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Discretization and Machine Learning Approximation of BSDEs with a Constraint on the Gains-Process

Idris Kharroubi* Thomas Lim† Xavier Warin‡

February 5, 2020

Abstract

We study the approximation of backward stochastic differential equations (BSDEs for short) with a constraint on the gains process. We first discretize the constraint by applying a so-called facelift operator at times of a grid. We show that this discretely constrained BSDE converges to the continuously constrained one as the mesh grid converges to zero. We then focus on the approximation of the discretely constrained BSDE. For that we adopt a machine learning approach. We show that the facelift can be approximated by an optimization problem over a class of neural networks under constraints on the neural network and its derivative. We then derive an algorithm converging to the discretely constrained BSDE as the number of neurons goes to infinity. We end by numerical experiments.

Mathematics Subject Classification (2010): 65C30, 65M75, 60H35, 93E20, 49L25.

Keywords: Constrained BSDEs, discrete-time approximation, neural networks approximation, facelift transformation.

1 Introduction

In this paper, we propose an algorithm for the numerical resolution of BSDEs with a constraint on the gains process. Namely, we consider the approximation of the minimal solution to the BSDE

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \quad t \leq T$$

with constraint

$$Z \in \sigma^\top(X)\mathcal{C}, \quad dt \otimes d\mathbb{P} - a.e.$$

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Here, \mathcal{C} is a closed convex set, K is a nondecreasing process, B is a d -dimensional Brownian motion and X solves the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t .$$

This kind of equation is related to the super-replication under portfolio constraints in mathematical finance (see e.g. [11]). A first approach to show existence of minimal solutions was done in [9] using a duality approach. As far as we know the most general result is given in [20] where the existence of a minimal solution is a byproduct of a general limit theorem for supersolutions of Lipschitz BSDE. In particular, the minimal solution is characterized as the limit of penalized BSDEs.

As far as we know, this characterization of the constrained solution as limit of penalized BSDEs is the wider one. In particular, we cannot express in a simple way how the constraint on the Z component acts on the process Y . Therefore, the construction of numerical scheme remains a challenging issue. A possible approach can be to use the penalized BSDEs to approximate the constrained solution. However, this leads to approximate BSDEs with exploding Lipschitz constant for the generator which gives a very slow and sometimes unstable converging scheme [12]. Therefore, one needs to focus on the structure of the constrained solution to set a stable numerical scheme.

Recently, [5] gives more insights on the minimal solutions of constrained BSDEs. The minimal solution is proved to satisfy a classical L^2 -type regularity -as for BSDEs without constraint- but only until T^- . At the terminal time T , the constraint leads to a boundary effect which consists in replacing the terminal value g by a functional transformation $F_{\mathcal{C}}[g]$ called facelift. This facelift transformation can be interpreted as the smallest function dominating the original function such that its derivative satisfies the constraint.

Taking advantage of those recent advances, we derive a converging approximation algorithm for constrained BSDEs.

To this end we proceed in two steps. We first provide a discrete time approximation of the constraint. Taking into account the boundary effect mentioned in [5], we apply the facelift operator to the Markov function relating Y to the underlying diffusion X , at the points of a given discrete grid. This leads to a new BSDE with a discrete-time constraint. Using the regularity property provided by [5], we prove a convergence result as the mesh of the constraint grid goes to zero. Let us mention the article [7] where a similar discretization is obtained for the super-replication price. However the approach used in [7] is different and consists in the approximation of the dual formulation by restricting it to stepwise processes.

We then provide a computable algorithm to approximate the BSDE with discrete-time constraint. The main issue here comes from the facelift transformation as it involves all the values of the Markov function linking Y to the underlying diffusion X . In particular, we cannot proceed as in the reflected case where the transformation on Y depends only on its value.

To overcome this issue we adopt a machine learning approach. More precisely, we compute the facelift by neural network approximators. Using the interpretation of the facelift as the smallest dominating function whose derivatives belong to the constraint set \mathcal{C} , we propose an approximation as a neural network minimizing the square error under the

constraint of having derivatives in \mathcal{C} and dominating the original function. We notice that this approximation turns the problem into a parametric one, which is numerically valuable.

Using the universal approximation property of neural networks up to order one, we show that this approximation converges to the facelift as the number of neurons goes to infinity. Combining our machine learning approximation of the facelift with recent machine learning approximations for BSDEs/PDEs described in [15], we are able to derive a fully computable algorithm for the approximation of BSDEs with constraints on the gain process.

The remainder of paper is organized as follows. In Section 2, we recall the main assumptions, definitions and results on BSDEs with constraints on the gains process. In Section 3, we introduce the discretely constraints and prove the convergence to the continuously constrained BSDEs as the mesh of the discrete constraint grid goes to zero. In Section 4, we present the neural network approximation of the facelift and propose a converging approximation scheme for discretely constrained BSDEs.

Finally, Section 5 is devoted to numerical experiments. At first, we show that the numerical approximation of the facelift by a neural network is not obvious using a simple minimization with penalization of the constraints. This simple approach numerically gives an upper bound of the facelift. We then derive an original iterative algorithm that we show on examples to converge to the facelift till dimension 10.

At last the whole algorithm including the facelift approximation and the BSDE resolution using the methodology in [15] is tested on some option pricing problems with differential interest rates.

2 BSDEs with a convex constraint on the gains-process

2.1 The constrained BSDE

Given a finite time horizon $T > 0$ and a finite dimension $d \geq 1$, we denote by Ω the space $C([0, T], \mathbb{R}^d)$ of continuous functions from $[0, T]$ to \mathbb{R}^d . We endow this space with the Wiener measure \mathbb{P} . We denote by B the coordinate process defined on Ω by $B_t(\omega) = \omega(t)$ for $\omega \in \Omega$. We then define on Ω the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ defined as the \mathbb{P} -completion of the filtration generated by B .

We are given two measurable functions $b, \sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \mathbb{R}^{d \times d}$ on which we make the following assumption.

(H b, σ)

- (i) The values of the function σ are invertible.
- (ii) The functions b, σ and σ^{-1} are bounded: there exists a constant $M_{b, \sigma}$ such that

$$|b(t, x)| + |\sigma(t, x)| + |\sigma^{-1}(t, x)| \leq M_{b, \sigma}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.

- (iii) The functions b and σ are Lipschitz continuous in their space variable uniformly in their time variable: there exists a constant $L_{b, \sigma}$ such that

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq L_{b, \sigma} |x - x'|$$

for all $t \in [0, T]$ and $x, x' \in \mathbb{R}^d$.

Under Assumption $(\mathbf{H}b, \sigma)$, we can define the process $X^{t,x}$ as the solution to the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \quad s \in [t, T],$$

and by classical estimates, there exists a constant C such that

$$\mathbb{E} \left[\sup_{s \in [t, T]} |X_s^{t,x}|^2 \right] \leq C$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and

$$\mathbb{E} \left[\sup_{s \in [t \vee t', T]} |X_s^{t,x} - X_s^{t',x'}|^2 \right] \leq C(|t - t'| + |x - x'|^2) \quad (2.1)$$

for all $t, t' \in [0, T]$ and $x, x' \in \mathbb{R}^d$.

We now define the backward equation. To this end, we consider two functions $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ on which we make the following assumption.

(Hf, g)

(i) The function g is bounded: there exists a constant M_g such that

$$|g(x)| \leq M_g$$

for all $x \in \mathbb{R}^d$.

(ii) The function f is continuous and satisfies the following growth property: there exists a constant M_f such that

$$|f(t, x, y, z)| \leq M_f(1 + |y| + |z|)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

(iii) The functions f and g are Lipschitz continuous in their space variables uniformly in their time variable: there exists two constants L_f and L_g such that

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| &\leq L_f(|x - x'| + |y - y'| + |z - z'|) \\ |g(x) - g(x')| &\leq L_g|x - x'| \end{aligned}$$

for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$.

We then fix a bounded convex subset \mathcal{C} of \mathbb{R}^d such that $0 \in \mathcal{C}$. For $t \in [0, T]$, we denote by $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \in [t, T]}$ the completion of the filtration generated by $(B_s - B_t)_{s \in [t, T]}$. We define $\mathbf{S}_{[t, T]}^2$ (resp. $\mathbf{H}_{[t, T]}^2$) as the set of \mathbb{R} -valued *càdlàg* \mathbb{F}^t -adapted (resp. \mathbb{R}^d -valued \mathbb{F}^t -predictable) processes U (resp. V) such that $\|U\|_{\mathbf{S}_{[t, T]}^2} := \mathbb{E}[\sup_{[t, T]} |U_s|^2] < +\infty$ (resp. $\|V\|_{\mathbf{H}_{[t, T]}^2} :=$

$\mathbb{E}[\int_t^T |V_s|^2 ds] < +\infty$). We also define $\mathbf{A}_{[t,T]}^2$ as the set of \mathbb{R} -valued nondecreasing *càdlàg* \mathbb{F}^t -adapted processes K such that $K_t = 0$ and $\mathbb{E}[|K_T|^2] < +\infty$.

A solution to the constrained BSDE with parameters $(t, x, f, g, \mathcal{C})$ is defined as a triplet of processes $(U, V, A) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2 \times \mathbf{A}_{[t,T]}^2$ such that

$$U_s = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, U_u, V_u) du - \int_s^T V_u dB_u + A_T - A_s \quad (2.2)$$

$$V_s \in \sigma(s, X_s^{t,x})^\top \mathcal{C} \quad (2.3)$$

for $s \in [t, T]$.

Under Assumptions $(\mathbf{H}b, \sigma)$ and $(\mathbf{H}f, g)$ and since $0 \in \mathcal{C}$, there exists a solution to (2.2)-(2.3) given by

$$U_s = (M_g + 1)e^{M_f(T-s)} - 1, \quad s \in [t, T], \quad U_T = g(X_T^{t,x}) \quad (2.4)$$

and

$$V_s = 0, \quad s \in [t, T], \quad (2.5)$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$. We therefore deduce from Theorem 4.2 in [20] that there exists a unique minimal solution $(Y^{t,x}, Z^{t,x}, K^{t,x})$ to (2.2)-(2.3) : for any other solution (U, V, A) to (2.2)-(2.3) we have

$$Y_s^{t,x} \leq U_s, \quad s \in [t, T].$$

The aim of this paper is to provide a numerical approximation of this minimal solution $(Y^{t,x}, Z^{t,x}, K^{t,x})$.

2.2 Related value function

Since $Y^{t,x}$ is \mathbb{F}^t -adapted, $Y_t^{t,x}$ is almost surely constant and we can define the function $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$v(t, x) = Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

From the uniqueness of the minimal solution to (2.2)-(2.3), we have

$$Y_s^{t,x} = v(s, X_s^{t,x})$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $s \in [t, T]$.

The aim of this paper is to provide a numerical approximation of this minimal solution $(Y^{t,x}, Z^{t,x}, K^{t,x})$ or equivalently an approximation of the function v .

We end this section by providing some properties of the function v . To this end, we define the facelift operator $F_{\mathcal{C}}$ defined by

$$F_{\mathcal{C}}[\varphi](x) = \sup_{y \in \mathbb{R}^d} \{\varphi(x+y) - \delta_{\mathcal{C}}(y)\}, \quad x \in \mathbb{R}^d,$$

for any function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, where $\delta_{\mathcal{C}}$ is the support function of the convex set \mathcal{C}

$$\delta_{\mathcal{C}}(y) = \sup_{z \in \mathcal{C}} z \cdot y, \quad y \in \mathbb{R}^d.$$

We recall that $\delta_{\mathcal{C}}$ is positively homogeneous and convex. As a consequence the facelift operator $F_{\mathcal{C}}$ satisfies

$$F_{\mathcal{C}}[F_{\mathcal{C}}[\varphi]] = F_{\mathcal{C}}[\varphi] \quad (2.6)$$

for any function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$.

We have the following properties for the function v .

Proposition 2.1. *The function v is locally bounded and satisfies the following properties.*

(i) *Time space regularity: there exists a constant L such that*

$$|v(t, x) - v(t', x')| \leq L(|t - t'|^{\frac{1}{2}} + |x - x'|) \quad (2.7)$$

for all $t, t' \in [0, T)$ and $x, x' \in \mathbb{R}^d$.

(ii) *Facelift identity*

$$v(t, x) = F_{\mathcal{C}}[v(t, \cdot)](x) \quad (2.8)$$

for all $(t, x) \in [0, T) \times \mathbb{R}^d$.

(iii) *Value at T^- :*

$$\lim_{t \rightarrow T^-} v(t, x) = F_{\mathcal{C}}[g](x) \quad (2.9)$$

for all $x \in \mathbb{R}^d$.

Proof. These results mainly rely on [5]. From classical estimates on BSDEs and the super-solution exhibited in (2.4)-(2.5), the function v is bounded.

The property (2.7) is a direct consequence of Theorem 2.1 (a) in [5]. We turn to the facelift identity. Fix $t \in [0, T)$, $\varepsilon > 0$ such that $t + \varepsilon < T$ and $x \in \mathbb{R}^d$. Since $(Y^{t,x}, Z^{t,x}, K^{t,x})$ is the minimal solution to (2.2)-(2.3), its restriction to $[t, t + \varepsilon]$ is also the minimal solution to

$$\begin{aligned} U_s &= v(t + \varepsilon, X_{t+\varepsilon}^{t,x}) + \int_s^{t+\varepsilon} f(u, X_u^{t,x}, U_u, V_u) du - \int_s^{t+\varepsilon} Z_u^{t,x} dB_u + A_T - A_s \\ V_s &\in \sigma(s, X_s^{t,x})^\top \mathcal{C} \end{aligned}$$

for $s \in [t, t + \varepsilon]$. From Theorem 2.1 (b) and (c) in [5], we deduce that

$$v(t + \varepsilon, X_{t+\varepsilon}^{t,x}) = F_{\mathcal{C}}[v(t + \varepsilon, \cdot)](X_{t+\varepsilon}^{t,x}).$$

From (2.7), we get (2.8) by sending ε to 0. The last property is a consequence of Theorem 2.1 (c) in [5]. \square

We end this section by a characterization of the minimal solution as the limit of penalized solutions. More precisely, we introduce the sequence $(Y^{n,t,x}, Z^{n,t,x}) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2$ which is defined for any $n \in \mathbb{N}^*$ as the solution of the following BSDE

$$\begin{aligned} Y_s^{n,t,x} &= g(X_T^{t,x}) \\ &+ \int_s^T \left(f(u, X_u^{t,x}, Y_u^{n,t,x}, Z_u^{n,t,x}) + n \max \{ -H(\sigma^\top(u, X_u^{t,x})^{-1} Z_u^{n,t,x}), 0 \} \right) du \\ &- \int_s^T Z_u^{n,t,x} dB_u, \quad s \in [t, T], \end{aligned} \quad (2.10)$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, where the operator H is defined by

$$H(p) = \inf_{|y|=1} (\delta_{\mathcal{C}}(y) - yp), \quad p \in \mathbb{R}^d.$$

We also introduce the related sequence of penalized PDEs

$$\begin{cases} -\partial_t v_n(t, x) - \mathcal{L}v_n(t, x) - f(t, x, v_n(t, x), \sigma(t, x)^\top Dv_n(t, x)) \\ -n \max \{ -H(Dv_n(t, x)), 0 \} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d \\ v_n(T, x) = g(x), \quad x \in \mathbb{R}^d \end{cases} \quad (2.11)$$

where the second order local operator \mathcal{L} related to the diffusion process X is defined by

$$\mathcal{L}\varphi(t, x) = b(t, x) \cdot D\varphi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D^2 \varphi(t, x)), \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

for any function $\varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ which is twice differentiable w.r.t. its space variable. As we use the notion of viscosity solution, we refer to [8] for its definition.

Proposition 2.2. (i) For $(t, x) \in [0, T] \times \mathbb{R}^d$ the BSDE (2.10) admits a unique solution $(Y^{n,t,x}, Z^{n,t,x}) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2$ and we have

$$Y_s^{n,t,x} = v_n(s, X_s^{t,x}), \quad s \in [t, T],$$

where v_n is the unique viscosity solution to (2.11) with polynomial growth.

(ii) The sequence $(Y^{n,t,x})_{n \geq 1}$ is nondecreasing and

$$\lim_{n \rightarrow +\infty} Y_s^{n,t,x} = Y_s^{t,x}, \quad \mathbb{P} - \text{a.s. for } s \in [t, T],$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$.

(iii) The sequence $(v_n)_{n \geq 1}$ is nondecreasing and converges pointwisely to the function v on $[0, T] \times \mathbb{R}^d$.

