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Regret analysis of the Piyavskii–Shubert algorithm for global Lipschitz optimization

Clément Bouttier

Airbus Group SAS

Tommaso R. Cesari

Artificial and Natural Intelligence
Toulouse Institute (ANITI)
Toulouse School of Economics
University of Milan

Sébastien Gerchinovitz

IRT Saint-Exupéry,
DEEL project
Institut de Mathématiques
de Toulouse

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Abstract

We consider the problem of maximizing a non-concave Lipschitz multivariate function f over a compact domain. We provide regret guarantees (i.e., optimization error bounds) for a very natural algorithm originally designed by Piyavskii and Shubert in 1972. Our results hold in a general setting in which values of f can only be accessed approximately. In particular, they yield state-of-the-art regret bounds both when f is observed exactly and when evaluations are perturbed by an independent subgaussian noise.

1 Introduction and related work.

The goal of online optimization is to find an approximate maximizer of a given function $f: D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ with as little evaluations of f as possible. In this paper we assume that the only access to the function f is through an oracle returning the (possibly) perturbed values of the function at the queried points. Perturbations can be deterministic or stochastic. No analytical expression of f or any of its derivatives is available.

At each round k the learner picks a new point $x_k \in D$ and the value $f(x_k)$ is revealed by the oracle, up to an additive perturbation ξ_k . After each evaluation, the learner can return a point $x_k^* \in D$, which may differ from the last queried point x_k .

We measure the accuracy of the approximation provided by the point x_n^* returned after the n -th evaluation of the function with the relative error

$$r_n := \sup_{x \in D} f(x) - f(x_n^*). \quad (1)$$

Following the bandit optimization literature, we call this error *simple regret* (or *regret* for short).

We consider two variants of the problem. In the first one, a budget n of evaluations is given to the learner. The goal is to find an $x_n^* \in D$ such that $r_n \leq \varepsilon$, with $\varepsilon > 0$ as small as possible. In the second one, a level of accuracy $\varepsilon > 0$ is given instead. In this case, the goal is to find an $x_n^* \in D$ such that $r_n \leq \varepsilon$, with n as small as possible.

Due to numerous practical applications, this black-box global optimization problem has received considerable attention over the past decades. Many different algorithms have been proposed in several communities such as concave optimization [Nesterov, 2004, Boyd and Vandenberghe, 2004, Bubeck, 2015], non-concave optimization [Hansen et al., 1992a, Jones et al., 1993, Jain and Kar, 2017, Basso, 1978, 1982, Shang et al., 2019], stochastic optimization or approximation [Spall, 2003, Shalev-Shwartz, 2011], Bayesian optimization [Brochu et al., 2010], and bandit optimization over metric spaces [Munos et al., 2014].

In this article, we focus on the case where f attains its maximum at some $x^* \in D$ and it is Lipschitz around x^* (Assumption 1). Unlike global Lipschitzness, this assumption does not imply continuity anywhere but at the maximizer x^* (Figure 1).

We consider two different settings. In the deterministic setting the values of f are observed up to deterministic (and possibly adaptive) adversarial perturbations (Section 3). In the stochastic setting the queried values are observed up to a subgaussian noise (Section 4).

We extend a classical algorithm originally designed by Piyavskii [1972] and Shubert [1972] for the one-dimensional deterministic setting with no perturbations. We call it *the Piyavskii–Shubert algorithm* in the sequel. The principle is simple: at each round k , the query point x_k is chosen as an approximate maximizer of a proxy function \hat{f}_{k-1} that provably satisfies $\hat{f}_{k-1}(x^*) \geq f(x^*)$. In the case of globally L -Lipschitz optimization without perturbations, the proxy at round k is the piecewise-conic function

$$\begin{aligned} \hat{f}_{k-1}: D &\rightarrow \mathbb{R} \\ x &\mapsto \min_{i \in \{1, \dots, k-1\}} \{f(x_i) + L\|x_i - x\|\} \end{aligned}$$

(see Figure 3) and it is the best upper bound on f given the available information. After observing the perturbed value of $f(x_k)$, the learner updates the proxy function \hat{f}_k and chooses the next point x_{k+1} as an approximate maximizer of \hat{f}_k .

1.1 Related works.

Several papers studied the Piyavskii–Shubert algorithm in the eighties and nineties (see, e.g., Mayne and Polak [1984], Mladineo [1986], Hansen et al. [1991] or the survey by Hansen et al. [1992a]). Despite this literature, little was known about the rate of convergence of its simple regret r_n as a function of the number n of evaluations of the function. A crude regret bound of the form $r_n = \mathcal{O}(n^{-1/d})$, where d is the ambient dimension, can be obtained from [Mladineo, 1986, Theorem 4.2] when f is globally Lipschitz. The authors show that the Piyavskii–Shubert algorithm is minimax optimal among all algorithms, and therefore superior to a uniform grid search ensuring a simple regret of order $n^{-1/d}$.

The analysis in Horn [2006] gives a regret rate that corresponds to our Corollary 1 in the special case in which the near-optimality dimension d^* (see Section 2.3 for a definition) is $d/2$.

In dimension $d = 1$ a bound on the sample complexity was derived by Hansen et al. [1991] for a variant of the Piyavskii–Shubert algorithm that stops automatically returning an ε -maximizer of f . More precisely, they proved that the number of iterations required by the algorithm to reach precision ε is at most proportional to $\int_0^1 (f(x^*) - f(x) + \varepsilon)^{-1} dx$. In this paper the authors rely heavily on the one-dimensional setting to study the proxy functions \hat{f}_k in an explicit manner. In the same paper, they claim “*Extending the results of this paper to the multivariate appears to be difficult*”.

In a recent paper, Malherbe and Vayatis [2017] studied a variant of the Piyavskii–Shubert algorithm called LIPO. Rather than maximizing the proxy function \hat{f}_{k-1} , LIPO queries the k -th point uniformly at random in the set of potential maximizers $\{x \in D : \hat{f}_{k-1}(x) \geq \max_{i=1, \dots, k-1} f(x_i)\}$. In their paper, they obtain upper bounds on the regret slightly worse (by a log factor) than our Corollary 1, under slightly stronger assumptions. In the same paper, they mention that a general regret analysis for the Piyavskii–Shubert algorithm was still missing (Table 1). Our results fill this theoretical gap.

In the context of global optimization, there is a line of research devoted to maximizing L -Lipschitz functions *without* knowing L . Algorithms for these problems typically sacrifice some of the available evaluations of f to compute an estimate of the Lipschitz constant L and use the remaining ones to optimize the objective. Such estimates \hat{L} of L can be defined (up to small margins) as $\hat{L} = \max_{i \neq j} \frac{f(x_i) - f(x_j)}{x_i - x_j}$, where the maximum is over all pairs of distinct values x_i, x_j that have been queried so far. Bubeck et al. [2011b] present a two-phase algorithm in which the estimation of the Lipschitz constant occurs in the first phase. Their results hold for twice differentiable objectives for which the eigenvalues of the Hessian are bounded everywhere by a constant M . Notably, the bounds they obtain are meaningless unless the time horizon is bigger than a function of M (i.e., in order to apply the result in practice one would need prior knowledge of M).

Malherbe and Vayatis [2017] introduce AdaLIPO, an adaptive version of LIPO. Their algorithm flips a biased coin at each round. Depending on the outcome, AdaLIPO either performs a LIPO step (using the current best estimate of the Lipschitz constant) or it evaluates the function at a point drawn uniformly in the domain of f . These uniform draws are used to obtain increasingly more accurate approximations of the Lipschitz constant. Again, the bounds they obtain become meaningless if the probability of getting a good estimate of the Lipschitz constant with only two uniform queries to f (what the authors call Γ , in Proposition 18) is too small. In some papers the Lipschitzness assumption is replaced by some other regularity assumption (e.g., Bartlett et al. [2019], Grill et al. [2015] use a notion of smoothness related to hierarchical partitioning that is comparable to our Assumption 1).

The idea of investing some of the available evaluations to approximate the Lipschitz constant could also be applied to the Piyavskii–Shubert algorithm with a budget of evaluations n . However, missing any prior quantitative information on the smoothness of f has a crucial limitation. It makes it *impossible* to design algorithms that take an accuracy ε as an input and stop automatically returning an ε -optimal point (without any other inputs). To see this, note that *all* algorithms will fail when the objective function is the “spike” $x \mapsto \max\{0, a - L\|x - x_0\|\}$ for some values of a , L , and x_0 that depend on the algorithm (typically a and L should be chosen large enough to force the algorithm to miss a tall and narrow spike).

It is worth noting that the Piyavskii–Shubert algorithm might not be computationally efficient in high dimensions. Indeed, the problem of optimizing f is replaced by the optimization of $n - 1$ other functions $\hat{f}_1, \dots, \hat{f}_{n-1}$ (the proxy functions used to compute the query points) which could be a demanding task on its own. To see this, note that finding the maximum of \hat{f}_k is related to the problem of determining a Voronoi diagram for a set of k points, which is known to have a computational complexity that is exponential in the dimension of the ambient space [Aurenhammer and Klein, 2000, Theorem 4.5, Section 4.3.2 “Power diagrams and convex hulls”]. Some algorithms get around these computational issues by replacing the piecewise-conic proxies of the Piyavskii–Shubert algorithm with looser but simpler functions. The DOO algorithm Munos [2011] (see also Perevozchikov [1990]), for example, uses piecewise-constant functions, with pieces that correspond to a predetermined hierarchical partition of D . We however conjecture that the associated sample complexity of such simplified algorithms is worse than that of the Piyavskii–Shubert algorithm (by a non-negligible multiplicative constant). This is supported by some low-dimensional experiments carried out in Hansen et al. [1992b].

The Piyavskii–Shubert algorithm can be applied to low-dimensional real-life problems (e.g., hyperparameter tuning), but also to maximization problems in which the computational cost is driven by the evaluation of the objective f rather than the dimension of the ambient space. Finally, we remark that the peeling technique that we introduce in this paper allows to write refined bounds that could be of theoretical interest on their own.

1.2 Main contributions.

In this paper we provide a theoretical analysis of the simple regret r_n of the Piyavskii–Shubert algorithm in arbitrary dimension $d \geq 1$, using recent concepts from the bandit optimization literature. Our only assumptions are that f is Lipschitz around a maximizer x^* (see Section 2 for a rigorous definition) and its domain D is compact.

Deterministic setting. We prove the first non-trivial upper bound on the simple regret r_n of the Piyavskii–Shubert algorithm in arbitrary dimension $d \geq 1$.¹ Rather than depending on the dimension d , our general result (Theorem 1) is expressed in terms of packing numbers that represent the hardness of the optimization of the objective function. In particular, this general result allows to bound the regret in terms of a quantity $d^* \in [0, d]$ usually referred to as *near-optimality dimension* of f (see Section 2.3 for a definition). When the values of f can be observed without perturbations, this bound is roughly

¹See discussion in previous section.

of the form

$$r_n \lesssim \begin{cases} n^{-1/d^*} & \text{if } d^* > 0 \\ \exp(-\Omega(n)) & \text{if } d^* = 0 \end{cases} \quad (2)$$

The regret bound (2) matches the best bound known so far in this deterministic setting.² Examples of other algorithms that attain this bound include the branch-and-bound algorithm [Perevozchikov \[1990\]](#), the DOO algorithm [Munos \[2011\]](#), and the LIPO algorithm [Malherbe and Vayatis \[2017\]](#).

To prove (2), we upper bound the sample complexity of the Piyavskii–Shubert algorithm. Our sample complexity bounds hold even if the values of f are perturbed, provided that the absolute values of the perturbations are bounded by a known constant α of order at most ε . More precisely, we derive the following equivalent upper bound: if $\alpha = \mathcal{O}(\varepsilon)$, then the number n of iterations needed to reach precision ε is at most $n = \mathcal{O}((1/\varepsilon)^{d^*})$ for $d^* > 0$ or $n = \mathcal{O}(\log(1/\varepsilon))$ for $d^* = 0$.

This result is satisfactory if an overall budget of evaluations n is imposed, but it might be impractical if a target precision ε is required instead. In this case, the knowledge of d^* is necessary to stop the algorithm at the right time, i.e., to compute the upper bound $n = \mathcal{O}((1/\varepsilon)^{d^*})$ or $n = \mathcal{O}(\log(1/\varepsilon))$ on the number of iterations needed to reach precision ε .

