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On Schur problem and Kostka numbers

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Abstract

We reconsider the two related problems: distribution of the diagonal elements of a Hermitian $n \times n$ matrix of known eigenvalues (Schur) and determination of multiplicities of weights in a given irreducible representation of $\mathrm{SU}(n)$ (Kostka). It is well known that the former yields a semi-classical picture of the latter. We present explicit expressions for low values of n that complement those given in the literature [11, 1], recall some exact (non asymptotic) relation between the two problems, comment on the limiting procedure whereby Kostka numbers are obtained from Littlewood–Richardson coefficients, and finally extend these considerations to the case of the B_2 algebra, with a few novel conjectures.

1 Introduction

1.1 Overview

It has been known for long that multiplicity problems in representation theory have an asymptotic limit that may be treated by semi-classical methods [13, 11]. For example, the behavior for large representations of generalized Littlewood–Richardson (LR) coefficients, i.e., coefficients of decomposition into irreducible representations (irreps) of the tensor product of two irreps, admits a semi-classical description in terms of Horn's problem. For a review and a list of references, see [6]. Similarly, one may consider the Kostka numbers $\mathrm{mult}_\lambda(\delta)$, i.e., the multiplicity of weight δ in the irrep of highest weight λ. As also well known [11, 1], the asymptotics (for large λ and δ) of those numbers is related to Schur's problem, which deals with the properties and distribution of diagonal elements of a Hermitian matrix of given spectrum.

In the following, we first reexamine that classical Schur problem: in the case where the original Hermitian matrix is taken at random uniformly on its orbit, we recall (sect. 2) how the probability density function (PDF) of its diagonal elements is determined by a *volume function* $\mathcal{I}$, which has as a support the permutahedron determined by the eigenvalues and is a piecewise polynomial function of these eigenvalues and of the diagonal elements, with non analyticities of a prescribed type on an a priori known locus. We give quite explicit expressions of that function for the cases of SU(3) and SU(4) (coadjoint) orbits (sect. 3). In sect. 4, we turn to the parallel representation-theoretic problem, namely the determination of Kostka numbers. We rederive (sect. 4.2) in this new context an exact relationship between these multiplicities and the aforementioned volume function, which was already discussed in the Horn problem [8, 6]. That relation leads in a natural way to the semi-classical asymptotic limit mentioned above (sect. 4.6). A guiding thread through this work is the (fairly obvious and well known) fact that Kostka numbers may be obtained as a certain limit of Littlewood–Richardson coefficients when two of their arguments grow large with a finite difference. In fact that limit is approached quite fast, and it is an intriguing problem to find values for the threshold value s_c of the scaling parameter beyond which the asymptotic Kostka number is reached. We address that question in sect. 4.4 and propose a (conservative) upper bound on that threshold in the case of SU(4). We also recall the combinatorial interpretation of Kostka's numbers, in terms of reduced Knutson–Tao honeycombs aka Gelfand–Tsetlin triangles, or of reduced O-blades (sect. 4.3, 4.5). The latter may be recast in a new picture of "forests of lianas", as discussed in the Appendix. Finally, sect. 5 is devoted to a discussion of what can be said or conjectured in the case of the B_2 case.

Several of these results have already appeared in some guise in the literature. The domains of polynomiality of the Duistermaat–Heckman measure and the transitions between them have been discussed for SU(n), with emphasis on $n = 3$ and 4, in [11]. The parallel analysis of multiplicities has been carried out in [1], using the method of vector partition functions. A general discussion making use of Littelmann's paths [16] has been done by Bliem [2], with an illustration in the case of the $B_2 = \mathfrak{so}(5)$ algebra. We believe, however, that several aspects of our approach are original, that our results for the case SU(4), resp. B_2, complement those of [11, 1], resp. [2], that the discussion of the threshold value s_c is novel, and that the liana picture may give a new insight in the combinatorial aspects of the problem.

This Schur problem –as its sibling the Horn problem– presents a unique and fascinating mix of various ingredients, algebraic, geometric and group theoretic. As such, we hope that our modest contribution would have pleased our distinguished colleague Boris Dubrovin, who, all his life, paid an acute attention to the interface between mathematics and physics.

1.2 Notations

In the following, we use two alternative notations for the objects pertaining to SU(n). First in the classical Schur problem, the ordered eigenvalues of a $n \times n$ Hermitian matrix will be denoted by $\alpha = (\alpha_1, \cdots, \alpha_n)$, with round brackets, and its diagonal elements by $\xi = (\xi_1, \cdots, \xi_n)$, with clearly $\sum_i \alpha_i = \sum_i \xi_i$. At the possible price of an overall shift of all α_i, one may assume they are all non negative. In the case they are all non negative integers, one may regard them as defining a partition and encode them in a Young diagram.

A vector λ of the weight lattice of SU(n) may be denoted by its $(n-1)$ (Dynkin) components in the fundamental weight basis: $\lambda = \{\lambda_1, \cdots, \lambda_{n-1}\}$, with curly brackets, or as a partition, with n "Young components" equal to the lengths of rows of the corresponding Young diagram: $\alpha = (\alpha_1, \cdots, \alpha_n)$. The former is recovered from the latter by $\lambda_i = \alpha_i - \alpha_{i+1}, \quad i = 1, \cdots, n-1$.

Conversely, when dealing with the highest weight λ of an irreducible representation (irrep) of SU(n), it is natural to define a decreasing partition α with $\alpha_i = \sum_{j=i}^{n-1} \lambda_j, \quad i = 1, \cdots, n-1$, and $\alpha_n = 0$. For a weight δ of that irrep, one defines the sequence $\xi = (\xi_1, \cdots, \xi_n)$, also called weight, with $\xi_i = \sum_{j=i}^{n-1} \delta_j + c, \quad i = 1, \cdots, n-1$, $\xi_n = c$, and one chooses $c = \frac{1}{n} \sum_{i=1}^{n-1} i(\lambda_i - \delta_i)$, an integer, in such a way that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \xi_i$. So ξ is a non necessarily decreasing partition of $\sum_{i=1}^{n} \alpha_i$. It is well known that $\mathrm{mult}_\lambda(\delta)$ is equal to the number of SU(n) semi-standard Young tableaux with fixed shape α and weight ξ. In what follows, we shall use both languages interchangeably, and use the notations (λ, δ) or (α, ξ) with the above meaning, without further ado, writing for instance $\mathrm{mult}_\lambda(\delta) = \mathrm{mult}_\alpha(\xi)$. Actually, we shall use the same notations and conventions, even if λ and δ are not integral (so that α and ξ are no longer partitions in the usual sense). We hope that the context will prevent possible confusions.

2 Schur's problem

Schur's problem deals with the following question: If A is a n-by-n Hermitian matrix with known eigenvalues $\alpha_1 \geqslant \ldots \geqslant \alpha_n$, what can be said about the diagonal elements $\xi_i := A_{ii}$, $i = 1, \cdots, n$? As shown by Horn [14], the ξ's lie in the closure of the *permutahedron* $\mathcal{P}_\alpha$, i.e., , the convex polytope in $\mathbb{R}^n$ whose vertices are the points $(\alpha_{P(1)}, \alpha_{P(2)}, \cdots, \alpha_{P(n)})$, $P \in \mathfrak{S}(n)$. Note that by a translation of A by a multiple of the identity matrix, we could always manage to have $\sum_i \xi_i = \sum_i \alpha_i = \mathrm{tr}\, A = 0$ [1], but we shall not generally assume this tracelessness in the following.

A more specific question is the following: if A is drawn randomly and uniformly on its SU(n)-orbit $\mathcal{O}_\alpha$, what is the PDF of the ξ's ? It turns out that this PDF is, up to a factor, the (inverse) Fourier transform of the orbital integral, i.e., the density of Duistermaat–Heckman's measure. To show that, we follow the same steps as in [21]. The characteristic function of the random variables ξ, i.e., the Fourier transform of the desired PDF, is

$$\varphi(x) = \mathbb{E}(e^{\mathrm{i} \sum_j x_j A_{jj}}) = \int dU \exp \mathrm{i} \sum_{j=1}^{n} x_j \, (U.\alpha.U^\dagger)_{jj}$$

with dU the normalized SU(n) Haar measure, and x belongs to the $\mathbb{R}^{n-1}$ hyperplane defined as $\sum_j x_j = 0$. From this $\varphi(x)$ we recover the PDF p of ξ by inverse Fourier transform

$$\mathrm{p}(\xi|\alpha) = \int \frac{d^{n-1}x}{(2\pi)^{n-1}} \, dU \, \exp \mathrm{i} \sum_j x_j ((U.\alpha.U^\dagger)_{jj} - \xi_j)$$

[1] This would be quite natural since the Lie algebra of SU(n) is the set of traceless skew-Hermitian matrices.

which is indeed the (inverse) Fourier integral of the orbital (HCIZ) integral $\mathcal{H}(\alpha, \mathrm{i}\, x) = \int dU e^{\mathrm{i}\,\mathrm{tr}\,(xU\alpha U^\dagger)}$

$$\begin{aligned} \mathrm{p}(\xi|\alpha) &= \int \frac{d^{n-1}x}{(2\pi)^{n-1}}\, e^{-\mathrm{i}\sum_j x_j\xi_j}\, \mathcal{H}(\alpha, \mathrm{i}\, x) && (1)\\ &= \frac{\Delta(\rho)}{\Delta(\alpha)} \int \frac{d^{n-1}x}{(2\pi)^{n-1}} \frac{1}{\Delta(\mathrm{i}\, x)} \sum_{w\in\mathfrak{S}(n)} \epsilon(w) e^{\mathrm{i}\sum_j x_j(\alpha_{w(j)}-\xi_j)}\,. && (2) \end{aligned}$$

In the above expressions $\epsilon(w)$ is the signature of w, Δ stands for the Vandermonde determinant, $\Delta(\alpha) = \prod_{1\leqslant i<j\leqslant n}(\alpha_i - \alpha_j)$ and ρ denotes the Weyl vector, written here in partition components $\rho = (n-1, n-2, \ldots, 0)$. In the present case of $su(n)$, $\Delta(\rho)$ is the superfactorial $\prod_{j=1}^{n-1} j!$. For a general simple Lie algebra, $\Delta(\xi) = \prod_{\alpha>0}\langle\alpha, \xi\rangle$, a product over the positive roots; in the simply-laced cases $\Delta(\rho)$ is the product[2] of factorials of the Coxeter exponents of the algebra.

It turns out that this may be recovered heuristically in a different way. Recall the connection between Schur's and Horn's problems [5]. Consider Horn's problem for two matrices $A \in \mathcal{O}_\alpha$ and $B \in \mathcal{O}_\beta$ and assume that $\alpha \ll \beta$ and that the β_j are distinct, $\beta_j - \beta_{j+1} \gg 1$. The eigenvalues of $C = A + B$, to the first order of perturbation theory, are of the form $\gamma_i = \beta_i + A_{ii} = \beta_i + \xi_i$. The β's being given, the PDF of the γ's is the PDF of the ξ's. The former, namely

$$\mathrm{p}(\gamma|\alpha, \beta) = \frac{1}{n!}\frac{\Delta(\gamma)^2}{\Delta(\rho)^2} \int \frac{d^{n-1}x}{(2\pi)^{n-1}}\, \Delta(x)^2\, \mathcal{H}(\alpha, \mathrm{i}\, x)\mathcal{H}(\beta, \mathrm{i}\, x)\mathcal{H}(\gamma, -\mathrm{i}\, x) \tag{3}$$

reduces in the limit to (1). To prove it, we expand as usual $\mathrm{p}(\gamma|\alpha, \beta)$ as

$$\mathrm{p}(\gamma|\alpha, \beta) = \frac{\Delta(\gamma)\Delta(\rho)}{\Delta(\alpha)\Delta(\beta)} \int \frac{d^{n-1}x}{(2\pi)^{n-1}} \frac{1}{\Delta(\mathrm{i}\, x)} \sum_{w,w'\in\mathfrak{S}(n)} \epsilon(w.w') e^{\mathrm{i}\sum_j x_j(\alpha_{w(j)}+\beta_{w'(j)}-\gamma_j)} \tag{4}$$

and notice that for $\alpha.x \sim O(1)$, $\beta.x \gg 1$, all terms $e^{\mathrm{i}\, x.\beta_{w'}}$, $w' \neq 1$ are rapidly oscillating and are suppressed, leaving only the term $w' = 1$ for which the exponential reduces to $e^{\mathrm{i}\sum_j x_j(\alpha_{w(j)}-\xi_j)}$ while $\Delta(\gamma)/\Delta(\beta) \approx 1$, thus reducing (4) to (2).

