or speed up ($\theta \leq 0$) the boundary dynamics relatively to the bulk dynamics. In this article we restrict ourselves to the case $\theta \geq 1$ and in a companion article [9], we look at the case $0 < \theta < 1$. Throughout the article, we therefore fix $\theta \geq 1$ and consider the Markov process $(\eta_t)_{t \geq 0}$ with infinitesimal generator given by \mathcal{L}_N .

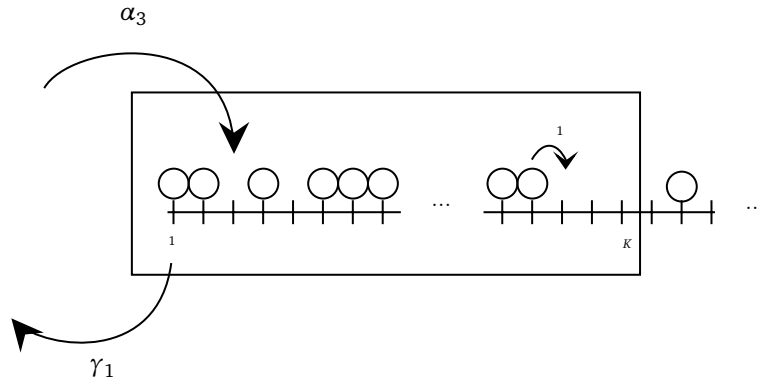


FIGURE 1. Left boundary dynamics.

2.2. Hydrodynamic equation and uniqueness. We now define the macroscopic limit of our model and its topological setup. We denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $L^2([0, 1])$ with respect to a measure μ defined in $[0, 1]$ and $\|\cdot\|_{L^2(\mu)}$ is the corresponding norm. When μ is the Lebesgue measure we write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_{L^2}$ for the corresponding norm.

Fix once and for all a finite time horizon $T > 0$. We denote by $C^{m,n}([0, T] \times [0, 1])$ the set of functions defined on $[0, T] \times [0, 1]$ that are m times differentiable on the first variable and n times differentiable

on the second variable, with continuous derivatives. For a function $G := G(s, u) \in C^{m,n}([0, T] \times [0, 1])$ we denote by $\partial_s G$ its derivative with respect to the time variable s .

Now we want to define the space where the solutions of the hydrodynamic equations will live on, namely the Sobolev space \mathcal{H}^1 on $[0, 1]$. For that purpose, we define the semi inner-product $\langle \cdot, \cdot \rangle_1$ on the set $C^\infty([0, 1])$ by $\langle G, H \rangle_1 = \langle \partial_u G, \partial_u H \rangle$ and the corresponding semi-norm is denoted by $\|\cdot\|_1$.

Definition 2.1. *The Sobolev space \mathcal{H}^1 on $[0, 1]$ is the Hilbert space defined as the completion of $C^\infty([0, 1])$ for the norm $\|\cdot\|_{\mathcal{H}^1}^2 := \|\cdot\|_{L^2}^2 + \|\cdot\|_1^2$. Its elements coincide a.e. with continuous functions. The space $L^2(0, T; \mathcal{H}^1)$ is the set of measurable functions $f : [0, T] \rightarrow \mathcal{H}^1$ such that $\int_0^T \|f_s\|_{\mathcal{H}^1}^2 ds < \infty$.*

We can now give the definition of the weak solution of the hydrodynamic equation that will be derived for the process described above when $\theta \geq 1$.

Definition 2.2. *Let $f_0 : [0, 1] \rightarrow [0, 1]$ be a measurable function, and recall from (2) the definition of the operator $D_{\lambda, \sigma}$. We say that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the heat equation with Robin boundary conditions (this will be obtained in the case $\theta = 1$)*

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ \partial_u \rho_t(0) = -D_{\alpha, \gamma} \rho_t(0), & t \in [0, T] \\ \partial_u \rho_t(1) = D_{\beta, \delta} \rho_t(1), & t \in [0, T] \\ \rho(0, \cdot) = f_0(\cdot), \end{cases} \quad (8)$$

if the following two conditions hold:

1. $\rho \in L^2(0, T; \mathcal{H}^1)$,
2. ρ satisfies the weak formulation:

$$\begin{aligned} F(\rho, G, t) &:= \langle \rho_t, G_t \rangle - \langle f_0, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_u^2 + \partial_s) G_s \rangle ds \\ &+ \int_0^t \{ \rho_s(1) \partial_u G_s(1) - \rho_s(0) \partial_u G_s(0) \} ds - \int_0^t G_s(1) (D_{\beta, \delta} \rho_s)(1) ds - \int_0^t G_s(0) (D_{\alpha, \gamma} \rho_s)(0) ds = 0, \end{aligned} \quad (9)$$

for all $t \in [0, T]$, any function $G \in C^{1,2}([0, T] \times [0, 1])$.

We say that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the heat equation with Neumann boundary conditions (this will be obtained in the case $\theta > 1$)

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ \partial_u \rho_t(0) = \partial_u \rho_t(1) = 0, & t \in [0, T] \\ \rho(0, \cdot) = f_0(\cdot), \end{cases} \quad (10)$$

if conditions 1. and 2. above hold, with (9) replaced by

$$F(\rho, G, t) := \langle \rho_t, G_t \rangle - \langle f_0, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_u^2 + \partial_s) G_s \rangle ds + \int_0^t \{ \rho_s(1) \partial_u G_s(1) - \rho_s(0) \partial_u G_s(0) \} ds = 0. \quad (11)$$

Throughout the present article we make the following assumption in the case $\theta = 1$ (Robin boundary conditions),

$$\text{The (finite) sequences } \alpha, \gamma, \beta \text{ and } \delta \text{ are non-increasing,} \quad (\text{H0})$$

which ensures uniqueness of the weak solutions of equation (8):

Lemma 2.3. *[Uniqueness of weak solutions] Consider the notion of weak solution introduced in Definition 2.2, and fix a measurable initial profile $f_0 : [0, 1] \rightarrow [0, 1]$. Assuming (H0), the weak solution of (8) is unique. Moreover, the weak solution of (10) is unique.*

The proof of the first statement is postponed to Appendix C. The Neumann case is classical and for that reason it is omitted, but the proof can be found in [13]. For the sake of concision, we do not recall for each of our main results that assumption (H0) is made, however since it guarantees uniqueness of weak solutions, this assumption is made throughout the article whenever $\theta = 1$.

Remark 2.4. For $\beta = \gamma \equiv j$ and $\delta = \alpha \equiv 0$ we recover the boundary conditions of [6].

Remark 2.5. For $K = 2$, (8) rewrites as

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 \rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ \partial_u \rho_t(1) = \beta_1 - (\beta_1 + \delta_1) \rho_t(1) + (\delta_2 - \beta_2) (\rho_t^2(1) - \rho_t(1)), \\ \partial_u \rho_t(0) = \rho_t(0) (\alpha_1 + \gamma_1) - \alpha_1 - (\gamma_2 - \alpha_2) (\rho_t^2(0) - \rho_t(0)), & t \in [0, T] \\ \rho(0, \cdot) = f_0(\cdot), \end{cases} \quad (12)$$

and for $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$ we recover the linear Robin boundary conditions as in [1] and when $\alpha_2 = \gamma_2 = \beta_2 = \delta_2 = 0$ and $\beta_1 = 1 - \delta_1 = \beta$ and $\alpha_1 = 1 - \alpha_1 = \alpha$ we deal with exactly the same model of [1] and we recover their result. Further note that the weak solution of (12) when $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$ (corresponding to linear Robin boundary conditions) is shown in [1] to be unique.

2.3. Hydrodynamic limit. In this section we state the hydrodynamic limit of the process $\{\eta_{tN^2}\}_{t \geq 0}$. Note the scaling factor N^2 whose purpose is to accelerate the process to a diffusive time scale. Let \mathcal{M}^+ be the space of positive measures on $[0, 1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$ we define the empirical measure $\pi^N(\eta, \cdot) \in \mathcal{M}^+$ on $[0, 1]$ as

$$\pi^N(\eta, du) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{\frac{x}{N}}(du), \quad (13)$$

where δ_a is a Dirac mass on $a \in [0, 1]$. Given the trajectory $\{\eta_{tN^2}\}_{t \geq 0}$ of the *accelerated* process, we further introduce $\pi_t^N(du) := \pi^N(\eta_{tN^2}, du)$ the empirical measure at the macroscopic time t . Below, and in what follows, we use the notation $\langle \pi_t^N, G \rangle$ to denote the integral of G w.r.t. the measure π_t^N . This notation should not be confused with the inner product in $L^2([0, 1])$. Fix $T > 0$ and $\theta \geq 0$. We denote by \mathbb{P}_{μ_N} the probability measure in the Skorohod space $\mathcal{D}([0, T], \Omega_N)$ induced by the Markov process $\{\eta_{tN^2}\}_{t \geq 0}$ and the initial probability measure μ_N and \mathbb{E}_{μ_N} denotes the expectation w.r.t. \mathbb{P}_{μ_N} .

Definition 2.6. We say that a sequence of probability measures $\{\mu_N\}_{N \geq 1}$ on Ω_N is associated with a profile $\rho_0 : [0, 1] \rightarrow [0, 1]$ if for any continuous function $G : [0, 1] \rightarrow \mathbb{R}$ and every $\delta > 0$

$$\lim_{N \rightarrow \infty} \mu_N \left(\eta \in \Omega_N : \left| \langle \pi^N, G \rangle - \langle G, \rho_0 \rangle \right| > \delta \right) = 0. \quad (14)$$

Our first result is the hydrodynamic limit for the process introduced above and it is stated as follows.

Theorem 2.7. Let $f_0 : [0, 1] \rightarrow [0, 1]$ be a measurable function and let $\{\mu_N\}_{N \geq 1}$ be a sequence of probability measures in Ω_N associated with f_0 in the sense of Definition 2.6. Then, for any $t \in [0, T]$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left(\left| \langle \pi_t^N, G \rangle - \langle G, \rho_t \rangle \right| > \delta \right) = 0,$$

where $\rho_t(\cdot)$ is the unique weak solution, in the sense of Definition 2.2, of (8) for $\theta = 1$, resp. (10) for $\theta > 1$.

Let $\{\mathbb{Q}_N\}_{N \geq 1}$ be the sequence of probability measures on $\mathcal{D}([0, T], \mathcal{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t \geq 0}$ and \mathbb{P}_{μ_N} , namely $\mathbb{Q}_N = \mathbb{P}_{\mu_N} \circ (\pi^N)^{-1}$. To prove Theorem 2.7 we first show that the sequence $\{\mathbb{Q}_N\}_{N \geq 1}$ is tight, and then prove that any of its limit points \mathbb{Q} is concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure (this is a consequence of the exclusion dynamics), whose density $\rho_t(u)$ is the unique (cf. Lemma 2.3) weak solution of the hydrodynamic equation. We prove Theorem 2.7 in Section 3.

2.4. Empirical currents. Our next result is a law of large numbers for the empirical currents of the process. Let $J_t^N(x)$ denote the process that counts the flux of particles (in the accelerated process $(\eta_{sN^2})_{s \geq 0}$) through the bond $\{x, x+1\}$ up to time t , i.e. the number of particles that jumped from the site x to the site $x+1$ minus the number of particles that jumped from the site $x+1$ to the site x during the time interval $[0, t]$. The empirical measure associated with this *conservative current* is defined as

$$J_t^N := \frac{1}{N^2} \sum_{x=1}^{N-2} J_t^N(x) \delta_{x/N}.$$

Notice the normalization factor N^2 which is taking into account the diffusive time rescaling and the space normalization. For $x \in I_{\pm}^K$, we denote by $K_t^N(x)$ the non-conservative current at the site x up to time t , that is, the number of particles that have been created minus the number of particles that have been removed of the system at site x . The corresponding empirical measure is given by

$$K_t^N := \frac{1}{N} \sum_{x \in I_{\pm}^K \cup I_{\pm}^K} K_t^N(x) \delta_{x/N}.$$

For a test function f we use the notation $\langle J_t^N, f \rangle$ and $\langle K_t^N, f \rangle$ to denote, respectively:

$$\langle J_t^N, f \rangle := \frac{1}{N^2} \sum_{x=1}^{N-2} J_t^N(x) f\left(\frac{x}{N}\right) \quad \text{and} \quad \langle K_t^N, f \rangle := \frac{1}{N} \sum_{x \in I_{\pm}^K \cup I_{\pm}^K} K_t^N(x) f\left(\frac{x}{N}\right).$$

Our second main result is a law of large numbers for the current fields.

Theorem 2.8 (Law of large Numbers for the current). *For any $t \in [0, T]$ and $f \in C^1([0, 1])$,*

$$\lim_{N \rightarrow +\infty} \mathbb{P}_{\mu_N} \left[\left| \langle J_t^N, f \rangle - \int_0^t \int_0^1 f(u) \partial_u \rho_s(u) du ds \right| > \delta \right] = 0,$$

$$\lim_{N \rightarrow +\infty} \mathbb{P}_{\mu_N} \left[\left| \langle K_t^N, f \rangle - \mathbf{1}_{\{\theta=1\}} \int_0^t f(0) (D_{\alpha, \gamma} \rho_s)(0) + f(1) (D_{\beta, \delta} \rho_s)(1) ds \right| > \delta \right] = 0,$$

where $\rho_t(\cdot)$ is the unique weak solution of (8) if $\theta = 1$ (resp. of (10) if $\theta > 1$). In other words, writing $j_t^N = J_t^N + K_t^N$, we have that j^N converges weakly to $j du$, where j is a weak solution to $j = -\nabla \rho$.

This theorem is proved in Section 4.

2.5. Hydrostatic limit. For any fixed N and $\theta \geq 1$, the Markov process η_t is irreducible on its finite state space Ω_N , it has a unique stationary measure that we denote by μ_N^{ss} . Our third main result concerns the hydrostatic limit for the dynamics, which gives the macroscopic behavior of our model starting from the stationary state μ_N^{ss} . One important ingredient in our proof is the uniqueness of the stationary solution of the hydrodynamic equation. In the regime $\theta = 1$, on the other hand, to establish the hydrostatic limit, it is sufficient to show that there is a unique stationary solution to the hydrodynamic equation (8). In the regime $\theta > 1$ any constant profile is a stationary solution to (10), however under suitable assumptions, this constant can be uniquely determined as the mass m^* to which the microscopic system relaxes on subdiffusive timescales.

