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L^q -weak solutions to the time-dependent Oseen system: decay estimates.

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Abstract

This article deals with L^q -weak solutions to the 3D time-dependent Oseen system. This type of solution is defined in terms of the velocity only. It is shown that the velocity may be represented by a sum of integrals none of which involves the pressure and without a surface integral of the spatial gradient of the velocity. On the basis of this representation formula, an estimate of the spatial decay of the velocity and its spatial gradient is derived. No boundary conditions need to be imposed for these results.

AMS subject classifications. 35Q30, 65N30, 76D05.

Key words. Oseen system, weak solutions, integral representation, spatial decay.

1 Introduction

In this work, we deal with the time-dependent Oseen system

$$u_t - \Delta_x u + \tau \partial_{x_1} u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, T_0), \quad (1.1)$$

with the exterior domain $\overline{\Omega}^c$ defined by $\overline{\Omega}^c := \mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is an open, bounded set in \mathbb{R}^3 with Lipschitz boundary and connected complement. System (1.1) arises as a linearization of the time-dependent Navier-Stokes system with Oseen term,

$$u_t - \Delta_x u + \tau \partial_{x_1} u + (u \cdot \nabla_x) u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, T_0). \quad (1.2)$$

The latter system is the usual model of the flow of a viscous incompressible fluid around a rigid body moving with constant velocity and without rotation. The parameter $\tau \in (0, \infty)$ corresponds to the Reynolds number of the fluid, and the function $f : \overline{\Omega}^c \times (0, T_0) \mapsto \mathbb{R}^3$ represents a volume force acting on the fluid. The unknowns in (1.1) are the velocity $u : \overline{\Omega}^c \times (0, T_0) \mapsto \mathbb{R}^3$ and the pressure $\pi : \overline{\Omega}^c \times (0, T_0) \mapsto \mathbb{R}$, hence $T_0 \in (0, \infty]$ is the life-span of the solution. We impose the initial condition $u(0) = U_0$ in $\overline{\Omega}^c$, where the initial velocity $U_0 : \overline{\Omega}^c \mapsto \mathbb{R}^3$ is given as well.

It was shown in [19] (see [19, Theorem 6.1] and the remark following it) that if u is the velocity part of a regular solution to (1.1) as specified in [19, Corollary 5.2 and Theorem 6.1], and if $|f(x, t)|$ and $|u(x, 0)|$ decrease sufficiently fast for $|x| \rightarrow \infty$, then the estimate

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} \left[\left(|x| \nu(x) \right)^{-(3+|\alpha|)/2+1/(2 \max\{\gamma'_1, \gamma'_2, \gamma'_3\})} + |x|^{-\gamma-|\alpha|} \right] \quad (1.3)$$

holds for $x \in B_{R_0}^c := \mathbb{R}^3 \setminus B_{R_0}$, $t \in (0, T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 1$. The parameter R_0 is some fixed positive real, sufficiently large so that $\overline{\Omega} \subset B_{R_0}$. The function $\nu : \mathbb{R}^3 \mapsto [1, \infty)$ is defined by $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$. Its presence in (1.3) should

be interpreted as a mathematical manifestation of the wake extending downstream behind the rigid body. The condition $|\alpha| \leq 1$ expresses the fact that inequality (1.3) yields an upper bound for the velocity u itself as well as for its spatial gradient $\nabla_x u$. The number γ on the right-hand side of (1.3) equals 3 if the zero flux condition

$$\int_{\partial\Omega} u(t) \cdot n^{(\Omega)} dx = 0 \quad (t \in (0, T_0)) \quad (1.4)$$

is fulfilled, with $n^{(\Omega)}$ denoting the outward unit normal to Ω . Otherwise γ takes the value 2. The parameters $\gamma_1, \gamma_2, \gamma_3 \in [0, \infty]$, $q \in (1, \infty)$ in (1.3) are introduced via the assumptions $u|_{Z_{R_0, T_0}} \in L^{\gamma_1}(0, T_0, L^q(\Omega_{R_0})^3)$, $\nabla_x u|_{Z_{R_0, T_0}} \in L^{\gamma_2}(0, T_0, L^q(\Omega_{R_0})^3)$ and $f|_{Z_{R_0, T_0}} \in L^{\gamma_3}(0, T_0, L^q(\Omega_{R_0})^3)$, where $\Omega_{R_0} := B_{R_0} \setminus \bar{\Omega}$ and $Z_{R_0, T_0} := \Omega_{R_0} \times (0, \infty)$. The appearance of these parameters γ_j in (1.3) means that the spatial decay of u and $\nabla_x u$ depends on L^p -integrability in time of $u|_{Z_{R_0, T_0}}$, $\nabla_x u|_{Z_{R_0, T_0}}$ and $f|_{Z_{R_0, T_0}}$. Another important feature of the theory in [19] is that no boundary conditions are imposed on u and π .

