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Subelliptic wave equations are never observable

Cyril Letrouit*[†]

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Abstract

It is well-known that observability (and, by duality, controllability) of the elliptic wave equation, i.e., with a Riemannian Laplacian, in time T_0 is almost equivalent to the Geometric Control Condition (GCC), which stipulates that any geodesic ray meets the control set within time T_0 . We show that in the subelliptic setting, GCC is never verified, and that subelliptic wave equations are never observable in finite time. More precisely, given any subelliptic Laplacian $\Delta = -\sum_{i=1}^m X_i^* X_i$ on a manifold M such that $\text{Lie}(X_1, \dots, X_m) = TM$ but $\text{Span}(X_1, \dots, X_m) \subsetneq TM$, we show that for any $T_0 > 0$ and any measurable subset $\omega \subset M$ such that $M \setminus \omega$ has nonempty interior, the wave equation with subelliptic Laplacian Δ is not observable on ω in time T_0 . The proof is based on the construction of sequences of solutions of the wave equation concentrating on spiraling geodesics (for the associated sub-Riemannian distance) spending a long time in $M \setminus \omega$. As a counterpart, we prove a positive result of observability for the wave equation in the Heisenberg group, where the observation set is a well-chosen part of the phase space.

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1 Introduction

1.1 Setting

This article focuses on the wave equation in sub-Riemannian manifolds, i.e., on subelliptic wave equations. Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n with a non-empty boundary ∂M . We consider a smooth horizontal distribution \mathcal{D} on M ,

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i.e., a smooth assignment $M \ni x \mapsto \mathcal{D}_x \subset T_x M$ (possibly with non-constant rank), and a Riemannian metric g on \mathcal{D} . We also assume that \mathcal{D} satisfies the Hörmander condition

$$\text{Lie}(\mathcal{D}) = TM \quad (1)$$

(see [Mon02]). The triple (M, \mathcal{D}, g) is called a sub-Riemannian structure. Additionally, we make the important assumption that the set of all $x \in M$ such that $\mathcal{D}_x \neq T_x M$ is dense in M ; in other words, (M, \mathcal{D}, g) is nowhere Riemannian. Finally, we assume that M is endowed with a smooth volume μ .

We consider the sub-Riemannian Laplacian $\Delta_{g,\mu}$ on $L^2(M, \mu)$, which only depends on g and μ , defined by

$$\Delta_{g,\mu} = - \sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m X_i^2 + \text{div}_\mu(X_i) X_i$$

where $(X_i)_{1 \leq i \leq m}$ denotes a local g -orthonormal frame such that $\mathcal{D} = \text{Span}(X_1, \dots, X_m)$ and the star designates the transpose in $L^2(M, \mu)$. The divergence with respect to μ is defined by $L_X \mu = (\text{div}_\mu X) \mu$, where L_X stands for the Lie derivative. Then $\Delta_{g,\mu}$ is hypoelliptic (see [Hör67]). In order to simplify notations, we set $\Delta = \Delta_{g,\mu}$ in the sequel, since g and μ are fixed once for all.

We consider Δ with Dirichlet boundary conditions and the domain $D(\Delta)$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\text{Id} - \Delta)u\|_{L^2}$. We also consider the operator $(-\Delta)^{\frac{1}{2}}$ with domain $D((-\Delta)^{\frac{1}{2}})$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\text{Id} - \Delta)^{\frac{1}{2}}u\|_{L^2}$.

Consider the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 & \text{in } (0, T) \times M \\ u = 0 & \text{on } (0, T) \times \partial M, \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \end{cases} \quad (2)$$

where $T > 0$. It is well-known (see for example [GR15, Theorem 2.1], [EN99, Chapter II, Section 6]) that for any $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, there exists a unique solution

$$u \in C^0(0, T; D((-\Delta)^{\frac{1}{2}})) \cap C^1(0, T; L^2(M)) \quad (3)$$

to (2) (in a mild sense).

We set

$$\|v\|_{\mathcal{H}} = \left(\int_M |\nabla^{\text{sR}} v(x)|^2 d\mu(x) \right)^{\frac{1}{2}}. \quad (4)$$

where, for any $\phi \in C^\infty(M)$,

$$\nabla^{\text{sR}} \phi = \sum_{i=1}^m (X_i \phi) X_i$$

is the horizontal gradient. Note that ∇^{sR} is the formal adjoint of $(-\text{div}_\mu)$ in $L^2(M, \mu)$, and that $\Delta = \text{div}_\mu \circ \nabla^{\text{sR}}$. Note also that $\|v\|_{\mathcal{H}} = \|(-\Delta)^{\frac{1}{2}}v\|_{L^2(M, \mu)}$.

The natural energy of a solution is

$$E(u(t, \cdot)) = \frac{1}{2} (\|\partial_t u(t, \cdot)\|_{L^2(M, \mu)}^2 + \|u(t, \cdot)\|_{\mathcal{H}}^2).$$

If u is a solution of (2), then

$$\frac{d}{dt} E(u(t, \cdot)) = 0,$$

and therefore the energy of u at any time is equal to

$$\|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2 = \|u_0\|_{\mathcal{H}}^2 + \|u_1\|_{L^2(M, \mu)}^2.$$

In this paper, we investigate exact observability for the wave equation (2).

Definition 1. Let $T_0 > 0$ and $\omega \subset M$ be a μ -measurable subset. The subelliptic wave equation (2) is exactly observable on ω in time T_0 if there exists a constant $C_{T_0}(\omega) > 0$ such that, for any $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, the solution u of (2) satisfies

$$\int_0^{T_0} \int_{\omega} |\partial_t u(t, x)|^2 d\mu(x) dt \geq C_{T_0}(\omega) \|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2. \quad (5)$$

1.2 Main result

Our main result is the following.

Theorem 1. Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset such that $M \setminus \omega$ has nonempty interior. Then the subelliptic wave equation (2) is not exactly observable on ω in time T_0 .

Consequently, using a duality argument (see Section 4.2), we obtain that exact controllability does not hold either in any finite time.

Definition 2. Let $T_0 > 0$ and $\omega \subset M$ be a measurable subset. The subelliptic wave equation (2) is exactly controllable on ω in time T_0 if for any $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, there exists $g \in L^2((0, T_0) \times M)$ such that the solution u of

$$\begin{cases} \partial_{tt}^2 u - \Delta u = \mathbf{1}_{\omega} g & \text{in } (0, T_0) \times M \\ u = 0 & \text{on } (0, T_0) \times \partial M, \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \end{cases} \quad (6)$$

satisfies $u(T_0, \cdot) = 0$.

Corollary 1. Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset such that $M \setminus \omega$ has nonempty interior. Then the subelliptic wave equation (2) is not exactly controllable on ω in time T_0 .

Remark 3. Theorem 1 holds under the two assumptions that \mathcal{D} satisfies the Hörmander condition (1) and that the set of $x \in M$ such that $\mathcal{D}_x \neq T_x M$ is dense in M . However, inspecting the proof, we see that the conclusion of Theorem 1 also holds under the weaker assumption that the set of $x \in M$ such that $\mathcal{D}_x \subsetneq \mathcal{D}_x + [\mathcal{D}_x, \mathcal{D}_x]$ is dense in M .

In the statement of Theorem 1, we assumed that the sub-Riemannian structure (M, \mathcal{D}, g) verifies $\mathcal{D}_x \neq T_x M$ for a dense set of $x \in M$. Let us explain how to adapt this result to the case of almost-Riemannian structures, i.e., sub-Riemannian structures which do not necessarily verify this assumption. A typical example is the Baouendi-Grushin case, for which $X_1 = \partial_{x_1}$ and $X_2 = x_1 \partial_{x_2}$ are vector fields on $(-1, 1)_{x_1} \times \mathbb{T}_{x_2}$. Then $\text{rank}(\mathcal{D}_x)$ is equal to 1 for $x_1 = 0$ and to 2 otherwise.

Theorem 2. Let $T_0 > 0$ and let $\omega \subset M$ be a measurable set such that $M \setminus \omega$ has an interior which is non-empty and which moreover contains a point x such that $\mathcal{D}_x \neq T_x M$. Then the subelliptic wave equation (2) is not exactly observable on ω in time T_0 .

Remark 4. In the Baouendi-Grushin case, the corresponding Laplacian is elliptic outside of the singular submanifold $S = \{x_1 = 0\}$. Therefore, in the Baouendi-Grushin case, the subelliptic wave equation is observable on any open subset containing S (with some finite minimal time of observability, see [BLR92]), but is not observable in any finite time on any subset ω such that the interior of $M \setminus \omega$ has a non-empty intersection with S .

Remark 5. The assumption of compactness on M is not necessary: we may remove it, and just require that the subelliptic wave equation (2) in M is well-posed. It is for example the case if M is complete for the sub-Riemannian distance induced by g since Δ is then essentially self-adjoint ([Str86]).

Remark 6. Theorem 1 remains true if M has no boundary. In this case, the equation (2) is well-posed in a space slightly smaller than (3): a condition of null average has to be added since non-zero constant functions on M are solutions of (2), see Section 1.5. The observability inequality of Theorem 1 remains true in this space of solutions: anticipating

the proof, we notice that the spiraling geodesics of Proposition 14 still exist (since their construction is purely local), and we subtract to the initial datum u_0^k of the localized solutions constructed in Proposition 13 their spatial average $\int_M u_0^k d\mu$.

Remark 7. Thanks to abstract results (see for example [Mil12]), Theorems 1 and 2 remain true when the subelliptic wave equation (2) is replaced by the subelliptic half-wave equation $\partial_t u + i\sqrt{-\Delta}u = 0$ with Dirichlet boundary conditions.

1.3 Ideas of the proof

The proof of Theorem 1 mainly requires two ingredients:

1. There exist solutions of the free subelliptic wave equation (2) whose energy concentrates along any given (normal) geodesic of (M, \mathcal{D}, g) ;
2. There exist normal geodesics of (M, \mathcal{D}, g) which “spiral” around curves transverse to \mathcal{D} , and which therefore remain arbitrarily close to their starting point on arbitrarily large-time intervals.

Combining these two facts, the proof of Theorem 1 is straightforward (see Section 4.1). Note that the first point follows from the general theory of propagation of complex Lagrangian spaces, while the second point is the main novelty of this paper.

Since our construction is purely local (meaning that it does not “feel” the boundary and only relies on the local structure of the vector fields), we can focus on the case where there is a (small) open neighborhood V of the origin such that $V \subset M \setminus \omega$. In the sequel, we assume it is the case.

Let us give an example of sub-Riemannian structure where the spiraling geodesics used in the proof of Theorem 1 are particularly simple. We consider the three-dimensional manifold with boundary $M_1 = (-1, 1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx (-1, 1)$ is the 1D torus. We endow M_1 with the vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1 \partial_{x_3}$ and we set $\mathcal{D}_1 = \text{Span}(X_1, X_2)$, with the metric g_1 being defined by the fact that (X_1, X_2) is a g_1 -orthonormal frame of \mathcal{D}_1 . Then, $(M_1, \mathcal{D}_1, g_1)$ is a sub-Riemannian structure, which we will call in the sequel the “Heisenberg manifold with boundary”. We endow it with an arbitrary smooth volume μ . The geodesics we consider are given by

$$\begin{aligned} x_1(t) &= \varepsilon \sin(t/\varepsilon) \\ x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon \\ x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned} \tag{7}$$

They spiral around the x_3 axis $x_1 = x_2 = 0$.

Here, one should think of ε as a small parameter. In the sequel, we denote by x_ε the geodesic with parameter ε .

Clearly, given any $T_0 > 0$, for ε sufficiently small, we have $x_\varepsilon(t) \in V$ for every $t \in (0, T_0)$. Our objective is to construct solutions u^k of the subelliptic wave equation (2) such that $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$ and the energy of $u^k(t, \cdot)$ outside of a ball $B_{g_1}(x_\varepsilon(t), r_k)$ centered at $x_\varepsilon(t)$ and with small radius $r_k > 0$

$$\int_{M_1 \setminus B_{g_1}(x_\varepsilon(t), r_k)} (|\partial_t u^k(t, x)|^2 + |\nabla^{\text{sR}} u^k(t, x)|^2) d\mu(x)$$

tends to 0 as $k \rightarrow +\infty$ uniformly with respect to $t \in (0, T_0)$. As a consequence, the observability inequality (5) fails.

The construction of solutions of the free wave equation whose energy concentrates on geodesics is classical in the elliptic (or Riemannian) case: these are the so-called Gaussian beams, for which a construction can be found for example in [Ral82]. Here, we adapt this construction to our subelliptic (sub-Riemannian) setting, which does not raise any problem since the geodesics we consider stay in the elliptic part of the operator Δ . It may also be directly justified with the theory of propagation of complex Lagrangian spaces (see Section 2).

In the general case where (M, \mathcal{D}, g) is not necessarily the Heisenberg manifold without boundary, the existence of spiraling geodesics also has to be justified. For that purpose, we first approximate (M, \mathcal{D}, g) by its *nilpotent approximation*, and we then prove that in the latter, it is possible to identify a ‘‘Heisenberg sub-structure’’, which gives the desired spiraling geodesics.

1.4 Sub-Riemannian geodesics

In this section, we recall a few basic facts about sub-Riemannian geodesics. In this paper, we just need to focus on *normal* geodesics, which are the natural extension of Riemannian geodesics since they are projections of bicharacteristics. Recall that there may also exist abnormal geodesics (see [Mon94]), but we did not address the problem of constructing solutions of (2) concentrating on these geodesics since it is not useful for our purpose.

We denote by $S_{\text{phg}}^m(T^*((0, T) \times M))$ the set of polyhomogeneous symbols of order m with compact support and by $\Psi_{\text{phg}}^m((0, T) \times M)$ the set of associated polyhomogeneous pseudodifferential operators of order m whose distribution kernel has compact support in $(0, T) \times M$ (see Appendix A).

We set $P = \partial_{tt}^2 - \Delta \in \Psi_{\text{phg}}^2((0, T) \times M)$, whose principal symbol is

$$p_2(t, \tau, x, \xi) = -\tau^2 + g^*(x, \xi)$$

with τ the dual variable of t and g^* the principal symbol of $-\Delta$. For $\xi \in T^*M$, we have (see Appendix A)

$$g^* = \sum_{i=1}^m h_{X_i}^2.$$

Here, given any smooth vector field X on M , we denote by h_X the Hamiltonian function (momentum map) on T^*M associated with X defined in local (x, ξ) -coordinates by $h_X(x, \xi) = \xi(X(x))$. Then g^* is both the principal symbol of $-\Delta$, and also the cometric associated with g .

