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Multidimensional inverse scattering for the Schrödinger equation

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Abstract. We give a short review of old and recent results on the multidimensional inverse scattering problem for the Schrödinger equation. A special attention is paid to efficient reconstructions of the potential from scattering data which can be measured in practice. In this connection our considerations include reconstructions from non-overdetermined monochromatic scattering data and formulas for phase recovering from phaseless scattering data. Potential applications include phaseless inverse X-ray scattering, acoustic tomography and tomographies using elementary particles. This paper is based, in particular, on results going back to M. Born (1926), L. Faddeev (1956, 1974), S. Manakov (1981), R. Beals, R. Coifman (1985), G. Henkin, R. Novikov (1987), and on more recent results of R. Novikov (1998 - 2019), A. Agaltsov, T. Hohage, R. Novikov (2019).

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1. Introduction

We consider the stationary Schrödinger equation:

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 1, \quad E > 0, \quad (1.1)$$

where v is a sufficiently regular function on \mathbb{R}^d with sufficient decay at infinity, for example:

$$\begin{aligned} v &\in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D, \\ D &\text{ is an open bounded domain in } \mathbb{R}^d, \end{aligned} \quad (1.2a)$$

or

$$|v(x)| \leq q(1 + |x|)^{-\sigma}, \quad x \in \mathbb{R}^d, \quad \text{for some } q \geq 0 \text{ and } \sigma > d. \quad (1.2b)$$

Equation (1.1), under assumptions (1.2a), can be used, in particular, for describing a quantum mechanical particle at fixed energy E interacting with a macroscopic object contained in D . In this case v is the potential of this interaction.

We recall also that the (time-dependent) Schrödinger equation is the quantum mechanical analogue of the Newton's second law.

Note that we assume that $\frac{\hbar^2}{2m} = 1$ and Δ is the standard Laplacian in x in the Schrödinger equation (1.1).

For equation (1.1) we consider the scattering eigenfunctions $\psi^+(x, k)$, $k \in \mathbb{R}^d$, $k^2 = E$, specified by the following asymptotics as $|x| \rightarrow \infty$:

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + o\left(\frac{1}{|x|^{(d-1)/2}}\right), \\ c(d, |k|) &= -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \end{aligned} \quad (1.3)$$

for some a priori unknown f . The function f arising in (1.3) is the scattering amplitude for equation (1.1) for fixed E and is defined on

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : k^2 = l^2 = E\} = \mathbb{S}_{\sqrt{E}}^{d-1} \times \mathbb{S}_{\sqrt{E}}^{d-1}. \quad (1.4)$$

We recall that function $\psi^+(x, k)$ at fixed k describes scattering of the incident plane wave described by e^{ikx} on the scatterer described by potential $v(x)$. In addition, the second term on the right-hand side of (1.3) describes the leading scattered spherical wave.

We also recall that in quantum mechanics the values of the functions ψ^+ and f with phase have no direct physical sense, whereas the phaseless values of $|\psi^+|^2$ and $|f|^2$ have probabilistic interpretations (the Born rule) and can be directly obtained in experiments; see [B], [FM]. In particular, $|f(k, l)|^2$ is differential scattering cross section, describing probability density of scattering of particle with initial impulse k into direction $l/|l| \neq k/|k|$.

We consider, in particular, the following problems for equation (1.1):

Problem 1.1. Given v , find ψ^+ and f .

Problem 1.2a. Reconstruct potential v from its scattering amplitude f .

Problem 1.2b. Reconstruct potential v , under assumptions (1.2a), from ψ^+ appropriately given outside of D .

Problem 1.3a. Reconstruct potential v from its phaseless scattering data $|f|^2$.

Problem 1.3b. Reconstruct potential v , under assumptions (1.2a), from its phaseless scattering data $|\psi^+|^2$ appropriately given outside of D .

Problem 1.1 is the direct scattering problem for equation (1.1). This problem can be solved via the Lippmann-Schwinger integral equation (2.1) and formula (2.3); see Section 2.

Problem 1.2a is the inverse scattering problem (from far field) for equation (1.1).

Problem 1.2b is the inverse scattering problem (from near field) for equation (1.1).

Problems 1.3a and 1.3b are the phaseless versions of the inverse scattering problems Problems 1.2a and 1.2b.

At this stage we do not specify which precise information about the functions f , ψ^+ , $|f|^2$, $|\psi^+|^2$ is used in each of Problems 1.2a, 1.2b, 1.3a, 1.3b and, in particular, we do not specify whether E is fixed.

Note that earlier studies on inverse scattering for the Schrödinger equation (in fact, on Problem 1.2a) were essentially stimulated by the Heisenberg's publications [H]; see related discussion in [F2]. In the mathematical literature the inverse scattering problem for the Schrödinger equation (in fact, Problem 1.2a) in dimension $d = 3$ without the

spherically symmetric assumption on v was posed for the first time in [Gel]. At present, there are many important results on Problem 1.2a; see [F1], [F2], [Mos], [Ber], [Re], [F3], [DT], [NM], [L], [Mar], [ChS], [HN], [New], [N1], [S], [N2], [N3], [EW], [Mel], [N4], [N6], [Gr], [HH], [Ho], [AW], [Buc], [N8], [ABR], [BAR], [E], [N9], [I], [IN], [N10], [N11], [AN2], [N16], [RS] and references therein. Formulas reducing Problem 1.2b to Problem 1.2a and vice versa are also known for a long time; see, for example, [Ber].

On the other hand, in view of the Born rule, from applied point of view, Problems 1.3a, 1.3b and similar phaseless inverse scattering problems are much more important than Problems 1.2a, 1.2b and similar phased inverse scattering problems for the quantum mechanical Schrödinger equation. However, until recently, the mathematics of inverse wave propagation problems without phase information, in general, and of Problems 1.3a, 1.3b, in particular, were much less developed than for the phased case and an essential progress in this direction (phaseless case) was done during recent years; see Chapter 10 of [ChS], [KS], [AS], [JL], [IK], [K1], [N12]-[N16], [KR], [Ro], [K2], [AN2], [AHN], [HoNo], [P], [N17], [NS] and references therein.

In the present paper we consider Problems 1.1- 1.3 mainly for the multidimensional case (i.e. $d \geq 2$) for fixed E and, especially, for $d = 2$ or $d = 3$. More precisely, our further presentation can be described as follows.