To establish the different uniqueness results above, we will need in the two cases ($\theta = 1$ and $\theta > 1$) further assumptions. For this reason, we introduce

$$\delta_1 \leq \alpha_1 \quad \text{and} \quad \beta_1 \leq \gamma_1, \tag{H1}$$

and

$$\alpha + \beta \quad \text{and} \quad \gamma + \delta \quad \text{are non increasing.} \tag{H2}$$

We are now ready to state our third main result.

Theorem 2.9. *For $\theta = 1$, assuming (H1) there exists a unique stationary solution ρ^* of (8), and μ_N^{ss} is associated with it in the sense of Definition 2.6, i.e.*

$$\lim_{N \rightarrow \infty} \mu_N^{ss} \left(\left| \langle \pi^N, G \rangle - \langle G, \rho^* \rangle \right| > \delta \right) = 0.$$

For $\theta > 1$, assuming (H2) there exists a unique constant $m^ \in [0, 1]$, such that μ_N^{ss} is associated with the constant profile $\rho^* \equiv m^*$.*

Remark 2.10 (On assumptions (H1) and (H2)). *Assumption (H1) is used in the case $\theta = 1$ to guarantee uniqueness of the stationary solution. Assumption (H2) is used for $\theta > 1$ to prove convergence of the mass of the system to a defined constant. As we will see through the article, one could weaken these assumptions, yet we elected to settle for assumptions (H1) and (H2) to provide the reader with a working case, since finding optimal bounds for both of the cases is a non-trivial algebraic problem that goes beyond the scope of this article.*

The proof of Theorem 2.9 is the purpose of Section 5.

3. PROOF OF THEOREM 2.7

In this section we present the proof of the hydrodynamic limit and we start by giving an heuristic argument in order to deduce the notion of weak solution given in Definition 2.2. To simplify the exposition we present the proof for the case $K = 2$ but emphasize that the general case follows straightforwardly.

3.1. Heuristic argument. We start by briefly outlining the argument before detailing the relevant steps of the proof in Section 3.3. Let us fix a test function $G \in C^{1,2}([0, T] \times [0, 1])$. Following from Dynkin's formula and simple computations,

$$\begin{aligned}
M_t^N(G) &:= \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, (\partial_s + N^2 \mathcal{L}_N) G_s \rangle ds \\
&= \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, (\partial_s + \Delta_N) G_s \rangle ds \\
&\quad - \int_0^t \nabla_N^+ G_s(0) \eta_{sN^2}(1) - \nabla_N^- G_s(1) \eta_{sN^2}(N-1) ds \\
&\quad - \frac{N^2}{N^\theta} \int_0^t G_s\left(\frac{1}{N}\right) \{\alpha_1 - \eta_{sN^2}(1)(\alpha_1 + \gamma_1)\} + G_s\left(\frac{N-1}{N}\right) \{\beta_1 - \eta_{sN^2}(N-1)(\beta_1 + \delta_1)\} ds \\
&\quad - \frac{N^2}{N^\theta} \int_0^t G_s\left(\frac{2}{N}\right) \{\alpha_2 \eta_{sN^2}(1) - \gamma_2 \eta_{sN^2}(2) - \eta_{sN^2}(1) \eta_{sN^2}(2)(\alpha_2 - \gamma_2)\} ds \\
&\quad - \frac{N^2}{N^\theta} \int_0^t G_s\left(\frac{N-2}{N}\right) \{\beta_2 \eta_{sN^2}(N-1) - \delta_2 \eta_{sN^2}(N-2) - \eta_{sN^2}(N-1) \eta_{sN^2}(N-2)(\beta_2 - \delta_2)\} ds
\end{aligned} \tag{15}$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where for each $t \geq 0$, $\mathcal{F}_t := \sigma(\eta_{sN^2} : s \leq t)$. Above, for $x \in \Lambda_N$, the discrete derivatives of G are defined by

$$\nabla_N^+ G_s\left(\frac{x}{N}\right) = N \left[G\left(\frac{x+1}{N}\right) - G\left(\frac{x}{N}\right) \right],$$

$\nabla_N^- G\left(\frac{x}{N}\right) = \nabla^+ G\left(\frac{x-1}{N}\right)$ and its discrete laplacian is defined by

$$\Delta_N G\left(\frac{x}{N}\right) = N^2 \left[G\left(\frac{x+1}{N}\right) - 2G\left(\frac{x}{N}\right) + G\left(\frac{x-1}{N}\right) \right].$$

Remark 3.1. For fixed $K \geq 2$, the expression above can be compactly written by introducing the operators $D_{\lambda, \sigma}^{N, \pm}$ defined by

$$\begin{aligned}
(D_{\lambda, \sigma}^{N, -} f)(x) &= \left\{ \lambda_x f(1) \dots f(x-1) (1-f(x)) - \sigma_x (1-f(1)) \dots (1-f(x-1)) f(x) \right\} 1_{x \in I_K^-} \\
(D_{\lambda, \sigma}^{N, +} f)(x) &= \left\{ \lambda_x (1-f(x)) f(x+1) \dots f(N-1) - \sigma_x f(x) (1-f(x+1)) \dots (1-f(N-1)) \right\} 1_{x \in I_K^+}
\end{aligned} \tag{16}$$

for $f : \mathbb{Z} \rightarrow \mathbb{R}$ and $\lambda = (\lambda_1, \dots, \lambda_K)$, $\sigma = (\sigma_1, \dots, \sigma_K)$. With this notation, Dynkin's formula takes the form

$$\begin{aligned}
M_t^N(G) &= \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, (\partial_s + \Delta_N) G_s \rangle ds \\
&\quad - \int_0^t \left\{ \nabla_N^+ G_s(0) \eta_{sN^2}(1) - \nabla_N^- G_s(1) \eta_{sN^2}(N-1) \right\} ds \\
&\quad - \frac{N^2}{N^\theta} \int_0^t \left\{ \langle \pi^N(D_{\alpha, \gamma}^{N, -} \eta_{sN^2}, \cdot), G_s \rangle + \langle \pi^N(D_{\beta, \delta}^{N, +} \eta_{sN^2}, \cdot), G_s \rangle \right\} ds.
\end{aligned}$$

In fact, the largest technical issue with the proof of the hydrodynamic limit is, as we will also see ahead, the proof of the replacement lemmas which roughly states $\frac{N^2}{N^\theta} \langle \pi^N(D_{\alpha, \gamma}^{N, -} \eta_{sN^2}, \cdot), G_s \rangle \xrightarrow{\mathbb{P}^{\mu_N}} (D_{\alpha, \gamma} \rho_s)(0) G_s(0)$ for $\theta = 1$, with $\rho(\cdot)$ being the unique weak solution to (8).

As we will show, $\mathbb{E}_{\mu_N} \left[(M_t^N(G))^2 \right]$ vanishes as $N \rightarrow \infty$. We now focus on the integral terms above. Let us start with the boundary term coming from the bulk dynamics, that is, the term on the second line of the previous display. By Theorem A.3, with the choice $\psi \equiv 1$, we are able to replace $\eta_s(1)$ (resp. $\eta_s(N-1)$) by the average in a box of size $\lfloor \varepsilon N \rfloor$ to the right of site 1 (resp. to the left of site $N-1$):

$$\overrightarrow{\eta}_s^{\varepsilon N}(1) := \frac{1}{\varepsilon N} \sum_{x=2}^{1+\varepsilon N} \eta_s(x), \quad \overleftarrow{\eta}_s^{\varepsilon N}(N-1) := \frac{1}{\varepsilon N} \sum_{x=N-2}^{N-1-\varepsilon N} \eta_s(x), \quad (17)$$

then, since, for N sufficiently big, $\overrightarrow{\eta}_s^{\varepsilon N}(1) \sim \rho_s(0)$ (resp.) $\overleftarrow{\eta}_s^{\varepsilon N}(N-1) \sim \rho_s(1)$ in a sense which will be explained later on, and by a Taylor expansion on the text function G , we arrive at

$$\int_0^t \partial_u G_s(0) \rho_s(0) - \partial_u G_s(1) \rho_s(1) ds.$$

which is exactly the fourth term at the right hand side of (9). By abuse of notation, above and below εN denotes $\lfloor \varepsilon N \rfloor$. Now we analyse the terms coming from the boundary dynamics. We start with the terms on the fourth line on the right hand-side of (15). Note that when $\theta > 1$, since G and η are bounded, these terms are of order $O(N^{1-\theta})$ and so they vanish as $N \rightarrow +\infty$. When $\theta = 1$ and using again Theorem A.3, with the choice $\psi \equiv 1$, those terms are going to contribute to the integral formulation with

$$\int_0^t G_s(0)(\alpha_1 - (\alpha_1 + \gamma_1)\rho_s(0)) + G_s(1)(\beta_1 - (\beta_1 + \delta_1)\rho_s(1)) ds.$$

Now we look at the fifth and sixth terms at the right hand-side of (15). We focus on the terms on the fifth line, but we note that the analysis is completely analogous for the terms in the sixth line. As before, for $\theta > 1$ those terms are of order $O(N^{1-\theta})$ and so they vanish as $N \rightarrow +\infty$. When $\theta = 1$, from Theorem A.2, with the choice $\varphi \equiv 1$, we can replace, for any term that does not involve the product of $\eta(1)$ and $\eta(2)$, $\eta(2)$ by $\eta(1)$ and from Theorem A.3 (with $\psi \equiv 1$), replace $\eta(1)$ by $\eta^{\varepsilon N}(1)$. For the quadratic terms in $\eta(1)\eta(2)$ we first apply Theorem A.3 (with $\psi(\eta) = \eta(1)$), to replace $\eta(2)$ by $\eta^{\varepsilon N}(1)$. In the resulting term $\eta(1)\eta^{\varepsilon N}(1)$, we then replace $\eta(1)$ by $\eta^{\varepsilon N}(1)$ by applying Theorem A.3 (with $\psi(\eta) = \eta^{\varepsilon N}(1)$). From this we conclude that the terms on the fifth line of (15) contribute to the integral formulation with

$$\int_0^t G_s(0)(\alpha_2 - \gamma_2)(\rho_s^2(0) - \rho_s(0)) ds.$$

Recall that we defined after Theorem 2.7 the distribution \mathbb{Q}_N of the trajectory of the empirical measure π^N . Assuming that one proves that the sequence $\{\mathbb{Q}_N\}_{N \geq 1}$ is tight (which is done in Section 3.2), the arguments above prove that any of its limit points is a Dirac measure supported on the trajectory $\pi_t(du) = \rho_t(u)du$ where $\rho_t(\cdot)$ is the unique weak solution of (8). These arguments are carried out in further detail in the next subsections.

3.2. Tightness.

Proposition 3.2. *The sequence $\{\mathbb{Q}_N\}_{N \geq 1}$ is tight under the Skorohod topology of $\mathcal{D}([0, T], \mathcal{M}^+)$.*

Proof. From Chapter 4 of [17], in order to prove tightness it is enough to show that

$$\lim_{\gamma \rightarrow 0} \limsup_{N \rightarrow +\infty} \sup_{\tau \in \mathcal{T}_N, \lambda \leq \gamma} \mathbb{P}_{\mu_N}^N (\eta \in \mathcal{D}([0, T], \Omega_N) : |\langle \pi_{\tau+\lambda}^N, G \rangle - \langle \pi_{\tau}^N, G \rangle| > \epsilon) = 0,$$

for any continuous function $G : [0, 1] \rightarrow \mathbb{R}$. Above \mathcal{T}_N is the set of stopping times bounded by T . In fact, we are going to prove the result for functions in $C^2([0, 1])$, but then, by an L^1 approximation it is simple to extend the result to continuous functions. By Proposition 4.1.7 in [17] it is enough to show the result for every function G in a dense subset of $C([0, 1])$, with respect to the uniform topology. From now on we assume that $G \in C^2([0, 1])$. From Dynkin's formula, plus Chebyshev's and Markov's inequality, we can bound the previous probability by

$$\frac{2}{\epsilon} \mathbb{E}_{\mu_N} \left[\left| \int_{\tau}^{\tau+\lambda} N^2 \mathcal{L}_N \langle \pi_s^N, G \rangle ds \right| \right] + \frac{4}{\epsilon^2} \mathbb{E}_{\mu_N} \left[(M_{\tau}^N(G) - M_{\tau+\lambda}^N(G))^2 \right].$$

Observe that since $G \in C^2([0, 1])$ we have that $|\Delta_N G(\frac{x}{N})| \leq 2\|G''\|_\infty$ and $|\nabla_N^\pm G(\frac{x}{N})| \leq \|G'\|_\infty$. In particular, since there is at most one particle per site, and recovering from (15) the expression for $\mathcal{L}_n \langle \pi_s^N, G \rangle$, we obtain straightforwardly

$$|N^2 \mathcal{L}_n \langle \pi_s^N, G \rangle| \lesssim \frac{1}{N^{\theta-1}} \|G'\|_\infty + \frac{1}{N^{\theta-1}} \|G''\|_\infty,$$

where the notation \lesssim means "less than a constant times". As a consequence, for $\theta \geq 1$

$$\lim_{\gamma \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{F}_T, \lambda \leq \gamma} \mathbb{E}_{\mu^N} \left[\left| \int_\tau^{\tau+\lambda} N^2 \mathcal{L}_N \langle \pi_s^N, G \rangle ds \right| \right] = 0.$$

Now we treat the remaining term. From Dynkin's formula, $(M_t^N(G))^2 - \langle M^N(G) \rangle_t$ is a (mean zero) martingale with respect to the natural filtration \mathcal{F}_t . From [17] (Appendix 1.6) one obtains that its quadratic variation is $\langle M^N(G) \rangle_t := \int_0^t B_s^N(G) ds$, where

$$B_s^N(G) := N^2 \left(\mathcal{L}_N \langle \pi^N(\eta_s), G \rangle^2 - 2 \langle \pi^N(\eta_s), G \rangle \mathcal{L}_N \langle \pi^N(\eta_s), G \rangle \right).$$

This yields

$$\mathbb{E}_{\mu^N} \left[(M_\tau^{N,H} - M_{\tau+\lambda}^{N,G})^2 \right] = \mathbb{E}_{\mu^N} \left[\int_\tau^{\tau+\lambda} B_s^N(G) ds \right].$$

We can split $B_s^N(G) := B_{s,-}^N(G) + B_{s,0}^N(G) + B_{s,+}^N(G)$, where each term corresponds to the contribution of $\mathcal{L}_{N,-}$, $\mathcal{L}_{N,0}$, $\mathcal{L}_{N,+}$, respectively. Now note that

$$\begin{aligned} B_{s,0}^N(G) &= N^2 \sum_{x \in \Lambda_N} \left(\langle \pi^N(\eta_s^{x,x+1}), H \rangle - \langle \pi^N(\eta_s), H \rangle \right)^2 \\ &= \sum_{x \in \Lambda_N} (\eta_s(x) - \eta_s(x+1))^2 \left(G(\frac{x+1}{N}) - G(\frac{x}{N}) \right)^2 \leq \frac{N-1}{N^2} \|G'\|_\infty^2. \end{aligned}$$

For the boundary dynamics, we bound the rates in the generator by a constant, which yields

$$B_{s,-}^N(G) \lesssim \frac{1}{N^{\theta-1}} \|G\|_\infty^2 \text{ and } B_{s,+}^N(G) \lesssim \frac{1}{N^{\theta-1}} \|G\|_\infty^2$$

and concludes the proof. \square

3.3. Characterization of the limit point. We now characterize the limit points of $\{\mathbb{Q}_N\}_{N \geq 1}$ and show that they concentrate on trajectories satisfying the weak form of the hydrodynamic equation.

