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# $L^{q}$-weak solutions to the 3 D time-dependent Oseen system: decay estimates. 

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#### Abstract

This article deals with $L^{q}$-weak solutions to the 3D time-dependent Oseen system. This type of solution is defined in terms of the velocity only. It is shown that the velocity may be represented by a sum of integrals none of which involves the pressure and without a surface integral of the spatial gradient of the velocity. On the basis of this representation formula, an estimate of the spatial decay of the velocity and its spatial gradient is derived. No boundary conditions have to be imposed for these results.


AMS subject classifications. 76D07, 35Q35, 35D30, 35C15, 35B40.
Key words. Oseen system, weak solutions, integral representation, spatial decay.

## 1 Introduction

In this work, we deal with the time-dependent Oseen system

$$
\begin{equation*}
u_{t}-\Delta_{x} u+\tau \partial x_{1} u+\nabla_{x} \pi=f, \quad \operatorname{div}_{x} u=0 \tag{1.1}
\end{equation*}
$$

in $\bar{\Omega}^{c} \times\left(0, T_{0}\right)$, where $\Omega$ is an open, bounded set in $\mathbb{R}^{3}$ with Lipschitz boundary. (Here and in the following, the notation $A^{c}$ for $A \subset \mathbb{R}^{3}$ stands for the complement $\mathbb{R}^{3} \backslash A$ of $A$ in $\mathbb{R}^{3}$. Thus $\bar{\Omega}^{c}$ is an exterior domain.) The preceding system arises as a linearization of the time-dependent Navier-Stokes system with Oseen term,

$$
\begin{equation*}
u_{t}-\Delta_{x} u+\tau \partial x_{1} u+\left(u \cdot \nabla_{x}\right) u+\nabla_{x} \pi=f, \quad \operatorname{div}_{x} u=0 \tag{1.2}
\end{equation*}
$$

The latter system is the usual model of the flow of a viscous incompressible fluid around a rigid body moving with constant velocity and without rotation. The parameter $\tau \in(0, \infty)$ corresponds to the Reynolds number of the fluid, and the function $f: \bar{\Omega}^{c} \times\left(0, T_{0}\right) \mapsto \mathbb{R}^{3}$ represents a volume force acting on the fluid. The unknowns in (1.1) are the velocity $u: \bar{\Omega}^{c} \times\left(0, T_{0}\right) \mapsto \mathbb{R}^{3}$ and the pressure $\pi: \bar{\Omega}^{c} \times\left(0, T_{0}\right) \mapsto \mathbb{R}$, hence $T_{0} \in(0, \infty]$ is the life-span of the solution.
In this work, we consider weak solutions of (1.1). Our aim is to derive bounds for $|u(x, t)|$ and $\left|\nabla_{x} u(x, t)\right|$ valid for large values of $|x|$ and showing how these quantities tend to zero for $|x| \rightarrow \infty$ ("decay estimates"). Such estimates are interesting because they are often associated with physical phenomena that can be observed macroscopically, for example the wake extending downstream behind the rigid body. Our decay bounds improve estimates established in previous articles.
We study a type of solution involving only the velocity $u$, which is supposed to fulfill the relations $u \in C^{0}\left(\left[0, T_{0}\right), L^{\varrho}\left(\bar{\Omega}^{c}\right)^{3}\right)$ and $\nabla_{x} u \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q}\left(\bar{\Omega}^{c}\right)^{9}\right)$, for some numbers
$\varrho, q \in(1, \infty)$. Equation (1.1) is supposed to be satisfied in the sense that

$$
\begin{align*}
& \int_{0}^{T_{0}} \int_{\bar{\Omega}^{c}}\left(-\gamma^{\prime}(t) u(t) \cdot \vartheta+\gamma(t)\left[\nabla_{x} u(t) \cdot \nabla \vartheta+\tau \partial x_{1} u(t) \cdot \vartheta-f(t) \cdot \vartheta\right]\right) d x d t  \tag{1.3}\\
& \quad-\gamma(0) \int_{\bar{\Omega}^{c}} U_{0} \cdot \vartheta d x=0 \quad \text { for } \gamma \in C_{0}^{\infty}\left(\left[0, T_{0}\right)\right), \vartheta \in C_{0, \sigma}^{\infty}\left(\bar{\Omega}^{c}\right), \quad \operatorname{div}_{x} u=0
\end{align*}
$$

where $f \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{r}\left(\bar{\Omega}^{c}\right)^{3}\right)$ and $U_{0} \in L^{\widetilde{p}}\left(\bar{\Omega}^{c}\right)^{3}$ for some $\widetilde{p}, r \in(1, \infty)$, and $\operatorname{div} U_{0}=0$ in the sense of distributions. (See Section 2 for the definition of our function spaces.)
We do not impose any boundary conditions on $u$. In fact, in concrete physical situations it is not always clear what is the right choice of such conditions. In particular, in some cases the usual no-slip condition is inappropriate. So it is an interesting feature of our theory that our decay estimates hold only on the basis of regularity assumptions on $u$, irrespective of any boundary conditions.

In order to present these estimates, we introduce two volume potentials, denoted by $\mathfrak{R}^{(\tau)}(g)$ and $\mathfrak{I}^{(\tau)}(V)$, mapping from $\mathbb{R}^{3} \times(0, \infty)$ into $\mathbb{R}^{3}$, and associated with functions $g \in L_{l o c}^{1}\left([0, T), L^{q}(A)^{3}\right)$ and $V \in L^{q}(A)^{3}$, where $A$ may be any measurable subset of $\mathbb{R}^{3}$, $q \in(1, \infty)$ and $T \in(0, \infty]$. These potentials are defined by

$$
\begin{align*}
& \mathfrak{R}^{(\tau)}(g)(x, t):=\int_{0}^{t} \int_{\mathbb{R}^{3}} \Lambda(x-y, t-s) \cdot \widetilde{g}(y, s) d y d s \quad\left(t \in(0, \infty), \text { a. e. } x \in \mathbb{R}^{3}\right)  \tag{1.4}\\
& \mathfrak{I}^{(\tau)}(V)(x, t):=\int_{\mathbb{R}^{3}} \Lambda(x-y, t) \cdot \widetilde{V}(y) d y \quad\left(t \in(0, \infty), x \in \mathbb{R}^{3}\right) \tag{1.5}
\end{align*}
$$

Here $\widetilde{g}$ and $\widetilde{V}$ stand for the zero extension of $g$ and $V$ to $\mathbb{R}^{3} \times(0, \infty)$ and $\mathbb{R}^{3}$, respectively. The function $\Lambda$, defined in (4.1), is a fundamental solution of the time-dependent Oseen system (1.1). Some key properties of these volume potentials are presented in Lemma 4.3 and 4.4. In view of stating our estimates, we further have to introduce some parameters and require some integrability conditions - local in space and global in time - on $u, \nabla_{x} u$ and $f$. In fact, we fix numbers $S_{0}, R_{0} \in(0, \infty)$ with $\bar{\Omega} \subset B_{S_{0}}$ and $S_{0}<R_{0}$, abbreviate $A_{R_{0}, S_{0}}:=$ $B_{R_{0}} \backslash \overline{B_{S_{0}}}$, and require there are parameters $\gamma_{1}, \gamma_{2}, \gamma_{3} \in[0, \infty], q \in(1, \infty)$ such that the restriction $u \mid A_{R_{0}, S_{0}} \times\left(0, T_{0}\right)$ belongs to $L^{\gamma_{1}}\left(0, T_{0}, L^{q}\left(A_{R_{0}, S_{0}}\right)^{3}\right), \nabla_{x} u \mid A_{R_{0}, S_{0}} \times\left(0, T_{0}\right)$ to $L^{\gamma_{2}}\left(0, T_{0}, L^{q}\left(A_{R_{0}, S_{0}}\right)^{3}\right)$, and $f \mid A_{R_{0}, S_{0}} \times\left(0, T_{0}\right)$ to $L^{\gamma_{3}}\left(0, T_{0}, L^{q}\left(A_{R_{0}, S_{0}}\right)^{3}\right)$. Actually our assumptions are somewhat more general (see at the beginning of Section 5 and Theorem 5.2 ), in view of applications in [18] to a nonlinear problem with (1.2) as special case. But with the preceding conditions, all the main difficulties of our proofs would already be present. We note that the parameter $S_{0}$ is introduced so that $\Omega$ need not be more regular than Lipschitz bounded. In fact, we will consider $u \mid{\overline{B_{S_{0}}}}^{c}$ instead of $u$. Finally define the function $\nu: \mathbb{R}^{3} \mapsto[1, \infty)$ by $\nu(x):=1+|x|-x_{1}$ for $x \in \mathbb{R}^{3}$. Our main decay estimate may then be stated as follows:

$$
\begin{align*}
& \left|\partial_{x}^{\alpha}\left[u-\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)-\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)\right](x, t)\right|  \tag{1.6}\\
& \leq \mathfrak{C}(|x| \nu(x))^{-3 / 2-|\alpha| / 2+1 /\left(2 \min \left\{\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \gamma_{3}^{\prime}\right\}\right)} \quad \text { for a. e. } t \in\left(0, T_{0}\right), \text { a. e. } x \in B_{R_{0}}^{c}
\end{align*}
$$

and $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha|:=\alpha_{1}+\alpha_{2}+\alpha_{3} \leq 1$, under the assumption that the zero flux condition

$$
\begin{equation*}
\int_{\partial \Omega} u(x, t) \cdot n^{(\Omega)}(x) d x=0 \quad\left(t \in\left(0, T_{0}\right)\right) \tag{1.7}
\end{equation*}
$$

is fulfilled, with $n^{(\Omega)}$ denoting the outward unit normal to $\Omega$. The presence of the function $\nu$ on the right-hand side of (1.6) may be interpreted as a mathematical manifestion of the wake. The appearance of the parameters $\gamma_{j}$ in (1.6) means that the spatial decay of $u$ and $\nabla_{x} u$ depends on $L^{p}$-integrability in time of $u, \nabla_{x} u$ and $f$; see the assumptions on $u$ and $f$ specified above. Here and everywhere else in this work, we write $\mathfrak{C}$ for constants which do not depend on the parameters whose range is indicated in the inequality under consideration (here: $x, t$ and $\alpha$ ). In particular these constants are always independent of $t$, wherever this latter parameter arises, but they frequently depend on $\tau$, as is the case here.
If (1.7) does not hold, we obtain that

$$
\begin{align*}
& \left|\partial_{x}^{\alpha}\left[u-\mathfrak{I}^{(\tau)}\left(U \mid{\overline{B_{S_{0}}}}^{c}\right)-\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)\right](x, t)\right|  \tag{1.8}\\
& \leq \mathfrak{C}\left[(|x| \nu(x))^{-3 / 2-|\alpha| / 2+1 /\left(2 \min \left\{\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \gamma_{3}^{\prime}\right\}\right)}+|x|^{-2-|\alpha|}\right] \quad \text { for } t, x, \alpha \text { as in (1.6). }
\end{align*}
$$

A more detailed version of these estimates may be found in Theorem 5.2, where we specify how the constants in (1.6) and (1.8) depend on $u, f$ and $U_{0}$.
The sum $\mathfrak{I}^{(\tau)}\left(U \mid{\overline{B_{0}}}^{c}\right)+\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)$ is the velocity part of a solution to (1.1) in $\mathbb{R}^{3} \times(0, \infty)$ with right-hand side $f$ replaced by the zero extension of $f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)$ to $\mathbb{R}^{3} \times(0, \infty)$, and with the solenoidal part of the zero extension of $U_{0} \mid \overline{S_{S_{0}}}{ }^{c}$ to $\mathbb{R}^{3}$ as initial data. We refer to [17, Corollary 3.5, 3.6] as concerns $\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)$, and to [10, Theorem 2.16, Lemma 2.11]) with respect to $\mathfrak{R}^{(\tau)}\left(f \mid{\overline{S_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)$. So this sum may be interpreted as a background flow independent of the presence of a rigid body, whereas the difference on the left-hand side of (1.6) and (1.8) corresponds to the perturbation generated in the flow field by the rigid body far from this body. It is the spatial decay of this perturbation which is of physical interest, and which we evaluate. Note that inequalities (1.6) and (1.8) hold without any assumptions on the spatial asymptotics of $f$ and $U_{0}$. On the other hand, the behavior for $|x| \rightarrow \infty$ of $\mathfrak{I}^{(\tau)}\left(U \mid{\overline{B_{S_{0}}}}^{c}\right)(x, t)$ and $\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)(x, t)$ is determined exclusively by the spatial asymptotics of $U_{0}$ and $f$. Thus the influence of the decay properties of $f$ and $U_{0}$ has effectively been seperated from the rest of the problem. Some results are available in literature with respect to the spatial decay of the two volume potentials ([12, Theorem 1.1], [13, Theorem 3.1], [17, Lemma 4.1, 4.2]). Here we contribute an improved version of [13, Theorem 3.1] as concerns $\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)\right)$ (Theorem 4.3), and a decay result for $\mathfrak{I}^{(\tau)}\left(U \mid{\overline{S_{S_{0}}}}^{c}\right)$ adapted to our situation (Theorem 4.4). Note that if $U_{0}$ and $f(t)$ have compact support, uniform with respect to $t$, and if $S_{0}$ is sufficiently large, inequalities (1.6) and (1.8) become estimates of $\left|\partial_{x}^{\alpha} u(x, t)\right|$.
Another main result of the work at hand is a representation formula for $L^{q}$-weak solutions to (1.3), stated in (5.21). It is by evaluating the integrals in this representation (Theorem 5.1) that we obtain (1.6) and (1.8).

Let us compare (1.6) and (1.8) with estimates available in literature. Reference [13] deals with $L^{2}$-weak solutions to (1.3) under Dirichlet boundary conditions with data and hence solutions - satisfying the zero flux condition (1.7). It is shown in this article that for a certain class of functions $f$ and $U_{0}$, the left-hand side of (1.6) is bounded by $\mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$ for $t, x, \alpha$ as in (1.6). It is further shown that $\left|\partial_{x}^{\alpha} u(x, t)\right|$ is majorized by the same term if $f$ and $U_{0}$ decay sufficiently fast for $|x| \rightarrow \infty$. In the work at hand, for the same type of solutions to the same problem, we deduce from (1.6) that the left-hand
side of this inequality admits the bound $\mathfrak{C}(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$, for a class of functions $f$ and $U_{0}$ much larger than the one considered in [13]. Moreover, for $\left|\partial_{x}^{\alpha} u(x, t)\right|$ we derive the bound $\mathfrak{C}|x|^{-5 / 4-|\alpha| / 2} \nu(x)^{-5 / 4-|\alpha| / 4}(\max \{1, \ln |x|\})^{|\alpha| n / 2}$, with some fixed $n \in \mathbb{N}$. The latter estimate is valid under assumptions on $f$ slightly weaker than in [13], and for functions $U_{0}$ with an asymptotic behavior for $|x| \rightarrow \infty$ adapted to the preceding bound on $\left|\partial_{x}^{\alpha} u(x, t)\right|$, which cannot be achieved under the weaker decay conditions on $U_{0}$ assumed in [13]. For more details we refer to the remarks at the end of Section 6 and to Theorem 6.1 and 6.2. The first of these two theorems evaluates $\left|\partial_{x}^{\alpha} u(x, t)\right|$ if $f=0, U_{0}=0$ (boundary-driven flow), and the second yields a bound of the left-hand side of (1.6) if $u$ fulfills homogeneous Dirichlet boundary conditions.
In [17], we derived inequalities (1.6) and (1.8) for solutions to (1.1) more regular than those considered here. In particular we imposed regularity conditions on the pressure ([17, Theorem 6.1]). The task we face here consists in extending these inequalities to $L^{q}$-weak solutions, which constitute a much larger class of solutions.
The present work is the basis of two further articles. In reference [18] we start from the representation formula (5.21), obtaining convergence estimates for $|x| \rightarrow \infty$ of $L^{2}$-strong solutions to a generalization of the nonlinear system (1.2). The focus in this latter work is on improving the decay results in [36], [42] and [14] without requiring specific boundary conditions. Reference [19] deals with the spatial asymptotics of mild solutions to (1.1) under homogeneous Dirichlet boundary conditions. It turned out these solutions are $L^{q_{-}}$ weak in the sense of the work at hand. It further turned out that if $f \in L^{1}\left(0, \infty, L_{\sigma}^{q}\left(\bar{\Omega}^{c}\right)\right)$ and $U_{0} \in L_{\sigma}^{q}\left(\bar{\Omega}^{c}\right)$ for some $q \in(1,3 / 2)$, inequality (1.6) may be applied with $\gamma_{j}=1$ for $j \in\{1,2,3\}$, leading to the bound $\mathfrak{C}(|x| \nu(x))^{-3 / 2-|\alpha| / 2}$, uniformly in $t$, for the left-hand side of (1.6).
This bound is also valid for $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|$ (Lemma 4.1), where $\Lambda$ is the fundamental solution to (1.1) mentioned above. It seems no stronger decay result may be obtained for $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|$. However, the estimate $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right| \leq \mathfrak{C}(R)(|x| \nu(x))^{-3 / 2-|\alpha| / 2}$, with $\alpha \in \mathbb{N}_{0}^{3},|\alpha| \leq 1, x \in$ $B_{R}^{c}$ for some $R>0$, does involve some loss of accuracy. In fact, the integral $\int_{\mathbb{R}^{3}} \Lambda(x, t) d t$ yields the standard fundamental solution of the stationary Oseen system (3.4), as may be seen by [22, Corollary 4.3, Theorem 4.2]. This latter solution is known to decay as $O\left([|x| \nu(x)]^{-1}\right)$ for $|x| \rightarrow \infty([27],[38])$, and this decay rate cannot be recovered via the integral $\int_{0}^{\infty}(|x| \nu(x)+t)^{-3 / 2} d t$, which only yields $O\left([|x| \nu(x)]^{-1 / 2}\right)$. An analogous remark is valid with respect to first-order space derivatives. But the integral $\int_{0}^{\infty}\left|\partial_{x}^{\alpha} \Lambda(x, t)\right| d t$ is bounded by $\mathfrak{C}(R)(|x| \nu(x))^{-1-|\alpha| / 2}$ if $|x| \geq R$ for some $R>0$ (Lemma 4.1). So in the estimate $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right| \leq \mathfrak{C}(R)(|x| \nu(x))^{-3 / 2-|\alpha| / 2}$, a structural element of $\Lambda$ gets lost, leading to the slower decrease of $\int_{0}^{\infty}(|x| \nu(x))^{-3 / 2} d t$ compared to that of the integral $\int_{0}^{\infty}|\Lambda(x, t)| d t$.
Since fundamental solutions play a key role in the study of spatial asymptotics, one may reasonably take the point of view that the decay rates provided by inequalities (1.6) and (1.8) may not be higher than the one corresponding to the best possible decay bound of $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|$. It is true that in view of the preceding remarks on the asymptotics of $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|$, the notion of "best possible decay bound of $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|$ " is somewhat ambiguous. Still we think that the inequality $\left|\partial_{x}^{\alpha} \Lambda(x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-3 / 2-|\alpha|}$ does constitute
a suitable yardstick, showing that inequalities (1.6) and (1.8) are indeed the best possible decay estimates valid for solutions to (1.1) independently of the choice of boundary conditions. This is analogous to the situation in the stationary case, where the best possible decay bound independent of boundary conditions is given by $\mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$ ([3], [27]), corresponding to the behavior mentioned above of the standard fundamental solution to the stationary Oseen system (3.4).
It is the choice of a representation formula for solutions to (1.1) that is a key difference between some of the studies mentioned above. References [36] and [42] use an equation that may be characterized as a Green's formula involving a fundamental solution of (1.1), like $\Lambda$. This formula has the drawback that it contains an integral on $\partial \Omega \times\left(0, T_{0}\right)$ in which the stress tensor appears. Such an integral requires a rather high regularity of the solution and is difficult to control. In [10], [13] and [14], we used an integral representation which does not contain such a term, but requires solving an integral equation on $\partial \Omega \times\left(0, T_{0}\right)$. The technical overhead related to this equation is considerable, and we were limited to what is essentially an $L^{2}$-framework. In [17], we at first proved the standard formula from [36] and [42] under assumptions as weak as possible ([17, Theorem 5.2, (5.6)]). Then, by means of some partial integrations, we obtained a new formula in which the critical term does not appear any more ([17, (5.7)]). This formula yields a representation of $u(x, t)$ only at points $x$ situated outside a ball around $\bar{\Omega}$. But since we are interested in the behavior of $u(x, t)$ for large values of $|x|$, this restriction does not matter in our context. However, the theory in $[17]$ is still inconvenient since the assumptions in the relevant corollary ([17, Corollary 5.2]) are too strong in order to allow applying [17, (5.7)] to the solution of (1.3) we consider here. In particular, [17, Corollary 5.2] imposes conditions on the pressure although this unknown does not appear in [17, (5.7)].

The main effort in the work at hand is directed at showing that the integral representation in $[17,(5.7)]$ may be extended to $L^{q}$-weak solutions of (1.1). As the key difficulty, we have to eliminate the conditions on the pressure just mentioned. To this end, we will use Friedrich's mollifier for functions with values in Banach spaces, smoothing weak solutions with respect to the time variable. We will consider these smoothed solutions as weak solutions of the stationary Oseen system (3.4), with the time derivative subsumed into the right-hand side. In this way we will be able to use the regularity theory of this latter system in order to construct an associated pressure. Once this result is available, the mollified version of the $L^{q}$-weak solution we started out with will turn out to be sufficiently regular so that we may apply the integral representation [17, (5.7)]; see Theorem 5.1. By letting certain parameters tend to zero in this formula, we will then obtain that [17, (5.7)] extends to our weak solution, in a slightly different form (Corollary 5.4).
In view of (1.3), the function $U_{0}$ seems to play the role of initial data. However, due to the lack of boundary conditions, the equation $u(0)=U_{0}$ need not hold in general (Lemma 5.4). But for the case $\operatorname{div} U_{0}=0$ in a distributional sense, we will show that $U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}$ is a harmonic function with $\left|\partial_{x}^{\alpha}\left(U_{0}-u(0)\right)(x)\right|=O\left(|x|^{-2-|\alpha|}\right)$ for $|x| \rightarrow \infty$, where $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. If $U_{0}-u(0)$ satisfies a zero flux condition on $\partial B_{R}$ for some $R>S_{0}$, we will even get $\left|\partial_{x}^{\alpha}\left(U_{0}-u(0)\right)(x)\right|=O\left(|x|^{-3-|\alpha|}\right)$ for $|x| \rightarrow \infty$ (Corollary 5.5).
Let us mention some references more distantly related to the work at hand. Takahashi [49] deals with spatial decay of solutions to (1.2) in the case $\Omega=\emptyset$ under a smallness
condition. In [4], [5], solutions to (1.1) and (1.2) are estimated in weighted $L^{p}$-norms, with the weights adapted to the wake in the flow field downstream to the rigid body. Reference [16] presents decay estimates in time and in space for solutions of (1.1) and (1.2), as a continuation of [13] (Oseen system (1.1)) and [14] (problem (1.2)), under the same assumptions and with the same methods as in these latter articles. Various technical aspects of the theory in [13], [14] and [16] are dealt with in predecessor papers [7] - [11]. Questions of existence, regularity and stability related to (1.1), (1.2) or generalizations of (1.2), in particular $L^{p}-L^{q}$-estimates of Oseen flows, are addressed in [25], [26], [30], [31], [32], [34], [35], [37], [40], [41], [45], [48].

## 2 Notation. Some auxiliary results.

The parameters $T_{0} \in(0, \infty]$ and $\tau \in(0, \infty)$ introduced in Section 1 are kept fixed throughout, as is the open, bounded set $\Omega \subset \mathbb{R}^{3}$ with Lipschitz boundary.
The symbol $\left|\mid\right.$ denotes the Euclidean norm of $\mathbb{R}^{n}$ for any $n \in \mathbb{N}$, as well as the length $\alpha_{1}+\alpha_{2}+\alpha_{3}$ of a multi-index $\alpha \in \mathbb{N}_{0}^{3}$. For $R \in(0, \infty), x \in \mathbb{R}^{3}$, put $B_{R}(x):=\left\{y \in \mathbb{R}^{3}\right.$ : $|x-y|<R\}$. In the case $x=0$, we write $B_{R}$ instead of $B_{R}(0)$.
Recall that in Section 1, we introduced the function $\nu: \mathbb{R}^{3} \mapsto[1, \infty)$ by setting $\nu(x):=$ $1+|x|-x_{1}$ for $x \in \mathbb{R}^{3}$.
We fix numbers $S_{0}, R_{0} \in(0, \infty)$ with $S_{0}<R_{0}$ and $\bar{\Omega} \subset B_{S_{0}}$, as well as a function $\varphi_{0} \in$ $C_{0}^{\infty}\left(B_{\left(R_{0}+S_{0}\right) / 2}\right)$ with $0 \leq \varphi_{0} \leq 1$ and $\varphi_{0} \mid B_{S_{0}+\left(R_{0}-S_{0}\right) / 4}=1$. We put $R_{1}:=\left(R_{0}+S_{0}\right) / 2$.
For $n \in \mathbb{N}, I \subset \mathbb{R}^{n}$, let $\chi_{I}$ stand for the characteristic function of $I$ in $\mathbb{R}^{n}$. If $A \subset \mathbb{R}^{3}$, we denote by $A^{c}$ the complement $\mathbb{R}^{3} \backslash A$ of $A$ in $\mathbb{R}^{3}$. Put $e_{l}:=\left(\delta_{j l}\right)_{1 \leq j \leq 3}$ for $1 \leq l \leq 3$ (unit vector in $\mathbb{R}^{3}$ ). If $A$ is an open bounded set in $\mathbb{R}^{3}$ with Lipschitz boundary, we write $n^{(A)}$ for the outward unit normal to $A$. If $R, S \in(0, \infty)$ with $S<R$, we write $A_{R, S}$ for the annular domain $B_{R} \backslash \overline{B_{S}}$.
Let $p \in[1, \infty)$ and $n \in \mathbb{N}$. For any open set $A \subset \mathbb{R}^{3}$, the norm of the Lebesgue space $L^{p}(A)$ is denoted by $\left\|\|_{p}\right.$, and the usual norm of the Sobolev space $W^{m, p}(A)$ of order $m$ and exponent $p$ is designated by $\left\|\|_{m, p}\right.$. Again for an open set $A \subset \mathbb{R}^{3}$, we define $C_{0, \sigma}^{\infty}(A):=\left\{V \in C_{0}^{\infty}(A)^{3}: \operatorname{div} V=0\right\}$, and write $L_{l o c}^{p}(A)$ and $W_{l o c}^{m, p}(A)$ for the set of all functions $V$ from $A$ into $\mathbb{R}$ such that $V \mid K \in L^{p}(K)$ and $V \mid K \in W^{1, p}(K)$, respectively, for any open, bounded set $K \subset \mathbb{R}^{3}$ with $\bar{K} \subset A$. We put $\nabla V:=\left(\partial_{k} V_{j}\right)_{1 \leq j, k \leq 3}$ for $V \in W_{l o c}^{1,1}(A)^{3}$.
Let $\mathcal{V}$ a normed space, and let the norm of $\mathcal{V}$ be denoted by $\|\|$. Take $n \in \mathbb{N}$. Then we will use the same notation $\left\|\|\right.$ for the norm on $\mathcal{V}^{n}$ defined by $\|\left(f_{1}, \ldots, f_{n}\right) \|:=\left(\sum_{j=1}^{n}\left\|f_{j}\right\|^{2}\right)^{1 / 2}$ for $\left(f_{1}, \ldots, f_{n}\right) \in \mathcal{V}^{n}$. The space $\mathcal{V}^{3 \times 3}$, as concerns its norm, is identified with $\mathcal{V}^{9}$. If $p \in(1, \infty), n \in\{1,3\}$ and $A \subset \mathbb{R}^{3}$ open, the dual space of $W_{0}^{1, p^{\prime}}(A)^{n}$ will be denoted by $W_{0}^{-1, p}(A)^{n}$, although in the case $n>1$ this dual space is not the Cartesian product of $W_{0}^{-1, p}(A)$. We write $L_{\sigma}^{p}(A)$ for the closure of $C_{0, \sigma}^{\infty}(A)$ with respect to the norm of $L^{p}(A)^{3}$.

We additionally introduce some weighted Sobolev spaces. To this end, for $x \in \mathbb{R}^{3}$ we put

$$
\begin{aligned}
& \omega_{p}^{(j)}(x):=(1+|x|)^{-j} \text { for } j \in\{1,2\}, \quad p \in(1, \infty) \backslash\{3 / 2,3\} \\
& \omega_{3 / 2}^{(1)}(x):=(1+|x|)^{-1}, \quad \omega_{3 / 2}^{(2)}(x):=(1+|x|)^{-2}(\ln (2+|x|))^{-1} \\
& \omega_{3}^{(j)}(x):=(1+|x|)^{-j}(\ln (2+|x|))^{-1} \text { for } j \in\{1,2\}
\end{aligned}
$$

Then we set

$$
\begin{aligned}
& W_{2}^{2, p}\left(\mathbb{R}^{3}\right):=\left\{V \in W_{l o c}^{2,1}\left(\mathbb{R}^{3}\right): \omega_{p}^{(2)} V, \quad \omega_{p}^{(1)} \partial_{l} V, \partial_{l} \partial_{k} V \in L^{p}\left(\mathbb{R}^{3}\right) \text { for } 1 \leq k, l \leq 3\right\} \\
& W_{1}^{1, p}\left(\mathbb{R}^{3}\right):=\left\{V \in W_{l o c}^{1,1}\left(\mathbb{R}^{3}\right): \omega_{p}^{(1)} V, \quad \partial_{l} V \in L^{p}\left(\mathbb{R}^{3}\right) \text { for } 1 \leq l \leq 3\right\} \quad(p \in(1, \infty))
\end{aligned}
$$

We will not work with a norm of $W_{2}^{2, p}\left(\mathbb{R}^{3}\right)$. However, the norm of $W_{1}^{1, p}\left(\mathbb{R}^{3}\right)$ defined by $\|V\|_{W_{1}^{1, p}\left(\mathbb{R}^{3}\right)}:=\left(\left\|\omega_{p}^{(1)} V\right\|_{p}^{p}+\|\nabla V\|_{p}^{p}\right)^{1 / p}$ will be relevant.
Let $p \in[1, \infty], \mathfrak{B}$ a Banach space and $J \subset \mathbb{R}$ an interval. Then the norm of the space $L^{p}(J, \mathfrak{B})$ is denoted by $\left\|\|_{L^{p}(J, \mathcal{B})}\right.$. Let $a, b \in \mathbb{R} \cup\{\infty\}$ with $a<b$, take $m \in \mathbb{N}$ and $q \in[1, \infty)$. Then we write $L^{p}(a, b, \mathcal{B})$ and $W^{1, q}(a, b, \mathcal{B})$ instead of $L^{p}((a, b), \mathcal{B})$ and $W^{1, q}((a, b), \mathcal{B})$, respectively. We use the expression $L_{l o c}^{p}([a, b), \mathcal{B})$ for the space of all functions $v:(a, b) \mapsto$ $\mathcal{B}$ such that $v \mid(a, T) \in L^{p}(a, T, \mathcal{B})$ for any $T \in(a, b)$. This space is to be distinguished from the space $L_{l o c}^{p}(a, b, \mathcal{B})$, defined in the usual way. Let $T \in(0, \infty], A \subset \mathbb{R}^{3}$ open, and $n \in \mathbb{N}$. Then we will write $\left\|\|_{q, p ; T}\right.$ and $\| \|_{q, p ; \mathbb{R}}$ instead of $\left\|\|_{L^{p}\left(0, T, L^{q}(A)^{n}\right)}\right.$ and $\| \|_{L^{p}\left(\mathbb{R}, L^{q}(A)^{n}\right)}$, respectively.
If $v \in W^{1,1}(a, b, \mathcal{B})$, then, possibly after a modification on a subset of $[a, b)$ with measure zero, the function $v$ belongs to $C^{0}([a, b), \mathcal{B})([50$, Lemma 3.1.1]). If the latter relation is already valid, we write $v \in W^{1,1}(a, b, \mathcal{B}) \cap C^{0}([a, b), \mathcal{B})$.
Of course, a function $v \in L^{p}\left(J, L^{q}(A)^{n}\right)$ may be considered also as a function on $A \times J$, although there is a minor issue with respect to measurability on $A \times J$, settled in [11, Lemma 2.1] and [17, Lemma 2.3]. We will write $v(t)(x)$ or $v(x, t)$, depending on whether we consider $v$ as a function on $J$ with values in $L^{q}(A)^{n}$, or as a function on $A \times J$. For an interval $J \subset \mathbb{R}$ and a function $v: J \mapsto W_{l o c}^{1,1}(A)^{3}$, the notation $\nabla_{x} v$ stands for the gradient of $v$ with respect to $x \in A$, in the sense that

$$
\nabla_{x} v: J \mapsto L_{l o c}^{1}(A)^{3 \times 3}, \nabla_{x} v(t)(x):=\left(\partial x_{k}\left(v_{j}(t)\right)(x)\right)_{1 \leq j, k \leq 3} \text { for } t \in J, x \in A
$$

(spatial gradient of $v$ ). Similar conventions are to be valid with respect to the expressions $\Delta_{x} v, \operatorname{div}_{x} v$ and $\partial x_{j} v$.
Concerning Bochner integrals, if $J \subset \mathbb{R}$ is open, $\mathcal{B}$ a Banach space and $w: J \mapsto \mathcal{B}$ an integrable function, it is convenient sometimes to write $\mathcal{B}-\int_{J} w(t) d t$ instead of $\int_{J} w(t) d t$ for the corresponding $\mathcal{B}$-valued Bochner integral. For the definition of Bochner integrals, we refer to [51, p. 132-133], or to [33, p. 78-80.].
We define the Fourier transform $\hat{f}$ of $f \in L^{1}\left(\mathbb{R}^{3}\right)$ by $\hat{f}(\xi):=(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{n}} e^{-i \xi \cdot z} f(z) d z$ for $\xi \in \mathbb{R}^{n}$. An analogous definition is to hold for functions belonging to $L^{2}\left(\mathbb{R}^{3}\right)$.
We write $C$ for numerical constants and $C\left(\gamma_{1}, \ldots, \gamma_{n}\right)$ for constants depending exclusively on paremeters $\gamma_{1}, \ldots, \gamma_{n} \in[0, \infty)$ for some $n \in \mathbb{N}$. However, such a precise bookkeeping
will be possible only at some places. Mostly we will use the symbol $\mathfrak{C}$, with the convention that it stands for a constant which does not depend on quantities pointed out by the word "for" in the context of the estimate under consideration. This concerns in particular the parameter $t$, which never enters in any constant. On the other hand, dependence on $\tau$ is frequent. At some places we state explicitly which are the dependencies of $\mathfrak{C}$, and sometimes we write $\mathfrak{C}\left(\gamma_{1}, \ldots, \gamma_{n}\right)$ in order to indicate that the constant in question is influenced in particular by the quantities $\gamma_{1}, \ldots, \gamma_{n}$.

