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Cyril Letrouit

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Subelliptic wave equations are never observable

Cyril Letrouit^{*†}

February 4, 2020

Abstract

It is well-known that observability (and, by duality, controllability) of the elliptic wave equation, i.e., with a Riemannian Laplacian, in time T_0 is almost equivalent to the Geometric Control Condition (GCC), which stipulates that any geodesic ray meets the control set within time T_0 . We show that in the subelliptic setting, GCC is never verified, and that subelliptic wave equations are never observable in finite time. More precisely, given any subelliptic Laplacian $\Delta = -\sum_{i=1}^m X_i^* X_i$ on a manifold M such that $\text{Lie}(X_1, \dots, X_m) = TM$ but $\text{Span}(X_1, \dots, X_m) \subsetneq TM$, we show that for any $T_0 > 0$ and any measurable subset $\omega \subset M$ such that $M \setminus \omega$ has nonempty interior, the wave equation with subelliptic Laplacian Δ is not observable on ω in time T_0 . The proof is based on the construction of sequences of solutions of the wave equation concentrating on spiraling geodesics (for the associated sub-Riemannian distance) spending a long time in $M \setminus \omega$. As a counterpart, we prove a positive result of observability for the wave equation in the Heisenberg group, where the observation set is a well-chosen part of the phase space.

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1 Introduction

1.1 Setting

This article focuses on the wave equation in sub-Riemannian manifolds, i.e., on subelliptic wave equations. Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n with a non-empty boundary ∂M . We consider a smooth horizontal distribution \mathcal{D} on M , i.e., a smooth assignment $M \ni x \mapsto \mathcal{D}_x \subset T_x M$ (possibly with non-constant rank), and a Riemannian metric g on \mathcal{D} . We also assume that \mathcal{D} satisfies the Hörmander condition

$$\text{Lie}(\mathcal{D}) = TM \tag{1}$$

(see [Mon02]). The triple (M, \mathcal{D}, g) is called a sub-Riemannian structure. Additionally, we make the important assumption that the set of all $x \in M$ such that $\mathcal{D}_x \neq T_x M$ is dense in M ; in other words, (M, \mathcal{D}, g) is nowhere Riemannian. Finally, we assume that M is endowed with a smooth volume μ .

We consider the sub-Riemannian Laplacian $\Delta_{g,\mu}$ on $L^2(M, \mu)$, which only depends on g and μ , defined by

$$\Delta_{g,\mu} = - \sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m X_i^2 + \text{div}_\mu(X_i) X_i$$

where $(X_i)_{1 \leq i \leq m}$ denotes a local g -orthonormal frame such that $\mathcal{D} = \text{Span}(X_1, \dots, X_m)$ and the star designates the transpose in $L^2(M, \mu)$. The divergence with respect to μ is defined by $L_X \mu = (\text{div}_\mu X) \mu$, where L_X stands for the Lie derivative. Then $\Delta_{g,\mu}$ is hypoelliptic (see [Hör67]). In order to simplify notations, we set $\Delta = \Delta_{g,\mu}$ in the sequel, since g and μ are fixed once for all.

We consider Δ with Dirichlet boundary conditions and the domain $D(\Delta)$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\text{Id} - \Delta)u\|_{L^2}$.

Consider the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 & \text{in } (0, T) \times M \\ u = 0 & \text{on } (0, T) \times \partial M, \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \end{cases} \tag{2}$$

where $T > 0$, and the initial data (u_0, u_1) are in an appropriate energy space. The natural energy of a solution is

$$E(u(t, \cdot)) = \frac{1}{2} \int_M (|\partial_t u(t, x)|^2 + |\nabla^{\text{sR}} u(t, x)|^2) d\mu(x)$$

where, for any $\phi \in C^\infty(M)$,

$$\nabla^{\text{sR}} \phi = \sum_{i=1}^m (X_i \phi) X_i$$

is the horizontal gradient. Note that ∇^{sR} is the formal adjoint of $(-\text{div}_\mu)$ in $L^2(M, \mu)$.

We denote by $\mathcal{H}(M)$ the completion of $C_c^\infty(M)$ for the norm

$$\|v\|_{\mathcal{H}} = \left(\int_M |\nabla^{\text{sR}} v(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.$$

and the space of initial data of (2) is naturally endowed with the norm

$$\|(u_0, u_1)\|_{\mathcal{H} \times L^2} = \left(\|u_0\|_{\mathcal{H}}^2 + \|u_1\|_{L^2(M, \mu)}^2 \right)^{\frac{1}{2}}.$$

It is well-known (see for example [GR15, Theorem 2.1], [EN99, Section II.6]) that for any $(u_0, u_1) \in \mathcal{H}(M) \times L^2(M)$, there exists a unique solution

$$u \in C^0(0, T; \mathcal{H}(M)) \cap C^1(0, T; L^2(M))$$

to (2) (in a mild sense). Moreover, if u is a solution of (2), then

$$\frac{d}{dt} E(u(t, \cdot)) = 0.$$

In this paper, we investigate exact observability for the wave equation (2).

Definition 1. *Let $T_0 > 0$ and $\omega \subset M$ be a μ -measurable subset. The subelliptic wave equation (2) is exactly observable on ω in time T_0 if there exists a constant $C_{T_0}(\omega) > 0$ such that, for any $(u_0, u_1) \in \mathcal{H}(M) \times L^2(M)$, the solution u of (2) satisfies*

$$\int_0^{T_0} \int_{\omega} |\partial_t u(t, x)|^2 d\mu(x) dt \geq C_{T_0}(\omega) \|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2. \quad (3)$$

1.2 Main result

Our main result is the following.

Theorem 1. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset such that $M \setminus \omega$ has nonempty interior. Then the subelliptic wave equation (2) is not exactly observable on ω in time T_0 .*

Consequently, using a duality argument (see Section 4.2), we obtain that exact controllability does not hold either in any finite time.

Definition 2. *Let $T_0 > 0$ and $\omega \subset M$ be a measurable subset. The subelliptic wave equation (2) is exactly controllable on ω in time T_0 if for any $(u_0, u_1) \in \mathcal{H}(M) \times L^2(M)$, there exists $g \in L^2((0, T_0) \times M)$ such that the solution u of*

$$\begin{cases} \partial_{tt}^2 u - \Delta u = \mathbf{1}_{\omega} g & \text{in } (0, T_0) \times M \\ u = 0 & \text{on } (0, T_0) \times \partial M, \\ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \end{cases} \quad (4)$$

satisfies $u(T_0, \cdot) = 0$.

Corollary 1. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset such that $M \setminus \omega$ has nonempty interior. Then the subelliptic wave equation (2) is not exactly controllable on ω in time T_0 .*

Remark 3. *Theorem 1 holds under the two assumptions that \mathcal{D} satisfies the Hörmander condition (1) and that the set of $x \in M$ such that $\mathcal{D}_x \neq T_x M$ is dense in M . However, inspecting the proof, we see that the conclusion of Theorem 1 also holds under the weaker assumption that the set of $x \in M$ such that $\mathcal{D}_x \subsetneq \mathcal{D}_x + [\mathcal{D}_x, \mathcal{D}_x]$ is dense in M .*

In the statement of Theorem 1, we assumed that the sub-Riemannian structure (M, \mathcal{D}, g) verifies $\mathcal{D}_x \neq T_x M$ for a dense set of $x \in M$. Let us explain how to adapt this result to the case of almost-Riemannian structures, i.e., sub-Riemannian structures which do not necessarily verify this assumption. A typical example is the Grushin case, for which $X_1 = \partial_{x_1}$ and $X_2 = x_1 \partial_{x_2}$ are vector fields on $(-1, 1)_{x_1} \times \mathbb{T}_{x_2}$. Then $\text{rank}(\mathcal{D}_x)$ is equal to 1 for $x_1 = 0$ and to 2 otherwise.

Theorem 2. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable set such that $M \setminus \omega$ has an interior which is non-empty and which moreover contains a point x such that $\mathcal{D}_x \neq T_x M$. Then the subelliptic wave equation (2) is not exactly observable on ω in time T_0 .*

Remark 4. *In the Grushin case, the corresponding Laplacian is elliptic outside of the singular submanifold $S = \{x_1 = 0\}$. According to the above result, in the Grushin case the subelliptic wave equation is observable on any subset containing S (with some finite minimal time of observability, see [BLR92]), but is not observable in any finite time on any subset ω such that the interior of $M \setminus \omega$ has a non-empty intersection with S .*

Remark 5. *The assumption of compactness on M is not necessary: we may remove it, and just require that the subelliptic wave equation (2) in M is well-posed. It is for example the case if M is complete for the sub-Riemannian distance induced by g since Δ is then essentially self-adjoint ([Str86]).*

Remark 6. *Theorem 1 remains true if M has no boundary. In this case, since non-zero constant functions on M are solutions of (2), one usually requires for the observability inequality (3) to hold only for solutions with initial data (u_0, u_1) verifying $\int_M u_0 d\mu = 0$ (i.e., we quotient by constant functions). Then, Theorem 1 remains true: anticipating the proof, we notice that the spiraling geodesics of Proposition 12 still exist (since their construction is purely local), and we subtract to the initial datum u_0^k of the localized solutions constructed in Proposition 11 their spatial average $\int_M u_0^k d\mu$.*

1.3 Ideas of the proof

The proof of Theorem 1 mainly requires two ingredients:

1. There exist solutions of the free subelliptic wave equation (2) whose energy concentrates along any given (normal) geodesic of (M, \mathcal{D}, g) ;
2. There exist normal geodesics of (M, \mathcal{D}, g) which “spiral” around curves transverse to \mathcal{D} , and which therefore remain arbitrarily close to their starting point on arbitrarily large-time intervals.

Combining these two facts, the proof of Theorem 1 is straightforward (see Section 4.1). Note that the first point follows from the general theory of propagation of complex Lagrangian spaces, while the second point is the main novelty of this paper.

Since our construction is purely local (meaning that it does not “feel” the boundary and only relies on the local structure of the vector fields), we can focus on the case where there is a (small) open neighborhood V of the origin such that $V \subset M \setminus \omega$. In the sequel, we assume it is the case.

Let us give an example of sub-Riemannian structure where the spiraling geodesics used in the proof of Theorem 1 are particularly simple. We consider the three-dimensional manifold with boundary $M_1 = (-1, 1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx (-1, 1)$ is the 1D torus. We endow M_1 with the vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1 \partial_{x_3}$ and we set $\mathcal{D}_1 = \text{Span}(X_1, X_2)$, with the metric g_1 being defined by the fact that (X_1, X_2) is a g_1 -orthonormal frame of \mathcal{D}_1 . Then, $(M_1, \mathcal{D}_1, g_1)$ is a sub-Riemannian structure, which we will call in the sequel the “Heisenberg manifold with boundary”. We endow it with an arbitrary smooth volume μ . The geodesics we consider are given by

$$\begin{aligned} x_1(t) &= \varepsilon \sin(t/\varepsilon) \\ x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon \\ x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned}$$

They spiral around the x_3 axis $x_1 = x_2 = 0$.

Here, one should think of ε as a small parameter. In the sequel, we denote by x_ε the geodesic with parameter ε .

Clearly, given any $T_0 > 0$, for ε sufficiently small, we have $x_\varepsilon(t) \in V$ for every $t \in (0, T_0)$. Our objective is to construct solutions u^k of the subelliptic wave equation (2) such that $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$ and the energy of $u^k(t, \cdot)$ outside of a ball $B_{g_1}(x_\varepsilon(t), r_k)$ centered at $x_\varepsilon(t)$ and with small radius $r_k > 0$

$$\int_{M_1 \setminus B_{g_1}(x_\varepsilon(t), r_k)} (|\partial_t u^k(t, x)|^2 + |\nabla^{\text{sR}} u^k(t, x)|^2) d\mu(x)$$

tends to 0 as $k \rightarrow +\infty$ uniformly with respect to $t \in (0, T_0)$. As a consequence, the observability inequality (3) fails.

The construction of solutions of the free wave equation whose energy concentrates on geodesics is classical in the elliptic (or Riemannian) case: these are the so-called Gaussian beams, for which a construction may be found for example in [Ral82]. Here, we adapt this construction to our subelliptic (sub-Riemannian) setting, which does not raise any problem since the geodesics we consider stay in the elliptic part of the operator Δ . It may also be directly justified with the theory of propagation of complex Lagrangian spaces (see Section 2).

In the general case where (M, \mathcal{D}, g) is not necessarily the Heisenberg manifold without boundary, the existence of spiraling geodesics also has to be justified. For that purpose, we first approximate (M, \mathcal{D}, g) by its *nilpotent approximation*, and we then prove that in the latter, it is possible to identify a ‘‘Heisenberg sub-structure’’, which gives the desired spiraling geodesics.