Proof. (i) By the definition of the operator H , the driver of BSDE (2.10) is globally Lipschitz continuous for all $n \geq 1$. From Theorem 1.1 in [17], there exists a unique $(Y^{n,t,x}, Z^{n,t,x}) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2$ solution to (2.10) for all $n \geq 1$. Then from Theorem 2.2 in [17], we get that the function $v_n : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$v_n(t, x) = Y_t^{n,t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

is a continuous viscosity solution to (2.11). By uniqueness to BSDE (2.10), we get

$$Y_s^{n,t,x} = v_n(s, X_s^{t,x}), \quad s \in [t, T].$$

Using Theorem 5.1 in [19], v_n is the unique viscosity solution to (2.11) with polynomial growth.

(ii) From Theorem 4.2 in [20], the sequence $(Y^{n,t,x})_{n \geq 1}$ is nondecreasing and converges pointwisely to $\tilde{Y}^{t,x}$ where $(\tilde{Y}^{t,x}, \tilde{Z}^{t,x})$ is the minimal solution to (2.2), with the constraint

$$H(\sigma^\top (X^{t,x})^{-1} \tilde{Z}^{t,x}) \geq 0,$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$. Since \mathcal{C} is closed, we get from Theorem 13.1 in [22]

$$\tilde{Z}^{t,x} \in \sigma(X^{t,x})^\top \mathcal{C}$$

and $(\tilde{Y}^{t,x}, \tilde{Z}^{t,x}) = (Y^{t,x}, Z^{t,x})$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

(iii) The nondecreasing convergence of v_n to v is an immediate consequence of (ii). \square

3 Discrete-time approximation of the constraint

3.1 Discretely constrained BSDE

We introduce in this section a BSDE with discretized constraint on the gains process. To this end, we first extend the definition of the facelift operator to random variables. More precisely, for $s \in [0, T]$ and $L > 0$, we denote by $\mathbf{D}_{L,s}$ the set of random flows $R = \{R^{t,x}, (t, x) \in [0, s] \times \mathbb{R}^d\}$ of the form

$$R^{t,x} = \varphi(X_s^{t,x}), \quad (t, x) \in [0, s] \times \mathbb{R}^d, \quad (3.12)$$

where $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is L -Lipschitz continuous. We also define the set \mathbf{D}_s by

$$\mathbf{D}_s = \bigcup_{L>0} \mathbf{D}_{L,s}.$$

We then define the operator $\mathfrak{F}_{\mathcal{C},s}$ on \mathbf{D}_s by

$$\mathfrak{F}_{\mathcal{C},s}[R]^{t,x} = F_{\mathcal{C}}[\varphi](X_s^{t,x}), \quad (t, x) \in [0, s] \times \mathbb{R}^d,$$

for $R \in \mathbf{D}_s$ of the form (3.12). We notice that the function φ appearing in the representation (3.12) is uniquely defined. Therefore the extended facelift operator $\mathfrak{F}_{\mathcal{C}}$ is well defined. Moreover, it satisfies the following stability property

$$R \in \mathbf{D}_{L,s} \Rightarrow \mathfrak{F}_{\mathcal{C},s}[R] \in \mathbf{D}_{L,s} \quad (3.13)$$

for all $s \in [0, T]$, $L > 0$ and $(t, x) \in [0, T] \times \mathbb{R}^d$. Hence $\mathfrak{F}_{\mathcal{C},s}$ maps \mathbf{D}_s into itself.

We then fix a grid $\mathcal{R} = \{r_0 = 0 < r_1 < \dots < r_n = T\}$, with $n \in \mathbb{N}^*$, of the time interval $[0, T]$ and we consider the discretely constrained BSDE: find $(Y^{\mathcal{R},t,x}, \tilde{Y}^{\mathcal{R},t,x}, Z^{\mathcal{R},t,x}, K^{\mathcal{R},t,x}) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2 \times \mathbf{A}_{[t,T]}^2$ such that

$$Y_T^{\mathcal{R},t,x} = \tilde{Y}_T^{\mathcal{R},t,x} = F_{\mathcal{C}}[g](X_T^{t,x}) \quad (3.14)$$

and

$$\tilde{Y}_u^{\mathcal{R},t,x} = Y_{r_{k+1}}^{\mathcal{R},t,x} + \int_u^{r_{k+1}} f(s, X_s, \tilde{Y}_s^{\mathcal{R},t,x}, Z_s^{\mathcal{R},t,x}) ds - \int_u^{r_{k+1}} Z_s^{\mathcal{R},t,x} dB_s \quad (3.15)$$

$$Y_u^{\mathcal{R},t,x} = \tilde{Y}_u^{\mathcal{R},t,x} \mathbf{1}_{(r_k, r_{k+1})}(u) + \mathfrak{F}_{\mathcal{C}, r_k}[\tilde{Y}_u^{\mathcal{R}}]^{t,x} \mathbf{1}_{\{r_k\}}(u) \quad (3.16)$$

for $u \in [r_k, r_{k+1}) \cap [t, T]$, $k = 0, \dots, n-1$, and

$$K_u^{\mathcal{R},t,x} = \sum_{k=0}^n (Y_{r_k}^{\mathcal{R},t,x} - \tilde{Y}_{r_k}^{\mathcal{R},t,x}) \mathbf{1}_{t \leq r_k \leq u \leq T}$$

for $u \in [t, T]$.

We also introduce the related PDE which takes the following form

$$v^{\mathcal{R}}(T, x) = \tilde{v}^{\mathcal{R}}(T, x) = F_{\mathcal{C}}[g](x), \quad x \in \mathbb{R}^d, \quad (3.17)$$

$$\begin{cases} -\partial_t \tilde{v}^{\mathcal{R}}(t, x) - \mathcal{L} \tilde{v}^{\mathcal{R}}(t, x) \\ -f(t, x, \tilde{v}^{\mathcal{R}}(t, x), \sigma(t, x)^\top D \tilde{v}^{\mathcal{R}}(t, x)) = 0, \quad (t, x) \in [r_k, r_{k+1}) \times \mathbb{R}^d \\ \tilde{v}^{\mathcal{R}}(r_{k+1}^-, x) = F_{\mathcal{C}}[\tilde{v}^{\mathcal{R}}(r_{k+1}, \cdot)](x), \quad x \in \mathbb{R}^d \end{cases} \quad (3.18)$$

and

$$v^{\mathcal{R}}(t, x) = \tilde{v}^{\mathcal{R}}(t, x) \mathbf{1}_{(r_k, r_{k+1})}(t) + F_{\mathcal{C}}[\tilde{v}^{\mathcal{R}}(t, \cdot)](x) \mathbf{1}_{\{t=r_k\}} \quad (3.19)$$

for $(t, x) \in [r_k, r_{k+1}) \times \mathbb{R}^d$, and $k = 0, \dots, n-1$.

We first show the well-posedness of the BSDE (3.14)-(3.15)-(3.16) and PDE (3.17)-(3.18)-(3.19), then we derive some regularity properties about the solutions.

Proposition 3.3. (i) For $(t, x) \in [0, T] \times \mathbb{R}^d$, the discretely constrained BSDE (3.14)-(3.15)-(3.16) admits a unique solution $(Y^{\mathcal{R},t,x}, \tilde{Y}^{\mathcal{R},t,x}, Z^{\mathcal{R},t,x}, K^{\mathcal{R},t,x}) \in \mathbf{S}_{[t,T]}^2 \times \mathbf{S}_{[t,T]}^2 \times \mathbf{H}_{[t,T]}^2 \times \mathbf{A}_{[t,T]}^2$.
(ii) The PDE (3.17)-(3.18)-(3.19) admits a unique bounded viscosity solution $(v^{\mathcal{R}}, \tilde{v}^{\mathcal{R}})$ and we have

$$Y_s^{\mathcal{R},t,x} = v^{\mathcal{R}}(s, X_s^{t,x}) \quad \text{and} \quad \tilde{Y}_s^{\mathcal{R},t,x} = \tilde{v}^{\mathcal{R}}(s, X_s^{t,x}), \quad s \in [t, T],$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$.

(iii) The family of functions $(v^{\mathcal{R}})_{\mathcal{R}}$ (resp. $(\tilde{v}^{\mathcal{R}})_{\mathcal{R}}$) is uniformly Lipschitz continuous in the space variable: there exists a constant L such that

$$|v^{\mathcal{R}}(t, x) - v^{\mathcal{R}}(t, x')| \leq L|x - x'|$$

for all $t \in [0, T]$ and $x, x' \in \mathbb{R}^d$.

(iv) The family of functions $(v^{\mathcal{R}})_{\mathcal{R}}$ (resp. $(\tilde{v}^{\mathcal{R}})_{\mathcal{R}}$) is uniformly $\frac{1}{2}$ -Hölder left-continuous (resp. right-continuous) in the time variable: there exists a constant L such that

$$\begin{aligned} |v^{\mathcal{R}}(t, x) - v^{\mathcal{R}}(r_{k+1}, x)| &\leq L\sqrt{r_{k+1} - t} \\ (\text{resp. } |\tilde{v}^{\mathcal{R}}(t, x) - \tilde{v}^{\mathcal{R}}(r_k, x)| &\leq L\sqrt{t - r_k}) \end{aligned}$$

for all $\mathcal{R} = \{r_0 = 0, r_1, \dots, r_n = T\}$ of $[0, T]$, $t \in (r_k, r_{k+1}]$ (resp. $t \in [r_k, r_{k+1})$), $k = 0, \dots, n-1$ and $x, x' \in \mathbb{R}^d$.

Proof. We fix a grid $\mathcal{R} = \{r_0 = 0 < r_1 < \dots < r_n = T\}$ of the time interval $[0, T]$.

Step 1. Existence and uniqueness to the BSDE and link with the PDE. We prove by a backward induction on k that (3.15)-(3.16) admits a unique solution on $[r_k, r_{k+1}]$ and that $\tilde{Y}_{r_k}^{\mathcal{R}, t, x}, Y_{r_k}^{\mathcal{R}, t, x} \in \mathbf{D}_s$ and that

$$Y^{\mathcal{R}, t, x} = v^{\mathcal{R}}(., X^{t, x}) \quad \text{and} \quad \tilde{Y}^{\mathcal{R}, t, x} = \tilde{v}^{\mathcal{R}}(., X^{t, x})$$

with $(v^{\mathcal{R}}, \tilde{v}^{\mathcal{R}})$ the unique viscosity solution to (3.17)-(3.18)-(3.19) with polynomial growth.

• $k = n - 1$. Since g is Lipschitz continuous, it is the same for $F_{\mathcal{C}}[g]$. From $(\mathbf{H}b, \sigma)$ and $(\mathbf{H}f, g)$ the BSDE admits a unique solution (see e.g. Theorem 1.1 in [17]). From Theorem 2.2 in [17], the functions $(v^{\mathcal{R}}, \tilde{v}^{\mathcal{R}})$ defined by

$$v^{\mathcal{R}}(t, x) = Y_t^{\mathcal{R}, t, x} \quad \text{and} \quad \tilde{v}^{\mathcal{R}}(t, x) = \tilde{Y}_t^{\mathcal{R}, t, x}, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

are the unique viscosity solution to (3.17)-(3.18)-(3.19) with polynomial growth. From the uniqueness to Lipschitz BSDEs (see e.g. Theorem 1.1 in [17]) we get

$$Y_s^{\mathcal{R}, t, x} = v^{\mathcal{R}}(s, X_s^{t, x}) \quad \text{and} \quad \tilde{Y}_s^{\mathcal{R}, t, x} = \tilde{v}^{\mathcal{R}}(s, X_s^{t, x}), \quad s \in [t, T],$$

for $(t, x) \in [r_{n-1}, r_n] \times \mathbb{R}^d$. Then, from Proposition A.5, we have $\tilde{Y}_{r_n}^{\mathcal{R}, t, x} \in \mathbf{D}_s$. By (3.13), $Y_{r_{n-1}}^{\mathcal{R}, t, x} \in \mathbf{D}_s$

• Suppose the property holds for $k + 1$. Then $\tilde{Y}_{r_{k+1}}^{\mathcal{R}, t, x} \in \mathbf{D}_s$. From Theorem 1.1 in [17], we get the existence and uniqueness of the solution on $[r_k, r_{k+1}]$. Then, from Theorem 2.2 in [17] and Theorem 5.1 in [19], the functions $(v^{\mathcal{R}}, \tilde{v}^{\mathcal{R}})$ defined by

$$v^{\mathcal{R}}(t, x) = Y_t^{\mathcal{R}, t, x} \quad \text{and} \quad \tilde{v}^{\mathcal{R}}(t, x) = \tilde{Y}_t^{\mathcal{R}, t, x}, \quad (t, x) \in [r_k, r_{k+1}] \times \mathbb{R}^d,$$

are the unique viscosity solution to (3.17)-(3.18)-(3.19) with polynomial growth. From The uniqueness to Lipschitz BSDEs (see e.g. Theorem 1.1 in [17]) we get

$$Y_s^{\mathcal{R}, t, x} = v^{\mathcal{R}}(s, X_s^{t, x}) \quad \text{and} \quad \tilde{Y}_s^{\mathcal{R}, t, x} = \tilde{v}^{\mathcal{R}}(s, X_s^{t, x}), \quad s \in [t, T],$$

for $(t, x) \in [r_k, r_{k+1}] \times \mathbb{R}^d$.

From Proposition A.5 we also have $\tilde{Y}_{r_k}^{\mathcal{R}, t, x} \in \mathbf{D}_s$. By (3.13), $Y_{r_k}^{\mathcal{R}, t, x} \in \mathbf{D}_s$.

Step 2. Uniform space Lipschitz continuity. From the definition of the function $v^{\mathcal{R}}$, (3.13) and Proposition A.5, we get a backward induction on k that

$$|v^{\mathcal{R}}(t, x) - v^{\mathcal{R}}(t, x')| \leq L_k |x - x'|$$

for all $t \in [r_k, r_{k+1}]$ and $x, x' \in \mathbb{R}^d$ with

$$L_n = L_g$$

and

$$L_{k-1} = e^{C(r_k - r_{k-1})} (1 + (r_k - r_{k-1}))^{\frac{1}{2}} (L_k^2 + C(r_k - r_{k-1}))^{\frac{1}{2}}$$

for $k = 0, \dots, n-1$, where $C = 2L_{b,\sigma} + L_{b,\sigma}^2 + (L_f \vee 2)^2$. We therefore get

$$\begin{aligned}
L_k^2 &= L_g^2 \prod_{j=k}^{n-1} (1 + r_{j+1} - r_j) e^{2C(r_{j+1} - r_j)} \\
&\quad + \sum_{\ell=k}^{n-1} C(r_{\ell+1} - r_\ell) \prod_{j=k}^{\ell} e^{2C(r_{j+1} - r_j)} (1 + r_{j+1} - r_j) \\
&\leq L_g^2 \prod_{j=0}^{n-1} (1 + r_{j+1} - r_j) + CT e^{CT} \prod_{j=0}^{n-1} (1 + r_{j+1} - r_j) \\
&\leq \left(L_g^2 + CT e^{2CT} \right) \left(1 + \frac{T}{n} \right)^n
\end{aligned}$$

for $k = 0, \dots, n-1$. Since the sequence $\left(\left(1 + \frac{T}{n} \right)^n \right)_{n \geq 1}$ is bounded we get the space Lipschitz property uniform in the grid \mathcal{R} .

Step 3. Uniform time Hölder continuity. From the previous step and Proposition A.7, we get the Hölder regularity uniform in the grid \mathcal{R} . □

3.2 Convergence of the discretely constrained BSDE

We fix a sequence $(\mathcal{R}^n)_{n \geq 1}$ of grids of the time interval $[0, T]$ of the form

$$\mathcal{R}^n := \{ r_0^n = 0 < r_1^n < \dots < r_{\kappa_n}^n = T \}, \quad n \geq 1.$$

We suppose this sequence is nondecreasing, that means $\mathcal{R}^n \subset \mathcal{R}^{n+1}$ for $n \geq 1$, and

$$|\mathcal{R}^n| := \max_{1 \leq k \leq \kappa_n} (r_k^n - r_{k-1}^n) \xrightarrow{n \rightarrow +\infty} 0.$$

Theorem 3.1. *The sequences of functions $(\tilde{v}^{\mathcal{R}^n})_{n \geq 1}$ and $(v^{\mathcal{R}^n})_{n \geq 1}$ are nondecreasing and converges to v*

$$\lim_{n \rightarrow +\infty} v^{\mathcal{R}^n}(t, x) = \lim_{n \rightarrow +\infty} \tilde{v}^{\mathcal{R}^n}(t, x) = v(t, x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

To prove Theorem 3.1 we need the following Lemma.