To address this issue, we study a version of the Piyavskii–Shubert algorithm that takes ε as input and stops automatically after a simple condition is satisfied ([Algorithm 2](#)), as in the one-dimensional study by [Hansen et al. \[1991\]](#). We prove that this algorithm is in a way adaptive to the near-optimality dimension d^* : it stops after $n = \mathcal{O}((1/\varepsilon)^{d^*})$ or $n = \mathcal{O}(\log(1/\varepsilon))$ iterations (if $d^* > 0$ or $d^* = 0$ respectively) without any prior knowledge on d^* , and guarantees $r_n \leq \varepsilon$. However, a closer look to the tighter bounds given in [Theorem 1](#) and [Theorem 2](#) reveals that the computational complexity of the automatically stopped version of the Piyavskii–Shubert algorithm is higher than its non-automatically stopped variant. The additional terms appearing in [Theorem 2](#) are not an artifact of the analysis, but evidence that stopping automatically is harder than exhausting a given budget of evaluations. After stating [Theorem 2](#), in [Section 3.2](#) we discuss why this is the case.

Stochastic setting. In [Section 4](#) we consider the case in which querying a point x returns a value $f(x) + \xi$, where ξ is an i.i.d. subgaussian noise. We design a natural extension of the Piyavskii–Shubert algorithm based on mini-batch sampling, a technique in which each value of f is queried a small number of times in a row in order to average out the random perturbations. We study the number n of evaluations of f needed by the algorithm to reach precision ε with probability at least $1 - \delta$. Note that the number of evaluations of f is now larger than the number of iterations of the algorithm because of the mini-batch sampling. We derive sample complexity bounds of the form $n \lesssim (1/\varepsilon)^{d^*+2}$ with high probability, where the \lesssim notation hides logarithmic terms and problem-dependent constants. These bounds correspond to regret upper bounds roughly of the form:

$$r_n \lesssim n^{-1/(d^*+2)},$$

which are known to be worst-case optimal as a consequence of the minimax lower bound of [Bubeck et al. \[2011a, Theorem 13\]](#) (this lower bound is stated for the cumulative regret, but it could be adapted for the simple regret).

Examples of other algorithms with a similar regret bound in the stochastic setting include, e.g., the zooming algorithm by [Kleinberg et al. \[2008\]](#), the HOO algorithm by [Bubeck et al. \[2011a\]](#) and the StoOO algorithm by [Munos et al. \[2014\]](#) (some of these papers derived upper bounds on the cumulative regret, which in turn imply upper bounds on the simple regret).

²Note that, when $d^* = 0$, different constants in $\Omega(n)$ lead to different rates. Also, we are not aware of lower bounds for all possible values of $d^* \in [0, d]$ except for $d^* = d$ [[Nesterov, 2004, Theorem 1.1.2](#)] or for $d^* = d/2$ [[Horn, 2006, Theorem 9](#)]. However we conjecture that the best sample complexity upper bounds known so far are minimax optimal (up to logarithmic factors).

1.3 Outline of the paper.

Our paper is organized as follows. We state our main assumptions on f and introduce all notation in Section 2. In Section 3 we revisit the Piyavskii–Shubert algorithm in the deterministic setting with perturbations. We prove a general upper bound on its regret involving packing numbers of suboptimal regions. This result implies state-of-the-art regret bounds in terms of the intrinsic dimension d^* of f , for an arbitrary dimension $d \geq 1$. We study two types of algorithms: the natural extension of the original Piyavskii–Shubert algorithm, whose optimal stopping would require the knowledge of d^* , and an adaptive variant that stops automatically guaranteeing precision ε when its halting criterion is met. In Section 4 we design an extension of the automatically stopped Piyavskii–Shubert algorithm that works in the stochastic setting. We prove an analogous general upper bound on the regret in terms of packing numbers and its consequent corollary expressed as a function of d^* . Finally, in the appendix, we recall a few important lemmas on packing numbers (Appendix A) and near-optimality dimension (Appendix B), we prove Corollaries 1 and 3 (Appendix C), and we discuss the connections of our results with a regret bound of Hansen et al. [1991] in dimension $d = 1$.

2 Assumption, definitions, and notation.

In this section we introduce a regularity assumption, all notation, and all definitions that will be used throughout the paper.

2.1 Assumption on f .

In all the sequel, D is any nonempty compact subset of \mathbb{R}^d , $f: D \rightarrow \mathbb{R}$ is any function, and $\|\cdot\|$ is any norm on \mathbb{R}^d . We make the following assumption, which is sometimes referred to as “local smoothness” Munos et al. [2014], and close in spirit to “calmness” Rockafellar [1985].

Assumption 1 (Lipschitzness around a maximum). *We assume that f attains its maximum at some $x^* \in D$ and that there exists a constant $L_0 > 0$ such that, for all $x \in D$,*

$$f(x) \geq f(x^*) - L_0 \|x^* - x\|.$$

Moreover, we assume that only D , $\|\cdot\|$, and an upper bound L_1 on L_0 are known to the learner.

Note that we do not even require this assumption to be true for *all* maximizers. To the best of our knowledge, all previous works on the Piyavskii–Shubert algorithm and variants thereof assume f to satisfy some *global* continuity condition (e.g., Lipschitzness, Hölderness, uniform continuity) Vanderbei [1999], Piyavskii [1972], Hansen et al. [1991], Rahal and Ziadi [2008], Lera and Sergeev [2002], Ellaia et al. [2012], Shen and Zhu [1987], Nešić et al. [2013]. However, recent contributions from the bandit optimization literature Munos et al. [2014] have shown that the behavior of this type of algorithms is usually driven by the regularity of f around its maxima Auer et al. [2007], Kleinberg et al. [2008], Bubeck et al. [2011a]. Our analysis will show that this is also the case for the Piyavskii–Shubert algorithm.

Lipschitzness around a maximum is a significantly weaker assumption than global Lipschitzness. The only constraint that it poses to the function f is for its graph to lie between the cone $\{(x, y) \in \mathbb{R}^{d+1} : y = f(x^*) - L_0 \|x^* - x\|\}$ and the hyperplane $\{(x, y) \in \mathbb{R}^{d+1} : y = f(x^*)\}$. Most of the good properties that globally Lipschitz functions enjoy are not guaranteed under this assumption. For example, f could be non-differentiable or even discontinuous *everywhere* on $D \setminus \{x^*\}$ (Figure 1).

2.2 Useful notation and definitions.

We denote the set of integers by \mathbb{Z} , the set of nonnegative integers $\{0, 1, 2, \dots\}$ by \mathbb{N} , and the set of positive integers $\{1, 2, \dots\}$ by \mathbb{N}^* . For all $x \in \mathbb{R}$, we write $\lceil x \rceil$ for the value $\min\{k \in \mathbb{Z} : k \geq x\}$ of the ceiling function at x . For all $\delta > 0$, we denote by $B_{\|\cdot\|}(\delta)$ the ball of radius δ in $(\mathbb{R}^d, \|\cdot\|)$ centered at the origin:

$$B_{\|\cdot\|}(\delta) := \{x \in \mathbb{R}^d : \|x\| \leq \delta\}.$$

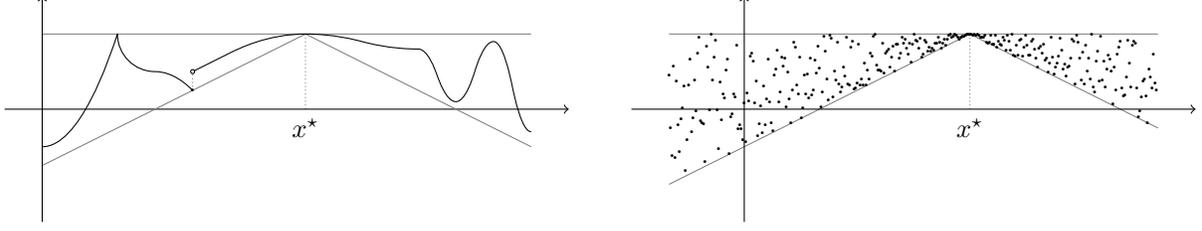


Figure 1: On the left, a typical example of a function Lipschitz around a maximum x^* : it has multiple maxima, points with vertical tangents, and discontinuities. On the right, a pathological example of function discontinuous everywhere but at a maximum x^* , but still Lipschitz around x^* . Our analysis holds even in this extreme case.

We now recall the definitions of packing and covering numbers. The former is of utmost importance to our analysis. For any bounded set $A \subset \mathbb{R}^d$ and any real number $r > 0$:

- the r -packing number of A is the largest number of r -separated points contained in A , that is,

$$\mathcal{N}(A, r) := \sup \left\{ k \in \mathbb{N}^* : \exists x_1, \dots, x_k \in A, \min_{i \neq j} \|x_i - x_j\| > r \right\}, \quad (3)$$

if A is nonempty, zero otherwise;

- the r -covering number of A is the smallest cardinality of an r -covering of A , that is,

$$\mathcal{M}(A, r) := \min \{ N \in \mathbb{N}^* : \exists x_1, \dots, x_N \in \mathbb{R}^d, \forall x \in A, \exists i \in \{1, \dots, N\}, \|x - x_i\| \leq r \},$$

if A is nonempty, zero otherwise.

In Appendix A we recall a few known inequalities about packing and covering numbers that will prove useful throughout the paper.

2.3 Sets of near-optimal points, with examples.

In this section we will introduce a few important definitions and properties that will play a crucial role in our analysis. For all $\varepsilon > 0$, we define the set of ε -optimal points of $f: D \rightarrow \mathbb{R}$ by

$$\mathcal{X}_\varepsilon := \{x \in D : f(x) \geq f(x^*) - \varepsilon\}.$$

We also denote its complement (i.e., the set of ε -suboptimal points) by $\mathcal{X}_\varepsilon^c$ and, for all $0 \leq a < b$, we define the $(a, b]$ -layer

$$\mathcal{X}_{(a,b]} := \mathcal{X}_a^c \cap \mathcal{X}_b = \{x \in D : a < f(x^*) - f(x) \leq b\} \quad (4)$$

(i.e., the set of points that are b -optimal but a -suboptimal). Whenever the explicit dependence on ε or $(a, b]$ can be omitted, we will simply refer to these sets as sets of *near-optimal* (resp., *suboptimal*) points or layers. In particular, we will say that an $(a, b]$ -layer is a *suboptimal layer* if $a > 0$.

Since f is L_0 -Lipschitz around x^* , every point in D is ε_0 -optimal with ε_0 defined by

$$\varepsilon_0 := L_0 \sup_{x, y \in D} \|x - y\|. \quad (5)$$

In other words, $\mathcal{X}_{\varepsilon_0} = D$. For this reason, without loss of generality we will only consider values of ε smaller than or equal to ε_0 .

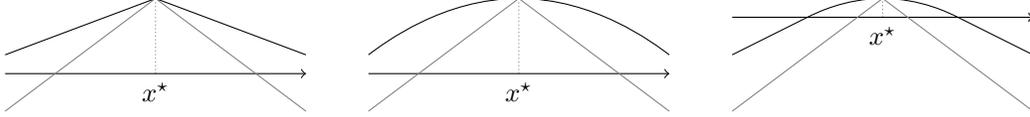


Figure 2: On the left, a linear function ($d^* = 0$). In the center, a quadratic function ($d^* = d/2$). On the right, a function that is linear outside of a neighborhood of x^* ($d^* = 0$ for large r), but quadratic inside ($d^* = d/2$ for small r).