1.4 Sub-Riemannian geodesics

In this section, we recall a few basic facts about sub-Riemannian geodesics. In this paper, we just need to focus on *normal* geodesics, which are the natural extension of Riemannian geodesics since they are projections of bicharacteristics. Recall that there may also exist abnormal geodesics (see [Mon94]), but we did not address the problem of constructing solutions of (2) concentrating on these geodesics since it is not useful for our purpose.

We denote by $S_{\text{phg}}^m(T^*((0, T) \times M))$ the set of polyhomogeneous symbols of order m with compact support and by $\Psi_{\text{phg}}^m((0, T) \times M)$ the set of associated polyhomogeneous pseudodifferential operators of order m whose distribution kernel has compact support in $(0, T) \times M$ (see Appendix A).

We set $P = \partial_{tt}^2 - \Delta \in \Psi_{\text{phg}}^2((0, T) \times M)$, whose principal symbol is

$$p_2(t, \tau, x, \xi) = -\tau^2 + g^*(x, \xi)$$

with τ the dual variable of t and g^* the principal symbol of $-\Delta$. For $\xi \in T^*M$, we have (see Appendix A)

$$g^*(x, \xi) = \sum_{i=1}^m h_{X_i}^2.$$

Here, given any smooth vector field X on M , we denoted by h_X the Hamiltonian function (momentum map) on T^*M associated with X defined in local (x, ξ) -coordinates by $h_X(x, \xi) = \xi(X(x))$. Then g^* is both the principal symbol of Δ , and also the cometric associated with g . Equivalently, $g^*(x, \xi) = \sup \{\xi(v)^2 \mid v \in T_x M, g_x(v) = 1\}$.

In $T^*(\mathbb{R} \times M)$, the Hamiltonian vector field \vec{H}_{p_2} associated with p_2 is given by $\vec{H}_{p_2} f = \{p_2, f\}$. Since $\vec{H}_{p_2} p_2 = 0$, we get that p_2 is constant along the integral curves of \vec{H}_{p_2} . Thus, the characteristic set $\mathcal{C}(p_2) = \{p_2 = 0\}$ is preserved by the flow of \vec{H}_{p_2} . Null-bicharacteristics are then defined as the maximal integral curves of \vec{H}_{p_2} which live in $\mathcal{C}(p_2)$. In other words, the null-bicharacteristics are the maximal solutions of

$$\begin{cases} \dot{t}(s) = -2\tau(s), \\ \dot{x}(s) = \nabla_{\xi} g^*(x(s), \xi(s)), \\ \dot{\tau}(s) = 0, \\ \dot{\xi}(s) = -\nabla_x g^*(x(s), \xi(s)), \\ \tau^2(0) = g^*(x(0), \xi(0)). \end{cases} \quad (5)$$

This definition needs to be adapted when the null-bicharacteristic meets the boundary ∂M , but in the sequel, we only consider solutions of (5) on time intervals where $x(t)$ does not reach ∂M .

In the sequel, we take $\tau = -1/2$, which gives $g^*(x(s), \xi(s)) = 1/4$. This also implies that $t(s) = s + t_0$ and, taking t as a time parameter, we are led to solve

$$\begin{cases} \dot{x}(t) = \nabla_{\xi} g^*(x(t), \xi(t)), \\ \dot{\xi}(t) = -\nabla_x g^*(x(t), \xi(t)), \\ g^*(x(0), \xi(0)) = \frac{1}{4}. \end{cases} \quad (6)$$

Remark 7. *Note that in the subelliptic setting, the co-sphere bundle S^*M may be decomposed as $S^*M = U^*M \cup S\Sigma$, where $U^*M = \{g^* = 1\}$ is a cylinder bundle, $\Sigma = \{g^* = 0\}$ is the characteristic cone and $S\Sigma$ is a twofold covering of M . As explained in [CdVHT18, Section 1], each fiber of S^*M is obtained by compactifying a cylinder with two points at infinity.*

We denote by $\phi_t : S^*M \rightarrow S^*M$ the (normal) geodesic flow defined by $\phi_t(x_0, \xi_0) = (x(t), \xi(t))$, where $(x(t), \xi(t))$ is a solution of the system given by the first two lines of (6) and initial conditions (x_0, ξ_0) . Note that any point in $S\Sigma$ is a fixed point of ϕ_t .

The curves $x(t)$ which solve (6) are the geodesics for the sub-Riemannian metric g . In other words, the projections of the null-bicharacteristics onto M , using the variable t as a parameter, coincide with the geodesics on M associated with the metric sub-Riemannian g traveled at speed one.

1.5 Observability in some regions of phase-space

We have explained in Section 1.3 that the existence of solutions of the subelliptic wave equation (2) concentrated on spiraling geodesics is an obstruction to observability in Theorem 1. Our goal in this section is to state a result ensuring observability if one “removes” in some sense these geodesics.

For this result, we focus on a version of the Heisenberg manifold described in Section 1.3 which has *no boundary*. This technical assumption avoids us using boundary microlocal defect measures in the proof, which, in this sub-Riemannian setting, are difficult to handle. As a counterpart, the observability inequality is written for functions with null initial average, since otherwise constant solutions would not be observable.

We consider the Heisenberg group G , that is \mathbb{R}^3 with the composition law

$$(x_1, x_2, x_3) \star (x'_1, x'_2, x'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 - x_1 x'_2).$$

Then $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1 \partial_{x_3}$ are left invariant vector fields on G . Since $\Gamma = \sqrt{2\pi}\mathbb{Z} \times \sqrt{2\pi}\mathbb{Z} \times 2\pi\mathbb{Z}$ is a co-compact subgroup of \mathbb{R}^3 , the left quotient $M_H = \Gamma \backslash G$ is a compact three dimensional manifold and, moreover, X_1 and X_2 are well-defined as vector fields on the quotient. Finally, we define the Heisenberg Laplacian $\Delta_H = X_1^2 + X_2^2$ on M_H . Since $[X_1, X_2] = -\partial_{x_3}$, it is a hypoelliptic operator. We set $\mathcal{D}_H = \text{Span}(X_1, X_2)$, with the metric g_H being defined by the fact that (X_1, X_2) is a g_H -orthonormal frame of \mathcal{D}_H . Then, $(M_H, \mathcal{D}_H, g_H)$ is a sub-Riemannian structure, which we call the “Heisenberg manifold without boundary”. We endow $(M_H, \mathcal{D}_H, g_H)$ with an arbitrary smooth volume μ .

We note that $g^*(x, \xi) = \xi_1^2 + (\xi_2 - x_1 \xi_3)^2$ and hence the null-bicharacteristics are solutions of

$$\begin{aligned} \dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1 \xi_3), \\ \dot{x}_2(t) &= 2(\xi_2 - x_1 \xi_3), & \dot{\xi}_2(t) &= 0, \\ \dot{x}_3(t) &= -2x_1(\xi_2 - x_1 \xi_3), & \dot{\xi}_3(t) &= 0. \end{aligned}$$

The spiraling geodesics described in Section 1.3 correspond to $\xi_1 = \cos(t/\varepsilon)/2$, $\xi_2 = 0$ and $\xi_3 = 1/(2\varepsilon)$. In particular, the constant ξ_3 is a kind of rounding number reflecting the fact that the geodesic spirals at a certain speed around the x_3 axis. Moreover, ξ_3 is preserved by the flow (somehow, the Heisenberg flow is completely integrable), and this property plays a key role in the proof of Theorem 3 below and justifies that we state it only for the Heisenberg manifold (without boundary).

As said above, geodesics corresponding to a large momentum ξ_3 are precisely the ones used to contradict observability in Theorem 1. We expect to be able to establish observability if we consider only solutions of (2) whose ξ_3 (in a certain sense) is not too large. This is the purpose of our second main result.

Set

$$V_\varepsilon = \left\{ (x, \xi) \in T^*M_H : |\xi_3| > \frac{1}{\varepsilon} (g_x^*(\xi))^{1/2} \right\}$$

Note that since ξ_3 is constant along null-bicharacteristics, the complementary V_ε^c of V_ε , is invariant under the geodesic flow, i.e., $\phi_t(V_\varepsilon^c) \subset V_\varepsilon^c$ (and in some sense V_ε is also invariant, but one should take care that the geodesic flow is not defined in $S\Sigma$, see Remark 7).

Theorem 3. *Let $B \subset M_H$ be an open sub-Riemannian ball and suppose that B is sufficiently small, so that $\omega = M_H \setminus B$ contains an horizontal strip. Let $a \in S_{phg}^0(T^*M_H)$, $a \geq 0$, such that, denoting by $j : T^*\omega \rightarrow T^*M_H$ the canonical injection,*

$$j(T^*\omega) \cup V_\varepsilon \subset \text{Supp}(a) \subset T^*M_H,$$

and in particular a does not depend on time. There exists $\kappa > 0$ such that for any $\varepsilon > 0$ and any $T \geq \kappa\varepsilon^{-1}$, there holds

$$\int_0^T |(Op(a)\partial_t u, \partial_t u)_{L^2(M_H, \mu)}| dt \geq C \|(u(0), \partial_t u(0))\|_{\mathcal{H} \times L^2}^2 \quad (7)$$

for some $C = C(\varepsilon, T) > 0$ and for any solution u of (2) which satisfies $\int_{M_H} u(0) d\mu = 0$.

1.6 Comments on the existing literature

The recent article [BS19] deals with the controllability of the Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$, where $u \in L^2((0, T) \times M_G)$, $M_G = (-1, 1)_x \times \mathbb{T}_y$ and $\Delta_G = \partial_x^2 + x^2 \partial_y^2$ is the Grushin Laplacian. Given a control set of the form $\omega = (-1, 1)_x \times \omega_y$, where ω_y is an open subset of \mathbb{T} , the authors prove the existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of a horizontal strip contained in $M_G \setminus \omega$. The intuition is that there are solutions of the Grushin Schrödinger equation which travel along the degenerate line $x = 0$ at a finite speed: in some sense, along this line, the Schrödinger equation behaves like a classical (half)-wave equation. What we want here is to explain in a few words why there is a minimal time of observability for the Schrödinger equation, while the wave equation is never observable in finite time as shown by Theorem 1.

The plane $\mathbb{R}_{x,y}^2$ endowed with the vector fields ∂_x and $x\partial_y$ also admits geodesics similar to the 1-parameter family q_ε , namely, for $\varepsilon > 0$,

$$\begin{aligned} x(t) &= \varepsilon \sin(t/\varepsilon) \\ y(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4) \end{aligned}$$

These geodesics, denoted by γ_ε , also “spiral” around the line $x = 0$ more and more quickly as $\varepsilon \rightarrow 0$, and so we might expect to construct solutions of the Grushin Schrödinger equation with energy concentrated along γ_ε , which would contradict observability when $\varepsilon \rightarrow 0$ as above for the Heisenberg wave equation.

However, we can convince ourselves that it is not possible to construct such solutions: in some sense, the dispersion phenomena of the Schrödinger equation exactly compensate the lengthening of the geodesics γ_ε as $\varepsilon \rightarrow 0$ and explain that even these Gaussian beams may be observed in ω from a certain minimal time $\mathcal{L}(\omega) > 0$ which is uniform in ε .

To put this argument into a more formal form, we consider the solutions of the bichar-

acteristic equations for the Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$ given by

$$\begin{aligned} x(t) &= \varepsilon \sin(\xi_y t) \\ y(t) &= \varepsilon^2 \xi_y \left(\frac{t}{2} - \frac{\sin(2\xi_y t)}{4\xi_y} \right) \\ \xi_x(t) &= \varepsilon \xi_y \cos(\xi_y t) \\ \xi_y(t) &= \xi_y. \end{aligned}$$

It follows from the hypoellipticity of Δ_G (see [BS19, Section 3] for a proof) that

$$|\xi_y|^{1/2} \lesssim \sqrt{-\Delta_G} = (|\xi_x|^2 + x^2 |\xi_y|^2)^{1/2} = \varepsilon |\xi_y|.$$

Therefore $\varepsilon^2 |\xi_y| \gtrsim 1$, and hence $|y(t)| \gtrsim t$, independently from ε and ξ_y . This heuristic gives the intuition that a minimal time $\mathcal{L}(\omega)$ is required to detect all solutions of the Grushin Schrödinger equation from ω , but that for $T_0 > \mathcal{L}(\omega)$, no solution is localized enough to stay in $M \setminus \omega$ during the time interval $(0, T_0)$. Roughly speaking, the frequencies of order ξ_y travel at speed $\sim \xi_y$, which is typical for a dispersion phenomenon. This picture is very different from the one for the wave equation (which we consider in this paper) for which no dispersion occurs.

Let us add some general bibliographical comments. Control of subelliptic PDEs has attracted much attention in the last decade. Most results in the literature deal with subelliptic parabolic equations, either the Grushin heat equation ([DK20], [Koe17]) or the heat equation in the Heisenberg group ([BC17], see also references therein). The paper [BS19] is the first to deal with a subelliptic Schrödinger equation and the present work is the first to handle exact controllability of subelliptic wave equations.