Lemma 3.1. *Let $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally bounded function such that*

$$u(t, x) = F_{\mathcal{C}}[u(t, \cdot)](x) \tag{3.20}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Then u is a viscosity supersolution to

$$H(Du) = 0.$$

Proof. Fix $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^d$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that

$$0 = (u - \varphi)(\bar{t}, \bar{x}) = \min_{[0, T] \times \mathbb{R}^d} (u - \varphi)(t, x) .$$

From (3.20) we get

$$\varphi(\bar{t}, \bar{x}) = F_{\mathcal{C}}[\varphi(\bar{t}, \cdot)](\bar{x}) . \quad (3.21)$$

Fix $y \in \mathcal{C}$. From Taylor formula we have

$$\varphi(\bar{t}, \bar{x} + y) = \varphi(\bar{t}, \bar{x}) + \int_0^1 D\varphi(\bar{t}, s\bar{x} + (1-s)(\bar{x} + y)) \cdot y ds$$

Since $0 \in \mathcal{C}$ we have $\varepsilon y \in \mathcal{C}$ for any $\varepsilon \in (0, 1)$. Since $\delta_{\mathcal{C}}$ is positively homogeneous, we get by taking εy in place of y

$$\varepsilon \left(\delta_{\mathcal{C}}(y) - \int_0^1 D\varphi(\bar{t}, \bar{x} + (1-s)\varepsilon y) \cdot y ds \right) = \varphi(\bar{t}, \bar{x}) - (\varphi(\bar{t}, \bar{x} + \varepsilon y) - \delta_{\mathcal{C}}(\varepsilon y)) .$$

Then from (3.21) we get

$$\delta_{\mathcal{C}}(y) - \int_0^1 D\varphi(\bar{t}, \bar{x} + (1-s)\varepsilon y) \cdot y ds \geq 0$$

for all $\varepsilon > 0$. Since $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$, we can apply the dominated convergence theorem and we get by sending ε to 0

$$\delta_{\mathcal{C}}(y) - D\varphi(\bar{t}, \bar{x}) \cdot y \geq 0 .$$

Since y is arbitrarily chosen in \mathcal{C} we get

$$H(D\varphi(\bar{t}, \bar{x})) \geq 0 .$$

□

Proof of Theorem 3.1. Fix $(t, x) \in [0, T] \times \mathbb{R}^d$. Since the sequence of grids $(\mathcal{R}^n)_{n \geq 1}$ is nondecreasing and $\mathfrak{F}_{\mathcal{C}, s}[Y] \geq Y$ for any $Y \in \mathbf{D}_s$, using the comparison Theorem 2.2 in [11], we get by induction that the sequences $(Y^{\mathcal{R}^n, t, x})_{n \geq 1}$ and $(\tilde{Y}^{\mathcal{R}^n, t, x})_{n \geq 1}$ are nondecreasing. Therefore the sequences of functions $(\tilde{v}^{\mathcal{R}^n})_{n \geq 1}$ and $(v^{\mathcal{R}^n})_{n \geq 1}$ are nondecreasing and we can define the limits

$$\begin{aligned} w(t, x) &= \lim_{n \rightarrow +\infty} v^{\mathcal{R}^n}(t, x) \\ \tilde{w}(t, x) &= \lim_{n \rightarrow +\infty} \tilde{v}^{\mathcal{R}^n}(t, x) \end{aligned}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. We proceed in four steps to prove that $v = w = \tilde{w}$.

Step 1. We have $\tilde{w} = w \leq v$. Still using the comparison Theorem 2.2 in [11] we get by induction

$$w(t, x) \geq \tilde{w}(t, x) , \quad (t, x) \in [0, T] \times \mathbb{R}^d .$$

Moreover, we get from Proposition 2.1 that $(Y^{t,x}\mathbf{1}_{[t,T]} + F_C[g](X_T^{t,x})\mathbf{1}_{\{T\}}, Z^{t,x})$ is a continuous supersolution to (3.15) on each interval $[r_k^n, r_{k+1}^n] \cap [t, T]$. Therefore, using Remark b. of Section 2.3 in [11], we get by induction $Y^{t,x} \geq Y^{\mathcal{R}^n, t, x}$ for all $n \geq 1$. Hence

$$v(t, x) \geq w(t, x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. We now prove $w = \tilde{w}$. Fix $n \geq 1$, $k \in \{0, \dots, \kappa_n - 1\}$, $t \in [r_k^n, r_{k+1}^n)$ and $x \in \mathbb{R}^d$. We have

$$\begin{aligned} |v^{\mathcal{R}^n}(t, x) - \tilde{v}^{\mathcal{R}^n}(t, x)| &\leq |v^{\mathcal{R}^n}(t, x) - v^{\mathcal{R}^n}(r_{k+1}^n, x)| + |v^{\mathcal{R}^n}(r_{k+1}^n, x) - \tilde{v}^{\mathcal{R}^n}(r_k^n, x)| \\ &\quad + |\tilde{v}^{\mathcal{R}^n}(r_k^n, x) - \tilde{v}^{\mathcal{R}^n}(t, x)|. \end{aligned}$$

From Proposition 3.3 (iii) we get

$$|v^{\mathcal{R}^n}(t, x) - \tilde{v}^{\mathcal{R}^n}(t, x)| \leq 2L\sqrt{|\mathcal{R}^n|} + |v^{\mathcal{R}^n}(r_{k+1}^n, x) - \tilde{v}^{\mathcal{R}^n}(r_k^n, x)|.$$

Since $v^{\mathcal{R}^n}$ coincides with $\tilde{v}^{\mathcal{R}^n}$ out of the grid \mathcal{R}^n we have

$$\begin{aligned} |v^{\mathcal{R}^n}(r_{k+1}^n, x) - \tilde{v}^{\mathcal{R}^n}(r_k^n, x)| &\leq |v^{\mathcal{R}^n}(r_{k+1}^n, x) - v^{\mathcal{R}^n}(\frac{r_k^n + r_{k+1}^n}{2}, x)| \\ &\quad + |\tilde{v}^{\mathcal{R}^n}(\frac{r_k^n + r_{k+1}^n}{2}, x) - \tilde{v}^{\mathcal{R}^n}(r_k^n, x)|. \end{aligned}$$

Still using Proposition 3.3 (iii) we get

$$|v^{\mathcal{R}^n}(r_{k+1}^n, x) - \tilde{v}^{\mathcal{R}^n}(r_k^n, x)| \leq 2L\sqrt{|\mathcal{R}^n|}$$

and

$$|v^{\mathcal{R}^n} - \tilde{v}^{\mathcal{R}^n}| \leq 4L\sqrt{|\mathcal{R}^n|} \xrightarrow{n \rightarrow +\infty} 0.$$

Step 2. The function w satisfies

$$w(t, x) = F_C[w(t, \cdot)](x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. We first prove

$$\lim_{n \rightarrow +\infty} F_C[v^{\mathcal{R}^n}(t, \cdot)](x) - v^{\mathcal{R}^n}(t, x) = 0.$$

Fix $n \geq 1$. If $t \in \mathcal{R}^n$, then $F_C[v^{\mathcal{R}^n}(t, \cdot)] - v^{\mathcal{R}^n}(t, \cdot) = 0$ from (2.6). Fix now $k \in \{0, \dots, \kappa_n - 1\}$ and $t \in (r_k^n, r_{k+1}^n)$. Then, still using (2.6), we have $v^{\mathcal{R}^n}(r_{k+1}^n, \cdot) = F_C[v^{\mathcal{R}^n}(r_{k+1}^n, \cdot)]$. Therefore we get

$$\begin{aligned} |F_C[v^{\mathcal{R}^n}(t, \cdot)] - v^{\mathcal{R}^n}(t, \cdot)| &\leq |F_C[v^{\mathcal{R}^n}(r_{k+1}^n, \cdot)](\cdot) - F_C[v^{\mathcal{R}^n}(t, \cdot)]| \\ &\quad + |v^{\mathcal{R}^n}(r_{k+1}^n, \cdot) - v^{\mathcal{R}^n}(t, \cdot)| \\ &\leq 2 \sup_{x \in \mathbb{R}^d} |v^{\mathcal{R}^n}(r_{k+1}^n, x) - v^{\mathcal{R}^n}(t, x)|. \end{aligned}$$

We deduce from Proposition 3.3 (ii) that

$$\sup_{x \in \mathbb{R}} |F_C[v^{\mathcal{R}^n}(t, \cdot)](x) - v^{\mathcal{R}^n}(t, x)| \leq 2L\sqrt{|\mathcal{R}^n|} \xrightarrow{n \rightarrow +\infty} 0.$$

Then we have

$$\begin{aligned} 0 \leq F_C[w(t, \cdot)](x) - w(t, x) &= F_C\left[\lim_{n \rightarrow +\infty} v^{\mathcal{R}^n}(t, \cdot)\right](x) - \lim_{n \rightarrow +\infty} v^{\mathcal{R}^n}(t, x) \\ &\leq \lim_{n \rightarrow +\infty} F_C[v^{\mathcal{R}^n}(t, \cdot)](x) - v^{\mathcal{R}^n}(t, x) = 0. \end{aligned}$$

Step 3. The function w is a viscosity supersolution to

$$\begin{cases} -\partial_t w(t, x) - \mathcal{L}w(t, x) \\ -f(t, x, w(t, x), \sigma(t, x)Dw(t, x)) = 0, & (t, x) \in [0, T] \times \mathbb{R}^d. \\ w(T, x) = g(x), & x \in \mathbb{R}^d, \end{cases} \quad (3.22)$$

We first prove that $v^{\mathcal{R}^n}$ is a viscosity supersolution to (3.22) for any $n \geq 1$. Fix $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^d$ and $n \geq 1$. If $\bar{t} = T$ then we have $v^{\mathcal{R}^n}(\bar{t}, \bar{x}) \geq g(\bar{x})$. If $\bar{t} \notin \mathcal{R}^n$ we deduce the viscosity supersolution property from (3.18). Suppose now that $\bar{t} = r_k$ for some $k = 0, \dots, n-1$. Fix $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that

$$0 = (v_*^{\mathcal{R}^n} - \varphi)(\bar{t}, \bar{x}) = \min_{[0, T] \times \mathbb{R}^d} (v_*^{\mathcal{R}^n} - \varphi).$$

We observe that the lsc envelope $v_*^{\mathcal{R}^n}$ of $v^{\mathcal{R}^n}$ is the function $\tilde{v}^{\mathcal{R}^n}$. We then have

$$0 = (\tilde{v}^{\mathcal{R}^n} - \varphi)(\bar{t}, \bar{x}) = \min_{[r_k^n, r_{k+1}^n] \times \mathbb{R}^d} (\tilde{v}^{\mathcal{R}^n} - \varphi).$$

From the viscosity property of $\tilde{v}^{\mathcal{R}^n}$, we deduce that

$$\partial_t \varphi(\bar{t}, \bar{x}) - \mathcal{L}\varphi(\bar{t}, \bar{x}) - f(\bar{t}, \bar{x}, \varphi(\bar{t}, \bar{x}), \sigma(\bar{t}, \bar{x})D\varphi(\bar{t}, \bar{x})) \geq 0$$

and $v^{\mathcal{R}}$ is a viscosity supersolution. We now turn to w . Since $v^{\mathcal{R}^n} \uparrow w$ as $n \uparrow +\infty$, we can apply stability results for semi-linear PDEs (see e.g. Theorem 4.1 in [1]) and we get the viscosity supersolution property of w .

Step 4. We have $w = v$. In view of Step 1, it suffices to prove that $w \geq v$. From Lemma 3.1 and Step 2, w is a viscosity supersolution to $H(Dw) \geq 0$. Then from Step 3, we deduce that w is a viscosity supersolution to (2.11). By Theorem 4.4.5 in [21] we get $w \geq v_n$ for all $n \geq 1$ and hence $w \geq v$ from Proposition 2.2 (iii). \square

Corollary 3.1. *We have the following uniform convergence*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times Q} |v^{\mathcal{R}^n}(t, x) - v(t, x)| &= \\ \lim_{n \rightarrow +\infty} \sup_{(t, x) \in [0, T] \times Q} |\tilde{v}^{\mathcal{R}^n}(t, x) - v(t, x)| &= 0 \end{aligned}$$

for every compact subset Q of \mathbb{R}^d .

Proof. We first define the function \hat{v} by

$$\hat{v}(t, x) = v(t, x)\mathbb{1}_{[0, T)}(t) + F_C[g](x)\mathbb{1}_{\{T\}}(t), \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

From Proposition 2.1, \hat{v} is continuous on $[0, T] \times \mathbb{R}^d$. Fix a compact Q of \mathbb{R}^d . Using Dini's Theorem we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup_{x \in Q} |v^{\mathcal{R}^n}(t, x) - \hat{v}(t, x)| &= \\ \lim_{n \rightarrow +\infty} \sup_{x \in Q} |\tilde{v}^{\mathcal{R}^n}(t, x) - \hat{v}(t, x)| &= 0 \end{aligned}$$

for every $t \in [0, T]$. In particular, if we define for $n \geq 1$ the functions $\Phi_n : [0, T] \rightarrow \mathbb{R}$ by

$$\Phi_n(t) = \sup_{x \in Q} |v^{\mathcal{R}^n}(t, x) - \hat{v}(t, x)|, \quad t \in [0, T],$$

then $(\Phi_n)_{n \geq 1}$ is a nonincreasing sequence of *càdlàg* functions such that

$$\lim_{n \rightarrow +\infty} \Phi_n(t) = 0$$

for all $t \in [0, T]$ and

$$\lim_{n \rightarrow +\infty} \Phi_n(t^-) = \lim_{n \rightarrow +\infty} \sup_{x \in Q} |\tilde{v}^{\mathcal{R}^n}(t, x) - \hat{v}(t, x)| = 0$$

for all $t \in (0, T]$. We then apply Dini's Theorem for *càdlàg* functions (see the Lemma in the proof of Theorem 2 Chapter VII Section 1 in [10]) and we get the uniform convergence of $(\Phi_n)_{n \geq 1}$ to 0. Since \hat{v} coincides with v on $[0, T) \times \mathbb{R}^d$, we get the desired result. \square

Corollary 3.2. *We have the following convergence result*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{[t, T)} |Y^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 \right] + \mathbb{E} \left[\sup_{[t, T)} |\tilde{Y}^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 \right] \\ + \mathbb{E} \left[\int_t^T |Z_s^{\mathcal{R}^n, t, x} - Z_s^{t, x}|^2 ds \right] = 0, \end{aligned}$$

for all $(t, x) \in [0, T) \times \mathbb{R}^d$.

Proof. We first write

$$\begin{aligned} \sup_{s \in [t, T)} |Y_s^{\mathcal{R}^n, t, x} - Y_s^{t, x}|^2 &= \sup_{s \in [t, T)} |v^{\mathcal{R}^n}(s, X_s^{t, x}) - v(s, X_s^{t, x})|^2, \\ \sup_{s \in [t, T)} |\tilde{Y}_s^{\mathcal{R}^n, t, x} - Y_s^{t, x}|^2 &= \sup_{s \in [t, T)} |\tilde{v}^{\mathcal{R}^n}(s, X_s^{t, x}) - v(s, X_s^{t, x})|^2. \end{aligned}$$

Since X has continuous paths, we get from Theorem 3.1

$$\lim_{n \rightarrow +\infty} \sup_{[t, T)} |Y^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 + \sup_{[t, T)} |\tilde{Y}^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 = 0, \quad \mathbb{P} - a.s.$$

By Lebesgue dominated convergence Theorem we get

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{[t, T)} |Y^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 \right] + \mathbb{E} \left[\sup_{[t, T)} |\tilde{Y}^{\mathcal{R}^n, t, x} - Y^{t, x}|^2 \right] = 0.$$

By classical estimates on BSDEs based on BDG and Young inequalities and Gronwall Lemma, we deduce

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\int_t^T |Z_s^{\mathcal{R}^n, t, x} - Z_s^{t, x}|^2 ds \right] = 0 .$$

□

4 Neural network approximation of the discretely constrained BSDE

4.1 Neural networks and approximation of the facelift

We first recall the definition of a neural network with single hidden layer. To this end, we fix a function $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$ called the activation function, and an integer $m \geq 1$, representing the number of neurons (also called nodes) on the hidden layer.

Definition 4.1. *The set \mathfrak{NN}_m^ρ of feedforward neural network with single hidden layer with m neurons and the activation function ρ is the set of functions*

$$x \in \mathbb{R}^d \mapsto \sum_{i=1}^m \lambda_i \rho(\alpha_i \cdot x) \in \mathbb{R} ,$$

where $\lambda_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}^d$, for $i = 1, \dots, m$.

For $m \geq 1$ we define the set Θ_m by

$$\Theta_m := \left\{ (\lambda_i, \alpha_i)_{i=1, \dots, m} : \lambda_i \in \mathbb{R} \text{ and } \alpha_i \in \mathbb{R}^d \text{ for } i = 1, \dots, m \right\} .$$

For $\theta = (\lambda_i, \alpha_i)_{i=1, \dots, m} \in \Theta_m$, we denote by NN^θ the function from \mathbb{R}^d to \mathbb{R} defined by

$$NN^\theta(x) = \sum_{i=1}^m \lambda_i \rho(\alpha_i \cdot x) \in \mathbb{R} , \quad x \in \mathbb{R}^d .$$

We also define the set \mathfrak{NN}^ρ by

$$\mathfrak{NN}^\rho := \bigcup_{m \geq 1} \mathfrak{NN}_m^\rho .$$

We suppose in the sequel that ρ is not identically equal to 0, belongs to $C^1(\mathbb{R}, \mathbb{R})$ and satisfies $\int_{\mathbb{R}} |\rho'(x)| dx < +\infty$. We denote by $C_b^1(\mathbb{R}^d, \mathbb{R})$ the set functions in $C_b^1(\mathbb{R}^d, \mathbb{R})$ with bounded derivative. We then have the following result from [13].