In $T^*(\mathbb{R} \times M)$, the Hamiltonian vector field \vec{H}_{p_2} associated with p_2 is given by $\vec{H}_{p_2}f = \{p_2, f\}$. Since $\vec{H}_{p_2}p_2 = 0$, we get that p_2 is constant along the integral curves of \vec{H}_{p_2} . Thus, the characteristic set $\mathcal{C}(p_2) = \{p_2 = 0\}$ is preserved by the flow of \vec{H}_{p_2} . Null-bicharacteristics are then defined as the maximal integral curves of \vec{H}_{p_2} which live in $\mathcal{C}(p_2)$. In other words, the null-bicharacteristics are the maximal solutions of

$$\begin{cases} \dot{t}(s) = -2\tau(s), \\ \dot{x}(s) = \nabla_{\xi} g^*(x(s), \xi(s)), \\ \dot{\tau}(s) = 0, \\ \dot{\xi}(s) = -\nabla_x g^*(x(s), \xi(s)), \\ \tau^2(0) = g^*(x(0), \xi(0)). \end{cases} \quad (8)$$

This definition needs to be adapted when the null-bicharacteristic meets the boundary ∂M , but in the sequel, we only consider solutions of (8) on time intervals where $x(t)$ does not reach ∂M .

In the sequel, we take $\tau = -1/2$, which gives $g^*(x(s), \xi(s)) = 1/4$. This also implies that $t(s) = s + t_0$ and, taking t as a time parameter, we are led to solve

$$\begin{cases} \dot{x}(t) = \nabla_{\xi} g^*(x(t), \xi(t)), \\ \dot{\xi}(t) = -\nabla_x g^*(x(t), \xi(t)), \\ g^*(x(0), \xi(0)) = \frac{1}{4}. \end{cases} \quad (9)$$

In other words, the t -variable parametrizes null-bicharacteristics in a way that they are traveled at speed 1.

Remark 8. *In the subelliptic setting, the co-sphere bundle S^*M can be decomposed as $S^*M = U^*M \cup S\Sigma$, where $U^*M = \{g^* = 1/4\}$ is a cylinder bundle, $S\Sigma = \{g^* = 0\}$ is the characteristic cone and $S\Sigma$ is the sphere bundle of Σ (see [CdVHT18, Section 1]).*

We denote by $\phi_t : S^*M \rightarrow S^*M$ the (normal) geodesic flow defined by $\phi_t(x_0, \xi_0) = (x(t), \xi(t))$, where $(x(t), \xi(t))$ is a solution of the system given by the first two lines of (9) and initial conditions (x_0, ξ_0) . Note that any point in $S\Sigma$ is a fixed point of ϕ_t , and that the other normal geodesics are traveled at speed 1 since we took $g^* = 1/4$ in U^*M (see Remark 8).

The curves $x(t)$ which solve (9) are geodesics (i.e. local minimizers) for the sub-Riemannian metric g . In other words, the projections of the null-bicharacteristics onto M , using the variable t as a parameter, are geodesics on M associated with the sub-Riemannian metric g (and traveled at speed one).

1.5 Observability in some regions of phase-space

We have explained in Section 1.3 that the existence of solutions of the subelliptic wave equation (2) concentrated on spiraling geodesics is an obstruction to observability in Theorem 1. Our goal in this section is to state a result ensuring observability if one “removes” in some sense these geodesics.

For this result, we focus on a version of the Heisenberg manifold described in Section 1.3 which has *no boundary*. This technical assumption avoids us using boundary microlocal defect measures in the proof, which, in this sub-Riemannian setting, are difficult to handle. As a counterpart, we need to consider solutions of the wave equation with null initial average, in order to get well-posedness.

We consider the Heisenberg group G , that is \mathbb{R}^3 with the composition law

$$(x_1, x_2, x_3) \star (x'_1, x'_2, x'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 - x_1 x'_2).$$

Then $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1 \partial_{x_3}$ are left invariant vector fields on G . Since $\Gamma = \sqrt{2\pi}\mathbb{Z} \times \sqrt{2\pi}\mathbb{Z} \times 2\pi\mathbb{Z}$ is a co-compact subgroup of G , the left quotient $M_H = \Gamma \backslash G$ is a compact three dimensional manifold and, moreover, X_1 and X_2 are well-defined as vector fields on the quotient. Finally, we define the Heisenberg Laplacian $\Delta_H = X_1^2 + X_2^2$ on M_H . Since $[X_1, X_2] = -\partial_{x_3}$, it is a hypoelliptic operator. We set $\mathcal{D}_H = \text{Span}(X_1, X_2)$, with the metric g_H being defined by the fact that (X_1, X_2) is a g_H -orthonormal frame of \mathcal{D}_H . Then, $(M_H, \mathcal{D}_H, g_H)$ is a sub-Riemannian structure, which we call the “Heisenberg manifold without boundary”. We endow $(M_H, \mathcal{D}_H, g_H)$ with an arbitrary smooth volume μ .

We introduce the space

$$L_0^2 = \left\{ u_0 \in L^2(M_H), \int_{M_H} u_0 d\mu = 0 \right\}$$

and we consider the operator Δ_H whose domain $D(\Delta_H)$ which is the completion in L_0^2 of the set of all $u \in C_c^\infty(M_H)$ with null-average for the norm $\|(\text{Id} - \Delta_H)u\|_{L^2}$. Then, $-\Delta_H$ is definite positive and we consider $(-\Delta_H)^{\frac{1}{2}}$ with domain $D((-\Delta_H)^{\frac{1}{2}}) = \mathcal{H}_0 := L_0^2 \cap \mathcal{H}(M_H)$. The wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta_H u = 0 & \text{in } \mathbb{R} \times M_H \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in D((-\Delta_H)^{\frac{1}{2}}) \times L_0^2 \end{cases} \quad (10)$$

admits a unique solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{\frac{1}{2}})) \cap C^1(\mathbb{R}; L_0^2)$.

We note that $-\Delta_H$ is invertible in L_0^2 . The space \mathcal{H}_0 is endowed with the norm $\|u\|_{\mathcal{H}}$ (defined in (4) and also equal to $\|(-\Delta_H)^{\frac{1}{2}}u\|_{L^2}$), and its topological dual \mathcal{H}'_0 is endowed with the norm $\|u\|_{\mathcal{H}'_0} := \|(-\Delta_H)^{-\frac{1}{2}}u\|_{L^2}$.

We note that $g^*(x, \xi) = \xi_1^2 + (\xi_2 - x_1 \xi_3)^2$ and hence the null-bicharacteristics are solutions of

$$\begin{aligned} \dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1 \xi_3), \\ \dot{x}_2(t) &= 2(\xi_2 - x_1 \xi_3), & \dot{\xi}_2(t) &= 0, \\ \dot{x}_3(t) &= -2x_1(\xi_2 - x_1 \xi_3), & \dot{\xi}_3(t) &= 0. \end{aligned} \quad (11)$$

The spiraling geodesics described in Section 1.3 correspond to $\xi_1 = \cos(t/\varepsilon)/2$, $\xi_2 = 0$ and $\xi_3 = 1/(2\varepsilon)$. In particular, the constant ξ_3 is a kind of rounding number reflecting the fact that the geodesic spirals at a certain speed around the x_3 axis. Moreover, ξ_3 is preserved by the flow (somehow, the Heisenberg flow is completely integrable), and this property plays a key role in the proof of Theorem 3 below and justifies that we state it only for the Heisenberg manifold (without boundary).

As said above, geodesics corresponding to a large momentum ξ_3 are precisely the ones used to contradict observability in Theorem 1. We expect to be able to establish observability if we consider only solutions of (2) whose ξ_3 (in a certain sense) is not too large. This is the purpose of our second main result.

Set

$$V_\varepsilon = \left\{ (x, \xi) \in T^*M_H : |\xi_3| > \frac{1}{\varepsilon} (g_x^*(\xi))^{1/2} \right\}$$

Note that since ξ_3 is constant along null-bicharacteristics, V_ε and its complementary V_ε^c are invariant under the bicharacteristic equations (11).

In the next statement, we call horizontal strip the periodization under the action of the co-compact subgroup Γ of a set of the form

$$\{(x_1, x_2, x_3) : (x_1, x_2) \in [0, \sqrt{2\pi})^2, x_3 \in I\}$$

where I is a strict open subinterval of $[0, 2\pi)$.

Theorem 3. *Let $B \subset M_H$ be an open sub-Riemannian ball and suppose that B is sufficiently small, so that $\omega = M_H \setminus B$ contains a horizontal strip. Let $a \in S_{\text{phg}}^0(T^*M_H)$, $a \geq 0$, such that, denoting by $j : T^*\omega \rightarrow T^*M_H$ the canonical injection,*

$$j(T^*\omega) \cup V_\varepsilon \subset \text{Supp}(a) \subset T^*M_H,$$

and in particular a does not depend on time. There exists $\kappa > 0$ such that for any $\varepsilon > 0$ and any $T \geq \kappa\varepsilon^{-1}$, there holds

$$C \|(u(0), \partial_t u(0))\|_{\mathcal{H}_0 \times L_0^2}^2 \leq \int_0^T |(Op(a)\partial_t u, \partial_t u)_{L^2}| dt + \|(u(0), \partial_t u(0))\|_{L_0^2 \times \mathcal{H}_0}^2 \quad (12)$$

for some $C = C(\varepsilon, T) > 0$ and for any solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{\frac{1}{2}})) \cap C^1(\mathbb{R}; L_0^2)$ of (10).

The term $\|(u_0, u_1)\|_{L^2 \times \mathcal{H}_0}^2$ in the right-hand side of (12) cannot be removed, i.e. our statement only consists in a *weak* observability inequality. Indeed, the usual way to remove such terms is to use a unique continuation argument for eigenfunctions φ of Δ , but here it does not work since $Op(a)\varphi = 0$ does not imply in general that $\varphi \equiv 0$ in the whole manifold, even if the support of a contains $j(T^*\omega)$ for some non-empty open set ω : in some sense, there is no ‘‘pseudodifferential unique continuation argument’’ available in the literature.

1.6 Comments on the existing literature

Elliptic and subelliptic waves. The exact controllability/observability of the elliptic wave equation is known to be almost equivalent to the so-called Geometric Control Condition (GCC) (see [BLR92]) that any geodesic enters the control set ω within time T . In some sense, our main result is that GCC is not verified in the subelliptic setting, as soon as $M \setminus \omega$ has non-empty interior. For the elliptic wave equation, in many geometrical situations, there exists a minimal time $T_0 > 0$ such that observability holds only for $T \geq T_0$: when there exists a geodesic $\gamma : (0, T_0) \rightarrow M$ traveled at speed 1 which does not meet $\bar{\omega}$, one constructs a sequence of initial data $(u_0^k, u_1^k)_{k \in \mathbb{N}^*}$ of the wave equation whose associated microlocal defect measure is concentrated on $(x_0, \xi_0) \in S^*M$ taken to be the initial conditions for the null-bicharacteristic projecting onto γ . Then, the associated sequence of solutions $(u^k)_{k \in \mathbb{N}^*}$ of the wave equation has an associated microlocal defect measure ν which is invariant under the

geodesic flow: $\vec{H}_p\nu = 0$ where \vec{H}_p is the Hamiltonian flow associated to the principal symbol p of the wave operator. In particular, denoting by $\pi : T^*M \rightarrow M$ the canonical projection, $\pi_*\nu$ gives no mass to ω since γ is contained in $M \setminus \bar{\omega}$, and this proves that observability cannot hold.

In the subelliptic setting, the invariance property $\vec{H}_p\nu = 0$ does not give any information on ν on the characteristic manifold Σ , since $\vec{H}_p = -2\tau\partial_t + \vec{g}^*$ vanishes on Σ . This is related to the lack of information on propagation of singularities in this characteristic manifold, see the main theorem of [Las82]. If one instead tries to use the propagation of the microlocal defect measure for subelliptic half-wave equations, one is immediately confronted with the fact that $\sqrt{-\Delta}$ is not a pseudodifferential operator near Σ .

This is why, in this paper, we used only the elliptic part of the symbol g^* (or, equivalently, the strictly hyperbolic part of p_2), where the propagation properties can be established, and then the problem is reduced to proving geometric results on geodesics of sub-Riemannian manifolds.

Subelliptic Schrödinger equations. The recent article [BS19] deals with the same observability problem, but for subelliptic Schrödinger equations: namely, the authors consider the (Baouendi)-Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$, where $u \in L^2((0, T) \times M_G)$, $M_G = (-1, 1)_x \times \mathbb{T}_y$ and $\Delta_G = \partial_x^2 + x^2\partial_y^2$ is the Baouendi-Grushin Laplacian. Given a control set of the form $\omega = (-1, 1)_x \times \omega_y$, where ω_y is an open subset of \mathbb{T} , the authors prove the existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of a horizontal strip contained in $M_G \setminus \omega$. The intuition is that there are solutions of the Baouendi-Grushin Schrödinger equation which travel along the degenerate line $x = 0$ at a finite speed: in some sense, along this line, the Schrödinger equation behaves like a classical (half)-wave equation. What we want here is to explain in a few words why there is a minimal time of observability for the Schrödinger equation, while the wave equation is never observable in finite time as shown by Theorem 1.

The plane $\mathbb{R}_{x,y}^2$ endowed with the vector fields ∂_x and $x\partial_y$ also admits geodesics similar to the 1-parameter family q_ε , namely, for $\varepsilon > 0$,

$$\begin{aligned} x(t) &= \varepsilon \sin(t/\varepsilon) \\ y(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4) \end{aligned}$$

These geodesics, denoted by γ_ε , also “spiral” around the line $x = 0$ more and more quickly as $\varepsilon \rightarrow 0$, and so we might expect to construct solutions of the Baouendi-Grushin Schrödinger equation with energy concentrated along γ_ε , which would contradict observability when $\varepsilon \rightarrow 0$ as above for the Heisenberg wave equation.

However, we can convince ourselves that it is not possible to construct such solutions: in some sense, the dispersion phenomena of the Schrödinger equation exactly compensate the lengthening of the geodesics γ_ε as $\varepsilon \rightarrow 0$ and explain that even these Gaussian beams may be observed in ω from a certain minimal time $\mathcal{L}(\omega) > 0$ which is uniform in ε .

To put this argument into a more formal form, we consider the solutions of the bicharacteristic equations for the Baouendi-Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$ given by

$$\begin{aligned} x(t) &= \varepsilon \sin(\xi_y t) \\ y(t) &= \varepsilon^2 \xi_y \left(\frac{t}{2} - \frac{\sin(2\xi_y t)}{4\xi_y} \right) \\ \xi_x(t) &= \varepsilon \xi_y \cos(\xi_y t) \\ \xi_y(t) &= \xi_y. \end{aligned}$$

It follows from the hypoellipticity of Δ_G (see [BS19, Section 3] for a proof) that

$$|\xi_y|^{1/2} \lesssim \sqrt{-\Delta_G} = (|\xi_x|^2 + x^2|\xi_y|^2)^{1/2} = \varepsilon|\xi_y|.$$

Therefore $\varepsilon^2|\xi_y| \gtrsim 1$, and hence $|y(t)| \gtrsim t$, independently from ε and ξ_y . This heuristic gives the intuition that a minimal time $\mathcal{L}(\omega)$ is required to detect all solutions of the Baouendi-Grushin Schrödinger equation from ω , but that for $T_0 > \mathcal{L}(\omega)$, no solution is localized enough to stay in $M \setminus \omega$ during the time interval $(0, T_0)$. Roughly speaking, the frequencies of order ξ_y travel at speed $\sim \xi_y$, which is typical for a dispersion phenomenon. This picture is very different from the one for the wave equation (which we consider in this paper) for which no dispersion occurs.