In Section 2 we recall potential applications of results on Problems 1.1-1.3 at fixed E for $d = 2$ or $d = 3$.

In Section 3 we recall well-known results on Problems 1.1.

In Section 4 we formalise the main objective of Problem 1.2a at fixed and sufficiently large E . In Section 5 we recall an old classical result to this objective. And in Sections 6, 8 we present results of [N5]-[N7], [N11] which achieve this objective.

In Section 9 we recall examples of non-uniqueness for Problem 1.3a in its initial formulation and in Section 10, 11 we present results of [N15], [AHN] on modified Problem 1.3a with background scatterers.

In Section 12 we present formulas of [N14], [N17] reducing Problem 1.3b to Problem 1.2a.

2. Potential applications

Results on Problems 1.1- 1.3 at fixed E for $d = 2$ or $d = 3$ admit potential applications, in particular, in the following domains:

- (i) Inverse problem of quantum scattering arising in nuclear physics and in tomographies using some elementary particles (see, for example, [ChS], [FO]);
- (ii) Acoustic tomography (see, for example, [ABR], [BAR]);
- (iii) Coherent x-ray imaging (see, for example, [JL], [HoNo]).

As regards to quantum scattering we assume that this scattering is modeled using the Schrödinger equation (1.1).

As regards to acoustic tomography and coherent x-ray imaging, we assume that direct scattering is modeled using the Helmholtz equation

$$-\Delta\psi = \left(\frac{\omega}{c(x)} + i\alpha(x, \omega)\right)^2\psi, \quad x \in \mathbb{R}^d, \quad (2.1)$$

with velocity of wave propagation $c(x)$, absorption coefficient $\alpha(x, \omega)$, at fixed frequency ω , where

$$c(x) \equiv c_0, \quad \alpha(x, \omega) \equiv 0 \text{ for } |x| \geq r. \quad (2.2)$$

Equation (2.1), under conditions (2.2), can be written in the form of the Schrödinger equation (1.1), where

$$v = \frac{\omega^2}{c_0^2} - \left(\frac{\omega}{c(x)} + i\alpha(x, \omega) \right)^2, \quad E = \frac{\omega^2}{c_0^2}, \quad (2.3)$$

$$v = v(x, \omega) \equiv 0 \text{ for } |x| \geq r.$$

Therefore, reconstruction methods for Problems 1.2, 1.3 at fixed E can be also used for inverse scattering for the Helmholtz equation (2.2) at fixed ω , under conditions (2.2), for $d = 2$ or $d = 3$.

As it was already mentioned in the Introduction, from applied point of view, the inverse problem of quantum scattering is the most important in its phaseless versions. On the other hand, in acoustic or electrodynamic experiments phased scattering data like ψ^+ and f can be directly measured, at least, in principle. However, in many important cases of monochromatic electro-magnetic wave propagation described using equation (2.1) (e.g., X-rays and lasers) the wave frequency is so great that only phaseless scattering data like $|\psi^+|$ and $|f|$ can be measured in practice by modern technical devices; see, e.g., [HoNo] and references therein.

3. Direct scattering

The scattering eigenfunctions ψ^+ satisfy the Lippmann-Schwinger integral equation

$$\psi^+(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x - y, k) v(y) \psi^+(y, k) dy, \quad (3.1)$$

$$G^+(x, k) \stackrel{\text{def}}{=} -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0} = G_0^+(|x|, |k|), \quad (3.2)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$. Note that

$$G^+(x, k) = -\frac{i}{4} H_0^1(|x| |k|) \text{ for } d = 2, \quad G^+(x, k) = -\frac{e^{i|k||x|}}{4\pi|x|} \text{ for } d = 3, \quad (3.3)$$

where H_0^1 is the Hankel function of the first type.

For the scattering amplitude f the following formula holds:

$$f(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ily} v(y) \psi^+(y, k) dy, \quad (3.4)$$

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where $k \in \mathbb{R}^d$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

Equation (3.1) and formula (3.4) are a particular case of the equation and formulas produced in [LS]. For basic mathematical results concerning (3.1), (3.4) we refer to [F3], [BSh], [V], [N11] and references therein.

Problem 1.1 can be solved via equation (3.1) and formula (3.4).

More precisely, we consider Problem 1.1 for those $E > 0$ that

$$\text{equation (3.1) is uniquely solvable for } \psi^+(\cdot, k) \in L^\infty(\mathbb{R}^d) \text{ for fixed } E > 0, \quad (3.5)$$

where $k \in \mathbb{R}^d$, $k^2 = E$. If, for example, v satisfies (1.2b) and is real-valued, then (3.5) is fulfilled automatically.

We also recall that for any $s > 1/2$ the following Agmon estimate holds:

$$\| \langle x \rangle^{-s} G_0^+(E) \langle x \rangle^{-s} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O(E^{-1/2}), \quad E \rightarrow +\infty, \quad (3.6)$$

where $\langle x \rangle$ denotes the multiplication operator by the function $(1 + |x|^2)^{1/2}$, $G_0^+(E)$ denotes the operator such that

$$G_0^+(E)u(x) = \int_{\mathbb{R}^d} G_0^+(|x - y|, E)u(y)dy, \quad (3.7)$$

where $G_0^+(|x|, E)$ is the function defined in (3.2), u is the test function. Estimate (3.6) was given implicitly in [A]. This estimate is very convenient for studies of equation (3.1) and formula (3.4) for large E ; see, e.g., [N11].

4. The main objective of Problem 1.2a at fixed and sufficiently large E

In order to explain and justify the main objective of Problem 1.2a at fixed and sufficiently large E , for $d \geq 2$, we consider, first, Problems 1.1 and 1.2a in the Born approximation for $q \rightarrow 0$, where q is the number in (1.2b). In this approximation we have, in particular:

$$\psi^+(x, k) \approx e^{ikx}, \quad f(k, l) \approx \hat{v}(k - l), \quad (4.1)$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (4.2)$$

Note that

$$(k, l) \in \mathcal{M}_E \Rightarrow k - l \in \mathcal{B}_{2\sqrt{E}}, \quad (4.3)$$

$$p \in \mathcal{B}_{2\sqrt{E}} \Rightarrow \exists (k, l) \in \mathcal{M}_E \text{ such that } p = k - l \text{ (for } d \geq 2), \quad (4.4)$$

where \mathcal{M}_E is defined by (1.4),

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : |p| \leq r\}. \quad (4.5)$$

Thus, in the Born approximation (for $q \rightarrow 0$), for $d \geq 2$, the scattering amplitude f on \mathcal{M}_E is reduced to the Fourier transform \hat{v} on $\mathcal{B}_{2\sqrt{E}}$.