Theorem 4.2. *\mathfrak{NN}^ρ is dense in $C_b^1(\mathbb{R}^d, \mathbb{R})$ for the topology of uniform convergence on compact sets: for any $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$ and for any compact Q of \mathbb{R}^d , there exists a sequence $(NN^{\theta_\ell})_{\ell \geq 1}$ of \mathfrak{NN}^ρ such that*

$$\sup_{x \in Q} |NN^{\theta_\ell}(x) - f(x)| + \sup_{x \in Q} |DNN^{\theta_\ell}(x) - Df(x)| \xrightarrow{\ell \rightarrow +\infty} 0 .$$

We turn to the facelift approximation by feedforward neural networks. We fix bounded and Lipschitz continuous functions φ and φ_ℓ , $\ell \geq 1$, from \mathbb{R}^d to \mathbb{R} and a random variable ξ . For $\varepsilon > 0$, we define the sequence of parameters $(\theta_{m,\varepsilon,\ell}^*)_{m,\varepsilon,\ell}$ by

$$\begin{aligned} \theta_{m,\varepsilon,\ell}^* &\in \arg \min_{\theta \in \Theta_m} \mathbb{E} \left[|(NN^\theta - \varphi_\ell)(\xi)|^2 \mathbf{1}_{B_\varepsilon}(\xi) \right] \\ \theta &\text{ s.t. } \mathbb{P}(DNN^\theta(\xi) \in \mathcal{C}_\varepsilon; (NN^\theta - \varphi_\ell)(\xi) \geq -\varepsilon \mid \xi \in B_\varepsilon) = 1 \end{aligned} \quad (4.23)$$

where DNN^θ denotes the gradient of NN^θ , \mathcal{C}_ε stands for the closed convex set defined by

$$\mathcal{C}_\varepsilon = \left\{ y \in \mathbb{R}^d : \exists x \in \mathcal{C}, |x - y| \leq \varepsilon \right\},$$

and B_ε stands for the ball $B(0, \frac{1}{\varepsilon})$.

Proposition 4.4. *Suppose $\text{Supp}(\mathbb{P}_\xi) = \mathbb{R}^d$ and that $\mathbb{E}[|\xi|^2] < +\infty$. Then, if φ_ℓ converges uniformly to φ on compact sets, we have*

$$\lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow +\infty} \lim_{\ell \rightarrow +\infty} \mathbb{E} \left[|(NN^{\theta_{m,\varepsilon,\ell}^*} - F_{\mathcal{C}}[\varphi])(\xi)|^2 \mathbf{1}_{B_\varepsilon}(\xi) \right] = 0. \quad (4.24)$$

Moreover, we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow +\infty} \lim_{\ell \rightarrow +\infty} \mathbb{E} \left[|(NN^{\theta_{m,\varepsilon,\ell}^*} \vee M \wedge (-M) - F_{\mathcal{C}}[\varphi])(\xi)|^2 \right] = 0. \quad (4.25)$$

for any constant $M > 0$ such that $|\varphi| \leq M$.

To prove this theorem we need the following Lemma.

Lemma 4.2. *Let X and $(X_n)_{n \geq 1}$ be positive integrable random variables such that*

$$\liminf_{n \rightarrow +\infty} X_n \geq X \geq 0, \quad (4.26)$$

and

$$\limsup_{n \rightarrow +\infty} \mathbb{E}[X_n] \leq \mathbb{E}[X]. \quad (4.27)$$

Then, we have

$$X_n \xrightarrow[\mathbb{P}\text{-p.s.}]{n \rightarrow +\infty} X.$$

Proof. We argue by contradiction. Suppose the \mathbb{P} -a.s. convergence of X_n to X does not hold. Then from (4.26), there exists some $\eta > 0$ and $\Omega_\eta \subset \Omega$ such that $\mathbb{P}(\Omega_\eta) > 0$ and

$$\liminf_{n \rightarrow +\infty} X_n \geq X + \eta \quad \text{on } \Omega_\eta. \quad (4.28)$$

From Fatou's Lemma and (4.27) we get

$$\mathbb{E}[X] \geq \mathbb{E}[\liminf_{n \rightarrow +\infty} X_n] \geq \mathbb{E}[X] + \eta \mathbb{P}(\Omega_\eta)$$

which contradicts $\mathbb{P}(\Omega_\eta) > 0$. □

Proof of Proposition 4.4. Step 1. We prove that for any $\varepsilon > 0$, there exists a sequence $(\theta'_{m,\varepsilon})_{m \geq 1}$ such that $\theta'_{m,\varepsilon} \in \Theta_m$ for $m \geq 1$, and

$$\limsup_{m \rightarrow +\infty} \mathbb{E} \left[|NN^{\theta'_{m,\varepsilon}}(\xi) - F_{\mathcal{C}}[\varphi](\xi)|^2 \mid \xi \in B_\varepsilon \right] \leq \varepsilon ,$$

and

$$\mathbb{P} \left(DNN^{\theta'_{m,\varepsilon}} \in \mathcal{C}_\varepsilon; (NN^{\theta'_{m,\varepsilon}} - \varphi_\ell)(\xi) \geq \varepsilon \mid \xi \in B_\varepsilon \right) = 1 ,$$

for ℓ and m large enough.

To this end, we introduce the sequence of mollifiers $\psi_n : \mathbb{R}^d \rightarrow \mathbb{R}_+$, $n \geq 1$, defined by

$$\psi_n(x) := n^d \psi(nx) , \quad x \in \mathbb{R} ,$$

where the function $\psi \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ has a compact support and is such that $\int_{\mathbb{R}^d} \psi(u) du = 1$. We then define the functions ϕ_n , $n \geq 1$, by

$$\phi_n(x) := \int_{\mathbb{R}^d} \psi_n(y) F_{\mathcal{C}}[\varphi](x - y) dy , \quad x \in \mathbb{R} .$$

Since φ is Lipschitz continuous and bounded, $F_{\mathcal{C}}[\varphi]$ is also Lipschitz continuous and bounded. From classical results, we know that ϕ_n converges to $F_{\mathcal{C}}[\varphi]$ as n goes to infinity uniformly on every compact subset of \mathbb{R} . Moreover, $\phi_n \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$. Since $F_{\mathcal{C}}[\varphi]$ is Lipschitz continuous it is almost everywhere differentiable by Rademacher Theorem and we get from the dominated convergence Theorem

$$D\phi_n(x) = \int_{\mathbb{R}^d} \psi_n(y) DF_{\mathcal{C}}[\varphi](x - y) dy .$$

From Lemma 3.1, we have $DF_{\mathcal{C}}[\varphi] \in \mathcal{C}$ almost everywhere on \mathbb{R}^d . Since \mathcal{C} is convex, we get

$$D\phi_n(x) \in \mathcal{C} \tag{4.29}$$

for all $x \in \mathbb{R}^d$ and all $n \geq 1$. Fix now $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$\sup_{x \in B_\varepsilon} |F_{\mathcal{C}}[\varphi](x) - \phi_{n_\varepsilon}(x)| \leq \frac{\varepsilon}{3} . \tag{4.30}$$

From Theorem 4.2, there exists a sequence $(\theta'_{m,\varepsilon})_{m \geq 1}$ such that

$$\sup_{B_\varepsilon} |NN^{\theta'_{m,\varepsilon}} - \phi_{n_\varepsilon}| + \sup_{B_\varepsilon} |DNN^{\theta'_{m,\varepsilon}} - D\phi_{n_\varepsilon}| \xrightarrow{m \rightarrow +\infty} 0 . \tag{4.31}$$

We therefore get from (4.29), (4.30) and (4.31)

$$\mathbb{P} \left(DNN^{\theta'_{m,\varepsilon}} \in \mathcal{C}_\varepsilon; (NN^{\theta'_{m,\varepsilon}} - \varphi)(\xi) \geq \frac{2\varepsilon}{3} \mid \xi \in B_\varepsilon \right) = 1 ,$$

for m large enough. From the local uniform convergence of φ_ℓ to φ , we get

$$\mathbb{P} \left(DNN^{\theta'_{m,\varepsilon}} \in \mathcal{C}_\varepsilon; (NN^{\theta'_{m,\varepsilon}} - \varphi_\ell)(\xi) \geq \varepsilon \mid \xi \in B_\varepsilon \right) = 1 ,$$

for ℓ large enough. Moreover, we have from (4.30) and (4.31)

$$\begin{aligned} \limsup_{m \rightarrow +\infty} \mathbb{E} \left[|NN^{\theta'_{m,\varepsilon}}(\xi) - F_{\mathcal{C}}[\varphi](\xi)|^2 \mid \xi \in B_\varepsilon \right] &\leq 2 \limsup_{m \rightarrow +\infty} \mathbb{E} \left[|NN^{\theta'_{m,\varepsilon}}(\xi) - \phi_{n_\varepsilon}(\xi)|^2 \mid \xi \in B_\varepsilon \right] \\ &\quad + 2\mathbb{E} \left[|\phi_{n_\varepsilon}(\xi) - F_{\mathcal{C}}[\varphi](\xi)|^2 \mid \xi \in B_\varepsilon \right] \\ &\leq \frac{\varepsilon^2}{2}. \end{aligned}$$

Step 2. From the definition (4.23) of $\theta_{m,\varepsilon,\ell}^*$ we get

$$\mathbb{E} \left[|(NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi_\ell)(\xi)|^2 \mid \xi \in B_\varepsilon \right] \leq \mathbb{E} \left[|(NN^{\theta'_{m,\varepsilon}} - \varphi_\ell)(\xi)|^2 \mid \xi \in B_\varepsilon \right].$$

By sending ℓ and m to ∞ , we get from Step 1

$$\begin{aligned} \limsup_{m \rightarrow +\infty} \limsup_{\ell \rightarrow +\infty} \mathbb{E} \left[|(NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi_\ell)(\xi)|^2 \mid \xi \in B_\varepsilon \right] &\leq \\ &\left(\mathbb{E} \left[|(F_{\mathcal{C}}[\varphi] - \varphi)(\xi)|^2 \mid \xi \in B_\varepsilon \right]^{\frac{1}{2}} + \sqrt{\frac{\varepsilon^2}{2}} \right)^2. \end{aligned} \quad (4.32)$$

Hence, we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow +\infty} \limsup_{\ell \rightarrow +\infty} \mathbb{E} \left[|(NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi_\ell)(\xi)|^2 \mid \xi \in B_\varepsilon \right] \leq \mathbb{E} \left[|(F_{\mathcal{C}}[\varphi] - \varphi)(\xi)|^2 \right]. \quad (4.33)$$

We now define the local facelift operator $F_{\mathcal{C}_\varepsilon}^\varepsilon$ by

$$F_{\mathcal{C}_\varepsilon}^\varepsilon[\phi](x) = \sup_{y \in \mathbb{R}^d : x+y \in B_\varepsilon} \{ \phi(x+y) - \delta_{\mathcal{C}_\varepsilon}(y) \}$$

for a locally bounded function ϕ and $x \in B_\varepsilon$. We observe that

$$NN^{\theta_{m,\varepsilon,\ell}^*} \geq F_{\mathcal{C}_\varepsilon}^\varepsilon[\varphi_\ell] - \varepsilon \quad \text{on } B_\varepsilon. \quad (4.34)$$

Indeed, from Taylor's formula and since $DN^{\theta_{m,\varepsilon,\ell}^*} \in \mathcal{C}_\varepsilon$ on B_ε we first have

$$NN^{\theta_{m,\varepsilon,\ell}^*}(x) - (NN^{\theta_{m,\varepsilon,\ell}^*}(x+y) - \delta_{\mathcal{C}_\varepsilon}(y)) = \int_0^1 (\delta_{\mathcal{C}_\varepsilon}(y) - DN^{\theta_{m,\varepsilon,\ell}^*}(x+sy) \cdot y) ds \geq 0$$

for all $x \in B_\varepsilon$ and $y \in \mathbb{R}^d$ such that $x+y \in B_\varepsilon$. Therefore

$$NN^{\theta_{m,\varepsilon,\ell}^*} = F_{\mathcal{C}_\varepsilon}^\varepsilon[NN^{\theta_{m,\varepsilon,\ell}^*}] \quad \text{on } B_\varepsilon.$$

Since $(NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi_\ell)(\xi) \geq -\varepsilon$ on B_ε we have

$$NN^{\theta_{m,\varepsilon,\ell}^*} = F_{\mathcal{C}_\varepsilon}^\varepsilon[NN^{\theta_{m,\varepsilon,\ell}^*}] \geq F_{\mathcal{C}_\varepsilon}^\varepsilon[\varphi_\ell] - \varepsilon \quad \text{on } B_\varepsilon. \quad (4.35)$$

From the uniform convergence of φ_ℓ to φ on compact sets, we have

$$(F_{\mathcal{C}_\varepsilon}^\varepsilon[\varphi_\ell] - F_{\mathcal{C}}[\varphi])(\xi) \mathbf{1}_{B_\varepsilon}(\xi) \xrightarrow[\varepsilon \rightarrow 0, \ell \rightarrow +\infty]{\mathbb{P}\text{-a.s.}} 0. \quad (4.36)$$

Therefore, we get

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{m \rightarrow +\infty} \liminf_{\ell \rightarrow +\infty} (NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi)(\xi) \mathbb{1}_{B_\varepsilon}(\xi) \geq (F_{\mathcal{C}}[\varphi] - \varphi)(\xi) \geq 0,$$

and

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{m \rightarrow +\infty} \liminf_{\ell \rightarrow +\infty} (NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi)^2(\xi) \mathbb{1}_{B_\varepsilon}(\xi) \geq (F_{\mathcal{C}}[\varphi] - \varphi)^2(\xi). \quad (4.37)$$

From (4.33), (4.37) and Lemma 4.2 we get

$$(NN^{\theta_{m,\varepsilon,\ell}^*} - \varphi)^2(\xi) \mathbb{1}_{B_\varepsilon}(\xi) \xrightarrow[\varepsilon \rightarrow 0, m \rightarrow +\infty, \ell \rightarrow +\infty]{\mathbb{P}\text{-p.s.}} (F_{\mathcal{C}}[\varphi] - \varphi)^2(\xi).$$

We deduce from (4.35) and (4.36)

$$(NN^{\theta_{m,\varepsilon,\ell}^*} - F_{\mathcal{C}}[\varphi])(\xi) \mathbb{1}_{B_\varepsilon}(\xi) \xrightarrow[\varepsilon \rightarrow 0, m \rightarrow +\infty, \ell \rightarrow +\infty]{\mathbb{P}\text{-p.s.}} 0.$$

We then notice that since $NN^{\theta_{m,\varepsilon,\ell}^*} \in \mathcal{C}_\varepsilon$ on B_ε and \mathcal{C} is bounded, the family $NN^{\theta_{m,\varepsilon,\ell}^*}$ satisfies a uniform linear growth property for ε in the neighborhood of 0^+ . Since $F_{\mathcal{C}}[\varphi]$ is bounded and $\mathbb{E}[|\xi|^2] < +\infty$, we can apply the dominated convergence Theorem and we get (4.24). The last result (4.4) is a consequence of (4.24), the square integrability of ξ and the bound $|F_{\mathcal{C}}[\varphi]| \leq M$. \square

4.2 The approximation scheme

We fix an initial condition X_0 at time $t = 0$ for the diffusion and we write X for X^{0,X_0} . We first fix two time grids

- a constraint grid $\mathcal{R} = \{r_0 = 0 < r_1 < \dots < r_\kappa = T\}$,
- a family of grids $\pi = \{\pi_k, k = 0, \dots, \kappa - 1\}$ where π_k is a grid of $[r_k, r_{k+1}]$ of the form $\pi_k = \{t_{k,0} = r_k < \dots < t_{k,n_k} = r_{k+1}\}$. We set $|\pi_k| = \max_{i=0, \dots, n_k-1} (t_{k,i+1} - t_{k,i})$.