As we will see in Sections 3 and 4, the size of the sets of near-optimal points and that of layers will be key quantities in our sample complexity bounds. As it turns out, the “correct” notion of size for this problem is the packing number (3). In particular, we will derive explicit and immediate corollaries whenever, for some $C^*, d^* \geq 0$, we have:³

$$\forall r \in (0, \varepsilon_0], \quad \mathcal{N}\left(\mathcal{X}_r, \frac{r}{2L_0}\right) \leq C^* \left(\frac{\varepsilon_0}{r}\right)^{d^*}. \quad (6)$$

The idea behind Inequality (6) is that ε -optimal points are hard to find if the corresponding set $\mathcal{X}_\varepsilon^c$ of ε -suboptimal points is large. Since for any increasing sequence $\varepsilon := r_0 < r_1 < r_2 < \dots$, the set of ε -suboptimal points $\mathcal{X}_\varepsilon^c$ can be decomposed into a union of suboptimal layers $\mathcal{X}_{(r_0, r_1]}, \mathcal{X}_{(r_1, r_2]}, \mathcal{X}_{(r_2, r_3]}, \dots$, and each of these layers $\mathcal{X}_{(r_{s-1}, r_s]}$ is included in \mathcal{X}_{r_s} (by definition of layer (4)), by controlling the size of each of these \mathcal{X}_{r_s} we can control the size of $\mathcal{X}_\varepsilon^c$. Therefore, by controlling how large the sets \mathcal{X}_r can be at all scales r , the parameters C^* and d^* quantify the difficulty of the optimization problem. (See the discussion in Appendix B about the related notions of *near-optimality dimension* Bubeck et al. [2011a] and *zooming dimension* Kleinberg et al. [2008].)

As noted in Lemma 7 (Appendix B), Inequality (6) is always true with $C^* = 9^d$ and $d^* = d$. However, depending on f , a significantly smaller value of the constant d^* may be picked. We provide three examples below. The first two are very classical. The last one indicates that a “worst-case” single value of d^* may be insufficient to give an appropriate description of the hardness of the optimization problem.

Example 1 (Linear Regime). Consider any norm $\|\cdot\|$ and the function $f(x) = 1 - L_0\|x - a\|$ on any compact domain $D \subset \mathbb{R}^d$, with $a \in D$ (Figure 2, left). Then f is L_0 -Lipschitz with respect to $\|\cdot\|$ and, for all $r \in (0, \varepsilon_0]$, we have $\mathcal{X}_r = \{x \in D : \|x - a\| \leq r/L_0\}$, which gives

$$\begin{aligned} \mathcal{N}\left(\mathcal{X}_r, \frac{r}{2L_0}\right) &\leq \mathcal{N}\left(B_{\|\cdot\|}(r/L_0), \frac{r}{2L_0}\right) \\ &\leq \mathcal{M}\left(B_{\|\cdot\|}(r/L_0), \frac{r}{4L_0}\right) && \text{(by (17))} \\ &\leq \left(1 + \frac{2(r/L_0)}{r/(4L_0)}\right)^d && \text{(by (18))} \\ &= 9^d, \end{aligned}$$

which does not depend on r . Thus Inequality (6) holds with $C^* = 9^d$ and $d^* = 0$.

Example 2 (Quadratic Regime). Fix any $\beta > 0$. Consider the Euclidean norm $\|\cdot\|_2$ on \mathbb{R}^d and the function $f(x) = 1 - \beta\|x - a\|_2^2$ on any compact domain $D \subset \mathbb{R}^d$, with $a \in D$ (Figure 2, center). Then, for all $r \in (0, \varepsilon_0]$,

³Property (6) could be rewritten somewhat equivalently as $\forall r \in (0, \varepsilon_0], \mathcal{N}(\mathcal{X}_r, r) \leq C^* \left(\frac{1}{r}\right)^{d^*}$. Our choice of normalizing r by $2L_0$ or ε_0 makes C^* more “intrinsic” (typically independent of L_0), as can be seen from the examples below.

$\mathcal{X}_r = \{x \in D : \|x - a\|_2 \leq \sqrt{r/\beta}\}$. Letting $\alpha = \sup_{x \in D} \|x - a\|_2$ and $L_0 = 2\alpha\beta$, note that f is L_0 -Lipschitz with respect to $\|\cdot\|_2$. Then, for all $r \in (0, \varepsilon_0]$,

$$\begin{aligned} \mathcal{N}\left(\mathcal{X}_r, \frac{r}{2L_0}\right) &\leq \mathcal{N}\left(B_{\|\cdot\|_2}(\sqrt{r/\beta}), \frac{r}{4\alpha\beta}\right) \\ &\leq \mathcal{M}\left(B_{\|\cdot\|_2}(\sqrt{r/\beta}), \frac{r}{8\alpha\beta}\right) && \text{(by (17))} \\ &\leq \left(1 + \frac{2\sqrt{r/\beta}}{r/(8\alpha\beta)}\right)^d = \left(1 + 16\alpha\sqrt{\frac{\beta}{r}}\right)^d && \text{(by (18))} \\ &\leq (1 + 8\sqrt{2})^d \left(\frac{\varepsilon_0}{r}\right)^{d/2}. && \text{(since } \varepsilon_0 = 2\alpha\beta \sup_{x,y \in D} \|x - y\|_2 \geq 2\alpha^2\beta) \end{aligned}$$

Thus Inequality (6) holds with $C^* = (1 + 8\sqrt{2})^d$ and $d^* = d/2$.

The next example shows that a unique value of d^* is sometimes insufficient to describe the shape of a function around a maximizer. Notably, our main regret bounds in Sections 3–4 will not depend on a single (worst-case) value of d^* , but on a suitable combination of such values at different scales r . This allows to give tighter bounds on the regret in cases like the following one.

Example 3 (Mixed Regime). Consider the Euclidean norm $\|\cdot\|_2$ on \mathbb{R}^d and the function $f: [-1, 1]^d \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/4 - \|x\|_2^2 & \text{if } \|x\|_2 \leq 1/2 \\ 1/2 - \|x\|_2 & \text{if } \|x\|_2 > 1/2 \end{cases}$$

(Figure 2, right). A direct verification shows that f is 1-Lipschitz with respect to $\|\cdot\|_2$ and attains its maximum at $x^* = 0$, with $f(x^*) = 1/4$. In this case, $\varepsilon_0 = \sup_{x,y} \|x - y\|_2 = 2\sqrt{d}$. Proceeding as in the previous two examples, we get, for all $r \in [1/4, 2\sqrt{d}]$,

$$\mathcal{N}\left(\mathcal{X}_r, \frac{r}{2L_0}\right) \leq 17^d,$$

but for all $r \in (0, 1/4)$,

$$\mathcal{N}\left(\mathcal{X}_r, \frac{r}{2L_0}\right) \leq \left(\frac{17}{2\sqrt{2}d^{1/4}}\right)^d \left(\frac{\varepsilon_0}{r}\right)^{d/2}.$$

Therefore, Inequality (6) holds with $d^* = 0$ for large values of r (linear regime) or with $d^* = d/2$ for small values of r (quadratic regime). Many different examples can be designed this way.

3 Deterministic perturbations.

In this section we provide a new regret analysis of the Piyavskii–Shubert algorithm [Piyavskii, 1972, Shubert, 1972] for the *deterministic setting*, i.e., when values of f are observed up to deterministic perturbations with absolute value bounded by a known constant. These perturbations can be chosen arbitrarily and even adaptively to the learner’s algorithm. In Section 4 we will study the *stochastic setting*, in which values of f are observed up to a subgaussian noise.

We consider two variants of the problem in the deterministic setting. When an overall budget n on the number of evaluations of f is fixed in advance, we analyze Algorithm 1 (Section 3.1), which makes n queries to f before returning a near-optimal point. When an accuracy level ε is imposed instead, we study Algorithm 2 (Section 3.2), which stops automatically and guarantees an ε -optimal solution after stopping. In the first case, we upper bound the number n of iterations required to reach any given precision $\varepsilon > 0$

(Theorem 1 and Corollary 1). This leads to an upper bound on the regret r_n as a function of the number n of queries to f (Corollary 2). In the second case, we upper bound the number n' of evaluations of f after which the algorithm automatically stops (Theorem 2 and Corollary 3). This second result might be more useful in practice, since the algorithm does not require the knowledge of n to guarantee an ε -optimal solution (see discussion at the beginning of Section 3.2).

3.1 Budget on queries.

In this section we derive new bounds for the simple regret r_n of the Piyavskii–Shubert algorithm. Our main result is a general bound on the sample complexity (Theorem 1), from which we derive a bound on r_n as a function of the number n of queries to f (Corollary 2). Formally, the algorithm applies to the following online optimization protocol.

A function $f: D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ that attains its maximum at some point $x^* \in D$ is fixed in advance. Only the compact domain D of the function, an upper bound L_1 on the Lipschitz constant L_0 (Assumption 1), an upper bound $\alpha \geq 0$ on the absolute value of the perturbations (see below), and the total number of queries $n \in \mathbb{N}^*$ are known to the learner in advance.

For each $k = 1, \dots, n$:

1. the learner chooses a point $x_k \in D$,
2. the environment picks a deterministic perturbation

$$\xi_k \in [-\alpha, \alpha], \quad (7)$$

depending on f and all points x_s chosen by the learner up to and including the current one x_k ,

3. the value $f(x_k) + \xi_k$ is revealed to the learner.

The learner then outputs a prediction x_n^* with the goal of minimizing the simple regret

$$r_n := f(x^*) - f(x_n^*). \quad (8)$$

In this section we extend the simpler original version of the Piyavskii–Shubert algorithm to our setting with perturbations. The behavior of the algorithm is illustrated in Figure 3. The Piyavskii–Shubert algorithm (Algorithm 1) works by maintaining a proxy \hat{f}_k of the objective function f that upper bounds f at the maximizer x^* (in case f is globally Lipschitz, we even have $\hat{f}_k(x) \geq f(x)$ for all $x \in D$). The next point x_{k+1} is chosen as an approximate maximizer of \hat{f}_k . The intuition is that such a point x_{k+1} may either correspond to an approximate maximizer of f (which reduces the regret immediately) or to a largely unexplored suboptimal region (which helps reduce the regret in the future, see Lemma 2). Note the similarity with the “optimism in the face of uncertainty” principle in stochastic multiarmed bandits Lattimore and Szepesvári [2018]. There, a proxy for the real mean of an arm is given by an upper confidence bound, and arms can have high upper confidence bounds only if they are either actually good, or they belong to an under-explored set of arms.

In Algorithm 1, we pick x_{k+1} as an α -optimal point of \hat{f}_k . The algorithm could be defined slightly more generally by picking x_{k+1} as an η -optimal point of \hat{f}_k , where $\eta \geq 0$ is an additional parameter independent of the bound α on $|\xi_k|$. Our choice to lighten the notation is due to the fact that we would have $2\alpha + \eta$ instead of 3α in the regret bound, therefore picking $\eta = \alpha$ would be a good choice anyway. Note also that picking $\eta > 0$ would allow to run the Piyavskii–Shubert algorithm on noncompact sets D (x_{k+1} would still be well defined); the regret analysis would be exactly the same. We only state our results for D compact and $\eta = \alpha \geq 0$ to simplify the presentation.

3.1.1 Main results

We first upper bound the minimum number of iterations (i.e., the sample complexity) needed for the simple regret $r_n = f(x^*) - f(x_n^*)$ to fall below some threshold ε . The following theorem is proved in Section 3.1.2.

Algorithm 1 Piyavskii–Shubert algorithm (with known query budget n)

Inputs: Lipschitz constant L_1 , number of iterations n , perturbation scale $\alpha \geq 0$, initial guess $x_1 \in D$

for $k = 1, \dots, n$ **do**
 observe $y_k = f(x_k) + \xi_k$
 update $\widehat{f}_k(x) = \min_{i \in \{1, \dots, k\}} \{y_i + L_1 \|x_i - x\| + \alpha\}$
 pick $x_{k+1} \in D$ such that $\widehat{f}_k(x_{k+1}) \geq \sup_{x \in D} \widehat{f}_k(x) - \alpha$
end for
 compute $i_n^* = \arg \max_{i \in \{1, \dots, n\}} y_i$
return $x_n^* := x_{i_n^*}$

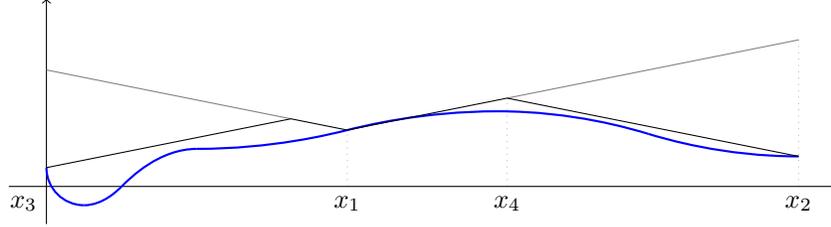


Figure 3: First three iterations of the Piyavskii–Shubert algorithm in dimension $d = 1$, with no perturbations ($\alpha = 0$). In blue the function f . In black the proxy function \widehat{f}_3 .