Moreover, in this approximation, for $d \geq 2$, the scattering amplitude f on $\mathcal{M}_{[E_0, E]}$, $0 < E_0 \leq E$, is also reduced to \hat{v} on $\mathcal{B}_{2\sqrt{E}}$, where $\mathcal{M}_{[E_0, E]} = \cup_{\zeta \in [E_0, E]} \mathcal{M}_\zeta$.

Therefore, the most natural way for solving Problem 1.2a at fixed and sufficiently large E in the Born approximation (for $q \rightarrow 0$), for $d \geq 2$, consists in the following formulas:

$$v(x) = v_{appr}^{lin}(x, E) + v_{err}^{lin}(x, E), \quad (4.6)$$

$$v_{appr}^{lin}(x, E) = \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \hat{v}(p) dp,$$

$$v_{err}^{lin}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \hat{v}(p) dp.$$

Here, $v_{appr}^{lin}(x, E)$ is an approximate but stable reconstruction from f on \mathcal{M}_E reduced to \hat{v} on $\mathcal{B}_{2\sqrt{E}}$ and $v_{err}^{lin}(x, E)$ is the reconstruction error.

In addition, if $v \in W^{m,1}(\mathbb{R}^d)$ (m -times smooth functions in $L^1(\mathbb{R}^d)$), $m > d$, then

$$\|v_{err}^{lin}(\cdot, E)\|_{L^\infty(\mathbb{R}^d)} = O(E^{-(m-d)/2}), \quad E \rightarrow +\infty. \quad (4.7)$$

De facto, the main objective of Problem 1.2a at fixed and sufficiently large E , for $d \geq 2$, consisted in finding analogs for the general non-linearized case of the reconstruction result (4.6), (4.7) existing for the linearised case near zero potential.

This objective was achieved in [N6], [N7], [N11]; see Sections 6, 8 below.

Of course, under condition (1.2a), $d \geq 2$, in the linearized case near zero potential, when f on \mathcal{M}_E is reduced to \hat{v} on $\mathcal{B}_{2\sqrt{E}}$, where $E \geq 0$, we have that \hat{v} on $\mathcal{B}_{2\sqrt{E}}$ uniquely determines \hat{v} on the entire \mathbb{R}^d via analytic continuation. And, therefore, in this case f on \mathcal{M}_E uniquely determines v . However, in contrast with (4.6) this reconstruction involves an analytic continuation and is rather unstable. For the general non-linearized case analogs of this uniqueness result were obtained in [N1], [N3], [Buc]. However, despite their mathematical importance, we do not consider uniqueness theorems without stable reconstruction as the main objective of inverse problems.

5. Old general result on Problem 1.2a for $d \geq 2$

If v satisfies (1.2a), then

$$f(k, l) = \hat{v}(k - l) + O(E^{-1/2}), \quad E \rightarrow +\infty, \quad (k, l) \in \mathcal{M}_E, \quad (5.1)$$

where \hat{v} is defined by (4.2). This result is known as the Born formula at high energies. As a mathematical theorem formula (5.1) goes back to [F1]. At present, one can prove (5.1) using estimate (3.6); see, e.g., [N11].

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Using (5.1) for $d \geq 2$ with

$$k = k_E(p) = \frac{p}{2} + \eta_E(p), \quad l = l_E(p) = -\frac{p}{2} + \eta_E(p), \quad (5.2)$$

where

$$\eta_E(p) = \left(E - \frac{p^2}{4}\right)^{1/2} \nu(p), \quad |\nu(p)| = 1, \quad \nu(p)p = 0,$$

one can reconstruct $\hat{v}(p)$ from f at high energies E for any $p \in \mathbb{R}^d$.

However, formula (5.1) gives no method to reconstruct v from f on \mathcal{M}_E with the error smaller than $O(E^{-1/2})$ even if $v \in S(\mathbb{R}^d)$, where S stands for the Schwartz class.

Applying the inverse Fourier transform F^{-1} to both sides of (5.1), one can obtain an explicit linear formula for $u_1 = u_1(x, E)$ in terms of f on \mathcal{M}_E , where

$$u_1(x, E) = v(x) + O(E^{-\alpha_1}), \quad E \rightarrow +\infty, \quad (5.3)$$

$$\alpha_1 = \frac{m-d}{2m} \quad \text{if } v \in W^{m,1}(\mathbb{R}^d).$$

One can see that

$$\alpha_1 \leq 1/2 \quad \text{even if } m \rightarrow +\infty.$$

Comparing (4.7) and (5.3) one can see that the approximate reconstruction $u_1(x, E)$ is not optimal and does not achieve yet the objective formulated in Section 4.

6. Results of [N6], [N7]

Let

$$W_s^{m,1}(\mathbb{R}^d) = \{u : (1 + |x|)^s \partial^J v(x) \in L^1(\mathbb{R}^d), |J| \leq m\}, \quad (6.1)$$

where $m \in \mathbb{N} \cup 0$, $s > 0$.

In [N6] for $v \in W_s^{m,1}(\mathbb{R}^2)$, $m > 2$, $s > 0$, for general nonlinearized case for $d = 2$, we succeeded, in particular, to give a stable reconstruction

$$f \text{ on } \mathcal{M}_E \xrightarrow{\text{stable reconstruction}} v_{appr}(\cdot, E) \text{ on } \mathbb{R}^2 \quad (6.2)$$

such that

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^2)} = O(E^{-(m-2)/2}) \text{ as } E \rightarrow +\infty. \quad (6.3)$$

For $d = 2$, this reconstruction result achieves the objective formulated in Section 4 in view of its stability and estimate (6.3) (which is similar to estimate (4.7) for $d = 2$).

Reconstruction (6.2) is based on Fredholm linear integral equations of the second type. Among these linear integral equations, the most important ones, historically, go back to the Gel'fand-Levitan integral equations of inverse scattering in dimension $d = 1$ and arise from a non-local Riemann-Hilbert problem for the Faddeev solutions ψ of the Schrödinger equation at fixed energy E . Riemann-Hilbert problems of such type go back to [M]. Definition of the Faddeev solutions ψ and some of their properties are recalled in Section 7.