The work at hand aims to extend the decay estimate (1.3) to L^q -weak solutions of (1.1). Solutions of this type verify a weak form (equation (5.1)) of (1.1) involving only the velocity u , which is supposed to be continuous as a function of $t \in [0, T_0]$ with values in $L^{q_0}(\bar{\Omega}^c)^3$, and satisfy the relation $\nabla_x u \in L^1_{loc}([0, T_0], L^{q_1}(\bar{\Omega}^c))$, for some numbers $q_0, q_1 \in (1, \infty)$. In addition, u and f are required to fulfill the conditions involving the parameters γ_j as stated above. Actually our assumptions are somewhat more general (see the beginning of Section 5 and Theorem 5.2), but the preceding conditions correspond to what we essentially have in mind when we use the term “ L^q -weak solutions”. Note that in particular nothing is supposed on the pressure. This point is the main difficulty of the theory presented in the work at hand. The greatest part of our proof of (1.3) consists in establishing the representation formula (5.24) for L^q -weak solutions (Section 5). This formula should be considered as one of the main results of this work. In particular it is an essential element of the theory developed in [20] on the nonlinear problem (1.6); see the remarks further below. Inequality (1.3), for its part, is the starting point of the theory in [21], where we study the spatial asymptotics of mild solutions to (1.1) under homogeneous Dirichlet boundary conditions.

Let us compare the results of the work at hand with what is already available in literature. Spatial decay of L^2 -weak solutions to (1.1) with Dirichlet boundary conditions was studied in [11] and [14], where these solutions were shown to satisfy the estimate

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1-|\alpha|/2} \quad (1.5)$$

for x, t, α as in (1.3), if f and U_0 decrease sufficiently fast. Inequality (1.3) applied in the situation considered in [11] and [14] yields the stronger decay bound $(|x| \nu(x))^{-5/4-|\alpha|/2}$ even though we do no longer impose a specific boundary condition. Again this result is valid under the usual condition that the data f and U_0 decrease in an appropriate way. These points are discussed in more detail in Section 6. In [15], we considered the nonlinear stability problem

$$\begin{aligned} \partial_t u - \Delta_x u + \tau \partial_{x_1} u + \tau (U \cdot \nabla_x) u + \tau (u \cdot \nabla) U + \tau (u \cdot \nabla_x) u + \nabla_x \pi &= f, \\ \operatorname{div}_x u &= 0 \quad \text{in } \bar{\Omega}^c \times (0, T_0), \end{aligned} \quad (1.6)$$

where $U : \overline{\Omega}^c \mapsto \mathbb{R}^3$ is the velocity part of a solution to the stationary Navier-Stokes system, and is considered as given. It was shown in that reference that inequality (1.5) remains valid for L^2 -strong solutions to (1.6) satisfying Dirichlet boundary conditions, again if the asymptotic behaviour of f and U_0 allows this rate of decay. In the successor paper [20] mentioned above, we will show that (1.3) with $p = 2$ is valid for these solution, so that the decay bound $\mathfrak{C}(|x|\nu(x))^{-1-|\alpha|/2}$ from [15] may be replaced by $\mathfrak{C}(|x|\nu(x))^{-5/4-|\alpha|/2}$, under the usual caveat on f and U_0 . No specific boundary conditions will be imposed in [20].

A remark is in order with respect to our proof of (1.3) for L^q -weak solutions. As already indicated, our main effort consists in showing that the integral representation (5.24) of the velocity is valid. This representation is a slightly modified version of equation [19, (5.7)] proved in [19] for regular solutions. Since all the integrals appearing in (5.24) were already estimated in [19], we obtain (1.3) almost immediately once this representation is available (Theorem 5.2). A major tool in the proof of (5.24) is Friedrich's mollifier for functions with values in Banach spaces, smoothing weak solutions with respect to the time variable. We will consider these smoothed solutions as weak solutions of the stationary Oseen system, with the time derivative subsumed into the right-hand side. In this way we will be able to use the regularity theory of this latter system in order to construct an associated pressure. Once this result is available, the mollified version of the L^q -weak solution we started out with will turn out to be a sufficiently regular solution of (1.1) so that we may apply formula [19, (5.7)] (Theorem 5.1). By letting certain parameters tend to zero in this formula, we will then obtain that (5.24) holds for the original weak solution; see Lemma 5.3 and Corollary 5.1. The mollifying procedure requires that $T_0 = \infty$. However, once (5.24) has been proved in the case $T_0 = \infty$, it is not difficult to handle the case $T_0 < \infty$ as well (Corollary 5.2).