In these expressions, the integration is over the hyperplane $\mathbb{R}^{n-1}$ defined by $\sum_i x_i = 0$ (in fact, the Cartan algebra of SU(n)). As usual, we change variables $u_i = x_i - x_{i+1}$, and denote $\tilde{\Delta}(u) := \Delta(x)$. We thus write

$$\begin{aligned} \mathrm{p}(\xi|\alpha) &= \frac{\Delta(\rho)}{\Delta(\alpha)}\mathcal{I}(\alpha; \xi) && (5)\\ \mathcal{I}(\alpha; \xi) &= \sum_{w\in\mathfrak{S}(n)} \epsilon(w) \int \frac{d^{n-1}u}{\tilde{\Delta}(\mathrm{i}\, u)} e^{\mathrm{i}\sum_{j=1}^n u_j \sum_{k=1}^j (\alpha_{w(k)}-\xi_k)}\,, && (6) \end{aligned}$$

and focus our attention on that function $\mathcal{I}$, that, for reasons explained later, we may call "the volume function of the Schur–Kostka problem". As everywhere in this paper we freely use notations that may refer either to Dynkin components (weights: λ, δ) or to Young components (partitions: α, ξ), so that with the conventions defined in (sect. 1.2) we may write, for instance, $\mathcal{I}(\lambda; \delta) = \mathcal{I}(\alpha; \xi)$.

It is first clear that, by definition, p and $\mathcal{I}$ must be symmetric functions of the ξ_i, $i = 1, \cdots, n$.

By *a priori* arguments [13, 9], or by explicit computation of (6), it is clear that $\mathcal{I}$ is a piece-wise polynomial of its arguments α and ξ, homogeneous of degree $(n-1)(n-2)/2$. By Riemann–Lebesgue theorem (i.e., looking at the decay of the integrand of (6) at large u), its differentiability class is C^{n-3}, just like in the parallel Horn's problem.

By using the same arguments as in [6], or by applying the previous limiting procedure to the $\mathcal{J}$ function of Horn's problem, we can assert *a priori* that the loci of non-analyticity of $\mathcal{I}$ (i.e., the places where its polynomial determination changes) are contained in the hyperplanes

$$\xi_i = \alpha_{w(i)}, \text{ or } \xi_i + \xi_j = \alpha_{w(i)} + \alpha_{w(j)}, \qquad w \in \mathfrak{S}(n)\,, \tag{7}$$

[2] In the non simply-laced cases one has to divide this product by an appropriate scaling coefficient (see for instance [6]) which is equal to 2^r for B_r, so that for B_2 considered in sect. 5, one gets $\Delta(\rho) = (1! \times 3!)(2^2) = 3/2$.

or more generally

$$\sum_{i\in I}\xi_i = \sum_{j\in J}\alpha_j \tag{8}$$

with $I, J \subset \{1, 2, \cdots, n\}$, $|I| = |J| \leqslant \lfloor \frac{n}{2} \rfloor$.

The relationship between the degree of non-differentiability and the cardinality of the sets I and J in 8 has been addressed in the fundamental work of Guillemin–Lerman–Sternberg [11] §3.5 . It is proven there that the "jump" (i.e., the change of determination) of $\mathcal{I}$ across a singular hyperplane is of the form

$$\Delta\mathcal{I} = \text{const.}\ (\sum_I \xi_i - \sum_J \alpha_j)^{m-1} + \cdots \tag{9}$$

where m is an integer depending on $k = |I| = |J|$, which we will now determine, thus showing that $\mathcal{I}$ is of class C^{m-2} across that singularity. Let's compute that number $m-1$ in the case of a singular hyperplane of the form (8). A generic orbit of $\mathrm{SU}(n)$ has dimension $d_n := (n^2-1)-(n-1) = n(n-1)$. Let $k = |I| = |J|$, with $k \leqslant \lfloor \frac{n}{2} \rfloor$, then

$$m - 1 = (d_n - d_k - d_{n-k})/2 - 1 = nk - k^2 - 1 = k(n-k) - 1\,. \tag{10}$$

Thus for $\mathrm{SU}(4)$, $k = 1$, resp. 2, leads to $m - 1 = 2$, resp. 3, i.e., corresponds to a C^1, resp. C^2 differentiability class; for $\mathrm{SU}(5)$, $k = 1$, resp. 2, leads to $m - 1 = 3$, resp. 5, and differentiability class C^2, resp. C^4, etc. This will be fully corroborated by the explicit expressions given below.

In parallel to the study of the "volume function" $\mathcal{I}$, one may consider the properties of the multiplicity function $\mathrm{mult}_\lambda(\delta)$. It is known that it is also a piece-wise polynomial function of λ and δ) [13]. Quite remarkably, it has been proved that its singularities (changes of polynomial determination) as a function of δ occur on the same locus as those of $\mathcal{I}(\alpha;\xi)$, see Theorem 3.2 in [1]. This change of determination is a product of $k(n-k)-1$ *distinct* factors of degree 1, to be compared with (9-10).

3 Explicit value for low n

3.1 $n = 2$

In that case, the PDF and the associated $\mathcal{I}$ functions are easily determined. Let $\alpha = (\alpha_1, -\alpha_1)$, then

$$A = U(\theta)\mathrm{diag}\,(\alpha_1, -\alpha_1)U^\dagger(\theta), \qquad (0 \leqslant \theta \leqslant \pi) \qquad \xi_1 = A_{11} = \alpha_1 \cos\theta\,,$$

whence a support $\xi_1 \in (-\alpha_1, \alpha_1)$ with a density

$$p(\xi|\alpha) = \frac{1}{2}\frac{\sin\theta\, d\theta}{d\xi_1} = \frac{1}{2\alpha_1}\,, \qquad \mathcal{I} = \frac{1}{2}\,.$$

As expected, the functions p and $\mathcal{I}$ are constant and discontinuous at the edges of their support.

Remark. The parallel computation in the case of a real symmetric matrix with action of $\mathrm{SO}(2)$ leads to a density singular on the edges of its support: $p = \frac{1}{\pi\sqrt{\alpha_1^2-\xi_1^2}}$.

3.2 $n = 3$

For $n = 3$, the function $\mathcal{I}$ is readily computed and given by the function displayed in red on Fig. 1. It is normalized by $\int d^2\xi\, \mathcal{I}(\alpha;\xi) = \Delta(\alpha)/\Delta(\rho)$ as it should, see (5). Equivalent formulae have been given in [11].

There is an alternative way of presenting this result. Instead of giving the value of $\mathcal{I}$ in each cell, start from the value 0 at the exterior of the permutahedron and give the rule for the change

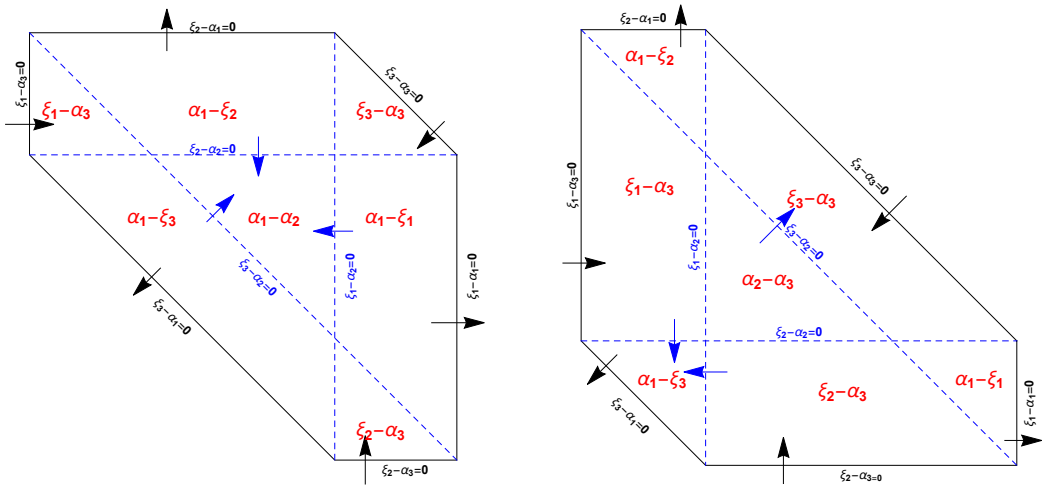


Figure 1: Piecewise polynomial determination of the function $\mathcal{I}_{su(3)} := \mathrm{PDF} \times \Delta(\alpha)$ of the $\xi_i := A_{ii}$, for SU(3). We assume $\alpha_1 > \alpha_2 > \alpha_3$. Left: $\alpha_2 > \frac{1}{2}(\alpha_1 + \alpha_3)$. Right: $\alpha_2 < \frac{1}{2}(\alpha_1 + \alpha_3)$.

of polynomial determination as one crosses a wall. The rule is as follows: if the equation of a wall is $\xi_i - \alpha_j = 0$, the function $\mathcal{I}$ is incremented by $\xi_i - \alpha_j$ when the wall is crossed in the direction of the arrow, see Fig. 1. (Here one should remember that $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \alpha_i$.)

We conclude that in the case $n = 3$, the changes of determination are by affine functions of ξ that vanish on the hyperplanes (here lines) of singularity, in accordance with the C^0 class of differentiability.

Remark. From the constraints on the honeycomb parameter, see below sect. 4.3, eq. (35), we may derive another expression of the same function

$$\begin{aligned} \mathcal{I}_{su(3)}(\alpha;\xi) &= \min(\alpha_1, \alpha_1 + \alpha_2 - \xi_1) - \max(\alpha_2, \xi_2, \xi_3, \alpha_1 + \alpha_3 - \xi_1) \\ &= \min(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_1 - \xi_i, \xi_i - \alpha_3) \qquad i = 1,2,3\,, \end{aligned} \tag{11}$$

which is manifestly non negative and fully symmetric in the ξ's.

Consider now two weights of SU(3), whose difference belongs to the root lattice. As explained above, we denote them either by the 2-dimensional vector of their (integral) Dynkin components $\lambda = \{\lambda_1, \lambda_2\}$ and $\delta = \{\delta_1, \delta_2\}$, or equivalently by their partition (Young) components α and ξ. Then the number of integer points in the interval (36) is the Kostka multiplicity, denoted $\mathrm{mult}_\alpha(\xi)$ or $\mathrm{mult}_\lambda(\delta)$, and reads

$$\mathrm{mult}_\alpha(\xi) = \mathcal{I}_{su(3)}(\alpha;\xi) + 1\,. \tag{12}$$

Expressed in Dynkin coordinates, this reads, if $(\delta_2 - \delta_1) \geqslant (\lambda_2 - \lambda_1)$

$$\mathrm{mult}_\lambda(\delta) = 1 + \min[\frac{1}{3}(2\lambda_1 + \lambda_2 + \delta_1 - \delta_2), \lambda_2, \frac{1}{3}(\lambda_1 + 2\lambda_2 + 2\delta_1 + \delta_2), \frac{1}{3}(\lambda_1 + 2\lambda_2 - \delta_1 - 2\delta_2)] \tag{13}$$

Otherwise, one should replace (λ, δ) by the conjugate pair $(\overline{\lambda}, \overline{\delta})$ in the above expression and use the fact that $\mathrm{mult}_{\overline{\lambda}}(\overline{\delta}) = \mathrm{mult}_\lambda(\delta)$.

These expressions are equivalent to those given in [1], sect. 7.1, for the multiplicities.