With similar ideas, in [LS20], the interplay between the subellipticity effects measured by the non-holonomic order of the sub-Riemannian distribution (see Section 3.1) and the strength of dispersion of Schrödinger-type equations was investigated. More precisely, for $\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ on $M = (-1, 1)_x \times \mathbb{T}_y$, and for $s \in \mathbb{N}$, the observability properties of the Schrödinger-type equation $(i\partial_t - (-\Delta_\gamma)^s)u = 0$ were shown to depend on the value $\kappa = 2s/(\gamma + 1)$. In particular it is proved that, for $\kappa < 1$, observability fails for any time, which is consistent with the present result, and that for $\kappa = 1$, observability holds only for sufficiently large times, which is consistent with the result of [BS19]. The results of [LS20] are somehow Schrödinger analogues of the results of [BCG14] which deal with a similar problem for the Baouendi-Grushin heat equation.

General bibliographical comments. Control of subelliptic PDEs has attracted much attention in the last decade. Most results in the literature deal with subelliptic parabolic equations, either the Baouendi-Grushin heat equation ([Koe17], [DK20], [BDE20]) or the heat equation in the Heisenberg group ([BC17], see also references therein). The paper [BS19] is the first to deal with a subelliptic Schrödinger equation and the present work is the first to handle exact controllability of subelliptic wave equations.

A slightly different problem is the *approximate* controllability of hypoelliptic PDEs, which has been studied in [LL20] for hypoelliptic wave and heat equations. Approximate controllability is weaker than exact controllability, and it amounts to proving “quantitative” unique continuation results for hypoelliptic operators. For the hypoelliptic wave equation, it is proved in [LL20] that for $T > 2 \sup_{x \in M}(\text{dist}(x, \omega))$ (here, dist is the sub-Riemannian distance), the observation of the solution on $(0, T) \times \omega$ determines the initial data, and therefore the whole solution.

1.7 Organization of the paper

In Section 2, we construct exact solutions of the subelliptic wave equation (2) concentrating on any given normal sub-Riemannian geodesic. First, in Section 2.1, we show that, given any normal sub-Riemannian geodesic $t \mapsto x(t)$ of (M, \mathcal{D}, g) (i.e., a projection of a null-bicharacteristic of the associated Hamiltonian system) which does not hit ∂M in the time interval $(0, T)$, it is possible to construct a sequence $(v_k)_{k \in \mathbb{N}}$ of *approximate solutions* of (2) whose energy concentrates along $t \mapsto x(t)$ during the time interval $(0, T)$ as $k \rightarrow +\infty$. By “approximate”, we mean here that $\partial_{tt}^2 v_k - \Delta v_k$ is small, but not necessarily exactly equal to 0. In Section 2.1, we provide a first proof for this construction using the classical propagation of complex Lagrangian spaces. An other proof using a Gaussian beam approach is provided in Appendix B. Then, in Section 2.2, using this sequence $(v_k)_{k \in \mathbb{N}}$, we explain how to construct a sequence $(u_k)_{k \in \mathbb{N}}$ of *exact* solutions of $(\partial_{tt}^2 - \Delta)u = 0$ in M with the same concentration property along the geodesic $t \mapsto x(t)$.

In Section 3, we prove the existence of geodesics which spiral in M , spending an arbitrarily large time in $M \setminus \omega$. These geodesics generalize the example described in Section 1.3 for the Heisenberg manifold with boundary. The proof proceeds in two steps, first proving the result in the so-called “nilpotent case” (Section 3.2), and then extending it to the general case (Section 3.3).

In Section 4.1, we use the results of Section 2 and Section 3 to conclude the proof of Theorem 1 and to prove Theorem 2. In Section 4.2, we deduce Corollary 1 by a duality argument. Finally, in Section 4.3, we prove Theorem 3.

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2 Gaussian beams along normal sub-Riemannian geodesics

2.1 Construction of sequences of approximate solutions

We consider a solution $(x(t), \xi(t))_{t \in [0, T]}$ of (9) on M . We shall describe the construction of solutions of

$$\partial_{tt}^2 u - \Delta u = 0 \quad (13)$$

on $[0, T] \times M$ with energy

$$E(u(t, \cdot)) := \frac{1}{2} \int_M (|\partial_t u(t, x)|^2 + |\nabla^{sR} u(t, x)|^2) d\mu(x)$$

concentrated along $x(t)$ for $t \in [0, T]$. The following proposition, which is inspired by [Ral82] and [MZ02], shows that it is possible, at least for approximate solutions of (13).

Proposition 9. *Fix $T > 0$ and let $(x(t), \xi(t))_{t \in [0, T]}$ be a solution of (9) (in particular $g^*(x(0), \xi(0)) = 1/4$) which does not hit the boundary ∂M in the time-interval $(0, T)$. Then there exist $a_0, \psi \in C^2((0, T) \times M)$ such that, setting, for $k \in \mathbb{N}$,*

$$v_k(t, x) = k^{\frac{n}{4}-1} a_0(t, x) e^{ik\psi(t, x)}$$

the following properties hold:

- v_k is an approximate solution of (13), meaning that

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1((0, T); L^2(M))} \leq Ck^{-\frac{1}{2}}. \quad (14)$$

- The energy of v_k is bounded below with respect to k and $t \in [0, T]$:

$$\exists A > 0, \forall t \in [0, T], \quad \liminf_{k \rightarrow +\infty} E(v_k(t, \cdot)) \geq A. \quad (15)$$

- The energy of v_k is small off $x(t)$: for any $t \in [0, T]$, we fix V_t an open subset of M for the initial topology of M , containing $x(t)$, so that the mapping $t \mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then

$$\sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t v_k(t, x)|^2 + |\nabla^{sR} v_k(t, x)|^2) d\mu(x) \xrightarrow{k \rightarrow +\infty} 0. \quad (16)$$

Remark 10. *The construction of approximate solutions such as the ones provided by Proposition 9 is usually done for strictly hyperbolic operators, that is operators with a principal symbol p_m of order m such that the polynomial $f(s) = p_m(t, q, s, \xi)$ has m distinct real roots when $\xi \neq 0$ (see for example [Ral82]). The operator $\partial_{tt}^2 - \Delta$ is not strictly hyperbolic because g^* is degenerate, but our proof shows that the same construction may be adapted without difficulty to this operator along normal bicharacteristics. This is due to the fact that along normal bicharacteristics, $\partial_{tt}^2 - \Delta$ is indeed strictly hyperbolic (or equivalently, Δ is elliptic). It was already noted by [Ral82] that the construction of Gaussian beams could be done for more general operators than strictly hyperbolic ones, and that the differences between the strictly hyperbolic case and more general cases arise while dealing with propagation of singularities. Also, in [Hör07, Chapter 24.2], it was noticed that “since only microlocal properties of p_2 are important, it is easy to see that hyperbolicity may be replaced by $\nabla_{\xi} p_2 \neq 0$ ”.*

Hereafter we provide two proofs of Proposition 9. The first proof is short and is actually quite straightforward for readers acquainted with the theory of propagation of complex Lagrangian spaces, once one has noticed that the solutions of (9) which we consider live in the *elliptic part* of the principal symbol of $-\Delta$. For the sake of completeness, and because this also has its own interest, we provide in Appendix B a second proof, longer but more elementary and accessible without any knowledge of complex Lagrangian spaces; it relies on the construction of Gaussian beams in the subelliptic context. The two proofs follow parallel paths, and indeed, the computations which are only sketched in the first proof are written in full details in the second proof, given in Appendix B.

First proof of Proposition 9. The construction of Gaussian beams, or more generally of a WKB approximation, is related to the transport of complex Lagrangian spaces along bicharacteristics, as reported for example in [Hör07, Chapter 24.2] and [Ivr19, Volume I, Part I, Chapter 1.2]. Our proof follows the lines of [Hör07, pages 426-428].

A usual way to solve (at least approximately) evolution equations of the form

$$Pu = 0 \tag{17}$$

where P is a hyperbolic second order differential operator with real principal symbol and C^∞ coefficients is to search for oscillatory solutions

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}. \tag{18}$$

In this expression as in the rest of the proof, we suppress the time variable t . Thus, we use $x = (x_0, x_1, \dots, x_n)$ where $x_0 = t$ in the earlier notations, and we set $x' = (x_1, \dots, x_n)$. Similarly, we take the notation $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ where $\xi_0 = \tau$ previously, and $\xi' = (\xi_1, \dots, \xi_n)$. The bicharacteristics are parametrized by s as in (8), and without loss of generality, we only consider bicharacteristics with $x(0) = 0$ at $s = 0$, which implies in particular $x_0(s) = s$ because of our choice $\tau^2(s) = g^*(x(s), \xi(s)) = 1/4$.

Taking charts of M , we can assume $M \subset \mathbb{R}^n$. The precise argument for reducing to this case is written at the end of Appendix B. Also, in the sequel, $P = \partial_t^2 - \Delta$.

Plugging the Ansatz (18) into (17), we get

$$Pv_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi} \tag{19}$$

with

$$\begin{aligned} A_1(x) &= p_2(x, \nabla\psi(x)) a_0(x) \\ A_2(x) &= La_0(x) \\ A_3(x) &= \partial_{tt}^2 a_0(x) - \Delta a_0(x). \end{aligned}$$

and L is a transport operator given by

$$La_0 = \frac{1}{i} \sum_{j=0}^n \frac{\partial p_2}{\partial \xi_j}(x, \nabla\psi(x)) \frac{\partial a_0}{\partial x_j} + \frac{1}{2i} \left(\sum_{j,k=0}^n \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) a_0. \tag{20}$$

In order for v_k to be an approximate solution of P , we are first led to cancel the higher order term in (19), i.e.,

$$f(x) := p_2(x, \nabla\psi(x)) = 0 \tag{21}$$

which we solve for initial conditions

$$\psi(0, x') = \psi_0(x'), \quad \nabla\psi_0(0) = \xi'(0) \quad \text{and} \quad \psi_0(0) = 0 \tag{22}$$

(i.e., we fix such a ψ_0 , and then we solve (21) for ψ). Indeed, it will be sufficient for our purpose for (21) to be verified at second order along the curve $x(s)$, i.e., $D_x^\alpha f(x(s)) = 0$ for any $|\alpha| \leq 2$ and any s . For that, we first notice that the choice $\nabla\psi(x(s)) = \xi(s)$ ensures

that (21) holds at orders 0 and 1 along the curve $s \mapsto x(s)$ (see Appendix B for detailed computations). Now, we explain how to choose $D^2\psi(x(s))$ adequately in order for (21) to hold at order 2.

We use the decomposition of p_2 into

$$p_2(x_0, x', \xi_0, \xi') = -(\xi_0 - r(x', \xi'))(\xi_0 + r(x', \xi')) + R(x', \xi')$$

where $r = \sqrt{g^*}$ in a conic neighborhood of $(0, \xi(0))$. Note that $\sqrt{g^*}$ is smooth in small conic neighborhoods of $(0, \xi(0))$ since $g^*(0, \xi(0)) = 1/4 \neq 0$. Indeed, g^* is elliptic along the whole bicharacteristic since $g^*(x(t), \xi(t)) = 1/4$ is preserved by the bicharacteristic flow. The rest term $R(x', \xi')$ is smooth and microlocally supported far from the bicharacteristic, i.e., $R(x', \xi') = 0$ for any $(x', \xi') \in T^*M$ in a conic neighborhood of $(x'(s), \xi'(s))$ for $s \in [0, T]$.

We consider the bicharacteristic γ_+ starting at $(0, 0, r(0, \xi'(0)), \xi'(0))$ and the bicharacteristic γ_- starting at $(0, 0, -r(0, \xi'(0)), \xi'(0))$.

We denote by $\Phi^\pm(x_0, y', \eta')$ the solution of the Hamilton equations with Hamiltonian $H_\pm(x_0, x', \xi') = \xi_0 \mp r(x', \xi')$ and initial datum $(x', \xi') = (y', \eta')$ at $x_0 = 0$. In other words, $\Phi^\pm(x_0, y', \eta') = e^{x_0 \bar{H}_\pm}(0, y', \eta')$. Then, for any s , $\Phi(s, \cdot)$ is well-defined and symplectic from a neighborhood of $(0, \xi'(0))$ to a neighborhood of $H_\pm(s, 0, \xi'(0))$.

The solution $\psi(s, \cdot)$ of (21) and (22) is equal to 0 on γ_\pm and $\nabla\psi(s, \cdot)$ is obtained by the transport of the values of $\nabla\psi_0$ by $\Phi^\pm(s, \cdot)$. In other words, to compute $\nabla\psi(s, \cdot)$, one transports the Lagrangian sub-space $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ along the Hamiltonian flow \bar{H}_\pm during a time s , which yields $\Lambda_s \subset T^*M$, and then, if possible, one writes Λ_s under the form $\{(x', \nabla_{x'}\psi(s, x'))\}$, which gives $\nabla_{x'}\psi(s, x')$. The trouble is that the solution is only local in time: when $x' \mapsto \pi(\Phi^\pm(s, x', \nabla\psi_0(x')))$ ceases to be a diffeomorphism (conjugate point), where $\pi : T^*M \rightarrow M$ is the canonical projection, we see that the process described above does not work (appearance of caustics). In the language of Lagrangian spaces, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\} \subset T^*M$ is a Lagrangian subspace and, since $\Phi^\pm(s, \cdot)$ is a symplectomorphism, $\Lambda_s = \Phi^\pm(s, \Lambda_0)$ is Lagrangian as well. If $\pi|_{\Lambda_s}$ is a local diffeomorphism, one can locally describe Λ_s by $\Lambda_s = \{(x', \nabla_{x'}\psi(s, x'))\} \subset T^*M$ for some function $\psi(s, \cdot)$, but blow-up happens when $\text{rank}(d\pi|_{\Lambda_s}) < n$ (classical conjugate point theory), and such a $\psi(s, \cdot)$ may not exist.

However, if the phase ψ_0 is complex, quadratic, and satisfies the condition $\text{Im}(D^2\psi_0) > 0$, where $D^2\psi_0$ denotes the Hessian, no blow-up happens, and the solution is global in time. Let us explain why. Indeed, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ then lives in the complexification of the tangent space T^*M , which may be thought of as $\mathbb{C}^{2(n+1)}$. We take coordinates (y, η) on $T^*\mathbb{R}^{n+1}$ or $T^*\overline{\mathbb{C}^{n+1}}$ and we consider the symplectic forms defined by $\sigma = \sum dy_j \wedge d\eta_j$ and $\sigma_{\mathbb{C}} = \sum dy_j \wedge d\eta_j$.