Proposition 3.3. *For any limit point \mathbb{Q} of $\{\mathbb{Q}_N\}_{N \geq 1}$, it holds*

$$\mathbb{Q} \left(\pi. \in \mathcal{D}([0, T], \mathcal{M}) : F(\rho, G, t) = 0 \right) = 1$$

where F is given in (9) for $\theta = 1$ and (11) for $\theta > 1$.

Proof. We present the proof for the case $\theta = 1$ and $K = 2$, since for $\theta > 1$ it is analogous. We present a remark at the end regarding the extension to other values of K . Fix a limit point \mathbb{Q} of $\{\mathbb{Q}_N\}_{N \geq 1}$. As a consequence of Corollary B.2, we have that $\mathbb{Q}(R_T) = 1$ where R_T is the event on which $\pi.$ is absolutely continuous w.r.t the Lebesgue measure and with density in \mathcal{H}^1 . To present the argument as simply and concisely as possible, assume that G is time independent, but the same arguments apply when this is not the case. To prove the Proposition, we show that for any $\delta > 0$ and any $G \in C^2([0, 1])$:

$$\mathbb{Q} \left(R_T \text{ and } \sup_{0 \leq t \leq T} |F(\rho, G, t)| > \delta \right) = 0, \quad (18)$$

that is

$$\begin{aligned} & \mathbb{Q}\left(R_T \text{ and } \left| \sup_{0 \leq t \leq T} |\langle \rho_t, G \rangle - \langle f_0, G \rangle + \int_0^t \langle \rho_s, \Delta G \rangle ds + \int_0^t \{ \rho_s(1) \partial_u G(1) - \rho_s(0) \partial_u G(0) \} ds \right. \right. \\ & \quad - \int_0^t \left\{ G(1)(\beta_1 - (\beta_1 + \delta_1) \rho_s(1) + (\delta_2 - \beta_2)(\rho_s^2(1) - \rho_s(1))) \right\} ds \\ & \quad \left. \left. - \int_0^t \left\{ G(0)(\alpha_1 - (\alpha_1 + \gamma_1) \rho_s(0) + (\gamma_2 - \alpha_2)(\rho_s^2(0) - \rho_s(0))) \right\} ds \right| > \delta \right) = 0. \end{aligned} \quad (19)$$

Due to the boundary terms, the set inside the probability above is not an open set in the Skorohod space. As a consequence, at this point, we cannot apply *Portmanteau's theorem*. To solve this problem, we take the following functions: $\overleftarrow{t}_\epsilon^u(v) = \frac{1}{\epsilon} 1_{(u-\epsilon, u]}(v)$ and $\overrightarrow{t}_\epsilon^u(v) = \frac{1}{\epsilon} 1_{[u, u+\epsilon)}(v)$, and we use the notation $\langle \pi_s, \overleftarrow{t}_\epsilon^u \rangle = \frac{1}{\epsilon} \int_{u-\epsilon}^u \rho_s(v) dv$ and $\langle \pi_s, \overrightarrow{t}_\epsilon^u \rangle = \frac{1}{\epsilon} \int_u^{u+\epsilon} \rho_s(v) dv$. Now observe that since $\rho \in L^2(0, T; \mathcal{H}^1)$, it is easy to prove for all $\epsilon > 0$ that $|\rho_s(u) - \langle \pi_s, \overleftarrow{t}_\epsilon^u \rangle| \leq \frac{1}{2} \epsilon \|\partial_u \rho\|_2^2$. As a consequence of the last result, we can bound the probability on the left-hand side of (19) by

$$\begin{aligned} & \mathbb{Q}\left(R_T \text{ and } \sup_{0 \leq t \leq T} \left| \langle \rho_t, G \rangle - \langle f_0, G \rangle - \int_0^t \langle \rho_s, \partial_u^2 G \rangle ds + \int_0^t \{ \langle \pi_s, \overleftarrow{t}_\epsilon^1 \rangle \partial_u G(1) - \langle \pi_s, \overrightarrow{t}_\epsilon^0 \rangle \partial_u G(0) \} ds \right. \right. \\ & \quad - \int_0^t \left\{ G(1)(\beta_1 - (\beta_1 + \delta_1) \langle \pi_s, \overleftarrow{t}_\epsilon^1 \rangle + (\delta_2 - \beta_2) \langle \pi_s, \overleftarrow{t}_\epsilon^1 \rangle (\langle \pi_s, \overleftarrow{t}_\epsilon^{1-\epsilon} \rangle - 1)) \right\} ds \\ & \quad \left. \left. - \int_0^t \left\{ G(0)(\alpha_1 - (\alpha_1 + \gamma_1) \langle \pi_s, \overrightarrow{t}_\epsilon^0 \rangle + (\gamma_2 - \alpha_2) \langle \pi_s, \overrightarrow{t}_\epsilon^0 \rangle (\langle \pi_s, \overrightarrow{t}_\epsilon^\epsilon \rangle - 1)) \right\} ds \right| > \delta/2 \right) + o_\epsilon(1). \end{aligned} \quad (20)$$

To finally apply Portmanteau's theorem, we argue that we can approximate $\overleftarrow{t}_\epsilon^u, \overrightarrow{t}_\epsilon^u$ by continuous functions in such a way that the error vanishes as $\epsilon \rightarrow 0$. Then, we apply Portmanteau's theorem and bound the first term in (20) from above by $\liminf_{N \rightarrow \infty} \mathbb{Q}_N(A(T, G, \delta, \epsilon))$, where we shortened $A(T, G, \delta, \epsilon)$ for the event in (20). Summing and subtracting $\int_0^t \mathcal{L}_N \langle \pi_s^N, G \rangle ds$ inside the absolute value in $A(T, G, \delta, \epsilon)$, recalling (15), we obtain that $\mathbb{Q}_N(A(T, G, \delta, \epsilon))$ is less than the sum of the following contributions

$$\begin{aligned} P_1 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} |M_t^N| \geq \frac{\delta}{14} \right), \\ P_2 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \langle \pi_s^N, \Delta_N G \rangle - \langle \pi_s^N, \partial_u^2 G \rangle ds \right| > \frac{\delta}{14} \right) \\ P_3 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \eta_{sN^2} (N-1) \nabla_N^- G(1) - \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle \partial_u G(1) ds \right| > \frac{\delta}{14} \right), \\ P_4 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t G\left(\frac{N-1}{N}\right) (\beta_1 - (\beta_1 + \delta_1) \eta_{sN^2} (N-1)) - G(1) (\beta_1 - (\beta_1 + \delta_1) \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle) ds \right| > \frac{\delta}{14} \right), \\ P_5 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t G\left(\frac{N-2}{N}\right) (\delta_2 \eta_{sN^2} (N-2) - \beta_2 \eta_{sN^2} (N-1)) - G(1) (\delta_2 - \beta_2) \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle ds \right| > \frac{\delta}{14} \right), \\ P_6 & := \mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t G\left(\frac{N-2}{N}\right) (\delta_2 - \beta_2) \eta_{sN^2} (N-1) \eta_{sN^2} (N-2) - \right. \right. \\ & \quad \left. \left. G(1) (\delta_2 - \beta_2) \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle (\langle \pi_s^N, \overleftarrow{t}_\epsilon^{1-\epsilon} \rangle - 1) ds \right| > \frac{\delta}{14} \right), \end{aligned}$$

and finally P_7, P_8, P_9 and P_{10} , which are the counterparts of P_3, P_4, P_5 and P_6 for the left boundary, which we do not explicitly write. The first term P_1 can be estimated with Doob's inequality,

$$\mathbb{P}_{\mu^N} \left(\sup_{0 \leq t \leq T} |M_t^N| > \frac{\delta}{14} \right) \leq \frac{C}{\delta} \mathbb{E}_{\mu^N} [|M_T^N|^2]^{\frac{1}{2}} = \frac{C}{\delta} \mathbb{E}_{\mu^N} \left[\int_0^T B_s^{N,H} ds \right]^{\frac{1}{2}},$$

where C is a constant and $B_s^{N,H}$ was introduced in Section 3.2. We can now proceed as in the proof of Proposition 3.2 to show that P_1 vanishes, as $N \rightarrow \infty$. The second term P_2 vanishes, for any N large enough, because G is smooth.

To estimate the remaining probabilities we apply the replacement lemmas that are stated and proved in Appendix A. To properly explain the procedure, recall from (17) the definition of $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ and $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1)$, since $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) = \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) = \langle \pi_s^N, \overrightarrow{t}_\epsilon^0 \rangle$) we have $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) \sim \rho_s(1)$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) \sim \rho_s(0)$), we show in Appendix A that we can exchange $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$) by the averages above, and $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$). In particular, to estimate P_3 , note that the difference between $\nabla_N^- G(1)$ and $\partial_u G(1)$ is of order N^{-1} . Furthermore, applying Lemma A.3 to $\psi(\eta) = 1$, and using Markov's inequality, we can replace $\eta_s(N-1)$ by $\overleftarrow{\eta}_s^{\epsilon N}(N-1)$ at the cost of an error of order N^{-1} . This proves that P_3 vanishes in the limit $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$. P_4 and P_5 are estimated in the exact same way as P_3 .

We now turn to P_6 : in it, we first replace $\eta_{sN^2}(N-2)$ by $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ by applying Lemma A.3 and Markov's inequality to $\psi(\eta) = \eta_{sN^2}(N-1)$. Now that we have the term $\eta_{sN^2}(N-1) \overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$, we apply Lemma A.3 and Markov's inequality a second time to $\psi(\eta) = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ which allows us to replace $\eta_{sN^2}(N-1)$ by $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ up to a vanishing error term. Noting that $\langle \pi_s^N, \overrightarrow{t}_\epsilon^0 \rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(1)$, and $\langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ and

$$\langle \pi_s^N, \overrightarrow{t}_\epsilon^0 \rangle \langle \pi_s^N, \overleftarrow{t}_\epsilon^1 \rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) \overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) + O((\epsilon N)^{-1}),$$

proves as wanted that P_6 vanishes, in the limit $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$.

The bounds for P_7, P_8, P_9 and P_{10} , are analogous to those on P_3, P_4, P_5 and P_6 . Together, all those bounds prove that $\limsup_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{Q}_N(A(T, G, \delta, \epsilon)) = 0$, so that (20) vanishes in the limit $\limsup_{N \rightarrow \infty}$ and then $\epsilon \rightarrow 0$. This proves Proposition 3.3. \square

Remark 3.4 (Case $K \geq 2$). *For the general case $K \geq 2$, the main problem are the terms of the form $\rho_s^{K-1}(0)$ and $(1 - \rho_s(0))^{K-1}$ (and similarly for the right boundary). A simple way to solve this is to proceed by induction. Since $a^2 = (a + b_1 - b_1)(a + b_2 - b_2) = (a - b_1)(a - b_2) + b_1(a - b_1) + b_2(a - b_2) + b_1 b_2$ and we have that $b_1 b_2 a = b_1 b_2(a + b_3 - b_3) = b_1 b_2(a - b_3) + b_1 b_2 b_3$, taking $a \equiv \rho_s(0)$ and $b_j \equiv \langle \pi_s, \overrightarrow{t}_\epsilon^{(j-1)\epsilon} \rangle$ for $j \geq 0$, we can replace $\rho_s^{K-1}(0)$ by $\prod_{j=0}^{K-2} \langle \pi_s, \overrightarrow{t}_\epsilon^{j\epsilon} \rangle$ plus a sum of terms that vanish when $\epsilon \rightarrow 0$. For the right boundary the argument is analogous.*

4. FICK'S LAW

In this section, we prove Theorem 2.8. Recall the notations set in Section 2.4. Fix $x \in \Lambda_N$. In order to apply Dynkin's formula to the current $J_t^N(x)$ and $K_t^N(x)$, we denote by $\tilde{\mathcal{L}}$ the joint generator of $\eta, J_t^N(x)$ and $K_t^N(x)$, given by

$$\tilde{\mathcal{L}}^x = \tilde{\mathcal{L}}_{N,0}^x + \frac{1}{N\theta} \tilde{\mathcal{L}}_{N,b}^x \quad (21)$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_{N,0}^x f(\eta, J) &= \sum_{z \in \Lambda_N \setminus \{x\}} (\eta(z) - \eta(z+1)) (f(\eta^{z,z+1}, J) - f(\eta, J)) \\ &\quad + \eta(x)(1 - \eta(x+1)) (f(\eta^{x,x+1}, J+1) - f(\eta, J)) + \eta(x+1)(1 - \eta(x)) (f(\eta^{x,x+1}, J-1) - f(\eta, J)). \end{aligned}$$

and $\tilde{\mathcal{L}}_{N,b}^x = \tilde{\mathcal{L}}_{N,+}^x + \tilde{\mathcal{L}}_{N,-}^x$, with

$$\begin{aligned} \tilde{\mathcal{L}}_{N,\pm}^x f(\eta, K) &= \sum_{z \in I_{\pm}^K \setminus \{x\}} c_z^{\pm}(\eta) (f(\eta^z, K) - f(\eta, K)) \\ &\quad + \mathbb{1}_{\{x \in I_{\pm}^K\}} \left[(1 - \eta(x)) c_x^{\pm}(\eta) (f(\eta^x, K+1) - f(\eta, K)) + \eta(x) c_x^{\pm}(\eta) (f(\eta^x, K-1) - f(\eta, K)) \right]. \end{aligned}$$

Recalling the definition of the operators $D_{\alpha,\gamma}^{N,\pm}$ and $D_{\beta,\delta}^{N,\pm}$ in (16),

$$\tilde{\mathcal{L}}_{N,0}^x J_s^N(x) = \eta_s(x) - \eta_s(x+1), \quad \tilde{\mathcal{L}}_{N,\pm}^x K_s^N(x) = (D_{\alpha,\gamma}^{N,-} \eta_s)(x) \quad \text{and} \quad \tilde{\mathcal{L}}_{N,b}^x K_s^N(x) = (D_{\beta,\delta}^{N,+} \eta_s)(x),$$

for $x \in \Lambda_N, I_-^K, I_+^K$, respectively.