Theorem 1. Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Define ε_0 as in Equation (5) and let $\varepsilon \in (0, \varepsilon_0)$. Assume also that the perturbations in (7) are bounded by $\alpha \in [0, \varepsilon/6)$, and that the Piyavskii–Shubert algorithm (Algorithm 1) is run with inputs L_1 , $n \in \mathbb{N}^*$, α , and $x_1 \in D$. Define

$$\tilde{n} := \sum_{s=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon} \rceil - 1} \mathcal{N} \left(\mathcal{X}_{(\varepsilon_0 2^{-s-1}, \varepsilon_0 2^{-s}]}, \frac{\varepsilon_0 2^{-s-1} - 3\alpha}{L_1} \right) + 1. \quad (9)$$

If $n \geq \tilde{n}$, then the simple regret (8) of the Piyavskii–Shubert algorithm satisfies $r_n \leq \varepsilon + 2\alpha$.

We now express the previous result in terms of the constants C^* , d^* introduced in (6). The following corollary is proved in Appendix C. It shows that when $d^* = 0$, the number of evaluations needed to reach an ε -optimizer is at most logarithmic in $1/\varepsilon$, while for $d^* > 0$, at most order of $(1/\varepsilon)^{d^*}$ evaluations are needed. Note that our bound is continuous with respect to d^* , i.e., for smaller and smaller values of d^* the polynomial bound approaches the logarithmic bound.

Corollary 1. Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Define ε_0 as in Equation (5) and let $\varepsilon \in (0, \varepsilon_0)$. Assume also that the perturbations in (7) are bounded by $\alpha \in [0, \varepsilon/9]$, and that the Piyavskii–Shubert algorithm (Algorithm 1) is run with inputs L_1 , $n \in \mathbb{N}^*$, α , and $x_1 \in D$. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6). Define

$$\bar{n} := 1 + C^* \left(1 + 28 \frac{L_1}{L_0} \mathbf{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0} \right)^d \times \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + \log_2 \left(\frac{18}{7} \right) & \text{if } d^* = 0 \\ \frac{\left(\frac{18}{7} \right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \quad (10)$$

If $n \geq \bar{n}$, then the simple regret (8) of the Piyavskii–Shubert algorithm satisfies $r_n \leq \varepsilon$.

Remark 1 (Comparison with earlier work). The above bound is of the same order as those proved for variants of the Piyavskii–Shubert algorithm, e.g., by [Perevozchikov \[1990\]](#) and [Munos et al. \[2014, Corollary 3.3\]](#) for

piecewise-constant proxy functions \widehat{f}_k , or by [Malherbe and Vayatis \[2017\]](#) when x_{k+1} is picked uniformly at random in the set $\{x \in D : \widehat{f}_k(x) \geq \max_{i \leq k} f(x_i)\}$. We therefore fill a gap in the study of the original (and very natural) Piyavskii–Shubert algorithm. Note also that the general bound of [Theorem 1](#) is much stronger than the more classical bound in [Corollary 1](#):

- As a simple example, when f is a constant function, only $n = 1$ evaluation is sufficient to maximize it. In this case [Theorem 1](#) gives the correct sample complexity $\tilde{n} = 1$, while [Corollary 1](#) yields a much worse upper bound \bar{n} of the order of $(1/\varepsilon)^d$ (since $d^* = d$ for constant functions).
- More importantly, [Theorem 1](#) implies meaningful upper bounds when different values of d^* appear at different scales ε (as in [Example 3](#) in [Section 2.3](#)). This case could be handled similarly to [Corollary 1](#); the resulting bound would involve different rates depending on the value of ε . The same comment applies to [Corollaries 2, 3, and 4](#).

The proof in [Appendix C](#) shows that the multiplicative term $(1 + 28 \frac{L_1}{L_0} \mathbb{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0})^d$ appears applying [Lemma 6](#) ([Appendix A](#)) because of the imperfect information on L_0 and the perturbations in the evaluations of f . If $L_0 = L_1$ and $\alpha = 0$ the term disappears. This suggests a discontinuity in the hardness of the problem with respect to the information available to the learner. A very accurate, but not exact, estimate of L_0 or evaluation of f might increase the sample complexity by a term exponential in the dimension of the ambient space. Moreover, we note that the constant 28 inside the exponential term is the result of upper bounding α with its maximum allowed value $\varepsilon/9$. A smaller α would result in a smaller constant. As can be seen in the second-to-last inequality in the proof of [Corollary 1](#) ([Appendix C](#)), the constant decreases monotonically when α decreases, and attains its minimum 4 when $\alpha = 0$. The same observation applies to [Corollary 3](#).

We conclude the section by upper bounding the regret r_n of the Piyavskii–Shubert algorithm as a function of n , when the values of f are observed exactly. This is obtained directly by letting $\alpha = 0$ and solving [Equation \(10\)](#) for ε .

Corollary 2. *Assume that f is L_0 -Lipschitz around x^* ([Assumption 1](#)) and let $L_1 \geq L_0$. Assume that the values of f are observed exactly, and that the Piyavskii–Shubert algorithm ([Algorithm 1](#)) is run with inputs L_1 , $n \in \mathbb{N}^*$, $\alpha = 0$, and $x_1 \in D$. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying [\(6\)](#). Then the simple regret [\(8\)](#) of the Piyavskii–Shubert algorithm satisfies*

$$r_n \leq \begin{cases} c_1 e^{-c_3 n} & \text{if } d^* = 0 \\ c_2 n^{-1/d^*} & \text{if } d^* > 0 \end{cases}$$

where $c_1, c_2 \geq 0$ and $c_3 > 0$ are constants that depend on $d, C^*, d^*, L_0, L_1, \varepsilon_0, \alpha$ but not on n .

3.1.2 Proof of [Theorem 1](#).

In this section we present a formal proof of [Theorem 1](#). The main argument, which consists in manipulating packing numbers at different scales, was already present in [[Munos et al., 2014](#), Chapter 3] (see also [Perevozchikov \[1990\]](#)). These earlier works however crucially relied on a prespecified hierarchical partitioning of the domain D (in particular the points x_k there are chosen in cells with highest upper-confidence values). In our case, the choice of the x_k is not based on any a priori hierarchical partitioning, which might be a reason why a tight analysis of the Piyavskii–Shubert algorithm for general dimensions $d \geq 1$ has been missing so far.

We begin by proving an important property of the proxy function \widehat{f}_k . When f is globally Lipschitz and $\alpha = 0$, the lower bound $\widehat{f}_k(x) \geq f(x)$ is well known and true not only for $x = x^*$ but for all $x \in D$.

Lemma 1. *Assume that f is L_0 -Lipschitz around x^* ([Assumption 1](#)) and let $L_1 \geq L_0$. Assume also that the perturbations in [\(7\)](#) are bounded by $\alpha \geq 0$, and that the Piyavskii–Shubert algorithm ([Algorithm 1](#)) is run with inputs L_1 , $n \in \mathbb{N}^*$, α , and $x_1 \in D$.*

Then, for all $k \in \{1, \dots, n\}$, the proxy function $\widehat{f}_k(x) = \min_{1 \leq i \leq k} \{f(x_i) + L_1 \|x_i - x\| + (\xi_i + \alpha)\}$ is L_1 -Lipschitz and satisfies, for all $j \in \{k, \dots, n\}$,

$$\widehat{f}_k(x^*) \geq f(x^*) \quad \text{and} \quad \widehat{f}_j(x_k) \leq f(x_k) + 2\alpha .$$

Proof. Fix any $k \geq 1$ and $j \geq k$. The fact that \widehat{f}_k is L_1 -Lipschitz is straightforward. Moreover,

$$\widehat{f}_k(x^*) = \min_{i \in \{1, \dots, k\}} \{f(x_i) + L_1 \|x_i - x^*\| + (\xi_i + \alpha)\} . \quad (11)$$

Since f is L_0 -Lipschitz around x^* and $L_1 \geq L_0$, we get for all $i \in \{1, \dots, k\}$

$$f(x_i) \geq f(x^*) - L_0 \|x_i - x^*\| \geq f(x^*) - L_1 \|x_i - x^*\| .$$

Plugging $f(x_i) + L_1 \|x_i - x^*\| \geq f(x^*)$ and $\xi_i \geq -\alpha$ in Equation (11) gives $\widehat{f}_k(x^*) \geq f(x^*)$. Furthermore, the definition of $\widehat{f}_j(x_k)$ and $\xi_k \leq \alpha$ imply, since $j \geq k$,

$$\widehat{f}_j(x_k) = \min_{i \in \{1, \dots, j\}} \{f(x_i) + L_1 \|x_i - x_k\| + (\xi_i + \alpha)\} \leq f(x_k) + L_1 \|x_k - x_k\| + (\xi_k + \alpha) \leq f(x_k) + 2\alpha ,$$

which concludes the proof. \square

Before proving the main result, we first state a key lemma which shows that if the Piyavskii–Shubert algorithm observes f at a Δ -suboptimal point x_i , then the next query points x_j are all distant from x_i by at least roughly Δ/L_1 . In other words, the Piyavskii–Shubert algorithm does not explore too much in suboptimal regions.

Lemma 2. *Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Assume also that the perturbations in (7) are bounded by $\alpha \geq 0$, and that the Piyavskii–Shubert algorithm (Algorithm 1) is run with inputs L_1 , $n \in \mathbb{N}^*$, α , and $x_1 \in D$.*

Fix any $\Delta > 0$ and assume that there exists $i \in \{1, \dots, n-1\}$ such that the point x_i queried by the Piyavskii–Shubert algorithm during the i -th iteration satisfies $x_i \in \mathcal{X}_\Delta^c$. Then, for all $j > i$, the j -th queried point satisfies

$$\|x_j - x_i\| > \frac{\Delta - 3\alpha}{L_1} .$$

Proof. Assume that $x_i \in \mathcal{X}_\Delta^c$ for some $i \in \{1, \dots, n-1\}$ and let $j > i$. Then

$$\begin{aligned} \widehat{f}_{j-1}(x_j) &\geq \widehat{f}_{j-1}(x^*) - \alpha && (x_j \text{ was selected at iteration } j) \\ &\geq f(x^*) - \alpha && (\text{by Lemma 1}) \\ &> f(x_i) + \Delta - \alpha && (x_i \in \mathcal{X}_\Delta^c) \\ &\geq \widehat{f}_{j-1}(x_i) + \Delta - 3\alpha && (\text{by Lemma 1}) \end{aligned}$$

Since \widehat{f}_{j-1} is L_1 -Lipschitz (Lemma 1), we have $L_1 \|x_j - x_i\| \geq |\widehat{f}_{j-1}(x_j) - \widehat{f}_{j-1}(x_i)| > \Delta - 3\alpha$. \square

We can now prove Theorem 1, the main result of Section 3.1. To this end, we use a peeling technique in which the input space D is partitioned in terms of the output values of f .