Because of the condition $\text{Im}(D^2\psi_0) > 0$, Λ_0 is called a “strictly positive Lagrangian space” (see [Hör07, Definition 21.5.5]), meaning that $i\sigma_{\mathbb{C}}(v, v) > 0$ for v in the tangent space to Λ_0 . For any s , the symplectic forms σ and $\sigma_{\mathbb{C}}$ are preserved by $\Phi(s, \cdot)$, meaning that $\Phi(s, \cdot)_*\sigma = \sigma$ and $\Phi(s, \cdot)_*\sigma_{\mathbb{C}} = \sigma_{\mathbb{C}}$, therefore $\sigma = 0$ on the tangent space to Λ_s , and $i\sigma_{\mathbb{C}}(v, v) > 0$ for v tangent to Λ_s . It precisely means that Λ_s is also a strictly positive Lagrangian space. Then, by [Hör07, Proposition 21.5.9], we know that there exists $\psi(s, \cdot)$ complex and quadratic with $\text{Im}(D^2\psi(s, \cdot)) > 0$ such that $\Lambda_s = \{(x', \nabla_{x'}\psi(s, x'))\}$ (to apply [Hör07, Proposition 21.5.9], recall that for $\varphi(x') = \frac{1}{2}(Ax', x')$, there holds $\nabla\varphi(x') = Ax'$). In other words, the key point in using complex phases is that strictly positive Lagrangian spaces are parametrized by complex quadratic phases φ with $\text{Im}(D^2\varphi) > 0$, whereas real Lagrangian spaces were not parametrized by real phases (see explanations above). This parametrization is a diffeomorphism from the Grassmannian of strictly positive Lagrangian spaces to the space of complex quadratic phases with φ with $\text{Im}(D^2\varphi) > 0$. Hence, the phase

$$\psi(s, y') = \nabla_{x'}\psi(x(s)) \cdot (y' - x'(s)) + \frac{1}{2}(y' - x'(s)) \cdot D_{x'}^2\psi(s, x'(s))(y' - x'(s))$$

for $s \in [0, T]$ and $y' \in \mathbb{R}^n$ is smooth and for this choice, (21) is satisfied at second order along $s \mapsto x(s)$ (the rest $R(x', \xi')$ plays no role since it vanishes in a neighborhood of $s \mapsto x(s)$).

Then, we note that A_2 vanishes along the bicharacteristic if and only if $La_0(x(s)) = 0$ (see also [Hör07, Equation (24.2.9)]). According to (20), this turns out to be a linear transport equation on $a_0(x(s))$, with leading coefficient $\nabla_\xi p_2(x(s), \xi(s))$ different from 0. Given $a \neq 0$ at $(t = 0, x' = x'(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a , and, by Cauchy uniqueness, $a_0(x(s)) \neq 0$ for any s . We can choose a_0 in a smooth (and arbitrary) way outside the bicharacteristic. We choose it to vanish outside a small neighborhood of this bicharacteristic, so that no boundary effect happens.

With these choices of ψ and a_0 , the bound (14) then follows from the following result whose proof is given in [Ral82, Lemma 2.8].

Lemma 11. *Let $c(x)$ be a function on \mathbb{R}^{n+1} which vanishes at order $S - 1$ on a curve Γ for some $S \geq 1$. Suppose that $\text{Supp } c \cap \{|x_0| \leq T\}$ is compact and that $\text{Im } \psi(x) \geq ad(x)^2$ on this set for some constant $a > 0$, where $d(x)$ denotes the distance from the point $x \in \mathbb{R}^{d+1}$ to the curve Γ . Then there exists a constant C such that*

$$\int_{|x_0| \leq T} |c(x)e^{ik\psi(x)}|^2 dx \leq Ck^{-S-n/2}.$$

Let us now sketch the end of the proof, which is given in Appendix B in full details. We apply Lemma 11 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1(0,T;L^2(M))} \leq C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (14). The bounds (15) and (16) follow from the facts that $\text{Im}(D^2\psi(s, \cdot)) > 0$ and $v_k(x) = k^{\frac{n}{4}-1}a_0(x)e^{ik\psi(x)}$. \square

Remark 12. *An interesting question would be to understand the delocalization properties of the Gaussian beams constructed along normal sub-Riemannian geodesics in Proposition 9. Compared with the usual Riemannian case done for example in [Ral82], there is a new phenomenon in the sub-Riemannian case since the geodesic $x(t)$ (or, more precisely, the associated momentum ξ) may approach the characteristic manifold $\Sigma = \{g^* = 0\}$ which is the set of directions in which Δ is not elliptic. In finite time T as in our case, the geodesic remains far from Σ , but it may happen as $T \rightarrow +\infty$ that it goes closer and closer to Σ . The question is then to understand the link between the delocalization properties of the Gaussian beams constructed along such a geodesic, and notably the interplay between the time T and the semi-classical parameter $1/k$.*

2.2 Construction of sequences of exact solutions in M

In this section, using the approximate solutions of Proposition 2.1, we construct *exact* solutions of (13) whose energy concentrates along a given normal geodesic of M which does not meet the boundary ∂M during the time interval $[0, T]$.

Proposition 13. *Let $(x(t), \xi(t))_{t \in [0, T]}$ be a solution of (9) in M (in particular $g^*(x(0), \xi(0)) = 1/4$) which does not meet ∂M . Let $\theta \in C_c^\infty([0, T] \times M)$ with $\theta(t, \cdot) \equiv 1$ in a neighborhood of $x(t)$ and such that the support of $\theta(t, \cdot)$ stays at positive distance of ∂M .*

Suppose $(v_k)_{k \in \mathbb{N}}$ is constructed along $x(t)$ as in Proposition 9 and u_k is the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2 - \Delta)u_k = 0 & \text{in } (0, T) \times M, \\ u_k = 0 & \text{in } (0, T) \times \partial M, \\ u_k|_{t=0} = (\theta v_k)|_{t=0}, \quad \partial_t u_k|_{t=0} = [\partial_t(\theta v_k)]|_{t=0}. \end{cases}$$

Then:

- *The energy of u_k is bounded below with respect to k and $t \in [0, T]$:*

$$\exists A > 0, \forall t \in [0, T], \quad \liminf_{k \rightarrow +\infty} E(u_k(t, \cdot)) \geq A. \quad (23)$$

- The energy of u_k is small off $x(t)$: for any $t \in [0, T]$, we fix V_t an open subset of M for the initial topology of M , containing $x(t)$, so that the mapping $t \mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then

$$\sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t u_k(t, x)|^2 + |\nabla^{sR} u_k(t, x)|^2) d\mu(x) \xrightarrow{k \rightarrow +\infty} 0. \quad (24)$$

Proof of Proposition 13. Set $h_k = (\partial_{tt}^2 - \Delta)(\theta v_k)$. We consider w_k the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2 - \Delta)w_k = h_k & \text{in } (0, T) \times M, \\ w_k = 0 & \text{in } (0, T) \times \partial M, \\ (w_k|_{t=0}, \partial_t w_k|_{t=0}) = (0, 0). \end{cases} \quad (25)$$

Differentiating $E(w_k(t, \cdot))$ and using Gronwall's lemma, we get the energy inequality

$$\sup_{t \in [0, T]} E(w_k(t, \cdot)) \leq C (E(w_k(0, \cdot)) + \|h_k\|_{L^1(0, T; L^2(M))}).$$

Therefore, using (14), we get $\sup_{t \in [0, T]} E(w_k(t, \cdot)) \leq Ck^{-1}$. Since $u_k = \theta v_k - w_k$, we obtain that

$$\lim_{k \rightarrow +\infty} E(u_k(t, \cdot)) = \lim_{k \rightarrow +\infty} E((\theta v_k)(t, \cdot)) = \lim_{k \rightarrow +\infty} E(v_k(t, \cdot))$$

for every $t \in [0, T]$ where the last equality comes from the fact that θ and its derivatives are bounded and $\|v_k\|_{L^2} \leq Ck^{-1}$ when $k \rightarrow +\infty$. Using (15), we conclude that (23) holds.

To prove (24), we observe similarly that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t u_k(t, x)|^2 + |\nabla^{sR} u_k(t, x)|^2) d\mu(x) \\ & \leq C \sup_{t \in [0, T]} \left(\int_{M \setminus V_t} (|\partial_t v_k(t, x)|^2 + |\nabla^{sR} v_k(t, x)|^2) d\mu(x) \right) + Ck^{-\frac{1}{2}} \\ & \rightarrow 0 \end{aligned}$$

as $k \rightarrow +\infty$, according to (16). It concludes the proof of Proposition 13. \square

3 Existence of spiraling geodesics

The goal of this section is to prove the following proposition, which is the second building block of the proof of Theorem 1, after the construction of localized solutions of the subelliptic wave equation (2) done in Section 2.

Proposition 14. *For any $T_0 > 0$, any $x \in M$ and any open neighborhood V of x in M (with the initial topology on M), there exists a non-stationary normal geodesic $t \mapsto x(t)$ (traveled at speed 1) of (M, \mathcal{D}, g) such that $x(t) \in V$ for any $t \in [0, T_0]$.*

In Section 3.1, we define the so-called nilpotent approximation $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ of (M, \mathcal{D}, g) at a point $q \in M$, a sub-Riemannian structure which is a first-order approximation of (M, \mathcal{D}, g) at point $q \in M$ whose associated Lie algebra $\text{Lie}(\widehat{\mathcal{D}}^q)$ is nilpotent. Roughly, the space $\widehat{\mathcal{D}}^q$ is spanned by vector fields $\widehat{X}_1^q, \dots, \widehat{X}_m^q$ such that $\widehat{X}_i^q \approx X_i^q$, but low order terms of X_i^q are not taken into account for defining \widehat{X}_i^q , so that the high order brackets of the \widehat{X}_i^q vanish (which is not generally the case for the X_i^q). The nilpotent approximation is a good local approximation of (M, \mathcal{D}, g) , but its study is much simpler.

The proof of Proposition 14 then splits into two steps, first proving the result for the nilpotent approximation (in Section 3.2) and then showing that the geodesics of (M, \mathcal{D}, g) are well approached by the geodesics of $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ (in Section 3.3) which is sufficient to conclude.

3.1 Nilpotent approximation

In this section, we recall the construction of the nilpotent approximation $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$. The definitions we give are classical, and the reader can refer to [ABB19, Chapter 10] and [Jea14, Chapter 2] for more material on this section.

Given a sub-Riemannian structure (M, \mathcal{D}, g) and a point $q \in M$, its tangent space at q in the Gromov-Hausdorff sense is the sub-Riemannian structure $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$, also called nilpotent approximation. It is defined intrinsically (meaning that it does not depend on a choice of coordinates or of local frame) as an equivalence class under the action of sub-Riemannian isometries (see [Bel96], [Jea14]).

Sub-Riemannian flag. We define the sub-Riemannian flag of (M, \mathcal{D}, g) as follows: we set $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and, for any $j \geq 1$, $\mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j]$. For any point $q \in M$, it defines a flag

$$\{0\} = \mathcal{D}_q^0 \subset \mathcal{D}_q^1 \subset \dots \subset \mathcal{D}_q^{r-1} \subsetneq \mathcal{D}_q^{r(q)} = T_q M.$$

The integer $r(q)$ is called the non-holonomic order of \mathcal{D} at q , and it is equal to 2 everywhere in the Heisenberg manifold for example. Note that it depends on q , see the Baouendi-Grushin example of Section 1.2.

For $0 \leq i \leq r(q)$, we set $n_i(q) = \dim \mathcal{D}_q^i$, and the sequence $(n_i(q))_{0 \leq i \leq r(q)}$ is called the growth vector at point q . We set $\mathcal{Q}(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q))$, which is generically the Hausdorff dimension of the metric space given by the sub-Riemannian distance on M (see [Mit85]). Finally, we define the non-decreasing sequence of weights $w_i(q)$ for $1 \leq i \leq n$ as follows. Given any $1 \leq i \leq n$, there exists a unique $1 \leq j \leq n$ such that $n_{j-1}(q) + 1 \leq i \leq n_j(q)$. We set $w_i(q) = j$. For example, for any q in the Heisenberg manifold, $w_1(q) = w_2(q) = 1$ and $w_3(q) = 2$: indeed, the coordinates x_1 and x_2 have “weight 1”, while the coordinate x_3 has “weight 2” since ∂_{x_3} requires a bracket to be generated.

Regular and singular points. We say that $q \in M$ is regular if the growth vector $(n_i(q'))_{0 \leq i \leq r(q')}$ at q' is constant for q' in a neighborhood of q . Otherwise, q is said to be singular. If any point $q \in M$ is regular, we say that the structure is equiregular. For example, the Heisenberg manifold is equiregular, but not the Baouendi-Grushin example.

Non-holonomic orders. The non-holonomic order of a smooth germ of function is given by the formula

$$\text{ord}_q(f) = \min\{s \in \mathbb{N} : \exists i_1, \dots, i_s \in \{1, \dots, m\} \text{ such that } (X_{i_1} \dots X_{i_s} f)(q) \neq 0\}$$

where we adopt the convention that $\min \emptyset = +\infty$.

The non-holonomic order of a smooth germ of vector field X at q , denoted by $\text{ord}_q(X)$, is the real number defined by

$$\text{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \text{ord}_q(Xf) \geq \sigma + \text{ord}_q(f), \forall f \in C^\infty(q)\}.$$

For example, there holds $\text{ord}_q([X, Y]) \geq \text{ord}_q(X) + \text{ord}_q(Y)$ and $\text{ord}_q(fX) \geq \text{ord}_q(f) + \text{ord}_q(X)$. As a consequence, every X which has the property that $X(q') \in \mathcal{D}_{q'}^i$ for any q' in a neighborhood of q is of non-holonomic order $\geq -i$.

Privileged coordinates. Locally around $q \in M$, it is possible to define a set of so-called “privileged coordinates” of M (see [Bel96]).

A family (Z_1, \dots, Z_n) of n vector fields is said to be adapted to the sub-Riemannian flag of (M, \mathcal{D}, g) at q if it is a frame of $T_q M$ at q and if $Z_i(q) \in \mathcal{D}_q^{w_i(q)}$ for any $i \in \{1, \dots, n\}$. In other words, for any $i \in \{1, \dots, r(q)\}$, the vectors $Z_1, \dots, Z_{n_i(q)}$ at q span \mathcal{D}_q^i .

A system of privileged coordinates at q is a system of local coordinates (x_1, \dots, x_n) such that $\text{ord}_q(x_i) = w_i$ for $1 \leq i \leq n$. In particular, for privileged coordinates, we have $\partial_{x_i} \in \mathcal{D}_q^{w_i(q)} \setminus \mathcal{D}_q^{w_i(q)-1}$ at q , meaning that privileged coordinates are adapted to the flag.