The following simple version of Young's inequality for integrals will be used frequently. Stated here for the convenience of the reader, we will refer to it as "Young's inequality".

Lemma 2.1 ([1, Corollary 2.25]) Let $n \in \mathbb{N}$ and $q \in[1, \infty]$. Then

$$
\left(\int_{\mathbb{R}^{n}}\left|\int_{\mathbb{R}^{n}} U(x-y) V(y) d y\right|^{q} d x\right)^{1 / q} \leq C\|U\|_{1}\|V\|_{q} \quad \text { for } U \in L^{1}\left(\mathbb{R}^{n}\right), V \in L^{q}\left(\mathbb{R}^{n}\right)
$$

The next theorem deals with solenoidal $W_{0}^{1, q}$-functions.
Theorem 2.1 ([29, Theorem III.4.2, III.6.1]) Let $n \in \mathbb{N}, q, r_{1}, \ldots, r_{n} \in(1, \infty), A \subset$ $\mathbb{R}^{3}$ open, bounded and with Lipschitz boundary. Let $V \in W_{0}^{1, q}\left(\bar{A}^{c}\right)^{3} \cap L^{r_{j}}\left(\bar{A}^{c}\right)^{3}$ for $1 \leq j \leq n$ with div $V=0$. Then there is a sequence $\left(\vartheta_{n}\right)$ in $C_{0, \sigma}^{\infty}\left(\bar{A}^{c}\right)$ such that $\left\|V-\vartheta_{n}\right\|_{1, q} \rightarrow 0$ and $\left\|V-\vartheta_{n}\right\|_{r_{j}} \rightarrow 0(n \rightarrow \infty)$ for $1 \leq j \leq n$.
The ensuing theorem presents a result on $L^{p}$-integrability of functions defined in an exterior domain and possessing an $L^{q}$-integrable gradient.
Theorem 2.2 Let $A \subset \mathbb{R}^{3}$ be open, bounded and with Lipschitz boundary. Let $q \in(1,3)$ and $V \in W_{l o c}^{1,1}\left(\bar{A}^{c}\right)$ with $\nabla V \in L^{q}\left(\bar{A}^{c}\right)^{3}$. Suppose there is some $\kappa \in(1, \infty)$ with $V \in L^{\kappa}\left(\bar{A}^{c}\right)$. Then $V \in L^{3 q /(3-q)}\left(\bar{A}^{c}\right)$ and $\|V\|_{3 q /(3-q)} \leq \mathfrak{C}\|V\|_{q}$.
Proof: This theorem may be deduced from [29, Theorem II.6.1]; see [15, Theorem 2.4] and its proof.

We mention some results about Bochner's integral. Our basic tool in this context is the following theorem (compatibility of bounded operators and Bochner integrals).

Theorem 2.3 Let $B_{1}, B_{2}$ be Banach spaces, $A: B_{1} \mapsto B_{2}$ a linear and bounded operator, $n \in \mathbb{N}, J \subset \mathbb{R}^{n}$ an open set and $f: J \mapsto B_{1}$ a Bochner integrable mapping. Then $A \circ f: J \mapsto B_{2}$ is Bochner integrable, too, and $A\left(B_{1}-\int_{J} f d x\right)=B_{2}-\int_{J} A \circ f d x$.
Proof: See [51, p. 134, Corollary 2], [33, Theorem 3.7.12 and the remark on p. 84].
Next we indicate a compatibility result for Bochner integrals with values in $L^{p}$-spaces.
Lemma 2.2 Let $J \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}, B \subset \mathbb{R}^{n}$ and $A \subset B$ open sets, $q_{1}, q_{2} \in$ $[1, \infty)$ and $f: J \mapsto L^{q_{1}}(B)^{3}$ a Bochner integrable mapping with $f(t) \mid A \in L^{q_{2}}(A)^{3}$ for $t \in J$ and $f \mid A: J \mapsto L^{q_{2}}(A)^{3}$ Bochner integrable as well. Then $\left(L^{q_{1}}(B)^{3}-\int_{J} f(s) d s\right) \mid A=$ $L^{q_{2}}(A)^{3}-\int_{J} f(s) \mid A d s$.
Proof: Use Theorem 2.3 and note that for $\psi \in C_{0}^{\infty}(A)^{3}$, the mapping $V \mapsto \int_{A} \psi \cdot V d x$ is linear and bounded as an operator of $V \in L^{q_{2}}(A)^{3}$ and as an operator of $V \in L^{q_{1}}(B)^{3}$, each time with values in $\mathbb{R}$.

Theorem 2.4 Let $B$ be a Banach space, $n \in \mathbb{N}, p \in[1, \infty)$ and $J \subset \mathbb{R}^{n}$ measurable.

Then the set of integrable functions from $J$ into $B$ only taking a finite number of values ("simple functions") is dense in $L^{p}(J, B)$.
Proof: See [24, Section 8.18.1 and Exercise 8.29]. On the basis of Lebesgue's theorem, the proof can be done by first approximating $f \in L^{p}(J, B)$ by functions with bounded support, and then by functions with bounded support and such that their range is bounded with respect to the norm of $B$. Functions of the latter kind belong to $L^{1}(J, B)$ and thus, by the definition of Bochner's integral, may be approximated in $L^{1}(J, B)$ by simple functions, which implies approximation in $L^{p}(J, B)$ in this situation, due to Lebesgue's theorem.
Corollary 2.1 Let $B$ be a Banach space, $A$ a dense subset of $B, p \in[1, \infty)$ and $J \subset \mathbb{R}^{n}$ open. Then the set $\left\{\sum_{j=1}^{k} \varphi_{j} a_{j}: k \in \mathbb{N}, \varphi_{j} \in C_{0}^{\infty}(J), a_{j} \in A\right.$ for $\left.1 \leq j \leq n\right\}$ is dense in $L^{p}(J, B)$. In particular the set of continuous functions $f: J \mapsto B$ with supp $(f)$ compact is dense in $L^{p}(J, B)$.
Proof: Use Theorem 2.4 and the density of $C_{0}^{\infty}(J)$ in $L^{p}(J)$.
In order to define Friedrich's mollifier for functions with values in Banach spaces, we fix a function $\rho \in C_{0}^{\infty}((-1,1))$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho(s) d s=1$, and put $\rho_{\delta}(r):=\delta^{-1} \rho\left(\delta^{-1} r\right)$ for $\delta \in(0, \infty), r \in \mathbb{R}$. If B is a Banach space and $f \in L_{l o c}^{1}(\mathbb{R}, B)$, define $f_{\delta}(t):=$ $B-\int_{\mathbb{R}} \rho_{\delta}(t-s) f(s) d s$ for $t \in \mathbb{R}, \delta \in(0, \infty)$.
Key properties of Friedrich's mollifier of functions with values in $\mathbb{R}$ carry over to functions with values in Banach spaces. Properties of this type needed in the work at hand are collected in the ensuing Theorem 2.5.
Theorem 2.5 Let $B$ be a Banach space and $f \in L_{\text {loc }}^{1}(\mathbb{R}, B)$. Then $f_{\delta} \in C^{\infty}(\mathbb{R}, B)$ and $f_{\delta}^{(n)}(t)=\int_{\mathbb{R}} \rho_{\delta}^{(n)}(t-s) f(s) d s(n \in \mathbb{N}, t \in \mathbb{R}, \delta \in(0, \infty))$. Moreover, if $f \in W_{l o c}^{1,1}(\mathbb{R}, B)$, then $\left(f_{\delta}\right)^{\prime}=\left(f^{\prime}\right)_{\delta}$.
Let $p \in[1, \infty)$ and $g \in L^{p}(\mathbb{R}, B)$. Then $\left\|g_{\delta}\right\|_{L^{p}(\mathbb{R}, B)} \leq\|g\|_{L^{p}(\mathbb{R}, B)}$ for $\delta \in(0, \infty)$ and $\left\|g_{\delta}-g\right\|_{L^{p}(\mathbb{R}, B)} \rightarrow 0(\delta \downarrow 0)$.
Let $h \in C^{0}(\mathbb{R}, B)$ and $t \in \mathbb{R}$. Then $\left\|\left(h_{\delta}-h\right)(t)\right\| \rightarrow 0(\delta \downarrow 0)$, where $\|\|$ denotes the norm of $B$.
Proof: The relation $f_{\delta} \in C^{\infty}(\mathbb{R}, B)$ and the equation for $f_{\delta}^{(n)}(t)$ follow from the relation $\rho_{\delta} \in C_{0}^{\infty}((-\delta, \delta))$. If $f \in W_{l o c}^{1,1}(\mathbb{R}, B)$ and $t \in \mathbb{R}$, the equation $\left(f_{\delta}\right)^{\prime}(t)=\left(f^{\prime}\right)_{\delta}(t)$ holds due to the above equation for $f_{\delta}^{(n)}(t)$ with $n=1$, and since the function $s \mapsto \rho_{\delta}(t-s)(s \in \mathbb{R})$ belongs to $C_{0}^{\infty}(\mathbb{R})$. The inequality $\left\|g_{\delta}\right\|_{L^{p}(\mathbb{R}, B)} \leq\|g\|_{L^{p}(\mathbb{R}, B)}$ for $\delta>0$ is an immediate consequence of Young's inequality and the choice of $\rho_{\delta}$. We further note that for $\epsilon \in(0, \infty)$, Corollary 2.1 yields existence of a function $g^{(\epsilon)} \in C^{0}(\mathbb{R}, B)$ with compact support such that $\left\|g-g^{(\epsilon)}\right\|_{L^{p}(\mathbb{R}, B)} \leq \epsilon / 2$. With this result available, the relation $\left\|g_{\delta}-g\right\|_{L^{p}(\mathbb{R}, B)} \rightarrow 0(\delta \downarrow 0)$ follows by the same arguments as in the case $B=\mathbb{R}$; see [1, p. 37-38] for example. The same reference yields the last claim of the theorem.

## 3 Some results on the Poisson equation and the stationary Oseen system.

In the ensuing theorem, we state some properties of the Newton potential. The proof of this theorem is well known (Hardy-Littlewood-Sobolev's inequality, Calderon-Zygmund's inequality and density arguments). We refer to [39], where the Stokes case is treated, with details elaborated in [23].
Theorem 3.1 Put $\mathfrak{N}(z):=(4 \pi|z|)^{-1}$ for $z \in \mathbb{R}^{3} \backslash\{0\}$ (fundamental solution of the Poisson equation). Let $q \in(1,3 / 2), F \in L^{q}\left(\mathbb{R}^{3}\right)$. Then $\int_{\mathbb{R}^{3}} \mathfrak{N}(x-y)|F(y)| d y<\infty$ for a. e. $x \in \mathbb{R}^{3}$. Put $(\mathfrak{N} * F)(x):=\int_{\mathbb{R}^{3}} \mathfrak{N}(x-y) F(y) d y$ for $x \in \mathbb{R}^{3}$ ("Newton potential"). Then $\mathfrak{N} * F \in W_{\text {loc }}^{2, q}\left(\mathbb{R}^{3}\right), \Delta(\mathfrak{N} * F)=-F, \mathfrak{N} * F \in L^{(1 / q-2 / 3)^{-1}}\left(\mathbb{R}^{3}\right), \partial_{k}(\mathfrak{N} * F) \in$ $L^{(1 / q-1 / 3)^{-1}}\left(\mathbb{R}^{3}\right)^{3}$ and $\partial_{k} \partial_{l}(\mathfrak{N} * F) \in L^{q}\left(\mathbb{R}^{3}\right)^{3}(1 \leq k, l \leq 3)$. If $F \in W^{1, q}\left(\mathbb{R}^{3}\right)$, then $\mathfrak{N} * F \in W_{l o c}^{3, q}\left(\mathbb{R}^{3}\right)$ and $\left\|\partial_{k} \partial_{l} \partial_{m}(\mathfrak{N} * F)\right\|_{q}+\left\|\partial_{l} \partial_{m}(\mathfrak{N} * F)\right\|_{q} \leq C(q)\|F\|_{1, q}$ for $1 \leq k, l, m \leq 3$. If $F \in C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$, the function $\mathfrak{N} * F$ belongs to $C^{\infty}\left(\mathbb{R}^{3}\right)$.
Next we present some technical results on harmonic functions.
Theorem 3.2 ("Weil's lemma", [46, Appendix]) Let $A \subset \mathbb{R}^{3}$ be open, $V \in L_{l o c}^{1}(A)$ and $\int_{A} V \Delta \varphi=0$ for $\varphi \in C_{0}^{\infty}(A)$. Then $V \in C^{\infty}\left(\mathbb{R}^{3}\right)$ and $\Delta V=0$.
Theorem 3.3 Let $A \subset \mathbb{R}^{3}$ be open and bounded, with $C^{1}$-boundary, $q \in(3, \infty), V \in$ $C^{\infty}\left(\bar{A}^{c}\right)$ with $\Delta V=0$ and $\nabla V \in L^{q}\left(\bar{A}^{c}\right)^{3}$. Then there is a number $a \in \mathbb{R}$ such that $V-a \in L^{q}\left(\bar{A}^{c}\right)$.

Proof: According to [29, Lemma II.6.1], we have $V \mid B_{R} \backslash \bar{A} \in W^{1, q}\left(B_{R} \backslash \bar{A}\right)$ for any $R \in(0, \infty)$ with $\bar{A} \subset B_{R}$. In particular the trace $V \mid \partial A$ is well defined and belongs to $W^{1-1 / q, q}(\partial A)$. In this situation, we refer to [46, Theorem I.10.6, 1), (10.43)] and Theorem 3.2 to obtain a function $W \in C^{\infty}\left(\bar{A}^{c}\right)$ such that $\Delta W=0, \nabla W \in L^{q}\left(\bar{A}^{c}\right)^{3}, W \mid B_{R} \backslash \bar{A} \in$ $L^{q}\left(B_{R} \backslash \bar{A}\right)$ for $R \in(0, \infty)$ with $\bar{A} \subset B_{R}$, and $W|\partial A=V| \partial A$, and such that there are numbers $c_{0}, r_{0} \in(0, \infty)$ with $\bar{A} \subset B_{r_{0}}$ and $|W(x)| \leq c_{0}|x|^{-1}$ for $x \in B_{r_{0}}^{c}$. It follows by [46, Theorem II.6.2, iv), vii)] that there is $a \in \mathbb{R}$ such that the difference $W-V-a$ possesses the same regularity and decay properties as $W$ does. The decay estimate satisfied by $W$ and $W-v-a$ yields there is $r_{1}>0$ with $\bar{A} \subset B_{r_{1}}$ and $V+a \mid B_{r_{1}}^{c} \in L^{p}\left(B_{r_{1}}^{c}\right)$ for $p \in(3, \infty)$. Since $q>3$, the theorem follows.
Lemma 3.1 Let $R, \widetilde{R}, S \in(0, \infty)$ with $S<\widetilde{R}<R, \varphi \in C_{0}^{\infty}\left(B_{R}\right)$ with $\varphi \mid B_{\widetilde{R}}=1$. Moreover let $k_{0} \in \mathbb{N}, V \in C^{\infty}\left({\overline{B_{S}}}^{c}\right)$ with $\Delta V=0, q_{j} \in(1, \infty)$ and $V^{(j)} \in L^{q_{j}}\left({\overline{B_{S}}}^{c}\right)$ for $1 \leq j \leq k_{0}$ such that $V=\sum_{j=1}^{k_{0}} V^{(j)}$. Put $F:=-2 \nabla(1-\varphi) \cdot \nabla V-\Delta(1-\varphi) V$.
Then $F \in C_{0}^{\infty}\left(B_{R}\right), V\left|{\overline{B_{R}}}^{c}=(\mathfrak{N} * F)\right|{\overline{B_{R}}}^{c}$ and $V \mid{\overline{B_{R}}}^{c} \in L^{p}\left({\overline{B_{R}}}^{c}\right)$ for $p \in(3, \infty)$.
Proof: Define $W:=(1-\varphi) V$. Then $W \in C^{\infty}\left(\mathbb{R}^{3}\right),-\Delta W=F$ and $W=\sum_{j=1}^{k_{0}}(1-\varphi) V^{(j)}$, where $(1-\varphi) V^{(j)} \in L^{q_{j}}\left(\mathbb{R}^{3}\right)$ for $1 \leq j \leq k_{0}$. Theorem 3.1 and the obvious relation $F \in C_{0}^{\infty}\left(B_{R}\right)$ imply that $\mathfrak{N} * F \in C^{\infty}\left(\mathbb{R}^{3}\right) \cap L^{p}\left(\mathbb{R}^{3}\right)$ for $p \in(3, \infty)$, and $-\Delta(\mathfrak{N} * F)=F$. We claim that $W=\mathfrak{N} * F$. To see this, we may for example reason as in [6, p. 15241525]: Since $W=\sum_{j=1}^{k_{0}}(1-\varphi) V^{(j)}$ and because $(1-\varphi) V^{(j)} \in L^{q_{j}}\left(\mathbb{R}^{3}\right)\left(1 \leq j \leq k_{0}\right)$ and $\mathfrak{N} * F \in L^{p}\left(\mathbb{R}^{3}\right)(p \in(3, \infty))$, we may apply Friedrich's mollifier to $W-\mathfrak{N} * F$, obtaining a bounded function $(W-\mathfrak{N} * F)_{\epsilon}$, defined in an obvious way for any $\epsilon \in(0, \infty)$.

Since $\Delta(W-\mathfrak{N} * F)=0$, we have $\Delta(W-\mathfrak{N} * F)_{\epsilon}=0$, also for any $\epsilon>0$. Thus Liouville's theorem yields $(W-\mathfrak{N} * F)_{\epsilon}=0(\epsilon \in(0, \infty))$. Again due to the equation $W=\sum_{j=1}^{k_{0}}(1-\varphi) V^{(j)}$ and the integrability properties of $(1-\varphi) V^{(j)}$ and $\mathfrak{N} * F$, we may conclude that $W-\mathfrak{N} * F=0$. So our claim is proved. Now the choice of $\varphi$ and the definition of $W$ imply $V\left|{\overline{B_{R}}}^{c}=(\mathfrak{N} * F)\right|{\overline{B_{R}}}^{c}$, so Theorem 3.1 yields $V \mid{\overline{B_{R}}}^{c} \in L^{p}\left({\overline{B_{R}}}^{c}\right)$ for $p \in(3, \infty)$.

In the next theorem, we introduce a pressure $\Pi$ associated with the velocity part $U$ of a weak solution to the Oseen system $(\lambda=0)$ or the Oseen resolvent system $(\lambda \neq 0)$ in $\mathbb{R}^{3}$. The case $\lambda \neq 0$ is included in view of an application in [18].
Theorem 3.4 Let $A \subset \mathbb{R}^{3}$ be open, $q \in(1, \infty), \lambda \in \mathbb{C}, U \in W_{\text {loc }}^{1, q}(A)^{3}$ with

$$
\begin{equation*}
\int_{A}\left(\nabla U \cdot \nabla \vartheta+\left(\tau \partial_{1} U+\lambda U-F\right) \cdot \vartheta\right) d x=0 \text { for } \vartheta \in C_{0, \sigma}^{\infty}(A), \quad \operatorname{div} U=0 \tag{3.1}
\end{equation*}
$$

Then there is a function $\Pi \in L_{l o c}^{q}(A)$ with

$$
\begin{equation*}
\int_{A}\left(\nabla U \cdot \nabla \vartheta+\left(\tau \partial_{1} U+\lambda U-F\right) \cdot \vartheta-\Pi \operatorname{div} \vartheta\right) d x=0\left(\vartheta \in C_{0}^{\infty}(A)^{3}\right), \operatorname{div} U=0 \tag{3.2}
\end{equation*}
$$

If $B \subset \mathbb{R}^{3}$ is open, bounded, with Lipschitz boundary and with $\bar{B} \subset A$, then

$$
\begin{equation*}
\|\Pi \mid B\|_{q} \leq \mathfrak{C}\left(\left\|\nabla U\left|B\left\|_{q}+|\lambda|\right\| U\right| B\right\|_{q}+\|F \mid B\|_{q}\right) \tag{3.3}
\end{equation*}
$$

for $\lambda, U, \Pi$ as before, with the additional assumption that and $\int_{B} \Pi=0$.
Proof: Let $B \subset \mathbb{R}^{3}$ be open and bounded, with $\bar{B} \subset A$. Since $F \in L_{l o c}^{q}(A)^{3}$, we have $F \mid B \in L^{q}(B)^{3}$, hence $F \mid B \in W_{0}^{-1, q}(B)$. Obviously $-\tau \partial_{1} U \mid B \in W_{0}^{-1, q}(B)^{3}$. Thus, by [29, Lemma IV.1.1], there is a function $\Pi \in L_{l o c}^{q}(A)$ such that (3.2) holds. The last statement of the theorem follows from [29, Lemma IV.1.1] once more and from Poincaré's inequality (estimate of $\left\|\lambda U+\tau \partial_{1} U\right\|_{-1, q}$ by $\left\|\lambda U+\tau \partial_{1} U\right\|_{q}$, with $\left\|\|_{-1, q}\right.$ defined as in [29, Lemma IV.1.1]).

Theorem 3.5 Let $A \subset \mathbb{R}^{3}$ be open, $\lambda \in \mathbb{C}, q, s \in(1, \infty), F \in L_{l o c}^{q}(A)^{3}, U \in W_{\text {loc }}^{1,1}(A)^{3}$ with $\nabla U \in L_{l o c}^{s}(A)^{9}$ such that (3.1) holds. Then $U \in W_{l o c}^{2, q}(A)^{3}$. Suppose in addition that $\Pi \in L_{l o c}^{s}(A)$ is such that the pair $(U, \Pi)$ satisfies (3.2). Then $\Pi \in W_{l o c}^{1, q}(A)$ and

$$
\begin{equation*}
-\Delta U+\tau \partial_{1} U+\lambda U+\nabla \Pi=F, \quad \operatorname{div} U=0 \tag{3.4}
\end{equation*}
$$

Proof: The theorem is a consequence of interior regularity of solutions to the Stokes system; see [15, Theorem 3.2] and its proof.
Theorem 3.6 Let $A \subset \mathbb{R}^{3}$ be open, $\lambda \in \mathbb{C}, F \in C^{\infty}(A)^{3}, q \in(1, \infty), U \in W_{\text {loc }}^{1,1}(A)^{3}$ with $\nabla U \in L_{l o c}^{q}(A)^{9}$, and $\Pi \in L_{\text {loc }}^{q}(A)$. Suppose that (3.2) holds. Then $U \in C^{\infty}(A)^{3}, \Pi \in$ $C^{\infty}(A)$, and (3.4) (Oseen system if $\lambda=0$ ) is valid.
Proof: [15, Corollary 3.2]. The assumption $v \in L^{q}(A)^{3}$ in that reference should read $v \in L_{l o c}^{q}(A)^{3}$.

We will need a solution theory for the Oseen system in weighted Sobolev spaces of functions defined in the whole space $\mathbb{R}^{3}$. This theory is useful for us because it holds in an $L^{q_{-}}$ framework with any $q \in(1, \infty)$ being admitted.

Theorem 3.7 Let $q \in(1, \infty)$. Then for any $F \in L^{q}\left(\mathbb{R}^{3}\right)^{3}$, there is a pair of functions $(U(F), \Pi(F))$ such that $U(F) \in W_{2}^{2, q}\left(\mathbb{R}^{3}\right)^{3}, \Pi(F) \in W_{1}^{1, q}\left(\mathbb{R}^{3}\right)$, the pair $(U(F), \Pi(F))$ solves (3.4) with $\lambda=0, \quad A=\mathbb{R}^{3}$, and $\left\|\partial_{l} \partial_{k} U(F)\right\|_{q}+\|\Pi\|_{W_{1}^{1, q}\left(\mathbb{R}^{3}\right)}+\left\|\partial_{1} U(F)\right\|_{q} \leq$ $C(q, \tau)\|F\|_{q}$ for $1 \leq k, l \leq 3$.
Proof: The theorem reproduces some of the statements of [2, Theorem 3.3].
In [15], we proved uniqueness of the velocity part $U$ of a weak solution to the Oseen system (equation (3.4) with $\lambda=0$ ) or to the Oseen resolvent problem (equation (3.4) with $\lambda \in \mathbb{C} \backslash\{0\}, \Re \lambda \geq 0)$ in the whole space $\mathbb{R}^{3}$, under the assumptions that $|\lambda| \leq(\tau / 2)^{2}$ and $U \mid B_{R}^{c} \in \sum_{j=1}^{3} L^{r_{j}}\left(B_{R}^{c}\right)^{3}, \nabla V \in \sum_{j=1}^{3} L^{q_{j}}\left(\mathbb{R}^{3}\right)^{9}$ for some $r_{j}, q_{j} \in(1, \infty)(j \in\{1,2,3\})$ and some $R \in(0, \infty)([15$, Theorem 5.1]). In the following theorem, we generalize this result in the sense that we admit functions $U$ growing polynomially for $|x| \rightarrow \infty$. However, such weak solutions need not vanish, but they turn out to be polynomials. The case $\lambda \neq 0$ will not be needed in what follows, but it is included because it can be handled without additional effort.
Theorem 3.8 Let $U \in W_{\text {loc }}^{1,1}\left(\mathbb{R}^{3}\right)^{3}$ satisfy (3.1) with $A=\mathbb{R}^{3}, F=0$ and $\lambda \in \mathbb{C}$ with $\Re \lambda \geq$ $0,|\lambda| \leq(\tau / 2)^{2}$. Suppose there are numbers $R \in(0, \infty), r \in[0, \infty), k_{0}, m_{0} \in \mathbb{N}, \varrho_{k}, q_{m} \in$ $(1, \infty)$ for $1 \leq k \leq k_{0}, 1 \leq m \leq m_{0}$, and functions $U^{(k)} \in L_{l o c}^{\varrho_{k}}\left(\mathbb{R}^{3}\right)^{3}, V^{(m)} \in L_{l o c}^{q_{m}}\left(\mathbb{R}^{3}\right)^{3 \times 3}$ for $k, m$ as before such that $U=\sum_{k=1}^{k_{0}} U^{(k)}$, $\partial_{\mu} U_{\sigma}=\sum_{m=1}^{m_{0}} V_{\mu, \sigma}^{(m)}$ for $1 \leq \mu, \sigma \leq 3$, and $\int_{B_{R}^{c}}\left(\left|U^{(k)}(x)\right||x|^{-r}\right)^{\varrho_{k}} d x<\infty, \int_{B_{R}^{c}}\left(\left|V^{(m)}(x)\right||x|^{-r}\right)^{q_{m}} d x<\infty$ for $1 \leq k \leq k_{0}, 1 \leq$ $m \leq m_{0}$. Then $U$ is a polynomial.