Proof of Theorem 1. Let $m_\varepsilon := \lceil \log_2(\varepsilon_0 \varepsilon^{-1}) \rceil \geq 1$. We use a peeling technique and partition the set $\mathcal{X}_\varepsilon^c$ of ε -suboptimal points into multiple layers (recall Equation (4)). Note that

$$(\varepsilon_0 2^{-m_\varepsilon}, +\infty) = (\varepsilon_0 2^{-m_\varepsilon}, \varepsilon_0 2^{-m_\varepsilon + 1}] \cup \dots \cup (\varepsilon_0/2, \varepsilon_0] \cup (\varepsilon_0, +\infty) .$$

Moreover $\mathcal{X}_{\varepsilon_0 2^{-m_\varepsilon}} \subset \mathcal{X}_\varepsilon$ and $\mathcal{X}_{\varepsilon_0}^c = \emptyset$ (by definition of ε_0 in Equation (5)). Thus we have

$$\mathcal{X}_\varepsilon^c \subset \mathcal{X}_{\varepsilon_0 2^{-m_\varepsilon}}^c \subset \left(\bigcup_{s=1}^{m_\varepsilon} A_s \right)$$

with $A_s := \mathcal{X}_{(\varepsilon_0 2^{-s}, \varepsilon_0 2^{-s+1}]}$ for all $1 \leq s \leq m_\varepsilon$. We remark that $(A_s)_{1 \leq s \leq m_\varepsilon}$ is a collection of disjoint sets that covers $\mathcal{X}_\varepsilon^c$. Therefore, for any $1 \leq k \leq n$, if $x_k \in \mathcal{X}_\varepsilon^c$, there exists a unique $s \in \{1, \dots, m_\varepsilon\}$ such that $x_k \in A_s$. For any $s \in \{1, \dots, m_\varepsilon\}$, by Lemma 2, the maximum number of rounds k at which x_k can be chosen in $A_s = \mathcal{X}_{\varepsilon_0 2^{-s}}^c \cap \mathcal{X}_{\varepsilon_0 2^{-s+1}}$ is upper bounded by $\mathcal{N}\left(A_s, \frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1}\right)$. Then

$$\left| \{k \in \{1, \dots, n\} : x_k \in \mathcal{X}_\varepsilon^c\} \right| \leq \sum_{s=1}^{m_\varepsilon} \left| \{k \in \{1, \dots, n\} : x_k \in A_s\} \right| \leq \sum_{s=1}^{m_\varepsilon} \mathcal{N}\left(A_s, \frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1}\right).$$

Therefore, if

$$n \geq \sum_{s=1}^{m_\varepsilon} \mathcal{N}\left(A_s, \frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1}\right) + 1$$

then there exists $k \in \{1, \dots, n\}$ such that $x_k \in \mathcal{X}_\varepsilon$, which implies (by definition of x_n^*) that

$$f(x_n^*) \geq f(x_k) + \xi_k - \xi_{i_n^*} \geq f(x^*) - \varepsilon - 2\alpha.$$

□

3.2 Automatic stopping.

In the previous section we derived bounds \tilde{n} and \bar{n} on the number n of iterations after which the Piyavskii–Shubert algorithm guarantees a regret r_n smaller than ε . However, these bounds can rarely be computed in practice, since they involve unknown quantities such as packing numbers or the constants C^* and d^* . We now study a version of the Piyavskii–Shubert algorithm that stops automatically outputting an ε -optimal solution after stopping, without requiring these unknown quantities (Algorithm 2). The only prior knowledge on the shape of f is an upper bound L_1 on the Lipschitz constant L_0 (Assumption 1). We derive an upper bound on the number n of queries to f needed to automatically stop; see Theorem 2 and Corollary 3 below.

Formally, the algorithm applies to the following online optimization protocol.

A function $f: D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ that attains its maximum at some point $x^* \in D$ is fixed in advance. Only the compact domain D of the function, an upper bound L_1 on the Lipschitz constant L_0 (Assumption 1), an upper bound $\alpha \geq 0$ on the absolute value of the perturbations (see below), and a prescribed accuracy $\varepsilon > 0$ are known to the learner in advance.

For each $k = 1, 2, \dots$:

1. the learner chooses a point $x_k \in D$,
2. the environment picks a deterministic perturbation

$$\xi_k \in [-\alpha, \alpha] \tag{12}$$

depending on f and all points x_s chosen by the learner up to and including the current one x_k ,

3. the value $y_k := f(x_k) + \xi_k$ is revealed to the learner.

At the end of each iteration n the learner can decide to interrupt the process and output a prediction x_n^* with the goal of returning an ε -optimal point. In other words, the regret $r_n := f(x^*) - f(x_n^*)$ at time n has to satisfy

$$r_n \leq \varepsilon.$$

Algorithm 2 ε -Piyavskii–Shubert algorithm (with known accuracy ε)

inputs: Lipschitz constant L_1 , accuracy $\varepsilon > 0$, perturbation scale $\alpha \geq 0$, initial guess $x_1 \in D$

initialization: $k \leftarrow 0$ and $\widehat{f}_0^* - f_0^* \leftarrow \varepsilon + 2\alpha + 1$

while $\widehat{f}_k^* - f_k^* > \varepsilon$ **do**
 update $k \leftarrow k + 1$
 observe $y_k \leftarrow f(x_k) + \xi_k$
 update $\widehat{f}_k(x) \leftarrow \min_{i \in \{1, \dots, k\}} \{y_i + L_1 \|x_i - x\| + \alpha\}$ for all $x \in D$
 pick $x_{k+1} \in D$ such that $\widehat{f}_k(x_{k+1}) \geq \sup_{x \in D} \widehat{f}_k(x) - \alpha$
 update $\widehat{f}_k^* \leftarrow \widehat{f}_k(x_{k+1})$ and $f_k^* \leftarrow \max_{i \in \{1, \dots, k\}} y_i$
end while
compute $i_k^* = \arg \max_{i \in \{1, \dots, k\}} y_i$
return $x_k^* \leftarrow x_{i_k^*}$

The ε -Piyavskii–Shubert algorithm (Algorithm 2, presented below) applies to our setting with perturbations. It behaves similarly to Algorithm 1. The only difference is that it stops as soon as the quantity $\widehat{f}_k^* - f_k^* \approx \max \widehat{f}_k - \max_{i \leq k} y_i$ (which controls the regret by Lemma 1) falls below ε .

The main result of the section (Theorem 2) shows that the number of iterations that the ε -Piyavskii–Shubert algorithm performs before stopping (and returning an ε -optimal point) can be controlled by packing numbers, similarly to Theorem 1.

The ε -Piyavskii–Shubert algorithm was extensively studied in dimension $d = 1$ when $\alpha = 0$. For example, Hansen et al. [1991] derived a tight upper bound on the number of evaluations of f needed before stopping, in terms of an integral involving the increments $f(x^*) - f(x)$. In Appendix D we discuss how to derive a regret bound similar to that of Corollary 3 from their upper bound. Our result could thus be interpreted as a generalization of this result to an arbitrary dimension $d \geq 1$.

We now state the main result of this section, which we prove in Section 3.2.1.

Theorem 2. *Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Define ε_0 as in Equation (5) and let $\varepsilon \in (0, \varepsilon_0)$. Assume also that the perturbations in (7) are bounded by $\alpha \in [0, \varepsilon/12]$, and that the ε -Piyavskii–Shubert algorithm (Algorithm 2) is run with inputs $L_1, \varepsilon, \alpha, x_1 \in D$. Then, the ε -Piyavskii–Shubert algorithm stops after $n \leq \widetilde{n}'$ iterations,⁴ where \widetilde{n}' is defined by*

$$\widetilde{n}' := \mathcal{N}\left(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon - 3\alpha}{L_1}\right) + \sum_{s=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon} \rceil} \mathcal{N}\left(\mathcal{X}_{(\varepsilon_0 2^{-s-1}, \varepsilon_0 2^{-s}]}, \frac{\varepsilon_0 2^{-s-1} - 3\alpha}{L_1}\right) \quad (13)$$

and its simple regret (8) satisfies $r_n \leq \varepsilon + 2\alpha$.

Note that \widetilde{n}' (Equation (13)) is very similar to \widetilde{n} (Equation (9)). The main difference is that the $+1$ term is replaced by $\mathcal{N}\left(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon - 3\alpha}{L_1}\right)$, which reveals a subtle but crucial difference between the Piyavskii–Shubert algorithm (Algorithm 1) and the ε -Piyavskii–Shubert algorithm (Algorithm 2). Consider a constant function f . Since all suboptimal layers are empty (because all points in D are optimal), Equation (9) reflects the fact that as little as $\widetilde{n} = 1$ evaluation is needed in order for Algorithm 1 to guarantee a near-optimal solution, since the first prediction is necessarily already optimal. However, with the same constant objective, the bound \widetilde{n}' on the number of iterations of Algorithm 2 is of order $\mathcal{N}\left(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon - 3\alpha}{L_1}\right) \gtrsim \left(\frac{1}{\varepsilon}\right)^d$, which is as big as it gets! This huge gap, which was already noticed by Hansen et al. [1991] in dimension $d = 1$, is unavoidable and due to the fact that Algorithm 2 is asked to complete a task that is significantly harder than that of Algorithm 1. The Piyavskii–Shubert algorithm simply runs for a prescribed amount of iterations, during which it happens to make good predictions if the objective is flat. In contrast, the ε -Piyavskii–Shubert

⁴This means that f is evaluated at most \widetilde{n}' times in total.

algorithm has to become aware that its predictions are accurate after they became so. If the function is very flat, a full grid search has to be performed in order to make sure that the proxy function at its highest value is close enough to the value of the objective. Note, however, that the term $\mathcal{N}(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon-3\alpha}{L_1})$ is not always this large. E.g., it is comparable to the rest of the bound if the objective function is $x \mapsto 1 - \|x\|$, as shown in Example 1.

As we did in the previous section, we now express the previous result in terms of the constants C^* , d^* introduced in (6). The following corollary is proved in Appendix C. It shows that the number of iterations before stopping and returning an ε -optimal point is logarithmic in $1/\varepsilon$ if $d^* = 0$, otherwise scales as $(1/\varepsilon)^{d^*}$. Note that as in Corollary 1, our bound is continuous in d^* .

Corollary 3. *Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Define ε_0 as in Equation (5) and let $\varepsilon \in (0, \varepsilon_0)$. Assume also that the perturbations in (7) are bounded by $\alpha \in [0, \varepsilon/15]$, and that the ε -Piyavskii–Shubert algorithm (Algorithm 2) is run with inputs $L_1, (13/15)\varepsilon, \alpha, x_1 \in D$. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6). Then, the ε -Piyavskii–Shubert algorithm stops after $n \leq \bar{n}'$ iterations, where \bar{n}' is defined by*

$$\bar{n}' := C^* \left(1 + 52 \frac{L_1}{L_0} \mathbb{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0} \right)^d \times \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + \log_2 \left(\frac{120}{13} \right) & \text{if } d^* = 0 \\ \frac{(4^{d^*} + 2^{d^*} - 1) \left(\frac{15}{13} \right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases}$$

and its simple regret (8) satisfies $r_n \leq \varepsilon$.

3.2.1 Proof of Theorem 2.

In this section we present a formal proof of Theorem 2. Before proving the main result, we first state a useful lemma analogous to Lemma 2. It shows that if the ε -Piyavskii–Shubert algorithm observes f at a suboptimal point x_i , then the next query points x_j are all distant from x_i and in any case it never queries points that are too close to each other. In other words, the ε -Piyavskii–Shubert algorithm does not explore too much in suboptimal regions and it does not waste evaluations on neighboring points.

Lemma 3. *Assume that f is L_0 -Lipschitz around x^* (Assumption 1) and let $L_1 \geq L_0$. Define ε_0 as in Equation (5) and let $\varepsilon \in (0, \varepsilon_0)$. Assume also that the perturbations in (7) are bounded by $\alpha \geq 0$, and that the ε -Piyavskii–Shubert algorithm (Algorithm 2) is run with inputs $L_1, \varepsilon, \alpha, x_1 \in D$ and stops after n iterations.⁵ Then, for all distinct $i, j \in \{1, \dots, n\}$,*

$$\|x_i - x_j\| > \frac{\varepsilon - 3\alpha}{L_1}.$$

Moreover, if $i < j$ and there exists $0 \leq k \leq m_\varepsilon := \lceil \log_2 \frac{\varepsilon_0}{\varepsilon} \rceil$ such that $x_i \in \mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}$, then

$$\|x_i - x_j\| > \frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}.$$

Proof. Proof of Lemma 3. Without loss of generality, assume $i < j$. Note that

$$\begin{aligned} \widehat{f}_{j-1}(x_j) &= \widehat{f}_{j-1}^* && \text{(by definition of } \widehat{f}_{j-1}^*) \\ &> f_{j-1}^* + \varepsilon && \text{(the algorithm ran for at least } j \text{ iterations)} \\ &\geq f(x_i) + \varepsilon - \alpha && \text{(by definition of } f^* \text{ and } \xi_i \geq -\alpha) \\ &\geq \widehat{f}_{j-1}(x_i) + \varepsilon - 3\alpha. && \text{(by Lemma 1)} \end{aligned}$$

⁵This means that $\widehat{f}_k^* - f_k^* > \varepsilon$ for all $k \in \{1, \dots, n-1\}$ and that $\widehat{f}_n^* - f_n^* \leq \varepsilon$.