3.3 $n = 4$

The permutahedron may be regarded as the convex polytope of points ξ satisfying the inequalities

$$\alpha_4 \leqslant \xi_i \leqslant \alpha_1\,,\ 1 \leqslant i \leqslant 4\,, \qquad \alpha_3 + \alpha_4 \leqslant \xi_i + \xi_j \leqslant \alpha_1 + \alpha_2,\ 1 \leqslant i < j \leqslant 4\,.$$

It has thus four pairs of hexagonal faces $\xi_i = \alpha_1$ or α_4, $i = 1, \cdots, 4$; and 3 pairs of rectangular faces $\xi_i + \xi_j = \alpha_1 + \alpha_2$ or $\alpha_3 + \alpha_4$, see Fig. 2. (Note that $\xi_i + \xi_j = \alpha_1 + \alpha_2 \Leftrightarrow \xi_k + \xi_\ell = \alpha_3 + \alpha_4$, i, j, k, ℓ all distinct, since $\sum_i \xi_i = \sum_i \alpha_i = \operatorname{tr} A$.) Moreover, we expect the function to be piecewise polynomial, with possible changes of determination across the hyperplanes of equation

$$\textbf{(i)}\colon\ \xi_i = \alpha_2 \text{ or } \alpha_3;$$

or

$$\textbf{(ii)}\colon\ \xi_i + \xi_j = \alpha_1 + \alpha_3 \text{ or } \alpha_1 + \alpha_4.$$

These hyperplanes yield a partition of the permutahedron into (open) polyhedral cells.

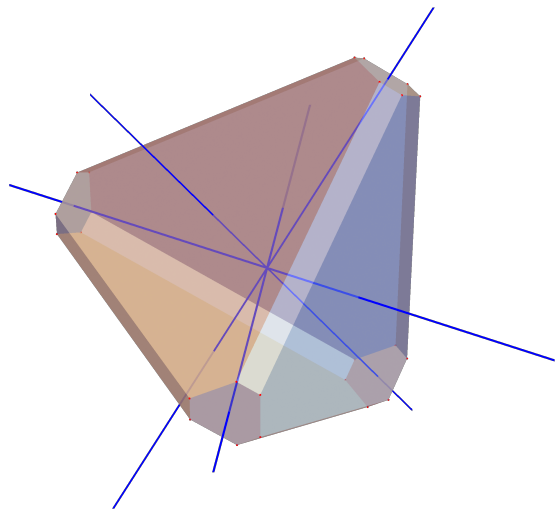


Figure 2: The permutahedron for $n = 4$ and $\alpha = (5, 4, 2, -11)$.
The four coordinates ξ_i run between -11 and 5 along the four blue axes.

From the expressions given in [21, 8] for the Horn volume, and taking the limit $\beta, \gamma \gg \alpha \approx \xi = \gamma - \beta$, one finds (notation $A_i := \alpha_{w(i)} - \xi_i,\ A_{ij} = A_i + A_j,\ A_{123} = A_1 + A_2 + A_3$)

$$\begin{aligned}\mathcal{I}_{su(4)}(\alpha;\xi) \quad = \quad & \sum_{w\in\mathfrak{S}(4)} \epsilon(w)\varepsilon(A_1) \\ & \Big(\varepsilon(A_2)\Big(\frac{1}{6}\big(\,|A_{123}|^3 - |A_{13}|^3 + |A_{23}|^3 - |A_3|^3\,\big) - \frac{1}{2}A_2(A_{123}\,|A_{123}| + A_3\,|A_3|)\Big) \\ & +\varepsilon(A_{12})\Big(\frac{1}{2}A_{12}(A_{123}\,|A_{123}| + A_3\,|A_3|) + \frac{1}{3}(|A_3|^3 - |A_{123}|^3)\Big)\Big)\,,\end{aligned} \tag{14}$$

where $\epsilon(w)$ is the signature of w and $\varepsilon(\cdot)$ is the sign function.

In principle, there is an alternative expression of $\mathcal{I}_{su(4)}$, though not explicit, coming from its interpretation as the volume of a 3d polytope, see sect. 4.3.

Finally we have yet another expression, which makes explicit the location of the singular (hyper)planes and the piecewise polynomial determinations. This will be described now.

In accordance with the results of [11], see above (9), we expect that the change of polynomial determination of $\mathcal{I}_{su(4)}$ is (at least) quadratic across the hyperplanes of type (i), and cubic across those of type (ii). This applies also to the vanishing of the function on the external faces, with a quadratic, resp. cubic behavior on hexagonal, resp. rectangular faces. This is confirmed by the detailed and explicit expression of the jumps of $\mathcal{I}_{su(4)}$ that we discuss now.

In the vicinity of an hexagonal face, internal or external, $\xi_i = \alpha_j\,, i,j = 1,\cdots,4$, $\mathcal{I}_{su(4)}$ undergoes a change of determination $\Delta\mathcal{I}_{su(4)}(\xi)$ of the form

$$\Delta\mathcal{I}_{su(4)}(\xi) = \frac{1}{2}(\xi_i - \alpha_j)^2\, p_c(\xi) \tag{15}$$

with $p_c(\xi)$ a degree 1 polynomial. $\Delta\mathcal{I}_{su(4)}(\xi)$ thus vanishes with its first order derivatives on that face, hence $\mathcal{I}_{su(4)}$ is of class C^1. More precisely, as $\xi_1 - \alpha_3$ increases through 0, for instance, which we denote by $\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_3$, $\mathcal{I}$ is incremented by

$$\Delta\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_3} = -\frac{1}{6}(\xi_1-\alpha_3)^2 \times \begin{array}{l} \\ \left\{\begin{array}{lll} 3(\xi_2-\alpha_4)+(\xi_1-\alpha_3) & \text{if} & \\ 3(\xi_3-\alpha_4)+(\xi_1-\alpha_3) & \text{if} & \\ 3(\xi_4-\alpha_4)+(\xi_1-\alpha_3) & \text{if} & \\ 3(\alpha_2-\alpha_4) & \text{if} & \\ 3(\alpha_1-\alpha_2) & \text{if} & \\ 3(\alpha_1-\xi_4)-(\xi_1-\alpha_3) & \text{if} & \\ 3(\alpha_1-\xi_3)-(\xi_1-\alpha_3) & \text{if} & \\ 3(\alpha_1-\xi_2)-(\xi_1-\alpha_3) & \text{if} & \end{array}\right. \end{array} \begin{array}{ccc} \xi_2-\alpha_2 & \xi_3-\alpha_2 & \xi_4-\alpha_2 \\ <0 & >0 & >0 \\ >0 & <0 & >0 \\ >0 & >0 & <0 \\ >0 & >0 & >0 \\ <0 & <0 & <0 \\ <0 & <0 & >0 \\ <0 & >0 & <0 \\ >0 & <0 & <0 \end{array} \tag{16}$$

or, in a more compact form,

$$\Delta\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_3} = -\frac{1}{2}(\xi_1-\alpha_3)^2\Big(\mathcal{I}_{su(3)}((\alpha_1,\alpha_2,\alpha_4);(\xi_2,\xi_3,\xi_4)) + \frac{\eta_{13}}{3}(\xi_1-\alpha_3)\Big)\,, \tag{17}$$

in terms of the function defined above in Fig. 1 or in eq. (11), and where $\eta_{13} = +1, 0, -1$ depending on the case, as read off the expression (16): $\eta_{13} = 1$ on the first three lines of (16), then 0 on the next two, and -1 on the last three. In other words, if

$$h((\beta_1,\beta_2,\beta_3);(\zeta_1,\zeta_2,\zeta_3)) := \begin{cases} +1 & \text{if } \operatorname{sign}(\zeta_1-\beta_2)+\operatorname{sign}(\zeta_2-\beta_2)+\operatorname{sign}(\zeta_3-\beta_2) = +1 \\ 0 & \text{if } \operatorname{sign}(\zeta_1-\beta_2)+\operatorname{sign}(\zeta_2-\beta_2)+\operatorname{sign}(\zeta_3-\beta_2) = \pm 3 \\ -1 & \text{if } \operatorname{sign}(\zeta_1-\beta_2)+\operatorname{sign}(\zeta_2-\beta_2)+\operatorname{sign}(\zeta_3-\beta_2) = -1 \end{cases}, \tag{18}$$

then $\eta_{13} = h((\alpha_1,\alpha_2,\alpha_4),(\xi_2,\xi_3,\xi_4))$, or more generally, $\eta_{ij} = h((\alpha_1,\cdots,\widehat{\alpha_j},\cdots,\alpha_4),(\xi_1,\cdots,\widehat{\xi_i},\cdots,\xi_4))$. (As usual, the caret means omission.)

The analogous increments through $\xi_1 = \alpha_2$ are given by similar expressions, where α_2 and α_3 have been swapped, **and** the overall sign changed, thus

$$\Delta\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_2} = \frac{1}{2}(\xi_1-\alpha_2)^2\Big(\mathcal{I}_{su(3)}((\alpha_1,\alpha_3,\alpha_4);(\xi_2,\xi_3,\xi_4)) + \frac{\eta_{12}}{3}(\xi_1-\alpha_2)\Big)\,, \tag{19}$$

with now η_{12} determined by the signs of the three differences $\xi_i - \alpha_3$, $i = 2,3,4$.

These formulae also apply to the external hexagonal faces, for example $\xi_1 = \alpha_4$, but now $\mathcal{I}$ vanishes on one side, thus the formula actually gives the value of $\mathcal{I}$:

$$\Delta\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_4} = \mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1=\alpha_4+0} = \frac{1}{2}(\xi_1-\alpha_4)^2\Big(\mathcal{I}_{su(3)}((\alpha_1,\alpha_2,\alpha_3);(\xi_2,\xi_3,\xi_4)) + \frac{\eta_{14}}{3}(\xi_1-\alpha_4)\Big)\,, \tag{20}$$

where η_{14} determined by the signs of the three differences $\xi_i - \alpha_2$, $i = 2,3,4$. Note that the overall sign is in agreement with the positivity of the functions $\mathcal{I}_{su(4)}$ and $\mathcal{I}_{su(3)}$. Likewise, across the "upper" face $\xi_1 = \alpha_1$,

$$\Delta\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1 \nearrow\!\!\!\!\!\!\neq \alpha_1} = -\mathcal{I}_{su(4)}(\xi)\Big|_{\xi_1=\alpha_1-0} = -\frac{1}{2}(\xi_1-\alpha_1)^2\Big(\mathcal{I}_{su(3)}((\alpha_2,\alpha_3,\alpha_4);(\xi_2,\xi_3,\xi_4)) + \frac{\eta_{11}}{3}(\xi_1-\alpha_1)\Big)\,. \tag{21}$$

Finally, since $\mathcal{I}_{su(4)}$ is a symmetric function of the ξ_i, $i = 1, \cdots, 4$, the changes of determination across walls of equation $\xi_{2,3,4} = \alpha_j$ are obtained from the previous expressions by a permutation of the ξ's.