For precise form of the equations and formulas involved into reconstruction (6.2) we refer to [N6]. The main idea of [N6] was published first in [N5].

Reconstruction (6.2) together with its multifrequency generalization was implemented numerically in [BAR].

In [N7] for $v \in W_s^{m,1}(\mathbb{R}^3)$, $m > 3$, $s > 0$, for general nonlinearized case for $d = 3$, we succeeded, in particular, to give a stable reconstruction

$$f \text{ on } \mathcal{M}_E \xrightarrow{\text{stable reconstruction}} v_{appr}(\cdot, E) \text{ on } \mathbb{R}^3 \quad (6.4)$$

such that

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^3)} = O(E^{-(m-3)/2} \ln E), \text{ as } E \rightarrow +\infty. \quad (6.5)$$

For $d = 3$, this reconstruction result achieves the objective formulated in Section 4 in view of its stability and estimate (6.5) (which is similar (in essence) to estimate (4.7) for $d = 3$).

Reconstruction (6.4) is based on linear and nonlinear integral equations. Among these integral equations, the most important are nonlinear ones arising from $\bar{\partial}$ -approach to 3D inverse scattering at fixed energy. This $\bar{\partial}$ -approach goes back to [BC], [HN] and involves different properties of the Faddeev generalized scattering amplitude h in complex domain at fixed energy E . Definition of the Faddeev generalized scattering amplitude h and some of its properties are recalled in Section 7.

For precise form of the equations and formulas involved into reconstruction (6.4) we refer to [N7], [N8].

Reconstruction (6.4) was implemented numerically in [ABR]. Some of these results of [ABR] are also presented in Section 4 of the survey paper [N8].

However, the main disadvantage of reconstruction (6.4) is the overdetermination of $f|_{\mathcal{M}_E}$ for $d = 3$, required for this reconstruction. Indeed, $f|_{\mathcal{M}_E}$ is a function of 4 variables for $d = 3$ ($\dim \mathcal{M}_E = 2d - 2 = 4$ for $d = 3$), whereas v is a function of 3 variables. This point was one of motivations for obtaining the results presented in Section 8.

7. Faddeev functions

The results of [N6], [N7] presented in Section 6 are based on properties of the Faddeev's functions ψ , h , ψ_γ , h_γ (see [F3], [BC], [HN], [N6], [N7]). Definitions and some properties of these functions are recalled below in this section.

The Faddeev solutions ψ of the Schrödinger equation are defined as the solutions of the integral equation (see [F3], [HN]):

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k)v(y)\psi(y, k)dy, \quad (7.1)$$

$$G(x, k) = e^{ikx}g(x, k), \quad g(x, k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (7.2)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ (and at fixed k equation (7.1) is an equation for $\psi = e^{ikx}\mu(x, k)$, where μ is sought in $L^\infty(\mathbb{R}^d)$). The Faddeev function h is defined by the formula (see [F3],

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[HN]):

$$h(k, l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (7.3)$$

where $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$, $Im k = Im l$.

Here $\psi(x, k)$ satisfies (1.1) for $E = k^2$, and ψ , G and h are (nonanalytic) continuations of ψ^+ , G^+ and f to the complex domain. In particular, $h(k, l)$ for $k^2 = l^2$ can be considered as the "scattering" amplitude in the complex domain for equation (1.1) for $E = k^2$.

Equation (7.1) and formulas (7.2), (7.3) are analogs in complex domain of equation (3.1) and formulas (3.2), (3.3).

The functions ψ_γ and h_γ are defined as follows (see [F3], [HN]):

$$\psi_\gamma(x, k) = \psi(x, k + i0\gamma), \quad h_\gamma(k, l) = h(k + i0\gamma, l + i0\gamma), \quad (7.4)$$

where $x, k, l, \gamma \in \mathbb{R}^d$, $|\gamma| = 1$. Note that

$$\psi^+(x, k) = \psi_{k/|k|}(x, k), \quad f(k, l) = h_{k/|k|}(k, l), \quad (7.5)$$

where $x, k, l \in \mathbb{R}^d$, $|k| > 0$.

The following relations are fulfilled (see [F3], [HN]):

$$\psi_\gamma(x, k) = \psi^+(x, k) + 2\pi i \int_{\mathbb{R}^d} h_\gamma(k, m) \theta((m - k)\gamma) \delta(m^2 - k^2) \psi^+(x, m) dm, \quad (7.6)$$

$$h_\gamma(k, l) = f(k, l) + 2\pi i \int_{\mathbb{R}^d} h_\gamma(k, m) \theta((m - k)\gamma) \delta(m^2 - k^2) f(m, l) dm, \quad (7.7)$$

where θ is the Heaviside step function, δ is the Dirac delta function, $x, k, l, \gamma \in \mathbb{R}^d$, $|\gamma| = 1$.

The following $\bar{\partial}$ - equations and asymptotics hold (see [BC], [HN]):

$$\frac{\partial}{\partial \bar{k}_j} \mu(x, k) = -2\pi \int_{\xi \in \mathbb{R}^d} \xi_j H(k, -\xi) e^{i\xi x} \delta(\xi^2 + 2k\xi) \mu(x, k + \xi) d\xi, \quad (7.8)$$

$$\mu(x, k) \rightarrow 1, \quad |k| \rightarrow \infty, \quad (7.9)$$

$$\frac{\partial}{\partial \bar{k}_j} H(k, p) = -2\pi \int_{\xi \in \mathbb{R}^d} \xi_j H(k, -\xi) H(k + \xi, p + \xi) \delta(\xi^2 + 2k\xi) d\xi, \quad (7.10)$$

$$H(k, p) \rightarrow \hat{v}(p), \quad |k| \rightarrow \infty, \quad (7.11)$$

where

$$\mu(x, k) = e^{-ikx} \psi(x, k), \quad H(k, p) = h(k, k - p), \quad (7.12)$$

\hat{v} is defined by (4.2), $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus \mathbb{R}^d$, $|k| = ((Rek)^2 + (Imk)^2)^{1/2}$, $p \in \mathbb{R}^d$, $j = 1, \dots, d$.