Proof: The theorem follows by the standard theory of topological vector spaces ([43, Section $7.3,7.11]$ ), as used in the proof of $[15$, Theorem 5.1$]$. The only point that should still be checked in more detail is whether the operators $T, S: \mathcal{S}\left(\mathbb{R}^{3}\right)^{3} \mapsto \mathbb{R}$ defined by $T(\phi):=\int_{\mathbb{R}^{3}}\left(\nabla U \cdot \nabla \phi+\tau \partial_{1} U \cdot \phi\right) d x$ and $S(\phi):=\int_{\mathbb{R}^{3}} U \cdot \phi d x$ for $\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}$, respectively, are tempered distributions, where $\mathcal{S}\left(\mathbb{R}^{3}\right)$ denotes the set of rapidly decreasing functions in $\mathbb{R}^{3}$, equipped in the usual way with a topology. So let us show that $T$ is a tempered distribution. To that end, put $\bar{R}:=\max \{1, R\}, \bar{r}:=\min \{n \in \mathbb{N}: n \geq r\}, p_{\alpha, \beta}(\phi):=$ $\sup \left\{\left|x^{\alpha} \partial^{\beta} \phi(x)\right|: x \in \mathbb{R}^{3}\right\}, G(\phi):=\sup \left\{|x|^{r+3} \sum_{\gamma \in \mathbb{N}_{0}^{3},|\gamma| \leq 1}\left|\partial^{\gamma} \phi(x)\right|: x \in B \frac{c}{R}\right\}$ for $\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}, \alpha, \beta \in \mathbb{N}_{0}^{3}$. Then we find for $\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}$ that

$$
G(\phi) \leq \mathfrak{C} \sup \left\{\sum_{\gamma \in \mathbb{N}_{0}^{3},|\gamma| \leq 1} \sum_{m=1}^{3}\left|x_{m}^{\bar{r}+3} \partial^{\gamma} \phi(x)\right|: x \in B \frac{c}{R}\right\}
$$

and thus $G(\phi) \leq \mathfrak{C} \sum_{\gamma \in \mathbb{N}_{0}^{3},|\gamma| \leq 1} \sum_{m=1}^{3} p_{(\bar{r}+3) e_{m}, \gamma}(\phi)$. On the other hand,

$$
\begin{aligned}
& \int_{B \frac{c}{R}}|x|^{-r-3} \sum_{\mu=1}^{3}\left|\partial_{\mu} U(x)\right| d x \leq C \sum_{m=1}^{m_{0}} \sum_{\mu, \sigma=1}^{3} \int_{B \frac{c}{R}}|x|^{-r-3}\left|V_{\mu \sigma}^{(m)}(x)\right| d x \\
& \leq C \sum_{m=1}^{m_{0}} \sum_{\mu, \sigma=1}^{3}\left(\int_{B \frac{c}{R}}\left|V_{\mu \sigma}^{(m)}(x)\right|^{q_{m}}|x|^{-r q_{m}} d x\right)^{1 / q_{m}}\left(\int_{B \frac{c}{R}}|x|^{-3 q_{m}^{\prime}} d x\right)^{1 / q_{m}^{\prime}}
\end{aligned}
$$

Since $\int_{B_{\bar{c}}^{c}}|x|^{-3 q_{m}^{\prime}} d x<\infty$ for $1 \leq m \leq m_{0}$, our assumptions on the functions $V^{(m)}$ imply that the right-hand side of the preceding estimate is finite. Thus we may conclude that
the integral $\int_{B \frac{c}{R}}|x|^{-r-3} \sum_{\mu=1}^{3}\left|\partial_{\mu} U(x)\right| d x$ is finite. As a consequence

$$
\left|\int_{B \frac{c}{R}}\left(\nabla U \cdot \nabla \phi+\tau \partial_{1} U \cdot \phi\right) d x\right| \leq \mathfrak{C} G(\phi) \int_{B \frac{c}{R}}|x|^{-r-3} \sum_{\mu=1}^{3}\left|\partial_{\mu} U(x)\right| d x \quad\left(\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}\right)
$$

with the right-hand side being bounded, in turn, by $\mathfrak{C} \sum_{\gamma \in \mathbb{N}_{3}^{3},|\gamma| \leq 1} \sum_{m=1}^{3} p_{(\bar{r}+3) e_{m}, \gamma}(\phi)$, for $\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}$. Since $\left|\int_{B_{\bar{R}}}\left(\nabla U \cdot \nabla \phi+\tau \partial_{1} U \cdot \phi\right) d x\right| \leq \mathfrak{C}\left\|U \mid B_{\bar{R}}\right\|_{1,1} p_{0,0}(\phi)$, we have thus found that $|T(\phi)|$ is bounded by $\mathfrak{C}\left(p_{0,0}(\phi)+\sum_{\gamma \in \mathbb{N}_{0}^{3},|\gamma| \leq 1} \sum_{m=1}^{3} p_{(\bar{r}+3) e_{m}, \gamma}(\phi)\right)$, for $\phi \in \mathcal{S}\left(\mathbb{R}^{3}\right)^{3}$. This shows our claim for $T$. A similar but simpler reasoning is valid for $S$.

In Section 5, when we exploit the preceding theorem, the next two lemmas will be useful.
Lemma 3.2 Let $m_{0} \in \mathbb{N}, p_{1}, \ldots, p_{m_{0}} \in(1, \infty), \widetilde{p}:=\max \left\{p_{m}: 1 \leq m \leq m_{0}\right\}, \epsilon_{0} \in$ $[0,1 / \widetilde{p}), \quad S_{1} \in(0, \infty), W^{(1)}, \ldots, W^{\left(m_{0}\right)} \in L_{l o c}^{1}\left(\mathbb{R}^{3}\right)^{3}$ such that $W:=\sum_{m=1}^{m_{0}} W^{(m)}$ is a polynomial and $\int_{B_{S_{1}}^{c}}\left(\left|W^{(m)}(x)\right||x|^{-2-\epsilon_{0}}\right)^{p_{m}} d x<\infty$ for $1 \leq m \leq m_{0}$. Then the degree of $W$ is at most 1.
Proof: Abbreviate $\mathfrak{A}(x):=\sum_{m=1}^{m_{0}}\left(\left|W^{(m)}(x)\right||x|^{-2-\epsilon_{0}}\right)^{p_{m}}$ for $x \in \mathbb{R}^{3}$. Suppose there is $\widetilde{R} \in\left[S_{1}, \infty\right)$ such that $\int_{\partial B_{R}} \mathfrak{A}(x) d o_{x} \geq 1 / R$ for $R \in[\widetilde{R}, \infty)$. Then $\int_{B_{\widetilde{R}}^{c}} \mathfrak{A}(x) d x=$ $\int_{\widetilde{R}}^{\infty} \int_{\partial B_{r}} \mathfrak{A}(x) d o_{x} d r \geq \int_{\widetilde{R}}^{\infty} r^{-1} d r=\infty$. Since $\int_{B_{S_{1}}^{c}} \mathfrak{A}(x) d x<\infty$ by our assumptions, we have arrived at a contradiction. Thus there is a sequence $\left(R_{n}\right)$ in $\left[\max \left\{S_{1}, 1\right\}, \infty\right)$ with $\int_{\partial B_{R_{n}}} \mathfrak{A}(x) d o_{x} \leq R_{n}^{-1}$ for $n \in \mathbb{N}$ and $R_{n} \rightarrow \infty$. It follows that $\mathfrak{K}_{n}:=\int_{\partial B_{1}} \mathfrak{A}\left(R_{n} x\right) d o_{x} \leq$ $R_{n}^{-3}$ for $n \in \mathbb{N}$. But $\left(R_{n}^{-2-\epsilon_{0}}\right)^{p_{m}}=\left(R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}}\right)^{p_{m}} R_{n}^{-p_{m} / \widetilde{p}} \geq\left(R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}}\right)^{p_{m}} R_{n}^{-1}$ for $n \in \mathbb{N}, 1 \leq m \leq m_{0}$, where the last inequality holds because $R_{n} \geq 1$ and $p_{m} \leq \widetilde{p}$. Setting $\widetilde{\mathfrak{K}}_{n}:=\int_{\partial B_{1}} \sum_{m=1}^{m_{0}}\left(\left|W^{(m)}\left(R_{n} y\right)\right| R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}}\right)^{p_{m}} d o_{y}$ for $n \in \mathbb{N}$, we get $\widetilde{\mathfrak{K}}_{n} \leq \mathfrak{K}_{n} R_{n} \leq R_{n}^{-2}$ by the preceding inequalities. On the other hand, for $m \in\left\{1, \ldots, m_{0}\right\}, n \in \mathbb{N}$, we find with Hölder's inequality that $R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}\left|W^{(m)}\left(R_{n} y\right)\right| d o_{y} \leq(4 \pi)^{1 / p_{m}^{\prime}} \widetilde{\mathfrak{K}}_{n}^{1 / p_{m}}$. It follows that $R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}} \sum_{m=1}^{m_{0}}\left|W^{(m)}\left(R_{n} y\right)\right| d o_{y} \rightarrow 0(n \rightarrow \infty)$, so we have found that $R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}\left|W\left(R_{n} y\right)\right| d o_{y} \rightarrow 0(n \rightarrow \infty)$.
Now suppose that the degree of $W$ is larger than 1 . Then there is $k_{0} \in \mathbb{N}$ and for any $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq k_{0}$ a number $a_{\alpha} \in \mathbb{R}$ such that $k_{0} \geq 2, W(x)=\sum_{\alpha \in \mathbb{N}_{0}^{3},|\alpha| \leq k_{0}} a_{\alpha} x^{\alpha}$ for $x \in \mathbb{R}^{3}$, and $a_{\alpha_{0}} \neq 0$ for some $\alpha_{0} \in \mathbb{N}_{0}^{3}$ with $\left|\alpha_{0}\right|=k_{0}$. Put $P(x):=\sum_{\alpha \in \mathbb{N}_{0}^{3},|\alpha|=k_{0}} a_{\alpha} x^{\alpha}\left(x \in \mathbb{R}^{3}\right)$. We distinguish two cases. In the first, we suppose that $\int_{\partial B_{1}}|P(y)| d o_{y}>0$. Since for $S \in(0, \infty)$

$$
\int_{\partial B_{1}}|W(S y)| d o_{y} \geq S^{k_{0}} \int_{\partial B_{1}}|P(y)| d o_{y}-\sum_{l=0}^{k_{0}-1} S^{l} \sum_{\alpha \in \mathbb{N}_{0}^{3},|\alpha|=l}\left|a_{\alpha}\right| \int_{\partial B_{1}}\left|y^{\alpha}\right| d o_{y}
$$

we may conclude there is $\widetilde{R} \in\left[S_{1}, \infty\right)$ such that $\int_{\partial B_{1}}|W(S y)| d o_{y} \geq S^{k_{0}} \int_{\partial B_{1}}|P(y)| d o_{y} / 2$ for $S \in[\widetilde{R}, \infty)$. Thus $R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}\left|W\left(R_{n} y\right)\right| d o_{y} \geq R_{n}^{k_{0}-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}|P(y)| d o_{y} / 2$ for $n \in \mathbb{N}$ with $R_{n} \geq \widetilde{R}$. On the other hand, since we assumed $k_{0} \geq 2$ and $\epsilon_{0} \in[0,1 / \widetilde{p})$, and because $R_{n} \rightarrow \infty$ and $\int_{\partial B_{1}}|P(y)| d o_{y}>0$, we get $R_{n}^{k_{0}-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}|P(y)| d o_{y} / 2 \rightarrow$
$\infty(n \rightarrow \infty)$. We may thus conclude that $R_{n}^{-2-\epsilon_{0}+1 / \widetilde{p}} \int_{\partial B_{1}}\left|W\left(R_{n} y\right)\right| d o_{y} \rightarrow \infty$ for $n$ tending to infinity, which is a contradiction to what was shown above.

In the second case, we assume that $\int_{\partial B_{1}}|P(y)| d o_{y}=0$. Then $P(y)=0$ for a. e. $y \in \partial B_{1}$, and hence for any $y$ on $\partial B_{1}$ by continuity. Since $P$ is homogeneous, it follows that $P=0$ in $\mathbb{R}^{3}$. For $x_{2}, x_{3} \in \mathbb{R}$, the function $P\left(\cdot, x_{2}, x_{3}\right)$ is a polynomial in one variable, so we may conclude that all coefficients of this polynomial vanish. But this polynomial may be written as $\sum_{l=0}^{k_{0}} P_{l}\left(x_{2}, x_{3}\right) r^{l}$ for $r \in \mathbb{R}$, with $P_{l}\left(x_{2}, x_{3}\right):=\sum_{m=0}^{k_{0}-l} a_{\left(l, m, k_{0}-l-m\right)} x_{2}^{m} x_{3}^{k_{0}-l-m}$ for $x_{2}, x_{3} \in \mathbb{R}, 0 \leq l \leq k_{0}$. The numbers $P_{l}\left(x_{2}, x_{3}\right)$ are the coefficients of $P\left(\cdot, x_{2}, x_{3}\right)$ and must therefore vanish $\left(x_{2}, x_{3} \in \mathbb{R}\right)$. But each function $P_{l}$ is a polynomial, too. So, in the next step, the same sort of reasoning may be applied to these polynomials, implying that their coefficients must vanish as well. In the end we get that $a_{\alpha}=0$ for $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha|=k_{0}$, which is a contradiction to the fact that $a_{\alpha_{0}} \neq 0$ for some $\alpha_{0} \in \mathbb{N}_{0}^{3}$ with $|\alpha|=k_{0}$. Thus we arrive at a contradiction in any case. This proves that the degree of $W$ cannot exceed 1.
Lemma 3.3 Let $m_{0} \in \mathbb{N}, p_{m} \in(1, \infty)$, $V^{(m)} \in L^{p_{m}}\left(\mathbb{R}^{3}\right)^{3}$ for $1 \leq m \leq m_{0}$. Suppose that $V:=\sum_{m=1}^{m_{0}} V^{(m)}$ is constant. Then $V=0$.
Proof: By an argument already used in the proof of [17, Lemma 5.2] and Lemma 3.2, we may choose a sequence $\left(R_{n}\right)$ in $(0, \infty)$ with $\int_{\partial B_{R_{n}}} \sum_{m=1}^{m_{0}}\left|V^{(m)}(x)\right|^{p_{m}} d o_{x} \leq R_{n}^{-1}$ for $n \in \mathbb{N}$ and with $R_{n} \rightarrow \infty$. Put $\mathfrak{A}_{n}:=\int_{\partial B_{1}} \sum_{m=1}^{m_{0}}\left|V^{(m)}\left(R_{n} y\right)\right|^{p_{m}} d o_{y}$ for $n \in \mathbb{N}$, so $\mathfrak{A}_{n} \leq R_{n}^{-3}$. But $\int_{\partial B_{1}}\left|V^{(m)}\left(R_{n} y\right)\right| d o_{y} \leq(4 \pi)^{1 / p_{m}^{\prime}} \mathfrak{A}_{n}^{1 / p_{m}}$ for $n \in \mathbb{N}, 1 \leq m \leq m_{0}$ by Hölder's inequality, hence $\int_{\partial B_{1}} \sum_{m=1}^{m_{0}}\left|V^{(m)}\left(R_{n} y\right)\right| d o_{y} \rightarrow 0(n \rightarrow \infty)$. Therefore $\int_{\partial B_{1}}\left|V\left(R_{n} y\right)\right| d o_{y} \rightarrow 0$ for $n$ tending to infinity. Since $V$ is constant, this means that $V=0$.

## 4 Some fundamental solutions and potential functions.

We recall that the fundamental solution $\mathfrak{N}$ of the Poisson equation ("Newton kernel") and a convolution ("Newton potential") with this fundamental solution was introduced in Theorem 3.1. We define the usual heat kernel in 3D by setting
$\mathfrak{H}(z, t):=(4 \pi t)^{-3 / 2} e^{-|z|^{2} /(4 t)}$ for $z \in \mathbb{R}^{3}, t \in(0, \infty), \quad \mathfrak{H}(z, 0):=0$ for $z \in \mathbb{R}^{3} \backslash\{0\}$.
Thus, in our context, $\mathfrak{H}$ is defined on $\mathfrak{B}:=\left(\mathbb{R}^{3} \times(0, \infty)\right) \cup\left(\left(\mathbb{R}^{3} \backslash\{0\}\right) \times\{0\}\right)$.
Theorem 4.1 The relations $\mathfrak{H} \in C^{\infty}(\mathfrak{B})$ and $\int_{\mathbb{R}^{3}} \mathfrak{H}(z, t) d t=1$ for $t \in(0, \infty)$ hold. If $\alpha \in \mathbb{N}_{0}^{3}, \sigma \in \mathbb{N}_{0}$, the inequality $\left|\partial_{z}^{\alpha} \partial_{t}^{\sigma} \mathfrak{H}(z, t)\right| \leq C(\alpha, \sigma)\left(|z|^{2}+t\right)^{-(3+|\alpha|+2 \sigma) / 2}$ is valid for $z \in \mathbb{R}^{3}, t \in(0, \infty)$.

Proof: See [47] for the preceding estimate.
The estimate in Theorem 4.1 in the case $|\alpha|=2, \sigma=0$ allows to define the velocity part $\Gamma$ of a fundamental solution to the time-dependent Stokes system,

$$
\Gamma_{j k}(z, t):=\mathfrak{H}(z, t) \delta_{j k}+\int_{t}^{\infty} \partial z_{j} \partial z_{k} \mathfrak{H}(z, s) d s \quad \text { for } \quad(z, t) \in \mathfrak{B}, j, k \in\{1,2,3\}
$$

([44, p. 310]), as well as the velocity part $\Lambda$ of a fundamental solution to the timedependent Oseen system (1.1),

$$
\begin{equation*}
\Lambda_{j k}(z, t):=\Gamma_{j k}\left(z-\tau t e_{1}, t\right) \quad \text { for } \quad(z, t) \in \mathfrak{B}, j, k \in\{1,2,3\} \tag{4.1}
\end{equation*}
$$

([42, p. 501]). We will need the following properties of $\Lambda$ and estimates related to $\Lambda$.
Lemma 4.1 The relations $\Lambda \in C^{\infty}(\mathfrak{B})^{3 \times 3}$ and $\sum_{k=1}^{3} \partial z_{k} \Lambda_{j k}(z, t)=0$ are valid for $1 \leq$ $j \leq 3, z \in \mathbb{R}^{3}, t \in(0, \infty)$.
Moreover $\left|\partial_{z}^{\alpha} \Lambda(z, t)\right| \leq C\left(\left|z-\tau t e_{1}\right|^{2}+t\right)^{-3 / 2-|\alpha| / 2}$ for $t, z$ as before and $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$.
Let $K>0$. Then $\left|\partial_{z}^{\alpha} \partial_{t}^{\sigma} \Lambda(z, t)\right| \leq C(K, \tau)(|z| \nu(z)+t)^{-3 / 2-|\alpha| / 2-\sigma / 2}$ for $z \in B_{K}^{c}, t \in$ $(0, \infty), \alpha \in \mathbb{N}_{0}^{3}, \sigma \in\{0,1\}$ with $|\alpha|+\sigma \leq 1$, and $\left|\partial_{z}^{\alpha} \Lambda(z, t)\right| \leq C(K, \tau)\left(|z|^{2}+t\right)^{-3 / 2-|\alpha| / 2}$ for $z \in B_{K}, t \in(0, \infty), \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$.
The estimate $\int_{\mathbb{R}^{3}}\left|\partial_{x}^{\alpha} \Lambda(x, t)\right|^{q} d x \leq C(\tau, q) t^{-(3+|\alpha|) q / 2+3 / 2}$ holds for $q \in(1, \infty), t \in(0, \infty)$ and $\alpha$ as before.
Let $R \in(0, \infty), n \in(2, \infty)$. Then the integral $\int_{0}^{\infty}\left(\left|z-\tau r e_{1}\right|^{2}+r\right)^{-n / 2} d r i s$ bounded by $C(R, n, \tau)(|z| \nu(z))^{-(n+1) / 2}$ for $z \in B_{R}^{c}$.
Proof: See [17, Lemma 3.3, Corollary 3.4, 3.5, Theorem 4.1].
The constant $C(K, \tau)$ in the second and third estimate of $\Lambda(x, t)$ may be specified in more detail as concerns its dependence on $\tau$. In fact, if for example $\sigma=0$ and $\alpha=0$, one may take $C(K, \tau)=C(K) \max \{1, \tau\}^{3 / 2}$, with the term $\nu(x)$ in the second estimate replaced by $1+\tau\left(|x|-x_{1}\right)([7$, Lemma 2]). The following technical observation on the function $\nu$ will be useful.

Lemma 4.2 ([20, Lemma 4.8]) The inequality $\nu(x) \leq C(1+|y|) \nu(x-y)$ holds for $x, y \in \mathbb{R}^{3}$.
We introduce the first of our potential functions.
Lemma 4.3 ([17, Corollary 3.5]) Let $A \subset \mathbb{R}^{3}$ be measurable, $q \in[1, \infty)$, $V \in L^{q}(A)^{3}$, and let $\widetilde{V}$ denote the zero extension of $V$ to $\mathbb{R}^{3}$. Then $\int_{\mathbb{R}^{3}}\left|\partial_{x}^{\alpha} \Lambda(x-y, t) \widetilde{V}(y)\right| d y<\infty$ for $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1, x \in \mathbb{R}^{3}, t \in(0, \infty)$. In particular we may define the function $\mathfrak{I}^{(\tau)}(V): \mathbb{R}^{3} \times(0, \infty) \mapsto \mathbb{R}^{3}$ as in $(1.5)$.
The derivative $\partial x_{l} \mathfrak{I}^{(\tau)}(V)(x, t)$ exists and equals $\int_{\mathbb{R}^{3}} \partial x_{l} \Lambda(x-y, t) \cdot \widetilde{V}(y) d y$ for $x, t$ as above, and for $l \in\{1,2,3\}$. The functions $\mathfrak{I}^{(\tau)}(V)$ and $\partial x_{l} \mathfrak{I}^{(\tau)}(V)$ are continuous in $\mathbb{R}^{3} \times(0, \infty)$. If $q>1$, then $\left\|\mathfrak{I}^{(\tau)}(V)(t)\right\|_{q} \leq C(q, \tau)\|V\|_{q}$ for $t \in(0, \infty)$.
If $m_{0} \in \mathbb{N}, \quad p_{l} \in[1, \infty), \quad V^{(l)} \in L^{p_{l}}(A)^{3}$ for $1 \leq l \leq m_{0}$, put $\mathfrak{I}^{(\tau)}\left(\sum_{l=1}^{m_{0}} V^{(l)}\right):=$ $\sum_{l=1}^{m_{0}} \mathfrak{I}^{(\tau)}\left(V^{(l)}\right)$.

We turn to the definition of another potential function.
Lemma 4.4 Let $T_{0} \in(0, \underset{\sim}{\infty}], A \subset \mathbb{R}^{3}$ measurable, $q \in[1, \infty)$ and $f$ a function from $L_{\text {loc }}^{1}\left(\left[0, T_{0}\right), L^{q}(A)^{3}\right)$. Let $\widetilde{f}$ denote the zero extension of $f$ to $\mathbb{R}^{3} \times(0, \infty)$. Then the integral $\int_{\mathbb{R}^{3}}\left|\partial_{x}^{\alpha} \Lambda(x-y, t-\sigma) \cdot \widetilde{f}(y, \sigma)\right| d y$ is finite for any $x \in \mathbb{R}^{3}, t \in(0, \infty), \sigma \in$ $(0, t), \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. Moreover, for a. e. $t \in(0, \infty)$ and for $\alpha$ as before, the integral
$\int_{0}^{t} \int_{\mathbb{R}^{3}}\left|\partial_{x}^{\alpha} \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma)\right| d y d \sigma$ is finite for a. e. $x \in \mathbb{R}^{3}$. Thus we may define the function $\mathfrak{R}^{(\tau)}(f): \mathbb{R}^{3} \times(0, \infty) \mapsto \mathbb{R}^{3}$ as in (1.4).
The relation $\mathfrak{R}^{(\tau)}(f)(t) \in W_{\text {loc }}^{1,1}\left(\mathbb{R}^{3}\right)^{3}$ holds for a. e. $t \in(0, \infty)$, and for such $t$ we have $\partial x_{l} \mathfrak{R}^{(\tau)}(f)(t)(x)=\int_{0}^{t} \int_{\mathbb{R}^{3}} \partial x_{l} \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma) d y d \sigma$ for $l \in\{1,2,3\}$ and $a$. $e$. $x \in \mathbb{R}^{3}$.
Moreover the integral $\int_{0}^{t}\left|\int_{\mathbb{R}^{3}} \Lambda(x-y, t-s) \cdot \tilde{f}(y, s) d y\right| d s$ is finite for any $t \in(0, \infty)$ and for a. e. $x \in \mathbb{R}^{3}$. Thus the function $\mathfrak{R}^{(\tau)}(f)$ is well defined even for any $t \in(0, \infty)$ (instead of only for a. e. $t \in(0, \infty)$ ) and for a. e. $x \in \mathbb{R}^{3}$. In addition the inequality $\left\|\mathfrak{R}^{(\tau)}(f)(t)\right\|_{q} \leq C(q)\left\|\widetilde{f} \mid \mathbb{R}^{3} \times(0, t)\right\|_{q, 1 ; t}$ holds for $t>0$.
Let $m_{0} \in \mathbb{N}, p_{l} \in(1, \infty)$ and $f^{(l)} \in L_{\text {loc }}^{1}\left(\left[0, T_{0}\right), L^{p_{l}}(A)^{3}\right)$ for $1 \leq l \leq m_{0}$. Then define $\mathfrak{R}^{(\tau)}\left(\sum_{j=1}^{m_{0}} f^{(l)}\right):=\sum_{j=1}^{m_{0}} \mathfrak{R}\left(f^{(l)}\right)$.
Proof: [17, Lemma 3.8, Corollary 3.7].
The next lemma deals with still another potential function, this one defined on the surface of a space-time cylinder.
Lemma 4.5 Let $q \in[1, \infty], T_{0} \in(0, \infty], A \subset \mathbb{R}^{3}$ open and bounded, with Lipschitz boundary, $\phi \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q}(\partial A)^{3}\right)$, $\widetilde{\phi}$ the zero extension of $\phi$ to $\partial A \times(0, \infty)$. For $t \in(0, \infty), x \in \mathbb{R}^{3} \backslash \partial A, \alpha \in \mathbb{N}_{0}^{3}$, the term $\left|\partial_{x}^{\alpha} \Lambda(x-y, t-s) \cdot \widetilde{\phi}(y, s)\right|$ is integrable as a function of $(y, s) \in \partial A \times(0, t)$. Define $\mathfrak{V}^{(\tau)}(\phi):=\mathfrak{V}^{(\tau, A)}(\phi):\left(\mathbb{R}^{3} \backslash \partial A\right) \times(0, \infty) \mapsto \mathbb{R}^{3}$ by

$$
\mathfrak{V}^{(\tau)}(\phi)(x, t):=\int_{0}^{t} \int_{\partial A} \Lambda(x-y, t-s) \cdot \tilde{\phi}(y, s) d o_{y} d s \quad \text { for } x \in \mathbb{R}^{3} \backslash \partial A, t \in(0, \infty) .
$$

Then, for any $t \in(0, \infty)$, the integral $\int_{0}^{t} \int_{\partial A} \Lambda(x-y, t-s) \cdot \widetilde{\phi}(y, s) d o_{y} d s$ as a function of $x \in$ $\mathbb{R}^{3} \backslash A$ belongs to $C^{\infty}\left(\mathbb{R}^{3} \backslash A\right)^{3}$, and $\partial_{x}^{\alpha} \mathfrak{J}^{(\tau)}(\phi)(x, t)=\int_{0}^{t} \int_{\partial A} \partial_{x}^{\alpha} \Lambda(x-y, t-s) \cdot \widetilde{\phi}(y, s) d o_{y} d s$ for $\alpha \in \mathbb{N}_{0}^{3}, x \in \mathbb{R}^{3} \backslash A$.
Proof: The function $\Lambda$ is $C^{\infty}$ on $\mathbb{R}^{3} \times(0, \infty)$ (Lemma 4.1), so the lemma follows from Lebesgue's theorem.
Lemma 4.6 Let $\phi \in L^{2}(\partial \Omega \times(0, \infty))^{3}$. Then the function $\mathfrak{V}^{(\tau)}(\phi) \mid B_{S_{0}}^{c} \times[0, \infty)$ belongs to $C^{0}\left([0, \infty), L^{4}\left(B_{S_{0}}^{c}\right)^{3}\right)$.
Proof: Since $\bar{\Omega} \subset B_{S_{0}}$, we have $\delta:=\operatorname{dist}\left(\partial \Omega, B_{S}^{c}\right)>0$. It follows for $y \in \partial \Omega, x \in B_{S_{0}}^{c}$ that $|x-y| \geq\left(1-\left(S_{0}-\delta / 2\right) / S_{0}\right)|x|$. Thus we may conclude from the second estimate in Lemma 4.1 that $\left|\partial_{r}^{\sigma} \Lambda(x-y, r)\right| \leq C\left(\tau, \delta, S_{0}\right)|x|^{-3 / 2-\sigma / 2}$ for $x \in B_{S_{0}}^{c}, y \in \partial \Omega, r \in[0, \infty), \sigma \in$ $\{0,1\}$. But the right-hand side of the preceding inequality, considered as a function of $x \in B_{S_{0}}^{c}$, belongs to $L^{4}\left(B_{S_{0}}^{c}\right)$. The lemma follows from the preceding observations.