We now turn to the rectangular faces. Across a rectangular face of equation $\xi_i + \xi_j = \alpha_k + \alpha_l$, $\mathcal{I}_{su(4)}$ is of class C^2, hence its change of determination $\Delta\mathcal{I}_{su(4)}$ vanishes cubically. We find

$$\Delta\mathcal{I}_{su(4)}\Big|_{\xi_i+\xi_j \nearrow \alpha_k+\alpha_l} = \pm\frac{1}{6}\big(\xi_i + \xi_j - \alpha_k - \alpha_l\big)^3$$

The overall sign is determined by the positivity for external faces $\xi_i+\xi_j = \alpha_1+\alpha_2$ or $\xi_i+\xi_j = \alpha_3+\alpha_4$

$$\Delta\mathcal{I}_{su(4)}\Big|_{\xi_i+\xi_j \nearrow \alpha_1+\alpha_2} = -\mathcal{I}(\xi)\Big|_{\xi_i+\xi_j=\alpha_1+\alpha_2-0} = \frac{1}{6}(\xi_i + \xi_j - \alpha_1 - \alpha_2)^3\,, \tag{22}$$

$$\Delta\mathcal{I}_{su(4)}\Big|_{\xi_i+\xi_j \nearrow \alpha_3+\alpha_4} = \mathcal{I}(\xi)\Big|_{\xi_i+\xi_j=\alpha_3+\alpha_4+0} = \frac{1}{6}(\xi_i + \xi_j - \alpha_3 - \alpha_4)^3\,. \tag{23}$$

Internal walls of type (ii) may be written as $\xi_i+\xi_j = \alpha_1+\alpha_3$ or $\xi_i+\xi_j = \alpha_2+\alpha_3$, with $1 \leqslant i < j \leqslant 4$. As $\xi_i + \xi_j - \alpha_{\frac{1}{2}} - \alpha_3$ increases through 0, the *change* of polynomial determination is given by

$$\Delta\mathcal{I}_{su(4)}\Big|_{\xi_i+\xi_j \nearrow \alpha_1+\alpha_3} = -\frac{1}{6}(\xi_i + \xi_j - \alpha_1 - \alpha_3)^3 \tag{24}$$

$$\Delta\mathcal{I}_{su(4)}\Big|_{\xi_i+\xi_j \nearrow \alpha_2+\alpha_3} = \begin{cases} \frac{1}{6}(\xi_i + \xi_j - \alpha_2 - \alpha_3)^3 & \text{if } \alpha_3 \leqslant \xi_i, \xi_j \leqslant \alpha_2 \\ 0 & \text{otherwise}\,. \end{cases} \tag{25}$$

Example. Fig. 3 displays a cross-section of the permutahedron for $\alpha = \{5, 4, 2, -11\}$ at $\xi_3 = (\alpha_1 + \alpha_2)/2$. The hyperplanes $\xi_i + \xi_j = \alpha_1 + \alpha_3$, resp. $= \alpha_2 + \alpha_3$ intersect these cross-sections along the green lines, resp. the orange lines. There are $1 + 3 \times (9 + 3) = 37$ cells of polynomiality for $\max(\alpha_2, (\alpha_1 - \alpha_2 + \alpha_3)) \leqslant \xi_3 \leqslant \alpha_1$.

Remarks.

1. The above expressions have been obtained in a semi-empirical way, checking on many cases their consistency with the original expression (6). A direct and systematic proof would clearly be desirable.

2. Denote by $\{i, j, k, \ell\}$ a permutation of $\{1, 2, 3, 4\}$. The first case (24) above, (green lines on Fig. 3), occurs only if $\alpha_3 \leqslant \xi_i, \xi_j \leqslant \alpha_1$ and $\alpha_4 \leqslant \xi_k, \xi_\ell \leqslant \alpha_2$. Indeed if $\xi_i + \xi_j = \alpha_1 + \alpha_3 \Leftrightarrow \xi_k + \xi_\ell = \alpha_2 + \alpha_4$, then $\xi_i = \underbrace{(\alpha_1 - \xi_j)}_{\geqslant 0} + \alpha_3 \geqslant \alpha_3$ and of course $\xi_i \leqslant \alpha_1$, while $\xi_k = \alpha_2 + \underbrace{(\alpha_4 - \xi_\ell)}_{\leqslant 0} \leqslant \alpha_2$ and of course $\xi_k \geqslant \alpha_4$. In other words, the planes $\xi_i + \xi_j = \alpha_1 + \alpha_3$ do not intersect the permutahedron if those conditions are not fulfilled.

3. In contrast, in the second case (25) (orange lines on Fig. 3), there is a change of polynomial determination only if $\alpha_3 \leqslant \xi_i, \xi_j \leqslant \alpha_2$ although the planes $\xi_i + \xi_j = \alpha_2 + \alpha_3$ intersect the permutahedron irrespective of that condition.

This illustrates the well known fact that the forms given above in (8) for the loci of change of polynomial determinations are only necessary conditions. It may be that the function $\mathcal{I}$ is actually regular across some of these (hyper)planes.

4 Relation of $\mathcal{I}$ with multiplicities

4.1 The Kostant multiplicity formula

The multiplicity of the weight δ in the irreducible representation of highest weight λ, aka the Kostka number, may be written in various ways, e.g. following Kostant [15]

$$\mathrm{mult}_\lambda(\delta) = \sum_{w\in W} \epsilon_w P(w(\lambda + \rho) - \delta - \rho) \tag{26}$$

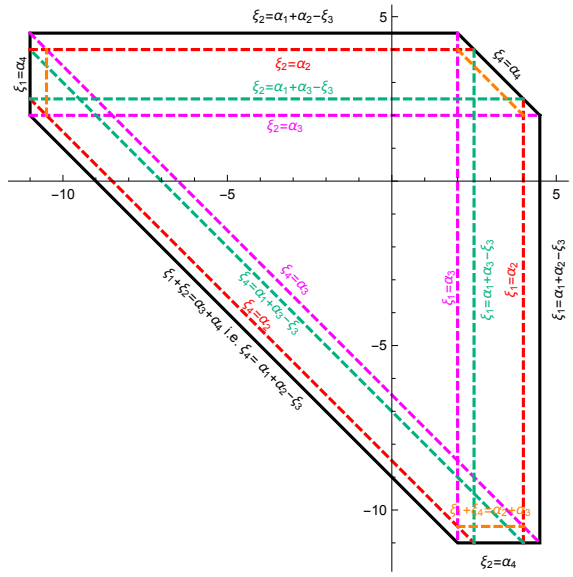


Figure 3: SU(4): A cross-section of the permutahedron, again for for $\alpha = \{5, 4, 2, -11\}$, along the plane at fixed $\xi_3 = (\alpha_1 + \alpha_2)/2$. (Recall that $\xi_4 = \sum_i \alpha_i - \xi_1 - \xi_2 - \xi_3$.) There are $1 + 3 \times (9 + 3) = 37$ cells of polynomiality. Note that the orange lines (of equation $\xi_i + \xi_j = \alpha_2 + \alpha_3$, $i, j = 1, 2, 4$) do not extend to the boundary of the permutahedron, reflecting the condition (25).

where P is Kostant's partition function ([12], Theorem 10.29). $P(\boldsymbol{\beta})$ gives the number of ways an element $\boldsymbol{\beta}$ of the root lattice may be decomposed as a non-negative integer linear combination of positive roots.

On the other hand, it has been pointed out by Heckman ([13], see also [11]) that *asymptotically*, for large weights λ and δ, one may write a semi-classical approximation of $\mathrm{mult}_\lambda(\delta)$. We shall recover below this result from the more general formula (39).

The Kostant partition functions for rank 2 Lie algebras are given in [20] and in [3]. The first reference also gives the partition function for A_3. The expression obtained by [3] for B_2 is more compact and this is the one that we give below. We used these formulae to check the consistency of expressions obtained by other means, using honeycombs or other pictographs, or counting the integer points in Berenstein-Zelevinsky (BZ) polytopes, see below.

If κ is a weight, with components k_i in the basis of simple roots ("Kac labels") (warning: *not* the basis of fundamental weights ("Dynkin labels")!), $P(\kappa)$ is given by:

SU(3): $$P(k_1, k_2) = \begin{cases} \min(k_1, k_2) + 1 & \text{if } k_1 \geqslant 0 \text{ and } k_2 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

SU(4): $$P(k_1, k_2, k_3) = \begin{pmatrix} 0 & k_1 < 0 \vee k_2 < 0 \vee k_3 < 0 \\ \frac{1}{6}(k_2+1)(k_2+2)(k_2+3) & k_2 \leqslant k_1 \wedge k_2 \leqslant k_3 \\ \frac{1}{6}(k_1+1)(k_1+2)(-2k_1+3k_2+3) & k_1 \leqslant k_2 \leqslant k_3 \\ \frac{1}{6}(k_1+1)(k_1+2)(-k_1+3k_3+3) & k_1 \leqslant k_3 \leqslant k_1+k_3 \leqslant k_2 \\ \frac{1}{6}(k_1+1)(k_1+2)(-k_1+3k_3+3) - \binom{k_1-k_2+k_3+2}{3} & k_1 \leqslant k_3 \leqslant k_2 \leqslant k_1+k_3 \\ \frac{1}{6}(k_3+1)(k_3+2)(3k_2-2k_3+3) & k_3 \leqslant k_2 \leqslant k_1 \\ \frac{1}{6}(k_3+1)(k_3+2)(3k_1-k_3+3) & k_3 \leqslant k_1 \leqslant k_1+k_3 \leqslant k_2 \\ \frac{1}{6}(k_3+1)(k_3+2)(3k_1-k_3+3) - \binom{k_1-k_2+k_3+2}{3} & k_3 \leqslant k_1 \leqslant k_2 \leqslant k_1+k_3 \end{pmatrix}.$$

Notice that for SU(4) there are seven cases (Kostant chambers of polynomiality) and that the last three cases are obtained from the previous three by exchanging k_1 and k_3.

$$B_2\colon\ P(k_1,k_2) = \begin{pmatrix} 0 & k_1<0 \vee k_2<0 \\ b(k_2) & k_2 \leqslant k_1 \\ \frac{1}{2}(k_1+1)(k_1+2) & 2k_1 \leqslant k_2 \\ \frac{1}{2}(k_1+1)(k_1+2) - b(2k_1-k_2-1) & k_1 \leqslant k_2 \leqslant 2k_1 \end{pmatrix}$$

Here the function b is defined by $b(x) = \left(-\lfloor\frac{x+1}{2}\rfloor + x + 1\right)\left(\lfloor\frac{x+1}{2}\rfloor + 1\right)$, in terms of the integer part function.

4.2 The $\mathcal{I}$–multiplicity relation

We follow here the same steps as in the case of the relation betweeen the Horn volume and LR coefficients [6, 5]. Take $\lambda' = \lambda + \rho$, where λ is the h.w. of the irrep V_λ and ρ is the Weyl vector.

$$\begin{aligned} \mathcal{I}(\lambda';\delta) &= \frac{\Delta(\lambda')}{\Delta(\rho)} \int_{\mathbb{R}^{n-1}} \frac{dx}{(2\pi)^{n-1}} \mathcal{H}(\lambda';\delta) e^{-\mathrm{i}\langle x,\delta\rangle} \\ &= \int_{\mathbb{R}^{n-1}} \frac{dx}{(2\pi)^{n-1}} \underbrace{\mathcal{H}(\lambda';\delta)\dim V_\lambda}_{\frac{\hat\Delta(e^{\mathrm{i}\,x})}{\Delta(\mathrm{i}\,x)}\chi_\lambda(e^{\mathrm{i}\,x})} e^{-\mathrm{i}\langle x,\delta\rangle} \\ &= \int_{\mathbb{R}^{n-1}/2\pi P^\vee} \frac{dx}{(2\pi)^{n-1}} \sum_{\psi\in 2\pi P^\vee} \frac{\hat\Delta(e^{\mathrm{i}\,(x+\psi)})}{\Delta(\mathrm{i}\,(x+\psi))} \chi_\lambda(e^{\mathrm{i}\,(x+\psi)}) e^{-\mathrm{i}\langle (x+\psi),\delta\rangle} \\ &= \int_{\mathbb{R}^{n-1}/2\pi P^\vee} \frac{dx}{(2\pi)^{n-1}} \sum_{\psi\in 2\pi P^\vee} e^{\mathrm{i}\langle \lambda+\rho-\delta,\psi\rangle} \frac{\hat\Delta(e^{\mathrm{i}\,x})}{\Delta(\mathrm{i}\,(x+\psi))} \chi_\lambda(e^{\mathrm{i}\,x}) e^{-\mathrm{i}\langle x,\delta\rangle} \end{aligned} \tag{27}$$

Following Etingof and Rains [10] one writes

$$\begin{aligned} \sum_{\psi\in 2\pi P^\vee} e^{\mathrm{i}\langle\rho,\psi\rangle} \frac{\hat\Delta(e^{\mathrm{i}\,x})}{\Delta(\mathrm{i}\,(x+\psi))} &= \sum_{\kappa\in K} r_\kappa \chi_\kappa(e^{\mathrm{i}\,x}) \\ \sum_{\psi\in 2\pi P^\vee} \frac{\hat\Delta(e^{\mathrm{i}\,x})}{\Delta(\mathrm{i}\,(x+\psi))} &= \sum_{\kappa\in \hat K} \hat r_\kappa \chi_\kappa(e^{\mathrm{i}\,x})\,, \end{aligned} \tag{28}$$

where the finite sets of weights K and $\hat K$ and the rational coefficients r_κ, $\hat r_\kappa$ have been defined in [6, 5], and one finds, with Q the root lattice,

$$\mathcal{I}(\lambda';\delta) = \begin{cases} \sum_{\kappa\in K} r_\kappa \sum_\tau C^\tau_{\lambda\kappa}\,\mathrm{mult}_\tau(\delta) & \text{if } \lambda-\delta\in Q \\ \sum_{\kappa\in \hat K} \hat r_\kappa \sum_\tau C^\tau_{\lambda\kappa}\,\mathrm{mult}_\tau(\delta) & \text{if } \lambda+\rho-\delta\in Q\,. \end{cases} \tag{29}$$

For su(3), the two formulae boil down to the same simple expression, since $\rho\in Q$ and $K=\hat K=\{0\}$, hence

$$\mathcal{I}_{su(3)}(\lambda';\delta) = \mathrm{mult}_\lambda(\delta)\,. \tag{30}$$

This expression is compatible with that given in (12), thanks to a peculiar identity that holds in su(3): $\mathcal{I}_{su(3)}(\lambda+\rho;\delta) = \mathcal{I}_{su(3)}(\lambda;\delta)+1$ or equivalently $\mathrm{mult}_{\lambda+\rho}(\delta) = \mathrm{mult}_\lambda(\delta)+1$. The latter is itself obtained in the large $\mu,\nu=\mu+\delta$ limit of the more general identity $C^{\nu+\rho}_{\lambda+\rho\,\mu+\rho} = C^\nu_{\lambda\mu}+1$ already mentioned in [7, 8].