We denote by X^π the Euler scheme of X related to the grid π . It is defined by $X_0^\pi = X_0$ and

$$\begin{cases} X_{t_{k,i+1}}^\pi &= X_{t_{k,i}}^\pi + b(t_{k,i}, X_{t_{k,i}}^\pi) \Delta t_{k,i} + \sigma(t_{k,i}, X_{t_{k,i}}^\pi) \Delta B_{t_{k,i}} \\ X_{t_{k+1,0}}^\pi &= X_{t_{k,n_k}}^\pi \end{cases}$$

with $\Delta t_{k,i} = t_{k,i+1} - t_{k,i}$ and $\Delta B_{t_{k,i}} = B_{t_{k,i+1}} - B_{t_{k,i}}$ for $k = 0, \dots, \kappa - 1$ and $i = 0, \dots, n_k - 1$. We then introduce the function $F : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$F(t, x, y, z, h, \Delta) := y - f(t, x, y, z)h + z \cdot \Delta$$

for $(t, x, y, z, h, \Delta) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d$. We fix two multi-parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_\kappa)$ and $m = (m_0^1, m_0^2, m_0^3, m_1^1, m_1^2, m_1^3, \dots, m_{\kappa-1}^1, m_{\kappa-1}^2, m_{\kappa-1}^3, m_\kappa^1)$ and two positive constants M and L . We define $\{\mathcal{V}_{k,i}^{\mathcal{R}, \pi, \varepsilon, m}\}_{0 \leq i \leq n_k}^{0 \leq k \leq \kappa-1}$ and $(\tilde{\mathcal{V}}_{k,i}^{\mathcal{R}, \pi, \varepsilon, m})_{0 \leq i \leq n_k}^{0 \leq k \leq \kappa-1}$ by

the following algorithm.

Algorithm 1: Global approximation scheme.

$$\mathcal{V}_{\kappa,0}^{\mathcal{R},\pi,\varepsilon,m} = \tilde{\mathcal{V}}_{\kappa,0}^{\mathcal{R},\pi,\varepsilon,m} = NN^{\theta_{\kappa,0}^*} \wedge (-M) \vee M$$

where

$$\begin{aligned} \theta_{\kappa,0}^* &\in \arg \min \mathbb{E} \left[\left| NN^{\theta}(X_T^{\pi}) - g(X_T^{\pi}) \right|^2 \mid X_T^{\pi} \in B_{\varepsilon_{\kappa}} \right] \\ \theta &\in \Theta_{m_{\kappa}^1} \text{ s.t. } \mathbb{P} \left(DNN^{\theta}(X_T^{\pi}) \in \mathcal{C}_{\varepsilon_{\kappa}}; NN^{\theta}(X_T^{\pi}) \geq g(X_T^{\pi}) - \varepsilon_{\kappa} \mid X_T^{\pi} \in B_{\varepsilon_{\kappa}} \right) = 1. \end{aligned}$$

for $k = \kappa - 1, \dots, 0$ **do**

$$\mathcal{V}_{k,n_k}^{\mathcal{R},\pi,\varepsilon,m} = \mathcal{V}_{k+1,0}^{\mathcal{R},\pi,\varepsilon,m} \quad \text{and} \quad \tilde{\mathcal{V}}_{k,n_k}^{\mathcal{R},\pi,\varepsilon,m} = \tilde{\mathcal{V}}_{k+1,0}^{\mathcal{R},\pi,\varepsilon,m}.$$

for $i = n_k - 1, \dots, 1$ **do**

$$\tilde{\mathcal{V}}_{k,i}^{\mathcal{R},\pi,\varepsilon,m} = \mathcal{V}_{k,i}^{\mathcal{R},\pi,\varepsilon,m} = NN^{\theta_{k,i}^*}$$

where

$$\begin{aligned} (\theta_{k,i}^*, \hat{\theta}_{k,i}^*) &\in \arg \min_{(\theta, \hat{\theta}) \in \Theta_{m_k^3} \times \Theta_{m_k^3}^d} \mathbb{E} \left[\left| NN^{\theta_{k,i+1}^*}(X_{t_{k,i+1}}^{\pi}) \right. \right. \\ &\quad \left. \left. - F(t_{k,i}, X_{t_{k,i}}^{\pi}, NN^{\theta}(X_{t_{k,i}}^{\pi}), NN^{\hat{\theta}}(X_{t_{k,i}}^{\pi}), \Delta t_{k,i}, \Delta B_{t_{k,i}}) \right|^2 \right]. \end{aligned}$$

end

$$\mathcal{V}_{k,0}^{\mathcal{R},\pi,\varepsilon,m} = NN^{\theta_{k,0}^*} \wedge (-M) \vee M \quad \text{and} \quad \tilde{\mathcal{V}}_{k,0}^{\mathcal{R},\pi,\varepsilon,m} = NN^{\tilde{\theta}_{k,0}^*}$$

where

$$\begin{aligned} \theta_{k,0}^* &\in \arg \min \mathbb{E} \left[\left| NN^{\theta}(X_{t_{k,0}}^{\pi}) - NN^{\tilde{\theta}_{k,0}^*}(X_{t_{k,0}}^{\pi}) \wedge (-M) \vee M \right|^2 \mid X_{t_{k,0}}^{\pi} \in B_{\varepsilon_k} \right] \\ \theta &\in \Theta_{m_k^1} \text{ s.t. } \mathbb{P} \left(DNN^{\theta}(X_{t_{k,0}}^{\pi}) \in \mathcal{C}_{\varepsilon_k}; \right. \\ &\quad \left. NN^{\theta}(X_{t_{k,0}}^{\pi}) \geq NN^{\tilde{\theta}_{k,0}^*}(X_{t_{k,0}}^{\pi}) \wedge (-M) \vee M - \varepsilon_k \mid X_{t_{k,0}}^{\pi} \in B_{\varepsilon_k} \right) = 1, \\ \tilde{\theta}_{k,0}^* &\in \arg \min \mathbb{E} \left[\left| NN^{\theta}(X_{t_{k,0}}^{\pi}) \wedge (-M) \vee M - NN^{\tilde{\theta}_{k,0}^*}(X_{t_{k,0}}^{\pi}) \right|^2 \mid X_{t_{k,0}}^{\pi} \in B_{\varepsilon_k} \right] \\ \theta &= (\lambda_i, \alpha_i)_{1 \leq i \leq m_k^2} \in \Theta_{m_k^2} \text{ s.t. } \left| \sum_{i=1}^{m_k^2} \lambda_i \alpha_i \right| \leq \frac{L+1}{|\rho'(0)|} \\ \tilde{\theta}_{k,0}^* &\in \arg \min_{(\theta, \hat{\theta}) \in \Theta_{m_k^3} \times \Theta_{m_k^3}^d} \mathbb{E} \left[\left| NN^{\theta_{k,1}^*}(X_{t_{k,1}}^{\pi}) \right. \right. \\ &\quad \left. \left. - F(t_{k,0}, X_{t_{k,0}}^{\pi}, NN^{\theta}(X_{t_{k,0}}^{\pi}), NN^{\hat{\theta}}(X_{t_{k,0}}^{\pi}), \Delta t_{k,0}, \Delta B_{t_{k,0}}) \right|^2 \right]. \end{aligned}$$

end

We choose the constants M and L such that the functions $v^{\mathcal{R}}$ are L -Lipschitz continuous and bounded by M . We recall that such constants exist from Proposition 3.3.

The sequences $\{\mathcal{V}_{k,i}^{\mathcal{R},\pi,\varepsilon,m}(X_{t_{k,i}}^{\pi})\}_{0 \leq i \leq n_k}^{0 \leq k \leq \kappa-1}$ and $\{\tilde{\mathcal{V}}_{k,i}^{\mathcal{R},\pi,\varepsilon,m}(X_{t_{k,i}}^{\pi})\}_{0 \leq i \leq n_k-1}^{0 \leq k \leq \kappa-1}$ play the role of approximations for $\{Y_{t_{k,i}}^{\mathcal{R}}\}_{0 \leq i \leq n_k}^{0 \leq k \leq \kappa-1}$ and $\{Y_{t_{k,i}}^{\mathcal{R}}\}_{0 \leq i \leq n_k}^{0 \leq k \leq \kappa-1}$ respectively. We then also define the approximation $\{\tilde{\mathcal{Z}}_{k,i}^{\mathcal{R},\pi}\}_{0 \leq i \leq n_k-1}^{0 \leq k \leq \kappa-1}$ of the process $Z^{\mathcal{R}}$ by

$$\tilde{\mathcal{Z}}_{k,i}^{\mathcal{R},\pi} = NN^{\hat{\theta}_{k,i}}(X_{t_{k,i}}^{\pi}),$$

for $k = 0, \dots, \kappa - 1$ and $i = 0, \dots, n_k - 1$.

4.3 Convergence of the approximation scheme

To study the behavior of the approximation Algorithm 1, we make the additional standing assumptions on the drift b , the diffusion coefficient σ and the driver f .

(Hb, σ)' There exists a constant $L_{b,\sigma} > 0$ such that

$$|b(t, x) - b(t', x')| + |\sigma(t, x) - \sigma(t', x')| \leq L_{b,\sigma} \left(|t - t'|^{\frac{1}{2}} + |x - x'| \right)$$

for all (t, x) and $(t', x') \in [0, T] \times \mathbb{R}^d$.

(Hf)' There exists a constant $L_f > 0$ such that

$$|f(t, x, y, z) - f(t', x', y', z')| \leq L_f \left(|t - t'|^{\frac{1}{2}} + |x - x'| + |y - y'| + |z - z'| \right)$$

for all (t, x, y, z) and $(t', x', y', z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

We next define the error $\text{Err}^{\pi, \mathcal{R}}$ related to the grids π and \mathcal{R}

$$\begin{aligned} \text{Err}_{\varepsilon,m}^{\pi, \mathcal{R}} &= \max_{k=0, \dots, \kappa-1} \max_{i=1, \dots, n_k} \mathbb{E} \left[|Y_{t_{k,i}}^{\mathcal{R}} - \mathcal{V}_{k,i}^{\mathcal{R},\pi,\varepsilon,m}(X_{t_{k,i}}^{\pi})|^2 \right] \\ &\quad + \max_{k=0, \dots, \kappa-1} \max_{i=0, \dots, n_k-1} \mathbb{E} \left[|\tilde{Y}_{t_{k,i}}^{\mathcal{R}} - \tilde{\mathcal{V}}_{k,i}^{\mathcal{R},\pi,\varepsilon,m}(X_{t_{k,i}}^{\pi})|^2 \right] \\ &\quad + \mathbb{E} \left[\sum_{i=0}^{n-1} \int_{t_{k,i}}^{t_{k,i+1}} |Z_t^{\mathcal{R}} - \tilde{\mathcal{Z}}_{k,i}^{\mathcal{R},\pi}|^2 dt \right]. \end{aligned}$$

We then have the following convergence result.

Theorem 4.3. *We have the following convergence*

$$\begin{aligned} &\lim_{n_0 \rightarrow +\infty} \lim_{m_0^3 \rightarrow +\infty} \lim_{\varepsilon_1 \rightarrow 0} \lim_{m_1^1 \rightarrow +\infty} \lim_{m_1^2 \rightarrow +\infty} \lim_{n_1 \rightarrow +\infty} \lim_{m_1^3 \rightarrow +\infty} \dots \\ &\dots \lim_{\varepsilon_{\kappa-1} \rightarrow 0} \lim_{m_{\kappa-1}^1 \rightarrow +\infty} \lim_{m_{\kappa-1}^2 \rightarrow +\infty} \lim_{n_{\kappa-1} \rightarrow +\infty} \lim_{m_{\kappa-1}^3 \rightarrow +\infty} \lim_{\varepsilon_{\kappa} \rightarrow 0} \lim_{m_{\kappa}^1 \rightarrow +\infty} \text{Err}_{\varepsilon,m}^{\pi, \mathcal{R}} = 0. \end{aligned}$$

To prove Theorem 4.3, we need the two following lemmata.

Lemma 4.3. *Let φ and φ_{ℓ} , $\ell \geq 1$, be functions from \mathbb{R}^d to \mathbb{R} . Suppose there exists constants L and M such that φ and φ_{ℓ} , $\ell \geq 1$, are L -Lipschitz continuous and bounded by M . Let ξ be a random variable such that $\text{Supp}(\mathbb{P}_{\xi}) = \mathbb{R}^d$ and suppose also that*

$$\mathbb{E}[|\varphi_{\ell}(\xi) - \varphi(\xi)|^2] \xrightarrow{\ell \rightarrow +\infty} 0. \quad (4.38)$$

Then φ_{ℓ} converges uniformly to φ on compact subsets of \mathbb{R}^d .

Proof. From Ascoli Theorem the sequence $(\varphi_\ell)_{\ell \geq 1}$ is compact for the convergence on compact subsets of \mathbb{R}^d . Let $\tilde{\varphi}$ be an adherence value. Then, up to a subsequence

$$\sup_K |\varphi_\ell - \tilde{\varphi}| \xrightarrow{\ell \rightarrow +\infty} 0$$

for any compact subset K of \mathbb{R}^d . From (4.38), we deduce that $\tilde{\varphi}(\xi) = \varphi(\xi)$ \mathbb{P} -a.s. and since $\text{Supp}(\mathbb{P}_\xi) = \mathbb{R}^d$ we get $\tilde{\varphi} = \varphi$ on \mathbb{R}^d . \square

The next results shows that for the approximation of a bounded and Lipschitz continuous function, we can restrict the neural network weights to a given bound.

Lemma 4.4. *Let φ and φ_ℓ , $\ell \geq 1$, be functions from \mathbb{R}^d to \mathbb{R} and ξ be a random variable satisfying conditions of Lemma 4.3. Suppose the activation function ρ is differentiable with $\rho'(0) \neq 0$. Define the sequence $(\theta_{m,\ell}^*)_{m,\ell \geq 1}$ by*

$$\begin{aligned} \theta_{m,\ell}^* &\in \arg \min \mathbb{E}[|NN^{\theta}(\xi) \vee (-M) \wedge M - \varphi_\ell(\xi)|^2] \\ \theta &= (\lambda_i, \alpha_i)_{1 \leq i \leq m} \in \Theta_m \text{ s.t. } \left| \sum_{i=1}^m \lambda_i \alpha_i \right| \leq \frac{L+1}{|\rho'(0)|}. \end{aligned}$$

Then

$$\lim_{m \rightarrow +\infty} \lim_{\ell \rightarrow +\infty} \mathbb{E}[|NN^{\theta_{m,\ell}^*}(\xi) \vee (-M) \wedge M - \varphi(\xi)|^2] = 0. \quad (4.39)$$

Proof. Using a mollification argument, we can assume w.l.o.g. that $\varphi \in C^1(\mathbb{R}^d, \mathbb{R})$. From Theorem 4.2, we can find a sequence $(\theta_m)_{m \geq 1}$ such that $\theta_m \in \Theta_m$ for $m \geq 1$ and $(NN^{\theta_m}, DNN^{\theta_m})_{m \geq 1}$ converges uniformly to $(\varphi, D\varphi)$ on compact sets. We therefore get for m large enough

$$\left| \sum_{i=1}^m \lambda_i^m \alpha_i^m \right| \leq \frac{L+1}{|\rho'(0)|}$$

where $\theta_m = (\lambda_i^m, \alpha_i^m)_{1 \leq i \leq m}$. From the definition of $\theta_{m,\ell}^*$ we have

$$\begin{aligned} \mathbb{E}[|NN^{\theta_{m,\ell}^*}(\xi) \vee (-M) \wedge M - \varphi(\xi)|^2] &\leq 2\mathbb{E}[|NN^{\theta_{m,\ell}^*}(\xi) \vee (-M) \wedge M - \varphi_\ell(\xi)|^2] \\ &\quad + 2\mathbb{E}[|\varphi_\ell(\xi) - \varphi(\xi)|^2] \\ &\leq 2\mathbb{E}[|NN^{\theta_m}(\xi) \vee (-M) \wedge M - \varphi_\ell(\xi)|^2] \\ &\quad + 2\mathbb{E}[|\varphi_\ell(\xi) - \varphi(\xi)|^2] \end{aligned}$$

which converges to zero as ℓ and m goes to ∞ . \square

Remark 4.1. *If we suppose the derivative of the activation function ρ is bounded by a constant C then, the condition $|\sum_{i=1}^m \lambda_i \alpha_i| \leq \frac{L+1}{|\rho'(0)|}$ restricts to neural networks that are $C \frac{L+1}{|\rho'(0)|}$ Lipschitz continuous.*