The derivation of reconstruction (6.2) involves relations (7.6), (7.7), the $\bar{\partial}$ - equation (7.8) and asymptotics (7.9), and some estimates on f and h , where $d = 2$.

The derivation of reconstruction (6.4) involves relations (7.7), the $\bar{\partial}$ - equation (7.10) and asymptotics (7.11), and some estimates on f and h , where $d = 3$.

8. Results of [N10], [N11]

Let

$$\Gamma_E^\delta = \{k = k_E(p), l = l_E(p) : p \in \mathcal{B}_{2\delta\sqrt{E}}\}, \quad 0 < \delta \leq 1, \quad (8.1)$$

where \mathcal{B}_r is defined by (4.5), $k_E(p)$ and $l_E(p)$ are defined as in (5.2), where ν is a piecewise continuous vector-function on \mathbb{R}^d , $d \geq 2$.

In this section we consider the following version of Problem 1.2a:

Reconstruct v on \mathbb{R}^d from f on Γ_E^δ .

One can see that

$$\Gamma_E^\delta \subset \mathcal{M}_E, \quad (8.2a)$$

$$\dim \mathcal{M}_E = 2d - 2, \quad \dim \Gamma_E^\delta = d \quad \text{for } d \geq 2, \quad (8.2b)$$

$$\dim \mathcal{M}_E > d \quad \text{for } d \geq 3. \quad (8.2c)$$

Due to (8.2a), any reconstruction of v from f on Γ_E^δ is also a reconstruction of v from f on \mathcal{M}_E . In addition, due to (8.2b), (8.2c), the problem of finding v from f on \mathcal{M}_E is overdetermined for $d \geq 3$, whereas the problem of finding v from f on Γ_E^δ is non-overdetermined.

In [N11], for $d \geq 2$, we succeeded, in particular, to give by explicit formulas a stable iterative reconstruction

$$f \text{ on } \Gamma_E^\delta \xrightarrow{\text{stable reconstruction}} u_j(\cdot, E) \text{ on } \mathbb{R}^d, \quad j = 1, 2, 3, \dots \quad (8.3)$$

such that

$$\|u_j(\cdot, E) - v\|_{L^\infty(D)} = O(E^{-\alpha_j}) \quad \text{as } E \rightarrow +\infty, \quad (8.4)$$

$$\alpha_j = \left(1 - \left(\frac{m-d}{m}\right)^j\right) \frac{m-d}{2d}, \quad j \geq 1,$$

under the assumptions that

$$v \text{ is a perturbation of some known background } v_0 \text{ satisfying (1.2b),} \quad (8.5)$$

$$\text{where } v - v_0 \in W^{m,1}(\mathbb{R}^d), \quad m > d, \quad \text{and } \text{supp}(v - v_0) \subset D,$$

where D is an open bounded domain (which is fixed a priori).

One can see that:

$$\alpha_1 = \frac{m-d}{2m}, \quad (8.6a)$$

$$\alpha_j \rightarrow \frac{j}{2} \quad \text{if } m \rightarrow +\infty, \quad (8.6b)$$

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$$\alpha_j \rightarrow \alpha_\infty = \frac{m-d}{2d} \quad \text{if } j \rightarrow +\infty, \quad (8.6c)$$

$$\alpha_\infty \rightarrow +\infty \quad \text{if } m \rightarrow +\infty. \quad (8.6d)$$

One can also see that α_1 is the number of (5.3) and that

$$\alpha_\infty < \frac{m-d}{2}, \quad (8.7)$$

where $(m-d)/2$ is the number of (4.7).

In view of (8.6), (8.7), the error estimate in (8.3), (8.4) is less perfect than in (6.2), (6.3) (for $d=2$) and in (6.4), (6.5) (for $d=3$). However, in (8.3) we reconstruct v from non-overdetermined data in contrast with (6.4) (for $d=3$). Besides, in fact, f on $\Gamma_E^{\delta(E)}$ only is used in (8.3), where

$$\delta(E) = \tau E^{-(d-1)/(2d)}, \quad \tau \in]0, 1], \quad (8.8)$$

$$\delta(E) \rightarrow 0 \quad \text{as } E \rightarrow +\infty.$$

In any case, α_∞ in (8.6c) is comparable with $(m-d)/2$ in (4.7). Therefore, reconstruction (8.3), (8.4) can be also considered as an analog for the general non-linearized case of the reconstruction result (4.6), (4.7) existing for the linearised case near zero potential.

For precise form of formulas involved into reconstruction (8.3) we refer to [N11].

Note that the iterative reconstruction (8.3), (8.4) of [N11] was preceded by the following stability estimate obtained in [N10] :

$$\|v_1 - v_2\|_{L^\infty(D)} \leq C_1(q, D, \sigma) E^{1/2} \|f_1 - f_2\|_{C(\Gamma_E^{\delta(E)})} + \quad (8.9)$$

$$C_2(q, r, D, \sigma, m) E^{-(m-d)/(2d)}, \quad E \geq 1,$$

under the assumptions that v_1, v_2 satisfy (1.2b), $\text{supp}(v_1 - v_2) \subset D$, $\|v_1 - v_2\|_{W^{m,1}(\mathbb{R}^d)} \leq r$, where q and σ are the numbers in (1.2b), $\delta(E)$ is defined by (8.8), $C_1 = C_1(q, D, \sigma)$, $C_2 = C_2(q, r, D, \sigma, m)$ are non-negative constants.

Note also that the proofs of (8.3), (8.4) and (8.9) are based on the the Lippmann-Schwinger integral equation (3.1) for ψ^+ , the integral formula (3.4) for the scattering amplitude f and the Agmon estimate (3.6) for the Green function G^+ . These proofs are much simpler than the proofs of (6.2), (6.3) and (6.4), (6.5) and, in view of their simplicity, can be relatively easy adopted for other problems. In particular, the iterative approximate reconstruction (8.3), (8.4) was extended to the phaseless case in [AHN], where the extended reconstruction was also implemented numerically (see also Section 11).

9. Examples of non-uniqueness for Problem 1.3a

It is known that $|f|^2$ even on $\mathcal{M}_{\mathbb{R}_+}$ does not determine v uniquely, in general, where $\mathcal{M}_{\mathbb{R}_+} = \cup_{E \in \mathbb{R}_+} \mathcal{M}_E$, $\mathbb{R}_+ =]0, +\infty[$.