In the rest of this section, we recapitulate results from [17] that will be needed in what follows. First we introduce another kernel function, which is a truncated version of $\Lambda$, and whose definition involves the numbers $S_{0}, R_{0}, R_{1}$ and the function $\varphi_{0}$ introduced at the beginning of Section 2. (Recall that $S_{0}<R_{0}, R_{1}=\left(R_{0}+S_{0}\right) / 2$ and $\varphi_{0} \in C_{0}^{\infty}\left(B_{R_{1}}\right)$ with $\varphi \mid B_{S_{0}+\left(R_{0}-S_{0}\right) / 4}=1,0 \leq \varphi_{0} \leq 1$.) However, since this definition would need some preparation and we will not work with it, we do not restate it, referring instead to [17, (3.13)]. In the ensuing theorem, we collect those properties of this kernel which will be
relevant here.
Theorem 4.2 There is a function $\mathfrak{G}:=\mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}: B_{R_{0}}^{c} \times B_{R_{1}} \times[0, \infty) \mapsto \mathbb{R}^{3 \times 3}$ with the following properties.
Let $x \in B_{R_{0}}^{c}, r \in[0, \infty)$. Then $\mathfrak{G}(x, \cdot, r) \in C_{0}^{\infty}\left(B_{R_{1}}\right)^{3 \times 3}, \sum_{k=1}^{3} \partial y_{k} \mathfrak{G _ { j k }}(x, y, r)=0$ for $1 \leq j \leq 3, y \in B_{R_{1}}$, and $\mathfrak{G}(x, y, r)=\Lambda(x-y, r)$ for $y \in B_{S_{0}+\left(R_{0}-S_{0}\right) / 4}$.
Let $x \in B_{R_{0}}^{c}, q \in(1, \infty)$. Then the mapping $r \mapsto \mathfrak{G}(x, \cdot, r)(r \in[0, \infty))$ belongs to $C^{1}\left([0, \infty), W^{1, q}\left(B_{R_{1}}\right)^{3 \times 3}\right)$. Thus a function $G^{\prime} \in C^{0}\left([0, \infty), W^{1, q}\left(B_{R_{1}}\right)^{3 \times 3}\right)$ may be defined by the condition $\left\|(\mathfrak{G}(x, \cdot, r+h)-\mathfrak{G}(x, \cdot, r)) / h-G^{\prime}(r)\right\|_{1, q} \rightarrow 0(h \rightarrow 0)$ for $r \in[0, \infty)$. We write $\partial_{r} \mathfrak{G}(x, y, r)$ instead of $G^{\prime}(r)(y)\left(r \in[0, \infty), y \in B_{R_{1}}\right)$.
Let $r \in[0, \infty), q \in(1, \infty)$.
Let $\sigma \in\{0,1\}$, and define $L(x): B_{R_{1}} \mapsto \mathbb{R}^{3 \times 3}$ by $L(x)(y):=\partial_{r}^{\sigma} \mathfrak{G}(x, y, r)$ for $x \in B_{R_{0}}^{c}, y \in$ $B_{R_{1}}$. Then $L(x) \in C_{0}^{\infty}\left(B_{R_{1}}\right)^{3 \times 3} \cap W^{1, q}\left(B_{R_{1}}\right)^{3 \times 3}$ for $x \in B_{R_{0}}^{c}$, and $L$ as a mapping from $B_{R_{0}}^{c}$ into $W^{1, q}\left(B_{R_{1}}\right)^{3 \times 3}$ is partially differentiable on ${\overline{B_{R_{0}}}}^{c}$. Thus we may define $D_{m} L:{\overline{B_{R_{0}}}}^{c} \mapsto W^{1, q}\left(B_{R_{1}}\right)^{3 \times 3}$ by the condition $\left\|\left(L\left(x+h e_{m}\right)-L(x)\right) / h-D_{m} L(x)\right\|_{1, q} \rightarrow$ $0(h \rightarrow 0)$, for $m \in\{1,2,3\}, x \in{\overline{B_{R_{0}}}}^{c}$. Instead of $D_{m} L(x)(y)$, we write $\partial x_{m} \partial_{r}^{\sigma} \mathfrak{G}(x, y, r)$. Let $l \in\{1,2,3\}$ and define $\widetilde{L}(x): B_{R_{1}} \mapsto \mathbb{R}^{3 \times 3}$ by $\widetilde{L}(x)(y):=\partial y_{l} \mathfrak{G}(x, y, r)$ for $x \in$ $B_{R_{0}}^{c}, y \in B_{R_{1}}$. Then $\widetilde{L}(x) \in C_{0}^{\infty}\left(B_{R_{1}}\right)^{3 \times 3} \cap L^{q}\left(B_{R_{1}}\right)^{3 \times 3}$ for $x \in B_{R_{0}}^{c}$, and $\widetilde{L}$ considered as an operator from $B_{R_{0}}^{c}$ into $L^{q}\left(B_{R_{1}}\right)^{3 \times 3}$ is partially differentiable on ${\overline{B_{R_{0}}}}^{c}$. Thus we may define $D_{m} \widetilde{L}:{\overline{B_{R_{0}}}}^{c} \mapsto L^{q}\left(B_{R_{1}}\right)^{3 \times 3}$ by the condition $\left\|\left(\widetilde{L}\left(x+h e_{m}\right)-\widetilde{L}(x)\right) / h-D_{m} \widetilde{L}(x)\right\|_{q} \rightarrow$ $0(h \rightarrow 0)\left(m \in\{1,2,3\}, x \in{\overline{B_{R_{0}}}}^{c}\right)$. Instead of $D_{m} \widetilde{L}(x)(y)$, we write $\partial x_{m} \partial y_{l} \mathfrak{G}(x, y, r)$.
Let $q \in(1, \infty), p \in[1, \infty]$. Then

$$
\begin{equation*}
\int_{B_{R_{1}}}\left|\partial_{x}^{\alpha} \partial_{t}^{\sigma} \partial_{y}^{\beta} \mathfrak{G}(x, y, t) \cdot V(y)\right| d y \leq \mathfrak{C}(|x| \nu(x))^{-(3+|\alpha|+\sigma) / 2}\|V\|_{q} \tag{4.2}
\end{equation*}
$$

for $V \in L^{q}\left(B_{R_{1}}\right)^{3}, t \in(0, \infty), x \in{\overline{B_{R_{0}}}}^{c}, \alpha, \beta \in \mathbb{N}_{0}^{3}, \sigma \in\{0,1\}$ with $|\alpha| \leq 1,|\beta|+\sigma \leq 1$, $\int_{0}^{t} \int_{B_{R_{1}}}\left|\partial_{x}^{\alpha} \partial_{t}^{\sigma} \partial_{y}^{\beta} \mathfrak{G}(x, y, t-s) \cdot v(y, s)\right| d y d s \leq \mathfrak{C}(|x| \nu(x))^{-(3+|\alpha|+\sigma) / 2+1 /\left(2 p^{\prime}\right)}\|v\|_{q, p ; t}$
for $t, x, \alpha, \beta, \sigma$ as in (4.2), and $v \in L^{p}\left(0, t, L^{q}\left(B_{R_{1}}\right)^{3}\right)$.
Proof: [17, (3.13), Lemma 3.11, 3.12, 3.13, Theorem 4.2].
We note a consequence of the preceding theorem.
Corollary 4.1 ([17, Corollary 4.2]) Let $\beta \in \mathbb{N}_{0}^{3}, \sigma \in\{0,1\}$ with $|\beta|+\sigma \leq 1$. Let $q \in(1, \infty)$, and let the function $v$ belong to $L_{\text {loc }}^{1}\left([0, \infty), L^{q}\left(B_{R_{1}}\right)^{3}\right)$ and the function $V$ to $L^{q}\left(B_{R_{1}}\right)^{3}$. Define

$$
F(x, t):=\int_{0}^{t} \int_{B_{R_{1}}} \partial_{s}^{\sigma} \partial_{y}^{\beta} \mathfrak{G}(x, y, t-s) \cdot v(y, s) d y d s, \quad H(x, t):=\int_{B_{R_{1}}} \mathfrak{G}(x, y, t) \cdot V(y) d y
$$

for $x \in{\overline{B_{R_{0}}}}^{c}, t \in[0, \infty)$. Take a number $l \in\{1,2,3\}$. Then the derivatives $\partial x_{l} F(x, t)$ and $\partial x_{l} H(x, t)$ exist pointwise, and they equal $\int_{0}^{t} \int_{B_{R_{1}}} \partial x_{l} \partial_{s}^{\sigma} \partial_{y}^{\beta} \mathfrak{G}(x, y, t-s) \cdot v(y, s) d y d s$ and $\int_{B_{R_{1}}} \partial x_{l} \mathfrak{G}(x, y, t) \cdot V(y) d y$, respectively, for $x \in{\overline{B_{R_{0}}}}^{c}, t \in[0, \infty)$

It will be convenient to subsume a number of terms in a single operator, which we define here, and whose definition makes sense due to the preceding Corollary 4.1. The parameters $T_{0}, S_{0}, R_{0}, R_{1}$ and the set $\Omega$ appearing in the following were fixed at the beginning of Section 2.

Let $A \subset B_{S_{0}}$ be open and bounded, with Lipschitz boundary. Put $A_{R}:=B_{R} \backslash \bar{A}, Z_{R, T}:=$ $A_{R} \times(0, T)$ for $R \in\left[S_{0}, \infty\right), T \in(0, \infty]$. Let $\mathfrak{A} \subset \mathbb{R}^{3} \times \mathbb{R}$ be such that $Z_{R_{1}, T_{0}} \subset \mathfrak{A}$. Let $q \in(1, \infty)$ and let $v: \mathfrak{A} \mapsto \mathbb{R}^{3}$ with $v\left|Z_{R_{1}, T_{0}} \in C^{0}\left(\left[0, T_{0}\right), L^{q}\left(A_{R_{1}}\right)^{3}\right), v(s)\right| A_{R_{1}} \in$ $W_{l o c}^{1,1}\left(A_{R_{1}}\right)^{3}$ for $s \in\left(0, T_{0}\right)$ and $\nabla_{x} v \mid Z_{R_{1}, T_{0}} \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q}\left(A_{R_{1}}\right)^{9}\right)$. Then, for $t \in\left(0, T_{0}\right)$ and $x \in{\overline{B_{R_{0}}}}^{c}$, we define

$$
\begin{align*}
& \mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, A, T_{0}}(v)(x, t):=\int_{0}^{t} \int_{A_{R_{1}}}\left(\sum_{l=1}^{3} \partial y_{l} \mathfrak{G}(x, y, t-s) \cdot \partial_{y_{l}} v(y, s)\right.  \tag{4.4}\\
& \left.-\tau \partial y_{1} \mathfrak{G}(x, y, t-s) \cdot v(y, s)-\partial_{s} \mathfrak{G}(x, y, t-s) \cdot v(y, s)\right) d y d s+\int_{A_{R_{1}}} \mathfrak{G}(x, y, 0) \cdot v(y, t) d y
\end{align*}
$$

Next we reproduce some decay estimates proved in [17], beginning with a decay estimate of $\mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, A, T_{0}}(v)$. We use the same notation as in (4.4).

Corollary 4.2 ([17, Corollary 4.3]) Let $p_{1}, p_{2} \in[1, \infty]$ and take $A, \mathfrak{A}, q$ as in (4.4). Then, for $v: \mathfrak{A} \mapsto \mathbb{R}^{3}$ such that $v\left|Z_{R_{1}, T_{0}} \in C^{0}\left(\left[0, T_{0}\right), L^{q}\left(A_{R_{1}}\right)^{3}\right), v(s)\right| A_{R_{1}} \in W_{\text {loc }}^{1,1}\left(A_{R_{1}}\right)^{3}$ $\left(s \in\left(0, T_{0}\right)\right)$ and $\nabla_{x} v \mid Z_{R_{1}, T_{0}} \in L^{p_{2}}\left(0, T_{0}, L^{q}\left(A_{R_{1}}\right)^{9}\right)$, and for $x \in{\overline{B_{R_{0}}}}^{c}, t \in\left(0, T_{0}\right), \alpha \in$ $\mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$, the term $\left|\partial_{x}^{\alpha} \mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, A, T_{0}}(v)(x, t)\right|$ is bounded by

$$
\mathfrak{C}\left(\left\|v\left|Z_{R_{1}, t}\left\|_{q, p_{1} ; t}+\right\| \nabla_{x} v\right| Z_{R_{1}, t}\right\|_{q, p_{2} ; t}+\left\|v(t) \mid \Omega_{R_{1}}\right\|_{q}\right) \max _{j \in\{1,2\}}(|x| \nu(x))^{-(3+|\alpha|) / 2+1 /\left(2 p_{j}^{\prime}\right)}
$$

Lemma 4.7 ([17, Lemma 4.3]) Let $A, \mathfrak{A}, q$ be given as in (4.4), and let $p_{1}, p_{2} \in[1, \infty]$. Then, for functions $v: \mathfrak{A} \mapsto \mathbb{R}^{3}$ with $v\left|Z_{R_{1}, T_{0}} \in L^{p_{1}}\left(0, T_{0}, L^{q}\left(A_{R_{1}}\right)^{3}\right), v(s)\right| A_{R_{1}} \in$ $W_{l o c}^{1,1}\left(A_{R_{1}}\right)^{3}\left(s \in\left(0, T_{0}\right)\right)$ and $\nabla_{x} v \mid Z_{R_{1}, T_{0}} \in L^{p_{2}}\left(0, T_{0}, L^{q}\left(A_{R_{1}}\right)^{9}\right)$, as well as for $x \in$ $B_{R_{0}}^{c}, t \in\left(0, T_{0}\right), \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 2, l \in\{1,2,3\}$, the term $\left|\partial_{x}^{\alpha} \mathfrak{V}^{(\tau)}\left(n_{l}^{(A)} v\right)(x, t)\right|$ is bounded by

$$
\mathfrak{C}\left(\left\|v\left|Z_{R_{1}, t}\left\|_{q, p_{1} ; t}+\right\| \nabla_{x} v\right| Z_{R_{1}, t}\right\|_{q, p_{2} ; t}\right) \max _{j \in\{1,2\}}(|x| \nu(x))^{-(3+|\alpha|) / 2+1 /\left(2 p_{j}^{\prime}\right)}
$$

where $\left(n_{l}^{(A)} v\right)(y, s):=n_{l}^{(A)}(y) v(y, s)$ for $y \in \partial A, s \in\left(0, T_{0}\right)$.
Lemma 4.8 ([17, Lemma 4.4]) Recall that the Newton kernel $\mathfrak{N}$ was introduced in Theorem 3.1. Let $q \in(1, \infty)$. Then the estimate $\left|\int_{\partial B_{S_{0}}}\left(\partial^{\alpha} \nabla \mathfrak{N}\right)(x-y) S_{0}^{-1} y \cdot V(y) d o_{y}\right| \leq$ $\mathfrak{C}|x|^{-2-|\alpha|}\|V\|_{q}$ holds for $V \in L^{q}\left(A_{R_{1}, S_{0}}\right)^{3} \cap W^{1,1}\left(A_{R_{1}, S_{0}}\right)^{3}$ with div $V=0$, and for $t \in(0, \infty), x \in B_{R_{0}}^{c}, \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. If $\int_{\partial B_{S_{0}}} y \cdot V(y) d o_{y}=0$, the factor $|x|^{-2-|\alpha|}$ may be replaced by $|x|^{-3-|\alpha|}$. The preceding results remain valid if $S_{0}$ and $R_{1}$ are replaced by $R_{1}$ and $S_{0}+3\left(R_{0}-S_{0}\right) / 4$, respectively.

The next two theorems provide conditions - the weakest we could find - on $f$ and $U_{0}$ such that $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{I}^{(\tau)}\left(U_{0}\right)$ decay with about the same rate for $|x| \rightarrow \infty$ as does the difference $u-\mathfrak{R}^{(\tau)}(f)-\mathfrak{I}^{(\tau)}\left(U_{0}\right)$ in the applications of our theory in Section 6 . We indicate that the ensuing theorem improves [13, Theorem 3.1].

Theorem 4.3 Let $f: \bar{\Omega}^{c} \times(0, \infty) \mapsto \mathbb{R}^{3}$ be measurable. Suppose there a numbers $q \in$ $(1, \infty), A, p_{0} \in(2, \infty), B \in[0,3 / 2]$ and a function $\gamma \in L^{2}((0, \infty)) \cap L^{p_{0}}((0, \infty))$ such that $A+\min \{1, B\}>3, A+B \geq 7 / 2, f \mid \Omega_{S_{0}} \times(0, \infty) \in L^{2}\left(0, \infty, L^{q}\left(\Omega_{S_{0}}\right)^{3}\right)$ and $|f(y, s)| \leq \gamma(s)|y|^{-A} \nu(y)^{-B}$ for $y \in{\overline{B_{S_{0}}}}^{c}, s \in(0, \infty)$. Then there is $n \in \mathbb{N}$ such that

$$
\begin{align*}
& \left|\partial_{x}^{\alpha} \Re^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)(x, t)\right|  \tag{4.5}\\
& \leq C\left(\tau, p_{0}, A, B, S_{0}, R_{0}\right)\left(\|\gamma\|_{2}+\|\gamma\|_{p_{0}}\right)|x|^{-5 / 4-|\alpha| / 2} \nu(x)^{-5 / 4-|\alpha| / 4}(\max \{1, \ln |x|\})^{|\alpha| n / 2}
\end{align*}
$$

for $t \in(0, \infty), x \in{\overline{B_{R_{0}}}}^{c}, \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. Moreover $\left|\partial_{x}^{\alpha} \mathfrak{R}^{(\tau)}\left(f \mid \Omega_{S_{0}} \times(0, \infty)\right)(x, t)\right| \leq$ $C\left(\tau, S_{0}, R_{0}\right)\left\|f \mid \Omega_{S_{0}} \times(0, \infty)\right\|_{2}(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$ for $t, x, \alpha$ as before.
Proof: Take $t, x, \alpha$ as in the theorem. It may be deduced from Lemma 4.4 and 4.1 that

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} \mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)(x, t)\right| \leq C\left(\tau, R_{0}\right)\left(\mathfrak{M}_{1}+\mathfrak{M}_{2}\right) \tag{4.6}
\end{equation*}
$$

with $\mathfrak{M}_{1}:=\int_{0}^{t} \int_{{\overline{B_{S_{0}}}}^{c} \cap B_{R_{0} / 2}(x)}\left(|x-y|^{2}+t-s\right)^{-3 / 2-|\alpha| / 2}|f(y, s)| d y d s$, and $\mathfrak{M}_{2}$ defined in the same way as $\mathfrak{M}_{1}$, except that the domain of integration ${\overline{B_{S_{0}}}}^{c} \cap B_{R_{0} / 2}(x)$ is replaced by ${\overline{B_{S_{0}}}}^{c} \backslash B_{R_{0} / 2}(x)$, and the term $|x-y|^{2}$ by $\left|x-y-\tau(t-s) e_{1}\right|^{2}$. Hölder's inequality and the assumptions on $f$ imply that $\mathfrak{M}_{1}$ is bounded by

$$
\begin{equation*}
\int{\overline{B_{S_{0}}}{ }^{c} \cap B_{R_{0} / 2}(x)}\left(\int_{0}^{t}\left(|x-y|^{2}+t-s\right)^{(-3 / 2-|\alpha| / 2) \cdot p_{0}^{\prime}} d s\right)^{1 / p_{0}^{\prime}}\|\gamma\|_{p_{0}}|y|^{-A} \nu(y)^{-B} d y \tag{4.7}
\end{equation*}
$$

Since $|x| \geq R_{0}$, we get for $y \in B_{R_{0} / 2}(x)$ that $|y| \geq|x|-|x-y| \geq|x| / 2$, and with Lemma 4.2 that $\nu(x) \leq C(1+|x-y|) \nu(y) \leq C\left(R_{0}\right) \nu(y)$. Moreover, since $p_{0}>2$, we have $-3-|\alpha|+2 / p_{0}^{\prime}>-3$. Therefore we may conclude from (4.7) that $\mathfrak{M}_{1} \leq$ $C\left(R_{0}, S_{0}\right)\|\gamma\|_{p_{0}}|x|^{-A} \nu(x)^{-B}$. Due to the assumptions $A>2, A+B \geq 7 / 2$ and because $|x| \geq C\left(R_{0}\right) \nu(x)$, it follows that $\mathfrak{M}_{1} \leq C\left(R_{0}, A, B\right)(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$. In order to find a bound for $\mathfrak{M}_{2}$, we use the last inequality in Lemma 4.1 , which yields

$$
\int_{0}^{t}\left(\left|x-y-\tau(t-s) e_{1}\right|^{2}+t-s\right)^{-3-|\alpha|} d s \leq C\left(R_{0}, \tau\right)(|x-y| \nu(x-y))^{-5 / 2-|\alpha|}
$$

for $y \in{\overline{B_{S_{0}}}}^{c} \backslash B_{R_{0} / 2}(x)$. Thus, proceeding as in the estimate of $\mathfrak{M}_{1}$, but with the exponent 2 instead of $p_{0}$, and recalling our assumptions on $f$, we get

$$
\mathfrak{M}_{2} \leq C\left(R_{0}, \tau\right)\|\gamma\|_{2} \int_{{\overline{B_{S_{0}}}}^{c} \backslash B_{R_{0} / 2}(x)}(|x-y| \nu(x-y))^{-5 / 4-|\alpha| / 2}|y|^{-A} \nu(y)^{-B} d y
$$

Obviously $|x-y| \geq C\left(R_{0}\right)(1+|x-y|)$ for $y \in B_{R_{0} / 2}(x)^{c}$, so we may conclude by Hölder's inequality that $\mathfrak{M}_{2} \leq C\left(R_{0}, \tau\right)\|\gamma\|_{2} \mathfrak{A}^{(1)}(\alpha)^{1 / 2} \mathfrak{A}^{(2)}(\alpha)^{1 / 2}$, where

$$
\mathfrak{A}^{(1)}(\alpha):=\int_{{\overline{B_{S_{0}}}}^{c} \backslash B_{R_{0} / 2}(x)}((1+|x-y|) \nu(x-y))^{-3 / 2-|\alpha| / 2}|y|^{-A} \nu(y)^{-B} d y
$$

and $\mathfrak{A}^{(2)}(\alpha)$ is defined in the same way as $\mathfrak{A}^{(1)}(\alpha)$, but with the exponent $-3 / 2-|\alpha| / 2$ replaced by $-1-|\alpha| / 2$. In the case $\alpha=0$, it follows by [38, Theorem 3.1 and 3.2] and
their proof, in particular [38, p. 87 above, (3.25)], and by our assumptions on $A$ and $B$ that $\mathfrak{A}^{(1)}(0) \leq C(A, B)(|x| \nu(x))^{-3 / 2}$ and $\mathfrak{A}^{(2)}(0) \leq C(A, B)(|x| \nu(x))^{-1}$, so $\mathfrak{M}_{2} \leq$ $C\left(R_{0}, S_{0}, A, B, \tau\right)\|\gamma\|_{2}(|x| \nu(x))^{-5 / 4}$. Suppose that $|\alpha|=1$. Then $\mathfrak{A}^{(2)}(\alpha)=\mathfrak{A}^{(1)}(0) \leq$ $C(A, B)(|x| \nu(x))^{-3 / 2}$. We further note that $\nu(x-y)^{-1} \nu(y)^{-1} \leq C \nu(x)^{-1}$ for any $y \in \mathbb{R}^{3}$, as follows by assuming for a contradiction that $|y|-y_{1} \leq\left(|x|-x_{1}\right) / 4$ and $|x-y|-(x-y)_{1} \leq$ $\left(|x|-x_{1}\right) / 4$. Thus

$$
\mathfrak{A}^{(1)}(\alpha) \leq C \nu(x)^{-1 / 2} \int_{{\overline{B_{S_{0}}}}^{c} \backslash B_{R_{0} / 2}(x)}(1+|x-y|)^{-2} \nu(x-y)^{-3 / 2}|y|^{-A} \nu(y)^{-B+1 / 2} d y .
$$

But $A+\min \{B-1 / 2,1\} \geq 3$ because $A>2$ and $A+B \geq 7 / 2$, so by [38, Theorem 3.3] and its proof, we obtain $\mathfrak{A}^{(1)}(\alpha) \leq C(A, B)|x|^{-2} \nu(x)^{-3 / 2}(\max \{\ln (|x|), 1\})^{n}$, for some $n \in \mathbb{N}$ independent of $x$ and $\alpha$. Thus, combining the preceding estimates, we get in the case $|\alpha|=1$ that $\mathfrak{M}_{2} \leq C\left(A, B, R_{0}, S_{0}, \tau\right)\|\gamma\|_{2}|x|^{-7 / 4} \nu(x)^{-3 / 2}(\max \{\ln (|x|), 1\})^{n / 2}$. Inequality (4.5) now follows from (4.6) and the preceding estimates of $\mathfrak{M}_{1}$ and $\mathfrak{M}_{2}$. As for an estimate of $\left|\partial_{x}^{\alpha} \mathfrak{R}^{(\tau)}\left(f \mid \Omega_{S_{0}} \times(0, \infty)\right)(x, t)\right|$, we refer to (4.3). Recall that $x \in{\overline{B_{R_{0}}}}^{c}$, so $\mathfrak{G}(x, y, r)=\Lambda(x-y, r)$ for $y \in \Omega_{S_{0}}, r \in(0, \infty)$.
Theorem 4.4 Let $q \in(1, \infty), V \in L_{\sigma}^{q}\left(\bar{\Omega}^{c}\right), \kappa_{0} \in(0,1 / 2), c_{0} \in(0, \infty)$ such that $V \mid{\overline{B_{S_{0}}}}^{c} \in W_{l o c}^{1,1}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}$ and $\left|\partial^{\alpha} V(y)\right| \leq c_{0}|y|^{-3 / 2-|\alpha| / 2-\kappa_{0}} \nu(y)^{-5 / 4-|\alpha| / 2-\kappa_{0}}$ for $y \in{\overline{B_{S_{0}}}}^{c}$ and $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. Then the estimate $\left|\partial_{x}^{\alpha} \mathfrak{I}^{(\tau)}(V)(x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$ holds for $t \in(0, \infty), x \in B_{R_{0}}^{c}$ and $\alpha$ as before.

Proof: For $\varrho \in\{0, \tau\}, p \in(1, \infty), W \in L^{p}\left(\mathbb{R}^{3}\right)^{3}$, define the function $\mathcal{H}^{(\varrho)}(W): \mathbb{R}^{3} \times$ $(0, \infty) \mapsto \mathbb{R}^{3}$ by $\mathcal{H}^{(\varrho)}(W)(x, t):=\int_{\mathbb{R}^{3}} \mathfrak{H}\left(x-y-t \varrho e_{1}, t\right) W(y) d y\left(x \in \mathbb{R}^{3}, t>0\right)$. Then $\mathfrak{I}^{(\tau)}(V)=\mathcal{H}^{(\tau)}(V)$ by [17, Corollary 3.5]. The spatial decay of $\mathcal{H}^{(\tau)}(V)$ is studied in [12]. Following the approach used in that reference, we start from a function $\widetilde{b} \in W_{l o c}^{1,1}\left(\mathbb{R}^{3}\right)^{3}$ verifying the estimate

$$
\left|\partial^{\alpha} \widetilde{b}(y)\right| \leq \gamma(1+|y|)^{-3 / 2-|\alpha| / 2-\kappa_{0}} \nu(y)^{-5 / 4-|\alpha| / 2-\kappa_{0}} \text { for } y \in \mathbb{R}^{3}, \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha| \leq 1, \text { (4.8) }
$$

where $\gamma$ is some positive real. Then we show that

$$
\begin{equation*}
\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(\widetilde{b})(z, t)\right| \leq \mathfrak{C}(|z| \nu(z))^{-5 / 4-|\alpha| / 2} \text { for } z \in \mathbb{R}^{3} \backslash\{0\}, t \in(0, \infty), \alpha \text { as before. } \tag{4.9}
\end{equation*}
$$

Note that in (4.9), it is $\mathcal{H}^{(0)}(\widetilde{b})$ and not $\mathcal{H}^{(\tau)}(\widetilde{b})$ which appears. The transition from (4.8) to (4.9) corresponds to the part of [12] which begins with the relation $\left|\partial^{\alpha} \widetilde{b}(y)\right| \leq$ $\gamma((1+|y|) \nu(y))^{-1 / 2-\kappa_{0}}($ see $[12,(4.6)])$, and ends with the conclusion $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(\widetilde{b})(z, t)\right| \leq$ $\mathfrak{C}(|z| \nu(z))^{-1 / 2-|\alpha| / 2}([12,(4.41)])$. The reasoning in [12] leading to this conclusion may be modified so as to provide a proof of (4.9) as a consequence of (4.8). However, some of these modifications are not completely obvious. Our proof of Theorem 4.4 consists in specifying these less obvious changes. Once (4.9) is established, Theorem 4.4 follows in almost the same way as the transition in [12] from the relation $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(\widetilde{b})(z, t)\right| \leq$ $\mathfrak{C}(|z| \nu(z))^{-1 / 2-|\alpha| / 2}$ for $z \in \mathbb{R}^{3}, t>0$ to the estimate of the term $\left|\partial_{x}^{\alpha} \mathcal{H}^{(\tau)}(V)(x, t)\right|$ by $\mathfrak{C}(|x| \nu(x))^{-1 / 2-|\alpha| / 2}$ for $x \in{\overline{B_{R_{0}}}}^{c}, t>0([12$, p. 322-323] $)$. This is why the passage from (4.9) to the estimate in Theorem 4.4 need not be elaborated here.

So let us turn to those parts of the proof of (4.9) that need some additional effort compared to the theory in [12]. To this end we introduce the notation $y^{\prime}=\left(y_{2}, y_{3}\right)$ for $y \in \mathbb{R}^{3}$, and we indicate that

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} \mathfrak{H}(x, t)\right| \leq \mathfrak{C}|x|^{-3-|\alpha|} \text { for } x \in \mathbb{R}^{3} \backslash\{0\}, t>0, \quad(1+|x|) \nu(x) \geq\left(1+\left|x^{\prime}\right|\right)^{2} / 8 \tag{4.10}
\end{equation*}
$$

for $x \in \mathbb{R}^{3}$; see Theorem 4.1 and $[12,(2.7)]$. Moreover we fix $t \in(0, \infty), z \in \mathbb{R}^{3} \backslash\{0\}$ and $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. The constants $\mathfrak{C}$ appearing in the rest of this proof are independent of $z, t$ and $\alpha$.
Our first aim is to derive the estimate $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(\widetilde{b})(z, t)\right| \leq \mathfrak{C}\left|z_{1}\right|^{-5 / 2-|\alpha|}$ in the case $z_{1}<$ 0 , a result which corresponds to [12, Lemma 4.7]. So let us suppose $z_{1}<0$. Put $H:=\left\{y \in \mathbb{R}^{3}: y_{1} \leq 0\right\}$ and consider the splitting $\left|\partial_{x}^{\alpha} \mathcal{H}^{(0)}(\widetilde{b})(x, t)\right| \leq \sum_{j=1}^{5} \mathfrak{A}_{j}$, with $\mathfrak{A}_{1}:=\left|\int_{\mathbb{R}^{3} \backslash H} \partial_{z}^{\alpha} \mathfrak{H}(z-y, t) \widetilde{b}(y) d y\right|, \mathfrak{A}_{2}:=\left|\int_{B_{|z| / 2}(z) \cap H} \mathfrak{H}(z-y, t) \partial^{\alpha} \widetilde{b}(y) d y\right|$ and $\mathfrak{A}_{3}:=\left|\int_{\partial B_{|z| / 2}(z) \cap H} \mathfrak{H}(z-y, t) \widetilde{b}(y)\left(\alpha \cdot 2|z|^{-1} y\right) d o_{y}\right|$. The term $\mathfrak{A}_{4}$ is defined as $\mathfrak{A}_{3}$, but with the domain of integration $\partial B_{|z| / 2}(z) \cap H$ replaced by $B_{|z| / 2}(z) \cap \partial H$, and the scalar product $\alpha \cdot 2|z|^{-1} y$ by $\alpha \cdot e_{1}$. Concerning $\mathfrak{A}_{5}$, it coincides with $\mathfrak{A}_{1}$ except that the set $H \backslash B_{|z| / 2}(z)$ takes the place of $\mathbb{R}^{3} \backslash H$ as domain of integration. Note that the terms $\mathfrak{A}_{2}$ to $\mathfrak{A}_{4}$ arise due to an integration by parts on $B_{|z| / 2}(z) \cap H$. In order to estimate $\mathfrak{A}_{1}$, we observe that inequality $(4.10)_{1}$ together with the condition $z_{1}<0$ and the definition of $H$ imply that $\left|\partial_{z}^{\alpha} \mathfrak{H}(z-y, t)\right| \leq \mathfrak{C}\left(\left|z_{1}\right|+y_{1}+\left|z^{\prime}-y^{\prime}\right|\right)^{-3-|\alpha|}$ for $y \in \mathbb{R}^{3} \backslash H$. Therefore we get with (4.8) and (4.10) $)_{2}$ that $\mathfrak{A}_{1} \leq \gamma\left|z_{1}\right|^{-5 / 2-|\alpha|} \int_{\mathbb{R}^{3} \backslash H} y_{1}^{-1 / 2}\left(1+\left|y_{1}\right|\right)^{-1 / 2-\kappa_{0} / 2}\left(1+\left|y^{\prime}\right|\right)^{-2-\kappa_{0}} d y$, hence $\mathfrak{A}_{1} \leq \mathfrak{C}\left|z_{1}\right|^{-5 / 2-|\alpha|}$. For $y \in \bar{H}$, we have $\nu(y) \geq 1+|y|$, so

$$
\begin{equation*}
\partial^{\beta} \widetilde{b}(y) \leq \gamma(1+|y|)^{-11 / 4-|\beta|-2 \kappa_{0}} \quad \text { for } y \in \bar{H}, \beta \in \mathbb{N}_{0}^{3},|\beta| \leq 1 . \tag{4.11}
\end{equation*}
$$