Let us mention some references more distantly related to the work at hand. Knightly [33] considered even the case that the velocity of the rigid body changes with time. However, his results are valid only under various smallness assumptions. Mizumachi [37] proved (1.5) with $\alpha = 0$, $f = 0$ for a certain class of solutions to the nonlinear system (1.2). Takahashi [43] deals with spatial decay of solutions to the Navier-Stokes system with Oseen term in the case $\Omega = \emptyset$ under a smallness condition. In [3], [4], solutions to (1.1) and (1.2) are estimated in weighted L^p -norms, with the weights adapted to the wake in the flow field downstream to the rigid body. Reference [18] by the present author combines decay estimates in time and in space, for solutions of (1.1) and (1.6), as a continuation of [14] (Oseen system (1.1)) and [15] (problem (1.6)), under the same assumptions and with the same methods as in these articles. Various technical aspects of the theory in [11], [14], [15] and [18] are dealt with in predecessor papers [7] – [10], [12], [13]. Questions of existence, regularity and stability related to (1.1), (1.2) or (1.6) are addressed in [25], [26], [27], [30], [31], [34], [35], [36], [40], [42].

2 Notation. Some auxiliary results.

The parameters $T_0 \in (0, \infty]$ and $\tau \in (0, \infty)$ introduced in Section 1 are kept fixed throughout, as is the open, bounded set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary.

The symbol $|\cdot|$ denotes the Euclidean norm of \mathbb{R}^n for any $n \in \mathbb{N}$, as well as the length $\alpha_1 + \alpha_2 + \alpha_3$ of a multi-index $\alpha \in \mathbb{N}_0^3$. For $R \in (0, \infty)$, $x \in \mathbb{R}^3$, put $B_R(x) := \{y \in \mathbb{R}^3 : |x - y| < R\}$. In the case $x = 0$, we write B_R instead of $B_R(0)$.

Recall that in Section 1, we introduced the function $\nu : \mathbb{R}^3 \mapsto [1, \infty)$ by setting $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$.

We fix numbers $S_0, R_0 \in (0, \infty)$ with $S_0 < R_0$ and $\bar{\Omega} \subset B_{S_0}$, as well as a function $\varphi_0 \in C_0^\infty(B_{(R_0+S_0)/2})$ with $0 \leq \varphi_0 \leq 1$ and $\varphi_0|_{B_{S_0+(R_0-S_0)/4}} = 1$. We put $R_1 := (R_0 + S_0)/2$.

For $n \in \mathbb{N}$, $I \subset \mathbb{R}^n$, let χ_I stand for the characteristic function of I in \mathbb{R}^n . If $A \subset \mathbb{R}^3$, we denote by A^c the complement $\mathbb{R}^3 \setminus A$ of A in \mathbb{R}^3 . Put $e_l := (\delta_{jl})_{1 \leq j \leq 3}$ for $1 \leq l \leq 3$ (unit vector in \mathbb{R}^3). If A is an open bounded set in \mathbb{R}^3 with Lipschitz boundary, we write $n^{(A)}$ for the outward unit normal to A . If $R, S \in (0, \infty)$ with $S < R$, we write $A_{R,S}$ for the annular domain $B_R \setminus \bar{B}_S$.

Let $p \in [1, \infty)$ and $n \in \mathbb{N}$. For any open set $A \subset \mathbb{R}^3$, the norm of the Lebesgue space $L^p(A)$ is denoted by $\|\cdot\|_p$, and the usual norm of the Sobolev space $W^{m,p}(A)$ of order m and exponent p is designated by $\|\cdot\|_{m,p}$. Again for an open set $A \subset \mathbb{R}^3$, we define $C_{0,\sigma}^\infty(A) := \{V \in C_0^\infty(A)^3 : \operatorname{div} V = 0\}$, and write $L_{loc}^p(A)$ and $W_{loc}^{m,p}(A)$ for the set of all functions V from A into \mathbb{R} such that $V|_K \in L^p(K)$ and $V|_K \in W^{1,p}(K)$, respectively, for any open, bounded set $K \subset \mathbb{R}^3$ with $\bar{K} \subset A$. We put $\nabla V := (\partial_k V_j)_{1 \leq j, k \leq 3}$ for $V \in W_{loc}^{1,1}(A)^3$.