By Dynkin's formula,

$$\widehat{M}_t^N(x) := J_t^N(x) - J_0^N(x) - \int_0^{tN^2} (\eta_s(x) - \eta_s(x+1)) ds,$$

is a martingale w.r.t. \mathcal{F}_t , so that for any test function $f \in C^1(0,1)$

$$\widetilde{M}_t^N(f) := J_t^N(f) - J_0^N(f) - \int_0^{tN^2} \sum_{x=1}^{N-2} f\left(\frac{x}{N}\right) (\eta_s(x) - \eta_s(x+1)) ds,$$

is a martingale as well. By summation by parts, the time integral above rewrites

$$\begin{aligned} &\int_0^{tN^2} \frac{1}{N} \sum_{x=1}^{N-1} \nabla_N^- f\left(\frac{x}{N}\right) \eta_s(x) + f\left(\frac{0}{N}\right) \eta_s(1) - f\left(\frac{N-1}{N}\right) \eta_s(N-1) ds \\ &= \int_0^t \langle \pi_t^N, \nabla_N^- f \rangle ds + \int_0^{tN^2} f\left(\frac{0}{N}\right) \eta_s(1) - f\left(\frac{N-1}{N}\right) \eta_s(N-1) ds. \end{aligned} \quad (22)$$

A simple computation also based on Dynkin's formula, shows that its quadratic variation is given by $\langle \widetilde{M}_t^N(f) \rangle_t = \int_0^t \frac{1}{N^2} \sum_{x \in \Lambda_N} f^2\left(\frac{x}{N}\right) (\eta_s(x) - \eta_s(x+1))^2 ds$,

so that the martingale $\widetilde{M}_t^N(f)$ vanishes in $\mathbb{L}^2(\mathbb{P}_{\mu_N})$, as $N \rightarrow \infty$.

Now we analyze the time integral above. From Theorem 2.7 and the Replacement Lemma A.3 the expression (22) converges, as $N \rightarrow \infty$, in \mathbb{P}_{μ_N} to

$$\int_0^t \int_0^1 \partial_u f(u) \rho_s(u) du ds + \int_0^t f(0) \rho_s(0) - f(1) \rho_s(1) ds = - \int_0^t \int_0^1 f(u) \partial_u \rho_s(u) ds. \quad (23)$$

Now we look at the non-conservative current. Using the same argument as above, we have that

$$K_t^N(f) - K_0^N(f) - N^{1-\theta} \int_0^{tN^2} \sum_{x \in I_-^K} f\left(\frac{x}{N}\right) (D_{\alpha,\gamma}^{N,-} \eta_s)(x) ds - N^{1-\theta} \int_0^{tN^2} \sum_{x \in I_+^K} f\left(\frac{x}{N}\right) (D_{\beta,\delta}^{N,+} \eta_s)(x) ds$$

is a martingale. As above it can be shown that this martingale vanishes in $\mathbb{L}^2(\mathbb{P}_{\mu_N})$, as $N \rightarrow \infty$. When $\theta > 1$ it is easy to see that the integral term above vanishes as $N \rightarrow +\infty$. In the case $\theta = 1$, from repeatedly applications of the replacement lemmas stated in Appendix A.2 the last expression converges, w.r.t. \mathbb{P}_{μ_N} , as $N \rightarrow \infty$, to

$$\int_0^t f(0) (D_{\alpha,\gamma} \rho_s)(0) + f(1) (D_{\beta,\delta} \rho_s)(1) ds,$$

which finishes the proof.

5. HYDROSTATIC LIMIT

5.1. Proof of Theorem 2.9. Given an integer N , and $\theta \geq 1$, define the distribution \mathcal{P}_N of the stationary empirical measure, on \mathcal{M}^+ , as $\mathcal{P}_N := \mu_N^{ss} \circ (\pi_N)^{-1}$, where both \mathcal{M}^+ and the empirical measure π_N were introduced at the beginning of Section 2.3. Recall from Definition 2.2 the definition of the functional $F(\rho, G, t)$ (depending on θ). Define

$$\mathcal{E} := \left\{ \pi \in \mathcal{M}^+ : \pi(du) = \rho^*(u)du, F(\rho^*, G, t) = 0, \forall t \in [0, T], \forall G \in C^{1,2}([0, T] \times [0, 1]) \right\},$$

which is the set of weak stationary solutions to the hydrodynamic equation. Now let d be the distance defined on the Skorohod space $\mathcal{D}([0, \infty), \mathcal{M}^+)$ under which this space is a Polish space (see [17], Chapter 4 for an example). The following result, which is analogous to e.g. Theorem 2.2 in [18], is the main ingredient to prove Theorem 2.9.

Proposition 5.1. $\{\mathcal{P}_N\}_{N \in \mathbb{N}}$ is concentrated in \mathcal{E} , i.e., $\forall \delta > 0$,

$$\lim_{N \rightarrow \infty} \mathcal{P}_N \left(\pi \in \mathcal{M}^+ : \inf_{\tilde{\pi} \in \mathcal{E}} d(\pi, \tilde{\pi}) \geq \delta \right) = 0.$$

To prove Proposition 5.1, one needs two ingredients:

- i) The empirical measure been macroscopically governed by a hydrodynamic equation (i.e. the hydrodynamic limit, Theorem 2.7, proved in Section 3).
- ii) The existence of a unique solution of (8) (cf. Lemma 2.3) and its convergence, w.r.t. the \mathbb{L}^2 norm, as time goes to infinity, to a stationary solution, which is a consequence of Proposition 5.10 below for $\theta = 1$. In the case $\theta > 1$ this convergence is classical, but the interested reader can straightforwardly adapt the argument we present below for the Robin case, derived when $\theta = 1$, to the Neumann case, derived when $\theta > 1$.

We will not prove this proposition, because once those two ingredients are obtained, it is a straightforward adaptation of Theorem 2.2 in [18].

We now use Proposition 5.1 to prove Theorem 2.9. In the case $\theta = 1$, under assumptions (H0) and (H1), we check in Section 5.2 that the set \mathcal{E} above is a singleton (more precisely, $\mathcal{E} = \{\rho^*(u)du\}$ where $\rho^*(u) = (1-u)\rho^*(0) + u\rho^*(1)$ with its value at the boundary determined by the unique solution of the nonlinear system of equations $\rho^*(1) - \rho^*(0) = -D_{\alpha, \gamma} \rho^*(0) = D_{\gamma, \alpha} \rho^*(1)$), so that the first assertion of Theorem 2.9 follows immediately from Proposition 5.1. For details we refer the reader to [20].

We now turn to the second assertion of Theorem 2.9, i.e. the case $\theta > 1$. In this case, any constant solution is a stationary solution to the hydrodynamic equation, so that

$$\mathcal{E} := \left\{ \pi \in \mathcal{M}^+ : \pi(du) = mdu, m \in [0, 1] \right\}.$$

We therefore further need to prove that \mathcal{P}_N only charges, in the limit $N \rightarrow \infty$, a single value m^* from all the possible constant values. For that purpose we use the method developed in [20], which consists in studying the evolution of the process started from its stationary state in a subdiffusive time scale $N^{1+\theta}$. In this subdiffusive time scale, the total mass of the system evolves via the boundary dynamics, and converges in time to a uniquely defined constant which is exactly the constant m^* we are looking for. The non-linear boundary terms pose some extra technical difficulties w.r.t. [20], which are solved in Corollary A.4.

To simplify the exposition we consider the case $K = 2$, and we make the appropriate remarks in the general case of K .

We now consider the process on the *subdiffusive time-scale* $\{\eta_t^N := \eta_{tN^{1+\theta}}\}_{t \geq 0}$, with initial distribution μ_N^{ss} . For each $t \geq 0$ and $\theta > 1$ fixed, we define the mass of the system as

$$m_t^N = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta_{tN^{1+\theta}}^N(x), \quad (24)$$

and for each $T > 0$ we let $\mathcal{D}([0, T], \mathbb{R})$ be the set of càdlàg trajectories $m : [0, T] \rightarrow \mathbb{R}$ with respect to the Skorohod topology. For each $N \in \mathbb{N}$, denote by \mathcal{Q}_N the distribution of the trajectory $(m_t^N)_{t \in [0, T]}$ on $\mathcal{D}([0, T], \mathbb{R})$, with η started from the stationary distribution μ_N^{ss} . A straightforward adaptation (to

the subdiffusive timescale) of Proposition 3.2 shows that the sequence $\{\mathcal{Q}_N\}_{N \geq 1}$ is tight. From the stationarity of μ_N^{ss} , for any $t \geq 0$, and any $I = (a, b) \subset [0, 1]$

$$\mathcal{P}_N(\pi^N : \langle \pi^N, 1 \rangle \in I) = Q_N(m^N : m_t \in I),$$

One may then bound, for any limit points \mathcal{P}^* , \mathcal{Q}^* of the sequences $\{\mathcal{P}_N\}_{N \geq 1}$, $\{\mathcal{Q}_N\}_{N \geq 1}$ (for more details, cf. [20], p.11) and for any fixed time $t \geq 0$,

$$\mathcal{P}^*(\pi(du) = mdu, m \in I) \leq \mathcal{Q}^*(m : m_t \in \bar{I}). \quad (25)$$

To conclude, we now only need to prove the following result

Lemma 5.2. *There exists $m^* \in [0, 1]$ such that , for any $\varepsilon > 0$*

$$\mathcal{Q}^*(m : |m_t - m^*| \geq \varepsilon) \xrightarrow[t \rightarrow \infty]{} 0 \quad (26)$$

This Lemma, together with (25), proves as wanted that $\mathcal{P}^*(\pi(du) = m^*du) = 1$, and concludes the proof of Theorem 2.9.

Proof of Lemma 5.2. We start by characterizing the typical trajectories of \mathcal{Q}^* . We apply Dynkin's formula as in (15) and we take $G \equiv 1$ to obtain that

$$\begin{aligned} m_t^N &= m_0^N + M_t^N(1) + \int_0^t \alpha_1 + \beta_1 + \eta_s^N(1)(\alpha_2 - (\alpha_1 + \gamma_1)) + \eta_s^N(N-1)(\beta_2 - (\beta_1 + \delta_1)) ds \\ &\quad - \int_0^t \gamma_2 \eta_s^N(2) + \delta_2 \eta_s^N(N-2) + \eta_s^N(1) \eta_s^N(2)(\alpha_2 - \gamma_2) + \eta_s^N(N-1) \eta_s^N(N-2)(\beta_2 - \delta_2) ds. \end{aligned} \quad (27)$$

Simple computations similar to the ones used in the proof of Proposition 3.2 also show that the quadratic variation of $M_t^N(1)$ is of order $O(N^{-1})$ so that $M_t^N(1)$ vanishes as $N \rightarrow \infty$, with respect to the $\mathbb{L}^2(\mathbb{P}_{\mu_N^{ss}})$ -norm. Moreover, from Corollary A.4 we are able to replace $\eta_s^N(1)$ and $\eta_s^N(2)$ (resp. $\eta_s^N(N-1)$ and $\eta_s^N(N-2)$) by m_s^N . From this we get that

$$\begin{aligned} m_t^N &= m_0^N + M_t^N(1) + \int_0^{tN^{1+\theta}} \alpha_1 + \beta_1 + m_s^N(\alpha_2 - (\alpha_1 + \gamma_1)) + m_s^N(\beta_2 - (\beta_1 + \delta_1)) ds \\ &\quad - \int_0^{tN^{1+\theta}} \gamma_2 m_s^N + \delta_2 m_s^N + (m_s^N)^2(\alpha_2 - \gamma_2) + (m_s^N)^2(\beta_2 - \delta_2) ds. \end{aligned} \quad (28)$$

To simplify notation let $i_x = \alpha_x + \beta_x$ and $o_x = \gamma_x + \delta_x$, for $x = \{1, \dots, K\}$. By taking the limit, when $N \rightarrow +\infty$, in the previous identity, we obtain that any limit point \mathcal{Q}^* is concentrated on solutions of the Ricatti Equation

$$\mathcal{Q}^* \left(m : m_t = m_0 + \int_0^t i_1 + (i_2 - o_2 - (i_1 + o_1))m_s - (i_2 - o_2)m^2(s) ds \right) = 1. \quad (29)$$

Remark 5.3. *Observe that the equation above is equivalent to $m_t = m_0 + \int_0^t D_{i,0} m_s ds$. In fact, as for the hydrodynamic limit, the same arguments show that for general values of $K > 2$ we have an analogous integral equation, where now $D_{i,0}$ induces a K -degree polynomial. For the case $K > 2$, the proof above is indeed identical in this case as long as we assume (H2). The only extra technical difficulty is that in (27) we shall have higher order polynomials in η , so that we have to apply Corollary A.4 a certain number of times to get closed equations in m_s^N .*

To conclude the proof of Lemma 5.2, we need to show the uniqueness of solutions for the Ricatti equation and that all solutions uniformly converge to the same constant. We first state the following technical lemma.