Since $|y| \geq|z| / 2$ for $y \in \overline{B_{|z| / 2}(z)}$, we obtain that $\mathfrak{A}_{2} \leq \mathfrak{C}|z|^{-11 / 4-|\alpha|-2 \kappa_{0}} \int_{\mathbb{R}^{3}} \mathfrak{H}(z-y, t) d y$ by (4.11). But $\int_{\mathbb{R}^{3}} \mathfrak{H}(z-y, t) d y=1$, so $\mathfrak{A}_{2} \leq \mathfrak{C}|z|^{-11 / 4-|\alpha|-2 \kappa_{0}}$. Because of (4.10) ${ }_{1}$ and (4.11), and since $|z-y|=|z| / 2,|y| \geq|z| / 2$ for $y \in \partial B_{|z| / 2}(z)$, the term $\mathfrak{A}_{3}$ is majorized by $\mathfrak{C}|z|^{-23 / 4-2 \kappa_{0}}$ times the surface measure of $H \cap \partial B_{|z| / 2}(z)$, hence $\mathfrak{A}_{3} \leq \mathfrak{C}|z|^{-15 / 4-2 \kappa_{0}}$. For $y \in \partial H \cap B_{|z| / 2}(z)$, we have $y_{1}=0$ and $|y| \geq|z| / 2$, hence again by (4.10) $)_{1}$ and (4.11), we obtain that $\mathfrak{A}_{4}$ is bounded by $\mathfrak{C}\left|z_{1}\right|^{-3}|z|^{-11 / 4-2 \kappa_{0}}$ times the surface measure of $\partial H \cap B_{|z| / 2}(z)$. Therefore $\mathfrak{A}_{4} \leq \mathfrak{C}\left|z_{1}\right|^{-15 / 4-2 \kappa_{0}}$. If $y \in H \backslash B_{|z| / 2}(z)$, then $|y-z| \geq|z| / 2$, so once more by $(4.10)_{1}$ and (4.11), $\mathfrak{A}_{5} \leq \mathfrak{C}|z|^{-5 / 2-|\alpha|} \int_{\mathbb{R}^{3}}|z-y|^{-1 / 2}(1+|y|)^{-11 / 4-2 \kappa_{0}} d y$. The preceding integral is bounded by $\mathfrak{C}|z|^{-1 / 4-2 \kappa_{0}}$, as may be seen by integrating seperately on $B_{2|z|}(z) \backslash B_{|z| / 2}(z)$ and $B_{2|z|}(z)^{c}$. Thus $\mathfrak{A}_{5} \leq \mathfrak{C}|z|^{-11 / 4-|\alpha|-2 \kappa_{0}}$. Altogether we obtain in the case $z_{1} \leq-1$ that $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq \mathfrak{C}\left|z_{1}\right|^{-5 / 2-|\alpha|}$. If $z_{1} \in(-1,0)$, this inequality follows immediately by a partial integration, estimate (4.8) and the equation $\int_{\mathbb{R}^{3}} \mathfrak{H}(z-y, t) d y=1$. So the inequality we looked for does indeed hold if $z_{1}<0$.
Next we assume that $\left|z^{\prime}\right| \geq|z| / 2$, we put $\widetilde{H}:=\left\{y \in \mathbb{R}^{3}:\left|y^{\prime}\right| \leq|z| / 4\right\}$, and we are going to show that $\widetilde{\mathfrak{A}}:=\left|\int_{\widetilde{H}} \partial_{z}^{\alpha} \mathfrak{H}(z-y, t) \widetilde{b}(y) d y\right| \leq|z|^{-5 / 2-|\alpha|}$; compare [12, (4.20)]. Due to the condition $\left|z^{\prime}\right| \geq|z| / 2$, we get $\left|y^{\prime}-z^{\prime}\right| \geq|z| / 4$ for $y \in \widetilde{H}$. Thus by (4.10) ${ }_{1},\left|\partial_{z}^{\alpha} \mathfrak{H}(z-y, t)\right| \leq$ $\mathfrak{C}\left|z_{1}-y_{1}\right|^{-1 / 2}|z|^{-5 / 2-|\alpha|}$ for $y \in \widetilde{H}$. It follows with (4.8) and (4.10) $)_{2}$ that $\widetilde{A}$ is bounded by $\mathfrak{C}|z|^{-5 / 2-|\alpha|}$ times the integral $\int_{\mathbb{R}^{3}}\left|z_{1}-y_{1}\right|^{-1 / 2}(1+|y|)^{-1 / 2-\kappa_{0} / 2}\left(1+\left|y^{\prime}\right|\right)^{-2-\kappa_{0}} d y$. This integral, in turn, may be majorized by $\mathfrak{C} \int_{\mathbb{R}}\left|z_{1}-y_{1}\right|^{-1 / 2}\left(1+\left|y_{1}\right|\right)^{-1 / 2-\kappa_{0} / 2} d y_{1}$. If
the domain $\mathbb{R}$ in that latter integral is replaced by $\left(z_{1}-1, z_{1}+1\right)$, the resulting integral is obviously bounded independently of $z_{1}$. An integration on $\mathbb{R} \backslash\left(z_{1}-1, z_{1}+1\right)$ may be performed via Hölder's inequality, with the exponent $p:=2\left(1+\kappa_{0} / 4\right) /\left(1+\kappa_{0}\right)$, say, acting on the term $\left(1+\left|y_{1}\right|\right)^{-1 / 2-\kappa_{0} / 2}$. Using the assumption $\kappa_{0}<1 / 2$, we again obtain a bound independent of $z_{1}$. Altogether we arrive at the desired estimate $\widetilde{\mathfrak{A}} \leq \mathfrak{C}|z|^{-5 / 2-|\alpha|}$.
Suppose that $z_{1}>0$ and $|z|-z_{1} \geq 1$. Under these conditions it is shown in [12, p. 317319] that $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq \mathfrak{C}(|z| \nu(z))^{-1-|\alpha| / 2}$. As our last aim in this proof, we want to show that in our situation, the stronger estimate $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq \mathfrak{C}(|z| \nu(z))^{-5 / 4-|\alpha| / 2}$ holds. To this end, put $G:=\left\{y \in \mathbb{R}^{3}:\left|y^{\prime}\right|>\left|z^{\prime}\right| / 2\right\}$. Integrating by parts twice, first on $\mathbb{R}^{3}$ and then on $G$, we may arrange things in such a way that $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq$ $\sum_{j=1}^{3} \mathfrak{B}_{j}$, where $\mathfrak{B}_{1}:=\left|\int_{\mathbb{R}^{3} \backslash G} \mathfrak{H}(y, t) \partial^{\alpha} \widetilde{b}(z-y) d y\right|, \mathfrak{B}_{2}:=\left|\int_{G} \partial_{y}^{\alpha} \mathfrak{H}(y, t) \widetilde{b}(z-y) d y\right|$ and $\mathfrak{B}_{3}:=\left.\left|\int_{\partial G} \mathfrak{H}(y, t) \widetilde{b}(z-y)\left(\left(0, y^{\prime}\right) \cdot \alpha\right) 2\right| z^{\prime}\right|^{-1} d o_{y} \mid$. For $y \in \mathbb{R}^{3} \backslash G$, we have $\left|y^{\prime}\right| \leq\left|z^{\prime}\right| / 2$, so with (4.10) 2 , $(1+|z-y|) \nu(z-y) \geq \mathfrak{C}\left|z^{\prime}-y^{\prime}\right|^{2} \geq \mathfrak{C}\left|z^{\prime}\right|^{2}$. Hence by (4.8) we get $\left|\partial^{\alpha} \widetilde{b}(z-y)\right| \leq \mathfrak{C}\left|z^{\prime}\right|^{-5 / 2-|\alpha|-2 \kappa_{0}}$ for such $y$. Since $\int_{\mathbb{R}^{3}} \mathfrak{H}(y, t) d y=1$, we may conclude that $\mathfrak{B}_{1} \leq \mathfrak{C}\left|z^{\prime}\right|^{-5 / 2-|\alpha|-2 \kappa_{0}}$. Using that $\left|y^{\prime}\right| \geq\left|z^{\prime}\right| / 2$ for $y \in G$, we deduce from (4.8) and (4.10) that $\mathfrak{B}_{2} \leq \mathfrak{C}\left|z^{\prime}\right|^{-5 / 2-|\alpha|} \int_{\mathbb{R}^{3}}\left|y_{1}\right|^{-1 / 2}(1+|z-y|)^{-1 / 2-\kappa_{0} / 2}\left(1+\left|z^{\prime}-y^{\prime}\right|\right)^{-2-\kappa_{0}} d y$. According to what we showed further above, this integral is bounded independently of $z$. Thus we may conclude that $\mathfrak{B}_{2} \leq \mathfrak{C}\left|z^{\prime}\right|^{-5 / 2-|\alpha|}$. Turning to $\mathfrak{B}_{3}$, we note that $\left|y^{\prime}\right|=$ $\left|z^{\prime}\right| / 2$ for $y \in \partial G$, so we obtain with (4.10) and (4.8) that $\mathfrak{H}(y, t) \leq \mathfrak{C}\left|z^{\prime}\right|^{-5 / 2}\left|y_{1}\right|^{-1 / 2}$ and in addition $|\widetilde{b}(z-y)| \leq \mathfrak{C}\left(1+\left|z_{1}-y_{1}\right|^{-1 / 2-\kappa_{0} / 2}\right)\left|z^{\prime}\right|^{-2-\kappa_{0}}$. As a consequence, $\mathfrak{B}_{3}$ is bounded by $\mathfrak{C}\left|z^{\prime}\right|^{-9 / 2-\kappa_{0}}$ times the integral $\int_{\mathbb{R}}\left|y_{1}\right|^{-1 / 2}\left(1+\left|y_{1}-z_{1}\right|\right)^{-1 / 2-\kappa_{0} / 2} d y_{1}$ times the arc length of $\left\{y^{\prime} \in \mathbb{R}^{2}:\left|y^{\prime}\right|=\left|z^{\prime}\right| / 2\right\}$. As was shown above, the preceding integral is bounded independently of $z_{1}$. Thus we get $\mathfrak{B}_{3} \leq \mathfrak{C}\left|z^{\prime}\right|^{-7 / 2-\kappa_{0}}$. Combining the estimates of $\mathfrak{B}_{1}, \mathfrak{B}_{2}$ and $\mathfrak{B}_{3}$, we may conclude that $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq \mathfrak{C}\left(\left|z^{\prime}\right|^{-5 / 2-|\alpha|}+\left|z^{\prime}\right|^{-7 / 2-\kappa_{0}}\right)$. But the assumptions $z_{1} \geq 0$ and $|z|-z_{1} \geq 1$ imply $|z| \geq 1$ and $\left|z^{\prime}\right|^{2}=\left(|z|+z_{1}\right)\left(|z|-z_{1}\right) \geq|z| \nu(z) / 2$. Therefore we arrive at the inequality $\left|\partial_{z}^{\alpha} \mathcal{H}^{(0)}(z, t)\right| \leq \mathfrak{C}(|z| \nu(z))^{-5 / 4-|\alpha| / 2}$ we wanted to show. This completes the proof of Theorem 4.4.

Next we restate the representation formula proved in [17] for regular solutions of (1.1).
Theorem 4.5 ([17, Corollary 5.2]) Let $A \subset B_{S_{0}}$ be open, bounded, with Lipschitz boundary, and put $A_{R}:=B_{R} \backslash \bar{A}$ for $R \in\left[S_{0}, \infty\right)$. Let $k_{0} \in \mathbb{N}$, and for $1 \leq k \leq k_{0}$, let $\varrho_{k} \in$ $(1, \infty)$ and suppose that $u^{(k)}$ belongs to $C^{0}\left(\left[0, T_{0}\right), L^{\varrho_{k}}\left(\bar{A}^{c}\right)^{3}\right) \cap W_{\text {loc }}^{1,1}\left(0, T_{0}, L^{\varrho_{k}}\left(\bar{A}^{c}\right)^{3}\right)$. Put $u=\sum_{k=1}^{k_{0}} u^{(k)}$. Let $\pi:\left(0, T_{0}\right) \mapsto W_{\text {loc }}^{1,1}\left(\bar{A}^{c}\right), n_{0} \in \mathbb{N}$. For $1 \leq j \leq n_{0}$, let $p_{j} \in(1, \infty)$ and $f^{(j)} \in L_{\text {loc }}^{1}\left(0, T_{0}, L^{p_{j}}\left(\bar{A}^{c}\right)^{3}\right)$. Suppose that $u(s) \in W_{\text {loc }}^{2,1}\left(\bar{A}^{c}\right)^{3}$ for a. e. $s \in\left(0, T_{0}\right)$, and

$$
\begin{equation*}
u^{\prime}(s)-\Delta_{x} u(s)+\tau \partial x_{1} u(s)+\nabla_{x} \pi(s)=f(s), \operatorname{div}_{x} u(s)=0 \text { for a. e. } s \in\left(0, T_{0}\right), \tag{4.12}
\end{equation*}
$$

with $f:=\sum_{j=1}^{n_{0}} f^{(j)}, u^{\prime}:=\sum_{k=1}^{k_{0}}\left(u^{(k)}\right)^{\prime}$.
In addition suppose there is $q_{1} \in(1, \infty)$ such that $\nabla_{y} u(s) \mid{\overline{B_{R_{0}}}}^{c} \in L^{q_{1}}\left({\overline{B_{R_{0}}}}^{c}\right)^{9}$ for a. e. $s \in\left(0, T_{0}\right)$ and $\nabla_{y} u \mid A_{R_{0}} \times\left(0, T_{0}\right) \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q_{1}}\left(A_{R_{0}}\right)^{9}\right)$. Assume that $u \mid A_{R_{0}} \times\left(0, T_{0}\right) \in$ $L_{l o c}^{1}\left(0, T_{0}, W^{2,1}\left(A_{R_{0}}\right)^{3}\right), \pi \mid A_{R_{0}} \times\left(0, T_{0}\right) \in L_{l o c}^{1}\left(0, T_{0}, W^{1,1}\left(A_{R_{0}}\right)\right)$, and there are numbers $a_{0} \in \mathbb{N}, \gamma_{j} \in(1, \infty)$ as well as functions $\pi^{(j)}:\left(0, T_{0}\right) \mapsto L_{\text {loc }}^{1}\left({\overline{B_{R_{0}}}}^{c}\right)$ for $1 \leq j \leq a_{0}$ such
that

$$
\pi(s) \mid{\overline{B_{R_{0}}}}^{c}=\sum_{j=1}^{a_{0}} \pi^{(j)}(s), \sum_{j=1}^{a_{0}} \int_{{\overline{B_{R_{0}}}}^{c}}\left(\left|\pi^{(j)}(y, s)\right|[(1+|y|) \ln (2+|y|)]^{-1}\right)^{\gamma_{j}} d y<\infty
$$

for a. e. $s \in\left(0, T_{0}\right)$. Then there is a subset $\mathfrak{T}_{T_{0}}$ of $\left(0, T_{0}\right)$ of measure zero such that for $t \in\left(0, T_{0}\right) \backslash \mathfrak{T}_{T_{0}}$, the equation

$$
\begin{align*}
& u(x, t)=\mathfrak{R}^{(\tau)}(f)(x, t)+\mathfrak{I}^{(\tau)}(u(0))(x, t)-\sum_{l=1}^{3} \partial x_{l} \mathfrak{V}^{(\tau)}\left(n_{l}^{(A)} u\right)(x, t)  \tag{4.13}\\
& \quad-\int_{\partial A}(\nabla \mathfrak{N})(x-y)\left(n^{(A)}(y) \cdot u(y, t)\right) d o_{y}+\mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, A, T_{0}}(u)(x, t) \\
& \quad-\int_{A_{R_{1}}} \mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}(x, y, t) \cdot u(y, 0) d y-\int_{0}^{t} \int_{A_{R_{1}}} \mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}(x, y, t-s) \cdot f(y, s) d y d s
\end{align*}
$$

holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$, where $N_{t}$ is some subset of ${\overline{B_{R_{0}}}}^{c}$ with measure zero. The function $\mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, A, T_{0}}(u)$ is defined in (4.4), and the function $\mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}$ in Theorem 4.2.
Finally we restate a tool which will help to extend the integral representation (4.13) to solutions of (4.12) that are more general than the ones in the preceding theorem.
Lemma 4.9 ([17, Lemma 5.3]) Let $A \subset B_{S_{0}}$ be open, bounded, with Lipschitz boundary. Let $k_{0} \in \mathbb{N}, \varrho_{k} \in(1, \infty)$, $u^{(k)} \in C^{0}\left(\left[0, T_{0}\right), L^{\varrho_{k}}\left(\bar{A}^{c}\right)^{3}\right)$ for $1 \leq k \leq k_{0}$, and put $u=\sum_{k=1}^{k_{0}} u^{(k)}$. Further suppose that $u(s) \in W_{l o c}^{1,1}\left(\bar{A}^{c}\right)^{3}$ for a. e. $s \in\left(0, T_{0}\right)$, div $v_{x} u=0$, and $\nabla_{x} u \mid A_{R_{1}} \times\left(0, T_{0}\right) \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q_{1}}\left(A_{R_{1}}\right)^{9}\right)$ for some $q_{1} \in(1, \infty)$. Furthermore let $n_{0} \in \mathbb{N}, p_{j} \in(1, \infty), f^{(j)} \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{p_{j}}\left(\bar{A}^{c}\right)^{3}\right)$ for $1 \leq j \leq n_{0}$, and put $f:=\sum_{j=1}^{n_{0}} f^{(j)}$.
Fix some function $\zeta \in C^{\infty}(\mathbb{R})$ with $\zeta|(-\infty, 1]=0, \zeta|[2, \infty)=1,0 \leq \zeta \leq 1$ and $\zeta^{\prime} \geq 0$. For $\epsilon \in(0, \infty), r \in \mathbb{R}$, define $\zeta_{\epsilon}(r):=\zeta(r / \epsilon)$. Put $u_{\epsilon}^{(k)}(s):=\zeta_{\epsilon}(s) u^{(k)}(s)$, $u_{\epsilon}(s):=$ $\zeta_{\epsilon}(s) u(s), f_{\epsilon}(s):=\zeta_{\epsilon}(s) f(s)$ and $g_{\epsilon}(s):=f_{\epsilon}(s)+\zeta_{\epsilon}^{\prime}(s) u(s)$ for $s \in\left(0, T_{0}\right), \epsilon \in(0, \infty), 1 \leq$ $k \leq k_{0}$.
Let $t \in\left(0, T_{0}\right)$. Suppose there is some $\epsilon_{0} \in(0, \infty)$ such that for $\epsilon \in\left(0, \epsilon_{0}\right]$, equation (4.13) holds with $u$, $f$ replaced by $u_{\epsilon}$ and $g_{\epsilon}$, respectively, if $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t, \epsilon}$ for some subset $N_{t, \epsilon}$ of ${\overline{B_{R_{0}}}}^{c}$ of measure zero. (This means in particular that the second from last term in (4.13) vanishes.) Then there is some zero-measure set $N_{t} \subset{\overline{B_{R_{0}}}}^{c}$ such that equation (4.13) remains valid for $u$ and $f$ if $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$.

## 5 Weak solutions of the Oseen system: representation formula and pointwise spatial decay.

We begin by introducing a $L^{q}$-weak solution to (1.1), with assumptions chosen in such a way that it will be possible to prove the representation formula (4.13) for this solution. In order to deduce a decay estimate from this formula, we will require somewhat stronger conditions (see Theorem 5.2). We again point out that $T_{0}, S_{0}, R_{0}, R_{1}, \varphi_{0}$ and $\Omega$ are introduced at the beginning of Section 2.

Fix parameters $n_{0}, m_{0} \in \mathbb{N}, \widetilde{p}, q_{0}, q_{1}, p_{1}, \ldots, p_{n_{0}}, \varrho_{1}, \ldots, \varrho_{m_{0}} \in(1, \infty)$, as well as functions $u:\left(0, T_{0}\right) \mapsto W_{l o c}^{1,1}\left(\bar{\Omega}^{c}\right)^{3}, f^{(j)} \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq j \leq n_{0}, G^{(l)} \in$ $C^{0}\left(\left[0, T_{0}\right), L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}, U_{0} \in L^{\widetilde{p}}\left(\bar{\Omega}^{c}\right)^{3}$ with the following properties:
$u \mid \Omega_{S_{0}} \times\left(0, T_{0}\right) \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q_{0}}\left(\Omega_{S_{0}}\right)^{3}\right), \operatorname{div}_{x} u(t)=0$ and $u(t) \mid{\overline{B_{S_{0}}}}^{c}=\sum_{l=1}^{m_{0}} G^{(l)}(t)$ for $t \in\left(0, T_{0}\right), \nabla_{x} u \in L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{3}\right)$, and equation (1.3) holds with $f=\sum_{j=1}^{n_{0}} f^{(j)}$.
The case $m_{0}>1$ or $n_{0}>1$ and the distinction between the functions $G^{(l)}$ on ${\overline{B_{S_{0}}}}^{c} \times(0, \infty)$ with $u(t) \mid{\overline{B_{S_{0}}}}^{c}=\sum_{l=1}^{m_{0}} G^{(l)}(t)$ on the one hand and $u \mid \Omega_{S_{0}} \times\left(0, T_{0}\right)$ on the other one are introduced in order to take account of some technical difficulties arising when the theory in this work is applied to a nonlinear problem in [18] and when it is used in the proof of Theorem 6.1, and in order to avoid any assumptions on $\partial \Omega$ stronger than Lipschitz regularity. Any reader who wants to avoid these technicalities may consider the case that $\partial \Omega$ is smooth, $u \in C^{0}\left(\left[0, T_{0}\right), L^{q_{0}}\left(\bar{\Omega}^{c}\right)^{3}\right), m_{0}=n_{0}=1$ and $p_{1}=q_{1}$. All the main difficulties of our proofs will then still be present.

Until further notice, we suppose that $T_{0}=\infty$. It is only at the end of the present section that we will turn to the case $T_{0}<\infty$ (Corollary 5.2).
In the ensuing lemma, we cut off the functions $u, f^{(j)}$ and $G^{(l)}$ near the instant $t=0$, and then present the version of (1.3) satisfied by the extension of these modified functions to the whole real axis. In the lemma after that, we apply Friedrich's mollifier to these extensions. Both lemmas mainly serve to introduce notation and collect obvious facts which constitute the basis for the rest of this section.
Lemma 5.1 Fix some $q \in(1, \infty)$ with

$$
q \leq \min \left(\left\{q_{0}, q_{1}, 5 / 4\right\} \cup\left\{\varrho_{l}: 1 \leq l \leq m_{0}\right\} \cup\left\{p_{j}: 1 \leq l \leq n_{0}\right\}\right) .
$$

Then the function $u \mid \Omega_{R} \times(0, \infty)$ belongs to $L_{\text {loc }}^{1}\left([0, \infty), L^{q}\left(\Omega_{R}\right)^{3}\right)$ and $u \mid A_{R, S_{0}} \times[0, \infty)$ to $C^{0}\left([0, \infty), L^{q}\left(A_{R, S_{0}}\right)^{3}\right)$, for $R \in\left[S_{0}, \infty\right)$.
Choose functions $\zeta_{\epsilon}$ for $\epsilon \in(0, \infty)$ as in Lemma 4.9 For s, $\epsilon \in(0, \infty)$, define $u_{\epsilon}(s):=$ $\zeta_{\epsilon}(s) u(s), f_{\epsilon}^{(j)}(s):=\zeta_{\epsilon}(s) f^{(j)}(s)\left(1 \leq j \leq n_{0}\right)$ and $G_{\epsilon}^{(l)}(s):=\zeta_{\epsilon}(s) G^{(l)}(s)\left(1 \leq l \leq m_{0}\right)$,
$p_{n_{0}+1}:=q, f_{\epsilon}^{\left(n_{0}+1\right)}:=\zeta_{\epsilon}^{\prime}(s) \chi_{\Omega_{S_{0}}} u(s), \quad p_{n_{0}+1+l}:=\varrho_{l}, f_{\epsilon}^{\left(n_{0}+1+l\right)}(s):=\zeta_{\epsilon}^{\prime}(s) \widetilde{G}^{(l)}(s)$ for $1 \leq l \leq m_{0}$, where $\widetilde{G}^{(l)}$ denotes the zero extension of $G^{(l)}:{\overline{B_{S_{0}}}}^{c} \times[0, \infty) \mapsto \mathbb{R}^{3}$ to $\bar{\Omega}^{c} \times[0, \infty)$. Then $f_{\epsilon}^{\left(n_{0}+k\right)} \in L_{l o c}^{1}\left([0, \infty), L^{p_{n_{0}+k}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq k \leq m_{0}+1$, and $\zeta_{\epsilon}^{\prime}(s) u(s)=$ $\sum_{k=1}^{m_{0}+1} f_{\epsilon}^{\left(n_{0}+k\right)}(s)(s \in(0, \infty))$.
The functions $u_{\epsilon}(t), f_{\epsilon}^{(j)}(t), G_{\epsilon}^{(l)}(t), \widetilde{G}_{\epsilon}^{(l)}(t)(t \in(0, \infty))$ are extended by zero to the real axis $\mathbb{R}\left(1 \leq j \leq n_{0}+m_{0}+1,1 \leq l \leq m_{0}\right)$, without change of notation. Then $f_{\epsilon}^{(j)} \in L_{l o c}^{1}\left(\mathbb{R}, L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $j \in\left\{1, \ldots, n_{0}+m_{0}+1\right\}, G_{\epsilon}^{(l)} \in C^{0}\left(\mathbb{R}, L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ and $\widetilde{G}_{\epsilon}^{(l)} \in C^{0}\left(\mathbb{R}, L^{\varrho_{l}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}$. In addition $u_{\epsilon} \mid \Omega_{R} \times \mathbb{R} \in L_{\text {loc }}^{1}\left(\mathbb{R}, L^{q}\left(\Omega_{R}\right)^{3}\right)$ and $u_{\epsilon} \mid A_{R, S_{0}} \times \mathbb{R} \in C^{0}\left(\mathbb{R}, L^{q}\left(A_{R, S_{0}}\right)^{3}\right)$ for $R \in\left[S_{0}, \infty\right)$, $u_{\epsilon}=\chi_{\Omega_{S_{0}}} u_{\epsilon}+\sum_{j=1}^{m_{0}} \widetilde{G}_{\epsilon}^{(l)}$ and $u_{\epsilon} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon}^{(l)}, u(s) \in W_{l o c}^{1,1}\left(\bar{\Omega}^{c}\right)^{3}$ for $s \in \mathbb{R}, \operatorname{div}_{x} u_{\epsilon}=0, \nabla_{x} u_{\epsilon} \in$
$L_{l o c}^{1}\left(\mathbb{R}, L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{9}\right)$,

$$
\begin{align*}
\int_{\mathbb{R}} & \int_{\bar{\Omega}^{c}}\left(-\gamma^{\prime}(t) u_{\epsilon}(t) \cdot \vartheta+\gamma(t)\left[\nabla_{x} u_{\epsilon}(t) \cdot \nabla \vartheta+\tau \partial x_{1} u_{\epsilon}(t) \cdot \vartheta\right.\right.  \tag{5.1}\\
& \left.\left.\quad-\sum_{j=1}^{n_{0}+m_{0}+1} f_{\epsilon}^{(j)}(t) \cdot \vartheta\right]\right) d x d t=0 \quad \text { for } \gamma \in C_{0}^{\infty}(\mathbb{R}), \vartheta \in C_{0, \sigma}^{\infty}\left(\bar{\Omega}^{c}\right)
\end{align*}
$$

We remark that the functions $f_{\epsilon}^{\left(n_{0}+1\right)}, \ldots, f_{\epsilon}^{\left(n_{0}+m_{0}+1\right)}$ are perturbation terms which arise in (5.1) due to the cut-off function $\zeta_{\epsilon}$ and the decomposition of $u(s)$ into the sum $\chi_{\Omega_{S_{0}}} u(s)+$ $\sum_{l=1}^{m_{0}} \widetilde{G}^{(l)}$. In the ensuing lemma, we use Friedrich's mollifier with respect to the time variable, as defined in the passage preceding Theorem 2.5. This mollifier is applied to functions with domain $\mathbb{R}$ and values in Banach spaces, in this context spaces $L^{r}(A)^{3}$, for certain $r \in(1, \infty)$ and certain open subsets $A$ of $\mathbb{R}^{3}$. The functions with index $\epsilon$ are those introduced in Lemma 5.1.

Concerning $u_{\epsilon}$, there is no Banach space $B$ which we deem useful and which is such that $u_{\epsilon} \in L_{l o c}^{1}(\mathbb{R}, B)$. So we cannot directly apply our definition of Friedrich's mollifier to $u_{\epsilon}$. Instead we use the equation $u_{\epsilon}=\chi_{\Omega_{S_{0}}} u_{\epsilon}+\sum_{j=1}^{m_{0}} \widetilde{G}_{\epsilon}^{(l)}$ in order to define a more regular version of $u_{\epsilon}$, denoted by $u_{\epsilon, \delta}$. Note that due to the relation $\nabla_{x} u_{\epsilon} \in L_{l o c}^{1}\left(\mathbb{R}, L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{3}\right)$, no such problem arises with $\nabla_{x} u_{\epsilon}$.
Lemma 5.2 Let $\epsilon, \delta \in(0, \infty)$. Put $f_{\epsilon, \delta}^{(j)}:=\left(f_{\epsilon}^{(j)}\right)_{\delta}\left(1 \leq j \leq n_{0}+m_{0}+1\right), G_{\epsilon, \delta}^{(l)}:=$ $\left(G_{\epsilon}^{(l)}\right)_{\delta}, \quad \widetilde{G}_{\epsilon, \delta}^{(l)}:=\left(\widetilde{G}_{\epsilon}^{(l)}\right)_{\delta}\left(1 \leq l \leq m_{0}\right)$. Then, for $j, l$ as before, the relations $f_{\epsilon, \delta}^{(j)} \in$
 fine $u_{\epsilon, \delta}:=\left(\chi_{\Omega_{S_{0}}} u_{\epsilon}\right)_{\delta}+\sum_{l=1}^{m_{0}} \widetilde{G}_{\epsilon, \delta}^{(l)}$. Then $u_{\epsilon, \delta} \mid \Omega_{R} \times \mathbb{R}=\left(u_{\epsilon} \mid \Omega_{R} \times \mathbb{R}\right)_{\delta} \in C^{\infty}\left(\mathbb{R}, L^{q}\left(\Omega_{R}\right)^{3}\right)$ for $R \in\left[S_{0}, \infty\right)$, and $u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon, \delta}^{(l)}$. Moreover $u_{\epsilon, \delta}(t) \in W_{l o c}^{1,1}\left(\bar{\Omega}^{c}\right)^{3}$ for $t \in \mathbb{R}, \partial x_{k} u_{\epsilon, \delta}=\left(\partial x_{k} u_{\epsilon}\right)_{\delta} \in C^{\infty}\left(\mathbb{R}, L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq k \leq 3$, and div$x_{x} u_{\epsilon, \delta}=0$.
Define $s_{0}:=n_{0}+2 m_{0}+2, f_{\epsilon, \delta}^{\left(n_{0}+m_{0}+2\right)}:=-\left(\chi_{\Omega_{S_{0}}} u_{\epsilon}\right)_{\delta}^{\prime}, p_{n_{0}+m_{0}+2}:=q, f_{\epsilon, \delta}^{\left(n_{0}+m_{0}+2+l\right)}:=$ $-\left(\widetilde{G}_{\epsilon, \delta}^{(l)}\right)^{\prime}$ and $p_{n_{0}+m_{0}+2+l}:=\varrho_{l}$ for $1 \leq l \leq m_{0}$. Then $f_{\epsilon, \delta}^{(j)} \in C^{\infty}\left(\mathbb{R}, L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $n_{0}+m_{0}+2 \leq j \leq s_{0}$.
For $t \in \mathbb{R}$, the function $u_{\epsilon, \delta}(t)$ satisfies (3.1) with $A, U, F$ replaced by $\bar{\Omega}^{c}, u_{\epsilon, \delta}(t)$ and $\sum_{j=1}^{s_{0}} f_{\epsilon, \delta}^{(j)}(t)$, respectively, and with $\lambda=0$.
The last statement of the lemma is the crucial one. It means that $u_{\epsilon, \delta}(t)$ is a weak solution in $\bar{\Omega}^{c}$ of the stationary Oseen system, with right-hand side $F$ as indicated.
Proof of Lemma 5.2: The regularity properties of $f_{\epsilon, \delta}^{(j)}, G_{\epsilon, \delta}^{(l)}, \widetilde{G}_{\epsilon, \delta}^{(l)}$ and $u_{\epsilon, \delta} \mid \Omega_{R} \times \mathbb{R}$ follow with Theorem 2.5 from the properties of $f_{\epsilon}^{(j)}, G_{\epsilon}^{(l)}, \widetilde{G}_{\epsilon}^{(l)}$ and $u_{\epsilon} \mid \Omega_{R} \times \mathbb{R}$, respectively, as stated in Lemma $5.1\left(1 \leq j \leq s_{0}, 1 \leq l \leq m_{0}\right.$ and $\left.R \in\left[S_{0}, \infty\right)\right)$. Lemma 2.2 and the equation $u_{\epsilon} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon}^{(l)}(t)$ (Lemma 5.1) yield that $\left(u_{\epsilon} \mid \Omega_{R} \times \mathbb{R}\right)_{\delta}=u_{\epsilon, \delta} \mid \Omega_{R} \times \mathbb{R}$ for $R \in\left[S_{0}, \infty\right)$ and $u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon, \delta}^{(l)}$.
Due to Theorem 2.3 and because the operator $\mathcal{A}_{p, \psi}: V \mapsto \int_{\bar{\Omega}^{c}} \psi \cdot V d x\left(V \in L^{p}\left(\bar{\Omega}^{c}\right)^{3}\right)$ is linear and bounded if $p \in(1, \infty)$ and $\psi \in C_{0}^{\infty}\left(\bar{\Omega}^{c}\right)^{3}$, it is not hard to show that
$u_{\epsilon, \delta}(t) \in W_{\text {loc }}^{1,1}\left(\bar{\Omega}^{c}\right)^{3}$ for $t \in \mathbb{R}$ and $\partial x_{k} u_{\epsilon, \delta}=\left(\partial x_{k} u_{\epsilon}\right)_{\delta}$ for $1 \leq k \leq 3$. It follows with Theorem 2.5 and Lemma 5.1 that $\partial x_{k} u_{\epsilon, \delta} \in C^{\infty}\left(\mathbb{R}, L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ and $\operatorname{div}_{x} u_{\epsilon, \delta}=0$. In particular equation $(3.1)_{2}$ holds.
Put $K^{(1)}:=\chi_{\Omega_{0}} \cdot u_{\epsilon}, K^{(l)}:=\widetilde{G}_{\epsilon}^{(l)}$ for $2 \leq l \leq m_{0}+1$. Using the equation $u_{\epsilon}=\sum_{l=1}^{m_{0}+1} K^{(l)}$ (Lemma 5.1), the linearity and boundedness of the operators $\mathcal{A}_{p, \psi}$ defined above, as well as Theorem 2.3, we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}} \int_{\bar{\Omega}^{c}} \partial_{s}\left(\rho_{\delta}(t-s)\right) u_{\epsilon}(s) \cdot \vartheta d x d s=-\sum_{l=1}^{m_{0}+1} \int_{\mathbb{R}} \int_{\bar{\Omega}^{c}} \rho_{\delta}^{\prime}(t-s) K^{(l)}(x, s) \cdot \vartheta(x) d x d s \\
& =-\sum_{l=1}^{m_{0}+1} \int_{\bar{\Omega}^{c}} \vartheta \cdot\left(L^{p_{m_{0}+n_{0}+1+j}}\left(\bar{\Omega}^{c}\right)^{3}-\int_{\mathbb{R}} \rho_{\delta}^{\prime}(t-s) K^{(l)}(s) d s\right) d x .
\end{aligned}
$$

But by Theorem 2.5 and the definition of $f_{\epsilon, \delta}^{(j)}$ for $m_{0}+n_{0}+2 \leq j \leq s_{0}$, the righthand side of the preceding equation equals $\sum_{j=m_{0}+n_{0}+2}^{s_{0}} \int_{\bar{\Omega}^{c}} f_{\epsilon, \delta}^{(j)}(t) \cdot \vartheta d x$. This explains the appearance of the terms $f_{\epsilon, \delta}^{(j)}(t)$ with $m_{0}+n_{0}+2 \leq j \leq s_{0}$ on the right-hand side of $(3.1)_{1}$ with the replacements indicated in the lemma. All the other terms, on the left- and right-hand side, arise from (5.1) in an obvious way, due to Theorem 2.3, the linearity and boundedness of the operators $\mathcal{A}_{p, \psi}$, and because of the equation $\partial x_{k} u_{\epsilon, \delta}=\left(\partial x_{k} u_{\epsilon}\right)_{\delta}$.