Since \widehat{f}_{j-1} is L_1 -Lipschitz (Lemma 1), we have $L_1\|x_i - x_j\| \geq |\widehat{f}_{j-1}(x_j) - \widehat{f}_{j-1}(x_i)| > \varepsilon - 3\alpha$, which gives the first inequality. As for the second inequality, it follows from Lemma 2 applied with $\Delta = \varepsilon_0 2^{-k-1}$ (note that $x_i \in \mathcal{X}_\Delta^c$ and that Algorithms 1 and 2 coincide until iteration n). \square

We can now prove Theorem 2. The proof proceeds similarly to that of Theorem 1: we rely on a peeling technique in which the input space D is partitioned in terms of the output values of f .

Proof of Theorem 2. Fix any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in [0, \varepsilon/12)$. Let $\mathcal{T} = \{1, \dots, n\}$ be the set of indices of all iterations performed by the ε -Piyavskii–Shubert algorithm (Algorithm 2) before stopping, $m_\varepsilon := \lceil \log_2 \frac{\varepsilon_0}{\varepsilon} \rceil \geq 1$, and for all $k \in \{0, \dots, m_\varepsilon\}$,

$$\begin{aligned} \mathcal{T}_k &:= \{t \in \mathcal{T} : x_t \in \mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}\}, \\ \mathcal{T}_{m_\varepsilon+1} &:= \{t \in \mathcal{T} : x_t \in \mathcal{X}_{\varepsilon/2}\}. \end{aligned}$$

Lemma 3 implies, for all $k \in \{0, \dots, m_\varepsilon\}$,

$$\begin{aligned} |\mathcal{T}_k| &\leq \mathcal{N}\left(\mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}, \frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}\right), \\ |\mathcal{T}_{m_\varepsilon+1}| &\leq \mathcal{N}\left(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon - 3\alpha}{L_1}\right). \end{aligned}$$

Since $\mathcal{X}_{(\varepsilon_0, +\infty)} = \emptyset$ (by definition of ε_0 in Equation (5)) and $[0, \varepsilon_0] = [0, \varepsilon/2] \cup (\varepsilon_0 2^{-m_\varepsilon-1}, \varepsilon_0 2^{-m_\varepsilon}) \cup \dots \cup (\varepsilon_0 2^{-1}, \varepsilon_0]$, we have $D = \mathcal{X}_{\varepsilon/2} \cup (\bigcup_{s=0}^{m_\varepsilon} \mathcal{X}_{(\varepsilon_0 2^{-s-1}, \varepsilon_0 2^{-s})})$. Hence, for all iterations $i \in \mathcal{T}$ there exists $k \in \{0, \dots, m_\varepsilon + 1\}$ such that $i \in \mathcal{T}_k$, which in turn gives

$$|\mathcal{T}| \leq \sum_{k=0}^{m_\varepsilon+1} |\mathcal{T}_k| \leq \mathcal{N}\left(\mathcal{X}_{\varepsilon/2}, \frac{\varepsilon - 3\alpha}{L_1}\right) + \sum_{k=0}^{m_\varepsilon} \mathcal{N}\left(\mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}, \frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}\right),$$

which shows that $n \leq \tilde{n}'$. To conclude, note that after the last iteration n performed by the algorithm before stopping, we have

$$r_n = f(x^*) - f(x_n^*) \leq \widehat{f}_n(x^*) - f(x_n^*) \leq \widehat{f}_n^* - f_n^* + 2\alpha \leq \varepsilon + 2\alpha,$$

where the first inequality follows by Lemma 1, the second by definition of \widehat{f}_n^* and f_n^* , and the last one by the fact that the condition in the while loop is false after the last iteration. \square

4 Stochastic perturbations.

In this section we show how to apply the results proven in Section 3 to an algorithm designed for the stochastic setting (Algorithm 3). More precisely, we assume that the values of f are observed up to subgaussian noise. We then use a mini-batch sampling technique to produce tight estimates of the value of f at each iteration of the algorithm.

Assumption 2. Let $\xi := (\xi_{k,i})_{k,i \in \mathbb{N}^*}$ be a sequence of independent random variables. We assume that there exists $\sigma_0 > 0$ such that all random variables in the sequence are σ_0 -subgaussian, i.e., for all $i, k \in \mathbb{N}^*$ and for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}(e^{\lambda \xi_{k,i}}) \leq e^{\lambda^2 \sigma_0^2 / 2}.$$

(In particular, the random variables $\xi_{k,i}$ are centered.) We say in this case that ξ is a σ_0 -subgaussian noise sequence. Moreover, we assume that an upper bound σ_1 on σ_0 is known to the learner.

We study directly the version of the problem in which an accuracy level ε is given and the algorithm stops automatically guaranteeing an ε -optimal solution. Formally, the algorithm applies to the following online optimization protocol.

A function $f: D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ that attains its maximum at some point $x^* \in D$ and a σ_0 -subgaussian noise sequence ξ (Assumption 2) are fixed in advance. Only the compact domain D of the function, an upper bound L_1 on the Lipschitz constant L_0 (Assumption 1), an upper bound σ_1 on σ_0 , and a prescribed accuracy $\varepsilon > 0$ are known to the learner in advance.

For each $k = 1, 2, \dots$:

1. the learner queries a point $x_k \in D$ for a finite amount of times,
2. for each query i of x_k , the value $f(x_k) + \xi_{k,i}$ is revealed to the learner.

At the end of each iteration n the learner can decide to interrupt the process and output a prediction x_n^* with the goal of returning an ε -optimal point. In other words, the regret $r_n := f(x^*) - f(x_n^*)$ at time n has to satisfy

$$r_n \leq \varepsilon. \quad (14)$$

The stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3) applies to our setting with stochastic perturbations. It behaves similarly to Algorithm 2. We maintain a proxy \widehat{f}_k of the objective function f that upper bounds f at the maximizer x^* with high probability. The proxy is updated during each iteration by querying the same value multiple times in order to build an estimate of the real value $f(x_k)$ of the function at that point up to a small perturbation $\bar{\xi}_k$. It then proceeds like Algorithm 2 with the guarantee that the perturbation will be controlled with high probability. All remarks made for Algorithms 1 and 2 still apply with high probability.

Algorithm 3 Stochastic ε -Piyavskii–Shubert algorithm

inputs: Lipschitz constant L_1 , subgaussian constant $\sigma_1 > 0$, accuracy $\varepsilon > 0$, confidence $1 - \delta \in (0, 1)$, initial guess $x_1 \in D$

initialization: $\varepsilon' \leftarrow (13/15)\varepsilon$, $\alpha \leftarrow \varepsilon/15$, $k \leftarrow 0$ and $\widehat{f}_0^* - f_0^* \leftarrow \varepsilon' + 1$

while $\widehat{f}_k^* - f_k^* > \varepsilon'$ **do**

update $k \leftarrow k + 1$ and $m_k = \left\lceil \frac{2\sigma_1^2}{\alpha^2} \ln \left(\frac{2k(k+1)}{\delta} \right) \right\rceil$

observe $f(x_k) + \xi_{k,1}, \dots, f(x_k) + \xi_{k,m_k}$ and compute the average $y_k \leftarrow f(x_k) + \bar{\xi}_k$, where

$$\bar{\xi}_k = \frac{1}{m_k} \sum_{i=1}^{m_k} \xi_{k,i}$$

update $\widehat{f}_k(x) \leftarrow \min_{i \in \{1, \dots, k\}} \{y_i + L_1 \|x_i - x\| + \alpha\}$ for all $x \in D$

pick $i_k^* \in \arg \max_{i \in \{1, \dots, k\}} y_i$ and $x_{k+1} \in D$ such that $\widehat{f}_k(x_{k+1}) \geq \sup_{x \in D} \widehat{f}_k(x) - \alpha$

update $f_k^* \leftarrow f(x_{i_k^*}) + \bar{\xi}_{i_k^*} = \max_{i \in \{1, \dots, k\}} y_i$ and $\widehat{f}_k^* \leftarrow \widehat{f}_k(x_{k+1}) \geq \sup \widehat{f}_k - \alpha$

end while

return $x_k^* \leftarrow x_{i_k^*}$

Before proving the main result of the section, we state a very standard concentration inequality for the empirical average of subgaussian random variables (see, e.g., [Boucheron et al., 2013, Section 2.3]).

Lemma 4. *Let ξ be a σ_0 -subgaussian noise sequence (Assumption 2), and $\sigma_1 \geq \sigma_0$. Then, for all $k, m \in \mathbb{N}^*$ and all $\alpha \geq 0$,*

$$\mathbb{P} \left(\left| \frac{1}{m} \sum_{i=1}^m \xi_{k,i} \right| \geq \alpha \right) \leq 2e^{-m\alpha^2/(2\sigma_0^2)} \leq 2e^{-m\alpha^2/(2\sigma_1^2)}. \quad (15)$$

Similarly to the previous section, we upper bound the number of evaluations of f after which the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3) automatically stops outputting an ε -optimal point. The proof of the next theorem follows by applying Theorem 2 (Section 3.2.1) on a “nice” event, the probability of which is controlled through multiple calls to Lemma 4.

Theorem 3. *Assume that f is L_0 -Lipschitz around x^* (Assumption 1), the noise ξ is σ_0 -subgaussian (Assumption 2), and let $L_1 \geq L_0$ and $\sigma_1 \geq \sigma_0$. Let also ε_0 be defined by Equation (5), $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, 1)$, $x_1 \in D$, and assume that the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3) is run with inputs $L_1, \sigma_1, \varepsilon, 1 - \delta, x_1$. Then, with probability at least $1 - \delta$, the stochastic ε -Piyavskii–Shubert algorithm stops after performing $N \leq \tilde{N}'$ evaluations of f , where*

$$\begin{aligned} \tilde{N}' &:= 900 \frac{\sigma_1^2}{\varepsilon^2} (\tilde{n}' + 1) \ln \left(\frac{4(\tilde{n}' + 1)}{\delta} \right) + \tilde{n}', \\ \tilde{n}' &:= \mathcal{N} \left(\mathcal{X}_{\frac{13}{30}\varepsilon}, \frac{2}{3} \frac{\varepsilon}{L_1} \right) + \sum_{k=0}^{\lceil \log_2(\frac{15}{13} \frac{\varepsilon_0}{\varepsilon}) \rceil} \mathcal{N} \left(\mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}, \frac{\varepsilon_0 2^{-k-1} - \varepsilon/5}{L_1} \right), \end{aligned}$$

and its simple regret (1) satisfies $r_N \leq \varepsilon$.

Note again the presence of the first term in the definition of \tilde{n}' . As highlighted before (see discussion after Theorem 2), this additional term is due to the price of automatic stopping and cannot be avoided.

Proof. Let $\varepsilon' = (13/15)\varepsilon$ and $\alpha = \varepsilon/15$ as in the initialization of the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3). Consider the “bad” event $E := \{\exists k \in \mathbb{N}^* : |\bar{\xi}_k| > \alpha\}$, where we recall that $\bar{\xi}_k = \frac{1}{m_k} \sum_{i=1}^{m_k} \xi_{k,i}$. By applying a union bound, Inequality (15), and the definition of $m_k = \lceil (2\sigma_1^2/\alpha^2) \ln(2k(k+1)/\delta) \rceil$, we get

$$\mathbb{P}(E) \leq \sum_{k=1}^{+\infty} \mathbb{P} \left(\left| \frac{1}{m_k} \sum_{i=1}^{m_k} \xi_{k,i} \right| > \alpha \right) \leq \sum_{k=1}^{+\infty} \frac{\delta}{k(k+1)} = \delta.$$

For each outcome belonging to the complement of E , during each iteration k the value $f(x_k)$ is observed up to a perturbation $\bar{\xi}_k$ with $|\bar{\xi}_k| \leq \alpha$. With probability at least $1 - \delta$ we can therefore apply Theorem 2 (with $\varepsilon' = (13/15)\varepsilon$ instead of ε) to upper bound the number of iterations before stopping by \tilde{n}' , which in turn gives that the total number of evaluations of f before stopping is at most

$$\sum_{i=1}^{\tilde{n}'} m_i \leq \sum_{i=1}^{\tilde{n}'} \left(\frac{4\sigma_1^2}{\alpha^2} \ln(i) + \frac{2\sigma_1^2}{\alpha^2} \ln \left(\frac{4}{\delta} \right) + 1 \right) \leq \frac{4\sigma_1^2}{\alpha^2} (\tilde{n}' + 1) \ln \left(\frac{4(\tilde{n}' + 1)}{\delta} \right) + \tilde{n}', \quad (16)$$

where in the last inequalities we used $2k(k+1) \leq 4k^2$ and

$$\sum_{k=1}^n \ln(k) \leq \int_1^{n+1} \ln(x) dx = (n+1) \ln(n+1) - n.$$

We conclude the proof by substituting the value of $\alpha = \varepsilon/15$ in (16). \square

Note that the large leading constant 900 depends on our choice of letting $\varepsilon' = (13/15)\varepsilon$ and $\alpha = \varepsilon/15$ in the initialization of the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3). Inequality (16) shows that this term appears because of the multiplicative constant $4/\alpha^2$. In principle, one could reduce it by picking a larger α . This, however, would result in larger packing numbers which could in turn increase the bound even further. The same observation applies to Corollary 4, in which a bigger α would translate into a constant larger than 52 in $(1 + 52 \frac{L_1}{L_0})^d$.