Lemma 5.4. *Let $\lambda = (\lambda_1, \dots, \lambda_K)$ and $\sigma = (\sigma_1, \dots, \sigma_K)$ and $D_{\lambda, \sigma}$ be the operator defined in (2) for $K \geq 1$ fixed. Then for $f_i : \mathbb{R} \rightarrow \mathbb{R}$ with $i = 1, 2$, we have*

$$D_{\lambda, \sigma} f_1 - D_{\lambda, \sigma} f_2 = -(f_1 - f_2) V_{\lambda, \sigma}(f_1, f_2), \quad (30)$$

where $V_{\lambda,\sigma}(f_1, f_2) = V_\lambda(f_1, f_2) + V_\sigma(1-f_1, 1-f_2)$ with the operator V_ϕ , for any sequence $\phi = (\phi_x)_{1 \leq x \leq K}$, is acting on functions $(f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, as

$$V_\phi(f_1, f_2)(u_1, u_2) = \sum_{\substack{x=1 \\ \phi_{K+1} := 0}}^K (\phi_x - \phi_{x+1}) \sum_{i=0}^{x-1} (f_1(u_1))^{x-1-i} (f_2(u_2))^i, \quad \text{with } u = (u_1, u_2).$$

In particular, if λ and σ are non-negative, non-increasing and $f_i \geq 0$ for $i = 1, 2$ then there is a constant $\underline{v}_K(\lambda, \sigma) > 0$, such that

$$V_{\lambda,\sigma} f \geq \underline{v}_K(\lambda, \sigma). \quad (31)$$

We can now conclude the proof of Lemma 5.2. From Lemma 5.4 it is simple to see that m is locally Lipschitz. By iteration we can extend to all times up to time $t > 0$, thus showing uniqueness. To see that there exists a unique solution $m \equiv m^* \in [0, 1]$ to the equation defined by the mass, $D_{i,\sigma} m$, observe that an immediate consequence of Lemma 5.4 is that $D_{i,\sigma} f$ is both continuous and monotone decreasing on $f \in [0, 1]$. In particular, letting $\mathbf{0}$ (resp. $\mathbf{1}$) be the constant 0 (resp. constant 1) function on $[0, 1]$, we have that

$$-\mathbf{0}_1 = D_{i,\sigma} \mathbf{1} \leq D_{i,\sigma} f \leq D_{i,\sigma} \mathbf{0} = \mathbf{0}_1,$$

thus, by the intermediate value theorem there exists a (unique) $f \equiv m \in [0, 1]$ such that $D_{i,\sigma} m = 0$.

Now fix a solution m to

$$m_t = m_0 + \int_0^t D_{i,\sigma} m_s ds,$$

and define $\widehat{m} = m - m^*$, (30) yields

$$\widehat{m}_t = \widehat{m}_0 - \int_0^t \widehat{m}_s V_{i,\sigma}(m_s, m^*) ds,$$

so that $\widehat{m}_t = \widehat{m}_0 e^{-\int_0^t V_{i,\sigma}(m_s, m^*) ds}$. Using (31) then yields $|\widehat{m}_t| \leq e^{-\underline{v}_K(i,\sigma)t}$ which proves (26). \square

We now prove the technical Lemma.

Proof of Lemma 5.4. For $u_1, u_2 \in \mathbb{R}$, let $y := f_1(u_1)$ and $z := f_2(u_2)$. Then

$$D_{\lambda,\sigma} y - D_{\lambda,\sigma} z = \sum_{x=1}^K \lambda_x \left((1-y)y^{x-1} - (1-z)z^{x-1} \right) - \sigma_x \left(y(1-y)^{x-1} - z(1-z)^{x-1} \right).$$

Observing that

$$\begin{aligned} (1-y)y^{x-1} - (1-z)z^{x-1} &= (y^{x-1} - z^{x-1}) - (y^x - z^x), \\ y(1-y)^{x-1} - z(1-z)^{x-1} &= -\left\{ ((1-y)^x - (1-z)^x) - ((1-y)^{x-1} - (1-z)^{x-1}) \right\}, \end{aligned}$$

we let $h_0(x) = y^x - z^x$ and $h_1(x) = (1-y)^x - (1-z)^x$. Performing a summation by parts, we have

$$\begin{aligned} D_{\lambda,\sigma} y - D_{\lambda,\sigma} z &= \sum_{x=0}^{K-1} \sigma_{x+1} (h_1(x+1) - h_1(x)) - \sum_{x=0}^{K-1} \lambda_{x+1} (h_0(x+1) - h_0(x)) \\ &= -(\lambda_K h_0(K) - \sigma_K h_1(K)) - \sum_{x=1}^{K-1} \left\{ (\lambda_x - \lambda_{x+1}) h_0(x) - (\sigma_x - \sigma_{x+1}) h_1(x) \right\}. \end{aligned} \quad (32)$$

Using the fact that for any integer $x \geq 1$ holds

$$a^x - b^x = (a-b) \sum_{i=0}^{x-1} b^i a^{x-1-i}, \quad a \geq b \geq 0,$$

we have the following decomposition

$$h_0(x) = (y-z) \sum_{i=0}^{x-1} z^i y^{x-1-i} \quad \text{and} \quad h_1(x) = -(y-z) \sum_{i=0}^{x-1} (1-z)^i (1-y)^{x-1-i}.$$

Replacing $h_0(x)$ and $h_1(x)$ in (32) and rearranging the terms, we end the proof of (30). To see (31), note that $\sum_{i=0}^{x-1} y^{x-1-i} z^i \geq 1$ for $x = 1, \dots, K$. Thus, if the sequences λ, σ are decreasing we can bound $V_{\lambda, \sigma}(y, z)$ from below by $\lambda_1 + \sigma_1 - (\lambda_K + \sigma_K)$. If the sequences are constant, i.e., $\lambda_x = \lambda_1, \sigma_x = \sigma_1$ for $x = 2, \dots, K$, then we can bound from below by $\lambda_1 + \sigma_1$. \square

We end the present section with some observations.

Remark 5.5 (Case $K = 2$.) *Fixing $K = 2$, the Riccati equation (29) is explicitly solvable by separation of variables. More precisely, we have two different scenarios: letting $i_2 = o_2$, we recover both the results of [1] and [20], where the mass stabilizes to $i_1(i_1 + o_1)^{-1}$. Otherwise, the equation has an oscillating solution, which we present below. Define $\Delta = -\{i_1^2 + 2i_1(i_2 + o_1 - o_2) + (o_1 + (i_2 - o_2))^2\}$. Then, we get*

$$m_t = 1/2 - \frac{i_1 + o_1}{2(i_2 - o_2)} + \frac{\Delta^{1/2}}{2(i_2 - o_2)} \tan\left(\frac{1}{2}\Delta^{1/2}(m_0 - t)\right).$$

In this way, imposing $\Delta = 0$ we have $m_t = m^* = 1/2 - \frac{i_1 + o_1}{2(i_2 - o_2)}$.

Remark 5.6 (Case $K > 2$.) *To finish, we observe that for the model introduced in [6], i.e., taking above $\beta_x = \gamma_x = j$ and $\alpha_x = \delta_x = 0$ for $x = \{1, \dots, K\}$, a simple computation shows that the solution of $D_{i,i}m^* = 0$ is, in fact, $m^* = 1/2$. The model in [6] is a particular case of considering $i = o$. For the latter, $D_{i,i}m^* = 0$ can be solved for $m \equiv m^*$ by noticing that $D_{i,i}m = D_{i,0}m - D_{i,0}(1 - m) = -(m - (1 - m))V_{i,0}(m, 1 - m) = 0 \Leftrightarrow m^* = 1/2$. Perhaps more interesting is the case when $i_x = i$, $o_x = o$ for $x \in \{1, \dots, K\}$, i.e., the rates are constant in x . Under these conditions, we have*

$$\frac{1 - (m^*)^K}{1 - (1 - m^*)^K} = \frac{o}{i}.$$

If $o = i$ then the mass stabilizes to the middle of point of $[0, 1]$, but for fixed i (resp. o), as the rate of removal (resp. injection) of particles increases (resp. decreases) that is $o \nearrow$ (resp. $i \searrow$), then the mass decreases exponentially in K .

5.2. On the uniqueness of the stationary macroscopic profile for $\theta = 1$. We now prove that, under our assumptions, the hydrodynamic limit for $\theta = 1$ admits a unique stationary solution.

The same idea used for $\theta > 1$ can be used now to guarantee uniqueness for the stationary solution of the heat equation with Robin boundary conditions. We will start by observing that for every $\rho^*(1)$ fixed, there exists a unique solution for $\rho^*(0)$ of the equation $-D_{\alpha, \gamma}\rho^*(0) = D_{\beta, \delta}\rho^*(1)$, which is equivalent to $D_{\alpha, \gamma}\rho^*(0) = D_{\delta, \beta}(1 - \rho^*(1))$. From the previous observations we know that $\forall f \in [0, 1]$ we have $D_{\delta, \beta}f \in [-\beta_1, \delta_1]$, and also that $\forall u \in [-\gamma_1, \alpha_1]$ there is one $f \in [0, 1] : D_{\alpha, \gamma}f = u$. In this way, there exists a unique $\rho^*(0) \in [0, 1]$ such that, for any fixed $\rho^*(1)$ with $D_{\delta, \beta}(1 - \rho^*(1)) = u \in [-\gamma_1 \wedge \beta_1, \alpha_1 \wedge \delta_1]$, we have $D_{\alpha, \gamma}\rho^*(0) = u$. More precisely, by monotonicity, we have that

$$\begin{aligned} \rho^*(0) &\in [D_{\alpha, \gamma}^{-1}(\alpha_1 \wedge \delta_1), D_{\alpha, \gamma}^{-1}(-(\gamma_1 \wedge \beta_1))] \subset [0, 1] \\ \rho^*(1) &\in [D_{\beta, \delta}^{-1}(\gamma_1 \wedge \beta_1), D_{\beta, \delta}^{-1}(-(\alpha_1 \wedge \delta_1))] \subset [0, 1] \end{aligned}$$

To study $\rho^*(1) - \rho^*(0) = D_{\beta, \delta}\rho^*(1)$, let us first define the (solution) map $\rho^*(1) \mapsto \phi_{\alpha, \gamma, \beta, \delta}(\rho^*(1))$ where $\phi_{\alpha, \gamma, \beta, \delta}(\rho^*(1)) := \rho^*(0)$ is the (unique for fixed $\rho^*(1)$) solution (function of $\rho^*(1)$) to the equation $-D_{\alpha, \gamma}\rho^*(0) = D_{\beta, \delta}\rho^*(1)$. Moreover, define the function T as

$$T : [D_{\beta, \delta}^{-1}(\gamma_1 \wedge \beta_1), D_{\beta, \delta}^{-1}(-(\alpha_1 \wedge \delta_1))] \rightarrow \mathbb{R}, \quad u \mapsto T(u) = u - \phi_{\alpha, \gamma, \beta, \delta}(u) - D_{\beta, \delta}u.$$

Recall from (H1) that $\alpha_1 \wedge \delta_1 = \delta_1$ and $\gamma_1 \wedge \beta_1 = \beta_1$. We claim that T is (Lipschitz) continuous and monotone increasing on $[0, 1]$. Assuming this, we conclude that T has domain $[0, 1]$ and therefore

$$-(\phi_{\alpha, \gamma, \beta, \delta}(0) + \beta_1) = T(0) \leq T(u) \leq T(1) = 1 + \delta_1 - \phi_{\alpha, \gamma, \beta, \delta}(1)$$

and thus there exists a unique $\rho^*(1)$ such that $T \circ \rho^*(1) = 0$, and we are done. To prove the claim, consider $g, h \in [0, 1]$ and shorten $g^* := \phi_{\alpha, \gamma, \beta, \delta}(g)$ and $h^* := \phi_{\alpha, \gamma, \beta, \delta}(h)$. Then,

$$T(g) - T(h) = g - h - (g^* - h^*) + (g - h)V_{\beta, \delta}(g, h). \quad (33)$$

We now relate $(g^* - h^*)$ and $(g - h)$. From the definition of ϕ we have that $D_{\alpha,\gamma}g^* + D_{\beta,\delta}g = 0$ and $D_{\alpha,\gamma}h^* + D_{\beta,\delta}h = 0$. Subtracting the two previous equations we observe that

$$(g^* - h^*)V_{\alpha,\gamma}(g^*, h^*) = -(g - h)V_{\beta,\delta}(g, h).$$

In this way, replacing in (33) the expression for $g^* - h^*$ given in last display we arrive at

$$T(g) - T(h) = (g - h) \left(1 + V_{\beta,\delta}(g, h) + \frac{V_{\beta,\delta}(g, h)}{V_{\alpha,\gamma}(g^*, h^*)} \right)$$

and since from Lemma 5.4 there are universal constants such that the $V_{\cdot,\cdot}$ terms above are bounded from below, and since $g, h, g^*, h^* \in [0, 1]$ they are also bounded from above, the claim is shown.

5.3. Uniqueness and convergence of (8) to stationary solutions. In this section, we assume (H1), and we show the convergence of the weak solution of (8) to the unique stationary solution, denoted here by ρ^* , investigated in the previous section. Existence and uniqueness of such a stationary solution is proved in Section 5.2. The main difficulty on showing this result lies on the fact that the weak solution is not regular enough w.r.t. time in order to have an integration by parts formula as we do w.r.t. space. To solve this issue, our approach is to relate the weak formulation and the mild formulation of (8). We then show, following [18], that mild and weak solutions are equivalent *in some sense*, which indirectly gives us a regular enough version of the weak solution to then proceed with energy estimates. With a few adjustments from [8] (Section 2.3), we first define the notion of mild solutions of (8).