Let \mathcal{V} a normed space, and let the norm of \mathcal{V} be denoted by $\|\cdot\|$. Take $n \in \mathbb{N}$. Then we will use the same notation $\|\cdot\|$ for the norm on \mathcal{V}^n defined by $\|(f_1, \dots, f_n)\| := (\sum_{j=1}^n \|f_j\|^2)^{1/2}$ for $(f_1, \dots, f_n) \in \mathcal{V}^n$. The space $\mathcal{V}^{3 \times 3}$, as concerns its norm, is identified with \mathcal{V}^9 . If $p \in (1, \infty)$, $n \in \{1, 3\}$ and $A \subset \mathbb{R}^3$ open, the dual space of $W_0^{1,p'}(A)^n$ will be denoted by $W_0^{-1,p}(A)^n$ (although in the case $n = 3$ this notation is not coherent with the usual custom of letting \mathfrak{A}^n stand for the Cartesian product of a given set \mathfrak{A}).

We additionally introduce some weighted Sobolev spaces. To this end, for $x \in \mathbb{R}^3$ we put

$$\begin{aligned} \omega_p^{(j)}(x) &:= (1 + |x|)^{-j} \text{ for } j \in \{1, 2\}, \quad p \in (1, \infty) \setminus \{3/2, 3\}, \\ \omega_{3/2}^{(1)}(x) &:= (1 + |x|)^{-1}, \quad \omega_{3/2}^{(2)}(x) := (1 + |x|)^{-2} (\ln(2 + |x|))^{-1}, \\ \omega_3^{(j)}(x) &:= (1 + |x|)^{-j} (\ln(2 + |x|))^{-1} \text{ for } j \in \{1, 2\}. \end{aligned}$$

Then we set

$$\begin{aligned} W_2^{2,p}(\mathbb{R}^3) &:= \{V \in W_{loc}^{2,1}(\mathbb{R}^3) : \omega_p^{(2)} V, \omega_p^{(1)} \partial_l V, \partial_l \partial_k V \in L^p(\mathbb{R}^3) \text{ for } 1 \leq k, l \leq 3\}, \\ W_1^{1,p}(\mathbb{R}^3) &:= \{V \in W_{loc}^{1,1}(\mathbb{R}^3) : \omega_p^{(1)} V, \partial_l V \in L^p(\mathbb{R}^3) \text{ for } 1 \leq l \leq 3\} \quad (p \in (1, \infty)). \end{aligned}$$

We will not work with a norm of $W_2^{2,p}(\mathbb{R}^3)$. However, the norm of $W_1^{1,p}(\mathbb{R}^3)$ defined by $\|V\|_{W_1^{1,p}(\mathbb{R}^3)} := (\|\omega_p^{(1)} V\|_p^p + \|\nabla V\|_p^p)^{1/p}$ will be relevant.

Let $p \in [1, \infty]$, \mathfrak{B} a Banach space and $J \subset \mathbb{R}$ an interval. Then the norm of the space $L^p(J, \mathfrak{B})$ is denoted by $\|\cdot\|_{L^p(J, \mathfrak{B})}$. Let $a, b \in \mathbb{R} \cup \{\infty\}$ with $a < b$, take $m \in \mathbb{N}$ and $q \in [1, \infty)$. Then we write $L^p(a, b, \mathfrak{B})$ and $W^{1,q}(a, b, \mathfrak{B})$ instead of $L^p((a, b), \mathfrak{B})$ and $W^{1,q}((a, b), \mathfrak{B})$, respectively. We use the expression $L_{loc}^p([a, b], \mathfrak{B})$ for the space of all functions $v : (a, b) \mapsto$

\mathcal{B} such that $v|(a, T) \in L^p(a, T, \mathcal{B})$ for any $T \in (a, b)$. This space is to be distinguished from the space $L^p_{loc}(a, b, \mathcal{B})$, defined in the usual way. Let $T \in (0, \infty]$, $A \subset \mathbb{R}^3$ open, and $n \in \mathbb{N}$. Then we will write $\|\cdot\|_{q,p;T}$ and $\|\cdot\|_{q,p;\mathbb{R}}$ instead of $\|\cdot\|_{L^p(0,T,L^q(A)^n)}$ and $\|\cdot\|_{L^p(\mathbb{R},L^q(A)^n)}$, respectively.

If $v \in W^{1,1}(a, b, \mathcal{B})$, then, possibly after a modification on a subset of $[a, b)$ with measure zero, the function v belongs to $C^0([a, b), \mathcal{B})$ ([44, Lemma 3.1.1]). If the latter relation is already valid, we write $v \in W^{1,1}(a, b, \mathcal{B}) \cap C^0([a, b), \mathcal{B})$.

