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Abstract

In this article we consider the inverse problem of determining the diffusion coefficient of the heat operator in an unbounded guide using a finite number of localized observations. For this problem, we prove a stability estimate in any finite portion of the guide using an adapted Carleman inequality. The boundary measurements are also realized on the boundary of a larger finite portion of the guide. A special care is required to avoid measurements on the cross-section boundaries which are inside the actual guide. This stability estimate uses a technical positivity assumption. Using arguments from control theory, we manage to remove this assumption for the inverse problem with non homogeneous given boundary conditions.

1 Introduction and main results

1.1 Introduction

Inverse problems associated with the heat operator have been frequently investigated as well on theoretical aspects as for applications, applications which cover large domains such medicine, ecology, biology, evolution of populations, physics ...

For such inverse problems there exist several methods involving different type of observations. In the case of finite number of observations we can consider for example boundary data, observation on all the domain at one time, spectral data, pointwise observation. On the other hand, nice results have been obtained using infinite number of measurements (e.g. methods using Dirichlet to Neumann map) but we don't consider such approaches here.

In this paper, we focus on the method by the Carleman estimates which allows to derive stability inequality in the form:

$$\|(\text{coefficient}_1) - (\text{coefficient}_2)\| \leq f\left(\|\text{observation}_1 - \text{observation}_2\|\right)$$

for a function f satisfying $\lim_{s\to 0} f(s) = 0$. This kind of inequality links the distance between two sets of coefficients to be reconstructed with the distance between two sets of observations in appropriate norms. Such stability inequalities lead to the uniqueness of the coefficient to be reconstructed given the observation in a good norm. They are also useful to improve the numerical reconstruction of the coefficients using noise-free observations [12]. The paper by [3] has initiated this method and a lot of authors have been inspired by the Carleman estimates to solve inverse problems. There is a huge literature on the determination of nonlinear spatially homogeneous terms or source terms in reaction-difusion equations from boundary measurements,

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see for instance [5, 9, 10, 13, 14, 16, 18, 20, 26] or [25] for a survey on this topic. These works provide Lipschitz stability, in addition to the uniqueness of the coefficients. However, this method requires, among other measurements, the knowledge of the solution u at some time $\theta > 0$ and for all x in the domain. Using a different approach, uniqueness results in the one dimensional case have been established by only one point measurement in space for different parabolic problems with strong nonlinearity [6, 22] or parabolic systems strongly coupled [23] or models with memory term [21] or simultaneous recovery of high order coefficients [11].
All the previous works concern bounded domain Ω and the case of unbounded domains for parabolic operators is less addressed (see for instance [4, 7]). Unbounded domain problems concentrate several difficulties: first, it is mandatory to rewrite or adapt carefully the existing Carleman estimates and in this paper we adapt ideas from [8] and [4] to write a new Carleman estimate for the parabolic operator in the divergence form in an infinite guide. Additively, this method involving some positivity assumption on the gradient of a solution of our problem, we develop a control approach in Sobolev norms for parabolic operators in unbounded domains.
Our manuscript is organized as follows. In section 2, we recall an adapted global Carleman estimate for our problem, in section 3 we prove our main result and in section 4, we carry out an adapted control in order to eliminate the positivity condition imposed by the inverse problem.

1.2 Settings and hypotheses

Let ω be a bounded domain in $\mathbb{R}^{n-1}$, $n \geq 2$ with C^2 boundary. Denote $\Omega := \mathbb{R} \times \omega$ and $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$. We consider the following problem

$$\begin{cases} \partial_t u - \nabla \cdot (c\nabla u) = 0 \ \text{ in } \ Q, \\ u = 0 \text{ on } \ \Sigma, \\ u(x,0) = u_0(x) \text{ in } \ \Omega, \end{cases} \tag{1.1}$$

where u_0 is a sufficiently smooth function and c is a bounded coefficient defined in Ω such that $c > 0$ and $c \in C^1(\overline{\Omega})$. Our problem can be stated as follows.
Let $l > 0$ and denote $\Omega_* = (-*, *) \times \omega$. We want to determine the coefficient c on Ω_l from a finite number of measurements of the solution u of the system (1.1) on a lateral subset of $\partial\Omega_L$ for some $L > l$ and from the knowledge of the solution at the time $\frac{T}{2}$. We stress out that the required measurements are not performed on all the boundary $\partial\Omega_L$ and that they avoid the cross-sections $\{\pm L\} \times \omega$.

Our proof will require the strong positivity condition (1.10). This condition, involved in almost all inverse problems dedicated to coefficients of the principal part of operators, is removed by the construction of an adapted control. The main difficulty is to prove a controllability result in H^1 norm on the unbounded domain Ω. The strategy is inspired by [1, Section 4] where the authors deal with a system of coupled equations but on a bounded domain.

We denote by Q_* the set $Q_* = \Omega_* \times (0,T) = (-*, *) \times \omega \times (0,T)$. For each $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$.

The cornerstone of Carleman estimates is to carry out special weight functions. In our inverse problem, the design of such weight function will allow us to eliminate observations on the cross-section of the wave guide. We follow some ideas from [8] mainly to eliminate the cross section observations since there is less constraints in the choice of the weight functions in the case of a parabolic operator. Nevertheless, we are going to detail the different steps which allow to get the main two inequalities (1.3). For this we choose $a \in \mathbb{R}^n \setminus \Omega_L$ such that if

$$d(x) = |x' - a'|^2 - x_1^2 \text{ for } x \in \Omega_L,$$

then

$$d > 0 \text{ in } \Omega_L, \quad |\nabla d| > 0 \text{ in } \overline{\Omega_L}. \tag{1.2}$$

From now on and for simplicity, we denote $\theta = \frac{T}{2}$. From [25, 26] we consider classical form of weight functions as follows, $t \in (0,T)$,

$$\psi(x,t) = d(x) - (t-\theta)^2 + M_1, \quad \text{where } M_1 > \sup_{0<t<T} (t-\theta)^2 = (T/2)^2,$$

and

$$\phi(x,t) = e^{\lambda\psi(x,t)}.$$

The constant $\lambda > 0$ will be set in Proposition 2.1. In view of the Carleman inequalities for the parabolic operators with regular weights, we need to use cut off functions in time. On the other hand, in view to manage our infinite wave guide we will need also to consider cut off functions in space but only in the infinite direction x_1. These cut-off functions will make appear additive terms coming from the commutator between the evolution operator and these cut-off functions. These residual terms will be estimated thanks to the following crucial properties of the weight function.

Proposition 1.1. *There exist $T > 0$, $L > l$ and $\epsilon > 0$ such that (1.2) holds and, setting*

$$O_{L,\epsilon} := \left\{\Omega_L \times \left((0,2\epsilon) \cup (T-2\epsilon, T)\right)\right\} \cup \left\{\left((-L,-L+2\epsilon) \cup (L-2\epsilon, L)\right) \times \omega \times (0,T)\right\},$$

we have

$$d_1 < d_0 < d_2 \tag{1.3}$$

where

$$d_0 = \inf_{\Omega_l} \phi(\cdot,\theta), \qquad d_1 = \sup_{O_{L,\epsilon}} \phi, \qquad d_2 = \sup_{\overline{\Omega_L}} \phi(\cdot,\theta).$$

These two estimates will be fruitful in Section 3 to solve our inverse problem.

Proof. First we define $\beta_0 := \inf_{x\in\Omega_l} \psi(x,\theta) = \inf_{x\in\Omega_l}(|x'-a'|^2 - x_1^2) + M_1$ and $\beta_1 > 0$ such that

$$\beta_1^2 := \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) - \inf_{x\in\Omega_l} (|x'-a'|^2 - x_1^2).$$

Note that $\beta_1^2 = \sup_{x'\in\omega} |x'-a'|^2 - \inf_{x'\in\omega} |x'-a'|^2 + l^2$.
Then, more precisely, we consider L and $T = 2L$ sufficiently large such that $L > l$ and $\beta_2 := T - \theta - \beta_1 > 0$.