Recall that the exact controllability of the elliptic wave equation is known to be almost equivalent to the so-called Geometric Control Condition (GCC) (see [BLR92]): this condition says that any geodesic enters the control set ω within time T . In some sense, our main result is that GCC is not verified in the subelliptic setting, as soon as $M \setminus \omega$ has non-empty interior.

A slightly different problem is the *approximate* controllability of hypoelliptic PDEs, which has been studied in [LL20] for hypoelliptic wave and heat equations. Approximate controllability is weaker than exact controllability, and it amounts to proving “quantitative” unique continuation results for hypoelliptic operators. For the hypoelliptic wave equation, it is proved in [LL20] that for $T > 2 \sup_{x \in M} (\text{dist}(x, \omega))$ (here, dist is the sub-Riemannian distance), the observation of the solution on $(0, T) \times \omega$ determines the initial data, and therefore the whole solution.

1.7 Organization of the paper

In Section 2, we construct exact solutions of the subelliptic wave equation (2) concentrating on any given normal sub-Riemannian geodesic. First, in Section 2.1, we show that, given any normal sub-Riemannian geodesic $t \mapsto x(t)$ of (M, \mathcal{D}, g) (i.e., a projection of a null-bicharacteristic of the associated Hamiltonian system) which does not hit ∂M in the time interval $(0, T)$, it is possible to construct a sequence $(v_k)_{k \in \mathbb{N}}$ of *approximate solutions* of (2) whose energy concentrates along $t \mapsto x(t)$ during the time interval $(0, T)$ as $k \rightarrow +\infty$. By “approximate”, we mean here that $\partial_{tt}^2 v_k - \Delta v_k$ is small, but not necessarily exactly equal to 0. In Section 2.1, we provide a first proof for this construction using the classical propagation of complex Lagrangian spaces. An other proof using a Gaussian beam approach is provided in Appendix B. Then, in Section 2.2, using this sequence $(v_k)_{k \in \mathbb{N}}$, we explain how to construct a sequence $(u_k)_{k \in \mathbb{N}}$ of *exact* solutions of $(\partial_{tt}^2 - \Delta)u = 0$ in M with the same concentration property along the geodesic $t \mapsto x(t)$.

In Section 3, we prove the existence of geodesics which spiral in M , spending an arbitrarily large time in $M \setminus \omega$. These geodesics generalize the example described in Section 1.3 for the Heisenberg manifold with boundary. The proof proceeds in two steps, first proving the

result in the so-called “nilpotent case” (Section 3.2), and then extending it to the general case (Section 3.3).

In Section 4.1, we use the results of Section 2 and Section 3 to conclude the proof of Theorem 1 and to prove Theorem 2. In Section 4.2, we deduce Corollary 1 by a duality argument. Finally, in Section 4.3, we prove Theorem 3.

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2 Gaussian beams along normal sub-Riemannian geodesics

2.1 Construction of sequences of approximate solutions

We consider a solution $(x(t), \xi(t))_{t \in [0, T]}$ of (6) on M . We shall describe the construction of solutions of

$$\partial_{tt}^2 u - \Delta u = 0 \quad (8)$$

on $[0, T] \times M$ with energy

$$E(u(t, \cdot)) := \frac{1}{2} \int_M (|\partial_t u(t, x)|^2 + |\nabla^{sR} u(t, x)|^2) d\mu(x)$$

concentrated along $x(t)$ for $t \in [0, T]$. The following proposition, which is inspired by [Ral82] and [MZ02], shows that it is possible, at least for approximate solutions of (8).

Proposition 8. *Fix $T > 0$ and let $(x(t), \xi(t))_{t \in [0, T]}$ be a solution of (6) which does not hit the boundary ∂M in the time-interval $(0, T)$. Then there exist $a_0, \psi \in C^2((0, T) \times M)$ such that, setting, for $k \in \mathbb{N}$,*

$$v_k(t, x) = k^{\frac{n}{4}-1} a_0(t, x) e^{ik\psi(t, x)}$$

the following properties hold:

- v_k is an approximate solution of (8), meaning that

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1((0, T); L^2(M))} \leq C k^{-\frac{1}{2}}. \quad (9)$$

- The energy of v_k is bounded below with respect to k and $t \in [0, T]$:

$$\exists A > 0, \forall t \in [0, T], \quad \liminf_{k \rightarrow +\infty} E(v_k(t, \cdot)) \geq A. \quad (10)$$

- The energy of v_k is small off $x(t)$: for any $t \in [0, T]$, we fix V_t an open subset of M for the initial topology of M , containing $x(t)$, so that the mapping $t \mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then

$$\sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t v_k(t, x)|^2 + |\nabla^{sR} v_k(t, x)|^2) d\mu(x) \xrightarrow{k \rightarrow +\infty} 0. \quad (11)$$

Remark 9. *The construction of approximate solutions such as the ones provided by Proposition 8 is usually done for strictly hyperbolic operators, that is operators with a principal symbol p_m of order m such that the polynomial $f(s) = p_m(t, q, s, \xi)$ has m distinct real roots when $\xi \neq 0$ (see for example [Ral82]). The operator $\partial_{tt}^2 - \Delta$ is not strictly hyperbolic because g^* is degenerate, but our proof shows that the same construction may be adapted without difficulty to this operator along normal bicharacteristics. It was already noted by [Ral82] that the construction of Gaussian beams could be done for more general operators than strictly hyperbolic ones, and that the differences between the strictly hyperbolic case and more general cases arise while dealing with propagation of singularities.*

Hereafter we provide two proofs of Proposition 8. The first proof is short and is actually quite straightforward for readers acquainted with the theory of propagation of complex Lagrangian spaces, once one has noticed that the solutions of (6) which we consider live in the *elliptic part* of the principal symbol of $\partial_{tt}^2 - \Delta$. For the sake of completeness, and because this also has its own interest, we provide in Appendix B a second proof, more elementary, using the concept of Gaussian beams in the subelliptic context.

First proof of Proposition 8. The usual construction of Gaussian beams, or more generally of a WKB approximation, is related to the transport of complex Lagrangian spaces along bicharacteristics, as reported for example in [Hör07, Chapter 24.2] and [Ivr19, Chapter 1.2].

The usual way to solve (at least approximately) evolution equations of the form

$$Pu = 0 \tag{12}$$

where P is a hyperbolic second order differential operator with real principal symbol p_2 and C^∞ coefficients is to search for oscillatory solutions $v_k(x) = k^{\frac{n}{4}-1}a_0(x)e^{ik\psi(x)}$. In this expression as in the whole proof, we suppress the time variable t . Thus we use $x = (x_0, x_1, \dots, x_n)$ where $x_0 = t$ in our earlier notations. We also set $x' = (x_1, \dots, x_n)$ and without loss of generality, we only consider bicharacteristics starting from 0 at time 0.

Plugging this Ansatz into (12), the leading term in Pv_k is

$$p_2(x, \nabla\psi(x)) = 0 \tag{13}$$

which we solve for initial conditions $\psi(0, x') = \psi_0(x')$, $\nabla\psi_0(0) = \xi(0)$ and $\psi_0(0) = 0$. We use the decomposition of P into two half-waves (see [Hör07, Lemma 23.2.8])

$$P(x, D) = (D_0 - \Gamma_+(x, D'))(D_0 - \Gamma_-(x, D')) + \omega(x, D').$$

Here, $\omega \in S^{-\infty}$, the principal symbol p_2 is written in coordinates as $p_2(x_0, x', \xi_0, \xi') = \xi_0^2 - r(x, \xi')$, and Γ_\pm has principal symbol $\gamma_\pm = \pm r^{1/2}$ in a conic neighborhood of $(0, \xi(0))$. This decomposition is justified because p_2 is elliptic in the direction $\xi(0)$.

Considering separately each half-wave, for example the one with the sign $+$, we are led to solve $\partial_t\psi - \gamma_+(x, \nabla\psi) = 0$, which is a Hamilton-Jacobi (or eikonal) equation. Its solution ψ_t is given by the transport of the values of ψ_0 by Φ_t , where $(\Phi_t)_{t \geq 0}$ is a family of symplectomorphisms on the phase space T^*M , and we have $\Phi_t = e^{t\vec{H}\gamma_+}$. The trouble is that the solution to this eikonal equation is only local in time, and blow-up (through caustics for example) may happen when $x' \mapsto \pi(\Phi_t(x', \nabla\psi_0(x')))$ ceases to be a diffeomorphism (conjugate point), where $\pi : T^*M \rightarrow M$ is the canonical projection. In the language of Lagrangian spaces, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\} \subset T^*M$ is a Lagrangian subspace and, since Φ_t is a symplectomorphism, $\Lambda_t = \Phi_t(\Lambda_0)$ is Lagrangian as well. If $\pi|_{\Lambda_t}$ is a local diffeomorphism, one can locally describe Λ_t by $\Lambda_t = \{(x', \nabla\psi_t(x'))\} \subset T^*M$ for some function ψ_t , but blow-up happens when $\text{rank}(d\pi|_{\Lambda_t}) < n$ (classical conjugate point theory), and such a ψ_t may not exist.

However, if the phase ψ is complex and satisfies the condition $\text{Im}(D^2\psi) > 0$, where $D^2\psi$ denotes the Hessian, no blow-up happens as we now explain. Indeed, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ then lives in the complexification of the tangent space T^*M , which may be thought of as \mathbb{C}^{2n} . Because of the condition $\text{Im}(D^2\psi_0) > 0$, Λ_0 is called a “strictly positive Lagrangian space” (see [Hör07, Definition 21.5.5]), meaning that $i\sigma_{\mathbb{C}}(v, v) > 0$ for v in the tangent space to Λ_0 ($\sigma_{\mathbb{C}}$ is defined just below). As in the real case, Λ_0 is transported by a “real dynamics”, which we wrote in (36) with coordinates (y, η) on \mathbb{R}^{2n} or \mathbb{C}^{2n} . The fact that the symplectic forms σ and $\sigma_{\mathbb{C}}$, defined by $\sigma = \sum dy_j \wedge d\eta_j$ and $\sigma_{\mathbb{C}} = \sum dy_j \wedge d\overline{\eta_j}$, are invariant under the dynamics (36) implies that $\sigma = 0$ on the tangent space to Λ_t , and that $i\sigma_{\mathbb{C}}(v, v) > 0$ for v tangent to Λ_t . It precisely means that Λ_t is also a strictly positive Lagrangian space. Then, by [HS13, Proposition 1.2.5] (or [Hör07, Chapter 21]), we know that there exists ψ_t with $\text{Im}(D^2\psi_t) > 0$ such that $\Lambda_t = \{(x', \nabla\psi_t(x'))\}$. This choice ensures that the terms in the expansion of Pv_k with a power $k^{\frac{n}{4}+1}$ vanish along the bicharacteristic. To cancel the terms with a power $k^{\frac{n}{4}}$,

it is sufficient to solve linear transport ODEs for a_0 along the bicharacteristic (see [Hör07, Equation (24.2.9)]).

Since we solved (12) at a sufficiently large order along the bicharacteristic starting from $(0, \xi(0))$, the bound (9) then follows from Lemma 20 in Appendix B. The bounds (10) and (11) follow from the facts that $v_k(x) = k^{\frac{n}{4}-1}a_0(x)e^{ik\psi(x)}$ and $\text{Im}(D^2\psi_t) > 0$. \square

Remark 10. *An interesting question would be to understand the delocalization properties of the Gaussian beams constructed along normal sub-Riemannian geodesics in Proposition 8. Compared with the usual Riemannian case done for example in [Ral82], there is a new phenomenon in the sub-Riemannian case since the geodesic $x(t)$ (or, more precisely, the associated momentum ξ) may approach the characteristic manifold $\Sigma = \{g^* = 0\}$ which is the set of directions in which Δ is not elliptic. In finite time T as in our case, the geodesic remains far from Σ , but it may happen as $T \rightarrow +\infty$ that it goes closer and closer to Σ . The question is then to understand the link between the delocalization properties of the Gaussian beams constructed along such a geodesic, and notably the interplay between the time T and the semi-classical parameter $1/k$.*

2.2 Construction of sequences of exact solutions in M

In this section, using the approximate solutions of Proposition 2.1, we construct *exact* solutions of (8) whose energy concentrates along a given normal geodesic of M which does not meet the boundary ∂M during the time interval $[0, T]$.