In particular, $|f|^2$ on $\mathcal{M}_{\mathbb{R}^d}$ is invariant with respect to translations of v . More precisely, we have the formulas:

$$f_y(k, l) = e^{i(k-l)y} f(k, l), \quad (9.1)$$

$$|f_y(k, l)|^2 = |f(k, l)|^2, \quad (9.2)$$

$$y \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_{\mathbb{R}^d},$$

where f is the scattering amplitude for v and f_y is the scattering amplitude for v_y , where

$$v_y(x) = v(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d. \quad (9.3)$$

Formula (9.1) is known, in particular, from the theory of solitons; see, e.g., [N9].

Besides, $|f|^2$ on $\mathcal{M}_{\mathbb{R}^d}$ does not determine uniquely even the point Bethe-Peierls-Fermi-Zel'dovich-Berezin-Faddeev potential $v_{0,\alpha}$ supported at $\{0\}$, for $d = 3$; see [N16]. More precisely, we have the formulas:

$$f_\alpha(k, l) = \frac{1}{2\pi^2} \frac{1}{\alpha + i\sqrt{E}}, \quad (9.4)$$

$$|f_\alpha(k, l)|^2 = |f_{-\alpha}(k, l)|^2, \quad (9.5)$$

$$\alpha \in \mathbb{R}, \quad (k, l) \in \mathcal{M}_{\mathbb{R}^d},$$

where f_α is the scattering amplitude for the point potential $v_{0,\alpha}$, for $d = 3$. An analogous result is also valid in dimension $d = 1$.

10. Results of [N15] on modified Problem 1.3a for $d \geq 2$

Let

$$S = \{|f|^2, |f_j|^2, j = 1, \dots, n\}, \quad (10.1)$$

where f is the initial scattering amplitude for v satisfying (1.2a), f_j is the scattering amplitude for

$$v_j = v + w_j, \quad j = 1, \dots, n, \quad (10.2)$$

where w_1, \dots, w_n are additional a priori known background scatterers such that

$$w_j \in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j, \quad (10.3)$$

Ω_j is an open bounded domain in \mathbb{R}^d , $\Omega_j \cap D = \emptyset$,

$w_j \neq 0$, $w_{j_1} \neq w_{j_2}$ for $j_1 \neq j_2$ in $L^\infty(\mathbb{R}^d)$,

$j, j_1, j_2 \in \{1, \dots, n\}$.

In other words, S consists of the phaseless scattering data $|f|^2, |f_1|^2, \dots, |f_n|^2$ measured sequentially for the unknown scatterer v and then for the unknown scatterer v in the presence of known scatterer w_j nonintersecting v for $j = 1, \dots, n$.

In this section we consider the following modified version of Problem 1.3a:

Problem 10.1. Reconstruct potential v from its phaseless scattering data S for some appropriate a priori known background scatterers w_1, \dots, w_n .

Problem 10.1 in dimension $d = 1$ for $n = 1$ was, actually, considered in [AS]. However, to our knowledge, Problem 10.1 in dimension $d \geq 2$ was not considered in the literature before work [N15].

Let the Fourier transforms \hat{v} and \hat{w}_j be represented as follows:

$$\hat{v}(p) = |\hat{v}(p)|e^{i\alpha(p)}, \quad \hat{w}_j(p) = |\hat{w}_j(p)|e^{i\beta_j(p)}, \quad (10.4)$$

where $p \in \mathbb{R}^d$, $j = 1, \dots, n$.

In [N15], for v , w_1 , w_2 satisfying (1.2a), (10.3), we give, in particular, the following formulas for solving Problem 10.1, for $d \geq 2$, $n = 2$:

$$\begin{aligned} |\hat{v}_j(p)|^2 &= |f_j(k, l)|^2 + O(E^{-1/2}) \quad \text{as } E \rightarrow +\infty, \\ p \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_E, \quad k - l = p, \quad j = 0, 1, 2, \end{aligned} \quad (10.5)$$

where $v_0 = v$, $f_0 = f$, v_j is defined by (10.2), $j = 1, 2$;

$$\begin{aligned} |\hat{v}| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} &= (2 \sin(\beta_2 - \beta_1))^{-1} \times \\ &\begin{pmatrix} \sin \beta_2 & -\sin \beta_1 \\ -\cos \beta_2 & \cos \beta_1 \end{pmatrix} \begin{pmatrix} |\hat{w}_1|^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2) \\ |\hat{w}_2|^{-1}(|\hat{v}_2|^2 - |\hat{v}|^2 - |\hat{w}_2|^2) \end{pmatrix}, \end{aligned} \quad (10.6)$$

where

$$\begin{aligned} |\hat{v}| &= |\hat{v}(p)|, \quad \alpha = \alpha(p), \quad \beta_j = \beta_j(p), \quad |\hat{w}_j| = |\hat{w}_j(p)|, \quad |\hat{v}_j| = |\hat{v}_j(p)|, \quad j = 1, 2, \\ \sin(\beta_2(p) - \beta_1(p)) &\neq 0, \quad |\hat{w}_1(p)| \neq 0, \quad |\hat{w}_2(p)| \neq 0, \quad p \in \mathbb{R}^d. \end{aligned} \quad (10.7)$$

Formulas (10.5) follow from (5.1) and are the Born formulas for $|f|^2$, $|f_1|^2$, $|f_2|^2$ at high energies. Using (10.5) we find $|\hat{v}|$, $|\hat{v}_1|$, $|\hat{v}_2|$ from $|f|^2$, $|f_1|^2$, $|f_2|^2$ at high energies.

Using (10.4), (10.6) for appropriate w_1 , w_2 we find \hat{v} from $|\hat{v}|$, $|\hat{v}_1|$, $|\hat{v}_2|$ and from a priori known \hat{w}_1 , \hat{w}_2 .