Now we are in a position to deal with the main difficulty of this section, that is, to show that $u_{\epsilon, \delta}$ satisfies the representation formula (4.13). The notation from Lemma 5.1 and 5.2 will be used without further notice.

Theorem 5.1 Let $\epsilon, \delta \in(0, \infty)$. Then, for any $t \in(0, \infty)$, there is a set $N_{t} \subset{\overline{B_{R_{0}}}}^{c}$ of measure zero such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$, with $T_{0}=\infty, A=B_{S_{0}}$ and $u$, $f$ replaced by $u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$ and $\sum_{j=1}^{n_{0}+m_{0}+1} f_{\epsilon, \delta}^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$, respectively.
Proof: Since $\bar{\Omega} \subset B_{S_{0}}$ by the choice of $S_{0}$ at the beginning of Section 2, we may fix $S_{1} \in\left(0, S_{0}\right)$ with $\bar{\Omega} \subset S_{S_{1}}$. Let $t \in \mathbb{R}$. For brevity, we set $U:=u_{\epsilon, \delta}(t), F^{(j)}:=f_{\epsilon, \delta}^{(j)}(t)$ for $1 \leq j \leq s_{0}$. Due to the choice of $q$ in Lemma 5.1 and by the properties of $u_{\epsilon, \delta}$ and $f_{\epsilon, \delta}^{(j)}\left(\right.$ Lemma 5.2), we have $U \in W_{l o c}^{1, q}\left(\bar{\Omega}^{c}\right)^{3}, \nabla U \in L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{9}, F^{(j)} \in L_{l o c}^{q}\left(\bar{\Omega}^{c}\right)^{3} \cap L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}$ for $1 \leq j \leq s_{0}$, and the function $U$ satisfies (3.1) with $\lambda=0$ and with $A, F$ replaced by $\bar{\Omega}^{c}$ and $\sum_{j=1}^{s_{0}} F^{(j)}$, respectively. Thus, by Theorem 3.4, there is $\Pi \in L_{l o c}^{p}\left(\bar{\Omega}^{c}\right)^{3}$ such that (3.2) holds with $A=\bar{\Omega}^{c}, \lambda=0$ and $F=\sum_{j=1}^{s_{0}} F^{(j)}$. Since the latter equation remains valid when we replace $\Pi$ by $\Pi+c$, for any number $c \in \mathbb{R}$, we may assume in addition that $\int_{A_{S_{0}, S_{1}}} \Pi d x=0$. Equation (3.2) is valid in particular for $\vartheta \in C_{0}^{\infty}\left(A_{S_{0}, S_{1}}\right)^{3}$. Therefore, according to Theorem 3.4, this function $\Pi$ satisfies the estimate

$$
\begin{equation*}
\left\|\Pi \mid A_{S_{0}, S_{1}}\right\|_{q} \leq C\left(q, S_{0}, S_{1}\right)\left(\left\|\nabla U\left|A_{S_{0}, S_{1}}\left\|_{q}+\right\| \sum_{j=1}^{s_{0}} F^{(j)}\right| A_{S_{0}, S_{1}}\right\|_{q}\right) . \tag{5.2}
\end{equation*}
$$

Due to (3.2) and the regularity properties indicated above for $U, F^{(j)}$ and $\Pi$, Theorem 3.5 now yields that $U \in W_{l o c}^{2, q}\left(\bar{\Omega}^{c}\right)^{3}, \Pi \in W_{l o c}^{1, q}\left(\bar{\Omega}^{c}\right)$ and the pair $(U, \Pi)$ solves (3.4) in $\bar{\Omega}^{c}$ with $\lambda=0$ and $F=\sum_{j=1}^{s_{0}} F^{(j)}$. We may choose a function $\widetilde{\varphi} \in C^{\infty}\left(\mathbb{R}^{3}\right)$ with $0 \leq \widetilde{\varphi} \leq$
$1, \widetilde{\varphi}\left|B_{S_{1}+\left(S_{0}-S_{1}\right) / 4}=0, \widetilde{\varphi}\right| B_{S_{1}+\left(S_{0}-S_{1}\right) / 2}^{c}=1$. Since $\bar{\Omega} \subset B_{S_{1}}$ and $\operatorname{supp}\left(\partial^{\alpha} \widetilde{\varphi}\right) \subset A_{S_{0}, S_{1}}$ for $\alpha \in \mathbb{N}_{0}^{3}$ with $\alpha \neq 0$, and by the regularity properties of $U$ and $\Pi$ just derived, the function

$$
\begin{equation*}
\widetilde{F}:=-\Delta \widetilde{\varphi} U-2\left(\nabla \widetilde{\varphi} \cdot \nabla U_{j}\right)_{1 \leq j \leq 3}+\tau \partial_{1} \widetilde{\varphi} U+\Pi \nabla \widetilde{\varphi} \tag{5.3}
\end{equation*}
$$

belongs to $W^{1, q}\left(\mathbb{R}^{3}\right)$, as does $\nabla \widetilde{\varphi} \cdot U$. In addition $\operatorname{supp}(\widetilde{F}) \cup \operatorname{supp}(\nabla \widetilde{\varphi} \cdot U) \subset A_{S_{0}, S_{1}}$. Put $\widetilde{W}:=\mathfrak{N} *(\nabla \widetilde{\varphi} \cdot U)$; see Theorem 3.1. Since $q<3 / 2$ by the choice of $q$ in Lemma 5.1 , we know by Theorem 3.1 that $\widetilde{W}$ is well defined and

$$
\begin{align*}
& \widetilde{W} \in W_{l o c}^{3, q}\left(\mathbb{R}^{3}\right), \quad \nabla \widetilde{W} \in L^{(1 / q-1 / 3)^{-1}}\left(\mathbb{R}^{3}\right)^{3}, \partial_{l} \partial_{m} \widetilde{W} \in W^{1, q}\left(\mathbb{R}^{3}\right)^{9}  \tag{5.4}\\
& \Delta \widetilde{W}=-\nabla \widetilde{\varphi} \cdot U, \quad\left\|\partial_{k} \partial_{l} \partial_{m} \widetilde{W}\right\|_{q}+\left\|\partial_{k} \partial_{l} \widetilde{W}\right\|_{q} \leq C(q)\|\nabla \widetilde{\varphi} \cdot U\|_{1, q} \quad(1 \leq k, l, m \leq 3)
\end{align*}
$$

We put

$$
\begin{align*}
& \widetilde{U}:=\widetilde{\varphi} U+\nabla \widetilde{W}, \widetilde{\Pi}:=\widetilde{\varphi} \Pi, \widetilde{F}^{(j)}:=\widetilde{\varphi} F^{(j)} \text { for } 1 \leq j \leq s_{0}  \tag{5.5}\\
& p_{s_{0}+1}:=q, \widetilde{F}^{\left(s_{0}+1\right)}:=\widetilde{F}-\Delta \nabla \widetilde{W}+\tau \partial_{1} \nabla \widetilde{W}
\end{align*}
$$

Note that $\operatorname{div} \nabla \widetilde{W}=\Delta \widetilde{W}=-\nabla \widetilde{\varphi} \cdot U$, hence $\operatorname{div} \widetilde{U}=0$. As a consequence of these observations and definitions, we have $\widetilde{U} \in W_{l o c}^{2, q}\left(\mathbb{R}^{3}\right)^{3}, \widetilde{\Pi} \in W_{l o c}^{1, q}\left(\mathbb{R}^{3}\right)$, $\widetilde{F}^{(j)} \in L^{p_{j}}\left(\mathbb{R}^{3}\right)^{3}$ for $1 \leq j \leq s_{0}+1, \widetilde{\varphi} U\left|B_{S_{0}}^{c}=U\right| B_{S_{0}}^{c}, \widetilde{\Pi}\left|B_{S_{0}}^{c}=\Pi\right| B_{S_{0}}^{c}$, and the pair $(\widetilde{U}, \widetilde{\Pi})$ solves (3.4) in $\mathbb{R}^{3}$ with $\lambda=0$ (stationary Oseen system) and with $F=\sum_{j=1}^{s_{0}+1} \widetilde{F}^{(j)}$. Since $\widetilde{F}^{(j)} \in L^{p_{j}}\left(\mathbb{R}^{3}\right)^{3}$ for $1 \leq j \leq s_{0}+1$, Theorem 3.7 yields functions $W^{(j)} \in W_{2}^{2, p_{j}}\left(\mathbb{R}^{3}\right)^{3}, \Pi^{(j)} \in W_{1}^{1, p_{j}}\left(\mathbb{R}^{3}\right)$ such that $\bar{\partial}_{1} W^{(j)} \in L^{p_{j}}\left(\mathbb{R}^{3}\right)^{3}$, equation (3.4) is satisfied in $\mathbb{R}^{3}$ with $U, \Pi, F$ replaced by $W^{(j)}, \Pi^{(j)}, \widetilde{F}^{(j)}$, respectively, and the inequality

$$
\begin{equation*}
\left\|\partial_{l} \partial_{m} W^{(j)}\right\|_{p_{j}}+\left\|\Pi^{(j)}\right\|_{W_{1}^{1, p_{j}}\left(\mathbb{R}^{3}\right)} \leq C\left(\tau, p_{j}\right)\left\|\widetilde{F}^{(j)}\right\|_{p_{j}}\left(1 \leq l, m \leq 3,1 \leq j \leq s_{0}+1\right) \tag{5.6}
\end{equation*}
$$

holds. Then $\bar{U}:=\widetilde{U}-\sum_{j=1}^{s_{0}+1} W^{(j)} \in W_{l o c}^{2, q}\left(\mathbb{R}^{3}\right)^{3}, \bar{\Pi}:=\widetilde{\Pi}-\sum_{j=1}^{s_{0}+1} \Pi^{(j)} \in W_{l o c}^{1, q}\left(\mathbb{R}^{3}\right)$, and $-\Delta \bar{U}+\tau \partial_{1} \bar{U}+\nabla \bar{\Pi}=0, \operatorname{div} \bar{U}=0$ in $\mathbb{R}^{3}$. It follows with Theorem 3.6 that $\bar{\Pi}$ is a $C^{\infty}$ function. Let us specify in which way the function $\bar{U}$ satisfies the assumptions of Theorem 3.8. Put $q_{j}(0):=q_{j}(1):=p_{j}$ for $j \in\left\{1, \ldots, s_{0}+1\right\}$. Then the definition of the space $W_{2}^{2, p_{j}}\left(\mathbb{R}^{3}\right)$ implies that $\int_{B_{R_{0}}^{c}}\left(\left|\partial^{\alpha} W^{(j)}(x)\right|\left[(1+|x|)^{2-|\alpha|} \ln (2+|x|)\right]^{-1}\right)^{q_{j}(|\alpha|)} d x<\infty$ for $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$ and $1 \leq j \leq s_{0}+1$. Put

$$
\begin{equation*}
\widetilde{q}:=\max \left(\left\{p_{j}: 1 \leq j \leq n_{0}\right\} \cup\left\{(1 / q-1 / 3)^{-1}\right\} \cup \max \left(\left\{\varrho_{l}: 1 \leq l \leq m_{0}\right\}\right)\right. \tag{5.7}
\end{equation*}
$$

and $\epsilon_{0}:=1 /(2 \widetilde{q})$. By the choice of $p_{n_{0}+1}, \ldots, p_{s_{0}+1}$ (Lemma 5.1, 5.2, (5.5)), this means in particular that $\widetilde{q} \geq p_{j}$ for $1 \leq j \leq s_{0}+1$, and $\widetilde{q} \geq q$. Obviously $1+|x| \leq C\left(R_{0}\right)|x|$ and $\ln (2+|x|) \leq C\left(R_{0}, \epsilon_{0}\right)|x|^{\epsilon_{0}}$ for $x \in B_{R_{0}}^{c}$. Thus we may conclude for $1 \leq j \leq s_{0}+1$ that

$$
\begin{equation*}
\int_{B_{R_{0}}^{c}}\left(\left|\partial^{\alpha} W^{(j)}(x)\right||x|^{-2-\epsilon_{0}}\right)^{q_{j}(|\alpha|)} d x<\infty \quad \text { for } \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha| \leq 1 \tag{5.8}
\end{equation*}
$$

Next we put $W^{\left(s_{0}+2\right)}:=-\nabla \widetilde{W}, q_{s_{0}+2}(0):=(1 / q-1 / 3)^{-1}, q_{s_{0}+2}(1):=q$. Then, due to (5.4), the relation in (5.8) is valid for $j=s_{0}+2$ as well. Since $\nabla U \in L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{3}$ as
mentioned above, $\widetilde{\varphi} \mid B_{S_{0}}^{c}=1$ and $R_{0}>S_{0}$, we get $\int_{B_{R_{0}}^{c}}\left(|\nabla(\widetilde{\varphi} U)(x)||x|^{-2-\epsilon_{0} \mid}\right)^{q_{1}} d x<\infty$. We further note that by the definition of $\widetilde{U}$ (see (5.5)), $\bar{U}$ and $W^{\left(s_{0}+2\right)}$, the equation $\nabla \bar{U}=$ $\nabla(\widetilde{\varphi} U)-\sum_{j=1}^{s_{0}+2} \nabla W^{(j)}$ holds. Thus, if in Theorem 3.8 we replace $U, m_{0},\left(V^{(m)}\right)_{1 \leq m \leq m_{0}}$ by $\bar{U}, s_{0}+3,\left(\left(\partial_{\mu}\left[\widetilde{\varphi} U_{\sigma}\right]\right)_{1 \leq \mu, \sigma \leq 3},\left(\partial_{\mu} W_{\sigma}^{(1)}\right)_{1 \leq \mu, \sigma \leq 3}, \ldots,\left(\partial_{\mu} W_{\sigma}^{\left(s_{0}+2\right)}\right)_{1 \leq \mu, \sigma \leq 3}\right)$, respectively, the assumptions on $\nabla U$ are satisfied with $r=2+\epsilon_{0}$. Concerning those on $U$, we recall that by the definition of $U$ and $u_{\epsilon, \delta}$, we have $\widetilde{\varphi} U=-\sum_{j=s_{0}+3}^{s_{0}+3+m_{0}} W^{(j)}$, with $W^{\left(s_{0}+3\right)}:=$ $-\widetilde{\varphi}\left(\chi_{\Omega_{S_{0}}} u_{\epsilon}\right)_{\delta}(t), W^{\left(s_{0}+3+l\right)}:=-\widetilde{\varphi} \widetilde{G}_{\epsilon, \delta}^{(l)}(t)$ for $1 \leq l \leq m_{0}$. Lemma 5.2 yields $W^{(j)} \in$ $L^{p_{j}(0)}\left(\mathbb{R}^{3}\right)^{3}$ for $s_{0}+3 \leq j \leq s_{0}+3+m_{0}$, with $p_{s_{0}+3}(0):=q, p_{s_{0}+3+l}(0):=\varrho_{l}$ for $1 \leq l \leq m_{0}$. It follows the relation in (5.8) holds for $s_{0}+3 \leq j \leq s_{0}+3+m_{0}$ with $\alpha=0$. Moreover, with the definition of $\widetilde{U}$ and $W^{\left(s_{0}+2\right)}$ we get that $\widetilde{U}=-\sum_{j=s_{0}+2}^{s_{0}+3+m_{0}} W^{(j)}$. Now the definition of $\bar{U}$ yields that $\bar{U}=-\sum_{j=1}^{s_{0}+m_{0}+3} W^{(j)}$. Therefore the assumptions on $U$ in Theorem 3.8 are satisfied with $\bar{U}, s_{0}+m_{0}+3,\left(-W^{(k)}\right)_{1 \leq k \leq s_{0}+m_{0}+3}$ in the role of $U, k_{0},\left(U^{(k)}\right)_{1 \leq k \leq k_{0}}$, respectively, and with $r=2+\epsilon_{0}$. At this point Theorem 3.8 yields that $\bar{U}$ is a polynomial. By the definitions in $(5.7)$ and the choice of the exponents $q_{j}(0)$, we have $\widetilde{q}=\max \left\{q_{j}(0): 1 \leq j \leq s_{0}+3+m_{0}\right\}$ and $\epsilon_{0} \in(0,1 / \widetilde{q})$. Thus it follows from (5.8) with $\alpha=0,1 \leq j \leq s_{0}+m_{0}+3$ and from Lemma 3.2 that the degree of $\bar{U}$ cannot exceed 1. As a first consequence of this result, we get $\partial^{\alpha} \bar{U}=0$ for $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha|=2$. Since $\bar{U}=\widetilde{\varphi} U-\sum_{j=1}^{s_{0}+2} W^{(j)}, \widetilde{\varphi} \mid B_{S_{0}}^{c}=1$ and $W^{\left(s_{0}+2\right)}=-\nabla \widetilde{W}$, we thus find

$$
\begin{equation*}
\partial^{\alpha} U\left|B_{S_{0}}^{c}=-\partial^{\alpha}(\nabla \widetilde{W})+\sum_{j=1}^{s_{0}+1} \partial^{\alpha} W^{(j)}\right| B_{S_{0}}^{c} \quad \text { for } \alpha \in \mathbb{N}_{0}^{3} \text { with }|\alpha|=2 \tag{5.9}
\end{equation*}
$$

It further follows there is $c_{1} \in \mathbb{R}$ with $\partial_{1} \bar{U}=c_{1}$. Again we recall that $\widetilde{\varphi} \mid B_{S_{0}}^{c}=1, S_{0}<R_{0}$ and $\nabla U \in L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{9}$, so $\partial_{1}(\widetilde{\varphi} U) \mid B_{R_{0}}^{c} \in L^{q_{1}}\left(B_{R_{0}}^{c}\right)^{3}$. Moreover $\partial_{1} W^{\left(s_{0}+2\right)}=\partial_{1} \nabla \widetilde{W} \in$ $L^{q}\left(\mathbb{R}^{3}\right)^{3}($ see $(5.4))$, and $\partial_{1} W^{(j)} \in L^{p_{j}}\left(\mathbb{R}^{3}\right)^{3}$ for $1 \leq j \leq s_{0}+1$ because $W^{(j)} \in W_{2}^{2, p_{j}}\left(\mathbb{R}^{3}\right)^{3}$. It follows from Lemma 3.3 and the equation $\bar{U}=\widetilde{\varphi} U-\sum_{j=1}^{s_{0}+2} W^{(j)}$ that $c_{1}$ vanishes. Recalling that $-\Delta \bar{U}+\tau \partial_{1} \bar{U}+\nabla \bar{\Pi}=0$, we thus obtain $\nabla \bar{\Pi}=0$. Since $\bar{\Pi} \in C^{\infty}\left(\mathbb{R}^{3}\right)$, as shown above, we may conclude there is $c \in \mathbb{R}$ with $\bar{\Pi}=c$. But $\bar{\Pi}=\widetilde{\varphi} \Pi-\sum_{j=1}^{s_{0}+1} \Pi^{(j)}$ by the definition of $\bar{\Pi}$, so we arrive at the equation

$$
\begin{equation*}
\Pi-c\left|{\overline{B_{S_{0}}}}^{c}=\sum_{j=1}^{s_{0}+1} \Pi^{(j)}\right|{\overline{B_{S_{0}}}}^{c} \tag{5.10}
\end{equation*}
$$

We have $\left\|\Pi^{(j)}\left|A_{R_{0}, S_{0}}\left\|_{1} \leq C\left(R_{0}, S_{0}, p_{j}\right)\right\| \Pi^{(j)}\right| A_{R_{0}, S_{0}}\right\|_{p_{j}}$ by Hölder's inequality, and obviously

$$
\left\|\Pi^{(j)} \mid A_{R_{0}, S_{0}}\right\|_{p_{j}} \leq C\left(R_{0}, S_{0}, p_{j}\right)\left(\int_{A_{R_{0}, S_{0}}}\left(\left|\Pi^{(j)}(x)\right|[(1+|x|) \ln (2+|x|)]^{-1}\right)^{p_{j}} d x\right)^{1 / p_{j}}
$$

so $\left\|\Pi^{(j)} \mid A_{R_{0}, S_{0}}\right\|_{1} \leq C\left(R_{0}, S_{0}, p_{j}\right)\left\|\Pi^{(j)}\right\|_{W_{1}^{1, p_{j}}\left(\mathbb{R}^{3}\right)}$, for $1 \leq j \leq s_{0}+1$. Similarly

$$
\left\|\nabla \Pi^{(j)}\left|A_{R_{0}, S_{0}}\left\|_{1} \leq C\left(R_{0}, S_{0}, p_{j}\right)\right\| \nabla \Pi^{(j)}\right| A_{R_{0}, S_{0}}\right\|_{p_{j}} \leq C\left(R_{0}, S_{0}, p_{j}\right)\left\|\nabla \Pi^{(j)}\right\|_{p_{j}}
$$

hence $\left\|\nabla \Pi^{(j)} \mid A_{R_{0}, S_{0}}\right\|_{1} \leq C\left(R_{0}, S_{0}, p_{j}\right)\left\|\Pi^{(j)}\right\|_{W_{1}^{1, p_{j}}\left(\underset{\left.\mathbb{R}^{3}\right)}{ }\right.}\left(1 \leq j \leq s_{0}+1\right)$. These estimates and (5.6) yield $\left\|\Pi^{(j)} \mid A_{R_{0}, S_{0}}\right\|_{1,1} \leq C\left(S_{0}, R_{0}, p_{j}, \tau\right)\left\|\widetilde{F}^{(j)}\right\|_{p_{j}}$ for $j$ as before. Due to (5.10),
we thus get

$$
\begin{equation*}
\left\|\Pi-c \mid A_{R_{0}, S_{0}}\right\|_{1,1} \leq C\left(S_{0}, R_{0}, p_{1}, \ldots, p_{s_{0}+1}, \tau\right) \sum_{j=1}^{s_{0}+1}\left\|\widetilde{F}^{(j)}\right\|_{p_{j}} \tag{5.11}
\end{equation*}
$$

In the rest of this proof, the constants denoted by $\mathfrak{C}$ only depend on $S_{1}, S_{0}, R_{0}, q, q_{1}, p_{1}$, $\ldots, p_{s_{0}}$ or $\tau$. Once more starting with Hölder's inequality, then using (5.4) and the relations $q \leq q_{1}$ (Lemma 5.1) and $\partial_{l} U=\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)$ (Lemma 5.2), we get

$$
\begin{align*}
& \left\|\partial_{k} \partial_{l} \partial_{m} \widetilde{W}\left|A_{R_{0}, S_{0}}\left\|_{1} \leq \mathfrak{C}\right\| \partial_{k} \partial_{l} \partial_{m} \widetilde{W}\right| A_{R_{0}, S_{0}}\right\|_{q} \leq \mathfrak{C}\left\|\partial_{k} \partial_{l} \partial_{m} \widetilde{W}\right\|_{q} \leq \mathfrak{C}\|\nabla \widetilde{\varphi} \cdot U\|_{1, q}  \tag{5.12}\\
& \leq \mathfrak{C}\left\|U \mid A_{S_{0}, S_{1}}\right\|_{1, q} \leq \mathfrak{C}\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\|\nabla U\|_{q_{1}}\right) \leq \mathfrak{C}\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}\right)
\end{align*}
$$

Obviously with (5.6), $\left\|\partial^{\alpha} W^{(j)}\left|A_{R_{0}, S_{0}}\left\|_{1} \leq \mathfrak{C}\right\| \partial^{\alpha} W^{(j)}\right| A_{R_{0}, S_{0}}\right\|_{p_{j}} \leq \mathfrak{C}\left\|\widetilde{F}^{(j)}\right\|_{p_{j}}$ for $1 \leq j \leq$ $s_{0}+1, \alpha \in \mathbb{N}_{0}^{3},|\alpha|=2$. Now we may conclude with (5.9) that

$$
\begin{equation*}
\left\|\partial^{\alpha} U \mid A_{R_{0}, S_{0}}\right\|_{1} \leq \mathfrak{C}\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}+\sum_{j=1}^{s_{0}+1}\left\|\widetilde{F}^{(j)}\right\|_{p_{j}}\right) \tag{5.13}
\end{equation*}
$$

for $\alpha$ as before. As in the last two inequalities in (5.12), we obtain that $\left\|U \mid A_{R_{0}, S_{0}}\right\|_{1,1}$ is bounded by the right-hand side of (5.12). This observation, (5.13) and (5.11) yield

$$
\begin{align*}
& \left\|U\left|A_{R_{0}, S_{0}}\left\|_{2,1}+\right\| \Pi-c\right| A_{R_{0}, S_{0}}\right\|_{1,1}  \tag{5.14}\\
& \leq C\left(\tau, S_{0}, R_{0}, q, q_{1}, p_{1}, \ldots, p_{s_{0}}\right)\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}+\sum_{j=1}^{s_{0}+1}\left\|\widetilde{F}^{(j)}\right\|_{p_{j}}\right)
\end{align*}
$$

Let us estimate $\left\|\widetilde{F}^{\left(s_{0}+1\right)}\right\|_{p_{s_{0}+1}}$. (See (5.5) for the definition of $\widetilde{F}^{\left(s_{0}+1\right)}$ and $p_{s_{0}+1}$. ) By (5.4) and the last four inequalities in (5.12), we get

$$
\begin{equation*}
\|\Delta \nabla \widetilde{W}\|_{q}+\left\|\partial_{1} \nabla \widetilde{W}\right\|_{q} \leq C(q)\|\nabla \widetilde{\varphi} \cdot U\|_{1, q} \leq \mathfrak{C}\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}\right) \tag{5.15}
\end{equation*}
$$

With $\widetilde{F}$ defined in (5.3), we further find that $\|\widetilde{F}\|_{q} \leq \mathfrak{C}\left(\left\|U\left|A_{S_{0}, S_{1}}\left\|_{1, q}+\right\| \Pi\right| A_{S_{0}, S_{1}}\right\|_{q}\right)$. The quantity $\left\|\Pi \mid A_{S_{0}, S_{1}}\right\|_{q}$ was evaluated in (5.2). In view of the right-hand side of this inequality, and by an estimate of $\left\|U \mid A_{S_{0}, S_{1}}\right\|_{1, q}$ as in the last two inequalities in (5.12), we may conclude that $\|\widetilde{F}\|_{q} \leq \mathfrak{C}\left(\left\|U\left|\Omega_{S_{0}}\left\|_{q}+\sum_{l=1}^{3}\right\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\left\|_{q_{1}}+\sum_{j=1}^{s_{0}}\right\| F^{(j)}\right| A_{S_{0}, S_{1}}\right\|_{q}\right)$. It follows with the preceding inequality, (5.15), (5.5) and the choice of $q$ in Lemma 5.1 that

$$
\begin{equation*}
\left\|\widetilde{F}^{\left(s_{0}+1\right)}\right\|_{p_{s_{0}+1}}=\left\|\widetilde{F}^{\left(s_{0}+1\right)}\right\|_{q} \leq \mathfrak{C}\left(\left\|U \mid \Omega_{S_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}+\sum_{j=1}^{s_{0}}\left\|F^{(j)}\right\|_{p_{j}}\right) \tag{5.16}
\end{equation*}
$$

Since $0 \leq \widetilde{\varphi} \leq 1$, we deduce from the definition of $\widetilde{F}^{(j)}$ in (5.5) that $\left\|\widetilde{F}^{(j)}\right\|_{p_{j}} \leq\left\|F^{(j)}\right\|_{p_{j}}$ for $1 \leq j \leq s_{0}$. Returning to the notation $u_{\epsilon, \delta}(t)$ and $f_{\epsilon, \delta}^{(j)}(t)$ introduced at the beginning of this proof to replace $U$ and $F^{(j)}$, respectively $\left(1 \leq j \leq s_{0}\right)$, and putting $\pi_{\epsilon, \delta}(t):=\Pi-c$, we thus obtain from (5.16) and (5.14) that

$$
\begin{align*}
& \left\|u_{\epsilon, \delta}(t)\left|A_{R_{0}, S_{0}}\left\|_{2,1}+\right\| \pi_{\epsilon, \delta}(t)\right| A_{R_{0}, S_{0}}\right\|_{1,1}  \tag{5.17}\\
& \leq C\left(\tau, S_{1}, S_{0}, R_{0}, q, q_{1}, p_{1}, \ldots, p_{s_{0}}\right)\left(\left\|u_{\epsilon, \delta}(t) \mid \Omega_{R_{0}}\right\|_{q}+\sum_{l=1}^{3}\left\|\left(\partial x_{l} u_{\epsilon}\right)_{\delta}(t)\right\|_{q_{1}}+\sum_{j=1}^{s_{0}}\left\|f_{\epsilon, \delta}^{(j)}(t)\right\|_{p_{j}}\right)
\end{align*}
$$

In addition, equation (5.10) takes the form $\pi_{\epsilon, \delta}(t)\left|{\overline{B_{S_{0}}}}^{c}=\sum_{j=1}^{s_{0}+1} \Pi^{(j)}\right|{\overline{B_{S_{0}}}}^{c}$. Since $\Pi^{(j)} \in$
 $\infty\left(1 \leq j \leq s_{0}+1\right)$. By what was shown for $U$ and $\Pi$ at the beginning of this proof, the relations $u_{\epsilon, \delta}(t) \in W_{l o c}^{2, q}\left(\bar{\Omega}^{c}\right)^{3}$ and $\pi_{\epsilon, \delta}(t) \in W_{l o c}^{1, q}\left(\bar{\Omega}^{c}\right)$ hold, and the pair $\left(u_{\epsilon, \delta}(t), \pi_{\epsilon, \delta}(t)\right)$ solves (3.4) in $\bar{\Omega}^{c}$, with $\lambda=0$ (stationary Oseen system) and $F=\sum_{j=1}^{s_{0}} f_{\epsilon, \delta}^{(j)}(t)$.
Recall that $G_{\epsilon, \delta}^{(l)} \in C^{\infty}\left(\mathbb{R}, L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}, u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{j=1}^{m_{0}} G_{\epsilon, \delta}^{(l)}$ and $f_{\epsilon, \delta}^{(j)} \in C^{\infty}\left(\mathbb{R}, L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq j \leq n_{0}+m_{0}+1$ (Lemma 5.2). Since $f_{\epsilon, \delta}^{\left(n_{0}+m_{0}+2\right)}(t) \mid{\overline{B_{S_{0}}}}^{c}=$ 0 and $f_{\epsilon, \delta}^{\left(n_{0}+m_{0}+2+l\right)}(t) \mid{\overline{B_{S_{0}}}}^{c}=-\left(G_{\epsilon, \delta}^{(l)}\right)^{\prime}(t)$ for $1 \leq l \leq m_{0}, t \in \mathbb{R}$ (Lemma 5.2), we obtain that $\left(u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}\right)^{\prime}(t)=-\sum_{j=n_{0}+m_{0}+2}^{n_{0}+2 m_{0}+2} f_{\epsilon, \delta}^{(j)}(t) \mid{\overline{B_{S_{0}}}}^{c}$. This equation, the relation $s_{0}=$ $n_{0}+2 m_{0}+2$ and the fact that the pair $\left(u_{\epsilon, \delta}(t), \pi_{\epsilon, \delta}(t)\right)$ solves (3.4) in $\bar{\Omega}^{c}$ with $\lambda=0$ and $F=\sum_{j=1}^{s_{0}} f_{\epsilon, \delta}^{(j)}(t)$ and with $t$ arbitrary in $\mathbb{R}$, allow us to conclude that the pair $\left(u_{\epsilon, \delta}\left|{\overline{B_{S_{0}}}}^{c} \times(0, \infty), \pi_{\epsilon, \delta}\right|{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)$ solves the time-dependent Oseen system (4.12) with $A=B_{S_{0}}$ and $f=\sum_{j=1}^{n_{0}+m_{0}+1} f_{\epsilon, \delta}^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$, without any exceptional values of $t$. In particular the equation $\operatorname{div}_{x} u(t)=0$ holds for any $t \in(0, \infty)$, without exceptional values.