Example: exponential coordinates of the second kind. Choose an adapted frame (Z_1, \dots, Z_n) at q . It is proved in [Jea14, Appendix B] that the inverse of the local diffeomorphism

$$(x_1, \dots, x_n) \mapsto \exp(x_1 Z_1) \circ \dots \circ \exp(x_n Z_n)(q)$$

defines privileged coordinates at q , called exponential coordinates of the second kind.

Dilations. We consider a chart of privileged coordinates at q given by a smooth mapping $\psi_q : U \rightarrow \mathbb{R}^n$, where U is a neighborhood of q in M , with $\psi_q(q) = 0$. For every $\varepsilon \in \mathbb{R} \setminus \{0\}$, we consider the dilation $\delta_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\delta_\varepsilon(x) = (\varepsilon^{w_i(q)}x_1, \dots, \varepsilon^{w_n(q)}x_n)$$

for every $x = (x_1, \dots, x_n)$. A dilation δ_ε acts also on functions and vector fields on \mathbb{R}^n by pull-back: $\delta_\varepsilon^*f = f \circ \delta_\varepsilon$ and δ_ε^*X is the vector field such that $(\delta_\varepsilon^*X)(\delta_\varepsilon^*f) = \delta_\varepsilon^*(Xf)$ for any $f \in C^1(\mathbb{R}^n)$. In particular, for any vector field X of non-holonomic order k , there holds $\delta_\varepsilon^*X = \varepsilon^{-k}X$.

Nilpotent approximation. Fix a system of privileged coordinates (x_1, \dots, x_n) at q . Given a sequence of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we define the weighted degree of $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ to be $w(\alpha) = w_1(q)\alpha_1 + \dots + w_n(q)\alpha_n$. Coming back to the vector fields X_1, \dots, X_m which span \mathcal{D} , we can write the Taylor expansion

$$X_i(x) \sim \sum_{\alpha, j} a_{\alpha, j} x^\alpha \partial_{x_j}.$$

Since $X_i \in \mathcal{D}$, its non-holonomic order is necessarily -1 , hence there holds $w(\alpha) \geq w_j(q) - 1$ if $a_{\alpha, j} \neq 0$. Therefore, we may write X_i as a formal series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s , meaning that

$$\delta_\varepsilon^*(\psi_q)_*X_i^{(s)} = \varepsilon^s(\psi_q)_*X_i^{(s)}.$$

We set $\widehat{X}_i^q = (\psi_q)_*X_i^{(-1)}$ for $1 \leq i \leq m$. Then \widehat{X}_i^q is homogeneous of degree -1 with respect to dilations, i.e., $\delta_\varepsilon^*\widehat{X}_i^q = \varepsilon^{-1}\widehat{X}_i^q$ for any $\varepsilon \neq 0$. Each \widehat{X}_i^q may be seen as a vector field on \mathbb{R}^n thanks to the coordinates (x_1, \dots, x_n) . Moreover,

$$\widehat{X}_i^q = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_\varepsilon^*(\psi_q)_*X_i$$

in C^∞ topology: all derivatives uniformly converge on compact subsets. For $\varepsilon > 0$ small enough we have

$$X_i^\varepsilon := \varepsilon \delta_\varepsilon^*(\psi_q)_*X_i = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology (see also [ABB19, Lemma 10.58]). An important property is that $(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ generates a nilpotent Lie algebra of step $r(q)$ (see [Jea14, Proposition 2.3]).

The nilpotent approximation $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ of (M, \mathcal{D}, g) at q is then defined as $\widehat{M} \simeq \mathbb{R}^n$ endowed with the sub-Riemannian distribution $\widehat{\mathcal{D}}^q = \text{Span}(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ and the sub-Riemannian metric \widehat{g}^q on $\widehat{\mathcal{D}}^q$ verifying $\widehat{g}^q(\widehat{X}_i^q, \widehat{X}_j^q) = g_q(X_i, X_j)$. For an explicit example of computation of nilpotent approximation, see [Jea14, Example 2.8].

3.2 The nilpotent case

We start the proof of Proposition 14. Let us first note that there exist $q \in V$ and $X_1, X_2 \in \mathcal{D}$ such that $[X_1, X_2](q) \notin \mathcal{D}_q$. This is because otherwise $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ in V , and therefore the rank condition (1) cannot hold in V . We are now reduced to prove the result for $x = q$.

We first prove Proposition 14 in the nilpotentization $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ of (M, \mathcal{D}, g) . In particular, $\text{Lie}(\widehat{\mathcal{D}}^q)$ is a nilpotent Lie algebra. The nilpotent Hamiltonian is denoted by

$$\widehat{H} = \sum_{i=1}^m h_{\widehat{X}_i^q}^2 \tag{26}$$

where we used the notation h_Z for vector fields Z which was introduced in Section 1.4. As explained in Section 1.3, the proof consists in identifying a “Heisenberg sub-structure” in $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$, and then to use the existence of spiraling geodesics in this sub-structure to conclude. In the next lemma, we shall use the notation U^*M introduced in Remark 8 for $M = \widehat{M}^q$.

Lemma 15. *For any $T_0 > 0$, and any open neighborhood V of 0 in \widehat{M}^q , there exists $\lambda_0 \in U^*\widehat{M}^q$ such that $\pi(e^{t\widehat{H}}\lambda_0) \in V$ for any $t \in [0, T_0]$.*

Proof of Lemma 15. For this proof, we simplify notations by replacing the vector fields \widehat{X}_i^q by the notation X_i , and denote $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ simply by (M, \mathcal{D}, g) .

We denote by r the non-holonomic order of \mathcal{D} at 0 (which is equal to the non-holonomic order at q before the nilpotentization).

The main idea of the proof is to use a desingularization $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$ of (M, \mathcal{D}, g) around 0 whose Lie algebra is *free up to some step r* (see [Jea14, Definition 2.13]) and nilpotent (the desingularization procedure is due to [RS76]).

Let us observe that it is not so easy to isolate a Heisenberg sub-structure in particular if there are relations such as $[X_1, X_3] = X_1$. This is why we introduce the desingularization $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$: its Lie algebra is free up to a step r , and therefore such relations cannot hold in $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$.

Thus, we introduce some notations related to free Lie algebras. Let $\mathcal{L} = \mathcal{L}(1, \dots, m)$ be the free Lie algebra generated by $\{1, \dots, m\}$. We denote by \mathcal{L}^s the subspace generated by elements of \mathcal{L} of length $\leq s$, and by \widetilde{n}_s the dimension of \mathcal{L}^s . Recall that in our notations M stands for \widehat{M}^q and we desingularize around 0.

Lemma 16 ([Bel96](Section 7.3), [Jea14](Lemma 2.5, Theorem 2.9 and Remark 2.9)). *Let $\widetilde{M} = M \times \mathbb{R}^{\widetilde{n}_r - n}$. Then there exist a neighborhood $\widetilde{U} \subset \widetilde{M}$ of $(0, 0)$, a neighborhood $U \subset M$ of 0 with $U \times \{0\} \subset \widetilde{U}$, local coordinates (u, v) on \widetilde{U} , and smooth vector fields on \widetilde{U} ,*

$$\widetilde{X}_i(u, v) = X_i(u) + \sum_{j=n+1}^{\widetilde{n}_r} b_{ij}(u, v)\partial_{v_j}, \quad i = 1, \dots, m, \quad (27)$$

such that

- Every \widetilde{q} in \widetilde{U} is regular;
- For any $1 \leq i \leq m$, we have $d\widetilde{\pi}(\widetilde{X}_i) = X_i$, where $\widetilde{\pi} : \widetilde{M} \rightarrow M$ is the canonical projection;
- $\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)$ is a free nilpotent Lie algebra of step r ;

For examples illustrating this desingularization procedure, see [Jea14, Examples 2.15 and 2.16].

We gather in the notations Y_1, \dots, Y_N the vector fields $\widetilde{X}_3, \dots, \widetilde{X}_m$ and all brackets involving the vector fields $\widetilde{X}_1, \dots, \widetilde{X}_m$ except $[\widetilde{X}_1, \widetilde{X}_2]$. In other words, if we set $\mathcal{Y} = \text{Span}(Y_1, \dots, Y_N)$, then \mathcal{Y} together with $\widetilde{X}_1, \widetilde{X}_2$ and $[\widetilde{X}_1, \widetilde{X}_2]$ form a basis of the free Lie algebra $\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)$.

We consider the coordinates given by

$$(x_1, x_2, x_3, y_1, \dots, y_N) \mapsto \exp(x_1\widetilde{X}_1)\exp(x_2\widetilde{X}_2)\exp(-x_3[\widetilde{X}_1, \widetilde{X}_2])\exp(y_1Y_1)\dots\exp(y_NY_N).$$

These are exponential coordinates of the second kind, hence privileged coordinates (see Section 3.1). The coordinates x_1 and x_2 have non-holonomic order 1, and x_3 has non-holonomic order 2. Due to the particular triangular structure of these coordinates, the

following relations hold

$$\begin{aligned}\tilde{X}_1 &= \partial_{x_1} \\ \tilde{X}_2 &= \partial_{x_2} && \text{on } x_1 = 0\end{aligned}\tag{28}$$

$$[\tilde{X}_1, \tilde{X}_2] = -\partial_{x_3} \quad \text{on } x_1 = x_2 = 0\tag{29}$$

$$Y_\ell = \partial_{y_\ell} \quad \text{on } x_1 = x_2 = x_3 = y_1 = \dots = y_{\ell-1} = 0.\tag{30}$$

Our first goal is to prove

$$\begin{aligned}\tilde{X}_1 &= \partial_{x_1} \\ \tilde{X}_2 &= \partial_{x_2} - x_1 \partial_{x_3} \\ \tilde{X}_j &= 0 \pmod{\mathcal{Y}} \text{ for any } j \geq 3.\end{aligned}\tag{31}$$

The first relation was already stated above. Let us prove the second relation. Due to (28) and the fact that \tilde{X}_2 and ∂_{x_2} have non-holonomic order -1 , there holds

$$\tilde{X}_2 = \partial_{x_2} + g(x_1)\partial_{x_3} + \sum_{\ell} f_\ell(x_1)\partial_{y_\ell}\tag{32}$$

where g and f_ℓ are functions of x_1 only and the sum is taken over coordinates y_ℓ with non-holonomic order 2. Since $g(x_1)\partial_{x_3}$ and $f_\ell(x_1)\partial_{y_\ell}$ have to be of non-holonomic order -1 , this imposes $g(x_1) = ax_1$ and $f_\ell(x_1) = b_\ell x_1$ for some $a, b_\ell \in \mathbb{R}$, and thanks to (29), we obtain that $b_\ell = 0$ for any ℓ , and $a = -1$. We finally prove (31). Fix $j \geq 3$. Reasoning with strong recurrence, we can assume that (31) holds for any $j' < j$ with $j' \geq 3$. Due to (30), there holds

$$\tilde{X}_j = g(x_1, x_2, x_3, y_1, \dots, y_{j-1}) \partial_{x_3} \pmod{\mathcal{Y}}$$

(again there are no terms ∂_{x_1} and ∂_{x_2} due to homogeneity of degree -1). Due to homogeneity of degree -1 , we know that

$$g(x_1, x_2, x_3, y_1, \dots, y_{j-1}) = ax_1 + bx_2 + \sum_{i=1}^{j-1} \alpha_i y_i$$

for some constants $a, b, \alpha_1, \dots, \alpha_{j-1} \in \mathbb{R}$. Using that $[\tilde{X}_1, \tilde{X}_j] = 0 \pmod{\mathcal{Y}}$ on $x_1 = x_2 = x_3 = y_1 = \dots = y_{j-1} = 0$ thanks to (30) yields $a = 0$. Similarly, using (32) and $[\tilde{X}_2, \tilde{X}_j] = 0 \pmod{\mathcal{Y}}$ on $x_1 = x_2 = x_3 = y_1 = \dots = y_{j-1} = 0$, we get $b = 0$. Using now (30) for $Y_\ell = [\tilde{X}_{j'}, \tilde{X}_j]$ and the strong recurrence assumption that (31) holds for j' , we get that $\alpha_{j'} = 0$ for any $1 \leq j' \leq j-1$. This ends the proof of (31).

Let us finish the proof of Lemma 15. We denote by \tilde{H} the Hamiltonian associated to $(\tilde{M}, \tilde{D}, \tilde{g})$. Due to the above considerations,

$$\tilde{H}(x, y, \xi, \eta) = \xi_1^2 + (\xi_2 - x_1 \xi_3)^2 + \sum_{i=1}^{m-2} \left(\sum_{\ell=3}^N \alpha_\ell^i \eta_\ell \right)^2\tag{33}$$

where $\xi_i = h_{\tilde{X}_i}$ and $\eta_\ell = h_{Y_\ell}$. We consider $\tilde{\lambda}_0 \in T_0^* \tilde{M}$ with

$$\langle \tilde{\lambda}_0, \tilde{X}_1 \rangle = 1/2\tag{34}$$

$$\langle \tilde{\lambda}_0, \tilde{X}_2 \rangle = 0\tag{35}$$

$$\langle \tilde{\lambda}_0, [\tilde{X}_1, \tilde{X}_2] \rangle = (2\varepsilon)^{-1}\tag{36}$$

$$\langle \tilde{\lambda}_0, Y \rangle = 0, \quad \forall Y \in \mathcal{Y}\tag{37}$$

We denote by $\tilde{x}(t) = \pi(e^{t\tilde{H}}\tilde{\lambda}_0)$ the corresponding normal geodesic in \tilde{M} for $t \in [0, T_0]$,

$$\tilde{x}(t) = (x_1(t), x_2(t), x_3(t), y_1(t), \dots, y_N(t)).$$

We note that (37) is preserved thanks to (33): for any t and any $Y \in \mathcal{Y}$, there holds $\langle e^{t\tilde{H}}\tilde{\lambda}_0, Y \rangle = 0$. Hence, for any $1 \leq \ell \leq N$, there holds $y_\ell(t) \equiv 0$. Thanks to (34), (35), (36) the coordinates $x_1(t), x_2(t)$ and $x_3(t)$ are exactly given by (7). In particular, $|x_i(t)| \leq C\varepsilon$ for $i = 1, 2, 3$ and any $t \in [0, T_0]$, where C only depends on T_0 . To sum up, we have identified a family of spiraling geodesics in $(\tilde{M}, \tilde{D}, \tilde{g})$, which stays at distance $\leq C\varepsilon$ from the origin over $[0, T_0]$. Indeed, we have proved that the Heisenberg geodesics are “embedded” in the geodesics of the desingularized nilpotent structure.