In connection with conditions (10.7), for simplicity, we can also assume that, for example:

$$w_2(x) = w_1(x - y), \quad y \in \mathbb{R}^d \setminus \{0\}. \quad (10.8)$$

In this case we have that

$$|\hat{w}_2(p)| = |\hat{w}_1(p)|, \quad \beta_2(p) = \beta_1(p) + py, \quad p \in \mathbb{R}^d, \quad (10.9)$$

and, therefore,

$$\begin{aligned} \text{conditions (10.7) are fulfilled if and only if } &p \in \mathbb{R}^d \setminus (A_y \cup Z), \\ A_y = \{p \in \mathbb{R}^d : e^{2ipy} = 1\}, \quad Z = \{p \in \mathbb{R}^d : |\hat{w}_1(p)| = 0\}. \end{aligned} \quad (10.10)$$

In addition:

$$A_y \text{ is closed and } Meas A_y = 0 \text{ in } \mathbb{R}^d, \quad y \neq 0, \quad (10.11)$$

$$Z \text{ is closed and } Meas Z = 0 \text{ in } \mathbb{R}^d, \quad (10.12)$$

where properties (10.12) follow from (10.3).

Thus, using (10.5)-(10.7) and assuming, for example, (10.8) one can see that the phaseless scattering data $S = \{|f|^2, |f_1|^2, |f_2|^2\}$ at high energies in dimension $d \geq 2$ and the background scatterers w_1, w_2 uniquely determine v . More precisely, one can reconstruct $\hat{v}(p)$ from S at high energies and from \hat{w}_1, \hat{w}_2 for any $p \in \mathbb{R}^d \setminus (A_y \cup Z)$, where A_y, Z are defined in (10.10), using formulas (10.5) with k, l as in (5.2) and then using formula (10.6).

Actually, the reconstruction result for Problem 10.1 consisted in formulas (10.5)-(10.7) is an analog of the old reconstruction result for Problem 1.2a consisted in formula (5.1).

11. Results of [AHN]

In [AHN], for $d \geq 2$, we succeeded, in particular, to give a stable iterative reconstruction

$$S \text{ on } \Gamma_E, \quad w_1, w_2 \xrightarrow{\text{stable reconstruction}} u_j(\cdot, E) \text{ on } \mathbb{R}^d, \quad j = 1, 2, 3, \dots \quad (11.1)$$

such that

$$\|u_j(\cdot, E) - v\|_{L^\infty(D)} = O(E^{-\alpha_j}) \text{ as } E \rightarrow +\infty, \quad (11.2)$$

$$\alpha_j = \frac{1}{2} \frac{m-d}{d+2\rho+\frac{m-d}{2N+1}} \left(1 - \left(\frac{m-d}{m+2\rho+\frac{m-d}{2N+1}}\right)^j\right), \quad j \geq 1,$$

under the assumptions that v satisfies (1.2a), $v \in W^{m,1}(\mathbb{R}^d)$, $m > d$, $N \in \mathbb{N}$, w_1, w_2 satisfy (10.3) and

$$w_1(x) = w(x - T_1), \quad w_2(x) = w(x - T_2), \quad x \in \mathbb{R}^d, \quad (11.3)$$

$$\begin{aligned} w &\in C(\mathbb{R}^d), \quad w = \bar{w}, \quad w(x) = 0 \quad \text{for } |x| > R, \\ \hat{w}(p) &= \overline{\hat{w}(p)} \geq \kappa(1 + |p|)^{-\rho}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (11.4)$$

for some fixed $T_1, T_2 \in \mathbb{R}^d$, $T_1 \neq T_2$, $R > 0$, $\kappa > 0$, $\rho > d$. (A broad class of w satisfying (11.4) was constructed in Lemma 1 of [AN1].)

Here, S is defined as in (10.1) for $n = 2$ and Γ_E is defined as in (8.1) for $\delta = 1$.

One can see that

$$\alpha_1 = \frac{1}{2} \frac{m-d}{m+2\rho+\frac{m-d}{2N+1}}, \quad (11.5a)$$

$$\alpha_j \rightarrow \frac{j}{2} \text{ if } m \rightarrow +\infty, \quad (11.5b)$$

$$\alpha_j \rightarrow \alpha_\infty = \frac{1}{2} \frac{m-d}{d+2\rho+\frac{m-d}{2N+1}} \text{ if } j \rightarrow +\infty, \quad (11.5c)$$

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$$\alpha_\infty \rightarrow +\infty \text{ if } m \rightarrow +\infty, \quad N \rightarrow +\infty. \quad (11.5d)$$

In [AHN], for $d \geq 2$, we also give, in particular, a stable iterative reconstruction

$$S \text{ on } \Gamma_E, \quad w_1, w_2 \xrightarrow{\text{stable reconstruction}} u_j(\cdot, E) \text{ on } \mathbb{R}^d, \quad j = 1, 2, 3, \dots \quad (11.6)$$

such that

$$\|u_j(\cdot, E) - v\|_{L^\infty(D)} = O(E^{-\alpha_j}) \text{ as } E \rightarrow +\infty, \quad (11.7)$$

$$\alpha_j = \frac{1}{2} \frac{m-d}{d+2\rho} \left(1 - \left(\frac{m-d}{m+2\rho} \right)^j \right), \quad j \geq 1,$$

under the assumptions that v satisfies (1.2a), $v \in W^{m,1}(\mathbb{R}^d)$, $m > d$, w_1, w_2 satisfy (10.3) and

$$w_1(x) = w(x - T_1), \quad w_2(x) = iw(x - T_1), \quad x \in \mathbb{R}^d, \quad (11.8)$$

where w satisfies (11.4).

Here, S is defined as in (10.1) for $n = 2$ and Γ_E is defined as in (8.1) for $\delta = 1$.

One can see that

$$\alpha_1 = \frac{1}{2} \frac{m-d}{m+2\rho}, \quad (11.9a)$$

$$\alpha_j \rightarrow \frac{j}{2} \text{ if } m \rightarrow +\infty, \quad (11.9b)$$

$$\alpha_j \rightarrow \alpha_\infty = \frac{1}{2} \frac{m-d}{d+2\rho} \text{ if } j \rightarrow +\infty, \quad (11.9c)$$

$$\alpha_\infty \rightarrow +\infty \text{ if } m \rightarrow +\infty. \quad (11.9d)$$

Note that an analog of u_1 of (11.1), (11.2) and an analog of u_1 of (11.6), (11.7) were constructed, first, in [AN1]. These u_1 are analogs for the phaseless case of u_1 of (5.3) for the phased case. In turn, reconstructions (11.1), (11.2) and (11.6), (11.7) are analogs for the phaseless case of reconstruction (8.3), (8.4) for the phased case.