As we have changed L we might have to push a' further away from ω to ensure simultaneously the condition (1.2) and $\beta_2 > 0$. In the case $n \geq 3$, we can push a' in a specific direction that does not modify the value of β_1. The case $n = 2$ can be dealt with explicit computations. Here we take advantage of the particular form of our weight function $d(x)$.

With these definitions, we get

$$\begin{aligned}(T-\theta)^2 \geq \beta_1^2 + \beta_2^2 &= \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) - \inf_{x\in\Omega_l} (|x'-a'|^2 - x_1^2) + \beta_2^2 \\ &\geq \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) + M_1 - \beta_0 + \beta_2^2.\end{aligned}$$

Then, for all $x \in \Omega_L$,

$$\psi(x,T) \leq |x'-a'|^2 - x_1^2 - \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) + \beta_0 - \beta_2^2 \leq \beta_0 - \beta_2^2.$$

As $\psi(x,0)=\psi(x,T)$, we deduce that there exists $\epsilon>0$ such that $\epsilon<\frac{T}{4}$ and:

$$\text{for all } x\in\Omega_L \text{ and } t\in((0,2\epsilon)\cup(T-2\epsilon,T)), \psi(x,t)<\beta_0-\frac{\beta_2^2}{2}.$$

Now, we choose ϵ small enough such that $l\leq L-2\epsilon$ and due to the symmetric role played by $t-\theta$ and x_1 in the formulation of $\psi(x,t)$ by the same way we have:

$$\forall x\in((-L,-L+2\epsilon)\cup(L-2\epsilon,L))\times\omega \text{ and } t\in(0,T),\ \psi(x,t)<\beta_0-\frac{\beta_2^2}{2}.$$

These two estimates end the proof of Proposition 1.1. □

Let us remark that the weight function is bounded by below (away from 0) et by above on Q_L. The fact that the weights do not explode will be convenient in various estimates.

Finally we define

$$\Gamma_L=\{x\in\partial\Omega_L, \langle x-a,\nu(x)\rangle\geq 0\},\qquad \gamma_L=\Gamma_L\cap\partial\Omega. \tag{1.4}$$

Therefore γ_L doesn't contain any cross section. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$ and $\nu(x)$ is the outward unit normal vector to $\partial\Omega_L$ at x.

Notice that, due to the geometry, $\nu_1(x)$ the first component of $\nu(x)$ vanishes on γ_L. Thus, we obtain

$$\sum_{j=1}^{n} c(x)\partial_j d(x)\nu_j(x)\leq 0,\qquad \forall x\in\partial\Omega_L\backslash\Gamma_L. \tag{1.5}$$

This is one of the conditions required for the weight function in the Carleman inequality proved in [26] that we will use in Section 2.

Let V_L be a neighbourhhod of the lateral boundary of Ω_L i.e. an open set such that $V_L\subset\subset\Omega$ and

$$\partial V_L\supset\Big(\partial\Omega_L\cap\partial\Omega\Big). \tag{1.6}$$

We also define

$$\widetilde{\Omega}_L=\Omega_L\backslash V_L. \tag{1.7}$$

Let $c^*\in C^1(\overline{\Omega})\cap L^\infty(\Omega)$, $c^*>C_{min}>0$ on $\overline{\Omega}$ and $\widetilde{M}>0$ a given constant. We will consider the following admissible set of diffusion coefficients

$$D=\left\{c\in C^1(\overline{\Omega})\cap L^\infty(\Omega) \text{ such that } \inf_{\overline{\Omega}} c>0,\ c=c^* \text{ on } V_L \text{ and } \|c\|_{C^1(\overline{\Omega})}<\widetilde{M}\right\}.$$

Notice that this means that the diffusion coefficient is supposed to be known in a neighbourhood of the lateral boundary of interest.

To ease the reading of Section 4, up to a restriction of V_L, we assume that there exists $r>0$ such that

$$V_L\cap\Omega_L=(-L,L)\times\left\{x'\in\omega\,;\ \text{dist}(x',\mathbb{R}^{n-1}\backslash\omega)<r\right\}. \tag{1.8}$$

1.3 Regularity assumptions for the inverse problem

The method of Carleman estimate used in this paper requires solutions of the problem (1.1) that are sufficiently regular. Indeed the Buckgheim-Klibanov method [3] implies several time differentiations of system (1.1).

We assume in the following that $c \in D$ and that u is an element of $\mathcal{H} = H^3(0,T,H^3(\Omega))$ satifying the a-priori bound

$$\|u\|_{\mathcal{H}} < M \qquad \text{for a given } M > 0.$$

More precisely, for the proof of Theorem 1.1, we assume that u is a strong solution of (1.1) enjoying the following regularity

$$u \in C^0(0,T,H^3(\Omega)) \cap C^1(0,T,H^1(\Omega)) \cap H^3(0,T,L^2(\Omega)) \cap L^2(0,T,H^3(\Omega)).$$

In order to obtain the regularity condition $u \in H^3(0,T,H^3(\Omega))$, we will use [15, Theorem 5.3]. Denote by A_0 the self-adjoint realization of $-\nabla(c\nabla\cdot)$ with domain

$$D(A_0) = H^2(\Omega) \cap H_0^1(\Omega). \tag{1.9}$$

By deriving u three times with respect to the time variable t, if we suppose that $u_0 \in D(A_0^3)$ is such that $A^3 u_0 \in H^2(\Omega)$, then we get that $u \in H^3(0,T,H^3(\Omega))$.

We will use the following notations. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $\alpha_i \in \mathbb{N}$, we set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and we define

$$H^{2,1}(Q_L) = \left\{ u \in L^2(Q_L)\,;\, \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \leq 2 \right\},$$

endowed with its norm

$$\|u\|^2_{H^{2,1}(Q_L)} = \sum_{|\alpha|+2\alpha_{n+1}\leq 2} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|^2_{L^2(Q_L)}.$$

1.4 Main results

The first main result of this article is the following global stability estimate.

Theorem 1.1. *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \backslash \Omega_L$ satisfying the conditions of Proposition 1.1. Assume that u_j for $j = 1, 2$ are solutions of (1.1) where c_j and $u_{0,j}$ are substituted respectively to c and u_0. Assume also that $c_1, c_2 \in D$. Then, if there exists $C > 0$ such that*

$$\left| \nabla d \cdot \nabla u_2 \left(x, \frac{T}{2} \right) \right| \geq C, \qquad \text{for a.e. } x \in \widetilde{\Omega}_L, \tag{1.10}$$

where $\widetilde{\Omega}_L$ is defined by (1.7), then,

$$\|c_1 - c_2\|^2_{H^1(\Omega_l)} \leq K \left(\|(u_1 - u_2)(\cdot, T/2)\|^2_{H^3(\Omega_L)} + \int_{\gamma_L \times (0,T)} \sum_{k=1}^{2} \left| \partial_\nu (\partial_t^k (u_1 - u_2)) \right|^2 \right)^\kappa$$

where $\partial_\nu u = \nu \cdot \nabla u$. Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending only on ω, l, M, $\widetilde{M}$, M_1, T and a.