Combining the proofs of Corollary 3 and Theorem 3 gives immediately the following upper bound on the number of evaluations of f after which the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3) automatically stops.

Corollary 4. Assume that f is L_0 -Lipschitz around x^* (Assumption 1), the noise ξ is σ_0 -subgaussian (Assumption 2), and let $L_1 \geq L_0$ and $\sigma_1 \geq \sigma_0$. Let also ε_0 be defined by Equation (5), $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, 1)$, $x_1 \in D$, and assume that the stochastic ε -Piyavskii–Shubert algorithm (Algorithm 3) is run with inputs $L_1, \sigma_1, \varepsilon, 1 - \delta, x_1$. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6). Then, with probability at least $1 - \delta$, the stochastic ε -Piyavskii–Shubert algorithm stops after performing $N \leq \overline{N}'$ evaluations of f , where

$$\begin{aligned} \overline{N}' &:= 900 \frac{\sigma_1^2}{\varepsilon^2} (\overline{n}' + 1) \ln \left(\frac{4(\overline{n}' + 1)}{\delta} \right) + \overline{n}' , \\ \overline{n}' &:= C^* \left(1 + 52 \frac{L_1}{L_0} \mathbb{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0} \right)^d \times \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + \log_2 \left(\frac{120}{13} \right) & \text{if } d^* = 0 \\ \frac{(4^{d^*} + 2^{d^*} - 1) \left(\frac{15}{13} \right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \end{aligned}$$

and its simple regret (1) satisfies $r_N \leq \varepsilon$.

The above high-probability bound is of the form $n \lesssim (1/\varepsilon)^{d^*+2}$, where the \lesssim notation hides logarithmic terms and problem-dependent constants. As mentioned in the introduction, if we switch the roles of n and ε , it corresponds to a regret bound roughly of the form:

$$r_n \lesssim n^{-1/(d^*+2)} ,$$

which is known to be worst-case optimal as a consequence of the minimax lower bound of [Bubeck et al. \[2011a, Theorem 13\]](#) (this lower bound is stated for the cumulative regret, but it could be adapted for the simple regret).

A Useful inequalities about packing and covering numbers.

Covering numbers and packing numbers (see Section 2.2) are closely related. In particular, the following well-known inequalities hold—see, e.g., [[Wainwright, 2019](#), Lemma 5.5 and Example 5.8, with permuted notation of \mathcal{M} and \mathcal{N}].⁶

Lemma 5. For any bounded set $A \subset \mathbb{R}^d$ and any real number $r > 0$,

$$\mathcal{N}(A, 2r) \leq \mathcal{M}(A, r) \leq \mathcal{N}(A, r) . \quad (17)$$

Furthermore, for all $\delta > 0$ and all $r > 0$,

$$\mathcal{M}(B_{\|\cdot\|}(\delta), r) \leq \left(1 + 2 \frac{\delta}{r} \mathbb{1}_{r < \delta} \right)^d . \quad (18)$$

We now state a known lemma about packing numbers at different scales. This result will be useful to control how an overestimation L_1 of the Lipschitz constant L_0 impacts the regret of our algorithms.

Lemma 6. For any bounded set $A \subset \mathbb{R}^d$ and any real numbers $r_1, r_2 > 0$,

$$\mathcal{N}(A, r_1) \leq \left(1 + 4 \frac{r_2}{r_1} \mathbb{1}_{r_2 > r_1} \right)^d \times \mathcal{N}(A, r_2) .$$

⁶The definition of r -covering number of a subset A of \mathbb{R}^d implied by [[Wainwright, 2019](#), Definition 5.1] is slightly stronger than the one used in our paper, because elements x_1, \dots, x_N of r -covers belong to A rather than just \mathbb{R}^d . Even if we do not need it for our analysis, Inequality (18) holds also in this stronger sense.

Proof. We can assume without loss of generality that A is nonempty and that $r_1 < r_2$. Then,

$$\begin{aligned}
\mathcal{N}(A, r_1) &\leq \mathcal{M}(A, r_1/2) && \text{(by (17))} \\
&\leq \mathcal{M}(A, r_2) \times \mathcal{M}(B_{\|\cdot\|}(r_2), r_1/2) && \text{(see below)} \\
&\leq \mathcal{N}(A, r_2) \times \mathcal{M}(B_{\|\cdot\|}(r_2), r_1/2) && \text{(by (17))} \\
&\leq \mathcal{N}(A, r_2) \times \left(1 + \frac{4r_2}{r_1}\right)^d. && \text{(by (18))}
\end{aligned}$$

The second inequality is obtained by building the $r_1/2$ -covering of A in two steps. First, we cover A with balls of radius r_2 . Second, we cover each ball of the first cover with balls of radius $r_1/2$. \square

B Simple bound on d^* , near-optimality dimension.

The next well-known lemma shows that Inequality (6) is always true with $C^* = 9^d$ and $d^* = d$ (though a significantly smaller value of d^* may exist, see Section 2.3). We recall that throughout the paper $D \subset \mathbb{R}^d$ is compact.

Lemma 7. *Assume that $f: D \rightarrow \mathbb{R}$ is L_0 -Lipschitz around a maximizer x^* (Assumption 1) and set $\varepsilon_0 = L_0 \sup_{x,y \in D} \|x - y\|$. Then, for all $\varepsilon \in (0, \varepsilon_0]$,*

$$\mathcal{N}\left(\mathcal{X}_\varepsilon, \frac{\varepsilon}{2L_0}\right) \leq 9^d \left(\frac{\varepsilon_0}{\varepsilon}\right)^d.$$

Proof. Let $\varepsilon \in (0, \varepsilon_0]$. Fix any $a \in D$ and set $B = B_{\|\cdot\|}(\sup_{y \in D} \|y - a\|)$. Then $\mathcal{X}_\varepsilon \subset D \subset a + B$, which in turn yields

$$\mathcal{N}\left(\mathcal{X}_\varepsilon, \frac{\varepsilon}{2L_0}\right) \leq \mathcal{N}\left(a + B, \frac{\varepsilon}{2L_0}\right) \leq \mathcal{M}\left(B, \frac{\varepsilon}{4L_0}\right) \leq \left(1 + \frac{8 \sup_{y \in D} \|y - a\| L_0}{\varepsilon}\right)^d \leq 9^d \left(\frac{\varepsilon_0}{\varepsilon}\right)^d,$$

where the second inequality follows by translation invariance and by (17), the third by (18), and the last one by definition of ε_0 . \square

When the minimum

$$d^*(L_0) := \min \left\{ d' \in \mathbb{R}^+ : \exists C > 0, \forall \varepsilon \in (0, \varepsilon_0], \mathcal{N}\left(\mathcal{X}_\varepsilon, \frac{\varepsilon}{2L_0}\right) \leq C \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d'} \right\}$$

exists, the quantity $d^*(L_0)$ is called the *near-optimality dimension* of f Bubeck et al. [2011a]. The *zooming dimension* is defined similarly by packing the layers $\mathcal{X}_{(\varepsilon/2, \varepsilon]}$ instead of the sets \mathcal{X}_ε (see Kleinberg et al. [2008]). When $d^*(L_0)$ is well defined, Inequality (6) is satisfied with $d^* = d^*(L_0)$ and the constant C^* equal to

$$C^*(L_0) := \min \left\{ C > 0 : \forall \varepsilon \in (0, \varepsilon_0], \mathcal{N}\left(\mathcal{X}_\varepsilon, \frac{\varepsilon}{2L_0}\right) \leq C \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*(L_0)} \right\}.$$

Note however that a small $d^*(L_0)$ does not necessarily imply that the sets \mathcal{X}_ε are small, since the constant $C^*(L_0)$ can be arbitrarily large. Consider, for example, a small $\rho > 0$ and the function $x \mapsto \max\{\rho - \|x - x_0\|, 0\}$. Its near-optimality dimension is 0 but picking $d^* = 0$ gives a constant C^* of the order of $(1/\rho)^d$.

Besides, in Example 3, we can check that $d^*(L_0) = d/2$, so that the near-optimality dimension corresponds here to the worst-case value of d^* among the linear and quadratic regimes. Our bounds in Sections 3 and 4 do not depend on this worst-case value but instead combine all best values of d^* at all scales ε .

C Proofs of Corollaries 1 and 3.

In this section we prove Corollaries 1 and 3 stated in Section 3. We begin by Corollary 1, which gives a sample complexity bound for the Piyavskii–Shubert algorithm in terms of the near-optimality dimension of f .

Proof of Corollary 1. Fix any $C^* \geq 0$, and $d^* \in [0, d]$ satisfying (6), $\varepsilon \in (0, \varepsilon_0)$, and $\alpha \in [0, \varepsilon/9]$. Let also $c := \mathbf{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0}$ and $\varepsilon' := (7/9)\varepsilon$. Note that $\varepsilon' \in (0, \varepsilon_0)$ and $\alpha \in [0, \varepsilon'/6]$. We can therefore apply Theorem 1 (with ε' instead of ε) to conclude that, if $n \geq \tilde{n}$, the simple regret of the Piyavskii–Shubert algorithm satisfies $r_n \leq \varepsilon' + 2\alpha \leq \varepsilon$, where

$$\begin{aligned}
\tilde{n} &= \sum_{s=1}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \mathcal{N} \left(\mathcal{X}_{(\varepsilon_0 2^{-s}, \varepsilon_0 2^{-s+1}]}, \frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1} \right) + 1 && \text{(by (9) in Theorem 1)} \\
&\leq \sum_{s=1}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \mathcal{N} \left(\mathcal{X}_{\varepsilon_0 2^{-s+1}}, \frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1} \right) + 1 \\
&\leq \sum_{s=1}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \left(1 + 4 \frac{\frac{\varepsilon_0 2^{-s}}{L_0}}{\frac{\varepsilon_0 2^{-s} - 3\alpha}{L_1}} c \right)^d \mathcal{N} \left(\mathcal{X}_{\varepsilon_0 2^{-s+1}}, \frac{\varepsilon_0 2^{-s}}{L_0} \right) + 1 && \text{(by Lemma 6 in Appendix A)} \\
&\leq \left(1 + 4 \frac{L_1}{L_0} \frac{\varepsilon'}{\varepsilon' - 6\alpha} c \right)^d \left(\sum_{s=1}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \mathcal{N} \left(\mathcal{X}_{\varepsilon_0 2^{-s+1}}, \frac{\varepsilon_0 2^{-s+1}}{2L_0} \right) \right) + 1 && (x \mapsto \frac{x}{x-3\alpha} \text{ is decreasing}) \\
&\leq C^* \left(1 + 4 \frac{L_1}{L_0} \frac{\varepsilon'}{\varepsilon' - 6\alpha} c \right)^d \left(\sum_{s=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil - 1} (2^{d^*})^s \right) + 1 && \text{(by (6))} \\
&\leq 1 + C^* \left(1 + 4 \frac{L_1}{L_0} \frac{\varepsilon'}{\varepsilon' - 6\alpha} c \right)^d \times \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon'} \right) + 1 & \text{if } d^* = 0 \\ \frac{2^{d^*} \left(\frac{\varepsilon_0}{\varepsilon'} \right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \\
&\leq 1 + C^* \left(1 + 28 \frac{L_1}{L_0} c \right)^d \times \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + \log_2 \left(\frac{18}{7} \right) & \text{if } d^* = 0 \\ \frac{\left(\frac{18}{7} \right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \\
&= \bar{n}
\end{aligned}$$

where the last inequality follows by $\alpha \leq \frac{\varepsilon}{9} = \frac{\varepsilon'}{7}$ and $\varepsilon' = \frac{7}{9}\varepsilon$. \square

We now show the analogous proof for automatic stopping, which concludes this section.