In addition, in [AHN] we implemented numerically a version of reconstruction (11.1), (11.2) for the case of three background scatterers w_1, w_2, w_3 and we implemented numerically reconstruction (11.6), (11.7); see Section 4 of [AHN].

12. Formulas of [N14], [N17] reducing Problem 1.3b to Problem 1.2a

Let

$$f_1(k, l) = c(d, |k|)f(k, l), \quad (k, l) \in \mathcal{M}_E, \quad (12.1)$$

$$a(x, k) = |x|^{(d-1)/2} (|\psi^+(x, k)|^2 - 1), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad k \in \mathbb{R}^d \setminus \{0\}, \quad (12.2)$$

where c is the constant of (1.3), ψ^+ , f are the function of (1.3), (3.1), (3.4), \mathcal{M}_E is defined by (1.4).

In [N14], for v satisfying (1.2a), we succeeded, in particular, to give the following formulas reducing Problem 1.3b to Problem 1.2a, for $d \geq 2$:

$$\begin{pmatrix} \operatorname{Re} f_1(k, l) \\ \operatorname{Im} f_1(k, l) \end{pmatrix} = M \left(\begin{pmatrix} a(x_1, k) \\ a(x_2, k) \end{pmatrix} - \begin{pmatrix} \delta a(x_1, k) \\ \delta a(x_2, k) \end{pmatrix} \right), \quad (12.3)$$

$$M = \frac{1}{2 \sin(\varphi_2 - \varphi_1)} \begin{pmatrix} \sin \varphi_2 & -\sin \varphi_1 \\ -\cos \varphi_1 & \cos \varphi_2 \end{pmatrix}, \quad (12.4)$$

$$x_1 = s\hat{l}, \quad x_2 = (s + \tau)\hat{l}, \quad \hat{l} = l/|l|, \quad (12.5)$$

$$\varphi_j = |k||x_j| - kx_j, \quad j = 1, 2, \quad (12.6)$$

$$\varphi_2 - \varphi_1 = \tau(|k| - k\hat{l}), \quad (12.7)$$

$$\delta a(x_1, k) = O(s^{-\alpha}), \quad \delta a(x_2, k) = O(s^{-\alpha}) \quad \text{as } s \rightarrow +\infty \quad (12.8)$$

uniformly in $\hat{k} = k/|k|$, $\hat{l} = l/|l|$ and τ at fixed $E > 0$,

$$\alpha = 1/2 \quad \text{for } d = 2, \quad \alpha = 1 \quad \text{for } d \geq 3, \quad (12.9)$$

where

$$\sin(\varphi_2 - \varphi_1) \neq 0, \quad (12.10)$$

$(k, l) \in \mathcal{M}_E$, $s > 0$, $\tau > 0$.

Formulas (12.1), (12.3)-(12.10) are explicit two-point formulas for approximate finding phased $f(k, l)$ at fixed $(k, l) \in \mathcal{M}_E$, $k \neq l$, from phaseless $|\psi^+(x, k)|^2$ at two points $x = x_1, x_2$ defined in (12.5), where s is sufficiently large.

In turn, article [N13] gives exact versions (without error terms) of formulas (12.1), (12.3)-(12.10) for the 3-point case for $d = 1$; see [N12], [N13] for details.

Detailed estimates for the error term $\delta a(x, k) = O(s^{-\alpha})$ in (12.3), (12.8), (12.9) are given in [NS], for $d = 3$ and for $d = 2$. However, the main drawback of the two-point formulas (12.1), (12.3)-(12.10) for finding phased f from phaseless $|\psi^+|^2$ is a slow decay of the error as $s \rightarrow +\infty$; see (12.3), (12.8), (12.9). This drawback motivated our considerations given in [N17].

In [N17], for v satisfying (1.2a), for fixed $(k, l) \in \mathcal{M}_E$, $k \neq l$, for $d = 3$ and $d = 2$, we succeeded, in particular, to give

$$\text{formulas for finding } f(k, l) \text{ up to } O(s^{-n}) \text{ as } s \rightarrow +\infty \quad (12.11)$$

from $|\psi^+(x, k)|^2$ given at $2n$ points $x = x_1(s), \dots, x_{2n}(s)$,

where

$$x_i(s) = r_i(s)\hat{l}, \quad i = 1, \dots, 2n, \quad \hat{l} = l/|l|, \quad (12.12)$$

$$r_{2j-1}(s) = \lambda_j s, \quad r_{2j}(s) = \lambda_j s + \tau, \quad j = 1, \dots, n,$$

$$\lambda_1 = 1, \quad \lambda_{j_1} < \lambda_{j_2} \quad \text{for } j_1 < j_2, \quad \tau = \tau_{\text{fixed}} > 0.$$

The point is that in (12.11) we have a rapid decay of the error as $s \rightarrow +\infty$ if n is sufficiently large. For $d = 3$, $n = 1$, formulas (12.11), (12.12) reduce to (12.1), (12.3)-(12.10).

The general idea of obtaining the $2n$ -point formulas (12.11), (12.12) can be described as follows.

We use that, under assumptions (1.2a), formula (1.3) admits the following much more precise version:

$$\psi^+(x, k) = e^{ikx} + \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} \left(\sum_{j=1}^n \frac{f_j(k, |k| \frac{x}{|x|})}{|x|^{j-1}} + O\left(\frac{1}{|x|^n}\right) \right) \quad \text{as } |x| \rightarrow \infty, \quad (12.13)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E > 0$, $n \in \mathbb{N}$.

Then, for fixed $(k, l) \in \mathcal{M}_E$, $k \neq l$, we look for formulas for finding $f_j(k, l)$ up to $O(s^{-(n-j+1)})$ as $s \rightarrow +\infty$, $j = 1, \dots, n$, from $|\psi^+(x, k)|^2$ given at $2n$ points $x = x_1(s), \dots, x_{2n}(s)$ of the form (12.12), where $f_j = f_j(k, l)$, $j = 1, \dots, n$, are the functions arising in (12.13).

Using the later formulas for f_1 and using (12.1) we obtain formulas (12.11), (12.12).

For precise form of formulas (12.11), (12.12) we refer to [N17].

The aforementioned formulas of [N13], [N14], [N17] reducing Problem 1.3b to Problem 1.2a permit to apply to the phaseless inverse scattering Problem 1.3b well developed methods existing for the inverse scattering Problem 1.2a with phase information; see Sections 4-8 for some of these methods.

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