Remark 1.1. *A careful inspection of the proof of Theorem 1.1 shows that the same result holds if we only assume that the diffusion coefficients c_1 and c_2 as well as their gradients are known on the boundary of Ω_L (instead of in a neighbourhood of this boundary) i.e. if $c_1, c_2 \in C^1(\overline{\Omega}) \cap L^\infty(\Omega)$ are such that*

$$\inf_{\overline{\Omega}} c_j > 0, \quad c_j = c^* \text{ and } \nabla c_j = \nabla c^* \text{ on } \partial\Omega \cap \partial\Omega_L, \quad \text{and } \|c_j\|_{C^1(\overline{\Omega})} < \widetilde{M}$$

and one replaces (1.10) by

$$\left| \nabla d \cdot \nabla u_2 \left(x, \frac{T}{2} \right) \right| \geq C, \qquad \text{for a.e. } x \in \Omega_L.$$

Remark 1.2. *Notice that, as already highlighted, the boundary observation is on γ_L and thus does not contain any cross-section in Ω_L. Though the equation is set on the unbounded guide Ω, the stability estimate on Ω_l is obtained with measurements on the finite portion Ω_L.*

Remark 1.3. *We look at the homogeneous Dirichlet case in order to simplify the regularity required for the solution u of (1.1) but we could obtain the same inverse result in the case of the inhomogeneous Dirichlet case*

$$\begin{cases} \partial_t u - \nabla \cdot (c\nabla u) = 0 \ \ in \ \ Q, \\ u = h \ \ on \ \ \Sigma, \\ u(x,0) = u_0(x) \ \ in \ \ \Omega. \end{cases} \tag{1.11}$$

The proof of Theorem 1.1 follows the ideas of [3] and is proved in Section 3. It strongly relies on the Carleman estimate given in Section 2. This strategy is quite classical.

The drawback is that, to develop the Carleman machinery, one needs the technical (not so easy to verify) assumption (1.10). In the following we propose a strategy to remove this assumption:

- we prove the existence of a boundary control such that the associated solution of (1.11) satisfies (1.10);
- we fix this boundary condition for all the required measurements that is we deal only with system (1.11) for a fixed h.

More precisely, the second main result of this article reads as follows

Theorem 1.2. *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \backslash \Omega_L$ satisfying the conditions of Proposition 1.1.*

For $j = 1, 2$ and $h \in L^2(0,T;\partial\Omega)$ we denote by u_j the solutions of (1.11) where c_j and $u_{0,j}$ are substituted respectively to c and u_0.

Let $c_2 \in D$ be such that c_2 and for every $j \in \{1, \dots, n\}$, $\partial_j c_2$, are uniformly continuous and bounded functions in Ω. Then there exists a control $h \in L^2(\gamma_L \times (0,T))$ depending on c_2 such that for any $c_1 \in D$,

$$\|c_1 - c_2\|^2_{H^1(\Omega_l)} \leq K \left(\|(u_1 - u_2)(.,T/2)\|^2_{H^3(\Omega_L)} + \int_{\gamma_L \times (0,T)} \sum_{k=1}^{2} \left| \partial_\nu (\partial_t^k (u_1 - u_2)) \right|^2 \right)^\kappa$$

where $\partial_\nu u = \nu \cdot \nabla u$. Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending only on ω, l, M, $\widetilde{M}$, M_1, T and a.

As noticed the boundary condition h exerted for the measurements associated to c_1 only depends on c_2. Morally, this means that the coefficient c_2 is known and that Theorem 1.2 is a local stability estimate around c_2.

Remark 1.4. *Notice that for any $\widetilde{\gamma}_L \subset \gamma_L$ the same statement holds true with a source-term h supported in $\widetilde{\gamma}_L$.*

This result is proved in Section 4. The main difficulty is to prove the existence of a control h such that the associated solution satisfies (1.10). This part of the proof is inspired by the strategy developed in [1] for systems on bounded domains.

2 Global Carleman estimate for a parabolic equation in a cylindrical domain

We start from a global Carleman-type estimate proved by Yuan and Yamamoto [26, Theorem 2.1, (2)] in a bounded domain. Its validity is ensured by the estimates (1.2) and (1.5) satisfied

by our weight functions. For more informations about Carleman estimates in a parabolic setting we refer to Yamamoto [25].

Let $s > 0$ and denote

$$LHS(u) := \int_{Q_L} \left(\frac{1}{s\phi}(|\partial_t u|^2 + |\Delta u|^2) + s\phi\ |\nabla u|^2 + s^3\phi^3|u|^2 \right) e^{2s\phi}.$$

In the following parts, C will be a generic positive constant.

Proposition 2.1. *There exist a value of $\lambda > 0$ and positive constants s_0 and $C = C(s_0)$ such that*

$$LHS(u) \leq C\|e^{s\phi} f\|^2_{L^2(Q_L)} + Cs \int_{\Gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi}, \tag{2.1}$$

for all $s > s_0$, and all $u \in H^{2,1}(Q_L)$ satisfying

$$\begin{cases} \partial_t u - \nabla \cdot (c\nabla u) = f, & \text{in } \Omega_L, \\ u(\cdot, 0) = u(\cdot, T) = 0 & \text{in } \Omega_L, \\ u = 0 & \text{on } \partial\Omega_L \times (0, T). \end{cases}$$

Let us remark that the Carleman inequality [26, Theorem 2.1, (2)] uses also λ as a second large parameter. As we will not use it, we now consider λ fixed in all the rest of the article such that Proposition 2.1 holds.

As in [4, Prop 4.2], we deduce the following Carleman inequality and we detail the proof for better understanding. The key difference with the Carleman inequality of Proposition 2.1 is to remove, on the cross-sections of Ω_L, the boundary condition and the observation.

Proposition 2.2. *There exist positive constants λ_0, s_0 and $C = C(\lambda_0, s_0)$ such that*

$$LHS(u) \leq C\|e^{s\phi} f\|^2_{L^2(Q_L)} + Cs^3 e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi}, \tag{2.2}$$

for all $s > s_0, \lambda > \lambda_0$ and all $u \in H^{2,1}(Q_L)$ satisfying

$$\begin{cases} \partial_t u - \nabla \cdot (c\nabla u) = f, & \text{in } \Omega_L, \\ u(\cdot, 0) = u(\cdot, T) = 0 & \text{in } \Omega_L, \\ u = 0 & \text{on } (\partial\Omega \cap \partial\Omega_L) \times (0, T). \end{cases}$$

Proof. Let χ, η be C^∞ cut-off functions defined by $0 \leq \chi \leq 1, 0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 0 \text{ if } t \in [0, \epsilon] \cup [T - \epsilon, T], \\ 1 \text{ if } t \in [2\epsilon, T - 2\epsilon], \end{cases} \quad \chi(x) = \begin{cases} 0 \text{ if } x_1 \in (-\infty, -L + \epsilon] \cup [L - \epsilon, +\infty), \\ 1 \text{ if } x_1 \in [-L + 2\epsilon, L - 2\epsilon]. \end{cases} \tag{2.3}$$

Recall that $\partial_t u - \nabla \cdot (c\nabla u) = f$. We consider $y = \eta\chi u$ and we get

$$\partial_t y - \nabla \cdot (c\nabla y) = h \text{ with } h = \eta\chi f + \eta R(u) + (\partial_t \eta)\chi u,$$

where R is the first order differential operator defined by $R(u) = -\nabla \cdot (cu\nabla\chi) - c\nabla\chi \cdot \nabla u$.