Proposition 11. *Let $(x(t), \xi(t))_{t \in [0, T]}$ be a solution of (6) in M which does not meet ∂M and $\theta \in C_c^\infty([0, T] \times M)$ with $\theta(t, \cdot) \equiv 1$ in a neighborhood of $x(t)$ and such that the support of $\theta(t, \cdot)$ stays at positive distance of ∂M .*

Suppose $(v_k)_{k \in \mathbb{N}}$ is constructed along $x(t)$ as in Proposition 8 and u_k is the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2 - \Delta)u_k = 0 & \text{in } (0, T) \times M, \\ u_k = 0 & \text{in } (0, T) \times \partial M, \\ u_k|_{t=0} = (\theta v_k)|_{t=0}, \quad \partial_t u_k|_{t=0} = [\partial_t(\theta v_k)]|_{t=0}. \end{cases}$$

Then:

- *The energy of u_k is bounded below with respect to k and $t \in [0, T]$:*

$$\exists A > 0, \forall t \in [0, T], \quad \liminf_{k \rightarrow +\infty} E(u_k(t, \cdot)) \geq A. \quad (14)$$

- *The energy of u_k is small off $x(t)$: for any $t \in [0, T]$, we fix V_t an open subset of M for the initial topology of M , containing $x(t)$, so that the mapping $t \mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then*

$$\sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t u_k(t, x)|^2 + |\nabla^{sR} u_k(t, x)|^2) d\mu(x) \xrightarrow{k \rightarrow +\infty} 0. \quad (15)$$

Proof of Proposition 11. Set $h_k = (\partial_{tt}^2 - \Delta)(\theta v_k)$. We consider w_k the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2 - \Delta)w_k = h_k & \text{in } (0, T) \times M, \\ w_k = 0 & \text{in } (0, T) \times \partial M, \\ (w_k|_{t=0}, \partial_t w_k|_{t=0}) = (0, 0). \end{cases} \quad (16)$$

Differentiating $E(w_k(t, \cdot))$ and using Gronwall's lemma, we get the energy inequality

$$\sup_{t \in [0, T]} E(w_k(t, \cdot)) \leq C (E(w_k(0, \cdot)) + \|h_k\|_{L^1(0, T; L^2(M))}).$$

Therefore, using (9), we get $\sup_{t \in [0, T]} E(w_k(t, \cdot)) \leq Ck^{-1}$. Since $u_k = \theta v_k - w_k$, we obtain that

$$\lim_{k \rightarrow +\infty} E(u_k(t, \cdot)) = \lim_{k \rightarrow +\infty} E((\theta v_k)(t, \cdot)) = \lim_{k \rightarrow +\infty} E(v_k(t, \cdot))$$

for every $t \in [0, T]$ where the last equality comes from the fact that θ and its derivatives are bounded and $\|v_k\|_{L^2} \leq Ck^{-1}$ when $k \rightarrow +\infty$. Using (10), we conclude that (14) holds.

To prove (15), we observe similarly that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{M \setminus V_t} (|\partial_t u_k(t, x)|^2 + |\nabla^{\text{sR}} u_k(t, x)|^2) d\mu(x) \\ & \leq C \sup_{t \in [0, T]} \left(\int_{M \setminus V_t} (|\partial_t v_k(t, x)|^2 + |\nabla^{\text{sR}} v_k(t, x)|^2) d\mu(x) \right) + Ck^{-\frac{1}{2}} \\ & \rightarrow 0 \end{aligned}$$

as $k \rightarrow +\infty$, according to (11). It concludes the proof of Proposition 11. \square

3 Existence of spiraling geodesics

The goal of this section is to prove the following proposition, which is the second building block of the proof of Theorem 1, after the construction of localized solutions of the subelliptic wave equation (2) done in Section 2.

Proposition 12. *For any $T_0 > 0$, any $x \in M$ and any open neighborhood V of x in M (with the initial topology on M), there exists a geodesic $t \mapsto x(t)$ of (M, \mathcal{D}, g) such that $x(t) \in V$ for any $t \in [0, T_0]$.*

In Section 3.1, we define the so-called nilpotent approximation $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ of (M, \mathcal{D}, g) at a point $q \in M$, a sub-Riemannian structure which is a first-order approximation of (M, \mathcal{D}, g) at point $q \in M$ whose associated Lie algebra $\text{Lie}(\widehat{\mathcal{D}}^q)$ is nilpotent. The proof of Proposition 12 then splits into two steps, first proving the result for the nilpotent approximation (in Section 3.2) and then showing that the geodesics of (M, \mathcal{D}, g) are well approached by the geodesics of $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ (in Section 3.3) which is sufficient to conclude.

3.1 Nilpotent approximation

Given a sub-Riemannian structure (M, \mathcal{D}, g) and a point $q \in M$, its tangent space at q in the Gromov-Hausdorff sense is the sub-Riemannian structure $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$, also called nilpotent approximation. It is defined intrinsically (meaning that it does not depend on a choice of coordinates or of local frame) as an equivalence class under the action of sub-Riemannian isometries (see [Bel96], [Jea14]).

Sub-Riemannian flag. We define the sub-Riemannian flag of (M, \mathcal{D}, g) as follows: we set $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and, for any $j \geq 1$, $\mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j]$. For any point $q \in M$, it defines a flag

$$\{0\} = \mathcal{D}_q^0 \subset \mathcal{D}_q^1 \subset \dots \subset \mathcal{D}_q^{r-1} \subsetneq \mathcal{D}_q^r = T_q M.$$

The integer $r(q)$ is called the non-holonomic order of \mathcal{D} at q , and it is equal to 2 in the Heisenberg manifold for example. For $0 \leq i \leq r(q)$, we set $n_i(q) = \dim \mathcal{D}_q^i$, and the sequence $(n_i(q))_{0 \leq i \leq r(q)}$ is called the growth vector at point q . We set $\mathcal{Q}(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q))$. Finally, we define the non-decreasing sequence of weights $w_i(q)$ for $1 \leq i \leq n$ as follows. Given any $1 \leq i \leq n$, there exists a unique $1 \leq j \leq n$ such that $n_{j-1}(q) + 1 \leq i \leq n_j(q)$. We set $w_i(q) = j$.

Regular and singular points. We say that $q \in M$ is regular if the growth vector $(n_i(q'))_{0 \leq i \leq r(q')}$ at q' is constant for q' in a neighborhood of q . Otherwise, q is said to be singular. If any point $q \in M$ is regular, we say that the structure is equiregular.

Non-holonomic orders. The non-holonomic order of a smooth germ of function is given by the formula

$$\text{ord}_q(f) = \min\{s \in \mathbb{N} : \exists i_1, \dots, i_s \in \{1, \dots, k\} \text{ such that } (X_{i_1} \dots X_{i_s} f)(q) \neq 0\}$$

where we adopt the convention that $\min \emptyset = +\infty$. The non-holonomic order of a smooth germ of vector field X at q , denoted by $\text{ord}_q(X)$, is the real number defined by

$$\text{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \text{ord}_q(Xf) \geq \sigma + \text{ord}_q(f), \quad \forall f \in C^\infty(q)\}.$$

For example, every X which has the property that $X(q') \in \mathcal{D}_q^i$ for any q' in a neighborhood of q is of non-holonomic order $\geq -i$.

Privileged coordinates. Locally around $q \in M$, it is possible to define a set of so-called “privileged coordinates” of M (see [Bel96]).

A family (Z_1, \dots, Z_n) of n vector fields is said to be adapted to the sub-Riemannian flag of (M, \mathcal{D}, g) at q if it is a frame of $T_q M$ at q and if $Z_i(q) \in \mathcal{D}_q^{w_i(q)}$ for any $i \in \{1, \dots, n\}$.

A system of privileged coordinates at q is a system of local coordinates (x_1, \dots, x_n) such that $\text{ord}_q(x_i) = w_i$ for $1 \leq i \leq n$. In particular, for privileged coordinates, we have $\partial_{x_i} \in \mathcal{D}_q^{w_i(q)} \setminus \mathcal{D}_q^{w_i(q)-1}$ at q , meaning that privileged coordinates are adapted to the flag.

Exponential coordinates of the second kind. Choose an adapted frame (Z_1, \dots, Z_n) at q . It is proved in [Jea14, Appendix B] that the inverse of the local diffeomorphism

$$(x_1, \dots, x_n) \mapsto \exp(x_1 Z_1) \circ \dots \circ \exp(x_n Z_n)(q)$$

defines privileged coordinates at q , called exponential coordinates of the second kind.

Dilations. We consider a chart of privileged at q given by a smooth mapping $\psi_q : U \rightarrow \mathbb{R}^n$, where U is a neighborhood of q in M , with $\psi_q(q) = 0$. For every $\varepsilon \in \mathbb{R} \setminus \{0\}$, we consider the dilation $\delta_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\delta_\varepsilon(x) = (\varepsilon^{w_i(q)} x_1, \dots, \varepsilon^{w_n(q)} x_n)$$

for every $x = (x_1, \dots, x_n)$. A dilation δ_ε acts also on functions and vector fields on \mathbb{R}^n by pull-back: $\delta_\varepsilon^* f = f \circ \delta_\varepsilon$ and $\delta_\varepsilon^* X$ is the vector field such that $(\delta_\varepsilon^* X)(\delta_\varepsilon^* f) = \delta_\varepsilon^*(Xf)$.

Nilpotent approximation. Fix a system of privileged coordinates (x_1, \dots, x_n) at q . Given a sequence of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we define the weighted degree of $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ to be $w(\alpha) = w_1(q)\alpha_1 + \dots + w_n(q)\alpha_n$. Every vector field X_i is of non-holonomic order ≥ -1 , hence it has a Taylor expansion

$$X_i(x) \sim \sum_{\alpha, j} a_{\alpha, j} x^\alpha \partial_{x_j}$$

where $w(\alpha) \geq w_j(q) - 1$ if $a_{\alpha, j} \neq 0$. Therefore, we may write X_i as a formal series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s . We set $\widehat{X}_i^q = X_i^{(-1)}$ for $1 \leq i \leq m$. Each \widehat{X}_i^q may be seen as a vector field on \mathbb{R}^n thanks to the coordinates (x_1, \dots, x_n) . Moreover,

$$\widehat{X}_i^q = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_\varepsilon^*(\psi_q)_* X_i$$

in C^∞ topology: all derivatives uniformly converge on compact subsets. For $\varepsilon > 0$ small enough we have

$$X_i^\varepsilon := \varepsilon \delta_\varepsilon^*(\psi_q)_* X_i = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology. Note also that \widehat{X}_i^q is homogeneous of degree -1 with respect to dilations, i.e., $\lambda \delta_\lambda^* \widehat{X}_i^q = \widehat{X}_i^q$ for any $\lambda \neq 0$. An important property is that $(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ generates a nilpotent Lie algebra of step $r = w_n$ (see [Jea14, Proposition 2.3]).

The nilpotent approximation $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ of (M, \mathcal{D}, g) at q is then defined as $\widehat{M} \simeq \mathbb{R}^n$ endowed with the sub-Riemannian distribution $\widehat{\mathcal{D}}^q = \text{Span}(\widehat{X}_1^q, \dots, \widehat{X}_k^q)$ and the sub-Riemannian metric \widehat{g}^q on $\widehat{\mathcal{D}}^q$ verifying $\widehat{g}^q(\widehat{X}_i^q, \widehat{X}_j^q) = g_q(X_i, X_j)$. Note that the nilpotent sub-Riemannian structure $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$ is homogeneous with respect to the dilations δ_ε .

3.2 The nilpotent case

We first prove Proposition 12 when (M, \mathcal{D}, g) is nilpotent, i.e., $\text{Lie}(\mathcal{D})$ is a nilpotent Lie algebra. The nilpotent Hamiltonian is denoted by

$$\widehat{H} = \sum_{i=1}^m h_{\widehat{X}_i}^2. \quad (17)$$

As explained in Section 1.3, the proof consists in identifying a ‘‘Heisenberg sub-structure’’ in (M, \mathcal{D}, g) , for which we know that there exist spiraling geodesics.

Lemma 13. *Let us assume that (M, \mathcal{D}, g) is nilpotent. Then, for any $T > 0$, any $x \in M$ and any open neighborhood V of x in M , there exists $\lambda_0 \in T^*M$ such that $\pi(e^{t\widehat{H}}\lambda_0) \in V$ for any $t \in [0, T_0]$.*

Proof of Lemma 13. For this proof, since all considered sub-Riemannian structures are nilpotent, we simplify notations by replacing the vector fields \widehat{X}_i by the notation X_i .

The main idea of the proof is to use a desingularization $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$ of (M, \mathcal{D}, g) around x whose Lie algebra is *free up to some step r* (see [Jea14, Definition 2.13]) and nilpotent. In this desingularized sub-Riemannian structure, we isolate a ‘‘Heisenberg sub-structure’’, and projecting its spiraling geodesics onto (M, \mathcal{D}, g) , we get the result. By Heisenberg sub-structure of $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$, we mean two vector fields $\widetilde{X}_1, \widetilde{X}_2$ which may be completed into a \widetilde{g} -orthonormal family, such that $[\widetilde{X}_1, \widetilde{X}_2] \notin \text{Span}(\widetilde{X}_1, \widetilde{X}_2)$ and the geodesics in \widetilde{M} of the Hamiltonian $h_{\widetilde{X}_1}^2 + h_{\widetilde{X}_2}^2$ are also geodesics of $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$.