For su(4), we have two distinct relations

$$\mathcal{I}_{su(4)}(\lambda';\delta) = \begin{cases} \frac{1}{24}\left(9\,\mathrm{mult}_\lambda(\delta) + \sum_\tau C^\tau_{\lambda\,\{1,0,1\}}\mathrm{mult}_\tau(\delta)\right) & \text{if } \lambda-\delta\in Q \\ \frac{1}{6}\sum_\tau C^\tau_{\lambda\,\{0,1,0\}}\mathrm{mult}_\tau(\delta) & \text{if } \lambda-\delta-\rho\in Q\,. \end{cases} \tag{31}$$

In particular for $\lambda=0$,

$$\mathcal{I}_{su(4)}(\rho;\delta) = \begin{cases} \frac{1}{24}\left(9\,\delta_{\delta 0} + \mathrm{mult}_{\{1,0,1\}}(\delta)\right) & \text{if } \delta\in Q \\ \frac{1}{6}\mathrm{mult}_{\{0,1,0\}}(\delta) & \text{if } \delta-\rho\in Q \end{cases} \tag{32}$$

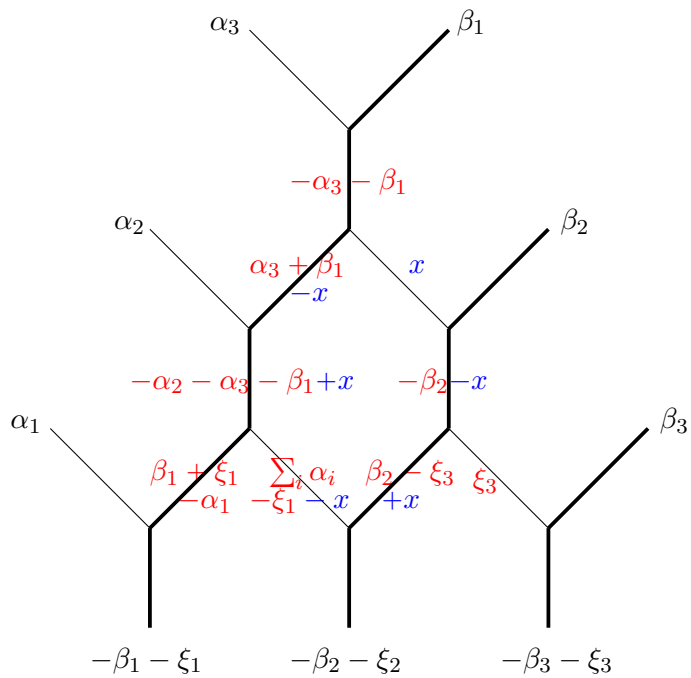


Figure 4: Knutson–Tao's honeycomb for $n = 3$ for $\gamma, \beta \gg \alpha, \xi = \gamma - \beta$. Thick lines carry the large values $\beta_1 > \beta_2 > \beta_3 \gg \alpha_i, \xi_i$.

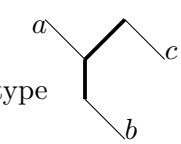
thus for $\delta = 0$, $\mathcal{I}_{su(4)}(\rho; 0) = \frac{1}{2}$ and for $\delta = \{1, 0, 1\}$ (or any of its Weyl images), $\mathcal{I}_{su(4)}(\rho; \delta) = \frac{1}{24}$, while for $\delta = \{0, 1, 0\}$ (or any of its Weyl images), $\mathcal{I}_{su(4)}(\rho; \delta) = \frac{1}{6}$.

Remark. Inverting the $\mathcal{I}$–multiplicity formula (29) is an interesting question that has been addressed in [17], sect. 6.

4.3 Polytopes and reduced KT honeycombs

Recall that Knutson–Tao (KT) honeycombs or other pictographs relevant for the LR coefficients of $su(n)$ depend on $(n-1)(n-2)/2$ parameters. For example, in the KT honeycombs, the $3n$ external edges carry the components of α, β, γ, while each internal line carries a number, such that at each vertex the sum of the incident numbers vanishes. Moreover, for each of the $3n(n+1)/2$ internal edges, one writes a certain inequality between those numbers. In the limit $\beta \sim \gamma \gg \alpha, \xi$, one third of these inequalities is automatically satisfied and one is left with $n(n+1)$ linear inequalities on the $(n-1)(n-2)/2$ parameters.

This is illustrated on Fig. 4 for $n = 3$. There, the choice of the parameters is such that the large numbers of order $\mathrm{O}(\beta, \gamma)$ are carried by North–South and NE–SO lines (heavy lines on the figure), while the NO–SE lines carry numbers of order $\mathrm{O}(\alpha, \xi)$. Inequalities attached to the latter are automatically satisfied.



In general, the surviving inequalities are of the type $c \leqslant a \leqslant b$ for all patterns of the type within the honeycomb.

One then sees that the KT honeycomb boils down to a Gelfand–Tsetlin (GT) triangle, see Fig.

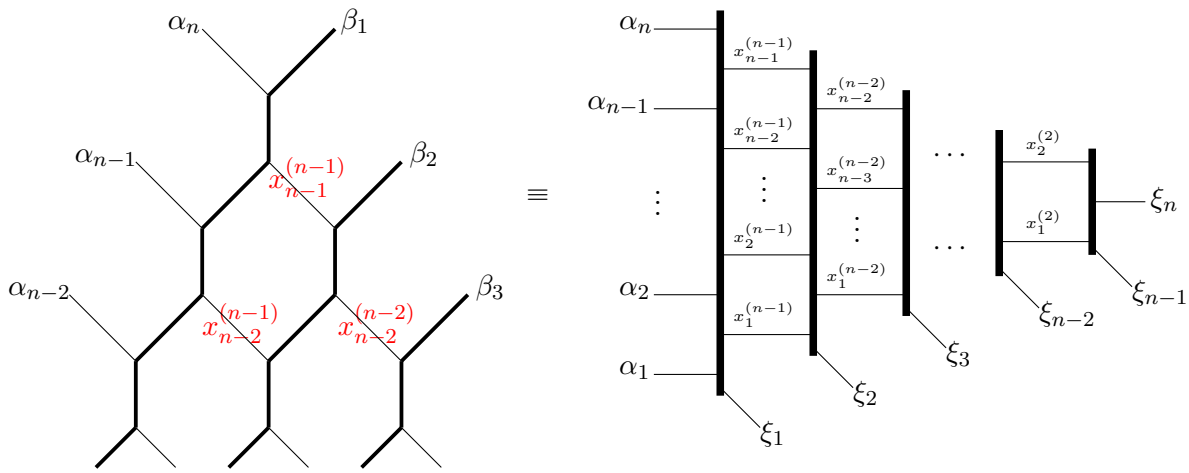


Figure 5: A Knutson–Tao's honeycomb for $\gamma, \beta \gg \alpha, \xi = \gamma - \beta$ reduced to a GT triangle.

5, and one is left with the GT inequalities

$$\begin{array}{ccccccc} \alpha_n \leqslant x_{n-1}^{(n-1)} \leqslant \alpha_{n-1} \leqslant & & \cdots & & \leqslant x_1^{(n-1)} \leqslant \alpha_1 & & \\ & x_{i+1}^{(j+1)} & \leqslant x_i^{(j)} \leqslant & x_i^{(j+1)} \,, & & 1 \leqslant i, j \leqslant n-2 \end{array} \tag{33}$$

together with the $n-2$ conservation laws

$$\sum_{i=1}^{n} \alpha_i = \xi_1 + \sum_{j=1}^{n-1} x_j^{(n-1)} = \xi_1 + \xi_2 + \sum_{j=1}^{n-2} x_j^{(n-2)} = \cdots = \sum_{i=1}^{n-2} \xi_i + x_1^{(2)} + x_2^{(2)} = \sum_{i=1}^{n} \xi_i \,. \tag{34}$$

According to the well-known rules, the semi-standard tableaux corresponding to that triangle must have $x_1^{(1)} = \xi_n$ boxes containing 1 (necessarily in the first row); $x_1^{(2)} + x_2^{(2)} - x_1^{(1)} = \xi_{n-1}$ boxes containing 2 (in the first two rows); etc; and ξ_1 boxes containing n.

Relations (33-34) define a polytope $\mathcal{P}(\alpha;\xi)$ in $\mathbb{R}^{(n-1)(n-2)/2}$, whose volume is the function $\mathcal{I}_{su(n)}(\alpha;\xi)$, as we show below in sect. 4.6.

Example. For $n = 3$, these relations reduce to

$$\begin{array}{rcl} \alpha_3 \leqslant x_2^{(2)} & \leqslant & \alpha_2 \leqslant x_1^{(2)} \leqslant \alpha_1 \\ x_2^{(2)} \leqslant \xi_3 & \equiv & x_1^{(1)} \leqslant x_1^{(2)} \\ \sum_{i=1}^{3} \alpha_i & = & \xi_1 + x_1^{(2)} + x_2^{(2)} = \sum_{i=1}^{3} \xi_i \,, \end{array} \tag{35}$$

hence to the following bounds on, say, $x_1^{(2)}$

$$\max(\alpha_2, \xi_2, \xi_3, \xi_2 + \xi_3 - \alpha_2) \leqslant x_1^{(2)} \leqslant \alpha_1 - \xi_1 + \min(\xi_1, \alpha_2) \tag{36}$$

which leads to the expression (11) of $\mathcal{I}_{su(3)}(\alpha;\xi)$.

A similar discussion may be carried out for the other algebras, based on the BZ inequalities. For example for B_2, see below sect. 5.

4.4 The Kostka multiplicity as a limit of LR coefficients

In the same spirit as the large β, γ limit above, we have

$$\mathrm{mult}_\lambda(\delta) = \lim_{s\to\infty} C_{\lambda\ \mu+s\rho}^{\mu+s\rho+\delta} \tag{37}$$

independently of the choice of μ. Actually we can choose $\mu = \rho$, so that the above formula reads

$$\mathrm{mult}_\lambda(\delta) = \lim_{s\to\infty} C_{\lambda\ s\rho}^{s\rho+\delta} \tag{38}$$

One can find an integer s_c such that $\forall s \geqslant s_c$, $\mathrm{mult}_\lambda(\delta) = C_{\lambda\ s\rho}^{s\rho+\delta}$. In the case A_3 for example, taking $\lambda = \{4, 5, 3\}$, $\delta = \{-4, -2, 5\}$, one has $\mathrm{mult}_\lambda(\delta) = 26$, whereas taking $s = 4, 5, 6, 7, 8, 9, \ldots$ one finds $C_{\lambda\ s\rho}^{s\rho+\delta} = 6, 19, 24, 26, 26, 26, \ldots$, so $s_c = 7$.