Then we can apply the previous Carleman estimate (2.1) and we deduce that there exists a positive constant C such that

$$LHS(y) \leq C\|e^{s\phi} h\|^2_{L^2(Q_L)} + Cs \int_{\Gamma_L \times (0,T)} |\partial_\nu y|^2 e^{2s\phi}.$$

Recall that γ_L is defined in (1.4). Then,

$$\int_{\Gamma_L\times(0,T)} |\partial_\nu y|^2 e^{2s\phi} \leq \int_{\gamma_L\times(0,T)} |\partial_\nu u|^2 e^{2s\phi}.$$

Moreover

$$\|e^{s\phi}\eta R(u)\|^2_{L^2(Q_L)} \leq Ce^{2sd_1}\|u\|^2_{L^2(0,T,H^1(\Omega_L))}$$
$$\text{and } \|e^{s\phi}(\partial_t\eta)\chi u\|^2_{L^2(Q_L)} \leq Ce^{2sd_1}\|u\|^2_{L^2(0,T,L^2(\Omega_L))}.$$

Then we obtain

$$LHS(y) \leq C\|e^{s\phi}f\|^2_{L^2(Q_L)} + Ce^{2sd_1}\|u\|^2_{L^2(0,T,H^1(\Omega_L))} + Cs\int_{\gamma_L\times(0,T)} |\partial_\nu u|^2 e^{2s\phi}. \tag{2.4}$$

Now we deal with $LHS(y)$. For $j = 0, 1, 2$, (with $\nabla^0 u = u$, $\nabla^1 u = \nabla u$, $\nabla^2 u = \Delta u$) since $\chi u = (1-\eta)\chi u + y$, it comes that

$$\|(s\phi)^{3/2-j}e^{s\phi}\nabla^j(\chi u)\|_{L^2(Q_L)} \leq \|(s\phi)^{3/2-j}e^{s\phi}(1-\eta)\nabla^j(\chi u)\|_{L^2(Q_L)} + \|(s\phi)^{3/2-j}e^{s\phi}\nabla^j y\|_{L^2(Q_L)}.$$

Thus,

$$\|(s\phi)^{3/2-j}e^{s\phi}\nabla^j(\chi u)\|_{L^2(Q_L)} \leq C(s^{3/2}e^{sd_1}\|u\|_{H^{2,1}(Q_L)} + \|(s\phi)^{3/2-j}e^{s\phi}\nabla^j y\|_{L^2(Q_L)}).$$

Doing the same for the term $\partial_t(\chi u) = (-\partial_t\eta)\chi u + (1-\eta)\chi\partial_t u + \partial_t y$ we deduce that

$$\|(s\phi)^{-1/2}e^{s\phi}\partial_t(\chi u)\|_{L^2(Q_L)} \leq e^{sd_1}\|u\|_{H^{2,1}(Q_L)} + \|(s\phi)^{-1/2}e^{s\phi}\partial_t y\|_{L^2(Q_L)}.$$

Thus since

$$LHS(\chi u) = \sum_{j=0}^{2} \|(s\phi)^{3/2-j}e^{s\phi}\nabla^j u\|^2_{L^2(Q_L)} + \|(s\phi)^{-1/2}e^{s\phi}\partial_t(\chi u)\|^2_{L^2(Q_L)}$$

we have

$$LHS(\chi u) \leq C(s^3e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + LHS(y)).$$

Then by the identities $\begin{cases} \partial_t u = \partial_t(\chi u) + (1-\chi)\partial_t u, \\ \nabla u = \nabla(\chi u) + (1-\chi)\nabla u - u\nabla\chi, \\ \Delta u = \Delta(\chi u) + (1-\chi)\Delta u - 2\nabla\chi\cdot\nabla u - u\Delta\chi, \end{cases}$ we get

$$LHS(u) \leq C(LHS(\chi u) + s^3e^{2sd_1}\|u\|_{H^{2,1}(Q_L)}) \leq C(s^3e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + LHS(y)).$$

Then, from (2.4), we finish the proof. □

3 Inverse Problem

3.1 Preliminary lemmas

We denote in the following $b(\theta) := b(\cdot,\theta)$ for any function b. As for the reconstruction of zeroth order coefficient (see [4, p.6]), we need to assume some hypothesis on the solution u at time θ on Ω_L. For the diffusion coefficient, the assumption is more involved and is exactly (1.10).

Now following an idea developed for example in [2, Lemma 2.4], we obtain the following result. Note that this key lemma is the one that requires the assumption (1.10). Thus, for the sake of completeness, we will give its proof.

Lemma 3.1. *Assume that* (1.10) *is satisfied and consider the first order partial differential operator* $Pf = \nabla \cdot (f \nabla u_2(\theta))$. *Then there exist positive constants* $s_1 > 0$ *and* $C_1 > 0$ *such that for all* $s \geq s_1$

$$s^2 \int_{\Omega_L} e^{2s\phi(\theta)} |f|^2 \leq C_1 \int_{\Omega_L} e^{2s\phi(\theta)} |Pf|^2,$$

for all $f \in H_0^1(\Omega_L)$ *such that* $f = 0$ *in* V_L *(see* (1.6)*).*

Proof. Let $f \in H_0^1(\Omega_L)$ be such that $f = 0$ in V_L. We denote $w = e^{s\phi(\theta)} f$ and $Qw = e^{s\phi(\theta)} P(e^{-s\phi(\theta)} w)$. So we get $Qw = Pw - sw\nabla\phi(\theta) \cdot \nabla u_2(\theta)$. Therefore we have

$$\begin{aligned}\int_{\Omega_L} |Qw|^2 &\geq s^2 \int_{\Omega_L} w^2 |\nabla\phi(\theta) \cdot \nabla u_2(\theta)|^2 - 2s \int_{\Omega_L} Pw \left(w\nabla\phi(\theta) \cdot \nabla u_2(\theta)\right) \\ &= s^2\lambda^2 \int_{\Omega_L} w^2 \phi(\theta)^2 |\nabla d \cdot \nabla u_2(\theta)|^2 - 2s\lambda \int_{\Omega_L} (Pw) w\phi(\theta)(\nabla d \cdot \nabla u_2(\theta)).\end{aligned}$$

As, $Pw = \nabla \cdot (w\nabla u_2(\theta)) = \nabla w \cdot \nabla u_2(\theta) + w\Delta u_2(\theta)$, we obtain,

$$\begin{aligned}\int_{\Omega_L} |Qw|^2 \geq s^2\lambda^2 \int_{\Omega_L} w^2 \phi(\theta)^2 |\nabla d \cdot \nabla u_2(\theta)|^2 - s\lambda \int_{\Omega_L} \phi(\theta)(\nabla(w^2) \cdot \nabla u_2(\theta))(\nabla d \cdot \nabla u_2(\theta)) \\ - 2s\lambda \int_{\Omega_L} w^2 \phi(\theta) \Delta u_2(\theta)(\nabla d \cdot \nabla u_2(\theta)).\end{aligned}$$

Thus integrating by parts the second term of the right-hand side we obtain

$$\begin{aligned}\int_{\Omega_L} |Qw|^2 \geq s^2\lambda^2 \int_{\Omega_L} w^2 \phi(\theta)^2 |\nabla d \cdot \nabla u_2(\theta)|^2 + s\lambda \int_{\Omega_L} w^2 \nabla \cdot \big(\phi(\theta)\big(\nabla d \cdot \nabla u_2(\theta)\big)\nabla u_2(\theta)\big) \\ - 2s\lambda \int_{\Omega_L} w^2 \phi(\theta) \Delta u_2(\theta)(\nabla d \cdot \nabla u_2(\theta)).\end{aligned}$$

Getting back to the original variables we obtain

$$\begin{aligned}\int_{\Omega_L} e^{2s\phi(\theta)} |Pf|^2 = \int_{\Omega_L} |Qw|^2 \geq s^2\lambda^2 \int_{\Omega_L} e^{2s\phi(\theta)} f^2 \, \phi(\theta)^2 |\nabla d \cdot \nabla u_2(\theta)|^2 \\ + s\lambda \int_{\Omega_L} e^{2s\phi(\theta)} f^2 \, \nabla \cdot \big(\phi(\theta)\big(\nabla d \cdot \nabla u_2(\theta)\big)\nabla u_2(\theta)\big) \\ - 2s\lambda \int_{\Omega_L} e^{2s\phi(\theta)} f^2 \, \phi(\theta) \Delta u_2(\theta)(\nabla d \cdot \nabla u_2(\theta)).\end{aligned}$$

Thus, using assumption (1.10), there exists a constant $C > 0$ depending on M such that

$$\int_{\Omega_L} e^{2s\phi(\theta)} |Pf|^2 \geq Cs^2 \int_{\widetilde{\Omega}_L} e^{2s\phi(\theta)} f^2 - Cs \int_{\Omega_L} e^{2s\phi(\theta)} f^2$$

where $\widetilde{\Omega}_L$ is defined by (1.7). As $f = 0$ in V_L,

$$\int_{\Omega_L} e^{2s\phi(\theta)} |Pf|^2 \geq Cs^2 \int_{\Omega_L} e^{2s\phi(\theta)} f^2 - Cs \int_{\Omega_L} e^{2s\phi(\theta)} f^2.$$

We conclude taking s sufficiently large. □

Moreover we recall the following classical result (see [8]).