Let us observe that, given a g -orthonormal frame (X_1, \dots, X_n) of (M, \mathcal{D}, g) , we cannot isolate a Heisenberg sub-structure by simply taking the quotient of $\text{Lie}(X_1, \dots, X_m)$ by its ideal $I(X_3, \dots, X_m)$, because $I(X_3, \dots, X_m)$ may be very large (even equal to $\text{Lie}(X_1, \dots, X_m)$) if there are relations such as $[X_1, X_3] = X_1$. This is why we introduce the desingularization $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$: its Lie algebra is free up to a step r , and therefore such relations cannot hold.

Let us first note that there exist $q_0 \in V$ and $X_1, X_2 \in \mathcal{D}$ such that $[X_1, X_2](q_0) \notin \mathcal{D}_{q_0}$. This is because otherwise $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ in V , and therefore the rank condition (1) cannot hold in V . We denote by r the non-holonomic order of \mathcal{D} at q_0 .

We now introduce some notations related to free Lie algebras. Let $\mathcal{L} = \mathcal{L}(1, \dots, m)$ be the free Lie algebra generated by $\{1, \dots, m\}$. We denote by \mathcal{L}^s the subspace generated by elements of \mathcal{L} of length $\leq s$, and by \tilde{n}_s the dimension of \mathcal{L}^s .

Lemma 14 ([Bel96], [Jea14](Lemma 2.5, Theorem 2.9 and Remark 2.9)). *Let $\widetilde{M} = M \times \mathbb{R}^{\tilde{n}_r - n}$. Then there exist a neighborhood $\widetilde{U} \subset \widetilde{M}$ of $(q_0, 0)$, a neighborhood $U \subset M$ of q_0 with $U \times \{0\} \subset \widetilde{U}$, local coordinates (x, y) on \widetilde{U} , and smooth vector fields $\widetilde{X}_1, \dots, \widetilde{X}_m$ on \widetilde{U} , such that*

- *Every \widetilde{q} in \widetilde{U} is regular;*
- *For any $1 \leq i \leq m$, we have $d\pi(\widetilde{X}_i) = X_i$, where $\pi : \widetilde{M} \rightarrow M$ is the canonical projection;*
- *$\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)$ is a free nilpotent Lie algebra of step r ;*
- *The image by the projection π of any geodesic (traveled at speed 1) of the non-holonomic system in \widetilde{U} defined by $\widetilde{X}_1, \dots, \widetilde{X}_m$ is a geodesic of (M, \mathcal{D}, g) (also traveled at speed 1).*

We set $\widetilde{\mathcal{D}} = \text{Span}(\widetilde{X}_1, \dots, \widetilde{X}_m)$ and we denote by \widetilde{g} the Riemannian metric on $\widetilde{\mathcal{D}}$ such that $(\widetilde{X}_1, \dots, \widetilde{X}_m)$ is \widetilde{g} -orthonormal. We denote by \mathcal{I} the ideal of $\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)$ generated by $\widetilde{X}_3, \dots, \widetilde{X}_m$ and all brackets of $\widetilde{X}_1, \dots, \widetilde{X}_m$ of length ≥ 3 . Then $\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)/\mathcal{I}$ has dimension 3.

We denote by \widetilde{X}_1^H and \widetilde{X}_2^H the images of \widetilde{X}_1 and \widetilde{X}_2 through this quotient. The Lie group G generated by $\exp(t\widetilde{X}_1^H)$ and $\exp(t\widetilde{X}_2^H)$ is nilpotent and free up to step 2, and therefore there are local coordinates (x_1, x_2, x_3) on G such that $\widetilde{X}_1^H = \partial_{x_1}$ and $\widetilde{X}_2^H = \partial_{x_2} + x_1\partial_{x_3}$. Then, the geodesics exhibited in Section 1.3 are spiraling geodesics in G . Since $\text{Lie}(\widetilde{X}_1, \dots, \widetilde{X}_m)$ is free, it gives spiraling geodesics in $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{g})$ (traveled at speed 1, i.e., $\widetilde{g}^* = 1$). Their

images by the canonical projection $\pi : \tilde{M} \rightarrow M$ are spiraling geodesics in (M, \mathcal{D}, g) (with $g^* = 1$). \square

Remark 15. *The geodesics constructed in Lemma 13 lose their optimality quickly, in the sense that their first conjugate point and their cut-point are close to q_0 . In local coordinates, they are closer to q_0 as the norm of λ_0 is larger.*

3.3 The general case

In this section, we conclude the proof of Proposition 12. We do not assume anymore that (M, \mathcal{D}, g) is nilpotent. We show that on finite time intervals, the geodesics of (M, \mathcal{D}, g) remain close to those of $(\widehat{M}^q, \widehat{\mathcal{D}}, \widehat{g})$. We set $Y_i = (\psi_q)_* X_i$ and $X_i^\varepsilon = \varepsilon \delta_\varepsilon^* Y_i$ which is a vector field on \mathbb{R}^n . Recall that

$$X_i^\varepsilon = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology. Therefore, using the homogeneity of \widehat{X}_i^q , we get

$$Y_i = \frac{1}{\varepsilon} (\delta_\varepsilon)_* X_i^\varepsilon = \frac{1}{\varepsilon} (\delta_\varepsilon)_* (\widehat{X}_i^q + \varepsilon R_i^\varepsilon) = \widehat{X}_i^q + (\delta_\varepsilon)_* R_i^\varepsilon. \quad (18)$$

Recall that the Hamiltonian \widehat{H} associated to the vector fields \widehat{X}_i^q is given by (17). Similarly, we set

$$H = \sum_{i=1}^m h_{Y_i}^2$$

where we used the notation h_Z for vector fields Z which was introduced in Section 1.4. We note that (18) gives

$$h_{Y_i} = h_{\widehat{X}_i^q} + h_{(\delta_\varepsilon)_* R_i^\varepsilon}.$$

Hence

$$\vec{H} = 2 \sum_{i=1}^m h_{Y_i} \vec{h}_{Y_i} = \vec{\widehat{H}} + \vec{\Theta},$$

where $\vec{\Theta}$ is a smooth vector field on $T^*\mathbb{R}^n$ such that $\|\vec{\Theta}(x, \xi)\| \rightarrow 0$ when $\|x\| \rightarrow 0$ (uniformly for $\|\xi\| \leq C$). This last point comes from the fact that $\vec{\Theta}$ has higher non-holonomic order than $\vec{\widehat{H}}$. In other words, it follows from the smooth dependence of R_i^ε on ε for the C^∞ topology (uniform convergence of all derivatives on compact subsets of \mathbb{R}^n). Together with Lemma (13) and using that $X_i = \psi_q^* Y_i$, it concludes the proof of Proposition 12.

4 Proofs

4.1 Proofs of Theorem 1 and Theorem 2

In this section, we conclude the proof of Theorem 1 and we prove Theorem 2.

Proof of Theorem 1. Fix a point $x \in M$ and an open neighborhood V of x in M such that $V \subset M \setminus \omega$. Such x and V exist since $M \setminus \omega$ has non-empty interior. Fix V' an open neighborhood of x in M such that $\overline{V'} \subset V$, and fix also $T_0 > 0$.

As already explained in Section 1.3, to conclude the proof of Theorem 1, we use Proposition 11 applied to the particular geodesics constructed in Proposition 12.

By Proposition 12, we know that there exists a normal geodesic $t \mapsto x(t)$ such that $x(t) \in V'$ for any $t \in (0, T_0)$. We denote by $(u_k)_{k \in \mathbb{N}}$ a sequence of solutions of (8) as in Proposition 11 whose energy at time t concentrates on $x(t)$ for $t \in (0, T_0)$. Because of (14), we know that

$$\|(u_k(0), \partial_t u_k(0))\|_{\mathcal{H} \times L^2} \geq c > 0$$

uniformly in k .

Therefore, in order to establish Theorem 1, it is sufficient to show that

$$\int_0^{T_0} \int_{\omega} |\partial_t u_k(t, x)|^2 d\mu(x) dt \xrightarrow{k \rightarrow +\infty} 0. \quad (19)$$

Since $x(t) \in V'$ for any $t \in (0, T_0)$, we get that for V_t chosen sufficiently small for any $t \in (0, T_0)$, the inclusion $V_t \subset V$ holds. Combining this last remark with (15), we get (19), which concludes the proof of Theorem 1. \square

Proof of Theorem 2. We first observe that the growth vector (defined in Section 3.1) of the nilpotent approximation $(\widehat{M}^x, \widehat{\mathcal{D}}^x, \widehat{g}^x)$ at 0 is equal to the growth vector of (M, \mathcal{D}, g) at x (see [Jea14, Lemma 2.1]). Hence, $(\widehat{M}^x, \widehat{\mathcal{D}}^x, \widehat{g}^x)$ is “strictly sub-Riemannian” at 0, in the sense that $\widehat{\mathcal{D}}_0^x \neq T_0 \widehat{M}^x$. The result then follows from Lemma 13 and the computations of Section 3.3. \square

4.2 Proof of Corollary 1

We endow the topological dual $\mathcal{H}(M)'$ with the norm $\|v\|_{\mathcal{H}(M)'} = \|(-\Delta)^{-1/2}v\|_{L^2(M)}$.

The following proposition is standard (see, e.g., [TW09], [LRLTT17]).

Lemma 16. *Let $T_0 > 0$, and $\omega \subset M$ be a measurable set. Then the following two observability properties are equivalent:*

(P1): *There exists C_{T_0} such that for any $(v_0, v_1) \in \mathcal{H}(M) \times L^2(M)$, the solution $v \in C^0(0, T_0; \mathcal{H}(M)) \cap C^1(0, T_0; L^2(M))$ of (2) satisfies*

$$\int_0^{T_0} \int_{\omega} |\partial_t v(t, q)|^2 d\mu(q) dt \geq C_{T_0} \|(v_0, v_1)\|_{\mathcal{H}(M) \times L^2(M)}. \quad (20)$$

(P2): *There exists C_{T_0} such that for any $(v_0, v_1) \in L^2(M) \times \mathcal{H}(M)'$, the solution $v \in C^0(0, T_0; L^2(M)) \cap C^1(0, T_0; \mathcal{H}(M)')$ of (2) satisfies*

$$\int_0^{T_0} \int_{\omega} |v(t, q)|^2 d\mu(q) dt \geq C_{T_0} \|(v_0, v_1)\|_{L^2 \times \mathcal{H}(M)'}^2. \quad (21)$$

Proof. Let us assume that (P2) holds. Let u be a solution of (2) with initial conditions $(u_0, u_1) \in \mathcal{H}(M) \times L^2(M)$. We set $v = \partial_t u$, which is a solution of (2) with initial data $v|_{t=0} = u_1 \in L^2(M)$ and $\partial_t v|_{t=0} = \Delta u_0 \in \mathcal{H}(M)'$. Since $\|(v_0, v_1)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_1, \Delta u_0)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_0, u_1)\|_{\mathcal{H}(M) \times L^2}$, applying the observability inequality (21) to $v = \partial_t u$, we obtain (20). The proof of the other implication is similar. \square

Finally, using Theorem 1, Lemma 16 and the standard HUM method ([Lio88]), we get Corollary 1.

4.3 Proof of Theorem 3

Let us first define spaces adapted to the fact that M_H has no boundary. We set

$$\mathcal{H}_0 = \left\{ u_0 \in \mathcal{H}, \int_{M_H} u_0 d\mu = 0 \right\}$$

and we denote its topological dual by \mathcal{H}'_0 .

Lemma 17. *The embeddings $\mathcal{H}_0 \hookrightarrow L^2$ and $L^2 \hookrightarrow \mathcal{H}'_0$ are compact.*

Proof. By duality, we only need to prove that $\mathcal{H}_0 \hookrightarrow L^2$ is compact. Using a classical subelliptic estimate (see [Hör67] and [RS76, Theorem 17]), we know that there exists $C > 0$ such that for any $u \in C^\infty(M_H)$,

$$\|u\|_{H^{\frac{1}{2}}(M_H)} \leq C(\|u\|_{L^2(M_H)} + \|u\|_{\mathcal{H}(M_H)})$$

Moreover, by the Poincaré inequality, we also have for any $u \in C^\infty(M_H)$ with $\int_{M_H} u d\mu = 0$ that $\|u\|_{L^2(M_H)} \leq C \|\partial_x u\|_{L^2(M_H)} \leq C \|u\|_{\mathcal{H}(M_H)}$. Therefore, $\mathcal{H}_0 \hookrightarrow H^{\frac{1}{2}}(M_H)$ with continuous embedding. The result then follows from the fact that $H^{\frac{1}{2}}(M_H)$ is compactly embedded in $L^2(M_H)$. \square

We then prove a weak observability inequality.