Finally, we notice that any normal geodesic of $(\tilde{M}, \tilde{D}, \tilde{g})$ yields a geodesic of (M, \mathcal{D}, g) just by taking in (27) all additional variables v_j and all variables dual to v_j equal to 0 (we use the “triangular structure” of the desingularization). Of course, the geodesic that we obtain in (M, \mathcal{D}, g) remains at distance $\leq C\varepsilon$ from the origin over $[0, T_0]$. Taking ε sufficiently small, this concludes the proof. \square

Remark 17. *The geodesics constructed in Lemma 15 lose their optimality quickly, in the sense that their first conjugate point and their cut-point are close to q . In local coordinates, they are closer to q as λ_0 approaches $S\Sigma$.*

3.3 The general case

In this section, we conclude the proof of Proposition 14. We do not assume anymore that (M, \mathcal{D}, g) is nilpotent. We show Proposition 14 using the pull-back by ψ_q (see Section 3.1) of the spiraling geodesics of $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ constructed in the previous section. We set $Y_i = (\psi_q)_* X_i$ and $X_i^\varepsilon = \varepsilon \delta_\varepsilon^* Y_i$ which is a vector field on \mathbb{R}^n . Recall that

$$X_i^\varepsilon = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology. Therefore, using the homogeneity of \widehat{X}_i^q , we get, for any $\varepsilon > 0$,

$$Y_i = \frac{1}{\varepsilon} (\delta_\varepsilon)_* X_i^\varepsilon = \frac{1}{\varepsilon} (\delta_\varepsilon)_* (\widehat{X}_i^q + \varepsilon R_i^\varepsilon) = \widehat{X}_i^q + (\delta_\varepsilon)_* R_i^\varepsilon. \quad (38)$$

The vector field $(\delta_\varepsilon)_* R_i^\varepsilon(x)$ does not depend on ε and has a size which tends uniformly to 0 as $x \rightarrow 0 \in \widehat{M}^q \simeq \mathbb{R}^n$. Recall that the Hamiltonian \widehat{H} associated to the vector fields \widehat{X}_i^q is given by (26). Similarly, we set

$$H = \sum_{i=1}^m h_{Y_i}^2.$$

We note that (38) gives

$$h_{Y_i} = h_{\widehat{X}_i^q} + h_{(\delta_\varepsilon)_* R_i^\varepsilon}.$$

Hence

$$\vec{H} = 2 \sum_{i=1}^m h_{Y_i} \vec{h}_{Y_i} = \vec{H} + \vec{\Theta},$$

where $\vec{\Theta}$ is a smooth vector field on $T^*\mathbb{R}^n$ such that $\|(\pi \circ \vec{\Theta})(x, \xi)\| \leq C\|x\|$ when $\|x\| \rightarrow 0$ (uniformly for $\|\xi\| \leq C$) where $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ is the canonical projection. This last point comes from the smooth dependence of R_i^ε on ε for the C^∞ topology (uniform convergence of all derivatives on compact subsets of \mathbb{R}^n). Together with Lemma 15 and using that $X_i = \psi_q^* Y_i$, it concludes the proof of Proposition 14.

4 Proofs

4.1 Proofs of Theorem 1 and Theorem 2

In this section, we conclude the proof of Theorem 1 and we prove Theorem 2.

Proof of Theorem 1. Fix a point $x \in M$ and an open neighborhood V of x in M such that $V \subset M \setminus \omega$. Such x and V exist since $M \setminus \omega$ has non-empty interior. Fix V' an open neighborhood of x in M such that $\overline{V'} \subset V$, and fix also $T_0 > 0$.

As already explained in Section 1.3, to conclude the proof of Theorem 1, we use Proposition 13 applied to the particular geodesics constructed in Proposition 14.

By Proposition 14, we know that there exists a normal geodesic $t \mapsto x(t)$ such that $x(t) \in V'$ for any $t \in (0, T_0)$. It is the projection of a bicharacteristic $(x(t), \xi(t))$ and since it is non-stationary and traveled at speed 1, there holds $g^*(x(t), \xi(t)) = 1/4$. We denote by $(u_k)_{k \in \mathbb{N}}$ a sequence of solutions of (13) as in Proposition 13 whose energy at time t concentrates on $x(t)$ for $t \in (0, T_0)$. Because of (23), we know that

$$\|(u_k(0), \partial_t u_k(0))\|_{\mathcal{H} \times L^2} \geq c > 0$$

uniformly in k .

Therefore, in order to establish Theorem 1, it is sufficient to show that

$$\int_0^{T_0} \int_{\omega} |\partial_t u_k(t, x)|^2 d\mu(x) dt \xrightarrow{k \rightarrow +\infty} 0. \quad (39)$$

Since $x(t) \in V'$ for any $t \in (0, T_0)$, we get that for V_t chosen sufficiently small for any $t \in (0, T_0)$, the inclusion $V_t \subset V$ holds (see Proposition 13 for the definition of V_t). Combining this last remark with (24), we get (39), which concludes the proof of Theorem 1. \square

Proof of Theorem 2. Let x be a point in the interior of $M \setminus \omega$ such that $\mathcal{D}_x \neq T_x M$. We fix an open neighborhood V of x whose closure is contained in $M \setminus \bar{\omega}$. Then there exist $q_0 \in V$ and $X_1, X_2 \in \mathcal{D}$ such that $[X_1 X_2](q_0) \notin \mathcal{D}_{q_0}$, otherwise there would hold $\mathcal{D}_x = T_x M$. We observe that the growth vector (defined in Section 3.1) of the nilpotent approximation $(\widehat{M}^{q_0}, \widehat{\mathcal{D}}^{q_0}, \widehat{g}^{q_0})$ at 0 is equal to the growth vector of (M, \mathcal{D}, g) at q_0 (see [Jea14, Lemma 2.1]). Following the proof of Lemma 15 for $(\widehat{M}^{q_0}, \widehat{\mathcal{D}}^{q_0}, \widehat{g}^{q_0})$, we obtain, for any neighborhood of 0, the existence of a spiraling geodesic of $(\widehat{M}^{q_0}, \widehat{\mathcal{D}}^{q_0}, \widehat{g}^{q_0})$ which is contained in this neighborhood for $t \in [0, T_0]$. Then, the computations of Section 3.3 show that there exists a geodesic of (M, \mathcal{D}, g) which is contained in V for $t \in [0, T_0]$. \square

4.2 Proof of Corollary 1

We endow the topological dual $\mathcal{H}(M)'$ with the norm $\|v\|_{\mathcal{H}(M)'} = \|(-\Delta)^{-1/2} v\|_{L^2(M)}$.

The following proposition is standard (see, e.g., [TW09], [LRLTT17]).

Lemma 18. *Let $T_0 > 0$, and $\omega \subset M$ be a measurable set. Then the following two observability properties are equivalent:*

(P1): *There exists C_{T_0} such that for any $(v_0, v_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, the solution $v \in C^0(0, T_0; D((-\Delta)^{\frac{1}{2}})) \cap C^1(0, T_0; L^2(M))$ of (2) satisfies*

$$\int_0^{T_0} \int_{\omega} |\partial_t v(t, q)|^2 d\mu(q) dt \geq C_{T_0} \|(v_0, v_1)\|_{\mathcal{H}(M) \times L^2(M)}. \quad (40)$$

(P2): *There exists C_{T_0} such that for any $(v_0, v_1) \in L^2(M) \times D((-\Delta)^{-\frac{1}{2}})$, the solution $v \in C^0(0, T_0; L^2(M)) \cap C^1(0, T_0; D((-\Delta)^{-\frac{1}{2}}))$ of (2) satisfies*

$$\int_0^{T_0} \int_{\omega} |v(t, q)|^2 d\mu(q) dt \geq C_{T_0} \|(v_0, v_1)\|_{L^2 \times \mathcal{H}(M)'}^2. \quad (41)$$

Proof. Let us assume that (P2) holds. Let u be a solution of (2) with initial conditions $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$. We set $v = \partial_t u$, which is a solution of (2) with initial data $v|_{t=0} = u_1 \in L^2(M)$ and $\partial_t v|_{t=0} = \Delta u_0 \in D((-\Delta)^{-\frac{1}{2}})$. Since $\|(v_0, v_1)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_1, \Delta u_0)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_0, u_1)\|_{\mathcal{H}(M) \times L^2}$, applying the observability inequality (41) to $v = \partial_t u$, we obtain (40). The proof of the other implication is similar. \square

Finally, using Theorem 1, Lemma 18 and the standard HUM method ([Lio88]), we get Corollary 1.

4.3 Proof of Theorem 3

We consider the space of functions $u \in C^\infty([0, T] \times M_H)$ such that $\int_{M_H} u(t, \cdot) d\mu = 0$ for any $t \in [0, T]$, and we denote by \mathcal{H}_T its completion for the norm $\|\cdot\|_{\mathcal{H}_T}$ induced by the scalar product

$$(u, v)_{\mathcal{H}_T} = \int_0^T \int_{M_H} (\partial_t u \partial_t v + (\nabla^{\text{sR}} u) \cdot (\nabla^{\text{sR}} v)) d\mu(q) dt.$$

We consider also the topological dual \mathcal{H}'_0 of the space \mathcal{H}_0 which was introduced in Section 1.5

Lemma 19. *The injections $\mathcal{H}_0 \hookrightarrow L^2(M_H)$, $L^2(M_H) \hookrightarrow \mathcal{H}'_0$ and $\mathcal{H}_T \hookrightarrow L^2((0, T) \times M_H)$ are compact.*

Proof. Let $(\varphi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of $L^2(M_H)$, labeled with increasing eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_k \rightarrow +\infty$, so that $-\Delta \varphi_k = \lambda_k \varphi_k$. The fact that $\lambda_1 > 0$, which will be used in the sequel, can be proved as follows: if $-\Delta \varphi = 0$ then $\int_{M_H} |\nabla^{\text{sR}} \varphi|^2 d\mu = 0$ and, since $\varphi \in C^\infty(M_H)$ by hypoelliptic regularity, we get $X_1 \varphi(x) = X_2 \varphi(x) = 0$ for any $x \in M_H$. Hence, $[X_1, X_2] \varphi \equiv 0$, and altogether, this proves that φ is constant, hence $\lambda_1 > 0$.

We prove the last injection. Let $u \in \mathcal{H}_T$. Writing $u(t, \cdot) = \sum_{k=1}^{\infty} a_k(t) \varphi_k(\cdot)$ (note that there is no 0-mode since $u(t, \cdot)$ has null average), we see that

$$\begin{aligned} \|u\|_{\mathcal{H}_T}^2 &\geq \|\nabla^{\text{sR}} u\|_{L^2((0, T) \times M_H)}^2 = \sum_{k=1}^{\infty} \lambda_k \|a_k\|_{L^2((0, T))}^2 \geq \lambda_1 \sum_{k=1}^{\infty} \|a_k\|_{L^2((0, T))}^2 \\ &= \lambda_1 \|u\|_{L^2((0, T) \times M_H)}^2, \end{aligned}$$

thus \mathcal{H}_T imbeds continuously into $L^2((0, T) \times M_H)$. Then, using a classical subelliptic estimate (see [Hör67] and [RS76, Theorem 17]), we know that there exists $C > 0$ such that

$$\|u\|_{H^{\frac{1}{2}}((0, T) \times M_H)} \leq C(\|u\|_{L^2((0, T) \times M_H)} + \|u\|_{\mathcal{H}_T}).$$

Together with the previous estimate, we obtain that for any $u \in \mathcal{H}_T$, $\|u\|_{H^{\frac{1}{2}}((0, T) \times M_H)} \leq C\|u\|_{\mathcal{H}_T}$. Then, the result follows from the fact that the injection $H^{\frac{1}{2}}((0, T) \times M_H) \hookrightarrow L^2((0, T) \times M_H)$ is compact.

The proof of the compact injection $\mathcal{H}_0 \hookrightarrow L^2(M_H)$ is similar, and the compact injection $L^2(M_H) \hookrightarrow \mathcal{H}'_0$ follows by duality. \square

Proof of Theorem 3. In this proof, we use the notation $P = \partial_{tt}^2 - \Delta_H$. For the sake of a contradiction, suppose that there exists a sequence $(u^k)_{k \in \mathbb{N}}$ of solutions of the wave equation such that $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$ for any $k \in \mathbb{N}$ and

$$\|(u_0^k, u_1^k)\|_{L^2 \times \mathcal{H}'_0} \rightarrow 0, \quad \int_0^T |(\text{Op}(a) \partial_t u^k, \partial_t u^k)_{L^2(M_H, \mu)}| dt \rightarrow 0 \quad (42)$$

as $k \rightarrow +\infty$. Following the strategy of [Tar90] and [Gér91], our goal is to associate a defect measure to the sequence $(u^k)_{k \in \mathbb{N}}$. Since the functional spaces involved in our result are unusual, we give the argument in detail.

First, up to extraction of a subsequence which we omit, (u_0^k, u_1^k) converges weakly in $\mathcal{H}_0 \times L^2(M_H)$ and, using the first convergence in (42) and the compact embedding $\mathcal{H}_0 \times L^2(M_H) \hookrightarrow L^2(M_H) \times \mathcal{H}'_0$, we get that $(u_0^k, u_1^k) \rightharpoonup 0$ in $\mathcal{H}_0 \times L^2_0$. Using the continuity of the solution with respect to the initial data, we obtain that $u^k \rightharpoonup 0$ weakly in \mathcal{H}_T . Using Lemma 19, we obtain $u^k \rightarrow 0$ strongly in $L^2((0, T) \times M_H)$.

Fix $B \in \Psi_{\text{phg}}^0((0, T) \times M_H)$. We have

$$\begin{aligned} (Bu^k, u^k)_{\mathcal{H}_T} &= \int_0^T \int_{M_H} (\partial_t(Bu^k)\partial_t u^k + (\nabla^{\text{sR}}(Bu^k)) \cdot (\nabla^{\text{sR}}u^k)) d\mu(q)dt \\ &= \int_0^T \int_{M_H} (([\partial_t, B]u^k)\partial_t u^k + ([\nabla^{\text{sR}}, B]u^k) \cdot (\nabla^{\text{sR}}u^k)) d\mu(q)dt \\ &\quad + \int_0^T \int_{M_H} ((B\partial_t u^k)(\partial_t u^k) + (B\nabla^{\text{sR}}u^k) \cdot (\nabla^{\text{sR}}u^k)) d\mu(q)dt \end{aligned} \quad (43)$$

Since $[\partial_t, B] \in \Psi_{\text{phg}}^0((0, T) \times M_H)$, $[\nabla^{\text{sR}}, B] \in \Psi_{\text{phg}}^0((0, T) \times M_H)$ and $u^k \rightarrow 0$ strongly in $L^2((0, T) \times M_H)$, the first one of the two lines in (43) converges to 0 as $k \rightarrow +\infty$. Moreover, the last line is bounded uniformly in k since $B \in \Psi_{\text{phg}}^0((0, T) \times M_H)$. Hence $(Bu^k, u^k)_{\mathcal{H}_T}$ is uniformly bounded. By a standard diagonal extraction argument (see [Gér91] for example), there exists a subsequence, which we still denote by $(u^k)_{k \in \mathbb{N}}$ such that (Bu^k, u^k) converges for any B of principal symbol b in a countable dense subset of $C_c^\infty((0, T) \times M_H)$. Moreover, the limit only depends on the principal symbol b , and not on the full symbol.