Definition 5.7 (Mild solution of (8)). *We call mild solution of (8) any function $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ satisfying $M(\rho, t) := \rho_t - S\rho_t = 0$, with*

$$S\rho_t(u) = \int_0^1 P_t(u, v) f_0(v) dv + \int_0^t \left\{ P_{t-s}(u, 0) (D_{\alpha,\gamma}\rho_s)(0) + P_{t-s}(u, 1) (D_{\beta,\delta}\rho_s)(1) \right\} ds = 0,$$

where $P_t(u, v) = \sum_{w \in \Psi^{-1}(v)} \Phi_t(u, w)$ is the density kernel generated by the Laplacian ∂_u^2 on $[0, 1]$ with reflecting Neumann boundary conditions, related to the heat kernel

$$\Phi_t(u, w) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{(u-w)^2}{4t}}, \quad (34)$$

by the reflection map $\psi : \mathbb{R} \rightarrow [0, 1]$ defined as

$$\psi(u+k) = \begin{cases} u, & u \in [0, 1], k \text{ even}, \\ 1-u, & u \in [0, 1], k \text{ odd}, \end{cases}$$

extended to \mathbb{R} by the symmetry $\psi(v) = \psi(-v)$, for $v \in \mathbb{R}$.

Remark 5.8. *Observe that fixed $u \in [0, 1]$, $S\rho_t(u)$ is differentiable w.r.t. time, and given a smooth initial data f_0 , we have that $S\rho_t \in C^\infty(0, 1)$. Moreover, there exists the limits $\lim_{u \rightarrow 0} \frac{d^n}{du^n} S\rho_t(u)$ and $\lim_{u \rightarrow 1} \frac{d^n}{du^n} S\rho_t(u)$ for any $n \in \mathbb{N}$, and for any $t \in [0, T]$ we have $S\rho_t \in C[0, 1]$.*

Following [18], with some adaptations to account for the fact that P_t involves here Neumann boundary conditions, we show below that if ρ is a weak solution of (8), then $\rho_t - S\rho_t = 0$ a.e. From the previous remark, $S\rho$ is regular enough to satisfy $F(\rho, S\rho, t) = 0$. Moreover, letting ρ^* be the stationary solution, as mentioned in Theorem 2.9, from simple energy estimates we can see that $F(S\rho, S\rho, t) - F(\rho^*, S\rho, t) = 0 \implies \|S\rho_t - \rho^*\|_{L^2} = O(e^{-Ct})$ for positive constant C , which implies weak convergence to the stationary profile. In this way, we have that $\pi_t \rightarrow \pi^*$ in \mathcal{M}^+ (endowed with the weak topology), since $\pi_t(du) = \rho_t(u)du$, with $\rho_t \xrightarrow{w} \rho^*$ and $\rho^*(u)du =: \pi^*(du)$, and thus we can apply Proposition 5.1.

Proposition 5.9. *If a function $\rho : [0, T] \rightarrow [0, 1]$ is a weak solution in the sense of Definition 2.2, where it satisfies $F(\rho, G, t) = 0$ for any $G \in C^{1,2}([0, T] \times [0, 1])$, then ρ also satisfies $M(\rho, t) = 0$ a.e. $\forall t > 0$. Moreover, if $\rho : [0, T] \times [0, 1]$ is a function satisfying $\langle M(\rho, t), G \rangle = 0$ for any $G \in C^{1,2}([0, T] \times [0, 1])$, then we have $F(S\rho, G, t) = 0$.*

Proof. Let us fix $g \in C^2([0, 1])$ and $T \geq t \geq 0$ and define $G^\epsilon : [0, T] \times [0, 1] \rightarrow [0, 1]$ as

$$G_s^\epsilon(u) = (P_{t-s}g)(u)1_{\{s \in [0, t]\}} + g(u) \frac{t + \epsilon - s}{\epsilon} 1_{\{s \in [t, t + \epsilon]\}} + (1 - 1_{\{s \in [t + \epsilon, T]\}}).$$

Above, for $t \geq 0$ and $u \in [0, 1]$, $(P_t f)(u) = \int_0^1 P_t(u, v) f(v) dv \equiv \langle P_t(u, \cdot), f \rangle$. Assume now that ρ is a weak solution in the sense of Definition 2.2. Recalling that for fixed v , $P_t(u, v)$ satisfies the heat equation, with Neumann boundary conditions, taking $G^\epsilon \in C^{1,2}([0, T] \times [0, 1])$ as our test function we have

$$\begin{aligned} F(\rho, G^\epsilon, T) &= -\langle f_0, P_t g \rangle - \int_0^t \left\{ (D_{\beta, \delta} \rho_s)(1)(P_{t-s}g)(1) + (D_{\alpha, \gamma} \rho_s)(0)(P_{t-s}g)(0) \right\} ds + \int_t^{t+\epsilon} \langle \rho_s, g \rangle \epsilon^{-1} ds \\ &\quad - \int_t^{t+\epsilon} \left\{ \langle \rho_s, \partial_u^2 g \rangle + (D_{\beta, \delta} \rho_s)(1)g + (D_{\alpha, \gamma} \rho_s)(0)g + \rho_s(1)\partial_u g(1) + \rho_s(0)\partial_u g(0) \right\} \epsilon^{-1} (t + \epsilon - s) ds. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, G^ϵ converges in the sup norm to $G_s(u) := P_{t-s}g(u)$. As a consequence, we have that $F(\rho, G^\epsilon, T)$ converges to

$$\langle \rho_t, g \rangle - \langle P_t f_0, g \rangle - \int_0^t \langle (D_{\beta, \delta} \rho_s)(1)P_{t-s}(\cdot, 1), g \rangle ds - \int_0^t \langle (D_{\alpha, \gamma} \rho_s)(0)P_{t-s}(\cdot, 0), g \rangle ds = 0.$$

Approximating G^ϵ by a sequence $(G_k^\epsilon)_{k \geq 1}$ in $C^{1,2}([0, T] \times [0, 1])$ w.r.t. the L^1 norm, and since $g \in C^2([0, 1])$ is arbitrary, proves as wanted that $M(\rho, t) = 0$ a.e.

For the converse, as in [18] we shall consider a test function g independent of time for a better exposition, since the extension to g time dependent is completely analogous. Let us assume that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ satisfies $\rho = S\rho$ a.e. $\forall t$. Then, for any $g \in C^2([0, 1])$ it must satisfy

$$\langle \rho_t, g \rangle = \langle P_t f_0, g \rangle + \int_0^t \langle P_{t-s}(\cdot, 1)(D_{\beta, \delta} \rho_s)(1) + P_{t-s}(\cdot, 0)(D_{\alpha, \gamma} \rho_s)(0), g \rangle ds.$$

In particular, differentiating the expression above with respect to time, we have

$$\begin{aligned} \frac{d}{dt} \langle \rho_t, g \rangle &= \langle (\partial_u^2 P_t)g, f_0 \rangle + D_{\beta, \delta} \rho_t(1)g(1) + D_{\alpha, \gamma} \rho_t(0)g(0) \\ &\quad + \int_0^t \langle \partial_t P_{t-s}(\cdot, 1)D_{\beta, \delta} \rho_s(1) + \partial_t P_{t-s}(\cdot, 0)D_{\alpha, \gamma} \rho_s(0), g \rangle ds. \end{aligned} \tag{35}$$

We now integrate by parts $(\partial_u^2 P_t)g(u)$ twice, that is:

$$(\partial_u^2 P_t)g(u) = \int_0^1 g(v) \partial_v^2 P_t(u, v) dv = \int_0^1 P_t(u, v) \partial_v^2 g(v) dv - P_t(u, 1) \partial_u g(1) + P_t(u, 0) \partial_u g(0).$$

Above, we used the fact that P_t satisfies Neumann boundary conditions. Since by assumption $M(\rho, t) = 0$ a.e., we have a.e. $P_t f_0 = \rho_t - \int_0^t \left\{ P_{t-s}(\cdot, 0)(D_{\alpha, \gamma} \rho_s)(0) + P_{t-s}(\cdot, 1)(D_{\beta, \delta} \rho_s)(1) \right\} ds$. In particular, replacing the expression above in (35), yields

$$\begin{aligned} \frac{d}{dt} \langle \rho_t, g \rangle &= \langle \rho_t, \partial_u^2 g \rangle + (D_{\beta, \delta} \rho_t)(1)g(1) + (D_{\alpha, \gamma} \rho_t)(0)g(0) + \partial_u g(0)\rho_t(0) - \partial_u g(1)\rho_t(1) \\ &\quad - \int_0^t \langle P_{t-s}(\cdot, 0)D_{\alpha, \gamma} \rho_s(0) + P_{t-s}(\cdot, 1)D_{\beta, \delta} \rho_s(1)g(1), \partial_u^2 g \rangle ds \\ &\quad - \partial_u g(0) \int_0^t P_{t-s}(0, 0)D_{\alpha, \gamma} \rho_s(0) + P_{t-s}(0, 1)D_{\beta, \delta} \rho_s(1) ds \\ &\quad + \partial_u g(1) \int_0^t P_{t-s}(1, 0)D_{\alpha, \gamma} \rho_s(0) + P_{t-s}(1, 1)D_{\beta, \delta} \rho_s(1) ds \\ &\quad + \int_0^t \langle \partial_t P_{t-s}(\cdot, 1)D_{\beta, \delta} \rho_s(1) + \partial_t P_{t-s}(\cdot, 0)D_{\alpha, \gamma} \rho_s(0), g \rangle ds. \end{aligned}$$

Integrating by parts the fourth line on the previous display twice, using the fact that P_t satisfies the heat equation with Neumann boundary conditions, yields

$$\frac{d}{dt} \langle \rho_t, g \rangle = \langle \rho_t, \partial_u^2 g \rangle + (D_{\beta, \delta} \rho_t)(1)g(1) + (D_{\alpha, \gamma} \rho_t)(0)g(0) + \partial_u g(0)\rho_t(0) - \partial_u g(1)\rho_t(1).$$

Integrating in time this identity yields as wanted that $F(\rho, g, t) = 0$ for any $g \in C^2([0, 1])$. The case when g is replaced by a time-dependent function G is similar, we omit it. \square

We now show that any weak solution converges exponentially to the stationary solution ρ^* .

Proposition 5.10. *There exists a constant $C > 0$ such that any weak solution ρ of (8) satisfies*

$$\|\rho - \rho^*\|_{L^2}^2 \leq e^{-2Ct},$$

where ρ^* is the unique stationary solution to (8).

Proof. From Lemma 5.4 we know that $D_{\alpha, \gamma} \rho_t(0) - D_{\alpha, \gamma} \rho^*(0) = -(\rho_t(0) - \rho^*(0))V_{\alpha, \gamma}(\rho_t, \rho^*)(0, 0)$. Let us write $V(0, t) := V_{\alpha, \gamma}(\rho_t, \rho^*)(0, 0)$ and $V(1, t) = V_{\beta, \delta}(\rho_t, \rho^*)(1, 1)$. Both $S\rho$ and ρ^* satisfy the weak formulation and from Remark 5.8 letting $w_t(u) := S\rho_t(u) - \rho^*(u)$ we have $F(S\rho, w, t) - F(\rho^*, w, t) = 0$, which rewrites

$$\begin{aligned} \langle w_t, w_t \rangle &= \langle w_0, w_0 \rangle + \int_0^t \langle w_s, (\partial_u^2 + \partial_s) w_s \rangle ds + \int_0^t w_s(0) (\partial_u w_s(0) - w_s(0) V_{\alpha, \gamma}(0, s)) ds \\ &\quad - \int_0^t w_s(1) (\partial_u w_s(1) + w_s(1) V_{\beta, \delta}(1, s)) ds. \end{aligned}$$

Differentiating w.r.t. time we have

$$\frac{d}{dt} \langle w_t, w_t \rangle = \langle w_t, (\partial_u^2 + \partial_t) w_t \rangle + w_t(0) (\partial_u w_t(0) - w_t(0) V(0, t)) - w_t(1) (\partial_u w_t(1) + w_t(1) V(1, t)).$$

Integrating by parts the first term on the r.h.s. of the previous display once in space, we obtain

$$\frac{d}{dt} \langle w_t, w_t \rangle = \langle w_t, \partial_t w_t \rangle - \langle \partial_u w_t, \partial_u w_t \rangle - (w_t(0))^2 V(0, t) - (w_t(1))^2 V(1, t).$$

Now note that $\langle w_t, \partial_t w_t \rangle = \frac{1}{2} \frac{d}{dt} \langle w_t, w_t \rangle$. From Poincaré's inequality, we know that there is a positive constant C such that $\langle \partial_u w_t, \partial_u w_t \rangle \geq C \langle w_t, w_t \rangle$ so that rearranging the terms yields

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dt} \langle w_t, w_t \rangle + \langle \partial_u w_t, \partial_u w_t \rangle + (w_t(0))^2 V(0, t) + (w_t(1))^2 V(1, t) \\ &\geq \frac{1}{2} \frac{d}{dt} \langle w_t, w_t \rangle + C \langle w_t, w_t \rangle = \frac{1}{2} \frac{d}{dt} \|w_t\|_{L^2}^2 + C \|w_t\|_{L^2}^2 \end{aligned}$$

where for the second inequality we used that from Lemma 5.4 we have $V(k, t) > 0$ for $k = 0, 1$. In particular, $\|w_t\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{-2Ct}$, follows from Gronwall's inequality, which concludes the proof since $\|w_0\|_{L^2}^2 \leq 1$. \square

APPENDIX A. REPLACEMENT LEMMAS

In this section we prove the replacements lemmas that are needed along the arguments presented above. We start by obtaining an estimate relating the Dirichlet form and the carré du champ operator for this model. As above, for simplicity of the presentation, we state and prove the results for the case $K = 2$, but the extension to the general case is completely analogous.