Proof of Theorem 4.3. We recall that for $(t, x) \in [0, T] \times \mathbb{R}^d$, $(Y^{\mathcal{R}, t, x}, Z^{\mathcal{R}, t, x})$ is defined by (3.14)-(3.15)-(3.16). From Proposition 2.1 and classical estimates on Euler scheme we have

$$\begin{aligned} & \max_{k=0, \dots, \kappa-1} \max_{i=1, \dots, n_k} \mathbb{E} \left[|Y_{t_k, i}^{\mathcal{R}} - Y_{t_k, i}^{\mathcal{R}, t_k, 0, X_{t_k, 0}^{\pi}}|^2 \right] \\ & + \max_{k=0, \dots, \kappa-1} \max_{i=0, \dots, n_k-1} \mathbb{E} \left[|\tilde{Y}_{t_k, i}^{\mathcal{R}} - \tilde{Y}_{t_k, i}^{\mathcal{R}, t_k, 0, X_{t_k, 0}^{\pi}}|^2 \right] \\ & + \sum_{k=0}^{\kappa-1} \mathbb{E} \left[\sum_{i=0}^{n_k-1} \int_{t_k, i}^{t_{k, i+1}} |Z_t^{\mathcal{R}} - Z_{t_k, i}^{\mathcal{R}, t_k, 0, X_{t_k, 0}^{\pi}}|^2 dt \right] \longrightarrow 0, \end{aligned} \quad (4.40)$$

as $\max_{0 \leq k \leq \kappa-1} |\pi_k| \rightarrow 0$. From Proposition 4.4 we have

$$\lim_{\varepsilon_{\kappa} \rightarrow 0} \lim_{m_{\kappa}^1 \rightarrow +\infty} \mathbb{E} \left[|(\mathcal{V}_{\kappa-1, n_{\kappa-1}}^{\mathcal{R}, \pi, \varepsilon, m} - F_C[g]) \mathbf{1}_{B_{\varepsilon_{\kappa}}} |^2 (X_{\kappa-1, n_{\kappa-1}}^{\pi}) \right] = 0.$$

Since $F_C[g]$ is Lipschitz continuous, we get from Theorem 4.1 in [15] and Corollary 2.2 in [14]

$$\begin{aligned} & \lim_{n_{\kappa-1} \rightarrow +\infty} \lim_{m_{\kappa-1}^3 \rightarrow +\infty} \lim_{\varepsilon_{\kappa} \rightarrow 0} \lim_{m_{\kappa}^1 \rightarrow +\infty} \max_{i=1, \dots, n_{\kappa-1}} \mathbb{E} \left[|Y_{t_{\kappa-1}, i}^{\mathcal{R}, t_{\kappa-1}, 0, X_{t_{\kappa-1}, 0}^{\pi}} - \mathcal{V}_{\kappa-1, i}^{\mathcal{R}, \pi, \varepsilon, m}(X_{t_{\kappa-1}, i}^{\pi})|^2 \right] \\ & + \max_{i=0, \dots, n_{\kappa-1}-1} \mathbb{E} \left[|\tilde{Y}_{t_{\kappa-1}, i}^{\mathcal{R}, t_{\kappa-1}, 0, X_{t_{\kappa-1}, 0}^{\pi}} - \tilde{\mathcal{V}}_{\kappa-1, i}^{\mathcal{R}, \pi, \varepsilon, m}(X_{t_{\kappa-1}, i}^{\pi})|^2 \right] \\ & + \mathbb{E} \left[\sum_{i=0}^{n_{\kappa-1}-1} \int_{t_{\kappa-1}, i}^{t_{\kappa-1}, i+1} |Z_t^{\mathcal{R}, t_{\kappa-1}, 0, X_{t_{\kappa-1}, 0}^{\pi}} - \tilde{Z}_{\kappa-1, i}^{\mathcal{R}, \pi}|^2 dt \right] = 0. \end{aligned}$$

From Proposition 4.4, Lemmata 4.3 and 4.4 and the previous convergence, we get

$$\begin{aligned} & \lim_{\varepsilon_{\kappa-1} \rightarrow 0} \lim_{m_{\kappa-1}^1 \rightarrow +\infty} \lim_{m_{\kappa-1}^2 \rightarrow +\infty} \lim_{n_{\kappa-1} \rightarrow +\infty} \mathbb{E} \left[|Y_{t_{\kappa-1}, 0}^{\mathcal{R}, t_{\kappa-1}, 0, X_{t_{\kappa-1}, 0}^{\pi}} - \mathcal{V}_{\kappa-1, 0}^{\mathcal{R}, \pi, \varepsilon, m}(X_{t_{\kappa-1}, 0}^{\pi})|^2 \right] = 0. \end{aligned}$$

Repeating this argument for each $k = \kappa - 2, \dots, 0$, and using (4.40), we get the result. \square

We end this section by a convergence result for the constrained solution. Take $(\mathcal{R}^{\ell})_{\ell \geq 1}$ a nondecreasing sequence such that

$$|\mathcal{R}^{\ell}| := \max_{1 \leq k \leq \kappa_{\ell}} (r_k^{\ell} - r_{k-1}^{\ell}) \xrightarrow{n \rightarrow +\infty} 0,$$

and define

$$\widehat{\text{Err}}_{\varepsilon, m}^{\pi, \ell} = \max_{i=0, \dots, n-1} \mathbb{E} \left[|Y_{t_i} - \mathcal{V}_i^{\mathcal{R}^{\ell}, \pi}(X_{t_i}^{\pi})|^2 \right] + \mathbb{E} \left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_t - \tilde{Z}_i^{\mathcal{R}^{\ell}, \pi}|^2 dt \right].$$

From Corollary 3.2 and Theorem 4.3 we obtain the following result.

Corollary 4.3. *We have the following convergence*

$$\begin{aligned} & \lim_{\ell \rightarrow +\infty} \lim_{n_0 \rightarrow +\infty} \lim_{m_0^3 \rightarrow +\infty} \lim_{\varepsilon_1 \rightarrow 0} \lim_{m_1^1 \rightarrow +\infty} \lim_{m_1^2 \rightarrow +\infty} \lim_{n_1 \rightarrow +\infty} \lim_{m_1^3 \rightarrow +\infty} \dots \\ & \dots \lim_{\varepsilon_{\kappa-1} \rightarrow 0} \lim_{m_{\kappa-1}^1 \rightarrow +\infty} \lim_{m_{\kappa-1}^2 \rightarrow +\infty} \lim_{n_{\kappa-1} \rightarrow +\infty} \lim_{m_{\kappa-1}^3 \rightarrow +\infty} \lim_{\varepsilon_{\kappa} \rightarrow 0} \lim_{m_{\kappa}^1 \rightarrow +\infty} \widehat{\text{Err}}_{\varepsilon, m}^{\pi, \ell} = 0. \end{aligned}$$

5 Numerical results

5.1 Neural network approximation

In the sequel we first show that we can approximate the facelift easily with neural networks. In a second part we test the global algorithm evaluating the BSDE with constraints.

5.1.1 Testing the facelift approximation of a function φ

Testing many penalizing function, it turns out that the use of simple Relu function is the best way to simply penalize the constraints introducing a second small parameters ϵ_1 . This function prevents the problem of vanishing gradient that may appear using some regularization of some heaviside function for example.

We propose to use a L_1 norm on the distance to the target and the penalty terms giving coefficients of the neural network satisfying

$$\theta_{m,\epsilon}^* \in \arg \min_{\theta \in \Theta_m} \mathbb{E} \left[|NN^\theta - \varphi|(\xi) + \min_{x \in \mathcal{C}} \frac{\|DNN^\theta(\xi) - x\|_1}{\epsilon_1} + \frac{((\varphi - NN^\theta)(\xi))^+}{\epsilon_1} \right] \quad (5.41)$$

where ξ is an uniform r.v. in B_ϵ .

Remark 5.2. *The use of a L_2 norm for the distance to the true function or/and the different constraints does not give results as good as with the objective function above.*

Using a neural network, we have no certainty to get the facelift of a function φ . The problem is not convex and we face a dilemma:

- either we use a rather high penalty coefficient ϵ_2 and may not satisfy the constraints,
- either we set a very small ϵ_2 and the distance between the estimated facelift and the function is only seen as some noise by the gradient descent.

As we want to use a rather small ϵ_2 parameter, we will get solutions above the real facelift. We then propose to use the iterative algorithm 2 that successively approximates the facelift by above.

Algorithm 2: Iterative algorithm for facelift calculation of a function φ .

Input: Function to facelift φ

$$\theta_{m,\varepsilon}^{*,0} \in \arg \min_{\theta \in \Theta_m} \mathbb{E} \left[|NN^\theta - \varphi|(\xi) + \min_{x \in \mathcal{C}} \frac{\|DNN^\theta(\xi) - x\|_1}{\epsilon_1} + \frac{((\varphi - NN^\theta)(\xi))^+}{\epsilon_1} \right]$$

for $k = 1, \dots, K$ **do**

$$\theta_{m,\varepsilon}^{*,k} \in \arg \min_{\theta \in \Theta_m} \mathbb{E} \left[|NN^\theta - \varphi|(\xi) + \min_{x \in \mathcal{C}} \frac{\|DNN^\theta(\xi) - x\|_1}{\epsilon_1} + \frac{((\varphi - NN^\theta)(\xi))^+}{\epsilon_1} + \frac{((NN^{\theta_{m,\varepsilon}^{*,k-1}} - \varphi)(\xi))^+}{\epsilon_1} \right]$$

end

Output: $\theta_{m,\varepsilon}^{*,K}$

We test three activation functions ReLU, tanh and ELU with the bounded set

$$\mathcal{C} = \{x \in \mathbb{R} / \|x\| \leq \hat{d}\},$$

for different values of \hat{d} .

ELU is the less effective while ReLU gives results slightly better than tanh. In the sequel ReLU is taken for numerical results. As for the number of hidden layers, one layer appears to be insufficient and 3 does not bring any improvement comparing to two hidden layers. We have to take at least 100 neurons per layer to get very good results. In the sequel we take 200 neurons.

In the numerical results we take mini batch of size 1000 with the Adam optimizer [16] using a learning rate equal to 0.001. We stop the algorithm after 100000 iterations and every hundred iterations we do a more accurate estimation of the loss with 10000 particles keeping the best network obtained during iterations.

We test the algorithm on a fixed convex set depending on the test case.

First case For the second test case we use the payoff of a butterfly function

$$\varphi(x) = (x - 0.8)^+ - 2(x - 1)^+ + (x - 1.2)^+.$$

The facelift function is peacewise linear given for $\hat{d} \leq 1$ by

$$\varphi_{\hat{d}}^A(x) = (1 - \hat{d}|x - 1|)^+.$$

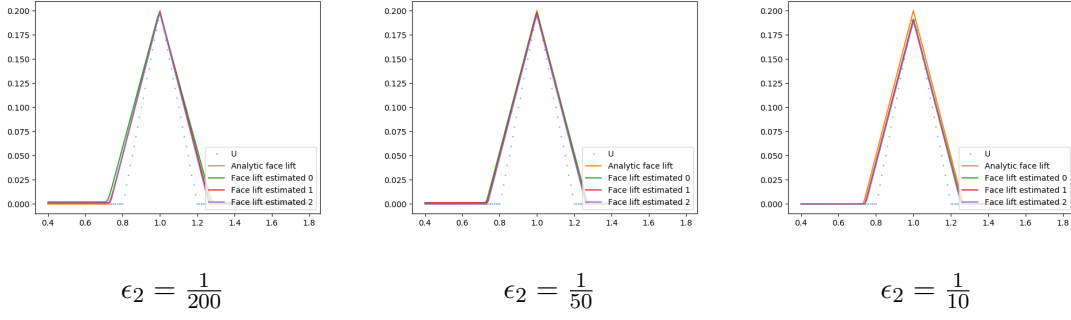


Figure 1: facelift approximation for different values of ϵ_2 for the first case with $\hat{d} = 0.75$, $K = 2$. On each figure, we give the results obtained for each iteration of the algorithm.

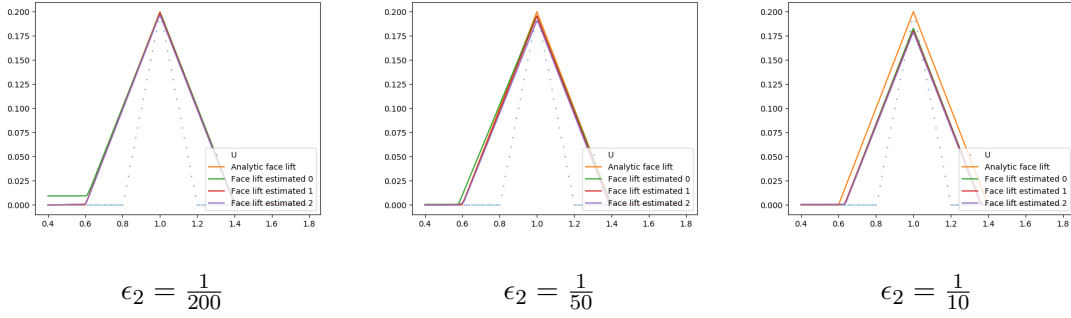


Figure 2: facelift approximation for different values of ϵ_2 for the first case with $\hat{d} = 0.5$, $K = 2$. On each figure, we give the results obtained for each iteration of the algorithm.

On Figures 1,2, we give the facelift obtained for different values of ϵ_2 and \hat{d} . For a small constraint ($\hat{d} = 7.5$), the facelift is calculated very well for all penalty even with one iteration of the algorithm meaning that a simple resolution of (5.41) is sufficient enough. For a smaller value \hat{d} a quite high penalty value is necessary to get a good result with at least two iterations of the algorithm.

Second case We want to calculate the facelift of the function

$$\varphi(x) = 4[(x - 0.8)^+ - (x - 1)^+] + (x - 1.2)^+ \quad (5.42)$$

on set $[0.6, 1.4]$. The facelift function is obviously piecewise linear and given for $\hat{d} \leq 4$ by

$$\varphi_{\hat{d}}^A(x) = \begin{cases} 4[(x - 0.8)^+ - (x_i - 1)^+] + (x_i - 1.2)^+, & x \geq 1 \\ (0.8 - \hat{d}|x - 1|)^+, & x < 1. \end{cases} \quad (5.43)$$

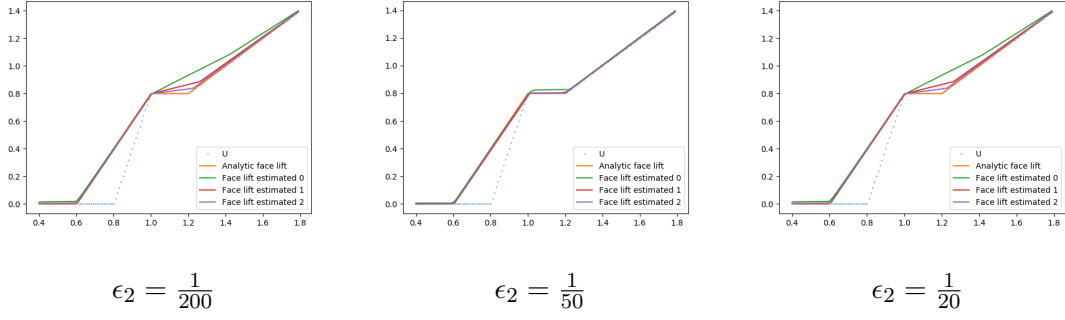


Figure 3: facelift approximation for different values of ϵ_2 for the first case with $\hat{d} = 2$, $K = 2$.

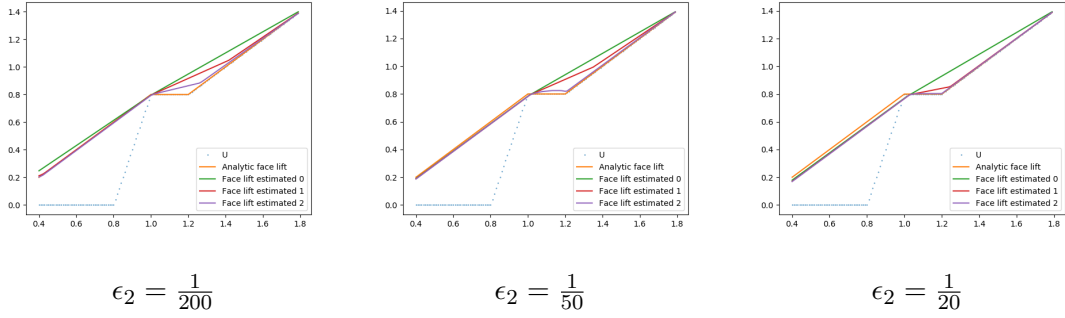


Figure 4: facelift approximation for different values of ϵ_2 for the first case with $\hat{d} = 1$, $K = 2$.

On this test case, at least 3 iterations of the algorithm are necessary to reach a good accuracy. As before, since the constraint is higher, the algorithm faces difficulty to reach a very good accuracy.

Third case For this third case, we take

$$\varphi(x) = \log(1 + e^x) + 4 \frac{\sin(2x)}{1 + 5x^2}.$$

On Figure 5, we give the function value obtained with different values of ϵ_2 using 3 iterations of the algorithm for different size \hat{d} .