Lemma 3.2. *There exist positive constants s_2 and C such that*

$$\int_{\Omega_L} e^{2s\phi(\theta)}|z(\theta)|^2 \leq Cs\int_{Q_L} e^{2s\phi}|z|^2 + \frac{C}{s}\int_{Q_L} e^{2s\phi}|\partial_t z|^2$$

for all $s \geq s_2$ and $z \in H^1(0,T;L^2(\Omega_L))$.

Proof. Recall that η is defined by (2.3). Consider any $w \in H^1(0,T;L^2(\Omega_L))$. We have

$$\begin{aligned}\int_{\Omega_L} |w(x,\theta)|^2 &= \int_{\Omega_L} |\eta(\theta)w(x,\theta)|^2\ dx \\ &= \int_{\Omega_L}\int_0^\theta \partial_t(\eta^2(t)|w(x,t)|^2)\ dt\ dx \\ &= 2\int_0^\theta \int_{\Omega_L} \eta^2(t)w(x,t)\partial_t w(x,t)\ dxdt + 2\int_0^\theta\int_{\Omega_L} \eta(t)\eta'(t)|w(x,t)|^2\ dxdt.\end{aligned}$$

As $0 \leq \eta \leq 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_{\Omega_L} |w(x,\theta)|^2 \leq C(s+1)\int_{Q_L} |w|^2 + \frac{C}{s}\int_{Q_L} |\partial_t w|^2.$$

Then we can conclude replacing w by $e^{s\phi}z$. □

3.2 Proof of Theorem 1.1

Now we study the linearized inverse problem associated with (1.1) and we evaluate some Sobolev norm of the conductivity in terms of suitable observations of the solution of (1.1) on a part of the lateral boundary γ_L. For this consider the following systems with $c_1, c_2 \in D$

$$\begin{cases} \partial_t u_1 - \nabla\cdot(c_1\nabla u_1) = 0 \ \text{ in } \ Q, \\ u_1 = 0 \text{ on } \ \Sigma \\ u_1(x,0) = u_{0,1}(x) \text{ in } \ \Omega, \end{cases}$$

and

$$\begin{cases} \partial_t u_2 - \nabla\cdot(c_2\nabla u_2) = 0 \ \text{ in } \ Q, \\ u_2 = 0 \text{ on } \ \Sigma \\ u_2(x,0) = u_{0,2}(x) \text{ in } \ \Omega. \end{cases}$$

Let

$$y = u_1 - u_2, \qquad c = c_1 - c_2. \tag{3.1}$$

We obtain

$$\partial_t y - \nabla\cdot(c_1\nabla y) = \nabla\cdot(c\nabla u_2) \ \text{ in } \ Q.$$

Remark 3.1. *In the rest of the proof we will deal with y only. Thus, even if we had started with* (1.11)*, y would still enjoy homogeneous Dirichlet boundary condition. This remark will be crucial in Section 4 to prove Theorem 1.2.*

We decompose the proof in three steps.

- First step: as in Proposition 2.2, we derive the equation satified by $z = \chi\eta y$ where the cut-off functions are defined by (2.3). This will allow to work on the bounded domain $\Omega_L \times (0,T)$.

As η only depends on t, we have

$$\partial_t(\eta y) - \nabla \cdot (c_1 \nabla(\eta y)) = \nabla \cdot (c \nabla(\eta u_2)) + y \partial_t \eta.$$

Thus,

$$\chi(\partial_t(\eta y) - \nabla \cdot (c_1 \nabla(\eta y))) = \chi(\nabla \cdot (c \nabla(\eta u_2)) + y \partial_t \eta).$$

Moreover

$$\begin{aligned} \nabla \cdot (c_1 \nabla z) &= \nabla \cdot (c_1 \nabla(\chi \eta y)) \\ &= \nabla \cdot (c_1 \chi \nabla(\eta y)) + \nabla \cdot (c_1 \eta y \nabla \chi) \\ &= \chi \nabla \cdot (c_1 \nabla(\eta y)) + 2 c_1 \nabla(\eta y) \cdot \nabla \chi + \eta y \nabla \cdot (c_1 \nabla \chi) \end{aligned}$$

and

$$\nabla \cdot (c \chi \nabla(\eta u_2)) = \chi \nabla \cdot (c \nabla(\eta u_2)) + c \nabla(\eta u_2) \cdot \nabla \chi.$$

So we get

$$\partial_t z - \nabla \cdot (c_1 \nabla z) = \nabla \cdot (c \chi \nabla(\eta u_2)) + \chi y \partial_t \eta - 2 c_1 \nabla(\eta y) \cdot \nabla \chi - \eta y \nabla \cdot (c_1 \nabla \chi) - c \nabla(\eta u_2) \cdot \nabla \chi. \tag{3.2}$$

- Second step: to estimate $\partial_t z$ (appearing in Lemma 3.2) we differentiate (3.2) with respect to t.

Let

$$z_1 = \partial_t z, \qquad z_2 = \partial_t^2 z.$$

Then,

$$\partial_t z_1 - \nabla \cdot (c_1 \nabla z_1) = f_1 \tag{3.3}$$

where

$$\begin{aligned} f_1 :&= \eta \nabla \cdot (c \chi \nabla(\partial_t u_2)) + \partial_t \eta \nabla \cdot (c \chi \nabla u_2)) - c \nabla(\partial_t(\eta u_2)) \cdot \nabla \chi + \chi \partial_t(y \partial_t \eta) \\ &\quad - 2 c_1 \nabla(\partial_t(\eta y)) \cdot \nabla \chi - \partial_t(\eta y) \nabla \cdot (c_1 \nabla \chi). \end{aligned}$$

And in the same way

$$\partial_t z_2 - \nabla \cdot (c_1 \nabla z_2) = f_2$$

where

$$\begin{aligned} f_2 :&= \eta \nabla \cdot (c \chi \nabla(\partial_t^2(u_2))) + 2 \partial_t \eta \nabla \cdot (c \chi \nabla(\partial_t u_2)) + \partial_t^2 \eta \nabla \cdot (c \chi \nabla u_2) \\ &\quad + \chi \partial_t^2(y \partial_t \eta) - 2 c_1 \nabla(\partial_t^2(\eta y)) \cdot \nabla \chi - \partial_t^2(\eta y) \nabla \cdot (c_1 \nabla \chi) - c \nabla(\partial_t^2(\eta u_2)) \cdot \nabla \chi. \end{aligned}$$

We evaluate (3.2) at $t = \theta$

$$\partial_t z(\theta) - \nabla \cdot (c_1 \nabla z(\theta)) = P(c\chi) - c \nabla u_2(\theta) \cdot \nabla \chi - 2 c_1 \nabla y(\theta) \cdot \nabla \chi - y(\theta) \nabla \cdot (c_1 \nabla \chi) \tag{3.4}$$

with P the operator defined in Lemma 3.1.