Let us now prove that

$$\liminf_{k \rightarrow +\infty} (Bu^k, u^k)_{\mathcal{H}_T} \geq 0 \quad (44)$$

when $b \geq 0$. With a bracket argument as in (43), we see that it is equivalent to proving that the liminf as $k \rightarrow +\infty$ of the quantity

$$Q_k(B) = (B\partial_t u^k, \partial_t u^k)_{L^2} + (B\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2} \quad (45)$$

is ≥ 0 . But there exists $B' \in \Psi_{\text{phg}}^0((0, T) \times M_H)$ such that $B' - B \in \Psi_{\text{phg}}^{-1}((0, T) \times M_H)$ and B' is positive (this is the so-called Friedrichs quantization, see for example [Tay74, Chapter VII]). Then, $\liminf_{k \rightarrow +\infty} Q_k(B') \geq 0$, and $Q_k(B' - B) \rightarrow 0$ since $(B' - B)\partial_t \in \Psi_{\text{phg}}^0((0, T) \times M_H)$ and $u^k \rightarrow 0$ strongly in $L^2((0, T) \times M_H)$. It immediately implies that (44) holds.

Therefore, setting $p = \sigma_p(P)$ and denoting by $\mathcal{C}(p)$ the characteristic manifold $\mathcal{C}(p) = \{p = 0\}$, there exists a non-negative Radon measure ν on $S^*(\mathcal{C}(p)) = \mathcal{C}(p)/(0, +\infty)$ such that

$$(\text{Op}(b)u^k, u^k)_{\mathcal{H}_T} \rightarrow \int_{S^*(\mathcal{C}(p))} b d\nu$$

for any $b \in S_{\text{phg}}^0((0, T) \times M_H)$.

Let $C \in \Psi_{\text{phg}}^{-1}((0, T) \times M_H)$ of principal symbol c . We have $\vec{H}_p c = \{p, c\} \in S_{\text{phg}}^0((0, T) \times M_H)$ and, for any $k \in \mathbb{N}$,

$$((CP - PC)u^k, u^k)_{\mathcal{H}_T} = (CPu^k, u^k)_{\mathcal{H}_T} - (Cu^k, Pu^k)_{\mathcal{H}_T} = 0 \quad (46)$$

since $Pu^k = 0$. To be fully rigorous, the identity of the previous line, which holds for any solution $u \in \mathcal{H}_T$ of the wave equation, is first proved for smooth initial data since $Pu \notin \mathcal{H}_T$ in general, and then extended to general solutions $u \in \mathcal{H}_T$. Taking principal symbols in (46), we get $\langle \nu, \vec{H}_p c \rangle = 0$.

Therefore, denoting by $(\psi_s)_{s \in \mathbb{R}}$ the maximal solutions of

$$\frac{d}{ds} \psi_s(\rho) = \vec{H}_p(\psi_s(\rho)), \quad \rho \in T^*(\mathbb{R} \times M_H)$$

(see (8)), we get that, for any $s \in (0, T)$,

$$0 = \langle \nu, \vec{H}_p c \circ \psi_s \rangle = \langle \nu, \frac{d}{ds} c \circ \psi_s \rangle = \frac{d}{ds} \langle \nu, c \circ \psi_s \rangle$$

and hence

$$\langle \nu, c \rangle = \langle \nu, c \circ \psi_s \rangle. \quad (47)$$

We note here that the precise homogeneity of c (namely $c \in S_{\text{phg}}^{-1}((0, T) \times M_H)$) does not matter since ν is a measure on the sphere bundle $S^*(\mathcal{C}(p))$. The identity (47) means that ν is invariant under the flow \vec{H}_p .

From the second convergence in (42), we can deduce that

$$\nu = 0 \text{ in } S^*(\mathcal{C}(p)) \cap T^*((0, T) \times \text{Supp}(a)). \quad (48)$$

The proof of this fact, which is standard (see for example [BG02, Section 6.2]), is given in Appendix C.

Let us prove that any geodesic of M_H with momentum $\xi \in V_\varepsilon^c$ enters ω in time at most $\kappa\varepsilon^{-1}$ for some $\kappa > 0$ which does not depend on ε . Indeed, the solutions of the bicharacteristic equations (11) with $g^* = 1/4$ and $\xi_3 \neq 0$ are given by

$$\begin{aligned} x_1(t) &= \frac{1}{2\xi_3} \cos(2\xi_3 t + \phi) + \frac{\xi_2}{\xi_3}, & x_2(t) &= B - \frac{\xi_2}{2\xi_3} \sin(2\xi_3 t + \phi) \\ x_3(t) &= C + \frac{t}{4\xi_3} + \frac{1}{16\xi_3^2} \sin(2(2\xi_3 t + \phi)) + \frac{\xi_2}{2\xi_3^2} \sin(2\xi_3 t + \phi) \end{aligned}$$

where B, C, ξ_2, ξ_3 are constants. Since $\xi \in V_\varepsilon^c$ and $g^* = 1/4$, there holds $\frac{1}{4|\xi_3|} \geq \frac{\varepsilon}{2}$. Hence, we can conclude using the expression for x_3 (whose derivative is roughly $(4|\xi_3|)^{-1}$) and the fact that $\omega = M_H \setminus B$ contains a horizontal strip. Note that if $\xi_3 = 0$, the expressions of $x_1(t), x_2(t), x_3(t)$ are much simpler and we can conclude similarly.

Hence, together with (48), the propagation property (47) implies that $\nu \equiv 0$. It follows that $\|u^k\|_{\mathcal{H}_T} \rightarrow 0$. By conservation of energy, it is a contradiction with the normalization $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$. Hence, (12) holds. \square

A Pseudodifferential calculus

We denote by Ω an open set of a d -dimensional manifold (typically $d = n$ or $d = n + 1$ with the notations of this paper) equipped with a smooth volume μ . We denote by q the variable in Ω , typically $q = x$ or $q = (t, x)$ with our notations. Let $\pi : T^*\Omega \rightarrow \Omega$ be the canonical projection. We recall briefly some facts concerning pseudodifferential calculus, following [Hör07, Chapter 18].

We denote by $S_{\text{hom}}^m(T^*\Omega)$ the set of homogeneous symbols of degree m with compact support in Ω . We also write $S_{\text{phg}}^m(T^*\Omega)$ the set of polyhomogeneous symbols of degree m with compact support in Ω . Hence, $a \in S_{\text{phg}}^m(T^*\Omega)$ if $a \in C^\infty(T^*\Omega)$, $\pi(\text{Supp}(a))$ is a compact of Ω , and there exist $a_j \in S_{\text{hom}}^{m-j}(T^*\Omega)$ such that for all $N \in \mathbb{N}$, $a - \sum_{j=0}^N a_j \in S_{\text{phg}}^{m-N-1}(T^*\Omega)$. We denote by $\Psi_{\text{phg}}^m(T^*\Omega)$ the space of polyhomogeneous pseudodifferential operators of order m on Ω , with a compactly supported kernel in $\Omega \times \Omega$. For $A \in \Psi_{\text{phg}}^m(\Omega)$, we denote by $\sigma_p(A) \in S_{\text{phg}}^m(T^*\Omega)$ the principal symbol of A . The sub-principal symbol is characterized by the action of pseudodifferential operators on oscillating functions: if $A \in \Psi_{\text{phg}}^m(\Omega)$ and $f(q) = b(q)e^{ikS(q)}$ with b, S smooth and real-valued, then

$$\int_{\Omega} A(f)\bar{f}d\mu = k^m \int_{\Omega} \left(\sigma_p(A)(q, S'(q)) + \frac{1}{k} \sigma_{\text{sub}}(A)(q, S'(q)) \right) |f(q)|^2 d\mu(q) + O(k^{m-2}).$$

A quantization is a continuous linear mapping

$$\text{Op} : S_{\text{phg}}^m(T^*\Omega) \rightarrow \Psi_{\text{phg}}^m(\Omega)$$

satisfying $\sigma_p(\text{Op}(a)) = a$. An example of quantization is obtained by using partitions of unity and, locally, the Weyl quantization, which is given in local coordinates by

$$\text{Op}^W(a)f(q) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_q^d \times \mathbb{R}_p^d} e^{i\langle q-q', p \rangle} a\left(\frac{q+q'}{2}, p\right) f(q') dq' dp.$$

We have the following properties:

1. If $A \in \Psi_{\text{phg}}^l(\Omega)$ and $B \in \Psi_{\text{phg}}^m(\Omega)$, then $[A, B] \in \Psi_{\text{phg}}^{l+m-1}(\Omega)$ and $\sigma_p([A, B]) = \frac{1}{i}\{\sigma_p(a), \sigma_p(b)\}$ where the Poisson bracket is taken with respect to the canonical symplectic structure of $T^*\Omega$.
2. If X is a vector field on Ω and X^* is its formal adjoint in $L^2(\Omega, \mu)$, then $X^*X \in \Psi_{\text{phg}}^2(\Omega)$, $\sigma_p(X^*X) = h_X^2$ and $\sigma_{\text{sub}}(X^*X) = 0$.
3. If $A \in \Psi_{\text{phg}}^m(\Omega)$, then A maps continuously the space $H^s(\Omega)$ to the space $H^{s-m}(\Omega)$.

B Proof of Proposition 9

In this Appendix, we give a second proof of Proposition 9 written in a more elementary form than the one of Section 2.1. Let us first prove the result when $M \subset \mathbb{R}^n$, following the proof of [Ral82]. The general case is addressed at the end of this section.

As in the proof of Section 2.1, we suppress the time variable t . Thus we use $x = (x_0, x_1, \dots, x_n)$ where $x_0 = t$. Similarly, $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ where $\xi_0 = \tau$ previously. Let Γ be the curve given by $x(s) \in \mathbb{R}^{n+1}$. We insist on the fact that in the proof the bicharacteristics are parametrized by s , as in (8). We consider functions of the form

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}.$$

We would like to choose $\psi(x)$ such that for all $s \in \mathbb{R}$, $\psi(x(s))$ is real-valued and $\text{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))$ is positive definite on vectors orthogonal to $\dot{x}(s)$. Roughly speaking, $|e^{ik\psi(x)}|$ will then look like a Gaussian distribution on planes perpendicular to Γ in \mathbb{R}^{n+1} .

We first observe that $\partial_{tt}^2 v_k - \Delta v_k$ can be decomposed as

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi} \quad (49)$$

with

$$\begin{aligned} A_1(x) &= p_2(x, \nabla \psi(x)) a_0(x) \\ A_2(x) &= L a_0(x) \\ A_3(x) &= \partial_{tt}^2 a_0(x) - \Delta a_0(x). \end{aligned}$$

Here we have set

$$L a_0 = \frac{1}{i} \sum_{j=0}^n \frac{\partial p_2}{\partial \xi_j}(x, \nabla \psi(x)) \frac{\partial a_0}{\partial x_j} + \frac{1}{2i} \left(\sum_{j,k=0}^n \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}(x, \nabla \psi(x)) \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) a_0 \quad (50)$$

(For general strictly hyperbolic operators, L contains a term with the sub-principal symbol of the operator, but here it is null, see Appendix A.)

In what follows, we construct a_0 and ψ so that $A_1(x)$ vanishes at order 2 along Γ and $A_2(x)$ vanishes at order 0 along the same curve. We will then be able to use Lemma 11 with $S = 3$ and $S = 1$ respectively.

Analysis of $A_1(x)$. Our goal is to show that, if we choose ψ adequately, we can make the quantity

$$f(x) = p_2(x, \nabla \psi(x)) \quad (51)$$

vanish at order 2 on Γ . For the vanishing at order 0, we prescribe that ψ satisfies $\nabla \psi(x(s)) = \xi(s)$, and then $f(x(s)) = 0$ since $(x(s), \xi(s))$ is a null-bicharacteristic. Note that this is possible since $x(s) \neq x(s')$ for any $s \neq s'$, due to $\dot{x}_0(s) = 1$ (bicharacteristics are traveled at

speed 1, see Section 1.4). For the vanishing at order 1, using (51) and (8), we remark that for any $0 \leq j \leq n$,

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x(s)) &= \frac{\partial p_2}{\partial x_j}(x(s)) + \sum_{k=0}^n \frac{\partial p_2}{\partial \xi_k}(x(s)) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\
&= -\dot{\xi}_j(s) + \sum_{k=0}^n \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\
&= -\frac{d}{ds} \left(\frac{\partial \psi}{\partial x_j}(x(s)) \right) + \sum_{k=0}^n \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\
&= 0.
\end{aligned} \tag{52}$$

Therefore, f vanishes automatically at order 1 along Γ (without making any particular choice for ψ): it just follows from (51) and the bicharacteristic equations (8). But for $f(x)$ to vanish at order 2 along Γ , it is required to choose a particular ψ . In the end, we will find that if ψ is given by the formula (58) below, with M being a solution of (53), then f vanishes at order 2 along Γ . Let us explain why.

Using the Einstein summation notation, we want that for any $0 \leq i, j \leq n$, there holds

$$\begin{aligned}
0 &= \frac{\partial^2 f}{\partial x_j \partial x_i} \\
&= \frac{\partial^2 p_2}{\partial x_j \partial x_i} + \frac{\partial^2 p_2}{\partial \xi_k \partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial^2 p_2}{\partial x_j \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_2}{\partial \xi_l \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} \frac{\partial^2 \psi}{\partial x_j \partial x_l} + \frac{\partial p_2}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_j \partial x_k \partial x_i}
\end{aligned}$$

along Γ . Introducing the matrices

$$\begin{aligned}
(M(s))_{ij} &= \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s)), & (A(s))_{ij} &= \frac{\partial^2 p_2}{\partial x_i \partial x_j}(x(s), \xi(s)), \\
(B(s))_{ij} &= \frac{\partial^2 p_2}{\partial \xi_i \partial x_j}(x(s), \xi(s)), & (C(s))_{ij} &= \frac{\partial^2 p_2}{\partial \xi_i \partial \xi_j}(x(s), \xi(s))
\end{aligned}$$

this amounts to solving the matricial Riccati equation

$$\frac{dM}{ds} + MCM + B^T M + MB + A = 0 \tag{53}$$

on a finite-length time-interval. While solving (53), we also require $M(s)$ to be symmetric, $\text{Im}(M(s))$ to be positive definite on the orthogonal complement of $\dot{x}(s)$, and $M(s)\dot{x}(s) = \dot{\xi}(s)$ to hold for all s due to (52).