Lemma 18. *There exists $\kappa > 0$ such that for any $\varepsilon > 0$ and any $T \geq \kappa \varepsilon^{-1}$, there holds*

$$C \|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2 \leq \int_0^T |(Op(a)\partial_t u, \partial_t u)_{L^2(M_H, \mu)}| dt + \|(u_0, u_1)\|_{L^2 \times \mathcal{H}'}^2 \quad (22)$$

for some $C(\varepsilon, T) > 0$ and any solution u of (2) such that $\int_{M_H} u_0 d\mu = 0$.

Proof of Lemma 18. In this proof, we use the notation $P = \partial_{tt}^2 - \Delta_H$. For the sake of a contradiction, suppose that there exists a sequence $(u^k)_{k \in \mathbb{N}}$ of solutions of the wave equation such that $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$ for any $k \in \mathbb{N}$ and

$$\|(u_0^k, u_1^k)\|_{L^2 \times \mathcal{H}'} \rightarrow 0, \quad \int_0^T |(Op(a)\partial_t u^k, \partial_t u^k)_{L^2(M_H, \mu)}| dt \rightarrow 0 \quad (23)$$

as $k \rightarrow \infty$. Following the strategy of [Tar90] and [Gér91], our goal is to associate a defect measure to the sequence $(u^k)_{k \in \mathbb{N}}$. Since the functional spaces involved in our result are unusual, we give the argument in detail.

We consider the space of functions $u \in C^\infty([0, T] \times M_H)$ such that $\int_{M_H} u(0, \cdot) d\mu = 0$, and we denote by \mathcal{H}_T its the completion for the norm $\|\cdot\|_{\mathcal{H}_T}$ induced by the scalar product

$$(u, v)_{\mathcal{H}_T} = \int_0^T \int_{M_H} (\partial_t u \partial_t v + (\nabla^{\text{sR}} u) \cdot (\nabla^{\text{sR}} v)) d\mu(q) dt$$

By conservation of energy, the quantity $\|u^k\|_{\mathcal{H}_T}$ is bounded independently of k . Together with the first convergence in (23), it implies that $u^k \rightharpoonup 0$ for the weak topology of \mathcal{H}_T .

Fix $B \in \Psi_{\text{phg}}^0((0, T) \times M_H)$. We have

$$\begin{aligned} (Bu^k, u^k)_{\mathcal{H}_T} &= \int_0^T \int_{M_H} (\partial_t (Bu^k) \partial_t u^k + (\nabla^{\text{sR}} (Bu^k)) \cdot (\nabla^{\text{sR}} u^k)) d\mu(q) dt \\ &= \int_0^T \int_{M_H} (([\partial_t, B]u^k) \partial_t u^k + ([\nabla^{\text{sR}}, B]u^k) \cdot (\nabla^{\text{sR}} u^k)) d\mu(q) dt \\ &\quad + \int_0^T \int_{M_H} ((B\partial_t u^k) (\partial_t u^k) + (B\nabla^{\text{sR}} u^k) \cdot (\nabla^{\text{sR}} u^k)) d\mu(q) dt \end{aligned} \quad (24)$$

Since $[\partial_t, B] \in \Psi_{\text{phg}}^0((0, T) \times M_H)$, $[\nabla^{\text{sR}}, B] \in \Psi_{\text{phg}}^0((0, T) \times M_H)$ and $u^k \rightarrow 0$ strongly in $L^2((0, T) \times M_H)$, the first one of the two lines in (24) converges to 0 as $k \rightarrow +\infty$. Moreover, the last line is bounded uniformly in k since $B \in \Psi_{\text{phg}}^0((0, T) \times M_H)$. Hence $(Bu^k, u^k)_{\mathcal{H}_T}$ is uniformly bounded. By a standard diagonal extraction argument (see [Gér91] for example), there exists a subsequence, which we still denote by $(u^k)_{k \in \mathbb{N}}$ such that (Bu^k, u^k) converges for any B of principal symbol b in a countable dense subset of $C_c^\infty((0, T) \times M_H)$. Moreover, the limit only depends on the principal symbol b , and not on the full symbol.

Let us now prove that

$$\liminf_{k \rightarrow +\infty} (Bu^k, u^k)_{\mathcal{H}_T} \geq 0 \quad (25)$$

when $b \geq 0$. With a bracket argument as in (24), we see that it is equivalent to proving

$$\liminf_{k \rightarrow +\infty} ((B\partial_t u^k, \partial_t u^k)_{L^2} + (B\nabla^{\text{sR}} u^k, \nabla^{\text{sR}} u^k)_{L^2}) \geq 0. \quad (26)$$

But there exists $B' \in \Psi_{\text{phg}}^0((0, T) \times M_H)$ such that $B' - B \in \Psi_{\text{phg}}^{-1}((0, T) \times M_H)$ and B' is positive (this is the so-called Friedrichs quantization, see for example [Tay74, Chapter VII]). It immediately implies that (26) and (25) hold.

Therefore, setting $p = \sigma_p(P)$ and denoting by $\mathcal{C}(p)$ the characteristic manifold $\mathcal{C}(p) = \{p = 0\}$, there exists a positive Radon measure ν on $S^*(\mathcal{C}(p)) = \mathcal{C}(p)/(0, +\infty)$ such that

$$(\text{Op}(b)u^k, u^k)_{\mathcal{H}_T} \rightarrow \int_{S^*(\mathcal{C}(p))} b d\nu$$

for any $b \in S_{\text{phg}}^0((0, T) \times M_H)$.

Let $C \in \Psi_{\text{phg}}^{-1}((0, T) \times M_H)$ of principal symbol c . We have $\vec{H}_p c = \{p, c\} \in S_{\text{phg}}^0((0, T) \times M_H)$ and, for any $k \in \mathbb{N}$,

$$((CP - PC)u^k, u^k)_{\mathcal{H}_T} = (CPu^k, u^k)_{\mathcal{H}_T} - (Cu^k, Pu^k)_{\mathcal{H}_T} = 0$$

since $Pu^k = 0$. Taking principal symbols, we get $\langle \nu, \vec{H}_p c \rangle = 0$.

Therefore, denoting by $(\phi_t)_{t \in \mathbb{R}}$ the sub-Riemannian normal flow introduced in Section 1.4,

$$0 = \langle \nu, \vec{H}_p c \circ \phi_t \rangle = \langle \nu, \frac{d}{dt} c \circ \phi_t \rangle = \frac{d}{dt} \langle \nu, c \circ \phi_t \rangle$$

and hence

$$\langle \nu, c \rangle = \langle \nu, c \circ \phi_t \rangle. \quad (27)$$

From the second convergence in (23), we can deduce that

$$\nu = 0 \text{ in } S^*(\mathcal{C}(p)) \cap T^*((0, T) \times \text{Supp}(a)). \quad (28)$$

The proof of this fact, which is standard (see for example [BG02, Section 6.2]), is given in Appendix C.

We note that any geodesic of M_H with momentum $\xi \in V_\varepsilon$ enters ω in time at most $\kappa\varepsilon^{-1}$ for some $\kappa > 0$ which does not depend on ε (this is because $\omega = M_H \setminus B$, where B is a small ball). Hence, together with (28), the propagation property (27) implies that $\nu \equiv 0$. It follows that $\|u^k\|_{\mathcal{H}_T} \rightarrow 0$. By conservation of energy, it is a contradiction with the normalization $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$. Hence, (22) holds. \square

Our goal is to remove the $\|(u_0, u_1)\|_{L^2 \times \mathcal{H}'}$ term in (22) in order to get (7). We follow the same scheme of proof as in [BS19, Section 8].

Let us denote by $S(t)$ the semigroup of the wave equation, which means that $S(t)(u_0, u_1)$ is the solution at time t of (2) associated to the initial data $(u_0, u_1) \in \mathcal{H} \times L^2$.

For $T > 0$, define the space of invisible solutions

$$\mathcal{N}_T = \left\{ (u_0, u_1) \in \mathcal{H}_0 \times L^2 : ((\text{Op}(a)\partial_t u, \partial_t u)_{L^2(M, \mu)})_{|[0, T]} = 0 \right\} \quad (29)$$

where u denotes the solution of (2) with initial data (u_0, u_1) .

Lemma 19. *For any $T > \kappa\varepsilon^{-1}$, there holds $\mathcal{N}_T = \{0\}$.*

Proof. Fix $T > \kappa\varepsilon^{-1}$. We will prove that there exists $\kappa\varepsilon^{-1} < T_1 \leq T$ such that $\mathcal{N}_{T_1} = \{0\}$, which will immediately imply that $\mathcal{N}_T = \{0\}$ since $\mathcal{N}_{T'} \subset \mathcal{N}_{T''}$ for $T' \geq T''$.

Let us remark that for any $T' > \kappa\varepsilon^{-1}$, Lemma 18 implies that there exists $C_{T'} > 0$ such that any $(u_0, u_1) \in \mathcal{N}_{T'}$ satisfies

$$C_{T'} \|(u_0, u_1)\|_{\mathcal{H} \times L^2} \leq \|(u_0, u_1)\|_{L^2 \times \mathcal{H}'}$$

Since $\mathcal{H} \times L^2$ is compactly embedded in $L^2 \times \mathcal{H}'$ by Lemma 17, we get that $\dim \mathcal{N}_{T'} < +\infty$. Consider the mapping $A(\delta) := \delta^{-1}(S(\delta) - \text{Id}) : \mathcal{N}_{T'} \rightarrow \mathcal{N}_{T'-\delta}$. Since $\dim \mathcal{N}_{T'}$ is an integer, there exist $\kappa\varepsilon^{-1} < T_1 \leq T$ and $\delta_0 > 0$ such that $\dim \mathcal{N}_{T_1-\delta} = \dim \mathcal{N}_{T_1}$ for any $\delta \in (0, \delta_0)$. Therefore, $A(\delta)$ is a linear map on \mathcal{N}_{T_1} . Letting $\delta \rightarrow 0$, we obtain that $\partial_t : \mathcal{N}_{T_1} \rightarrow \mathcal{N}_{T_1}$

which associates to $(u_0, u_1) \in \mathcal{N}_{T_1}$ the couple $(\partial_t u|_{t=0}, \partial_{tt}^2 u|_{t=0})$ with $u = S(t)(u_0, u_1)$ is a well-defined linear operator.

For the sake of a contradiction, assume that $\dim \mathcal{N}_{T_1} \geq 1$. Then there exists $\lambda \in \mathbb{C}$ an eigenvalue of $\partial_t : \mathcal{N}_{T_1} \rightarrow \mathcal{N}_{T_1}$ and a corresponding eigenfunction $(u_0, u_1) \in \mathcal{N}_{T_1}$. We consider $u = S(t)(u_0, u_1)$. We have $\Delta_H u = \partial_{tt}^2 u = \lambda^2 u$, which means that u is an eigenfunction of Δ_H . Since $u|_\omega = 0$, a result of Bony [Bon69, Corollaire 4.1] ensures that $u = 0$, which is a contradiction. Therefore, $\mathcal{N}_{T_1} = \{0\}$ and $\mathcal{N}_T = \{0\}$. \square

End of the proof of Theorem 3. We argue again by contradiction. Let us assume that there exists a sequence $(u^k)_{k \in \mathbb{N}}$ of solutions of the wave equation such that $(u_0^k, u_1^k) \in \mathcal{H}_0 \times L^2$, $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$ for any $k \in \mathbb{N}$ and

$$\int_0^T |(\text{Op}(a)\partial_t u^k, \partial_t u^k)_{L^2(M_H, \mu)}| dt \rightarrow 0 \quad (30)$$

as $k \rightarrow +\infty$. The sequence (u_0^k, u_1^k) is bounded in $\mathcal{H}_0 \times L^2$, and therefore, extracting if necessary a subsequence, it converges weakly to some $(u_0, u_1) \in \mathcal{H}_0 \times L^2$. Denoting by u the associated solution of (2), it follows that $\partial_t u^k \rightharpoonup \partial_t u$ for the weak topology of $L^2((0, T) \times M_H)$.