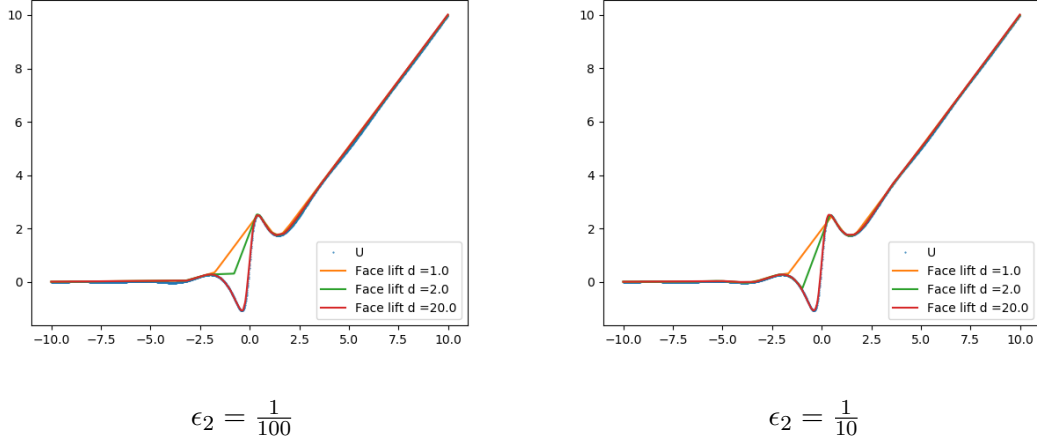


Figure 5: facelift approximation for different values of \hat{d} for the third case.

As we can see on Figure 6, constraints are well respected for test case 3.

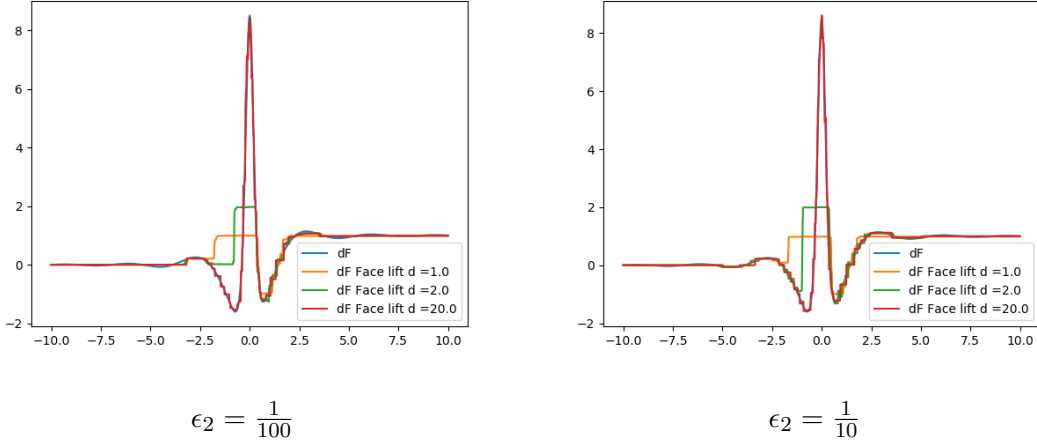


Figure 6: Derivative of the facelift approximation for different values of ϵ_2 for the third case

5.2 Results in higher dimension

We extend the φ function given by (5.42) in higher dimension by

$$\varphi(x) = \frac{1}{d} \sum_{i=1}^d 4[(x_i - 0.8)^+ - (x_i - 1)^+] + (x_i - 1.2)^+, \quad x \in \mathbb{R}^d. \quad (5.44)$$

As before the facelift can be calculated analytically for $d\hat{d} \leq 4$ as

$$\varphi^{A,\hat{d}}(x) = \frac{1}{d} \sum_{i=1}^d \varphi_{\hat{d}}^A(x_i)$$

where $\varphi_{\hat{d}}^A$ is given by equation (5.43).

We test the accuracy of the facelift calculated $NN^{\theta_{m,\varepsilon}^{*,k}}$ in different dimension by plotting

$$\mathbb{E}[(NN^{\theta_{m,\varepsilon}^{*,k}} - \varphi^A)^2(\xi)]$$

with respect to k for $\epsilon_2 = \frac{1}{4000}$ for different values of d and \hat{d} .

Remark 5.3. *Taking a very small value permits to get better results in high dimension but increases the number of iterations for easier cases.*

On Figure 7, we plot the error due to the algorithm with respect to number of iterations for different dimensions. Iterations are stopped below 10 when errors starts increasing meaning that the solution estimated is below the true one. In real application, a check on the L_2 difference between the estimation and the function to facelift is used to stop the iterations.

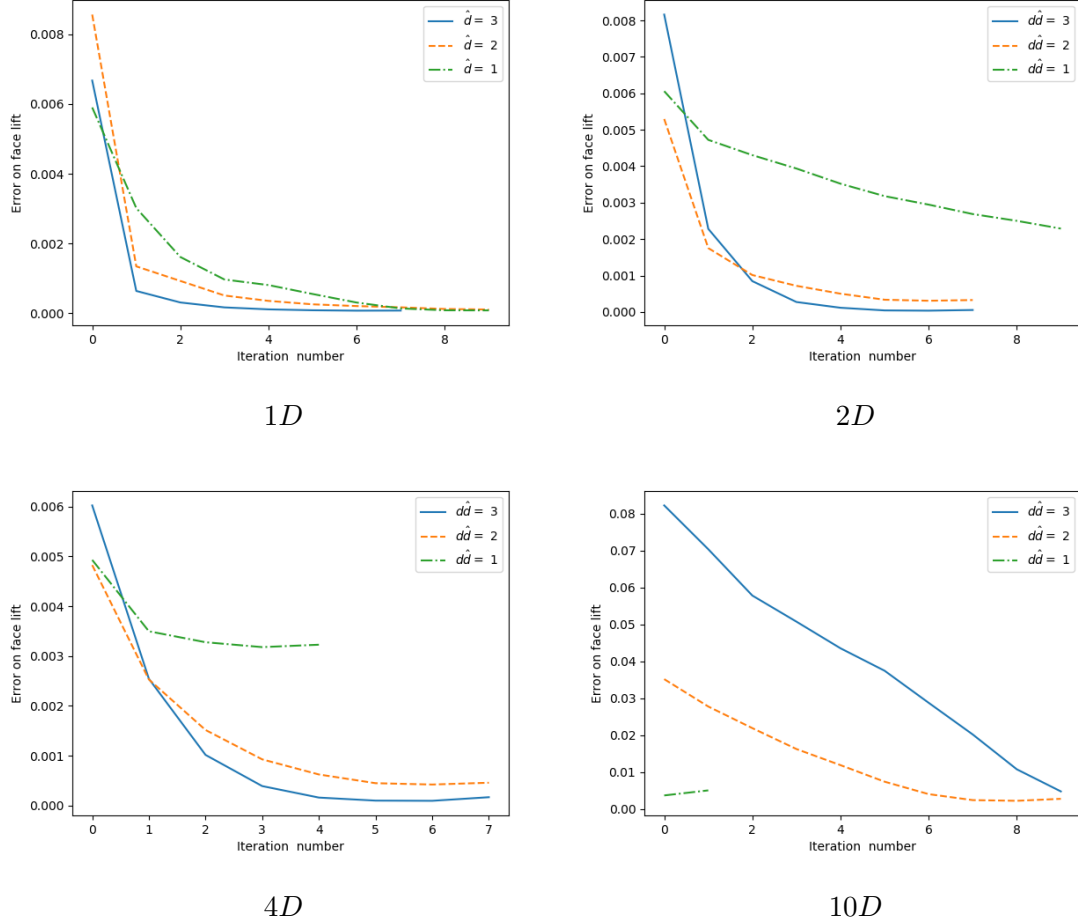


Figure 7: Error with respect to the number of iterations

As expected, the convergence in dimension 10D is harder to achieve and hard constraints (small \hat{d}) are difficult to solve.

5.2.1 Solving the BSDE with constraints

In this section we propose to solve the problem of option pricing with differential interest rates [3] adding a constraint on the number of shares held in the portfolio. The forward process is given by the Black Scholes model

$$X_s^{t,x} = x + \int_t^s \mu X_u^{t,x} du + \int_t^s \sigma X_u^{t,x} dB_u, \quad s \in [t, T].$$

The driver is given by

$$f(t, x, y, z) = -ry - z.(\sigma^{-1}(\mu - r)x) - (R - r)(y - z.(\sigma^{-1}x)).$$

As the facelift is calculated by a neural network, it seems natural to solve the transition problem between two time steps by the same methodology. Currently two effective methods have been developed to solve this problem [15] and [2]. It turns out that the method given in [15] is more accurate than the method given in [2]. Then we apply the method given in [15] as described in the previous section to our problem. We decide to apply the constraint after each resolution so we take $n_k = 1$ in the implemented algorithm.

The parameters are taken as follows: we keep as for the facelift calculation two hidden layers with 200 neurons. For the activation function we keep the tanh function used in [15]. The size of mini batch is taken equal to 1000, and we check the convergence every 100 epoch iterations. When reduction of the loss is not effective enough we reduce the learning rate with the methodology explained in [6]. Total number of iterations is limited to 50000 for each time step. Numerical test show that the number of neurons could be lower and the activation function could be a ReLU or ELU : taking 50 and 100 neurons gives very similar results for activation functions listed above.

In one dimension, we give the results obtained for the second payoff function used in Section 5.1.1. We take $T = 1$, $r = 0.05$, $\mu = 0.07$, $\sigma = 0.3$, $x = 1$ for the initial asset value. The convex set is a ball of radius \hat{d} , ϵ_2 taken equal to $\frac{1}{50}$ and the number of iterations K in algorithm 2 equal to 2. We give results obtained for different value of R . Taking $R = r$, we get a semi analytical value by taking the expectation of the facelift payoff of the process under the risk neutral measure as explained in [4]. This expectation is calculated by taking $1e7$ trajectories.

When $R \neq r$, no solution is available for this non linear problem. In Tables 1,2, we give the results obtained with 20 time steps for different values of \hat{d} . We give the average of 5 calculations and the standard deviation of the results. Notice that without constraints and $R = r$, the analytical solution is 0.558.

$\hat{d} = 3$				$\hat{d} = 2$			
R	0.05	0.07	0.09	R	0.05	0.07	0.09
Analytical	0.591			Analytical	0.648		
Numerical	0.598	0.612	0.627	Numerical	0.653	0.661	0.669
Std	0.004	0.002	0.0008	Std	0.001	0.002	0.003

Table 1: Results obtained by neural network algorithm for payoff 5.42 using 20 time steps ($n = 21$)

R	0.05	0.07	0.09
Analytical	0.736		
Numerical	0.742	0.743	0.744
Std	0.003	0.002	0.0009

Table 2: Results obtained by neural network algorithm for payoff 5.42 using 20 time steps ($n = 21$), $\hat{d} = 1$.

5.2.2 Results in higher dimension

In this section we take $R = r$ such that we get an analytical solution and we use the previous algorithm with the payoff (5.44).

Table 3: Results obtained by neural network algorithm for payoff 5.44 using 20 time steps ($n = 21$) with $\epsilon_2 = \frac{1}{50}$

d	2			4			6		
$d\hat{d}$	3	2	1	3	2	1	3	2	1
Ref	0.591	0.648	0.736	0.591	0.648	0.736	0.591	0.648	0.736
Num	0.592	0.644	0.739	0.591	0.637	0.722	0.591	0.631	0.707
Std	0.002	0.002	0.001	0.002	0.003	0.003	0.001	0.002	0.002

Table 4: Results obtained by neural network algorithm for payoff 5.44 using 20 time steps ($n = 21$) with $\epsilon_2 = \frac{1}{250}$

d	2			4			6		
$d\hat{d}$	3	2	1	3	2	1	3	2	1
Ref	0.591	0.648	0.736	0.591	0.648	0.736	0.591	0.648	0.736
Num	0.598	0.6550	0.760	0.602	0.653	0.749	0.607	0.654	0.749
Std	0.005	0.003	0.006	0.002	0.001	0.013	0.003	0.002	0.008

As noticed before, since the constraints gets tighter, the results are not as good. Taking a very small ε gives results with a higher standard deviation.

A Regularity estimates on solutions to parabolic semi-linear PDEs

We recall in this appendix an existence and uniqueness results for viscosity solution to semi-linear PDEs. We also give a regularity property with an explicit form for the Lipschitz and Hölder constants. Although, this regularity is classical in PDE theory, we choose to provide such a result as we did not find any explicit mention of the dependence of the regularity coefficient in the literature.

We fix $\underline{t}, \bar{t} \in [0, T]$ and we consider a PDE of the form

$$\begin{cases} -\partial_t w(t, x) - \mathcal{L}w(t, x) \\ -h(t, x, w(t, x), \sigma(t, x)^\top Dw(t, x)) = 0, \quad (t, x) \in [\underline{t}, \bar{t}] \times \mathbb{R}^d \\ w(\bar{t}, x) = m(x), \quad x \in \mathbb{R}^d \end{cases} \quad (\text{A.45})$$

We make the following assumption on the coefficients m and h .

(Hh, m)

(i) The function m is bounded: there exists a constant M_m such that

$$|m(x)| \leq M_m$$

for all $x \in \mathbb{R}^d$.

(ii) The function h is continuous and satisfies the following growth property: there exists a constant M_h such that

$$|h(t, x, y, z)| \leq M_h(1 + |y| + |z|)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

(iii) The functions h and m are Lipschitz continuous in their space variables uniformly in their time variable: there exist two constants L_h and L_m such that

$$\begin{aligned} |m(x) - m(x')| &\leq L_m |x - x'| \\ |h(t, x, y, z) - h(t, x', y', z')| &\leq L_h(|x - x'| + |y - y'| + |z - z'|) \end{aligned}$$

for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$.

Proposition A.5. *Suppose (\mathbf{Hb}, σ) and (\mathbf{Hh}, m) hold. The PDE (A.45) admits a unique viscosity solution w with polynomial growth: there exist an integer $p \geq 1$ and a constant C such that*

$$|w(t, x)| \leq C(1 + |x|^p), \quad (t, x) \in [\underline{t}, \bar{t}] \times \mathbb{R}^d.$$

Moreover, w satisfies the following space regularity property

$$|w(t, x) - w(t, x')| \leq e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} |x - x'|$$

for all $t \in [\underline{t}, \bar{t}]$ and $x, x' \in \mathbb{R}^d$.

We first need the following lemma.

Lemma A.5. *Under (\mathbf{Hb}, σ) we have the following estimate*

$$\begin{aligned} \sup_{s \in [t \vee t', \bar{t}]} \mathbb{E} \left[|X_s^{t,x} - X_s^{t',x'}|^2 \right] &\leq \\ e^{(2L_{b,\sigma} + L_{b,\sigma}^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t})) &\left(|x - x'| + M_{b,\sigma} \sqrt{|t - t'|} \right)^2 \end{aligned} \quad (\text{A.46})$$

for $t, t' \in [\underline{t}, \bar{t}]$ and $x, x' \in \mathbb{R}^d$.

Proof. Fix $t, t' \in [\underline{t}, \bar{t}]$ such that $t' \leq t$ and $x, x' \in \mathbb{R}^d$. From Itô's formula and $(\mathbf{H}b, \sigma)$ we have

$$\mathbb{E}[|X_s^{t,x} - X_s^{t',x'}|^2] \leq \mathbb{E}[|X_t^{t,x} - X_t^{t',x'}|^2] + (2L_{b,\sigma} + L_{b,\sigma}^2) \int_t^s \mathbb{E}[|X_u^{t,x} - X_u^{t',x'}|^2] du$$

for $s \in [t, \bar{t}]$. By Gronwall's Lemma we get

$$\sup_{s \in [t, \bar{t}]} \mathbb{E}[|X_s^{t,x} - X_s^{t',x'}|^2] \leq \mathbb{E}[|X_t^{t,x} - X_t^{t',x'}|^2] e^{(2L_{b,\sigma} + L_{b,\sigma}^2)(\bar{t}-t)}.$$

Moreover, we have

$$\begin{aligned} \mathbb{E}[|X_t^{t,x} - X_t^{t',x'}|^2] &= \mathbb{E}\left[\left|x - x' - \int_{t'}^t b(s, X_s^{t',x'}) ds - \int_{t'}^t \sigma(s, X_s^{t',x'}) dB_s\right|^2\right] \\ &\leq |x - x'|^2 + M_{b,\sigma}^2 |t - t'|^2 + M_{b,\sigma}^2 |t - t'| + 2M_{b,\sigma} |x - x'| |t - t'| \\ &\leq \left(|x - x'| + M_{b,\sigma} \sqrt{|t - t'|}\right)^2 (1 + (\bar{t} - \underline{t})). \end{aligned}$$

Which give the result. \square

Proof of Proposition A.5. For $(t, x) \in [\underline{t}, \bar{t}] \times \mathbb{R}^d$, we introduce the following BSDE: find $(\mathcal{Y}^{t,x}, \mathcal{Z}^{t,x}) \in \mathbf{S}_{[\underline{t}, \bar{t}]}^2 \times \mathbf{H}_{[\underline{t}, \bar{t}]}^2$ such that

$$\mathcal{Y}_u^{t,x} = m(X_{\bar{t}}^{t,x}) + \int_u^{\bar{t}} h(s, X_s^{t,x}, \mathcal{Y}_s^{t,x}, \mathcal{Z}_s^{t,x}) ds - \int_u^{\bar{t}} \mathcal{Z}_s^{t,x} dB_s, \quad u \in [t, \bar{t}].$$

From Theorem 1.1 in [17], we get the existence and uniqueness of the solution to this BSDE for all $(t, x) \in [\underline{t}, \bar{t}] \times \mathbb{R}^d$. From Theorem 2.2 in [17] and Theorem 5.1 in [19], the function w defined by

$$w(t, x) = \mathcal{Y}_t^{t,x}, \quad (t, x) \in [\underline{t}, \bar{t}] \times \mathbb{R}^d,$$

is continuous and is the unique viscosity solution to (A.45) with polynomial growth.