Let M_0 be a symmetric $(n+1) \times (n+1)$ matrix with $\text{Im}(M_0) > 0$ on the orthogonal complement of $\dot{x}(0)$ and $M_0 \dot{x}(0) = \dot{\xi}(0)$ (in particular $\text{Im}(M_0) \dot{x}(0) = 0$). It is shown in [Ral82] that there exists a global solution $M(s)$ on $[0, T]$ of (53) which satisfies all the above conditions and such that $M(0) = M_0$. The proof just requires that A, C are symmetric, but does not need anything special about p_2 (in particular, it applies to our sub-Riemannian case where p_2 is degenerate). For the sake of completeness, we recall the proof here.

We consider $(Y(s), N(s))$ the matrix solution with initial data $(Y(0), N(0)) = (\text{Id}, M_0)$ (where Id is the $(n+1) \times (n+1)$ identity matrix) to the linear system

$$\begin{cases} \dot{Y} = BY + CN \\ \dot{N} = -AY - B^T N. \end{cases} \tag{54}$$

We note that $(Y(s)\dot{x}(0), N(s)\dot{x}(0))$ then also solves (54), with Y and N being this time vectorial. One can check that $(\dot{x}(s), \dot{\xi}(s))$ is the solution of the same linear system with same initial data, and therefore, for any $s \in \mathbb{R}$,

$$\dot{x}(s) = Y(s)\dot{x}(0), \quad \dot{\xi}(s) = N(s)\dot{x}(0). \tag{55}$$

All the coefficients in (54) are real and A and C are symmetric, and it follows that the flow defined by (54) *on vectors* preserves both the real symplectic form acting on pairs $(y, \eta) \in (\mathbb{R}^{n+1})^2$ and $(y', \eta') \in (\mathbb{R}^{n+1})^2$ given by

$$\sigma((y, \eta), (y', \eta')) = y \cdot \eta' - \eta \cdot y'$$

and the complexified form $\sigma_{\mathbb{C}}((y, \eta), (y', \eta')) = \sigma((y, \eta), (\overline{y'}, \overline{\eta'}))$ for $(y, \eta) \in (\mathbb{C}^{n+1})^2$ and $(y', \eta') \in (\mathbb{C}^{n+1})^2$. When we say that $\sigma_{\mathbb{C}}$ is invariant under (54), it means that we allow complex vectorial initial data in (54).

Let us prove that $Y(s)$ is invertible for any s . Let $v \in \mathbb{C}^{n+1}$ and $s_0 \in \mathbb{R}$ be such that $Y(s_0)v = 0$. We set $y(s_0) = Y(s_0)v$ and $\eta(s_0) = N(s_0)v$ and consider $\chi(s_0) = (y(s_0), \eta(s_0))$. From the conservation of $\sigma_{\mathbb{C}}$, we get

$$0 = \sigma_{\mathbb{C}}(\chi(s_0), \chi(s_0)) = \sigma_{\mathbb{C}}(\chi(0), \chi(0)) = v \cdot \overline{M_0 v} - \overline{v} \cdot M_0 v = -2i\overline{v} \cdot (\text{Im}(M_0))v.$$

Since $\text{Im}(M_0)$ is positive definite on the orthogonal complement to $\dot{x}(0)$, there holds $v = \lambda \dot{x}(0)$ for some $\lambda \in \mathbb{C}$. Hence

$$0 = Y(s_0)v = \lambda Y(s_0)\dot{x}(0) = \lambda \dot{x}(s_0)$$

where the last equality comes from (55). Since $\dot{x}_0(s_0) = \frac{\partial p_2}{\partial \xi_0}(s_0) = -2\xi_0(s_0) = 1$, there holds $\dot{x}(s_0) \neq 0$, hence $\lambda = 0$. It follows that $v = 0$ and $Y(s_0)$ is invertible.

Now, for any $s \in \mathbb{R}$, we set

$$M(s) = N(s)Y(s)^{-1}$$

which is a solution of (53) with $M(0) = M_0$. It verifies $M(s)\dot{x}(s) = \dot{\xi}(s)$ thanks to (55). Moreover, it is symmetric: if we denote by $y^i(s)$ and $\eta^i(s)$ the column vectors of Y and N , by preservation of σ , for any $0 \leq i, j \leq n$, the quantity

$$\sigma((y^i(s), \eta^i(s)), (y^j(s), \eta^j(s))) = y^i(s) \cdot M(s)y^j(s) - y^j(s) \cdot M(s)y^i(s)$$

is equal to the same quantity at $s = 0$, which is equal to 0 since M_0 is symmetric.

Let us finally prove that for any $s \in \mathbb{R}$, $\text{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$. Let $y(s_0) \in \mathbb{C}^{n+1}$ be in the orthogonal complement of $\dot{x}(s_0)$. We decompose $y(s_0)$ on the column vectors of $Y(s_0)$:

$$y(s_0) = \sum_{i=0}^n b_i y^i(s_0), \quad b_i \in \mathbb{C}.$$

For $s \in \mathbb{R}$, we consider $y(s) = \sum_{i=0}^n b_i y^i(s)$ and we set $\chi(s) = \sum_{i=0}^n b_i (y^i(s), \eta^i(s))$. Then,

$$\sigma_{\mathbb{C}}(\chi(s), \chi(s)) = -2i\overline{y(s)} \cdot \text{Im}(M(s))y(s). \quad (56)$$

By preservation of $\sigma_{\mathbb{C}}$ and using (56), we get that

$$\overline{y(s_0)} \cdot \text{Im}(M(s_0))y(s_0) = \overline{y(0)} \cdot \text{Im}(M_0)y(0). \quad (57)$$

But $y(0)$ cannot be proportional to $\dot{x}(0)$ otherwise, using (55), we would get that $y(s_0)$ is proportional to $\dot{x}(s_0)$. Hence, the right hand side in (57) is > 0 , which implies that $\text{Im}(M(s_0))$ is positive definite on the orthogonal complement to $\dot{x}(s_0)$.

Therefore, we found a choice for the second order derivatives of ψ along Γ which meets all our conditions. For $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$ and s such that $t = t(s)$, we set

$$\psi(x) = \xi'(s) \cdot (x' - x'(s)) + \frac{1}{2}(x' - x'(s)) \cdot M(s)(x' - x'(s)), \quad (58)$$

and for this choice of ψ , f vanishes at order 2 along Γ .

To sum up, as in the Riemannian (or “strictly hyperbolic”) case handled by Ralston in [Ral82], the key observation is that the invariance of σ and $\sigma_{\mathbb{C}}$ prevents the solutions of (53) with positive imaginary part on the orthogonal complement of $\dot{x}(0)$ to blowup.

Analysis of $A_2(x)$. We note that A_2 vanishes along Γ if and only if $La_0(x(s)) = 0$. According to (50), this turns out to be a linear transport equation on $a_0(x(s))$. Moreover, the coefficient of the first-order term, namely $\nabla_{\xi} p_2(x(s), \xi(s))$, is different from 0. Therefore, given $a_0 \neq 0$ at $(t = 0, x = x(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a_0 , and, by Cauchy uniqueness, $a_0(x(s)) \neq 0$ for any s . Note that we have prescribed a_0 only along Γ , and we may choose a_0 in a smooth (and arbitrary) way outside Γ . We choose it to vanish outside a small neighborhood of Γ .

Proof of (14). We use (49) and we apply Lemma 11 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1(0,T;L^2(M))} \leq C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (14).

Proof of (15). We first observe that since $\text{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$ and continuous as a function of s , there exist $\alpha, C > 0$ such that for any $t(s) \in [0, T]$ and any $x' \in M$,

$$|\partial_t v_k(t(s), x')|^2 + |\nabla^{\text{sR}} v_k(t(s), x')|^2 \geq \left(C|a_0(t(s), x')|^2 k^{\frac{n}{2}} + O(k^{2(\frac{n}{2}-1)}) \right) e^{-\alpha k d(x', x'(s))^2}$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in \mathbb{R}^n . We denote by ℓ_n the Lebesgue measure on \mathbb{R}^n . Using the observation that for any function f ,

$$\int_M f(x') e^{-\alpha k d(x', x'(s))^2} d\mu(x') \sim \frac{\pi^{n/2}}{k^{n/2} \sqrt{\alpha}} f(x'(s)) \frac{d\mu}{d\ell_n}(x'(s)) \quad (59)$$

as $k \rightarrow +\infty$, and the fact that $a_0(x(s)) \neq 0$, we obtain (15).

Proof of (16). We observe that since $\text{Im}(M(s))$ is positive definite (uniformly in s) on the orthogonal complement of $\dot{x}(s)$, there exist $C, \alpha' > 0$ such that for any $t \in [0, T]$, for any $x' \in M$, $|\partial_t v_k(t(s), x')|$ and $|\nabla^{\text{sR}} v_k(t(s), x')|$ are both bounded above by $Ck^{\frac{n}{4}} e^{-\alpha' k d(x', x'(s))^2}$. Therefore

$$\begin{aligned} & \int_{M \setminus V_{t(s)}} (|\partial_t v_k(t(s), x')|^2 + |\nabla^{\text{sR}} v_k(t(s), x')|^2) d\mu(x') \\ & \leq Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\mu(x') \\ & \leq Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\ell_n(x') + o(1) \end{aligned} \quad (60)$$

where, in the last line, we used the fact that $|d\mu/d\ell_n| \leq C$ in a fixed compact subset of M (since μ is a smooth volume), and the $o(1)$ comes from the eventual blowup of μ at the boundary of M .

Now, $M \subset \mathbb{R}^n$, and there exists $r > 0$ such that $B_d(x(s), r) \subset V_{t(s)}$ for any s such that $t(s) \in (0, T)$, where $d(\cdot, \cdot)$ still denotes the Euclidean distance in \mathbb{R}^n . Therefore, we bound above the integral in (60) by

$$Ck^{n/2} \int_{\mathbb{R}^n \setminus B_d(x(s), r)} e^{-2\alpha' k d(x', x'(s))^2} d\ell_n(x') \quad (61)$$

Making the change of variables $y = k^{-1/2}(y - x(s))$, we bound above (61) by

$$C \int_{\mathbb{R}^n \setminus B_d(0, rk^{1/2})} e^{-2\alpha' \|y\|^2} d\ell_n(y)$$

with $\|\cdot\|$ the Euclidean norm. This last expression is bounded above by

$$C e^{-\alpha' r^2 k} \int_{\mathbb{R}^n} e^{-\alpha' \|y\|^2} d\ell_n(y)$$

which implies (16).

Extension of the result to any manifold M . In the case of a general manifold M , not necessarily included in \mathbb{R}^n , we use charts together with the above construction. We cover M by a set of charts $(U_\alpha, \varphi_\alpha)$, where (U_α) is a family of open sets of M covering M and $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ is an homeomorphism U_α onto an open subset of \mathbb{R}^n . Take a solution $(x(t), \xi(t))_{t \in [0, T]}$ of (9). It visits a finite number of charts in the order $U_{\alpha_1}, U_{\alpha_2}, \dots$, and we choose the charts and a_0 so that $v_k(t, \cdot)$ is supported in a unique chart at each time t . The above construction shows how to construct a_0 and ψ as long as $x(t)$ remains in the same chart. For any $l \geq 1$, we choose t_l so that $x(t_l) \in U_{\alpha_l} \cap U_{\alpha_{l+1}}$ and $a_0(t_l, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Since there is a (local) solution v_k for any choice of initial $a_0(t_l, x(t_l))$ and $\text{Im} \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j} \right) (t_l, x(t_l))$ in Proposition 9, we see that v_k may be continued from the chart U_{α_l} to the chart $U_{\alpha_{l+1}}$. This continuation is smooth since the two solutions coincide as long as $a_0(t, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Patching all solutions on the time intervals $[t_l, t_{l+1}]$ together, it yields a global in time solution v_k , as desired.

C Proof of (48)

Because of the second convergence in (42) and the non-negativity of a , it amounts to proving that

$$\int_0^T |(\nabla^{\text{sR}} \text{Op}(a) u^k, \nabla^{\text{sR}} u^k)_{L^2(M_H, \mu)}| dt \rightarrow 0.$$

Now, we notice that for any $B \in \Psi_{\text{phg}}^0((0, T) \times M_H)$, there holds

$$(B u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} \xrightarrow{k \rightarrow +\infty} 0 \quad \text{and} \quad (B u^k, \partial_t u^k)_{L^2((0, T) \times M_H)} \xrightarrow{k \rightarrow +\infty} 0 \quad (62)$$

since $u^k \rightarrow 0$ strongly in $L^2((0, T) \times M_H)$ and both $\nabla^{\text{sR}} u^k$ and $\partial_t u^k$ are bounded in $L^2((0, T) \times M_H)$. We apply this to $B = [\nabla^{\text{sR}}, \text{Op}(a)]$, and then, also using (62), we see that we can replace $\text{Op}(a)$ by its Friedrichs quantization $\text{Op}^F(a)$, which is positive (see [Tay74, Chapter VII]). In other words, we are reduced to prove

$$(\text{Op}^F(a) \nabla^{\text{sR}} u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} \xrightarrow{k \rightarrow +\infty} 0. \quad (63)$$

Let $\delta > 0$ and $\tilde{a} \in S_{\text{phg}}^0((-\delta, T + \delta) \times M_H)$, $0 \leq \tilde{a} \leq \sup(a)$ and such that $\tilde{a}(t, \cdot) = a(\cdot)$ for $0 \leq t \leq T$. Making repeated use of (62) and of integrations by parts, we have

$$\begin{aligned} (\text{Op}^F(\tilde{a}) \nabla^{\text{sR}} u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} &= (\nabla^{\text{sR}} \text{Op}^F(\tilde{a}) u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &= -(\text{Op}^F(\tilde{a}) u^k, \Delta u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &= -(\text{Op}^F(\tilde{a}) u^k, \partial_t^2 u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &= (\partial_t \text{Op}^F(\tilde{a}) u^k, \partial_t u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &= (\text{Op}^F(\tilde{a}) \partial_t u^k, \partial_t u^k)_{L^2((0, T) \times M_H)} + o(1). \end{aligned}$$

Finally we note that since Op^F is a positive quantization, we have

$$\begin{aligned} (\text{Op}^F(a) \nabla^{\text{sR}} u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} &\leq (\text{Op}^F(\tilde{a}) \nabla^{\text{sR}} u^k, \nabla^{\text{sR}} u^k)_{L^2((0, T) \times M_H)} \\ &= (\text{Op}^F(\tilde{a}) \partial_t u^k, \partial_t u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &\leq C\delta + (\text{Op}^F(a) \partial_t u^k, \partial_t u^k)_{L^2((0, T) \times M_H)} + o(1) \\ &\leq C\delta + o(1) \end{aligned}$$

where C does not depend on δ . Making $\delta \rightarrow 0$, it concludes the proof of (63), and consequently (48) holds.

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