It is interesting to find a general upper bound for s_c (we are not aware of any attempt of this kind in the literature). We shall find such a bound in the case of the Lie algebra A_3. As the reader will see, our method relies on a brute-force calculation; it would be nice to find a more elegant approach that could be generalized to A_n, or more generally, to any simple Lie algebra. For a given highest weight λ and any weight δ belonging to its weight system, we calculate the Littlewood-Richardson coefficient $S(\lambda, \delta, s) = C_{\lambda\ s\rho}^{s\rho+\delta}$ from the Kostant-Steinberg formula, using the Kostant partition function for A_3 given in sect. 4.1, and show explicitly (this is indeed a brute-force calculation !) that the difference $S(\lambda, \delta, s+1) - S(\lambda, \delta, s)$ vanishes, equivalently, that $S(\lambda, \delta, s)$ is stationary, for values of λ, δ, s obeying the following set of constraints:

$$\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_3 \geqslant 0, \lambda_1+\lambda_2+\lambda_3 > 0, s > 0, 4(s+1) > X_1 \geqslant 0, 2(s+1) > X_2 \geqslant 0, 4(s+1) > X_3 \geqslant 0.$$

where

$$\begin{aligned} X_1 &= (\lambda_1 + 2\lambda_2 + 3\lambda_3) - (\delta_1 + 2\delta_2 + 3\delta_3), \\ X_2 &= (\lambda_1 + 2\lambda_2 + \lambda_3) - (\delta_1 + 2\delta_2 + \delta_3), \\ X_3 &= (3\lambda_1 + 2\lambda_2 + \lambda_3) - (3\delta_1 + 2\delta_2 + \delta_3). \end{aligned}$$

The s-independent inequalities relating λ and δ are nothing else than the Schur inequalities; equivalently, they can be obtained by writing that the partition α is larger than the weight ξ for the dominance order on partitions — as everywhere in this paper, α and ξ refer to the Young components (partitions) associated with the weights λ and δ. In the previous set of constraints, one can replace the s-independent inequalities relating λ and δ by the following ones: $-(\lambda_1 + \lambda_2 + \lambda_3) \leqslant \phi \leqslant (\lambda_1 + \lambda_2 + \lambda_3)$, where ϕ can be δ_1, δ_2, δ_3 or $(\delta_1 + \delta_2 + \delta_3)$. The latter, namely the one with $\phi = (\delta_1 + \delta_2 + \delta_3)$, expresses the fact that λ (resp. $-\overline{\lambda}$) is the highest weight (resp. is the lowest weight). The obtained s-dependent inequalities imply $4(s_c + 1) < \max(X_1, 2X_2, X_3) + 1$.

For a given highest weight λ this allows us to obtain a bound independent of the choice of the weight δ. One finds $s_c \leqslant 2(\lambda_1 + \lambda_2 + \lambda_3)$. Indeed, one can check explicitly that, given λ, and for any δ of its weight system, the function $S(\lambda, \delta, s)$ is stationary for $s \geqslant 2\sum_j \lambda_j$.

4.5 Reduced O-blades and reduced isometric honeycombs

The general discussion carried out in section 4.3 could be expressed in terms of other pictographs, for instance BZ-triangles, O-blades, or isometric honeycombs (see our discussion in [7] or [8] for a presentation of the last two). We have seen how KT honeycombs are "reduced" to GT patterns when one moves from the Horn problem to the Schur problem. An analogous reduction holds if we use isometric honeycombs or rather, their O-blades partners. We shall illustrate this with SU(4) by choosing the (dominant) weight $\lambda = \{4, 5, 3\}$ and the weight $\delta = \{-4, -2, 5\}$; here components are

expressed in terms of Dynkin labels[3]. Equivalently, in terms of partitions, we have $\alpha = (12, 8, 3, 0)$, and $\xi = (-1, 3, 5, 0) + (4, 4, 4, 4) = (3, 7, 9, 4)$.

λ is the highest weight of an irreducible representation (of dimension 16500), and the weight subspace associated with δ has dimension 26. There is a basis in this representation space for which every basis vector can be attached to a semi-standard Young tableau with filling $1, 2, 3, 4$ and shape (partition) α, or, equivalently, to a Gelfand–Tsetlin pattern. Here is one of them, and one sees immediately that its associated basis vector indeed belongs to the weight subspace defined by δ (or ξ):

$$\text{Young tableau}: \begin{array}{cccccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & & & & \\ 3 & 4 & 4 & & & & & & & & & \end{array} \qquad \text{with GT pattern}: \begin{array}{c} 12, 8, 3, 0 \\ 12, 6, 1 \\ 7, 3 \\ 3 \end{array}$$

Remember that the sequence of lines of the GT pattern is obtained from the chosen Young tableau by listing the shapes of the tableaux obtained by removing successive entries, starting from the largest one (here 4): $\{\{1,1,1,2,2,2,2,3,3,3,3,3\},\{2,2,2,3,3,3\},\{3\}\}, \{\{1,1,1,2,2,2,2\},\{2,2,2\}\}, \{\{1,1,1\}\}$.

Using $s = 1000$ in (38), we display in (Fig. 6, Left) one[4] of the O-blades describing the space of intertwiners for the triple[5] $((s\rho, s\rho + \delta) \to \lambda))$. It is a member of a one-parameter (s) family of O-blades for which the integers carried by the horizontal edges are s-dependent and for which the other edge values stay constant as soon as $s > s_c$. What remains in the limit of large s (meaning $s > s_c$) is only the "reduced O-blade" given in (Fig. 6, Middle) where we have removed the 0's and the values carried by the s-dependent *horizontal* edges since they are irrelevant.
The net result is that we have as many distinct reduced O-blades (for instance those obtained by reducing the ones associated with the triple $((s\rho, s\rho + \delta) \to \lambda))$ as we have distinct GT patterns, namely, a number equal to the dimension $\text{mult}_\lambda(\delta)$ of the weight subspace defined by δ. Those (26 of them) associated with the example chosen above are displayed on Fig. 12.
Isometric honeycombs cannot be displayed on a page if s is large (because they are isometric !), so we choose $s = 8$ and display on Fig. 6 (Right) the isometric honeycomb partner of the O-blade obtained, in the same one-parameter family, for this value of s (here $s_c = 7$ so that taking $s = 8$ is enough). Now the integers carried by the *vertical* edges are irrelevant (they would change with s): what matters are the integers carried by the non-vertical edges of the –possibly degenerate– parallelo-hexagons.

The particular reduced O-blade displayed in Fig. 6 (Middle) can also be obtained directly, by a simple combinatorial rule, from the Young tableau displayed previously, or from the corresponding GT pattern, without any appeal to the limit procedure (38) using Littlewood-Richardson coefficients, and without considering their associated pictographs: see the Appendix "Lianas and forests".

4.6 Asymptotics of the $\mathcal{I}$–multiplicity formula: $\mathcal{I}$ as a volume and stretching polynomials.

We may repeat the same chain of arguments as in Horn'problem [8, 6]: upon scaling by $p \gg 1$ of λ, δ in (29)

[3]Remember that external sides of KT honeycombs are labelled by integer partitions whereas external sides of O-blades or of isometric honeycombs are labelled by the Dynkin labels of the chosen weights.

[4]As we shall see below, this choice (among the 26 O-blades obtained when $s > s_c$) is not arbitrary but dictated by our wish to establish a link with the previously chosen Young tableau.

[5]In order to ease the discussion of the correspondence with Young tableaux, and also to draw the weight λ on the bottom of each pictograph, it is better to display the O-blades (or their isometric honeycombs partners) associated with the Littlewood-Richardson coefficient $C^{\lambda}_{s\rho+\delta, s\rho}$ or with $C^{\lambda}_{s\rho, s\rho+\delta}$ rather than those associated with $C^{s\rho+\delta}_{\lambda, s\rho}$. Notice that the numbers of such pictographs are all equal to $\text{mult}_\lambda(\delta)$ when s is big enough. The choice made in Fig. 6, 12 corresponds to the triple $((s\rho, s\rho + \delta) \mapsto \lambda)$.

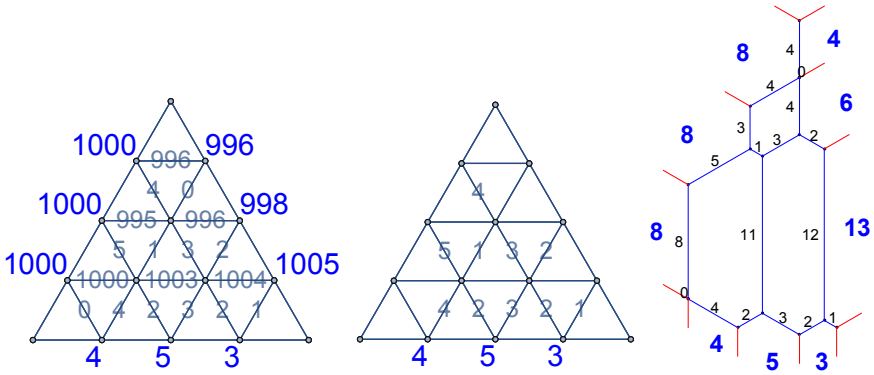


Figure 6: Left : One of the 26 O-blades associated with $C^{\lambda}_{s\rho+\delta,s\rho}$ for $\lambda = \{4,5,3\}$, $\delta = \{-4,-2,5\}$, $s = 1000$. Middle: same, with the 0's and the s-dependent labeling of horizontal lines removed. Right: The corresponding isometric honeycomb for the choice $s = 8 > s_c$.

$$
\begin{aligned}
\mathcal{I}(p\lambda+\rho,p\delta) &\approx \mathcal{I}(p\lambda,p\delta) = p^{(n-1)(n-2)/2}\mathcal{I}(\lambda,\delta) \\
&= \sum_{\kappa\in K} r_\kappa \sum_\tau C^{\tau}_{p\lambda\,\kappa}\mathrm{mult}_\tau(p\delta) \\
&\approx \sum_{\kappa\in K} r_\kappa \sum_\tau \sum_{k\in[\kappa]} \delta_{\tau,p\lambda+k}\mathrm{mult}_\tau(p\delta) \\
&= \sum_{\kappa\in K} r_\kappa \sum_{k\in[\kappa]} \mathrm{mult}_{p\lambda+k}(p\delta) \\
&= \underbrace{\sum_{\kappa\in K} r_\kappa \dim_\kappa}_{=1} \mathrm{mult}_{p\lambda}(p\delta) \\
&= \mathrm{mult}_{p\lambda}(p\delta) = p^d \mathrm{vol}_d(\mathcal{P}(\lambda;\delta)) + \cdots
\end{aligned}
\tag{39}
$$

where $\mathcal{P}(\lambda;\delta)$ is the polytope defined in sect. 4.3 or associated with one of the pictographs mentioned in sect. 4.5. Thus for generic cases for which $d = (n-1)(n-2)/2$, we have the identification

$$
\mathcal{I}(\lambda,\delta) = \mathrm{vol}_d(\mathcal{P}(\lambda;\delta))\,, \tag{40}
$$

(while for the non generic cases, both $\mathcal{I}$ and $\mathrm{vol}_{(n-1)(n-2)/2}$ vanish).

For illustration we consider the weights $\lambda = \{4,5,3\}$ *and* $\delta = \{-4,-2,5\}$ of SU(4) already chosen in a previous section. The multiplicities obtained by scaling them by a common factor $p = 1,2,3,\ldots$ lead to the sequence $26, 120, 329, 699, 1276, 2106, 3235, 4709, 6574, 8876, \ldots$, which can be encoded by the cubic polynomial $\frac{1}{6}(6 + 35p + 69p^2 + 46p^3)$ whose dominant term is $23/3$, which is indeed the value of the Schur volume function $\mathcal{I}(\lambda,\delta)$.