Using Lemma 3.1, we obtain an upper bound for a weighted L^2-norm of c. Notice that indeed $c\chi \in H_0^1(\Omega_L)$. The homogeneous Dirichlet boundary condition on the cross-section is ensured by the presence of the cut-off function χ. As $c_1, c_2 \in D$, we also get that $c\chi = 0$ in V_L. Thus,

$$s^2 \int_{\Omega_L} e^{2 s \phi(\theta)} c^2 \chi^2 \leq C \int_{\Omega_L} e^{2 s \phi(\theta)} (P(c\chi))^2.$$

Then, using also (3.4), there exists a positive constant C such that for s sufficiently large

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}c^2\chi^2 \le C\int_{\Omega_L} e^{2s\phi(\theta)}c^2|\nabla\chi|^2 + C\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z(\theta)|^2 \\ + C\int_{\Omega_L} e^{2s\phi(\theta)}(|\Delta z(\theta)|^2 + |\nabla z(\theta)|^2 + |\nabla y(\theta)|^2 + |y(\theta)|^2).$$

Recall that $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z(\theta)|^2 = \int_{\Omega_L} e^{2s\phi(\theta)}|z_1(\theta)|^2$. Thus, applying Lemma 3.2 to z_1 yields

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}c^2\chi^2 \le Ce^{2sd_1} + Ce^{2sd_2}B_1(\theta) + Cs\int_{Q_L} e^{2s\phi}|z_1|^2 + \frac{C}{s}\int_{Q_L} e^{2s\phi}|z_2|^2 \quad (3.5)$$

with $B_1(\theta) = \|z(\theta)\|^2_{H^2(\Omega_L)} + \|y(\theta)\|^2_{H^1(\Omega_L)}$.
Moreover by the Carleman inequality (2.2), for s sufficiently large, we have for $i = 1, 2$,

$$\int_{Q_L} e^{2s\phi}|z_i|^2 \le \int_{Q_L} \phi^3 e^{2s\phi}|z_i|^2 \\ \le \frac{C}{s^3}\int_{Q_L} e^{2s\phi}|f_i|^2 + Ce^{2sd_1}\|z_i\|^2_{H^{2,1}(Q_L)} + \frac{C}{s^2}\int_{\gamma_L\times(0,T)} |\partial_\nu z_i|^2 e^{2s\phi}. \quad (3.6)$$

Thus from (3.5) and (3.6) we obtain

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}c^2\chi^2 \le Ce^{2sd_1} + Ce^{2sd_2}B_1(\theta) + \frac{C}{s^2}\int_{Q_L} e^{2s\phi}(|f_1|^2 + |f_2|^2) \\ + Cse^{2sd_1}\left(\|z_1\|^2_{H^{2,1}(Q_L)} + \|z_2\|^2_{H^{2,1}(Q_L)}\right) + \frac{C}{s}\int_{\gamma_L\times(0,T)} \left(|\partial_\nu z_1|^2 + |\partial_\nu z_2|^2\right) e^{2s\phi}.$$

Notice that, as $\|u_i\|_{\mathcal{H}} < M$, the term $\|z_1\|^2_{H^{2,1}(Q_L)} + \|z_2\|^2_{H^{2,1}(Q_L)}$ is bounded.
Now let us deal with the terms $\int_{Q_L} e^{2s\phi}|f_i|^2$. The first term in f_1 (see (3.3)) can be controlled by $(c\chi)^2 + |\nabla(c\chi)|^2$. The other terms involve derivatives of the cut-off functions (and bounded quantities depending on M and $\widetilde{M}$). Thus, since $e^{2s\phi} \le e^{2s\phi(\theta)}$, we get

$$\int_{Q_L} |f_1|^2 e^{2s\phi} \le Ce^{2sd_1} + C\int_{\Omega_L} ((c\chi)^2 + |\nabla(c\chi)|^2)e^{2s\phi(\theta)}.$$

The same computations on f_2 finally imply

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}c^2\chi^2 \le Ce^{2sd_2}B_1(\theta) + \frac{C}{s^2}\int_{\Omega_L} ((c\chi)^2 + |\nabla(c\chi)|^2)e^{2s\phi(\theta)} + Cse^{2sd_1} \\ + \frac{C}{s}e^{2sd_2}\int_{\gamma_L\times(0,T)} (|\partial_\nu z_1|^2 + |\partial_\nu z_2|^2). \quad (3.7)$$

- Third step: we apply the same strategy to derive an estimate of a weighted H^1-norm of c. For any integer $1 \le i \le n$, taking the space derivative with respect to x_i in (3.4), we obtain

$$\partial_t(\partial_{x_i} z(\theta)) - \nabla\cdot(c_1\nabla\partial_{x_i} z(\theta)) = \nabla\cdot((\partial_{x_i} c_1)\nabla z(\theta)) + \nabla\cdot(\partial_{x_i}(c\chi)\nabla u_2(\theta)) \\ +\nabla\cdot(c\chi\nabla(\partial_{x_i} u_2(\theta))) - \partial_{x_i}(c\nabla u_2(\theta)\cdot\nabla\chi) - \partial_{x_i}(2c_1\nabla y(\theta)\cdot\nabla\chi) - \partial_{x_i}(y(\theta)\nabla\cdot(c_1\nabla\chi)) \quad (3.8)$$

Notice that the second term of the right-hand side can be expressed as

$$\nabla \cdot (\partial_{x_i}(c\chi)\nabla u_2(\theta)) = P(\partial_{x_i}(c\chi)).$$

As previously, the fact that $c_1, c_2 \in D$ and the definition of χ imply that we can apply again Lemma 3.1: there exists a positive constant C such that for s sufficiently large,

$$s^2 \int_{\Omega_L} e^{2s\phi(\theta)} (\partial_{x_i}(c\chi))^2 \leq C \int_{\Omega_L} e^{2s\phi(\theta)} (P(\partial_{x_i}(c\chi)))^2.$$

Thus, expressing $P(\partial_{x_i}(c\chi))$ from (3.8) we obtain

$$\begin{aligned} s^2 \int_{\Omega_L} e^{2s\phi(\theta)} (\partial_{x_i}(c\chi))^2 \leq Ce^{2sd_2} B_2(\theta) + C \int_{\Omega_L} e^{2s\phi(\theta)} |\partial_t(\partial_{x_i} z(\theta))|^2 + C \int_{O_{L,\epsilon}} e^{2s\phi(\theta)} \\ + C \int_{\Omega_L} e^{2s\phi(\theta)} (|c\chi|^2 + |\nabla(c\chi)|^2) \end{aligned}$$

with $B_2(\theta) = \|z(\theta)\|^2_{H^3(\Omega_L)} + \|y(\theta)\|^2_{H^2(\Omega_L)}$.
Then,

$$\begin{aligned} s^2 \int_{\Omega_L} e^{2s\phi(\theta)} (\partial_{x_i}(c\chi))^2 \leq Ce^{2sd_2} B_2(\theta) + C \int_{\Omega_L} e^{2s\phi(\theta)} |\partial_{x_i} z_1(\theta))|^2 \\ + C \int_{\Omega_L} e^{2s\phi(\theta)} (|c\chi|^2 + |\nabla(c\chi)|^2) + Ce^{2sd_1}. \end{aligned}$$

So using Lemma 3.2

$$\begin{aligned} s^2 \int_{\Omega_L} e^{2s\phi(\theta)} (\partial_{x_i}(c\chi))^2 \leq Ce^{2sd_2} B_2(\theta) + C \int_{\Omega_L} e^{2s\phi(\theta)} \left(|c\chi|^2 + |\nabla(c\chi)|^2\right) + Ce^{2sd_1} \\ + Cs \int_{Q_L} e^{2s\phi} (\partial_{x_i} z_1)^2 + \frac{C}{s} \int_{Q_L} e^{2s\phi} (\partial_{x_i} z_2)^2. \end{aligned} \tag{3.9}$$