A.1. Dirichlet forms. Let $\rho : [0, 1] \rightarrow [0, 1]$ be a measurable profile and let $\nu_{\rho(\cdot)}^N$ be the Bernoulli product measure on Ω_N defined by

$$\nu_{\rho(\cdot)}^N(\eta : \eta(x) = 1) = \rho\left(\frac{x}{N}\right). \quad (36)$$

For a probability measure μ on Ω_N and a density $f : \Omega_N \rightarrow \mathbb{R}$ with respect to μ , the Dirichlet form is defined as

$$\langle f, -\mathcal{L}_N f \rangle_\mu = \langle f, -\mathcal{L}_{N,0} f \rangle_\mu + \frac{j}{N^\theta} \langle f, -\mathcal{L}_{N,b} f \rangle_\mu, \quad (37)$$

and the carré du champ is defined by:

$$D_N(\sqrt{f}, \mu) := D_{N,0}(\sqrt{f}, \mu) + \frac{j}{N^\theta} D_{N,b}(\sqrt{f}, \mu), \quad (38)$$

where

$$\begin{aligned} D_{N,0}(\sqrt{f}, \mu) &:= \sum_{x=1}^{n-2} \int_{\Omega_N} \left[\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right]^2 d\mu, \\ D_{N,\pm}(\sqrt{f}, \mu) &= \sum_{x \in I_\pm^k} \int c_x^\pm(\eta) \left[\sqrt{f(\eta^x)} - \sqrt{f(\eta)} \right]^2 d\mu, \end{aligned} \quad (39)$$

where we recall the rates c_x^\pm defined in (6), and $D_{N,b} = D_{N,-} + D_{N,+}$. We claim that for $\theta \geq 1$ and for $\rho : [0, 1] \rightarrow [0, 1]$ a constant profile equal to, for example, α , the following bound holds

$$\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\alpha^N} \lesssim -D_N(\sqrt{f}, \nu_\alpha^N) + O\left(\frac{1}{N}\right). \quad (40)$$

From Lemma 5.1 and Lemma 5.2 of [1] it is only necessary to control the contribution from the non-linear part of the boundary dynamics. To do that, it is enough to apply Lemma 5.1 of [3] and the result follows. We leave these computations to the reader.

A.2. Replacement Lemmas. We start this section by proving the next lemma which is the basis for the replacement lemmas that are presented next.

Lemma A.1. *Let $x < y \in \Lambda_N$ and let $\varphi : \Omega \rightarrow \Omega$ be a positive and bounded function which satisfies $\varphi(\eta) = \varphi(\eta^{z,z+1})$ for any $z = x, \dots, y-1$. For any density f with respect to ν_α and any positive constant A , it holds that*

$$\left| \langle \varphi(\eta)(\eta(x) - \eta(y)), f \rangle_{\nu_\alpha^N} \right| \lesssim \frac{1}{A} D_N(\sqrt{f}, \nu_\alpha^N) + A(y-x).$$

Proof. By summing and subtracting appropriate terms, we have that

$$\begin{aligned} \left| \langle \varphi(\eta)(\eta(x) - \eta(y)), f \rangle_{\nu_\alpha^N} \right| &\leq \frac{1}{2} \sum_{z=x}^{y-1} \left| \int \varphi(\eta)(\eta(z) - \eta(z+1)) [f(\eta) - f(\eta^{z,z+1})] d\nu_\alpha^N \right| \\ &\quad + \frac{1}{2} \sum_{z=x}^{y-1} \left| \int \varphi(\eta)(\eta(z) - \eta(z+1)) [f(\eta) + f(\eta^{z,z+1})] d\nu_\alpha^N \right|. \end{aligned}$$

Note that since φ satisfies $\varphi(\eta) = \varphi(\eta^{z,z+1})$ for any $z = x, \dots, y-1$, by a change of variables, we conclude that the last term in the previous display is equal to zero. Now, we treat the remaining term. Using the equality $(a-b) = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and then Young's inequality, the first term at the right side of last display is bounded from above by a constant times

$$\sum_{z=x}^{y-1} A \int (\varphi(\eta)(\eta(z) - \eta(z+1)))^2 \left(\sqrt{f(\eta^{z,z+1})} + \sqrt{f(\eta)} \right)^2 d\nu_\alpha^N + \frac{1}{A} D_N(\sqrt{f}, \nu_\alpha^N).$$

The fact that φ is bounded, $|\eta(x)| \leq 1$ and f is a density, the term on the left-hand side of last expression is bounded from above by a constant. This ends the proof. \square

We are now able to show the first Replacement Lemma.

Lemma A.2. Fix $x, y \in \Lambda_N$ such that $|x - y| = o(N)$. Let $\varphi : \Omega \rightarrow \Omega$ be a positive and bounded function which satisfies $\varphi(\eta) = \varphi(\eta^{z, z+1})$ for any $z = x, \dots, y - 1$. For any $t \in [0, T]$ we have that

$$\limsup_{N \rightarrow +\infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \varphi(\eta_{sN^2})(\eta_{sN^2}(x) - \eta_{sN^2}(y)) ds \right| \right] = 0. \quad (41)$$

Proof. From entropy inequality, Jensen's inequality and Feynman-Kac's formula, for any positive constant B , the expectation in the statement of the lemma is bounded from above by

$$\frac{H(\mu_N | \nu_\alpha^N)}{BN} + t \sup_f \left\{ \langle \varphi(\eta)(\eta(x) - \eta(y)), f \rangle_{\nu_\alpha^N} + \frac{N}{B} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\alpha^N} \right\}. \quad (42)$$

Now we observe that $H(\mu_N | \nu_\alpha^N) \leq NC_{\alpha, \beta}$. The remark that the supremum above is over densities f with respect to ν_α . By Lemma A.1 with the choice $A = \frac{B}{N}$ we have that

$$\left| \langle \varphi(\eta)(\eta(x) - \eta(y)), f \rangle_{\nu_\alpha^N} \right| \lesssim \frac{N}{B} D_N(\sqrt{f}, \nu_\alpha^N) + \frac{B}{N} |y - x|$$

From (40) and the inequality above, the term on the right-hand side of (42), is bounded from above by $\frac{B}{N} |y - x| + \frac{1}{N}$. Taking $N \rightarrow \infty$ and then $B \rightarrow +\infty$ we are done. \square

Lemma A.3 (Replacement Lemma). Let $\psi : \Omega \rightarrow \Omega$ be a positive and bounded function which satisfies $\psi(\eta) = \psi(\eta^{z, z+1})$ for any $z = x + 1, \dots, x + \varepsilon N - 1$. For any $t \in [0, T]$ and $x \in \Lambda_N$ such that $x \in \{1, \dots, N - \varepsilon N - 2\}$ we have that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow +\infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \psi(\eta_{sN^2})(\eta_{sN^2}(x) - \overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x)) ds \right| \right] = 0. \quad (43)$$

Note that for $x \in \Lambda_N$ such that $x \in \{N - \varepsilon N - 1, N - 1\}$ the previous result is also true, but we replace in the previous expectation $\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(x)$ by $\overleftarrow{\eta}_{sN^2}^{\varepsilon N}(x)$, where both averages were defined in (17).

Proof. We present the proof for the case when $x \in \{1, \dots, N - \varepsilon N - 2\}$ but we note that the other case is completely analogous. By applying the same arguments as in the proof of the previous lemma, we can bound from above the previous expectation by

$$\frac{C_{\alpha, \beta}}{B} + t \sup_f \left\{ \langle \psi(\eta)(\eta(x) - \overrightarrow{\eta}^{\varepsilon N}(x)), f \rangle_{\nu_\alpha^N} + \frac{N}{B} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\alpha^N} \right\}. \quad (44)$$

where B is a positive constant. The supremum above is over densities f with respect to ν_α . The first term in the supremum above can be written as

$$\frac{1}{\varepsilon N} \sum_{y=x+1}^{x+\varepsilon N} \langle \psi(\eta)(\eta(x) - \eta(y)), f \rangle_{\nu_\alpha^N}$$

By Lemma A.1 with the choice $A = \frac{B}{N}$ and from (40), the term on the right-hand side of (44), is bounded from above by $B\varepsilon + \frac{1}{N}$. Taking $N \rightarrow \infty$, $\varepsilon \rightarrow 0$ and then $B \rightarrow +\infty$ we are done. \square

Now we state the replacement lemma that we need when the process is speeded up in the subdiffusive time scale.

Corollary A.4. Recall from (24) the definition of the mass of the system m_t^N at the subdiffusive time scale $tN^{1+\theta}$. For any $\theta > 1$ and $t \in [0, T]$ and $x \neq z \in \Lambda_N$ we have that

$$\limsup_{N \rightarrow +\infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \eta_{sN^{1+\theta}}(z)(\eta_{sN^{1+\theta}}(x) - m_s^N) ds \right| \right] = 0. \quad (45)$$

The proof follows exactly the same strategy as above, the only difference being that when we use Lemma A.1 the function $\varphi(\eta) = \eta(z)$ is not invariant under the exchanges in the bulk. Nevertheless, by observing that the integrand function above can be written as

$$\eta(z)(\eta(x) - \langle \pi^N, 1 \rangle) = \frac{1}{N} \eta(z)(\eta(x) - \eta(z)) + \frac{1}{N} \eta(z) \sum_{y \neq x, z} \eta(x) - \eta(y) \quad (46)$$

and thanks to the exclusion rule, we can reduce the proof of this corollary to the proof of Lemma A.3. To do that, go to the proof of Lemma A.3 take $\varepsilon = 1$ and since the time scale is now subdiffusive, choose $A = BN^{-\theta}$.

APPENDIX B. ENERGY ESTIMATE

Now we prove that the density $\rho(t, u)$ belongs to the space $L^2(0, T; \mathcal{H}^1)$, see Definition 2.1. Define the linear functional ℓ_ρ defined in $C_c^{0,1}([0, T] \times (0, 1))$ by

$$\ell_\rho(G) = \int_0^T \int_0^1 \partial_u G_s(u) \rho(s, u) du ds = \int_0^T \int_0^1 \partial_u G_s(u) \pi(s, du) ds.$$

Proposition B.1. *There exist positive constants C and c such that*

$$\mathbb{E} \left[\sup_{G \in C_c^{0,1}([0, T] \times (0, 1))} \left\{ \ell_\rho(G) - c \|G\|_2^2 \right\} \right] \leq C < \infty.$$

Above $\|G\|_2$ denotes the norm of a function $G \in L^2([0, T] \times (0, 1))$.

Before proving this result, we state and prove an energy estimate for the macroscopic density.

Corollary B.2. *Any limit point \mathbb{Q} of the sequence $(\mathbb{Q}_N)_{N \geq 1}$ satisfies*

$$\mathbb{Q}(\pi \in \mathcal{D}([0, T], \mathcal{M}), \pi_t := \rho_t(u) du, \rho \in L^2(0, T; \mathcal{H}^1)) = 1.$$

We denote R_T the event above.

Proof of Corollary B.2. We first note that because of the exclusion between particles, every limit point \mathbb{Q} is concentrated on trajectories of measure that are absolutely continuous with respect to the Lebesgue measure (see e.g. [17], p.57, last paragraph for more details).

From Proposition B.1, ℓ_ρ is \mathbb{Q} -almost surely continuous and therefore we can extend this linear functional to $L^2([0, T] \times (0, 1))$. As a consequence of the Riesz's Representation Theorem there exists $H \in L^2([0, T] \times (0, 1))$ such that

$$\ell_\rho(G) = - \int_0^T \int_0^1 G_s(u) H_s(u) du ds$$

for all $G \in C_c^{0,1}([0, T] \times (0, 1))$. From this we conclude that $\rho \in L^2(0, T; \mathcal{H}^1)$. \square

Proof of Proposition B.1. By density and by the Monotone Convergence Theorem it is enough to prove that for a countable dense subset $\{G_m\}_{m \in \mathbb{N}}$ on $C_c^{0,2}([0, T] \times (0, 1))$ it holds that

$$\mathbb{E} \left[\max_{k \leq m} \left\{ \ell_\rho(G^k) - c \|G^k\|_2^2 \right\} \right] \leq C_0,$$

for any m and for C_0 independent of m . Note that the function that associates to a trajectory $\pi \in \mathcal{D}([0, T], \mathcal{M}^+)$ the number $\max_{k \leq m} \left\{ \ell_\rho(G^k) - c \|G^k\|_2^2 \right\}$, is continuous and bounded w.r.t. the Skorohod topology of $\mathcal{D}([0, T], \mathcal{M}^+)$ and for that reason, the expectation in the previous display is equal to the next limit

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\max_{k \leq m} \left\{ \int_0^T \frac{1}{N-1} \sum_{x=1}^{N-1} \partial_u G_s^k \left(\frac{x}{N} \right) \eta_{sN^2}(x) ds - c \|G^k\|_2^2 \right\} \right].$$

By entropy and Jensen's inequalities plus the fact that $e^{\max_{k \leq m} a_k} \leq \sum_{k=1}^m e^{a_k}$ the previous display is bounded from above by

$$C_0 + \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\sum_{k=1}^m e^{\int_0^T \sum_{x \in \Lambda_N} \partial_u G_s^k \left(\frac{x}{N} \right) \eta_{sN^2}(x) ds - cN \|G^k\|_2^2} \right],$$

By linearity of the expectation, to treat the second term in the previous display it is enough to bound the term

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[e^{\int_0^T \sum_{x \in \Lambda_N} \partial_u G_s \left(\frac{x}{N} \right) \eta_{sN^2}(x) ds - cN \|G\|_2^2} \right],$$

for a fixed function $G \in C_c^{0,2}([0, T] \times (0, 1))$, by a constant independent of G . By Feynman-Kac's formula, the expression inside the limsup is bounded from above by

$$\int_0^T \sup_f \left\{ \frac{1}{N} \int_{\Omega_N} \sum_{x \in \Lambda_N} \partial_u G_s(\frac{x}{N}) \eta(x) f(\eta) d\nu_\alpha^N - c \|G\|_2^2 + N \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\alpha^N} \right\} ds \quad (47)$$

where the supremum is carried over all the densities f with respect to ν_α^N . Note that by a Taylor expansion on G , it is easy to see that we can replace its space derivative by the discrete gradient $\nabla_N^+ G_s(\frac{x-1}{N})$ by paying an error of order $O(\frac{1}{N})$. Then, from a summation by parts, we obtain

$$\int_{\Omega_N} \sum_{x=1}^{N-2} G_s(\frac{x}{N}) (\eta(x) - \eta(x+1)) f(\eta) d\nu_\alpha^N$$