Another way to obtain this value is to use the volume function *for the Horn problem*, that we called $\mathcal{J}(\lambda,\mu,\nu)$ in refs [8] and [6]. In the generic case this function gives the volume of the hive polytope associated with the triple $(\lambda,\mu) \to \nu$ i.e., the polytope whose $C^{\nu}_{\lambda,\mu}$ integer points label the honeycombs for this specific space of intertwiners. It also gives the leading coefficient of the Littlewood-Richardson polynomial (a polynomial[6] in the variable p) giving, when p is a non-negative integer, the LR coefficient $C^{p\nu}_{p\lambda,p\mu}$ for highest weights scaled by p. Since the Kostka numbers (multiplicities of weights) can be obtained as a limit of LR coefficients for special arguments

[6]We assume here that the chosen Lie group is SU(n), otherwise, this object may be a quasi-polynomial.

(see (38)), the same is true under scaling. For $s > s_c$ (determined by the choice of λ and δ), we have therefore $\mathcal{I}(\lambda, \delta) = \mathcal{J}(\lambda, s\rho, s\rho + \delta)$. We check, on the same example as before, that we have indeed, $\mathcal{J}(\{4,5,3\}, \{s,s,s\}, \{s-4, s-2, s+5\}) = 13/3, 7, 23/3, 23/3, 23/3, 23/3, \ldots$ for $s = 5, 6, 7, 8, \ldots$ As expected, the values of $\mathcal{J}$ stabilize, and the asymptotics, i.e., $\mathcal{I}(\lambda, \delta)$, is reached for $s = 7$. Notice the various kinds of scalings involved here: 1) a scaling of ρ by s, with the weights λ and δ remaining constant, 2) a scaling of λ and δ by the non-negative integer p, 3) a simultaneous scaling of the three arguments of $\mathcal{J}$ by a non-negative integer p giving rise to a polynomial (in p) encoding the LR coefficients $C_{p\lambda, ps\rho}^{ps\rho + p\delta}$.

5 The case of B_2

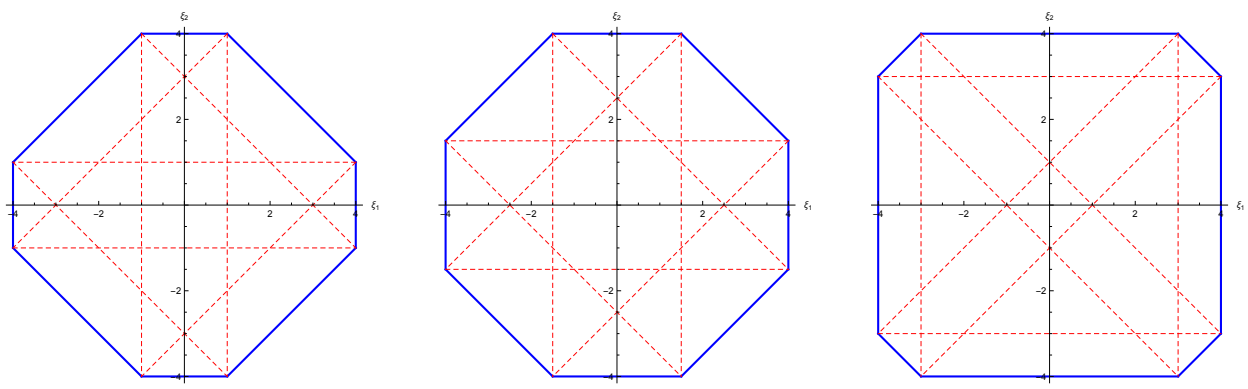


Figure 7: The Schur polytope $\mathcal{O}$ in the B_2 case, with its singular lines, in the three cases $\alpha = (4, 1)$, $\alpha = (4, \frac{3}{2})$ and $\alpha = (4, 3)$.

5.1 The Schur octagon

Consider the orbit $\mathcal{O}_\alpha$ of the group SO(5) acting by conjugation on a block-diagonal skew symmetric matrix
$A = \operatorname{diag}\left(\begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}_{i=1,2}, 0\right)$ with real α_i. Note that one may choose $0 \leqslant \alpha_2 \leqslant \alpha_1$. Schur's problem reads: what can be said about the projections ξ_i of a matrix of $\mathcal{O}_\alpha$ on an orthonormal basis X_i of the Cartan algebra, $\xi_i = \operatorname{tr} O.A.O^T.X_i$, $i = 1, 2$, where $O \in \mathrm{SO}(5)$? More specifically, if the matrix O is taken randomly and uniformly distributed in SO(5), what is the PDF of the ξ's? There is again a connection with the corresponding Horn problem, as discussed for example in [6]. Taking the limit $\beta, \gamma \gg \alpha$, $\xi = \gamma - \beta$ finite, in the expressions and figures of [6], we find that the support of the Schur volume is the octagon

$$\mathcal{O} \quad : \qquad -\alpha_1 \leqslant \xi_1, \xi_2 \leqslant \alpha_1 \,, \qquad -(\alpha_1 + \alpha_2) \leqslant \xi_1 \pm \xi_2 \leqslant \alpha_1 + \alpha_2 \,, \tag{41}$$

and that the singular lines of the PDF, or of the associated volume function, are

$$\xi_1, \xi_2 = \pm \alpha_2 \,, \qquad \xi_1 \pm \xi_2 = \pm(\alpha_1 - \alpha_2) \,,$$

see Fig. 7. Three cases occur, depending on whether the ratio α_2/α_1 belongs to $(0, 1/3)$, $(1/3, 1/2)$ or $(1/2, 1)$. For a given value of that ratio, $1 + 4 \times 6 = 25$ cells occur, and in total, one should have $1 + 4 \times 8 = 33$ possible cells, in accordance with Bliem's result [2].

Working out this same limit in the expressions of the Horn volume given in [6], we find a (relatively) simple expression for the Schur volume function

$$\begin{aligned} \mathcal{I}_{B_2}(\alpha;\xi) &= 0 \qquad \text{if } \xi \notin \mathcal{O} \\ \Delta\mathcal{I}_{B_2}(\alpha;\xi) &= \begin{cases} \pm\frac{1}{2}(\xi_i-\alpha_j)^2 & \text{across any line } \xi_i=\alpha_j \qquad i,j=1,2 \\ \pm\frac{1}{4}(\xi_1+\epsilon\xi_2-\epsilon''(\alpha_1+\epsilon'\alpha_2))^2 & \text{across any line } \xi_1+\epsilon\xi_2=\epsilon''(\alpha_1+\epsilon'\alpha_2) \end{cases} \end{aligned} \tag{42}$$

$\epsilon,\epsilon',\epsilon''=\pm1$, and with the overall sign of the change determined by the prescription of Fig. 8.

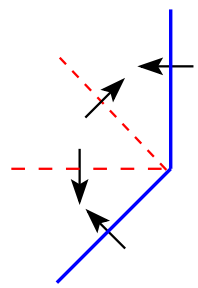


Figure 8: Prescriptions for the changes of $\mathcal{I}_{B_2}$ across the lines emanating from a vertex: the sign in (42) is + if the line is crossed along the direction of the arrow, − otherwise. This holds for any rotated configuration of that type.

One may finally plot the resulting PDF $\mathrm{p}(\xi|\alpha)=\frac{3}{2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)}\mathcal{I}_{B_2}(\alpha,\xi)$ for a given value of α and compare it with the "experimental" histogram of a large random sampling of matrices of $\mathcal{O}_\alpha$, see Fig. 9.

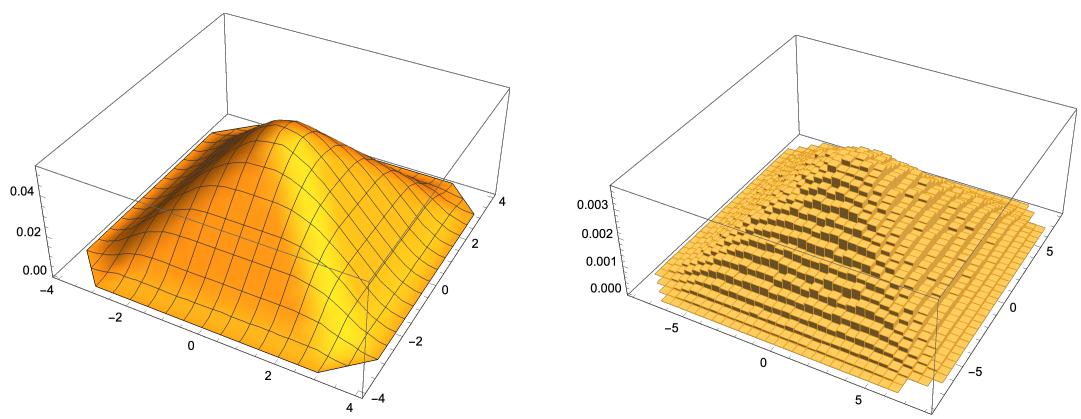


Figure 9: Comparing the plot of the PDF with the histogram of ξ values obtained from 10^6 matrices of the orbit $\mathcal{O}_\alpha$, for $\alpha=(4,3)$.

Repeating the steps followed in sect. 4.2 and making use of results in sect. 3 of [6], one derives a relation between $\mathcal{I}_{B2}$ and multiplicities:

$$\mathcal{I}_{B_2}(\lambda';\delta)=\begin{cases} \frac{1}{8}\left(3\,\mathrm{mult}_\lambda(\delta)+\sum_\tau C^\tau_{\lambda\,\{1,0\}}\mathrm{mult}_\tau(\delta)\right) & \text{if } \lambda-\delta\in Q \\ \frac{1}{4}\sum_\tau C^\tau_{\lambda\,\{0,1\}}\mathrm{mult}_\tau(\delta) & \text{if } \lambda-\delta-\rho\in Q\,. \end{cases} \tag{43}$$

In particular for $\lambda=0$,

$$\mathcal{I}_{B_2}(\rho;\delta)=\begin{cases} \frac{1}{8}\left(3\,\delta_{\delta 0}+\mathrm{mult}_{\{1,0\}}(\delta)\right) & \text{if } \delta\in Q \\ \frac{1}{4}\mathrm{mult}_{\{0,1\}}(\delta) & \text{if } \delta-\rho\in Q \end{cases} \tag{44}$$

For example, take for δ the short simple root, $\delta = (0,1) = \{-1,2\} \in Q$, $\mathcal{I}_{B_2}(\rho;\delta) = \frac{1}{8}$, $\mathrm{mult}_{\{1,0\}}(\delta) = 1$.

Remarks.
1. A general piece-wise polynomial expression of $\mathrm{mult}_\lambda(\delta)$ has been given by Bliem [2]. His expressions should be consistent with the relation (43) and our implicit expression of $\mathcal{I}_{B_2}$ in (42).
2. One should also notice that (38) holds true in general, and thus in the current B_2 case. For example, $\mathrm{mult}_{\{20,12\}}(\{18,-6\}) = 56$ may be recovered asymptotically as $C^{\{s+18,s-6\}}_{\{20,12)\}\,\{s,s\}}$, which takes the values $0,0,0,0,0,3,8,14,20,26,31,36,40,44,47,50,52,54,55,56,56,56,56,56,56$ as s grows from 1 to 25. Hence here $s_c = 20$.