Moreover by the Carleman inequality (2.2), we have for $j = 1, 2$,

$$\begin{aligned} s \int_{Q_L} e^{2s\phi} |\partial_{x_i} z_j|^2 &\leq C \int_{Q_L} s\phi e^{2s\phi} |\partial_{x_i} z_j|^2 \\ &\leq C \int_{Q_L} e^{2s\phi} |f_j|^2 + Cs^3 e^{2sd_1} \|z_j\|^2_{H^{2,1}(Q_L)} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu z_j|^2 e^{2s\phi} \end{aligned}$$

The term in f_j in the right-hand side is estimated as in the previous step. This leads to

$$s \int_{Q_L} e^{2s\phi} |\partial_{x_i} z_j|^2 \leq C \int_{Q_L} e^{2s\phi} ((c\chi)^2 + |\nabla(c\chi)|^2) + Cs^3 e^{2sd_1} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu z_j|^2 e^{2s\phi}. \tag{3.10}$$

From (3.9) and (3.10) we deduce

$$\begin{aligned} s^2 \int_{\Omega_L} e^{2s\phi(\theta)} (\partial_{x_i}(c\chi))^2 \leq Ce^{2sd_2} B_2(\theta) + C \int_{\Omega_L} e^{2s\phi(\theta)} \left(|c\chi|^2 + |\nabla(c\chi)|^2\right) + Cs^3 e^{2sd_1} \\ + Cs \int_{\gamma_L \times (0,T)} \left(|\partial_\nu z_1|^2 + |\partial_\nu z_2|^2\right) e^{2s\phi} \end{aligned} \tag{3.11}$$

which ends this step.

Using inequalities (3.11) for $1 \leq i \leq n$ and gathering with (3.7), we get for s sufficiently large

$$s^2 \int_{\Omega_L} e^{2s\phi(\theta)} \left((c\chi)^2 + |\nabla(c\chi)|^2 \right) \leq CsB_3(\theta) + Cs^3 e^{2sd_1}$$

with $B_3(\theta) = B_1(\theta) + B_2(\theta) + \int_{\gamma_L \times (0,T)} (|\partial_\nu z_1|^2 + |\partial_\nu z_2|^2)$.
Therefore

$$\|c\|^2_{H^1(\Omega_l)} \leq C \left(e^{2s(d_2-d_0)} B_3(\theta) + se^{2s(d_1-d_0)} \right).$$

As $d_1 - d_0 < 0$ and $d_2 - d_0 > 0$, optimizing this last inequality with respect to s, we complete the proof.

4 Removing the technical assumption by an adapted control

In Theorem 1.1, we assume (1.10) satisfied. However since this hypothesis concerns the gradient of one solution of (1.1) in $\widetilde{\Omega}_L$, it is difficult to verify it in the case of real applications. Thus, we develop ideas based on control theory to ensure such strong hypothesis.

The goal of this section is to prove that for any $c \in D$ sufficiently regular, there exists a control h such that the associated solution of (1.11) satisfies (1.10). Then, the proof of Theorem 1.2 will follow directly (see Section 4.4). As the condition (1.10) involves a property for the gradient, we will seek for controllability results in stronger norms, namely $H^1(\Omega_L)$. The strategy is inspired by [1] and is as follows:

- Assume that there is a function u_b sufficiently smooth and satisfying (1.10) i.e. there exists $\delta > 0$ such that

  $$|\nabla u_b \cdot \nabla d| \geq \delta > 0 \quad \text{a.e. in } \widetilde{\Omega}_L.$$

  The existence and regularity of such function is detailed in Section 4.3.
- Let β a positive constant be such that $|\nabla d| \leq \beta$ in Ω_L. Then, if there exists h such that the associated solution of (1.11), denoted by $u(x,t;h)$, satisfies

  $$\|\nabla u(.,\theta,h) - \nabla u_b\|_{L^\infty(\Omega_L)} \leq \frac{\delta}{2\beta},$$

  it comes that $u(\cdot,\cdot;h)$ satisfies (1.10).
- To do so, we will prove the following result: for any u_b sufficiently regular, any $\varepsilon > 0$ and any $\tau > 0$, there exists $h \in L^2(\gamma_L \times (0,T))$ such that

  $$\|u(\cdot,\tau;h) - u_b\|_{H^2(\Omega)} \leq \varepsilon.$$

  A more precise statement is given in Proposition 4.5. This allows to construct a sequence of controls h_n such that

  $$\nabla u(x,\theta,h_n) \underset{n\to+\infty}{\longrightarrow} \nabla u_b(x), \quad \text{for a.e. } x \in \Omega,$$

  and thus proves the previous item. To prove this result, first, we use classical results from control theory to establish this approximate controllability in $L^2(\Omega)$-norm. Then, using the regularity properties of system (1.1), we extend this result to the $H^2(\Omega)$-norm. To do so, we extend to unbounded domains the strategy given in [1]: after reaching approximately (in the weak norm) the target, let the system evolves freely to benefit from the regularizing properties but not too long to stay close to the target (in strong norms). We start this section recalling those regularization properties we will need.

Assume in the rest of this section that c satisfies the assumptions given in Theorem 1.2 for c_2.

4.1 Analytic properties of the elliptic operator

First, notice that the $H^2(\Omega)$-norm we are interested in is related to the operator A_0 (see (1.9)). Indeed from classical elliptic results (see for instance [17, Theorem 3.1.1]), one has the following result.

Proposition 4.1. *There exists a positive constant C such that for any $u \in D(A_0)$,*

$$\|u\|_{H^2(\Omega)} \leq C\|A_0 u\|_{L^2(\Omega)}. \tag{4.1}$$

Then, we notice that the following propositions (see [17, Theorems 3.1.2 ii) and 3.1.3 ii)]), imply that $-A_0$ generates an analytic semigroup on $L^2(\Omega)$.

Proposition 4.2. *There exists $\omega_0 \in \mathbb{R}$ such that for any $\Re(\lambda) \geq \omega_0$, for any $f \in L^2(\Omega)$ the problem*

$$\begin{cases} \lambda u - \nabla(c\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u \in H^2(\Omega)$ which continuously depends on f. Moreover, the resolvent set $\rho(A_0)$ satisfies $\{\lambda \in \mathbb{C}, \Re(\lambda) \geq \omega_0\} \subset \rho(A_0)$.

Proposition 4.3. *There exist $\omega \geq \omega_0$ and $C > 0$ such that if $\Re(\lambda) \geq \omega$, for any $u \in D(A_0)$, then*

$$|\lambda|\|u\|_{L^2(\Omega)} \leq C\|\lambda u - A_0 u\|_{L^2(\Omega)}.$$

From Proposition 4.2 and Proposition 4.3 we deduce that the operator $-A_0$ is sectorial and thus generates an analytic semigroup. Therefore (see for instance [19, (6.7) p.70]) there exists a positive constant C such that for all $t > 0$ and for all $u \in D(A_0)$

$$\|A_0 e^{-tA_0} u\|_{L^2(\Omega)} \leq \frac{C}{t}\|u\|_{L^2(\Omega)}. \tag{4.2}$$

Finally, from (4.1) and (4.2) we get that there exists a positive constant C such that for all $t > 0$ and for all $u \in D(A_0)$

$$\|e^{-tA_0} u\|_{H^2(\Omega)} \leq \frac{C}{t}\|u\|_{L^2(\Omega)}. \tag{4.3}$$

This is the key inequality we will use.

4.2 Approximate controllability

First we give a classical result of approximate controllability in $L^2(\Omega)$.