We now turn to the regularity estimate. We first check the regularity w.r.t. the variable x .

Fix $t \in [\underline{t}, \bar{t}]$ and $x, x' \in \mathbb{R}^d$. By Itô's formula we have

$$\begin{aligned} |\mathcal{Y}_s^{t,x} - \mathcal{Y}_s^{t,x'}|^2 &= |m(X_{\bar{t}}^{t,x}) - m(X_{\bar{t}}^{t,x'})|^2 \\ &\quad + \int_s^{\bar{t}} (h(u, X_u^{t,x}, \mathcal{Y}_u^{t,x}, \mathcal{Z}_u^{t,x}) - h(u, X_u^{t,x'}, \mathcal{Y}_u^{t,x'}, \mathcal{Z}_u^{t,x'})) (\mathcal{Y}_u^{t,x} - \mathcal{Y}_u^{t,x'}) du \\ &\quad - \int_s^{\bar{t}} |\mathcal{Z}_u^{t,x} - \mathcal{Z}_u^{t,x'}|^2 du - \int_s^{\bar{t}} (\mathcal{Y}_u^{t,x} - \mathcal{Y}_u^{t,x'}) (\mathcal{Z}_u^{t,x} - \mathcal{Z}_u^{t,x'}) dB_u \end{aligned}$$

for $s \in [t, \bar{t}]$. Using Lipschitz properties of h and m and Young inequality we get

$$\begin{aligned} \mathbb{E}[|\mathcal{Y}_s^{t,x} - \mathcal{Y}_s^{t,x'}|^2] &\leq L_m^2 \mathbb{E}[|X_{\bar{t}}^{t,x} - X_{\bar{t}}^{t,x'}|^2] + L_h^2 \int_s^{\bar{t}} \mathbb{E}[|X_u^{t,x} - X_u^{t,x'}|^2] du \\ &\quad + \left(\frac{L_h^2}{4} + L_h + 1\right) \int_s^{\bar{t}} \mathbb{E}[|\mathcal{Y}_u^{t,x} - \mathcal{Y}_u^{t,x'}|^2] du, \quad s \in [t, \bar{t}]. \end{aligned}$$

Since $(\frac{x^2}{4} + x + 1) \leq x^2$ for $x \geq 2$ we get

$$\begin{aligned} \mathbb{E}[|\mathcal{Y}_s^{t,x} - \mathcal{Y}_s^{t,x'}|^2] &\leq L_m^2 \mathbb{E}[|X_{\bar{t}}^{t,x} - X_{\bar{t}}^{t,x'}|^2] + (L_h \vee 2)^2 \int_s^{\bar{t}} \mathbb{E}[|X_u^{t,x} - X_u^{t,x'}|^2] du \\ &\quad + (L_h \vee 2)^2 \int_s^{\bar{t}} \mathbb{E}[|\mathcal{Y}_u^{t,x} - \mathcal{Y}_u^{t,x'}|^2] du, \quad s \in [t, \bar{t}]. \end{aligned}$$

Then using (A.46), we get

$$\begin{aligned} \mathbb{E}[|\mathcal{Y}_s^{t,x} - \mathcal{Y}_s^{t,x'}|^2] &\leq e^{(2L_{b,\sigma} + L_{b,\sigma}^2)(\bar{t}-t)} (1 + (\bar{t} - t)) (L_m^2 + (\bar{t} - t)(L_h \vee 2)^2) |x - x'|^2 \\ &\quad + (L_h \vee 2)^2 \int_s^{\bar{t}} \mathbb{E}[|\mathcal{Y}_u^{t,x} - \mathcal{Y}_u^{t,x'}|^2] du, \quad s \in [t, \bar{t}]. \end{aligned}$$

Using Gronwall's Lemma we get

$$\mathbb{E}[|\mathcal{Y}_t^{t,x} - \mathcal{Y}_t^{t,x'}|^2] \leq e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t}-t)} (1 + (\bar{t} - t)) (L_m^2 + (\bar{t} - t)(L_h \vee 2)^2) |x - x'|^2.$$

Therefore, we get

$$|w(t, x) - w(t, x')| \leq e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t}-t)} (1 + (\bar{t} - t))^{\frac{1}{2}} (L_m^2 + (\bar{t} - t)(L_h \vee 2)^2)^{\frac{1}{2}} |x - x'|.$$

□

In this last result we prove that under our assumptions the Z component of a solution to a BSDE is bounded. We recall that $(\mathcal{Y}^{t,x}, \mathcal{Z}^{t,x}) \in \mathbf{S}_{[t,\bar{t}]^2}^2 \times \mathbf{H}_{[t,\bar{t}]}^2$ denotes the solution to

$$\mathcal{Y}_u^{t,x} = m(X_{\bar{t}}^{t,x}) + \int_u^{\bar{t}} h(s, X_s^{t,x}, \mathcal{Y}_s^{t,x}, \mathcal{Z}_s^{t,x}) ds - \int_u^{\bar{t}} \mathcal{Z}_s^{t,x} dB_s, \quad u \in [t, \bar{t}],$$

for $(t, x) \in [t, \bar{t}] \times \mathbb{R}^d$.

Proposition A.6. *Under $(\mathbf{H}b, \sigma)$ and $(\mathbf{H}h, m)$, the process $\mathcal{Z}^{t,x}$ satisfies*

$$|\mathcal{Z}^{t,x}| \leq M_{b,\sigma} e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t}-t)} (1 + (\bar{t} - t))^{\frac{1}{2}} (L_m^2 + (\bar{t} - t)(L_h \vee 2)^2)^{\frac{1}{2}}$$

$d\mathbb{P} \otimes dt$ a.e. on $\Omega \times [t, \bar{t}]$.

Proof. By a mollification argument, we can find regular functions b_n and σ_n satisfying $(\mathbf{H}b, \sigma)$ with same constants as b, σ, h_n and m_n satisfying $(\mathbf{H}h, m)$ with same constants as h and m for $n \geq 1$ such that

$$(b_n, \sigma_n, h_n, m_n) \xrightarrow{n \rightarrow +\infty} (b, \sigma, h, m), \quad (\text{A.47})$$

uniformly on compact sets. We fix now $(t, x) \in [t, \bar{t}] \times \mathbb{R}^d$ and we denote by $(X^{t,x,n}, \mathcal{Y}^{t,x,n}, \mathcal{Z}^{t,x,n}) \in \mathbf{S}_{[t,\bar{t}]^2}^2 \times \mathbf{S}_{[t,\bar{t}]}^2 \times \mathbf{H}_{[t,\bar{t}]}^2$ the solution to

$$\begin{aligned} X_u^{t,x,n} &= x + \int_t^u b_n(s, X_s^{t,x,n}) ds + \int_t^u \sigma_n(s, X_s^{t,x,n}) dB_s, \quad u \in [t, \bar{t}], \\ \mathcal{Y}_u^{t,x,n} &= m_n(X_{\bar{t}}^{t,x,n}) + \int_u^{\bar{t}} h_n(s, X_s^{t,x,n}, \mathcal{Y}_s^{t,x,n}, \mathcal{Z}_s^{t,x,n}) ds - \int_u^{\bar{t}} \mathcal{Z}_s^{t,x,n} dB_s, \quad u \in [t, \bar{t}]. \end{aligned}$$

From (A.47) we get

$$\|Y^{t,x} - Y^{t,x,n}\|_{\mathbf{S}_{[t,\bar{t}]}^2} + \|Z^{t,x} - Z^{t,x,n}\|_{\mathbf{H}_{[t,\bar{t}]}^2} \xrightarrow{n \rightarrow +\infty} 0. \quad (\text{A.48})$$

From Theorem 3.2 in [18], we have

$$Y_s^{t,x,n} = w_n(s, X^{t,x,n}) \quad s \in [t, \bar{t}],$$

where w_n is a regular solution to

$$\begin{cases} -\partial_t w_n - \mathcal{L}w_n - h_n(\cdot, w_n, \sigma_n^\top Dw_n) = 0, & \text{on } [\underline{t}, \bar{t}] \times \mathbb{R}^d, \\ w_n(\bar{t}, \cdot) = m_n, & \text{on } \mathbb{R}^d. \end{cases}$$

From the uniqueness of solutions to Lipschitz BSDEs we get by applying Itô's formula

$$Z_s^{t,x,n} = (\sigma_n^\top Dw_n)(s, X_s^{t,x}), \quad s \in [t, \bar{t}].$$

Since σ_n , m_n and h_n satisfy $(\mathbf{H}h, m)$, we get from Proposition A.5

$$\sup_{[\underline{t}, \bar{t}] \times \mathbb{R}^d} |Dw_n| \leq e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}}.$$

Therefore, we have

$$|Z_s^{t,x,n}| \leq M_{b,\sigma} e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}}$$

$d\mathbb{P} \otimes dt$ a.e. on $\Omega \times [t, \bar{t}]$. We then conclude using (A.48). \square

Proposition A.7. *Under $(\mathbf{H}b, \sigma)$ and $(\mathbf{H}h, m)$ the unique viscosity solution with linear growth w (A.45) satisfies the following time regularity property*

$$\begin{aligned} |w(t, x) - w(t', x)| &\leq \\ &e^{(3L_{b,\sigma} + 2L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{3}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} M_{b,\sigma} \sqrt{t - t'} \\ &\quad + M_h (M_m + M_h(\bar{t} - \underline{t})) e^{M_h(\bar{t} - \underline{t})} (t' - t) \\ &+ M_h M_{b,\sigma} e^{(2L_{b,\sigma} + L_{b,\sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} (t' - t) \end{aligned}$$

for all $t, t' \in [\underline{t}, \bar{t}]$ and $x \in \mathbb{R}^d$.

Proof. We take the same notations as in the proof of Proposition A.5. We fix $t, t' \in [\underline{t}, \bar{t}]$ such that $t' \leq t$ and $x \in \mathbb{R}^d$. We have

$$\begin{aligned} |w(t, x) - w(t', x)| &= |\mathcal{Y}_t^{t,x} - \mathcal{Y}_{t'}^{t',x}| \\ &= \left| \mathcal{Y}_t^{t,x} - \mathbb{E} \left[\mathcal{Y}_t^{t, X_t^{t',x}} + \int_{t'}^t f(s, X_s^{t',x}, \mathcal{Y}_s^{t',x}, \mathcal{Z}_s^{t',x}) ds \right] \right| \\ &\leq \mathbb{E} \left[|\mathcal{Y}_t^{t,x} - \mathcal{Y}_t^{t, X_t^{t',x}}| \right] + M_h \int_{t'}^t (1 + \mathbb{E}[|\mathcal{Y}_s^{t',x}|] + \mathbb{E}[|\mathcal{Z}_s^{t',x}|]) ds \end{aligned}$$

By a classical argument using $(\mathbf{H}h, m)$, Young's inequality and Grownwall's Lemma we have

$$\sup_{s \in [\underline{t}, \bar{t}]} \mathbb{E}[|\mathcal{Y}_s^{t', x}|^2] \leq M_m^2 + e^{4(M_h + M_h^2)(\bar{t} - \underline{t})}.$$

Then using Proposition A.6 we have

$$\mathbb{E}[|\mathcal{Z}_s^{t', x}|] \leq M_{b, \sigma} e^{(2L_{b, \sigma} + L_{b, \sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}}$$

for $s \in [t', t]$. From the regularity w.r.t. the variable x given in Proposition A.5 we get

$$\begin{aligned} |w(t, x) - w(t', x)| &\leq \\ e^{(2L_{b, \sigma} + L_{b, \sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} &\mathbb{E}[|X_t^{t, x} - X_t^{t', x}|] \\ &+ M_h (M_m^2 + e^{4(M_h + M_h^2)(\bar{t} - \underline{t})} + 1) (t' - t) \\ &+ M_h M_{b, \sigma} e^{(2L_{b, \sigma} + L_{b, \sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} (t' - t). \end{aligned}$$

From (A.46) we get

$$\begin{aligned} |w(t, x) - w(t', x)| &\leq \\ e^{(3L_{b, \sigma} + 2L_{b, \sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t})) (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} &M_{b, \sigma} \sqrt{t - t'} \\ &+ M_h (M_m^2 + e^{4(M_h + M_h^2)(\bar{t} - \underline{t})} + 1) (t' - t) \\ &+ M_h M_{b, \sigma} e^{(2L_{b, \sigma} + L_{b, \sigma}^2 + (L_h \vee 2)^2)(\bar{t} - \underline{t})} (1 + (\bar{t} - \underline{t}))^{\frac{1}{2}} (L_m^2 + (\bar{t} - \underline{t})(L_h \vee 2)^2)^{\frac{1}{2}} (t' - t). \end{aligned}$$

□

References

- [1] Guy Barles. *Solution de viscosités des équations d'Hamilton Jacobi*, volume 17 of *Mathématiques et Applications*. Springer Verlag, 1994.
- [2] Christian Beck, Sebastian Becker, Patrick Cheridito, Arnulf Jentzen, and Ariel Neufeld. Deep splitting method for parabolic pdes. *arXiv preprint arXiv:1907.03452*, 2019.
- [3] Yaacov Z Bergman. Option pricing with differential interest rates. *The Review of Financial Studies*, 8(2):475–500, 1995.
- [4] Mark Broadie, Jaksa Cvitanic, and Halil Mete Soner. Optimal replication of contingent claims under portfolio constraints. *The Review of Financial Studies*, 11:59–79, 1998.
- [5] Bouchard Bruno, Romuald Elie, and Ludovic Moreau. Regularity of bsdes with a convex constraint on the gains-process. *Bernoulli*, 24(3):1613–1635, 2018.
- [6] Quentin Chan-Wai-Nam, Joseph Mikael, and Xavier Warin. Machine learning for semi linear pdes. *Journal of Scientific Computing*, 79(3):1667–1712, 2019.

- [7] Jean-François Chassagneux, Romuald Elie, and Kharroubi Idris. A numerical probabilistic scheme for super-replication with convex constraints on the delta. *Forthcoming*, 2020.
- [8] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.
- [9] Jakša Cvitanić, Ioannis Karatzas, and H. Mete Soner. Backward stochastic differential equations with constraints on the gains-process. *The Annals of Probability*, 26(4):1522–1551, 1998.
- [10] Claude Dellacherie and Paul-André Meyer. Probability and potential. *Paris: Hermann*, 1978.
- [11] Nicole El Karoui, Shige Peng, and Marie-Claire Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.
- [12] Emmanuel Gobet and Jean-Philippe Lemor. Numerical simulation of bsdes using empirical regression methods: theory and practice. *arXiv preprint arXiv:0806.4447*, 2008.
- [13] Kurt Hornik, Maxwell Stinchcombe, and Halbert White. Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks. *Neural Networks*, 3(5):551–560, 1990.
- [14] Kurt Hornik, Maxwell Stinchcombe, and Halbert White. Multilayer feedforward networks are universal approximators. *Neural Networks*, 2(5):359–366, 1989.
- [15] Côme Huré, Huyên Pham, and Xavier Warin. Some machine learning schemes for high-dimensional nonlinear pdes. *Mathematics of Computation*, Forthcoming.
- [16] Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.
- [17] Étienne Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic pdes of second order. In *Stochastic Analysis and Related Topics VI*, pages 79–127. Springer, 1998.
- [18] Etienne Pardoux and Shige Peng. Backward stochastic differential equations and quasi-linear parabolic partial differential equations. In Boris L. Rozovskii and Richard B. Sowers, editors, *Stochastic Partial Differential Equations and Their Applications*, volume 176 of *Lecture Notes in Control and Information Sciences*, pages 200–217. Springer, Berlin, Heidelberg, 1991.
- [19] Etienne Pardoux, Frédéric Pradeilles, and Zusheng Rao. Probabilistic interpretation of a system of semi-linear parabolic partial differential equations. *Annales de l’I. H. P., section B*, 33(4):467–490, 1997.

- [20] Shige Peng. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyers type. *Probability Theory and Related Fields*, 113(4):473–499, 1999.
- [21] Huy  n Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*, volume 61 of *Stochastic modelling and applied probability*. Springer-Verlag Berlin Heidelberg, 2009.
- [22] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton Mathematical Series. Princeton University Press, 1970.