5.2 A_3 versus B_2

It is well known that the B_2 root system may be obtained by folding that of A_3. It is thus suggested to compare Kostka multiplicities for cases that enjoy some symmetry in that folding. Consider in particular the special case of $\lambda = \{\lambda_1, 1, \lambda_1\}$. Inequalities on the three parameters i,j,k of A_3 oblades for $\mathrm{mult}_{\{\lambda_1,\lambda_2=1,\lambda_1\}}(\{\delta_1,\delta_2,\delta_1\})$ reduce to

$$\begin{aligned}&\{i \leqslant \lambda_1,\ \delta_2 \leqslant 2k+1,\ \delta_1 + k \leqslant i,\ \delta_1 + j + k + \lambda_1 \geqslant i,\ -1 \leqslant \delta_2 - 2(j+k) \leqslant 1, \\ &i \geqslant 0,\ i+j \geqslant 0,\ \delta_1 + j + k \leqslant i,\ \delta_1 + j + 2k \leqslant i,\ \delta_2 + 1 \geqslant 2j,\ i \geqslant k\}\end{aligned} \tag{45}$$

Consideration of a large number of examples then suggest the following
Conjecture 1:
i) the number of triplets (i,j,k) satisfying these inequalities is a square integer, viz

$$\mathrm{mult}_{\{\lambda_1,1,\lambda_1\}}(\{\delta_1,\delta_2,\delta_1\}) = m^2\,;$$

ii) the corresponding B_2 multiplicity is then $\mathrm{mult}_{\{\lambda_1,1\}}(\{\delta_1,\delta_2\}) = m\,.$

5.3 Stretching (quasi)polynomial for the B_2 multiplicity

Let $\kappa = \lambda - \delta$. As well known, $\mathrm{mult}_\lambda(\delta)$ vanishes if $\kappa \notin Q$, the root lattice, i.e., if κ_2 (in Dynkin indices) is odd. Now consider the stretched multiplicity $\mathrm{mult}_{s\lambda}(s\delta)$, $s \in \mathbb{N}$. It is in general a quasi-polynomial of s. Again, we have found a fairly strong evidence for, and we propose the following
Conjecture 2:
i) for λ_2 and δ_2 both even, $\mathrm{mult}_{s\lambda}(s\delta)$ *is a polynomial of s for κ_1 even, and, except for a finite number of cases, a quasi-polynomial for κ_1 odd;*
ii) for λ_2 and δ_2 both odd, generically it is a quasi-polynomial, except if $2\mathcal{I}_{B_2}$ (twice the Schur volume) is an integer.

5.4 Pictographs for B_2 ?

A combinatorial algorithm based on Littelmann's paths [16] has been proposed by Bliem[2]. We tried (hard) to invent a B_2 analog of O-blades, either degenerate (Kostka coefficients) or not (LR coefficients). It seems that it cannot be done without introducing edges carrying both positive and negative integers, and the result is not particularly appealing, so we leave this as an open problem !

Acknowledgements

It is a great pleasure to thank Colin McSwiggen for several enlightening discussions and suggestions.

Appendix : Lianas and forests

We now explain how to obtained directly a reduced O-blade from an SU(n) Young tableau. We do not assume that the given tableau is semi-standard, but its entries along any chosen column should increase when going down.

Draw an equilateral triangle with sides of length n, choose one side ("ground level") and mark the inner points $1, \ldots, n-1$. To every column (with j elements) of the chosen tableau, associate a zigzag line (a *liana*) going upward, but only north-west or north-east, from the marked point j located on the ground level of the triangle. When going up, there are $n+1$ consecutive levels for SU(n), the ground level being the first. An entry marked p in the chosen column makes the associated liana to grow upward to the left between levels $(n+1)-p$ and $(n+1)-(p-1)$, otherwise, it grows upward to the right. When it reaches the boundary of the surrounding triangle, the liana continues upward, following the boundary. One finally superimposes the lianas rooted in the same points (the positions $1, 2, \ldots, n-1$), formally adding the characteristic functions of their sets, when they grow from the same point, and obtain in this way a *liana forest*.

As in sect. 4.5 we consider the SU(4) example $\begin{array}{cccccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & & & & \\ 3 & 4 & 4 & & & & & & & & & \end{array}$. We have 12 lianas with five levels. Reading this Young tableau from right to left we see that left-going directions occur between levels specified by $\{\{3\},\{3\},\{3\},\{3\},\{3,4\},\{2,4\},\{2,3\},\{2,3\},\{2,3\},\{1,2,4\},\{1,2,4\},\{1,2,3\}\}$, to which we associate the lianas displayed in Fig. 10. For instance the 6th liana, which is rooted in the second point, is described by $\{2,4\}$, so it grows upward to the left above levels $3 = 5-2$ and $1 = 5-4$, otherwise it grows to the right.

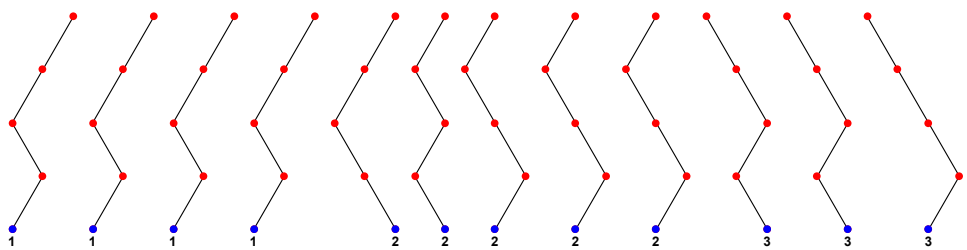


Figure 10: SU(4): Lianas associated with the Young tableau chosen in the text.

When superimposing the lianas rooted in the same points we recover the reduced O-blade displayed of Fig. 6, (Middle), as a liana forest: see Fig. 11. Notice that the labels 9, 3, 5, 3, 1, on the boundary of the triangle given in Fig. 11 (Right), are absent in the reduced O-blade, but they can be obtained immediately from the latter by using the property that when a liana reaches the boundary, it continues upward, following the boundary.

The chosen dominant weight (the highest weight $\lambda = \{4, 5, 3\}$), with associated Young diagram of shape $(12 = 4+5+3, 8 = 5+3, 3, 0)$, a (decreasing) partition of 23, appears at the first level of the liana forest. The weight ξ can be read from the sequence giving the total number of left-going edges, when one moves downward, level after level; namely $(3, 7, 9 = 8+1, 4)$, a (non-decreasing) partition of 23, for which the associated weight is indeed $\delta = \{-4, -2, 5\} = \{3-7, 7-9, 9-4\}$. This could also be read from the Young tableau given at the beginning of sect. 4.5. Notice that the sequence of lianas (given by Fig. 10) or the *indexed liana forest* (that keeps track of the origin of each liana, see Fig. 11, Left) has the same information contents as the Young tableau we started from. Part of this information is lost when we superimpose the lianas and remove the indices that give the point they grow from (Fig. 11, Right): several distinct sequences of lianas (i.e., distinct Young tableaux, not necessarily semi-standard, or distinct indexed liana forests) may give rise to the same (unindexed) liana forest. If we are only interested in multiplicities and not in the construction of explicit basis vectors in representation spaces, one may forget about *indexed* liana forests because of

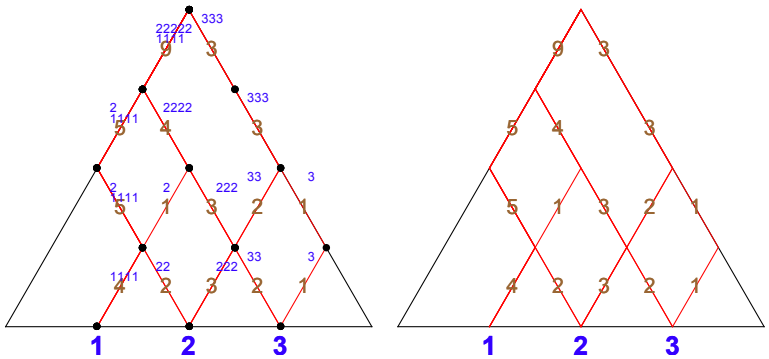


Figure 11: SU(4): Liana forest associated with the Young tableau chosen in the text.
Left: With the origin of lianas marked as a superscript.
Right: With the origin removed. The corresponding O-blade is in position 19 of Fig. 12.

the existence of a one-to-one correspondence between reduced O-blades and liana forests (compare for instance Fig. 6, (Middle) and Fig. 11, (Right)). The number of liana forests being the same as the number of reduced O-blades (in our example they are all displayed on Fig. 12), it is also equal to the dimension of the weight subspace defined by δ in the representation space of highest weight λ. A more detailed analysis of the combinatorics underlying these constructions clearly falls beyond the scope of the present paper, and we shall stop here.

This way of encoding Young tableaux was explained to one of us (R.C.), more than ten years ago, by A. Ocneanu [18], who also invented the "O-blades" to display the intertwiners that appear in the combinatorics of LR coefficients. Other aspects of the above forests are summarized in a video lecture: see [19]. The terminology "lianas" is ours. It was a pleasure (but not so much of a surprise) to rediscover this particular kind of graphical encoding in our discussion of the Schur problem.

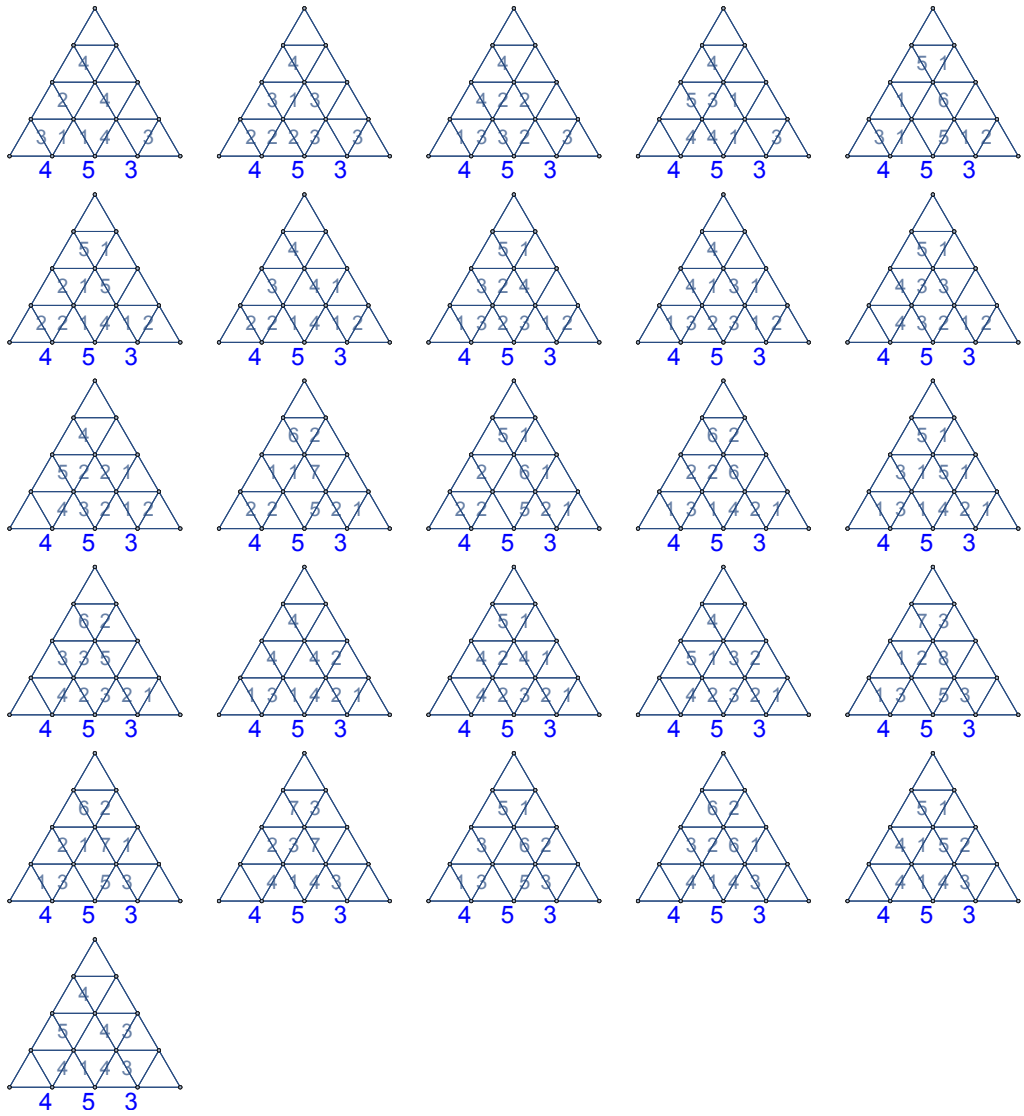


Figure 12: SU(4): All the reduced O-blades for the example discussed in the text

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