Let us prove that

$$\int_0^T |(\text{Op}(a)\partial_t u, \partial_t u)_{L^2(M_H, \mu)}| dt = 0. \quad (31)$$

We write $\text{Op}(a) = A^*A + B$ with $A \in \Psi_0^{\text{phg}}(T^*M)$ and $B \in \Psi_{-1}^{\text{phg}}(T^*M)$ and we have

$$0 \leq \int_0^T ((A^*A + B)\partial_t u^k, \partial_t u^k)_{L^2(M_H, \mu)} dt + o(1) \xrightarrow{k \rightarrow +\infty} 0$$

where the $o(1)$ comes from Gårding inequality ($\text{Op}(a)$ is asymptotically non-negative). It follows that $\int_0^T (B\partial_t u^k, \partial_t u^k)_{L^2(M_H, \mu)} dt \rightarrow 0$, and since $B\partial_t u^k \rightarrow B\partial_t u$ strongly in $L^2((0, T) \times M_H)$, we get that

$$\int_0^T (B\partial_t u, \partial_t u)_{L^2(M_H, \mu)} dt = 0.$$

We also have

$$\|A\partial_t u\|_{L^2((0, T) \times M_H)} \leq \liminf_{k \rightarrow +\infty} \|A\partial_t u^k\|_{L^2((0, T) \times M_H)} = 0.$$

These last two lines imply (31).

From (31), we deduce that $u \in \mathcal{N}_T$. From Lemma 19, we deduce that $u = 0$. In particular, we have $(u_0, u_1) = (0, 0)$, and (u_0^k, u_1^k) converges to $(0, 0)$ for the weak topology of $\mathcal{H}_0 \times L^2$, and hence for the strong topology of $L^2 \times \mathcal{H}'_0$ by Lemma 17. Together with (30) and the fact that $\|(u_0^k, u_1^k)\|_{\mathcal{H} \times L^2} = 1$, this contradicts (22). It concludes the proof of Theorem 3. \square

A Pseudodifferential calculus

We denote by Ω an open set of a d -dimensional manifold (typically $d = n$ or $d = n + 1$ with the notations of this paper) equipped with a smooth volume μ . We denote by q the variable in Ω , typically $q = x$ or $q = (t, x)$ with our notations. Let $\pi : T^*\Omega \rightarrow \Omega$ be the canonical projection. We recall briefly some facts concerning pseudodifferential calculus, following [Hör07, Chapter 18].

We denote by $S_{\text{hom}}^m(T^*\Omega)$ the set of homogeneous symbols of degree m with compact support in Ω . We also write $S_{\text{phg}}^m(T^*\Omega)$ the set of polyhomogeneous symbols of degree m with compact support in Ω . Hence, $a \in S_{\text{phg}}^m(T^*\Omega)$ if $a \in C^\infty(T^*\Omega)$, $\pi(\text{Supp}(a))$ is a compact of Ω , and there exist $a_j \in S_{\text{hom}}^{m-j}(T^*\Omega)$ such that for all $N \in \mathbb{N}$, $a - \sum_{j=0}^N a_j \in S_{\text{phg}}^{m-N-1}(T^*\Omega)$. We denote by $\Psi_{\text{phg}}^m(T^*\Omega)$ the space of polyhomogeneous pseudodifferential operators of order m on Ω , with a compactly supported kernel in $\Omega \times \Omega$. For $A \in \Psi_{\text{phg}}^m(\Omega)$, we denote by

$\sigma_p(A) \in S_{\text{phg}}^m(T^*\Omega)$ the principal symbol of A . The sub-principal symbol is characterized by the action of pseudodifferential operators on oscillating functions: if $A \in \Psi_{\text{phg}}^m(\Omega)$ and $f(q) = b(q)e^{ikS(q)}$ with b, S smooth and real-valued, then

$$\int_{\Omega} A(f)\bar{f}d\mu = k^m \int_{\Omega} \left(\sigma_p(A)(q, S'(q)) + \frac{1}{k}\sigma_{\text{sub}}(A)(q, S'(q)) \right) |f(q)|^2 d\mu(q) + O(k^{m-2}).$$

A quantization is a continuous linear mapping

$$\text{Op} : S_{\text{phg}}^m(T^*\Omega) \rightarrow \Psi_{\text{phg}}^m(\Omega)$$

satisfying $\sigma_p(\text{Op}(a)) = a$. An example of quantization is obtained by using partitions of unity and, locally, the Weyl quantization, which is given in local coordinates by

$$\text{Op}^W(a)f(q) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_q^d \times \mathbb{R}_p^d} e^{i\langle q-q', p \rangle} a\left(\frac{q+q'}{2}, p\right) f(q') dq' dp.$$

We have the following properties:

1. If $A \in \Psi_{\text{phg}}^l(\Omega)$ and $B \in \Psi_{\text{phg}}^m(\Omega)$, then $[A, B] \in \Psi_{\text{phg}}^{l+m-1}(\Omega)$ and $\sigma_p([A, B]) = \frac{1}{i}\{\sigma_p(a), \sigma_p(b)\}$ where the Poisson bracket is taken with respect to the canonical symplectic structure of $T^*\Omega$.
2. If X is a vector field on Ω and X^* is its formal adjoint in $L^2(\Omega, \mu)$, then $X^*X \in \Psi_{\text{phg}}^2(\Omega)$ and $\sigma_p(X^*X) = h_X^2$.
3. If $A \in \Psi_{\text{phg}}^m(\Omega)$, then A maps continuously the space $H^s(\Omega)$ to the space $H^{s-m}(\Omega)$.

B Proof of Proposition 8

Let us first prove the result when $M \subset \mathbb{R}^n$, following the proof of [Ral82].

In this proof, we suppress the time variable t . Thus we use $x = (x_0, x_1, \dots, x_n)$ where $x_0 = t$ in our earlier notations. Similarly, $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ where $\xi_0 = \tau$ previously. Let Γ be the projection a null-bicharacteristic curve, i.e., a solution of (5) given by $x(s) \in \mathbb{R}^{n+1}$. We insist on the fact that the bicharacteristics are parametrized by s , as in (5). We consider functions of the form

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}.$$

We would like to choose $\psi(x)$ such that for all $s \in \mathbb{R}$, $\psi(x(s))$ is real-valued and $\text{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))$ is positive definite on vectors orthogonal to $\dot{x}(s)$. Roughly speaking, $|e^{ik\psi(x)}|$ will then look like a Gaussian distribution on planes perpendicular to Γ in \mathbb{R}^{n+1} .

We first observe that $\partial_{tt}^2 v_k - \Delta v_k$ may be decomposed as

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi}$$

with

$$A_1(x) = p_2 \left(x, \frac{\partial \psi(x)}{\partial x} \right) a_0(x)$$

$$A_2(x) = L a_0(x)$$

$$A_3(x) = \partial_{tt}^2 a_0(x) - \Delta a_0(x).$$

Here we have set

$$L a_0 = \frac{1}{i} \left(\frac{\partial p_2}{\partial \xi_j} \left(x, \frac{\partial \psi}{\partial x} \right) \frac{\partial a_0}{\partial x_j} \right) + \left(\frac{1}{2i} \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k} \left(x, \frac{\partial \psi}{\partial x} \right) \frac{\partial^2 \psi}{\partial x_j \partial x_k} + p_1 \left(x, \frac{\partial \psi}{\partial x} \right) \right) a_0 \quad (32)$$

and p_1 is the sub-principal symbol of $\partial_{tt}^2 - \Delta$ (see Appendix A).

We will use several times the following classical lemma. It is particularly useful to estimate oscillating integrals and its proof is given in [Ral82, Lemma 2.8].

Lemma 20. *Let $c(x)$ be a function on \mathbb{R}^{n+1} which vanishes at order $S-1$ on a curve Γ for some $S \geq 1$. Suppose that $\text{Supp } c \cap \{|x_0| \leq T\}$ is compact and that $\text{Im } \psi(x) \geq ad^2(x)$ on this set for some constant $a > 0$, where $d(x)$ denotes the distance from the point $x \in \mathbb{R}^{d+1}$ to the curve Γ . Then there exists a constant C such that*

$$\int_{|x_0| \leq T} \left| c(x) e^{ik\psi(x)} \right|^2 dx \leq Ck^{-S-n/2}.$$

In what follows, we construct a_0 and ψ so that $A_1(x)$ vanishes at order 2 along Γ and $A_2(x)$ vanishes at order 0 along the same curve. We will then be able to use Lemma 20 with $S = 3$ and $S = 1$ respectively.

Analysis of $A_1(x)$. Our goal is to show that, if we choose ψ adequately, we can make the quantity

$$f(x) = p_2 \left(x, \frac{\partial \psi(x)}{\partial x} \right)$$

vanish at order 2 on Γ . For the vanishing at order 0, we prescribe $\frac{\partial \psi}{\partial x}(x(s)) = \xi(s)$, and then $f(x(s)) = 0$ since $(x(s), \xi(s))$ is a null-bicharacteristic. For the vanishing at order 1, using (5), we remark that

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x(s)) &= \frac{\partial p_2}{\partial x_j}(x(s)) + \sum_{k=0}^n \frac{\partial p_2}{\partial \xi_k}(x(s)) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\dot{\xi}_j(s) + \sum_{k=0}^n \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\frac{d}{ds} \left(\frac{\partial \psi}{\partial x_j}(x(s)) \right) + \sum_{k=0}^n \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= 0. \end{aligned} \tag{33}$$

Therefore, f vanishes automatically at order 1 along Γ (without making any particular choice for ψ): it just follows from the bicharacteristic equations (5). But for $f(x)$ to vanish at order 2 along Γ , it is required to choose a particular ψ . Using the Einstein summation notation, we want

$$\begin{aligned} 0 &= \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= \frac{\partial^2 p_2}{\partial x_j \partial x_i} + \frac{\partial^2 p_2}{\partial \xi_k \partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial^2 p_2}{\partial x_j \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_2}{\partial \xi_l \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} \frac{\partial^2 \psi}{\partial x_j \partial x_l} + \frac{\partial p_2}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_j \partial x_k \partial x_i} \end{aligned}$$

to hold along Γ for $i, j = 0, 1, \dots, n$. Introducing the matrices

$$\begin{aligned} (M(s))_{ij} &= \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s)), & (A(s))_{ij} &= \frac{\partial^2 p_2}{\partial x_i \partial x_j}(x(s), \xi(s)), \\ (B(s))_{ij} &= \frac{\partial^2 p_2}{\partial \xi_i \partial x_j}(x(s), \xi(s)), & (C(s))_{ij} &= \frac{\partial^2 p_2}{\partial \xi_i \partial \xi_j}(x(s), \xi(s)) \end{aligned}$$

this amounts to solving the matricial Riccati equation

$$\frac{dM}{ds} + MCM + B^T M + MB + A = 0 \tag{34}$$

on the time-interval $(0, T)$. While solving (34), we also require $M(s)$ to be symmetric, $\text{Im}(M(s))$ to be positive definite on the orthogonal complement of $\dot{x}(s)$, and $M(s)\dot{x}(s) = \dot{\xi}(s)$ to hold for all $s \in \mathbb{R}$ due to (33).

It is shown in [Ral82] that for any $(n+1) \times (n+1)$ matrix M_0 with $\text{Im}(M_0) > 0$ on the orthogonal complement of $\dot{x}(0)$ and $M_0\dot{x}(0) = \dot{\xi}(0)$, there exists a global solution $M(s)$ on

$[0, T]$ of (34) which satisfies all these conditions and such that $M(0) = M_0$. The proof just requires that A, C are symmetric, but does not need anything special about p_2 (in particular, it applies to our sub-Riemannian case where p_2 is degenerate). For the sake of completeness, we recall the proof below.

We first extend $y^0 = \dot{x}(0)$ to a basis y^0, \dots, y^n of \mathbb{R}^{n+1} , and we define $\eta^i = M(0)y^i$. Finally, $M(s)$ is defined by the equations

$$M(s)y^i(s) = \eta^i(s), \quad i = 0, \dots, n, \quad (35)$$

where $(y^i(s), \eta^i(s))$ is the solution with initial data (y^i, η^i) to the linear system

$$\begin{cases} \dot{y} = By + C\eta, & y^i(0) = y^i, \\ \dot{\eta} = -Ay - B^T\eta & \eta^i(0) = \eta^i. \end{cases} \quad (36)$$

All the coefficients in (36) are real and A and C are symmetric, therefore the flow defined by (36) preserves both the real symplectic form acting on pairs $(y, \eta) \in (\mathbb{R}^n)^2$ and $(y', \eta') \in (\mathbb{R}^n)^2$ given by

$$\sigma((y, \eta), (y', \eta')) = y \cdot \eta' - \eta \cdot y'$$

and the complexified form $\sigma_{\mathbb{C}}((y, \eta), (y', \eta')) = \sigma((y, \eta), (\overline{y'}, \overline{\eta'}))$ for $(y, \eta) \in (\mathbb{C}^n)^2$ and $(\tilde{y}, \tilde{\eta}) \in (\mathbb{C}^n)^2$. When we say that $\sigma_{\mathbb{C}}$ is invariant under (36), it means that we allow complex initial data in (36). Note that, decomposing $y = y_1 + iy_2$ (with $y_1, y_2 \in \mathbb{R}^n$) and similarly for η , we have in particular

$$\sigma_{\mathbb{C}}((y, \eta), (y, \eta)) = -2i(y_1 \cdot \eta_2 - y_2 \cdot \eta_1).$$

Thus, if $\sum_{i=0}^n b_i y^i(s_0) = 0$ with $b_i \in \mathbb{R}$, and if we set $v = \sum_{i=0}^n b_i (y^i(s_0), \eta^i(s_0))$, then $0 = \sigma_{\mathbb{C}}(v, v)$, and therefore

$$-2iy \cdot \text{Im}(M(0))\overline{y} = 0$$

where $y = \sum_{i=0}^n b_i y^i(0)$. From this, we deduce that $b_i = 0$ for $i = 0, \dots, n$, and $M(s)$ is well-defined by (35). One may check directly thanks to (36) that M satisfies (34). Moreover, the invariance of σ implies that $M(s)$ is symmetric. We may also check that $(y^0(s), \eta^0(s)) = (\dot{x}(s), \dot{\xi}(s))$, which implies that $M(s)\dot{x}(s) = \dot{\xi}(s)$. Together with the invariance of $\sigma_{\mathbb{C}}$, it also implies that $\text{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$. Therefore, we found a choice for the second order derivatives of ψ along Γ which meets all our conditions. For $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$ and s such that $t = t(s)$, we set

$$\psi(x) = \xi'(s) \cdot (x' - x'(s)) + \frac{1}{2}(x' - x'(s)) \cdot M(s)(x' - x'(s)). \quad (37)$$

To sum up, as in the Riemannian (or “strictly hyperbolic”) case handled by Ralston in [Ral82], the key observation is that the invariance of σ and $\sigma_{\mathbb{C}}$ prevents the solutions of (34) with positive imaginary part on the orthogonal complement of $\dot{x}(0)$ to blowup.