By writing the previous term as one half of it plus one half of it and in one of the halves we swap the occupation variables $\eta(x)$ and $\eta(x+1)$, for which the measure ν_α is invariant, the last display becomes equal to

$$\frac{1}{2} \int_{\Omega_N} \sum_{x=1}^{N-2} G_s(\frac{x}{N}) (\eta(x) - \eta(x+1)) (f(\eta) - f(\eta^{x,x+1})) d\nu_\alpha^N. \quad (48)$$

Repeating similar arguments to those used in the proof of Lemma A.1, the last term is bounded from above by

$$\begin{aligned} & \frac{1}{4N} \int_{\Omega_N} \sum_{x=1}^{N-2} (G_s(\frac{x}{N}))^2 (\sqrt{f(\eta)} + \sqrt{f(\eta^{x,x+1})})^2 d\nu_\alpha^N + \frac{1}{4N} \int_{\Omega_N} \sum_{x=1}^{N-2} (\sqrt{f(\eta)} - \sqrt{f(\eta^{x,x+1})})^2 d\nu_\alpha^N \\ & \leq \frac{C}{N} \sum_{x \in \Lambda_N} (G_s(\frac{x}{N}))^2 + \frac{1}{4N} D_{0,N}(\sqrt{f}, \nu_\alpha^N) \end{aligned}$$

for some $C > 0$. From (40) we get that (47) is bounded from above by

$$C \int_0^T \left[1 + \frac{1}{N} \sum_{x \in \Lambda_N} (G_s(\frac{x}{N}))^2 \right] ds - c \|G\|_2^2$$

plus an error of order $O(\frac{1}{N})$. Above C is a positive constant independent of G . Since $\frac{1}{N} \sum_{x \in \Lambda_N} (G_s(\frac{x}{N}))^2$ converges, as $N \rightarrow +\infty$, to $\|G\|_2^2$, then it is enough to choose $c > C$ to conclude that

$$\limsup_{N \rightarrow \infty} \left\{ C \int_0^T \left[1 + \frac{1}{N} \sum_{x \in \Lambda_N} (G_s(\frac{x}{N}))^2 \right] ds - c \|G\|_2^2 \right\} \lesssim 1$$

and we are done. \square

APPENDIX C. UNIQUENESS OF WEAK SOLUTIONS OF (8)

We start this section by recalling from Section 7.2 of [2] the next two lemmas, which will be used in our proof. The first one concerns uniqueness of the strong solutions of the heat equation with *linear* Robin boundary conditions.

Lemma C.1. *For any $t \in (0, T]$, the following problem with Robin boundary conditions*

$$\begin{cases} \partial_s \varphi(s, u) + a \partial_u^2 \varphi(s, u) = \lambda \varphi(s, u), & (s, u) \in [0, t] \times (0, 1), \\ \partial_u \varphi(s, 0) = b(s, 0) \varphi(s, 0), & s \in [0, t], \\ \partial_u \varphi(s, 1) = -b(s, 1) \varphi(s, 1), & s \in [0, t], \\ \varphi(t, u) = h(u), & u \in (0, 1), \end{cases} \quad (49)$$

with $h \equiv h(u) \in C_0^2([0, 1])$, $\lambda \geq 0$, $0 < a \equiv a(u, t) \in C^{2,2}([0, T] \times [0, 1])$, and for $u \in \{0, 1\}$, $0 < b \equiv b(u, t) \in C^2[0, T]$, has a unique solution $\varphi \in C^{1,2}([0, t] \times [0, 1])$. Moreover, if $h \in [0, 1]$ then we have $\forall (s, u) \in [0, t] \times [0, 1]$:

$$0 \leq \varphi(s, u) \leq e^{-\lambda(t-s)}.$$

The second lemma is a technical regularization result on the coefficients $b(s, \cdot)$.

Lemma C.2. *Let $0 \leq b$ be a bounded measurable function in $[0, T]$, $A = \{t \in [0, T] : b(t) > 0\}$ and $p \in [1, \infty)$. Then there is a sequence $(b_k)_{k \geq 0}$ of positive functions in $C^\infty[0, T]$ such that $b_k \xrightarrow{k \rightarrow \infty} b$ in $L^p([0, T])$ and*

$$\left\| \frac{b}{b_k} - 1 \right\|_{L^p(A)} \xrightarrow{k \rightarrow \infty} 0.$$

For the proof of Lemma 2.3, that is of uniqueness of weak solutions of (8), we will follow Filo's method [12], but mostly as presented in Section 7.2 of [2]. The main idea is to choose a particular test function for the weak formulation satisfied by $w := \rho^{(1)} - \rho^{(2)}$, where $\rho^{(1)}$ and $\rho^{(2)}$ are two weak solutions with the same initial data. Although we do not have as much work to treat the bulk terms as in [2], our main issue is the non linearity of the boundary conditions.

Recalling the weak formulation in (9) and Lemma 5.4, since

$$D_{\lambda, \sigma} \rho_s^{(1)}(v) - D_{\lambda, \sigma} \rho_s^{(2)}(v) = -w_s(v) V_{\lambda, \sigma}(\rho_s^{(1)}, \rho_s^{(2)})(v, v) := w_s(v) V_{\lambda, \sigma}(v, s)$$

for $v = 0, 1$ and $(\lambda, \sigma) = (\alpha, \gamma), (\beta, \delta)$, we have:

$$\begin{aligned} \langle w_t, G_t \rangle &= \int_0^t \langle w_s, (\partial_u^2 + \partial_s) G_s \rangle ds + \int_0^t w_s(0) (\partial_u G_s(0) - G_s(0) V_{\alpha, \gamma}(0, s)) ds \\ &\quad - \int_0^t w_s(1) (\partial_u G_s(1) + G_s(1) V_{\beta, \delta}(1, s)) ds. \end{aligned} \quad (50)$$

Now we choose our test functions. Since $V_{\cdot, \cdot}$ does not have enough regularity, we have to overcome this problem by using Lemma C.2. We focus on the left boundary, since for the right boundary the computations are analogous. Let $A_0 = \{t \in [0, T] : V_{\alpha, \gamma}(0, t) > 0\}$ (similarly, we define A_1 with respect to the right boundary). From Lemma 5.4 we have $V_{\alpha, \gamma}(0, s) > 0$ and we may therefore exchange $[0, t]$ by A_0 (resp. A_1) and apply Lemma C.2. As a consequence of Lemma C.2, for k large enough, there exists $b_k(s, 0)$ close to $V_{\alpha, \gamma}(0, s)$ in $L^p([0, T])$ for $p \in [1, +\infty)$:

$$\left\| \frac{V_{\alpha, \gamma}(0, \cdot)}{b_k(\cdot, 0)} - 1 \right\|_{L^p(A_0)} \leq \epsilon$$

for $\epsilon > 0$ and $A_0 = \{s \in [0, t] : V_{\alpha, \gamma}(0, s) > 0\}$. Now we choose the space of test functions as a sequence φ_k , where for each k , the function φ_k solves (49) with $b_k(\cdot, 0)$ given above and with $\lambda = 0$. From the boundary conditions of (49), second term in (50) writes as

$$\int_0^t \varphi_k(0, s) w_s(0) (b_k(0, s) - V_{\alpha, \gamma}(0, s)) ds. \quad (51)$$

Exchanging $[0, t]$ by A_0 , the last display can be bounded from above by

$$\left| \int_{A_0} \varphi_k(0, s) w_s(0) b_k(0, s) \left(1 - \frac{V_{\alpha, \gamma}(0, s)}{b_k(0, s)} \right) ds \right| \leq 2 \|b_k(0, s)\|_{L^1(A_0)} \left\| \frac{V_{\alpha, \gamma}(0, s)}{b_k(0, s)} - 1 \right\|_{L^1(A_0)} \lesssim \epsilon,$$

where we used that $\varphi_k(\cdot)$ and $w(\cdot)$ are bounded functions. For the right boundary the argument is completely analogous.

Now we treat the bulk term. From our choice of test function we have

$$\int_0^t \langle w_s, (\partial_u^2 + \partial_s) \varphi_k \rangle ds = \int_0^t \langle w_s, (1-a) \partial_u^2 \varphi_k(\cdot, s) \rangle ds.$$

Letting $a = 1$, we thus have that $\langle w_t, \varphi_t \rangle \lesssim \epsilon$. Since $\varphi_t = h$, it is enough to take $h \equiv h_k \in C_0^2([0, 1])$ such that $h_k(\cdot) \xrightarrow{k \rightarrow \infty} 1_{\{u \in [0, 1] : w_t(u) > 0\}}(t, \cdot)$ in $L^2([0, 1])$. The conclusion follows straightforwardly.

Remark C.3. *We remark that for the model in [8] the lower bound takes the form of the term $x = K$ in the sum above, with $\gamma = 0$. For $K = 1$, that is, the case studied in [1], we have $V = \rho^{(1)} + \rho^{(2)}$, and thus $V = 0 \implies w = 0$.*

REFERENCES

- [1] Baldasso, R., Menezes, O., Neumann, A., Souza, R. R.: Exclusion Process with Slow Boundary, *Journal of Statistical Physics*, Volume 167, no. 5, 1112–1142 (2017).
- [2] Bonorino, L., De Paula, R., Gonçalves, P., Neumann, A.: Hydrodynamics for the porous medium model with slow reservoirs, arXiv:1904.10374 (2019).
- [3] Bernardin, C., Gonçalves, P., Jiménez-Oviedo, B.: Slow to fast infinitely extended reservoirs for the symmetric exclusion process with long jumps, *Markov Processes and Related Fields*, no. 25, 217–274 (2019).
- [4] Bernardin, C., Gonçalves, P., Jiménez-Oviedo, B.: A microscopic model for a one parameter class of fractional laplacians with Dirichlet boundary conditions, arXiv:1803.00792 (2018).
- [5] B. Derrida: Non-equilibrium steady states: Fluctuations and large deviations of the density and of the current. *J. Stat. Mech. Theory Exp.*, P07023 (2007).
- [6] De Masi, A., Presutti, E., Tsagkarogiannis, D., Vares, M.: Truncated correlations in the stirring process with births and deaths, *Electronic Journal of Probability*, Volume 17, no. 6 (2012).
- [7] De Masi, A., Presutti, E., Tsagkarogiannis, D., Vares, M.: Non-equilibrium Stationary States in the Symmetric Simple Exclusion with Births and Deaths, *Journal of Statistical Physics*, Volume 147, no. 3, 519–528 (2012).
- [8] De Masi, A., Presutti, E., Tsagkarogiannis, D., Vares, M.: Current Reservoirs in the Simple Exclusion Process, *Journal of Statistical Physics*, Volume 144, no. 3, 519–528 (2011).
- [9] Erignoux, C., Gonçalves, P., Nahum, G.: *Hydrodynamics for SSEP with non-reversible slow boundary dynamics: Part II, the critical regime and beyond*, arxiv.org (2020).
- [10] Erignoux, C.: Hydrodynamic limit of boundary driven exclusion processes with nonreversible boundary dynamics, *Journal of Statistical Physics*, Volume 172, Issue 5, 1327–1357 (2018).
- [11] Erignoux, C., Landim, C., Xu, T.: Stationary states of boundary driven exclusion processes with nonreversible boundary dynamics, *Journal of Statistical Physics*, Volume 171, number 4, 599–631 (2018).
- [12] Filo, J.: A nonlinear diffusion equation with nonlinear boundary conditions: methods of lines. *Mathematica Slovaca*, Volume 38, Issue 3, 273–296, (1988).
- [13] Franco, T., Gonçalves, P., Neumann, A.: Hydrodynamical Behavior of Symmetric Exclusion With Slow Bonds, *Annales de l’Institut Henri Poincaré - Probabilités et Statistiques*, Volume 49, no. 2 (2013)
- [14] Franco, T., Gonçalves, P., Schutz, G.: Scaling limits for the exclusion process with a slow site, *Stochastic Processes and their applications*, Volume 126, Issue 3, 800–831 (2016).
- [15] Gonçalves, P.: Hydrodynamics for symmetric exclusion in contact with reservoirs, *Stochastic Dynamics Out of Equilibrium Springer Lecture Notes*, Institut Henri Poincaré, Paris, France (2019).
- [16] Guo, M. Z., Papanicolaou, G. C., Varadhan, S. R. S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. *Commun. Math. Phys.* 118, 31–59, (1988).
- [17] Kipnis, C., Landim, C.: *Scaling limits of interacting particle systems*, Springer-Verlag (1999).
- [18] Landim, C., Tsunoda, K.: Hydrostatics and Dynamical Large Deviations for a Reaction-Diffusion Model, *Annals Henri Poincaré*, Volume 54, no. 1, 51–74 (2018).
- [19] Sethuraman, S., Shahar, D.: Hydrodynamic limits for long-range asymmetric interacting particle systems, *Elec. J. Probab.*, Volume 23, no. 130, 1–54 (2018).
- [20] Tsunoda, K.: Hydrostatic Limit for Exclusion Process With Slow Boundary Revisited, accepted for publication in *RIMS Kôkyûroku Bessatsu*.

CLÉMENT ERIGNOUX, DIP. MATEMATICA E FISICA, UNIV. DEGLI STUDI ROMA TRE, LARGO SAN LEONARDO MURIALDO, 1 - 00146 ROMA, ITALY

Email address: clement.erignoux@gmail.com

PATRÍCIA GONÇALVES, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL.

Email address: pgoncalves@tecnico.ulisboa.pt

GABRIEL NAHUM, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL.

Email address: gabriel.nahum@tecnico.ulisboa.pt