Proposition 4.4. *For any $u_b \in L^2(\Omega)$, any $\varepsilon > 0$ and any $\tau > 0$, there exists $h \in L^2(\gamma_L \times (0, \tau))$ such that*

$$\|u(\cdot, \tau; h) - u_b\|_{L^2(\Omega)} \leq \varepsilon.$$

Proof. This result is quite classical in control theory. Let us sketch its proof for the sake of completeness. Let $\widehat{\Omega}$ be an open set in $\mathbb{R}^n$ satisfying

$$\Omega \subset \widehat{\Omega} \quad \text{and} \quad \left(\partial\Omega \cap \left(\mathbb{R}^n \backslash \partial\widehat{\Omega}\right)\right) \subset \gamma_L$$

and let $\hat{\omega}$ be a domain such that $\hat{\omega} \subset\subset \left(\widehat{\Omega} \cap \left(\mathbb{R}^n \backslash \Omega\right)\right)$. From [24] and a classical duality argument, we have the approximate controllability in $L^2(\widehat{\Omega} \times (0,\tau))$ with a localized control $g \in L^2(\hat{\omega} \times (0,\tau))$ for the auxiliary problem

$$\begin{cases} \partial_t u - \nabla \cdot (c \nabla u) = 1_{\hat{\omega}} g \text{ in } \widehat{\Omega} \times (0,\tau), \\ u = 0 \text{ on } \partial\widehat{\Omega} \times (0,\tau), \\ u(.,0) = u_0 \text{ in } \widehat{\Omega}. \end{cases}$$

We conclude the proof of this proposition by taking h as the trace of u on γ_L. □

We now turn to approximate controllability in more regular norms.

Proposition 4.5. *For any $u_b \in D(A_0^2)$, any $\varepsilon > 0$ and any $\tau > 0$, there exists $h \in L^2(\gamma_L \times (0,\tau))$ such that*

$$\|u(\cdot,\tau;h) - u_b\|_{H^2(\Omega)} \leq \varepsilon.$$

Proof. Let $u_b \in D(A_0^2)$ and $\varepsilon > 0$. From Proposition 4.4, for any $\tau_1 \in (0,\tau)$ and any $\delta > 0$, there exists $h \in L^2(\gamma_L \times (0,\tau_1))$ such that

$$\|u(\cdot,\tau_1;h) - u_b\|_{L^2(\Omega)} \leq \delta.$$

In what follows, we extend h by 0 on (τ_1,τ). Thus,

$$u(\cdot,\tau;h) = e^{-(\tau-\tau_1)A_0} u(\cdot,\tau_1;h).$$

First, the regularizing properties will allow to obtain estimates in stronger norms. Indeed, from (4.3) we get

$$\begin{aligned} \left\| u(\cdot,\tau;h) - e^{-(\tau-\tau_1)A_0} u_b \right\|_{H^2(\Omega)} &= \left\| e^{-(\tau-\tau_1)A_0} \left(u(\cdot,\tau_1;h) - u_b \right) \right\|_{H^2(\Omega)} \\ &\leq \frac{C}{(\tau-\tau_1)} \|u(\cdot,\tau_1;h) - u_b\|_{L^2(\Omega)} \\ &\leq \frac{C}{(\tau-\tau_1)} \delta. \end{aligned} \tag{4.4}$$

We now prove that for τ_1 small enough, $e^{-(\tau-\tau_1)A_0} u_b$ is not far from u_b. Indeed, from [19, Theorem 6.13], we have

$$\begin{aligned} \left\| A_0 \left(e^{-(\tau-\tau_1)A_0} u_b - u_b \right) \right\|_{L^2(\Omega)} &= \left\| (e^{-(\tau-\tau_1)A_0} - I) A_0 u_b \right\|_{L^2(\Omega)} \\ &= \left\| \int_0^{\tau-\tau_1} \frac{d}{d\eta} (e^{-\eta A_0}) A_0 u_b \mathrm{d}\eta \right\|_{L^2(\Omega)} \\ &= \left\| \int_0^{\tau-\tau_1} e^{-\eta A_0} A_0^2 u_b \mathrm{d}\eta \right\|_{L^2(\Omega)} \\ &\leq (\tau-\tau_1) \left\| A_0^2 u_b \right\|_{L^2(\Omega)}. \end{aligned} \tag{4.5}$$

From (4.1) and (4.5), we deduce that

$$\left\| e^{-(\tau-\tau_1)A_0} u_b - u_b \right\|_{H^2(\Omega)} \leq C(\tau-\tau_1) \left\| A_0^2 u_b \right\|_{L^2(\Omega)}. \tag{4.6}$$

So using the triangle inequality, estimates (4.4) and (4.6) we obtain

$$\|u(\cdot,\tau;h)-u_b\|_{H^2(\Omega)} \leq \frac{\mathsf{C}}{(\tau-\tau_1)}\delta + \mathsf{C}(\tau-\tau_1)\left\|A_0^2 u_b\right\|_{L^2(\Omega)},$$

for a specific constant C. Finally, choosing τ_1 sufficiently close to τ such that

$$\mathsf{C}(\tau-\tau_1)\|A_0^2 u_b\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2}$$

then, δ sufficiently small such that

$$\frac{\mathsf{C}}{(\tau-\tau_1)}\delta \leq \frac{\varepsilon}{2}$$

end the proof. □

4.3 Construction of an appropriate target

In the above proof, to obtain approximate controllability in $H^2(\Omega)$-norms we need targets that are sufficiently regular, namely in $D(A_0^2)$. To apply our strategy, we now prove that there exists some function $u_b \in D(A_0^2)$ and a constant $\delta > 0$ such that

$$|\nabla d(x)\cdot\nabla u_b(x)| \geq \delta, \quad \text{for a.e. } x \in \widetilde{\Omega}_L.$$

Let $\xi_1 \in C^\infty(\mathbb{R};\mathbb{R})$ such that $0 \leq \xi_1 \leq 1$ and

$$\begin{cases} \xi_1(x_1) = 0, & \text{if } |x_1| \geq 2L, \\ \xi_1(x_1) = 1, & \text{if } |x_1| \leq L. \end{cases}$$

Let $\xi' \in C^\infty(\mathbb{R}^{n-1};\mathbb{R})$ such that $0 \leq \xi' \leq 1$ and

$$\begin{cases} \xi'(x') = 0, & \text{if } \operatorname{dist}(x',\mathbb{R}^{n-1}\backslash\omega) \leq \dfrac{r}{2}, \\ \xi'(x') = 1, & \text{if } \operatorname{dist}(x',\mathbb{R}^{n-1}\backslash\omega) \geq r, \end{cases}$$

where r is defined in (1.8). Let

$$u_b(x) := d(x)\xi_1(x_1)\xi'(x'), \quad \forall x = (x_1,x') \in \Omega.$$

Thus, $u_b \in D(A_0)$ and $\nabla u_b(x) = \nabla d(x)$, for any $x \in \widetilde{\Omega}_L$. Due to (1.2), this implies that (1.10) is satisfied. As $u_b \in C^\infty(\Omega)$ and u_b identically vanishes near the boundary we also obtain that $u_b \in D(A_0^2)$.

Remark 4.1. *Finding such u_b in $C^\infty(\Omega)$ such that*

$$\inf_{\Omega_L} |\nabla u_b \cdot \nabla d| > 0,$$

can easily be done taking for instance $u_b = d$. The main difficulty is to ensure all the boundary conditions so that $u_b \in D(A_0^2)$. This is precisely why we assumed that c is known in V_L. With this assumption, there is no requirement on ∇u_b near the boundary which allows to use cut-off functions to design u_b.

4.4 Conclusion

We now have all the ingredients to prove Theorem 1.2. Let u_b as constructed in Section 4.3. Then, from Proposition 4.5, there exists h depending on c_2 such that the solution u_2 satisfies (1.10).

As proved in Section 3, the stability derives from the Carleman inequality applied to y defined by (3.1). As, u_1 and u_2 are both solutions of (1.11) with the same boundary condition h it comes that y solves (1.1) that is with homogeneous Dirichlet boundary conditions. The rest of the proof remains unchanged.

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