Analysis of $A_2(x)$. We note that A_2 vanishes along Γ if and only if $La_0(x(s)) = 0$. According to (32), this turns out to be a linear transport equation on a_0 , which has a solution. Note that, given a_0 at $(t = 0, x = x(0))$, this determines a_0 only along Γ , and we may choose a_0 in a smooth (and arbitrary) way outside Γ . We choose it to vanish outside a small neighborhood of Γ .

Proof of (9). We apply Lemma 20 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1(0, T; L^2(M))} \leq C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (9).

Proof of (10). We first observe that since $\text{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$ and continuous as a function of s , there exist $\alpha, C > 0$ such that for any $t(s) \in [0, T]$ and any $x' \in M$,

$$|\partial_t v_k(t(s), x')|^2 + |\partial_t v_k(t(s), x')|^2 \geq Ck^{\frac{n}{2}} e^{-\alpha k d(x', x'(s))^2}$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in \mathbb{R}^n . Using the observation that for any function f ,

$$\int_M f(x') e^{-\alpha k d(x', x'(s))^2} d\mu(x') \sim \frac{\pi^{n/2}}{k^{n/2} \sqrt{\alpha}} f(x'(s)) \frac{d\mu}{d\ell_n}(x'(s)) \quad (38)$$

as $k \rightarrow +\infty$, we obtain (10). Here ℓ_n denotes the Lebesgue measure on \mathbb{R}^n .

Proof of (11). We observe that since $\text{Im}(M(s))$ is positive definite (uniformly in s) on the orthogonal complement of $\dot{x}(s)$, there exist $C, \alpha' > 0$ such that for any $t \in [0, T]$, for any $x' \in M$, $|\partial_t v_k(t(s), x')|$ and $|\nabla^{\text{sR}} v_k(t(s), x')|$ are both bounded above by $Ck^{\frac{n}{4}} e^{-\alpha' k d(x', x'(s))^2}$. Therefore

$$\begin{aligned} & \int_{M \setminus V_{t(s)}} (|\partial_t v_k(t(s), x')|^2 + |\nabla^{\text{sR}} v_k(t(s), x')|^2) d\mu(x') \\ & \leq Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\mu(x') \\ & \leq Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\ell_n(x') + o(1) \end{aligned} \quad (39)$$

where, in the last line, we used the fact that $|d\mu/d\ell_n| \leq C$ in a fixed compact subset of M (since μ is a smooth volume), and the $o(1)$ comes from the eventual blowup of μ at the boundary of M .

Now, $M \subset \mathbb{R}^n$, and there exists $r > 0$ such that $B_d(x(s), r) \subset V_{t(s)}$ for any s such that $t(s) \in (0, T_0)$, where $d(\cdot, \cdot)$ still denotes the Euclidean distance in \mathbb{R}^n . Therefore, we bound above the integral in (39) by

$$Ck^{n/2} \int_{\mathbb{R}^n \setminus B_d(x(s), r)} e^{-2\alpha' k d(x', x'(s))^2} d\ell_n(x') \quad (40)$$

Making the change of variables $y = k^{-1/2}(y - x(s))$, we bound above (40) by

$$C \int_{\mathbb{R}^n \setminus B_d(0, rk^{1/2})} e^{-2\alpha' \|y\|^2} d\ell_n(y)$$

with $\|\cdot\|$ the Euclidean norm. This last expression is bounded above by

$$C e^{-\alpha' r^2 k} \int_{\mathbb{R}^n} e^{-\alpha' \|y\|^2} d\ell_n(y)$$

which implies (11).

Extension of the result to any manifold M . In the case of a general manifold M , not necessarily included in \mathbb{R}^n , we use charts together with the above construction. We cover M by a set of charts $(U_\alpha, \varphi_\alpha)$, where (U_α) is a family of open sets of M covering M and $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ is an homeomorphism U_α onto an open subset of \mathbb{R}^n . Take a solution $(x(t), \xi(t))_{t \in [0, T]}$ of (6). It visits a finite number of charts in the order $U_{\alpha_1}, U_{\alpha_2}, \dots$, and we choose the charts and a_0 so that $v_k(t, \cdot)$ is supported in a unique chart at each time t . The above construction shows how to construct a_0 and ψ as long as $x(t)$ remains in the same chart. For any $l \geq 1$, we choose t_l so that $x(t_l) \in U_{\alpha_l} \cap U_{\alpha_{l+1}}$ and $a_0(t_l, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Since there is a (local) solution v_k for any choice of initial $a_0(t_l, x(t_l))$ and $\text{Im}\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right)(t_l, x(t_l))$ in Proposition 8, we see that v_k may be continued from the chart U_{α_l} to the chart $U_{\alpha_{l+1}}$. This continuation is smooth since the two solutions coincide as long as $a_0(t, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Patching all solutions on the time intervals $[t_l, t_{l+1}]$ together, it yields a global in time solution v_k , as desired.

C Proof of (28)

Because of the second convergence in (23), it amounts to proving that

$$\int_0^T |(\text{Op}(a)\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2(M_H, \mu)}| dt \rightarrow 0.$$

Replacing $\text{Op}(a)$ by its Friedrichs quantization $\text{Op}^F(a)$, which is positive (see [Tay74, Chapter VII]), we see that it is sufficient to prove

$$(\text{Op}^F(a)\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2((0,T)\times M_H)} \rightarrow 0. \quad (41)$$

Let $\delta > 0$ and $\tilde{a} \in S_{\text{phg}}^0((-\delta, T+\delta)\times M_H)$, $0 \leq \tilde{a} \leq \sup(a)$ and such that $\tilde{a} = a$ for $0 \leq t \leq T$. Using integrations by parts, we have

$$\begin{aligned} (\text{Op}^F(\tilde{a})\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2((0,T)\times M_H)} &= -(\text{div}_\mu(\text{Op}^F(\tilde{a})\nabla^{\text{sR}}u^k), u^k)_{L^2((0,T)\times M_H)} \\ &= -(\text{Op}^F(\tilde{a})\Delta u^k, u^k)_{L^2((0,T)\times M_H)} + o(1) \\ &= -(\text{Op}^F(\tilde{a})\partial_t^2 u^k, u^k)_{L^2((0,T)\times M_H)} + o(1) \\ &= -(\partial_t \text{Op}^F(\tilde{a})\partial_t u^k, u^k)_{L^2((0,T)\times M_H)} + o(1) \\ &= (\text{Op}^F(\tilde{a})\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + o(1) \end{aligned}$$

where the $o(1)$ comes from direct commutator estimates together with the fact that $\|u^k\|_{\mathcal{H}_T} = 1$. Finally we note that since Op^F is a positive quantization, we have

$$\begin{aligned} (\text{Op}^F(a)\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2((0,T)\times M_H)} &\leq (\text{Op}^F(\tilde{a})\nabla^{\text{sR}}u^k, \nabla^{\text{sR}}u^k)_{L^2((0,T)\times M_H)} \\ &= (\text{Op}^F(\tilde{a})\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + o(1) \\ &\leq C\delta + (\text{Op}^F(a)\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + o(1) \end{aligned}$$

where C does not depend on δ . Making $\delta \rightarrow 0$, it concludes the proof of (41), and consequently (28) holds.

References

- [BC17] Karine Beauchard and Piermarco Cannarsa. Heat equation on the Heisenberg group: Observability and applications. *Journal of Differential Equations*, 262(8):4475–4521, 2017.
- [Bel96] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, pages 1–78. Springer, 1996.
- [BG02] Nicolas Burq and Patrick Gérard. *Contrôle optimal des équations aux dérivées partielles*. Ecole polytechnique, Département de mathématiques, 2002.
- [BLR92] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM Journal on Control and Optimization*, 30(5):1024–1065, 1992.
- [Bon69] Jean-Michel Bony. Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. In *Annales de l'Institut Fourier*, volume 19, pages 277–304, 1969.
- [BS19] Nicolas Burq and Chenmin Sun. Time optimal observability for Grushin Schrödinger equation. *arXiv preprint arXiv:1910.03691*, 2019.
- [CdVHT18] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case. *Duke Mathematical Journal*, 167(1):109–174, 2018.

- [DK20] Michel Duprez and Armand Koenig. Control of the Grushin equation: non-rectangular control region and minimal time. *ESAIM: Control, Optimisation and Calculus of Variations*, 2020.
- [EN99] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194. Springer Science & Business Media, 1999.
- [Gér91] Patrick Gérard. Mesures semi-classiques et ondes de Bloch. *Séminaire Équations aux dérivées partielles (Polytechnique)*, pages 1–19, 1991.
- [GR15] Claudia Garetto and Michael Ruzhansky. Wave equation for sums of squares on compact lie groups. *Journal of Differential Equations*, 258(12):4324–4347, 2015.
- [Hör67] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119(1):147–171, 1967.
- [Hör07] Lars Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*. Springer Science & Business Media, 2007.
- [HS13] Michael Hitrik and Johannes Sjöstrand. Two minicourses on analytic microlocal analysis. In *Algebraic and Analytic Microlocal Analysis*, pages 483–540. Springer, 2013.
- [Ivr19] Victor Ivrii. *Microlocal Analysis, Sharp Spectral Asymptotics and Applications I: Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics*. Springer International Publishing, 2019.
- [Jea14] Frédéric Jean. *Control of nonholonomic systems: from sub-Riemannian geometry to motion planning*. Springer, 2014.
- [Koe17] Armand Koenig. Non-null-controllability of the Grushin operator in 2D. *Comptes Rendus Mathématique*, 355(12):1215–1235, 2017.
- [Lio88] Jacques-Louis Lions. Contrôlabilité exacte, stabilisation et perturbations de systemes distribués. Tome 1. Contrôlabilité exacte. *Rech. Math. Appl*, 8, 1988.
- [LL20] Camille Laurent and Matthieu Léautaud. Tunneling estimates and approximate controllability for hypoelliptic equations. *To appear in Mem. Am. Math. Soc.*, 2020.
- [LRLTT17] Jérôme Le Rousseau, Gilles Lebeau, Peppino Terpolilli, and Emmanuel Trélat. Geometric control condition for the wave equation with a time-dependent observation domain. *Analysis & PDE*, 10(4):983–1015, 2017.
- [Mon94] Richard Montgomery. Abnormal minimizers. *SIAM Journal on Control and Optimization*, 32(6):1605–1620, 1994.
- [Mon02] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc., 2002.
- [MZ02] Fabricio Macia and Enrique Zuazua. On the lack of observability for wave equations: a Gaussian beam approach. *Asymptotic Analysis*, 32(1):1–26, 2002.
- [Ral82] James Ralston. Gaussian beams and the propagation of singularities. *Studies in partial differential equations*, 23(206):C248, 1982.
- [RS76] Linda Preiss Rothschild and Elias M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Mathematica*, 137(1):247–320, 1976.
- [Str86] Robert S. Strichartz. Sub-Riemannian geometry. *Journal of Differential Geometry*, 24(2):221–263, 1986.
- [Tar90] Luc Tartar. H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 115(3-4):193–230, 1990.

- [Tay74] Michael Taylor. Pseudodifferential operators. *Lecture notes in Mathematics*, 1974.
- [TW09] Marius Tucsnak and George Weiss. *Observation and control for operator semi-groups*. Springer Science & Business Media, 2009.