Lemma 5.2 states in particular that the function $\nabla_{x} u_{\epsilon, \delta}$ belongs to $C^{\infty}\left(\mathbb{R}, L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{9}\right)$ and $u_{\epsilon, \delta} \mid \Omega_{R_{0}} \times \mathbb{R}$ to $C^{\infty}\left(\mathbb{R}, L^{q}\left(\Omega_{R_{0}}\right)^{3}\right)$. Thus the right-hand side of (5.17) is integrable with respect to $t \in J$ for any bounded interval $J \subset \mathbb{R}$. Therefore we may conclude from (5.17) that $u_{\epsilon, \delta} \mid A_{R_{0}, S_{0}} \times(0, \infty) \in L_{l o c}^{1}\left([0, \infty), W^{2,1}\left(A_{R_{0}, S_{0}}\right)^{3}\right)$ and $\pi_{\epsilon, \delta} \mid A_{R_{0}, S_{0}} \times(0, \infty) \in$ $L_{l o c}^{1}\left([0, \infty), W^{1,1}\left(A_{R_{0}, S_{0}}\right)\right)$.
Thus all assumptions of Theorem 4.5 are verified with obvious replacements, in particular with $B_{S_{0}}$ in the role of $A$. The theorem now follows from Theorem 4.5.

Next we show that (4.13) remains valid when $\delta$ tends to zero.
Lemma 5.3 Let $\epsilon, t \in(0, \infty)$. Then there is a set $N_{t, \epsilon} \subset{\overline{B_{R_{0}}}}^{c}$ of measure zero such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t, \epsilon}$, with $T_{0}=\infty, A=B_{S_{0}}$ and $u$, $f$ replaced by $u_{\epsilon} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$ and $\sum_{j=1}^{n_{0}+m_{0}+1} f_{\epsilon}^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$, respectively.
Proof: Abbreviate $\mathfrak{G}:=\mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}$, with $\mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}$ defined in Theorem 4.2. Recall that $R_{1}:=\left(S_{0}+R_{0}\right) / 2$, as defined at the beginning of Section 2. Let $l \in\left\{1, \ldots, m_{0}\right\}, k \in$ $\{1,2,3\}, j \in\left\{1, \ldots, n_{0}+m_{0}+1\right\}, v \in\left\{G_{\epsilon}^{(l)}, \partial x_{k} u_{\epsilon}, f_{\epsilon}^{(j)}\right\}$. Put $p:=\varrho_{l}, A:={\overline{B_{S_{0}}}}^{c}$ if $v=G_{\epsilon}^{(l)}, p:=q_{1}, A:=\bar{\Omega}^{c}$ if $v=\partial x_{k} u_{\epsilon}$, and $p:=p_{j}, A:=\bar{\Omega}^{c}$ if $v=f_{\epsilon}^{(j)}$. Due to this choice of $p$ and $A$, and by Lemma 5.1, the relation $v \in L_{l o c}^{1}\left(\mathbb{R}, L^{p}(A)^{3}\right)$ holds. As a consequence $\chi_{(-1, t+1)} v \in L^{1}\left(\mathbb{R}, L^{p}(A)^{3}\right)$. Therefore we get by Theorem 2.5 that $\left\|\chi_{(-1, t+1)} v-\left(\chi_{(-1, t+1)} v\right)_{\delta}\right\|_{p, 1 ; \mathbb{R}} \rightarrow 0(\delta \downarrow 0)$. But for $\delta \in(0,1], s \in(0, t)$, we have $\left(\chi_{(-1, t+1)} v\right)_{\delta}(s)=v_{\delta}(s)$, so $\left\|v_{\delta}-v \mid A \times(0, t)\right\|_{p, 1 ; t} \rightarrow 0(\delta \downarrow 0)$. Thus we have shown that

$$
\begin{align*}
& \left\|G_{\epsilon, \delta}^{(l)}-G_{\epsilon}^{(l)}\left|{\overline{B_{S_{0}}}}^{c} \times(0, t)\left\|_{\varrho_{l}, 1 ; t} \rightarrow 0,\right\|\left(\partial x_{k} u_{\epsilon}\right)_{\delta}-\partial x_{k} u_{\epsilon}\right| \bar{\Omega}^{c} \times(0, t)\right\|_{q_{1}, 1 ; t} \rightarrow 0  \tag{5.18}\\
& \left\|f_{\epsilon, \delta}^{(j)}-f_{\epsilon}^{(j)} \mid \bar{\Omega}^{c} \times(0, t)\right\|_{p_{j}, 1 ; t} \rightarrow 0(\delta \downarrow 0) \text { for } 1 \leq l \leq m_{0}, 1 \leq k \leq 3,1 \leq j \leq n_{0}+m_{0}+1
\end{align*}
$$

In particular, since $\partial x_{k} u_{\epsilon, \delta}=\left(\partial x_{k} u_{\epsilon}\right)_{\delta}$ according to Lemma 5.2, we deduce from (5.18) that $\left\|\partial x_{k} u_{\epsilon, \delta}-\partial x_{k} u_{\epsilon} \mid \bar{\Omega}^{c} \times(0, t)\right\|_{q_{1}, 1 ; t} \rightarrow 0(\delta \downarrow 0)$ for $1 \leq k \leq 3$. At this point, recall that
$u_{\epsilon} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon}^{(l)}$ (Lemma 5.1), and $u_{\epsilon, \delta} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=\sum_{l=1}^{m_{0}} G_{\epsilon, \delta}^{(l)}$ (Lemma 5.2). Using the parameter $q$ introduced in Lemma 5.1, and observing that $q \leq \varrho_{l}$ for $1 \leq l \leq m_{0}$, we may thus further conclude from (5.18) that $\left\|u_{\epsilon, \delta}-u_{\epsilon} \mid A_{R, S_{0}} \times(0, t)\right\|_{q, 1 ; t} \rightarrow 0(\delta \downarrow 0)$ for any $R \in\left[S_{0}, \infty\right)$. For $l \in\left\{1, \ldots, m_{0}\right\}$, since $\widetilde{G}_{\epsilon}^{(l)}(t)=u_{\epsilon}(0)=0$ if $t \in(-\infty, \epsilon]$, we have $\widetilde{G}_{\epsilon, \delta}^{(l)}(t)=\left(\chi_{\Omega_{S_{0}}} u_{\epsilon}\right)_{\delta}(t)=0$ for $t \in(-\infty, \epsilon / 2], \delta \in(0, \epsilon / 2]$, hence $\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(0)=0$, and thus $\mathfrak{I}^{(\tau)}\left(\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(0) \mid{\overline{B_{S_{0}}}}^{c}\right)=0$, for $\delta \in(0, \epsilon / 2]$. Since $G^{(l)} \in C^{0}\left(\mathbb{R}, L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$, the last statement in Theorem 2.5 yields that $\left\|\left(G_{\epsilon, \delta}^{(l)}-G_{\epsilon}^{(l)}\right)(t)\right\|_{\varrho_{l}} \rightarrow 0(\delta \downarrow 0)$ for $1 \leq l \leq m_{0}$. Again because $q \leq \varrho_{l}$, it follows that $\left\|\left(G_{\epsilon, \delta}^{(l)}-G_{\epsilon}^{(l)}\right)(t) \mid A_{R, S_{0}}\right\|_{q} \rightarrow 0(\delta \downarrow 0)$ for $1 \leq l \leq$ $m_{0}, R \in\left[S_{0}, \infty\right)$, and thus with Lemma $5.2\left\|\left(u_{\epsilon, \delta}-u_{\epsilon} \mid A_{R, S_{0}} \times(0, \infty)\right)(t)\right\|_{q} \rightarrow 0(\delta \downarrow 0)$ for the same range of $R$.
Let $x \in B_{R_{0}}^{c}$. By Theorem 4.2, the function $y \mapsto \mathfrak{G}(x, y, 0)\left(y \in B_{R_{1}}\right)$ belongs to $L^{q^{\prime}}\left(B_{R_{1}}\right)^{3}$, so we may conclude that $\int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, 0) \cdot\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(y, t) d y \rightarrow 0$ for $\delta \downarrow 0$. Since $\left\|u_{\epsilon, \delta}-u_{\epsilon} \mid \Omega_{R} \times(0, t)\right\|_{q, 1 ; t} \rightarrow 0$ and $\left\|\partial y_{k} u_{\epsilon, \delta}-\partial y_{k} u_{\epsilon} \mid \bar{\Omega}^{c} \times(0, t)\right\|_{q_{1}, 1 ; t} \rightarrow 0$ for $\delta \downarrow 0$ if $1 \leq k \leq 3, R \in\left[S_{0}, \infty\right)$, as shown above, inequality (4.3) in Theorem 4.2 yields that $\int_{0}^{t} \int_{A_{R_{1}, S_{0}}} \partial y_{k}^{\sigma} \partial_{s}^{1-\sigma} \mathfrak{G}(x, y, t-s) \cdot\left(\partial y_{k}^{\mu} u_{\epsilon, \delta}-\partial y_{k}^{\mu} u_{\epsilon}\right)(y, s) d y d s \rightarrow 0(\delta \downarrow 0)$ for $\sigma, \mu \in\{0,1\}, k \in\{1,2,3\}$. Altogether we arrive at the relation $\mathfrak{K}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(x, t) \rightarrow$ $0(\delta \downarrow 0)$, with $\mathfrak{K}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)=\mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, B_{S_{0}}, T_{0}}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)$ defined in (4.4). Recall that $\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(0)=0$ for $\delta \in(0, \epsilon / 2]$, so $\int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, t) \cdot\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(y, 0) d y \rightarrow 0(\delta \downarrow 0)$. Since $\left\|f_{\epsilon, \delta}^{(j)}-f_{\epsilon}^{(j)}\right\|_{p_{j}, 1 ; t} \rightarrow 0(\delta \downarrow 0)$ (see above), we may apply (4.3) again, to obtain that $\int_{0}^{t} \int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, t-s) \cdot\left(f_{\epsilon, \delta}^{(j)}-f_{\epsilon}^{(j)}\right)(y, s) d y d s \rightarrow 0(\delta \downarrow 0)\left(j \in\left\{1, \ldots, n_{0}+m_{0}+1\right\}\right)$. Using the function $\varphi_{0} \in C_{0}^{\infty}\left(B_{R_{1}}\right)$ fixed at the beginning of this section, we set $\mathfrak{E}(y):=$ $\left(-\left(\partial_{j} \mathfrak{N}\right)(x-y) \partial_{k} \varphi_{0}(y)+\left(\partial_{k} \partial_{j} \mathfrak{N}\right)(x-y) \varphi_{0}(y)\right)_{1 \leq j, k \leq 3}$ for $y \in \overline{B_{R_{1}}}$. Since $x \in B_{R_{0}}^{c}$, this function $\mathfrak{E}$ is well defined and belongs to $C^{1}\left(\overline{B_{R_{1}}}\right)^{3 \times 3}$. Hence, by the Divergence theorem and because $\operatorname{div}_{x}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)=0$,

$$
\int_{\partial S_{0}}(\nabla \mathfrak{N})(x-y) \cdot\left(S_{0}^{-1} y \cdot\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(y, t)\right) d o_{y}=\int_{A_{R_{1}, S_{0}}} \mathfrak{E}(y) \cdot\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(y, t) d y
$$

Again referring to the relation $\left\|\left(u_{\epsilon, \delta}-u_{\epsilon} \mid A_{R_{1}, S_{0}} \times(0, \infty)\right)(t)\right\|_{q} \rightarrow 0(\delta \downarrow 0)$, it follows that $\int_{\partial S_{0}}(\nabla \mathfrak{N})(x-y) \cdot\left(S_{0}^{-1} y \cdot\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(y, t)\right) d o_{y} \rightarrow 0(\delta \downarrow 0)$. By Lemma 4.7, we find

$$
\begin{aligned}
& K_{\delta}:=\left|\sum_{l=1}^{3} \partial x_{l} \mathfrak{V}^{\left(\tau, B_{S_{0}}\right)}\left(n_{l}^{\left(B_{S_{0}}\right)}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)\right)(x, t)\right| \\
& \leq \mathfrak{C}\left(\left\|u_{\epsilon, \delta}-u_{\epsilon}\left|A_{R_{1}, S_{0}} \times(0, t)\left\|_{q, 1 ; t}+\right\| \nabla_{y}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)\right| A_{R_{1}, S_{0}} \times(0, t)\right\|_{q, 1 ; t}\right)
\end{aligned}
$$

for $\delta \in(0, \infty)$, with the constant $\mathfrak{C}$ independent of $u$ (and also of $x$ and $t$, but this is not relevant here). Hence $K_{\delta} \rightarrow 0(\delta \downarrow 0)$ by what we have proved above for the convergence of $u_{\epsilon, \delta}-u_{\epsilon}$ and $\nabla_{x}\left(u_{\epsilon, \delta}-u_{\epsilon}\right)$, and because $q \leq q_{1}$.
Up to this point, $x$ was arbitrary but fixed in $B_{R_{0}}^{c}$. Since $\left\|f_{\epsilon, \delta}^{(j)}-f_{\epsilon}^{(j)} \mid \bar{\Omega}^{c} \times(0, t)\right\|_{p_{j}, 1 ; t} \rightarrow 0$ for $\delta \downarrow 0$ by (5.18), Lemma 4.4 yields $\left\|\Re^{(\tau)}\left(f_{\epsilon, \delta}^{(j)}-f_{\epsilon}^{(j)}\right)(t)\right\|_{p_{j}} \rightarrow 0(\delta \downarrow 0)$, for $1 \leq$ $j \leq n_{0}+m_{0}+1$. Recalling that $\left\|\left(u_{\epsilon, \delta}-u_{\epsilon} \mid A_{R, S_{0}} \times(0, \infty)\right)(t)\right\|_{q} \rightarrow 0(\delta \downarrow 0)$ for $R \in$ $\left[S_{0}, \infty\right)$, we see there is a zero-measure set $M_{t, \epsilon} \subset \mathbb{R}^{3}$ and a sequence $\left(\delta_{n}\right)$ in $(0, \infty)$ with
$\delta_{n} \rightarrow 0$ such that $\mathfrak{R}^{(\tau)}\left(f_{\epsilon, \delta_{n}}^{(j)}-f_{\epsilon}^{(j)}\right)(x, t) \rightarrow 0(n \rightarrow \infty)$ and $\left(u_{\epsilon, \delta}-u_{\epsilon}\right)(x, t) \rightarrow 0$ for $x \in{\overline{B_{S_{0}}}}^{c} \backslash M_{t, \epsilon}, 1 \leq j \leq n_{0}+m_{0}+1$. By Theorem 5.1, for any $n \in \mathbb{N}$, there is a set $N_{t, n} \subset{\overline{B_{R_{0}}}}^{c}$ of measure zero such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t, n}$, with $A=B_{S_{0}}$, with $u_{\epsilon, \delta_{n}}\left|{\overline{B_{S_{0}}}}^{c} \times(0, \infty), \pi_{\epsilon, \delta_{n}}\right|{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$ in the role of $u$ and $\pi$, respectively, with $n_{0}$ replaced by $n_{0}+m_{0}+1$, and the functions $f^{(j)}\left(1 \leq j \leq n_{0}\right)$ by $f_{\epsilon, \delta_{n}}^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times$ $(0, \infty)\left(1 \leq j \leq n_{0}+m_{0}+1\right)$. (The function $\pi$ only appears in the assumptions of equation (4.13), not in the equation itself.) Letting $n$ tend to zero in that equation, we may conclude by the preceding convergence results that (4.13) is satisfied by $u_{\epsilon}, \pi_{\epsilon}, f_{\epsilon}^{(j)}$ as well, in the way stated in the lemma.

Finally we let $\epsilon$ tend to zero in (4.13).
Corollary 5.1 Let $t \in(0, \infty)$. Then there is a zero-measure set $N_{t} \subset{\overline{B_{R_{0}}}}^{c}$ such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$, with $T_{0}=\infty, A=B_{S_{0}}$, and with $u$, $f$ replaced by $u\left|{\overline{B_{S_{0}}}}^{c} \times(0, \infty), f=\sum_{j=1}^{n_{0}} f^{(j)}\right|{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$, respectively,
Proof: By definitions in Lemma 5.1, the equation $\zeta_{\epsilon}^{\prime} u(s)\left|{\overline{B_{S_{0}}}}^{c}=\sum_{j=n_{0}+1}^{n_{0}+m_{0}+1} f_{\epsilon}^{(j)}(s)\right|{\overline{B_{S_{0}}}}^{c}$ holds for $s \in(0, \infty)$. (Actually $f_{\epsilon}^{\left(n_{0}+1\right)} \mid{\overline{B_{S_{0}}}}^{c} \times \mathbb{R}=0$, but this does not matter here.) Therefore Lemma 5.3 implies that for any $\epsilon \in(0, \infty)$, there is a set $N_{t, \epsilon} \subset{\overline{B_{R_{0}}}}^{c}$ of measure zero such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t, \epsilon}$ with $T_{0}=\infty, A=B_{S_{0}}$ and with $u, f$ replaced by $u_{\epsilon} \mid \overline{B_{S_{0}}} \times(0, \infty)$ and $g_{\epsilon}$, respectively, where the function $g_{\epsilon}$ is defined by $g_{\epsilon}(s):=\sum_{j=1}^{n_{0}} f_{\epsilon}^{(j)}(s)+\zeta_{\epsilon}^{\prime}(s) u(s) \mid{\overline{B_{S_{0}}}}^{c}$ for $s \in(0, \infty)$. We recall that $G^{(l)} \in$ $C^{0}\left([0, \infty), L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}$ and $u(s) \mid{\overline{B_{S_{0}}}}^{c}=\sum_{l=1}^{m_{0}} G^{(l)}(s)$ for $s \in(0, \infty)$. Thus we see that all assumptions of Lemma 4.9 are valid with $A$, $u, f^{(j)}\left(1 \leq j \leq n_{0}\right)$ replaced by $B_{S_{0}}, u \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)$ and $f^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\left(1 \leq j \leq n_{0}\right)$, respectively, with $m_{0}$ in the role of $k_{0}$ and $G^{(l)}\left(1 \leq l \leq m_{0}\right)$ in that of $u^{(l)}\left(1 \leq l \leq k_{0}\right)$. Lemma 4.9 implies that there is a measurable set $N_{t} \subset{\bar{B} R_{0}}^{c}$ with properties as stated in the corollary.

Up to now, we considered the case $T_{0}=\infty$ in our assumptions at the beginning of this section. But in the present context, the transition from this case to the case $T_{0}<\infty$ is easy to perform:

Corollary 5.2 Suppose that $T_{0}<\infty$. Then, for $t \in\left(0, T_{0}\right)$, there is a zero-measure set $N_{t} \subset{\overline{B_{R_{0}}}}^{c}$ such that equation (4.13) holds for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$, with $A=B_{S_{0}}$, $f=$ $\sum_{j=1}^{n_{0}} f^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)$, and with $u$ replaced by $u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)$.
Proof: Let $T^{\prime} \in\left(0, T_{0}\right)$, and choose a function $\zeta \in C^{\infty}(\mathbb{R})$ such that $\zeta$ equals 1 on the interval $\left(-\infty, T^{\prime}+\left(T_{0}-T^{\prime}\right) / 4\right)$ and vanishes on $\left(T^{\prime}+\left(T_{0}-T^{\prime}\right) / 2, \infty\right)$. Define $\widetilde{u}(s):=$ $\zeta(s) u(s), \widetilde{f}^{(j)}(s):=\zeta(s) f^{(j)}(s), \widetilde{G}^{(l)}(s):=\zeta(s) G^{(l)}(s), \widetilde{f}^{\left(n_{0}+1\right)}(s):=\zeta^{\prime}(s) \chi_{\Omega_{S_{0}}} u(s)$ and $\widetilde{f}^{\left(n_{0}+1+l\right)}(s):=\zeta^{\prime}(s) \bar{G}^{(l)}(s)$, for $1 \leq j \leq n_{0}, 1 \leq l \leq k_{0}, s \in\left(0, T_{0}\right)$, where $\bar{G}^{(l)}$ denotes the zero extension of $G^{(l)}$ to $\bar{\Omega} \times\left(0, T_{0}\right)$. On $\left[T_{0}, \infty\right)$, the value zero is assigned to these functions. Then all assumptions listed at the beginning of this section are valid with $T_{0}, n_{0}$ replaced by $\infty$ and $n_{0}+m_{0}+1$, respectively, and with $\widetilde{u}, \widetilde{f}^{(j)}(1 \leq j \leq$ $\left.n_{0}+m_{0}+1\right), \widetilde{G}^{(l)}\left(1 \leq l \leq m_{0}\right)$ in the role of $u, f^{(j)}\left(1 \leq j \leq n_{0}\right), G^{(j)}\left(1 \leq l \leq m_{0}\right)$, respectively. Now Corollary 5.1 yields Corollary 5.2 with $T^{\prime}$ in the place of $T_{0}$. Since $T^{\prime}$ was taken arbitrarily in $\left(0, T_{0}\right)$, the proof is complete.

Let us consider the relationship between $U_{0}$ and $u(0)$.
Corollary 5.3 For $x \in{\overline{B_{R_{0}}}}^{c}, t \in(0, \infty)$, the equation

$$
\begin{equation*}
\mathfrak{I}^{(\tau)}\left(u(0)-U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)(x, t)+\int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, t) \cdot\left(U_{0}(y)-u(y, 0)\right) d y=0 \tag{5.19}
\end{equation*}
$$

and the inequality

$$
\left|\partial_{x}^{\alpha}\left[\mathfrak{J}^{(\tau)}\left(U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}\right)\right](x, t)\right| \leq \mathfrak{C}\left\|U_{0}-u(0) \mid A_{R_{1}, S_{0}}\right\|_{\min \{\widetilde{p}, q\}}(|x| \nu(x))^{-3 / 2-|\alpha| / 2}
$$

hold, with the parameter $q$ defined in Lemma 5.1, and with $\widetilde{p}$ introduced at the beginning of this section in the assumption $U_{0} \in L^{\widetilde{p}}\left(\bar{\Omega}^{c}\right)^{3}$. The constant $\mathfrak{C}$ is independent of $u, U_{0}, x$ and $t$. There is a function $\Pi \in W_{\text {loc }}^{1,1}\left({\overline{B_{S_{0}}}}^{c}\right)$ such that $U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}=\nabla \Pi$.
Proof: Let $\vartheta \in C_{0, \sigma}^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$. We show that $\int_{\overline{B_{S_{0}}}}{ }^{c} U_{0} \cdot \vartheta d x=\int_{\overline{B_{S_{0}}}}{ }^{c} u(0) \cdot \vartheta d x$. Take $T \in$ $\left(0, T_{0}\right)$. Since $\nabla_{x} u$ is in $L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{q_{1}}\left(\bar{\Omega}^{c}\right)^{9}\right)$ and $f^{(j)}$ in $L_{l o c}^{1}\left(\left[0, T_{0}\right), L^{p_{j}}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for $1 \leq$ $j \leq n_{0}$, the function $K(s):=\int_{\overline{B_{S_{0}}}} c\left(\nabla_{x} u(s) \cdot \nabla \vartheta+\tau \partial x_{1} u(s) \cdot \vartheta-f(s) \cdot \vartheta\right) d x\left(s \in\left(0, T_{0}\right)\right)$, with $f:=\sum_{j=1}^{n_{0}} f^{(j)}$, belongs to $L_{l o c}^{1}\left(\left[0, T_{0}\right)\right)$, in particular $K \mid(0, T) \in L^{1}((0, T))$. On the other hand, since $u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)=\sum_{l=1}^{m_{0}} G^{(l)}$ and $G^{(l)} \in C^{0}\left(\left[0, T_{0}\right), L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}$, the function $H(s):=\int_{\overline{B_{S_{0}}}} u(s) \cdot \vartheta d x$ with $s \in\left[0, T_{0}\right)$ is in $C^{0}\left(\left[0, T_{0}\right)\right)^{3}$. Moreover, due to equation (1.3), we get

$$
\begin{equation*}
\int_{0}^{\infty}\left(-\gamma^{\prime}(s) H(s)+\gamma(s) K(s)\right) d s=\gamma(0) \int_{\overline{B_{S_{0}}}} U_{0} \cdot \vartheta d x \quad \text { for } \gamma \in C_{0}^{\infty}\left(\left[0, T_{0}\right)\right) \tag{5.20}
\end{equation*}
$$

This equation for $\gamma \in C_{0}^{\infty}((0, T))$ and the relations $H \mid[0, T] \in C^{0}([0, T])$ and $K \mid(0, T) \in$ $L^{1}((0, T))$ yield that $H \mid(0, T) \in W^{1,1}((0, T))$ and $H^{\prime}(s)=-K(s)$ for a. e. $s \in(0, T)$. Let $\gamma \in C_{0}^{\infty}([0, T))$. It follows that $\gamma H \in C^{0}([0, T]) \cap W^{1,1}((0, T))$ and $(\gamma H)^{\prime}=-\gamma K+\gamma^{\prime} H$, hence $\int_{0}^{T}\left(\gamma K-\gamma^{\prime} H\right) d s=\gamma(0) H(0)$. Comparing this equation with (5.20) and recalling that $H(0)=\int_{\overline{B_{S_{0}}}} c u(0) \cdot \vartheta d x$, we arrive at the result $\int_{\overline{B_{S_{0}}}} c U_{0} \cdot \vartheta d x=\int_{\overline{S_{S_{0}}}} c u(0) \cdot \vartheta d x$ we wanted to show. According to [29, Lemma III.1.1], the preceding equation implies existence of a function $\Pi$ with properties as stated at the end of the corollary.
Now let $t \in(0, \infty)$ and $x \in{\overline{B_{R_{0}}}}^{c}$, and put $Y(y):=\Lambda(x-y, t)-\mathfrak{G}(x, y, t)$ for $y \in B_{R_{1}}$. Recall that $\mathfrak{G}(x, \cdot, t) \in C_{0}^{\infty}\left(B_{R_{1}}\right)^{3 \times 3}, \mathfrak{G}(x, y, t)=\Lambda(x-y, t)$ for $y \in B_{S_{0}+\left(R_{0}-S_{0}\right) / 4}$ (Theorem 4.2), and $\Lambda \mid \mathbb{R}^{3} \times(0, \infty) \in C^{\infty}\left(\mathbb{R}^{3} \times(0, \infty)\right)^{3 \times 3}($ Lemma 4.1). Thus we get $\operatorname{supp}(Y) \subset$ $B_{S_{0}+\left(R_{0}-S_{0}\right) / 4}^{c}$ and $Y \in C^{\infty}\left(\mathbb{R}^{3}\right)^{3 \times 3}$. Moreover $\sum_{k=1}^{3} \partial y_{k} Y_{j k}(y)=0$ for $y \in \mathbb{R}^{3}, 1 \leq j \leq 3$ (Lemma 4.1, Theorem 4.2), and $Y \in W^{1, r}\left(\mathbb{R}^{3}\right)^{3 \times 3}$ for $r \in(1, \infty)$ by Lemma 4.1 and because $\mathfrak{G}(x, \cdot, t) \in C_{0}^{\infty}\left(B_{R_{1}}\right)^{3 \times 3}$. Therefore, for $1 \leq \mu \leq 3$, Theorem 2.1 provides a sequence $\left(\vartheta_{n}^{(\mu)}\right)_{n \geq 1}$ in $C_{0, \sigma}^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$ such that in particular $\left\|\left(Y_{\mu k}\right)_{1 \leq k \leq 3}-\vartheta_{n}^{(\mu)}\right\|_{r} \rightarrow 0(n \rightarrow \infty)$ for $r=\widetilde{p}$ and $r \in\left\{\varrho_{m}: 1 \leq m \leq m_{0}\right\}$. But $\int_{\overline{B_{S_{0}}}} c \vartheta_{n}^{(\mu)} \cdot U_{0} d x=\int_{\overline{B_{S_{0}}}}{ }^{c} \vartheta_{n}^{(\mu)} \cdot u(0) d x(1 \leq$ $\mu \leq 3, n \in \mathbb{N}$ ) by what has already been proved, and $u(0) \mid{\overline{B_{S_{0}}}}^{c}=\sum_{l=1}^{m_{0}} G^{(l)}(0), G^{(l)}(0) \in$ $L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\left(1 \leq l \leq m_{0}\right)$ by our assumptions. Hence, by letting $n$ tend to infinity, we obtain $\int_{\overline{B_{S_{0}}}} c Y \cdot U_{0} d y=\int_{\overline{B_{S_{0}}}} c Y \cdot u(0) d y$. Due to the definition of $Y$, this implies (5.19). The estimate stated in the corollary follows from (5.19) and (4.2).
Corollary 5.1, 5.2 and 5.3 yield the final form of our representation formula:

Corollary 5.4 Let $t \in\left(0, T_{0}\right)$. Then there is a zero-measure set $N_{t} \subset{\overline{B_{S_{0}}}}^{c}$ such that

$$
\begin{align*}
& u(x, t)=\mathfrak{R}^{(\tau)}(f)(x, t)+\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)(x, t)  \tag{5.21}\\
& \quad-\sum_{l=1}^{3} \partial x_{l} \mathfrak{V}^{\left(\tau, B_{S_{0}}\right)}\left(n_{l}^{\left(S_{0}\right)} u\right)(x, t)-\int_{\partial B_{S_{0}}}(\nabla \mathfrak{N})(x-y)\left(n^{\left(S_{0}\right)}(y) \cdot u(y, t)\right) d o_{y} \\
& \quad+\mathfrak{K}(u)(x, t)-\int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, t) \cdot U_{0}(y) d y-\int_{0}^{t} \int_{A_{R_{1}, S_{0}}} \mathfrak{G}(x, y, t-s) \cdot f(y, s) d y d s
\end{align*}
$$

for $x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$, with $f=\sum_{j=1}^{n_{0}} f^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)$, where $\mathfrak{G}=\mathfrak{G}_{R_{0}, S_{0}, \varphi_{0}}$ was introduced in Theorem 4.2, and $\mathfrak{K}(u)=\mathfrak{K}_{R_{0}, S_{0}, \varphi_{0}, B_{S_{0}}, T_{0}}(u)$ in (4.4).
Let us consider the relation between $U_{0}$ and $u(0)$.
Corollary 5.5 Suppose that $\int_{\overline{B_{S_{0}}}}{ }^{c} U_{0} \cdot \nabla \varphi=0$ for $\varphi \in C_{0}^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$. Then $U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}$ belongs to $C^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}$, and the inequality $\left|\partial_{x}^{\alpha}\left(U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}\right)(x)\right| \leq \mathfrak{C}|x|^{-2-|\alpha|}$ holds for $x \in{\overline{B_{R_{0}}}}^{c}, \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$. If $\int_{\partial B_{R}}\left(U_{0}(x)-u(x, 0)\right) \cdot x d o_{x}=0$ for some $R \in\left(S_{0}, \infty\right)$, the factor $(|x| \nu(x))^{-2-|\alpha|}$ may be replaced by $(|x| \nu(x))^{-3-|\alpha|}$.
Proof: Since $\operatorname{div}_{x} u=0$, we have $\int_{{\overline{S_{0}}}^{c}} u(t) \cdot \nabla \varphi d x=0$ for $t \in\left(0, T_{0}\right), \varphi \in C_{0}^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$. By the assumptions on $U_{0}$, the continuity of the functions $G^{(l)}:\left[0, T_{0}\right) \mapsto L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}$ and because $u \mid{\overline{B_{S_{0}}}}^{c} \times\left[0, T_{0}\right)=\sum_{j=1}^{m 0} G^{(l)}$, we may conclude that $\int_{\overline{B_{S_{0}}}}{ }^{c}\left(U_{0}-u(0)\right) \cdot \nabla \varphi d x=0$ for $\varphi$ as before. It further follows that $U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}=U_{0}-\sum_{j=1}^{m_{0}} G^{(l)}(0)$. Choose $\Pi$ as in Corollary 5.3. Then $\int_{\overline{B_{S_{0}}}} \Pi \Delta \varphi d x=0$ for $\varphi$ as before, so Theorem 3.2 yields $\Pi \in C^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$ and $\Delta \Pi=0$. As a consequence $\partial_{j} \Pi=\left(U_{0}-\sum_{j=1}^{m_{0}} G^{(l)}(0)\right)_{j}, \quad \partial_{j} \Pi \in$ $C^{\infty}\left({\overline{B_{S_{0}}}}^{c}\right)$ and $\Delta \partial_{j} \Pi=0$ for $1 \leq j \leq 3$. Due to Lemma 3.1 with $S, \widetilde{R}, R, \varphi, V$ replaced by $S_{0}, S_{0}+\left(R_{0}-S_{0}\right) / 4, R_{1}, \varphi_{0}, \partial_{j} \Pi$, respectively, where $R_{0}, R_{1}, \varphi_{0}$ were introduced at
 Since $\Pi$ is a $C^{\infty}$-function, it follows with the abbreviation $B:={\overline{B_{S_{0}+\left(R_{0}-S_{0}\right) / 8}}}^{c}$ that $\nabla \Pi \mid B \in L^{p}(B)^{3}$ for $p>3$.
At this point we may apply Theorem 3.3 to obtain $\Pi \mid B \in L^{p}(B)$ for $p>3$, possibly after subtraction of a constant from $\Pi$. Again by Lemma 3.1, this time with $S, \widetilde{R}, R, \varphi, V$ replaced by $S_{0}+\left(R_{0}-S_{0}\right) / 8, S_{0}+\left(R_{0}-S_{0}\right) / 4, R_{1}, \varphi_{0}, \Pi$, respectively, we may conclude that $\Pi\left|{\overline{B_{R_{1}}}}^{c}=(\mathfrak{N} * F)\right|{\overline{B_{R_{1}}}}^{c}$, where $F:=-2 \nabla\left(1-\varphi_{0}\right) \cdot \nabla \Pi-\Delta\left(1-\varphi_{0}\right) \Pi$. Note that $F \in C_{0}^{\infty}\left(B_{R_{1}}\right)$. By the properties of $\varphi_{0}$ and because of the equations $\Delta \Pi=0, \Delta \mathfrak{N}=0$ and $\Pi\left|{\overline{B_{R_{1}}}}^{c}=(\mathfrak{N} * F)\right|{\overline{B_{R_{1}}}}^{c}$, some integrations by parts on $B_{R_{1}} \backslash \overline{B_{S_{0}}}$ yield that $\Pi(x)=$ $\int_{\partial R_{1}}\left(\mathcal{Z}^{(1)}+\mathcal{Z}^{(2)}\right)(x, y) d o_{y}$, with $\mathcal{Z}^{(1)}(x, y):=-\mathfrak{N}(x-y) \nabla \Pi(y) \cdot R_{1}^{-1} y d o_{y}$ and $\mathcal{Z}^{(2)}(x, y):=$ $\nabla_{y} \mathfrak{N}(x-y) \cdot R_{1}^{-1} y \Pi(y)$ for $x \in B_{R_{0}}^{c}, y \in \partial B_{R_{1}}$. Since $U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}=\nabla \Pi$, we thus get $\partial_{x}^{\alpha}\left(U_{0}-u(0)\right)(x)=\int_{\partial B_{R_{1}}} \partial_{x}^{\alpha} \nabla_{x}\left(\mathcal{Z}^{(1)}+\mathcal{Z}^{(2)}\right)(x, y) d y$. On the other hand,

$$
\begin{align*}
& \left|\partial_{x}^{\alpha} \nabla_{x} \mathcal{Z}^{(1)}(x, y)\right| \leq C\left(R_{0}, R_{1}\right)|x|^{-2-|\alpha|}|\nabla \Pi(x)|,  \tag{5.22}\\
& \left|\partial_{x}^{\alpha} \nabla_{x} \mathcal{Z}^{(2)}(x, y)\right| \leq C\left(R_{0}, R_{1}\right)|x|^{-3-|\alpha|}|\Pi(x)| \quad \text { for } x \in{\overline{B_{R_{0}}}}^{c}, y \in \partial B_{R_{1}} \alpha \in \mathbb{N}_{0}^{3},|\alpha| \leq 1 .
\end{align*}
$$

Thus $\left|\partial_{x}^{\alpha}\left(U_{0}-u(0)\right)(x)\right| \leq \mathfrak{C}|x|^{-2-|\alpha|}$ for $x, \alpha$ as in (5.22). Suppose there is some $R \in$ $\left(S_{0}, \infty\right)$ such that $\int_{\partial B_{R}}\left(U_{0}(x)-u(x, 0)\right) \cdot x d o_{x}$ vanishes. Since $\operatorname{div}\left(U_{0}-u(0)\right) \mid{\overline{B_{S_{0}}}}^{c}=$
$\Delta \Pi=0$, we see that $U_{0}-u(0)$ has zero flux also on $\partial B_{R_{1}}$. Thus, referring to Lemma 4.8 with $q$, $V$ replaced by $\min \{q, \widetilde{p}\}, U_{0}-u(0) \mid B_{S_{0}+3\left(R_{0}-S_{0}\right) / 4} \backslash \overline{B_{R_{1}}}(q$ chosen in Lemma 5.1), and recalling that $\nabla \Pi=U_{0}-u(0) \mid{\overline{B_{S_{0}}}}^{c}$, we obtain that $\left|\int_{\partial B_{R_{1}}} \partial_{x}^{\alpha} \nabla_{x} \mathcal{Z}^{(1)}(x, y) d o_{y}\right| \leq$ $\mathfrak{C}|x|^{-3-|\alpha|}$ for $x, \alpha$ as in (5.22). Due to the second inequality in (5.22), the preceding estimate also holds with $\mathcal{Z}^{(2)}$ in the role of $\mathcal{Z}^{(1)}$. Thus the last claim of the corollary follows from the equation for $\partial_{x}^{\alpha}\left(U_{0}-u(0)\right)(x)$ in the line preceding (5.22).

In Section 1 we indicated that the equation $u(0)=U_{0}$ need not hold in general. This may be seen by the ensuing lemma.
Lemma 5.4 Let $\varrho, q, \widetilde{p}, u, f$ and $U_{0}$ be given as in (1.3). Suppose in addition that $\varrho>$ $3 / 2$. Then there is function $v$ having exactly the same properties as $u$ - in particular satisfying (1.3) with $U_{0}$ unchanged - , but with $v(0) \neq u(0)$.
Proof: Take $\varphi \in C_{0}^{\infty}(\Omega)$ with $\varphi \neq 0, \varphi \geq 0$. Put $\psi:=\mathfrak{N} * \varphi$; see Theorem 3.1. That latter theorem yields that $\psi \in C^{\infty}\left(\mathbb{R}^{3}\right),-\Delta \psi=\varphi, \partial^{\alpha} \psi \in L^{p}\left(\mathbb{R}^{3}\right)$ for $p \in(3, \infty)$ if $\alpha=(0,0,0)$, for $p \in(3 / 2, \infty)$ in the case $\alpha \in \mathbb{N}_{0}^{3},|\alpha|=1$, and for $p \in(1, \infty)$ if $\alpha \in \mathbb{N}_{0}^{3},|\alpha|=2$. Put $v(x, t):=u(x, t)+\nabla \psi(x)$ for $x \in \bar{\Omega}^{c}, t \in\left(0, T_{0}\right)$, and note that $\int_{\bar{\Omega}^{c}} \partial^{\alpha} \nabla \psi \cdot \partial^{\beta} \vartheta d x=0$ for $\vartheta \in C_{0, \sigma}^{\infty}\left(\bar{\Omega}^{c}\right), \alpha, \beta \in \mathbb{N}_{0}^{3}$. The lemma follows from this observation.

In order to exploit our representation formula (5.21) for the purpose of decay estimates, we need somewhat stronger assumptions on $u$. The ensuing theorem gives the details.
Theorem 5.2 Abbreviate $\mathfrak{A}:=A_{R_{1}, S_{0}} \times\left(0, T_{0}\right), f:=\sum_{j=1}^{n_{0}} f^{(j)} \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)$. In addition to the assumptions at the beginning of this section, suppose that the function $u \mid \mathfrak{A}$ belongs to $L^{\infty}\left(0, T_{0}, L^{q}\left(A_{R_{1}, S_{0}}\right)^{3}\right)$ and to $L^{\gamma_{1}}\left(0, T_{0}, L^{q}\left(A_{R_{0}, S_{0}}\right)^{3}\right)$, the function $\nabla_{x} u \mid \mathfrak{A}$ to $L^{\gamma_{2}}\left(0, T_{0}, L^{q}\left(A_{R_{1}, S_{0}}\right)^{9}\right)$, and $f \mid \mathfrak{A}$ is in the space $L^{\gamma_{3}}\left(0, T_{0}, L^{q}\left(A_{R_{1}, S_{0}}\right)^{3}\right)$, for certain parameters $\gamma_{1}, \gamma_{2}, \gamma_{3} \in[1, \infty]$, with $q$ chosen in Lemma 5.1. Then there is a zero-measure set $\mathfrak{S}_{T_{0}} \subset\left(0, T_{0}\right)$ such that

$$
\begin{align*}
& \mid \partial_{x}^{\alpha}\left[u-\mathfrak{R}^{(\tau)}\left(f \mid{\left.\left.\overline{{B_{S_{0}}}^{c}} \times\left(0, T_{0}\right)\right)-\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)\right](x, t) \mid}_{\leq}^{\leq} \mathfrak{C}\left(\left\|u\left|\mathfrak{A}\left\|_{q, \infty ; T_{0}}+\right\| u\right| \mathfrak{A}\right\|_{q, \gamma_{1} ; T_{0}}+\left\|\nabla_{x} u\left|\mathfrak{A}\left\|_{q, \gamma_{2} ; T_{0}}+\right\| f\right| \mathfrak{A}\right\|_{q, \gamma_{3} ; T_{0}}+\left\|U_{0} \mid A_{R_{1}, S_{0}}\right\|_{\widetilde{p}}\right)\right.\right.  \tag{5.23}\\
& \left.\quad\left((|x| \nu(x))^{-3 / 2-|\alpha| / 2+1 /\left(2 \min \left\{\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \gamma_{3}^{\prime}\right\}\right)}+|x|^{-\gamma-|\alpha|}\right)\right)
\end{align*}
$$

for $t \in\left(0, T_{0}\right) \backslash \mathfrak{S}_{T_{0}}, x \in{\overline{B_{R_{0}}}}^{c} \backslash N_{t}$ with some set $N_{t} \subset{\overline{B_{R_{0}}}}^{c}$ of measure zero, and for $\alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$, where $\gamma=3$ if condition (1.7) is valid, and $\gamma=2$ else. If $\gamma=3$, the last line in (5.23) may be replaced by $(|x| \nu(x))^{-3 / 2-|\alpha| / 2+1 /\left(2 \min \left\{\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \gamma_{3}^{\prime}\right\}\right)}$. The constant $\mathfrak{C}$ in (5.23) is independent of $u, f, U_{0}, x, t$ and $\alpha$.
Proof: Since $q \leq \min \left\{\varrho_{l}: 1 \leq l \leq m_{0}\right\}, u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right)=\sum_{l=1}^{m_{0}} G^{(l)}$ and $G^{(l)} \in$ $C^{0}\left(\left[0, T_{0}\right), L^{\varrho_{l}}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$ for $1 \leq l \leq m_{0}$, we have $u \mid \mathfrak{A} \in C^{0}\left(\left[0, T_{0}\right), L^{q}\left(A_{R_{1}, S_{0}}\right)^{3}\right)$. As a consequence, we get $\left\|u(t)\left|A_{R_{1}, S_{0}}\left\|_{q} \leq\right\| u\right| \mathfrak{A}\right\|_{q, \infty ; T_{0}}$ for $t \in\left[0, T_{0}\right)$. Thus inequality (5.23) follows from Corollary 5.4, 4.1, (4.2), (4.3), Corollary 4.2, Lemma 4.7 and 4.8. Note that since $u(t) \mid \Omega_{S_{0}} \in W^{1, q}\left(\Omega_{S_{0}}\right)^{3}$ for $t \in\left(0, T_{0}\right)$ and $\operatorname{div}_{x}=0$, condition (1.7) holds if and only $\int_{\partial B_{S_{0}}} u(t) \cdot n^{(\Omega)} d o_{x}=0$ for such $t$. Obviously $|x| \geq C(R) \nu(x)$ for $x \in B_{R}^{c}, R \in(0, \infty)$, so in the case $\gamma=3$, inequality (5.23) holds without the term $|x|^{-\gamma-|\alpha|}$ in the last line.

We note that if $T_{0}<\infty$, the parameters $\gamma_{1}, \gamma_{2}$ and $\gamma_{3}$ may be taken equal to 1 . Thus, if
$\gamma=3$, we obtain the optimal decay bound $-3 / 2-|\alpha| / 2$.

## 6 Some applications

In this section we use Theorem 5.2 in order to determine the spatial asymptotics of $L^{2}$ weak solutions to the Oseen system (1.1) under Dirichlet boundary conditions. This type of solutions are studied in [13], so the results in this section may be compared with what is obtained in this reference. Let us first consider the case $f=0, U_{0}=0$ (boundary-driven flow).
Theorem 6.1 For $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{4}\right)^{3}$ with $\varphi \mid \mathbb{R}^{3} \times(-\infty, 0]=0$, define the fractional derivative $\partial_{t}^{1 / 2} \varphi$ by $\left(\partial_{t}^{1 / 2} \varphi\right)(x, t):=\pi^{-1 / 2} \partial_{t}\left(\int_{0}^{t}(t-s)^{-1 / 2} \varphi(x, s) d s\right)$ for $t \in(0, \infty), x \in \mathbb{R}^{3}$.
Put $S_{T_{0}}:=\partial \Omega \times\left(0, T_{0}\right)$. Let $b \in L^{2}\left(S_{T_{0}}\right)^{3}$ with $\int_{\partial \Omega} b(t) \cdot n^{(\Omega)}$ dox $=0$ for $t \in\left(0, T_{0}\right)$. Suppose there is a sequence $\left(\varphi_{n}\right)$ in $C_{0}^{\infty}\left(\mathbb{R}^{4}\right)^{3}$ such that $\varphi_{n} \mid \mathbb{R}^{3} \times(-\infty, 0]=0$ for $n \in \mathbb{N},\left\|b-\varphi_{n}\right\|_{2} \rightarrow$ $0(n \rightarrow \infty)$ and $\int_{0}^{T_{0}}\left\|\left(\varphi_{n}-\varphi_{m}\right)(t)\left|\partial \Omega\left\|_{H^{1}(\partial \Omega)^{3}}^{2} d t \rightarrow 0, \int_{0}^{T_{0}}\right\| \partial_{t}^{1 / 2}\left(\varphi_{n}-\varphi_{m}\right)(t)\right| \partial \Omega\right\|_{2}^{2} d t \rightarrow$ $0, \int_{0}^{T_{0}}\left\|\partial_{t}\left(\varphi_{n}-\varphi_{m}\right)(t) \cdot n^{(\Omega)}\right\|_{H^{1}(\partial \Omega)^{\prime}}^{2} d t \rightarrow 0$ for $m, n \rightarrow \infty$. Here the space $H^{1}(\partial \Omega)$ is to be defined in the usual way, and the symbol $\left\|\|_{H^{1}(\partial \Omega)^{3}}\right.$ denotes the usual norm of $H^{1}(\partial \Omega)^{3}$ with respect to some local coordinates (see [28, Section III.6.7] for example). The symbol $\left\|\|_{H^{1}(\partial \Omega)^{\prime}}\right.$ stands for the canonical norm of the dual space of $H^{1}(\partial \Omega)$.
Then there is a unique function $u \in L_{l o c}^{2}\left(\left[0, T_{0}\right), H^{1}\left(\bar{\Omega}^{c}\right)^{3}\right)$ such that div${ }_{x} u(t)=0$ and $u(t) \mid \partial \Omega=b(t)$ for $t \in\left(0, T_{0}\right)$, and such that equation (1.3) is satisfied with $f=0$ and $U_{0}=0$. Moreover $u \in L^{\infty}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{3}\right)$ and $\nabla_{x} u \in L^{2}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{9}\right)$. In addition $\left|\partial_{x}^{\alpha} u(x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$ for $t, x, \alpha$ as in (5.23).
Proof: The uniqueness statement follows as in the Stokes case; see [11, Theorem 3.7] and its proof. For the existence result we refer to [13, Theorem 2.26], which yields a function $u \in L_{l o c}^{2}\left(\left[0, T_{0}\right), H^{1}\left(\bar{\Omega}^{c}\right)^{3}\right)$ such that $\operatorname{div} u(t)=0, u(t) \mid \partial \Omega=b(t)$ for $t \in$ $\left(0, T_{0}\right)$ and equation (1.3) is valid with $f=0, U_{0}=0$. According to [13, (2.13)] and [8, Theorem 2.3], this function $u$ additionally belongs to $L^{\infty}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{3}\right)$, and $\nabla_{x} u \in$ $L^{2}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{9}\right)$. Reference [13] reduces its existence result to a solution theory for a boundary integral equation related to the time-dependent Stokes system. The key point of this theory, which is due to Shen [44], states that for any $b \in L^{2}\left(S_{T_{0}}\right)^{3}$ satisfying the conditions in Theorem 6.1, the integral equation in question admits a unique solution in the space $\left\{\varphi \in L^{2}\left(S_{T_{0}}\right)^{3}: \int_{\partial \Omega} \varphi(t) \cdot n^{(\Omega)} d o_{x}=0\right.$ for a. e. $\left.t \in\left(0, T_{0}\right)\right\}([44$, p. 365]). This explains the choice of assumptions imposed on $b$ in Theorem 6.1.
We still have to consider the estimate stated at the end of Theorem 6.1. Theorem 2.2 implies that $u \in L^{2}\left(0, T_{0}, L^{6}(\bar{\Omega})^{3}\right)$. In addition, by $[13,(2.17)]$ and Lemma 4.6 we have $u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right) \in C^{0}\left(\left[0, T_{0}\right), L^{4}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}\right)$. Therefore the estimate in question follows from Theorem 5.2 with $U_{0}=0, n_{0}=1, p_{1}=2, f^{(1)}=0, m_{0}=1, \varrho_{1}=4, G^{(1)}=$ $u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right), q_{0}=q_{1}=\widetilde{p}=\gamma_{j}=2$ for $j \in\{1,2,3\}$.

In the situation of Theorem 6.1, and for $t, x, \alpha$ as in (5.23), reference [13] yields the estimate $\left|\partial_{x}^{\alpha} u(x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$; see [13, (2.17), Lemma 3.2].

As another application of Theorem 5.2, we consider standard $L^{2}$-weak solutions to (1.1) with homogeneous Dirichlet boundary conditions, under assumptions that are only slightly more restrictive than the conditions needed for the usual existence result. In fact, we suppose $f \in L^{2}\left(0, T_{0}, L_{\sigma}^{q}\left(\bar{\Omega}^{c}\right)\right)$ for some $q \in[6 / 5,2]$ instead of $f \in L^{2}\left(0, T_{0}, \mathcal{V}^{\prime}\right)$, with $\mathcal{V}^{\prime}$ defined below. Here are the details.
Theorem 6.2 Put $\mathcal{V}:=\left\{V \in W_{0}^{1,2}\left(\bar{\Omega}^{c}\right)^{3}: \operatorname{div} V=0\right\}$, equip $\mathcal{V}$ with the norm of $W^{1,2}\left(\bar{\Omega}^{c}\right)^{3}$, and let $\mathcal{V}^{\prime}$ denote the dual space to $\mathcal{V}$. Let $U_{0} \in L_{\sigma}^{2}\left(\bar{\Omega}^{c}\right)$ and $f \in L^{2}\left(0, T_{0}, \mathcal{V}^{\prime}\right)$. Then there is a unique function $u:\left(0, T_{0}\right) \mapsto \mathcal{V}$ such that $u \in L^{\infty}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{3}\right), \nabla u \in$ $L^{2}\left(0, T_{0}, L^{2}\left(\bar{\Omega}^{c}\right)^{9}\right)$, and equation (1.3) is fulfilled with the modification that the term $f(t) \cdot \vartheta$ is dropped, and instead the term $f(t)(\vartheta)$ is added outside the integral over $\bar{\Omega}^{c}$. This function $u$ is in $C^{0}\left(\left[0, T_{0}\right), L^{2}\left(\bar{\Omega}^{c}\right)^{3}\right)$, and $u(0)=U_{0}$.
If $f \in L^{2}\left(0, T_{0}, L^{r}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for some $r \in[6 / 5,2]$, then $f \in L^{2}\left(0, T_{0}, \mathcal{V}^{\prime}\right)$, and we have for $x, t, \alpha$ as in (5.23) that

$$
\begin{equation*}
\left|\partial_{x}^{\alpha}\left[u-\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)-\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)\right](x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-5 / 4-|\alpha| / 2} \tag{6.1}
\end{equation*}
$$

Proof: As in the proof of the preceding theorem, we refer to [13, Theorem 3.7] and its proof as regards uniqueness, which follows as in the Stokes case. Concerning existence, the argument is also the same as in the Stokes case. We refer to [50, p. 171-176 and p. 180], in particular [50, p. 175, (1.65)]. The equation $\int_{\bar{\Omega}^{c}} \partial_{1} V \cdot V d x=0$ valid for $V \in W_{0}^{1,2}\left(\bar{\Omega}^{c}\right)^{3}$ is the reason why the Oseen term does not generate a major problem. Note that according to [29, Theorem III.4.2], the space $V$ in [50, Section3.1] (closure of $C_{0, \sigma}^{\infty}\left(\bar{\Omega}^{c}\right)$ with respect to the norm of $\left.W^{1,2}\left(\bar{\Omega}^{c}\right)^{3}\right)$ coincides with the space $\mathcal{V}$ defined in the theorem. In order to show that $u \in C^{0}\left(\left[0, T_{0}\right), L^{2}\left(\bar{\Omega}^{c}\right)^{3}\right)$, we consider $v$ as a weak solution of the time-dependent Stokes system, in an analogous way as stated in (1.3) for the Oseen system, but now with the right-hand side $f-\tau \partial x_{1} u \in L^{2}\left(0, T_{0}, \mathcal{V}^{\prime}\right)$. Then [50, Theorem 3.1.1] yields continuity of $u$ on $\left[0, T_{0}\right)$ with values in $L^{2}\left(\bar{\Omega}^{c}\right)^{3}$, and the equation $u(0)=U_{0}$. Suppose that $f \in L^{2}\left(0, T_{0}, L^{r}\left(\bar{\Omega}^{c}\right)^{3}\right)$ for some $r \in[6 / 5,2]$. Since Theorem 2.2 with $\kappa=q=2$ implies $u \in L^{2}\left(0, T_{0}, L^{6}\left(\bar{\Omega}^{c}\right)^{3}\right)$, we see that all the assumptions imposed at the beginning of Section 5 and in Theorem 5.2 are satisfied if we take $n_{0}=1, p_{1}=$ $r, f^{(1)}=f, m_{0}=1, \varrho_{1}=2, G^{(1)}=u \mid{\overline{B_{S_{0}}}}^{c} \times\left(0, T_{0}\right), q_{0}=q_{1}=\widetilde{p}=\gamma_{j}=2$ for $1 \leq j \leq 3, q=\min \{r, 5 / 4\}$. Therefore inequality (6.1) follows from (5.23).

In order to compare the preceding theorem with the results in [13], we indicate that $\mathfrak{I}^{(\tau)}\left(U_{0}\right)(x, t)=\int_{\bar{\Omega}^{c}} \mathfrak{H}\left(x-y-\tau t e_{1}, t\right) U_{0}(y) d y$ for $U_{0} \in L_{\sigma}^{2}\left(\bar{\Omega}^{c}\right), x \in \mathbb{R}^{3}, t \in(0, \infty)$ ([17, (3.9)]). If this equation is taken into account, the theory in [13] ([13, Theorem 2.26, Corollary 2.28, inequality (3.7)]) yields the estimate $\left|\partial_{x}^{\alpha}\left[u-\mathfrak{R}^{(\tau)}(f)-\mathfrak{I}^{(\tau)}\left(U_{0}\right)\right](x, t)\right| \leq$ $\mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$ for $t, x, \alpha$ as in (5.23), under conditions on $f$ and $U_{0}$ that are more restrictive than those in Theorem 6.2. The preceding estimate is equivalent to the inequality $\left|\partial_{x}^{\alpha}\left[u-\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)-\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)\right](x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$, again for $t, x, \alpha$ as in (5.23). This equivalence follows from the second estimate in Theorem 4.3 and because $\left|\partial_{x}^{\alpha} \mathfrak{I}^{(\tau)}\left(U_{0} \mid \Omega_{S_{0}}\right)(x, t)\right| \leq \mathfrak{C}(|x| \nu(x))^{-3 / 2-|\alpha| / 2}$ for $x \in{\overline{B_{R_{0}}}}^{c}, t \in(0, \infty), \alpha \in \mathbb{N}_{0}^{3}$ with $|\alpha| \leq 1$ ([17, Lemma 4.1]). So, compared to the theory in [13], Theorem 6.2 yields stronger convergence for $|x| \rightarrow \infty$ under weaker assumptions on $U_{0}$ and $f$.

As for $\partial_{x}^{\alpha} \mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)(x, t)$, Theorem 4.3 improves the decay bound in [13, Theorem 3.1] - even under slightly more general assumptions on $f$ - to almost the same level as that of $\partial_{x}^{\alpha}\left[u-\mathfrak{R}^{(\tau)}\left(f \mid{\overline{B_{S_{0}}}}^{c} \times(0, \infty)\right)-\mathfrak{I}^{(\tau)}\left(U_{0} \mid{\overline{B_{S_{0}}}}^{c}\right)\right](x, t)$ in the preceding theorem.
If $U_{0} \in L_{\sigma}^{2}\left(\bar{\Omega}^{c}\right)$, the function $\mathfrak{I}^{(\tau)}\left(U_{0}\right)$ maps continuously from $[0, \infty)$ into $L^{2}\left(\bar{\Omega}^{c}\right)^{3}$, with $\mathfrak{I}^{(\tau)}\left(U_{0}\right)(0)=U_{0}$ ([17, Corollary 3.6]). Therefore it cannot be expected that the term $\left|\partial_{x}^{\alpha} \mathfrak{I}^{(\tau)}\left(U_{0}\right)(x, t)\right|$ converges more rapidly for $|x| \rightarrow \infty$ than $\left|\partial^{\alpha} U_{0}(x)\right|$ does, if the convergence of the former term is to be uniform with respect to $t$. In [13] it is assumed that $U_{0} \mid{\overline{B_{S_{0}}}}^{c} \in W_{l o c}^{1,1}\left({\overline{B_{S_{0}}}}^{c}\right)^{3}$ and there is $\kappa_{0} \in(0,1]$ with $\left|\partial_{x}^{\alpha} U_{0}(x)\right|=O\left([|x| \nu(x)]^{-1-|\alpha| / 2-\kappa_{0}}\right)$ for $|x| \rightarrow \infty$. This allows to apply [12, Theorem 1.1], which yields that $\left|\partial_{x}^{\alpha} \mathfrak{J}^{(\tau)}\left(U_{0}\right)(x, t)\right| \leq$ $\mathfrak{C}(|x| \nu(x))^{-1-|\alpha| / 2}$ for $x \in{\overline{B_{R_{0}}}}^{c}, t \in(0, \infty), \alpha \in \mathbb{N}_{0}^{3},|\alpha| \leq 1$. In Theorem 4.4 this result is adapted to our situation: If $\left|\partial_{x}^{\alpha} U_{0}(x)\right|=O\left(|x|^{-3 / 2-\kappa_{0}} \nu(x)^{-1-|\alpha| / 2-\kappa_{0}}\right)$ for $|x| \rightarrow \infty$, then the term $\left|\partial_{x}^{\alpha} \mathfrak{I}^{(\tau)}\left(U_{0}\right)(x, t)\right|$ is bounded by $\mathfrak{C}(|x| \nu(x))^{-5 / 4-|\alpha| / 2}$ for $t, x, \alpha$ as before. It seems that in the assumptions on $U_{0}$, the factor $|x|^{-3 / 2-\kappa_{0}}$ cannot be replaced by $|x|^{-5 / 4-\kappa_{0}}$, and $\kappa_{0}$ cannot be taken as zero.

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