Proof of Corollary 3. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6), $\varepsilon \in (0, \varepsilon_0)$, and $\alpha \in [0, \varepsilon/15]$. Let also $c = \mathbf{1}_{L_1 \neq L_0 \text{ or } \alpha \neq 0}$, $\varepsilon' := (13/15)\varepsilon$, and $m_{\varepsilon'} = \lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil$. Note that $\varepsilon' \in (0, \varepsilon_0)$ and $\alpha \in [0, \varepsilon'/12]$. We can therefore apply Theorem 2 (with ε' instead of ε) to conclude that the simple regret of the ε -Piyavskii–Shubert

algorithm after the last iteration n satisfies $r_n \leq \varepsilon' + 2\alpha \leq \varepsilon$. Moreover, the total number n of iterations is bounded by

$$\begin{aligned} \tilde{n}' &= \mathcal{N}\left(\mathcal{X}_{\varepsilon'/2}, \frac{\varepsilon' - 3\alpha}{L_1}\right) + \sum_{k=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \mathcal{N}\left(\mathcal{X}_{(\varepsilon_0 2^{-k-1}, \varepsilon_0 2^{-k})}, \frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}\right) \quad (\text{by Theorem 2}) \\ &=: \text{(I)} + \text{(II)} \end{aligned}$$

For the first term, we have

$$\text{(I)} \leq \mathcal{N}\left(\mathcal{X}_{\varepsilon'}, \frac{\varepsilon' - 3\alpha}{L_1}\right) \leq \left(1 + 4 \frac{\frac{\varepsilon'}{2L_0}}{\frac{\varepsilon' - 3\alpha}{L_1}} c\right)^d \mathcal{N}\left(\mathcal{X}_{\varepsilon'}, \frac{\varepsilon'}{2L_0}\right) \leq \left(1 + 2 \frac{L_1}{L_0} \frac{\varepsilon'}{\varepsilon' - 3\alpha} c\right)^d C^* \left(\frac{\varepsilon_0}{\varepsilon'}\right)^{d^*}$$

where the first inequality follows by the fact that $A \mapsto \mathcal{N}(A, r)$ is nondecreasing (with respect to the inclusion) for all $r > 0$, the second by Lemma 6, and the last inequality follows by (6). Combining the fact that $\alpha \leq \varepsilon/15 = \varepsilon'/13$ with $x \mapsto \frac{x}{x-3\alpha}$ being decreasing on $(3\alpha, +\infty)$, and finally plugging $\varepsilon' = (13/15)\varepsilon$ in the last bound above, we get

$$\text{(I)} \leq C^* \left(1 + \frac{13}{5} \frac{L_1}{L_0} c\right)^d \left(\frac{15}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*}.$$

The second term (II) can be upper bounded by following the same lines as in the proof of Corollary 1. More precisely, we have

$$\begin{aligned} \text{(II)} &\leq \sum_{k=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \mathcal{N}\left(\mathcal{X}_{\varepsilon_0 2^{-k}}, \frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}\right) \leq \sum_{k=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} \left(1 + 4 \frac{\frac{\varepsilon_0 2^{-k}}{2L_0}}{\frac{\varepsilon_0 2^{-k-1} - 3\alpha}{L_1}} c\right)^d \mathcal{N}\left(\mathcal{X}_{\varepsilon_0 2^{-k}}, \frac{\varepsilon_0 2^{-k}}{2L_0}\right) \\ &\leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \sum_{k=0}^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil} (2^{d^*})^k \leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon}\right) + \log_2 \left(\frac{60}{13}\right) & \text{if } d^* = 0 \\ \frac{(2^{d^*})^{\lceil \log_2 \frac{\varepsilon_0}{\varepsilon'} \rceil + 1} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \\ &\leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon}\right) + \log_2 \left(\frac{60}{13}\right) & \text{if } d^* = 0 \\ \frac{\left(\frac{60}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} \tilde{n}' &= \text{(I)} + \text{(II)} \\ &\leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \left(\left(\frac{15}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} + \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon}\right) + \log_2 \left(\frac{60}{13}\right) & \text{if } d^* = 0 \\ \frac{\left(\frac{60}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \right) \\ &\leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon}\right) + \log_2 \left(\frac{120}{13}\right) & \text{if } d^* = 0 \\ \frac{(2^{d^*} - 1) \left(\frac{15}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} + \left(\frac{60}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \\ &\leq \left(1 + 52 \frac{L_1}{L_0} c\right)^d C^* \begin{cases} \log_2 \left(\frac{\varepsilon_0}{\varepsilon}\right) + \log_2 \left(\frac{120}{13}\right) & \text{if } d^* = 0 \\ \frac{(4^{d^*} + 2^{d^*} - 1) \left(\frac{15}{13}\right)^{d^*} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{d^*} - 1}{2^{d^*} - 1} & \text{if } d^* > 0 \end{cases} \end{aligned}$$

□

D Connections with a known regret bound in dimension $d = 1$.

Hansen et al. [1991] provided an extensive study of the ε -Piyavskii–Shubert algorithm without perturbations (Algorithm 2 with $\alpha = 0$) for Lipschitz functions defined on a compact interval. Theorems 4 and 5 in Hansen et al. [1991] imply the following upper bound on the number n_{Py} of iterations performed by the ε -Piyavskii–Shubert algorithm before stopping.

Theorem 4 (Hansen et al. [1991]). *Assume that $D = [0, 1]$, f is globally L_0 -Lipschitz on $[0, 1]$, and let $L_1 \geq L_0$. Let also $\varepsilon > 0$, $\alpha = 0$, $x_1 \in D$, and assume that the ε -Piyavskii–Shubert algorithm (Algorithm 2) is run with inputs $L_1, \varepsilon, 0, x_1$. Then, the ε -Piyavskii–Shubert algorithm stops after at most n_{Py} iterations, where*

$$n_{\text{Py}} := 1 + \frac{2L_0}{\ln(1 + L_0/L_1)} \int_0^1 \frac{dx}{f(x^*) - f(x) + \varepsilon}. \quad (19)$$

Using (6), we can further upper bound n_{Py} in terms of the pair (d^*, C^*) , as shown below.

Corollary 5. *Assume that $D = [0, 1]$, f is globally L_0 -Lipschitz on $[0, 1]$, and let $L_1 \geq L_0$. Let also $\varepsilon > 0$, $\alpha = 0$, $x_1 \in D$, and assume that the ε -Piyavskii–Shubert algorithm (Algorithm 2) is run with inputs $L_1, \varepsilon, 0, x_1$. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6). Then, the ε -Piyavskii–Shubert algorithm stops after at most \bar{n}_{Py} iterations, where*

$$\bar{n}_{\text{Py}} := 1 + \frac{v_1 C^*}{\ln(1 + L_0/L_1)} \times \begin{cases} 2 \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + 3 & \text{if } d^* = 0 \\ \left(\frac{2^{d^*+1}}{2^{d^*} - 1} + 1 \right) \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} & \text{if } d^* > 0 \end{cases}$$

and v_1 is the volume $\int_{\mathbb{R}} \mathbb{1}_{\|x\| \leq 1} dx$ of the real unit ball with respect to $\|\cdot\|$.

Proof. We start from Equation (19). This proof relies on a peeling technique similar to that of Theorems 1 and 2. Fix any $C^* \geq 0$ and $d^* \in [0, d]$ satisfying (6), and without loss of generality let $\varepsilon \in (0, \varepsilon_0)$. Defining again $m_\varepsilon := \lceil \log_2(\varepsilon_0 \varepsilon^{-1}) \rceil \geq 1$, we have $[0, \varepsilon_0] = [0, \varepsilon] \cup (\varepsilon_0 2^{-m_\varepsilon}, \varepsilon_0 2^{-m_\varepsilon+1}] \cup (\varepsilon_0 2^{-m_\varepsilon+1}, \varepsilon_0 2^{-m_\varepsilon+2}] \cup \dots \cup (\varepsilon_0/2, \varepsilon_0]$ which in turn yields

$$\begin{aligned} \int_0^1 \frac{dx}{f(x^*) - f(x) + \varepsilon} &\leq \sum_{i=1}^{m_\varepsilon} \int_{\mathcal{X}_{(\varepsilon_0 2^{-i}, \varepsilon_0 2^{-i+1})}} \frac{dx}{f(x^*) - f(x) + \varepsilon} + \int_{\mathcal{X}_\varepsilon} \frac{dx}{f(x^*) - f(x) + \varepsilon} \\ &\leq \sum_{i=1}^{m_\varepsilon} \int_{\mathcal{X}_{(\varepsilon_0 2^{-i}, \varepsilon_0 2^{-i+1})}} \frac{dx}{\varepsilon_0 2^{-i}} + \int_{\mathcal{X}_\varepsilon} \frac{dx}{\varepsilon}. \end{aligned}$$

Let v_1 be the volume $\int_{\mathbb{R}} \mathbb{1}_{\|x\| \leq 1} dx$ of the real unit ball with respect to $\|\cdot\|$. For any nonempty subset A of $[0, 1]$, any radius $\alpha > 0$, and any collection \mathcal{B} of $\|\cdot\|$ -balls of radius α covering A , we have

$$\int_A dx \leq \sum_{B \in \mathcal{B}} \int_B dx \leq \alpha v_1 \cdot |\mathcal{B}|.$$

Plugging this in the previous inequality gives

$$\int_0^1 \frac{dx}{f(x^*) - f(x) + \varepsilon} \leq \sum_{i=1}^{m_\varepsilon} \frac{1}{2^{-i} \varepsilon_0} \frac{2^{-i} \varepsilon_0}{L_0} v_1 \cdot \mathcal{M} \left(\mathcal{X}_{\left(\frac{\varepsilon_0}{2^i}, \frac{\varepsilon_0}{2^{i-1}}\right)}, \frac{2^{-i} \varepsilon_0}{L_0} \right) + \frac{1}{\varepsilon} \frac{\varepsilon}{2L_0} v_1 \cdot \mathcal{M} \left(\mathcal{X}_\varepsilon, \frac{\varepsilon}{2L_0} \right),$$

where the covering number $\mathcal{M}(A, \delta)$ is the minimum number of δ -balls required to cover A . Using the

inclusion $\mathcal{X}_{(\varepsilon_0 2^{-i}, \varepsilon_0 2^{-i+1}]} \subset \mathcal{X}_{\varepsilon_0 2^{-i+1}}$, and applying (17) and (6), we obtain

$$\begin{aligned}
\int_0^1 \frac{dx}{f(x^*) - f(x) + \varepsilon} &\leq \frac{v_1}{L_0} \sum_{i=1}^{m_\varepsilon} C^* \left(\frac{\varepsilon_0}{2^{-i+1} \varepsilon_0} \right)^{d^*} + \frac{v_1}{2L_0} C^* \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} \\
&\leq \frac{v_1 C^*}{L_0} \sum_{i=1}^{m_\varepsilon} 2^{(i-1)d^*} + \frac{v_1 C^*}{2L_0} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} \\
&\leq \begin{cases} \frac{v_1 C^*}{2L_0} (2m_\varepsilon + 1) & \text{if } d^* = 0 \\ \frac{v_1 C^*}{L_0} \frac{2^{m_\varepsilon d^*} - 1}{2^{d^*} - 1} + \frac{v_1 C^*}{2L_0} \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} & \text{if } d^* > 0 \end{cases} \\
&\leq \begin{cases} \frac{v_1 C^*}{2L_0} \left(2 \log_2 \left(\frac{\varepsilon_0}{\varepsilon} \right) + 3 \right) & \text{if } d^* = 0 \\ \frac{v_1 C^*}{2L_0} \left(\frac{2^{d^*+1}}{2^{d^*} - 1} + 1 \right) \left(\frac{\varepsilon_0}{\varepsilon} \right)^{d^*} & \text{if } d^* > 0 \end{cases}
\end{aligned}$$

which gives the result. □

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