Of course, a function $v \in L^p(J, L^q(A)^n)$ may be considered also as a function on $A \times J$, although there is a minor issue with respect to measurability on $A \times J$, settled in [12, Lemma 2.1] and [19, Lemma 2.3]. We will write $v(t)(x)$ or $v(x, t)$, depending on whether we consider v as a function on J with values in $L^q(A)^n$, or as a function on $A \times J$. For an interval $J \subset \mathbb{R}$ and a function $v : J \mapsto W^{1,1}_{loc}(A)^3$, the notation $\nabla_x v$ stands for the gradient of v with respect to $x \in A$, in the sense that

$$\nabla_x v : J \mapsto L^1_{loc}(A)^{3 \times 3}, \quad \nabla_x v(t)(x) := \left(\partial x_k(v_j(t))(x) \right)_{1 \leq j, k \leq 3} \text{ for } t \in J, x \in A$$

(spatial gradient of v). Similar conventions are to be valid with respect to the expressions $\Delta_x v$, $\operatorname{div}_x v$ and $\partial x_j v$.

Concerning Bochner integrals, if $J \subset \mathbb{R}$ is open, \mathcal{B} a Banach space and $w : J \mapsto \mathcal{B}$ an integrable function, it is convenient sometimes to write $\mathcal{B} - \int_J w(t) dt$ instead of $\int_J w(t) dt$ for the corresponding \mathcal{B} -valued Bochner integral. For the definition of the Bochner integral, we refer to [46, p. 132-133], or to [32, p. 78-80].

We define the Fourier transform \hat{f} of $f \in L^1(\mathbb{R}^3)$ by $\hat{f}(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\xi \cdot z} f(z) dz$ for $\xi \in \mathbb{R}^n$. An analogous definition is to hold for functions belonging to $L^2(\mathbb{R}^3)$.

We write C for numerical constants and $C(\gamma_1, \dots, \gamma_n)$ for constants depending exclusively on parameters $\gamma_1, \dots, \gamma_n \in [0, \infty)$ for some $n \in \mathbb{N}$. However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol \mathfrak{C} for constants whose dependence on parameters must be traced from context. Sometimes we write $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ in order to indicate that the constants in question is influenced by the quantities $\gamma_1, \dots, \gamma_n$. But in such cases, this constant depends on other parameters as well.

The following simple version of Young's inequality for integrals will be used frequently. Stated her for the convenience of the reader, we will refer to it as "Young's inequality".

Lemma 2.1 ([1, Corollary 2.25]) *Let $n \in \mathbb{N}$ and $q \in [1, \infty]$. Then*

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} U(x-y) V(y) dy \right|^q dx \right)^{1/q} \leq C \|U\|_1 \|V\|_q \quad \text{for } U \in L^1(\mathbb{R}^n), V \in L^q(\mathbb{R}^n).$$

We will use Minkowski's inequality for integrals, which we restate, too.

Theorem 2.1 ([1, Theorem 2.9]) *Let $m, n \in \mathbb{N}$, $p \in [1, \infty)$, $F : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ a measurable function. Then*

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(x, y)| dy \right)^p dx \right)^{1/p} \leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(x, y)|^p dx \right)^{1/p} dy.$$

The next theorem deals with solenoidal $W_0^{1,q}$ -functions.

Theorem 2.2 ([29, Theorem III.4.2, III.6.1]) *Let $n \in \mathbb{N}$, $q, r_1, \dots, r_n \in (1, \infty)$, $A \subset \mathbb{R}^3$ open, bounded and with Lipschitz boundary. Let $V \in W_0^{1,q}(\overline{A}^c)^3 \cap L^{r_j}(\overline{A}^c)^3$ for $1 \leq j \leq n$ with $\operatorname{div} V = 0$. Then there is a sequence (ϑ_n) in $C_{0,\sigma}^\infty(\overline{A}^c)$ such that $\|V - \vartheta_n\|_{1,q} \rightarrow 0$ and $\|V - \vartheta_n\|_{r_j} \rightarrow 0$ ($n \rightarrow \infty$) for $1 \leq j \leq n$.*

The ensuing theorem presents a result on L^p -integrability of functions defined in an exterior domain and possessing an L^q -integrable gradient.

Theorem 2.3 *Let $A \subset \mathbb{R}^3$ be open, bounded and with Lipschitz boundary. Let $q \in (1, 3)$ and $V \in W_{loc}^{1,1}(\overline{A}^c)$ with $\nabla V \in L^q(\overline{A}^c)^3$. Suppose there is some $\kappa \in (1, \infty)$ with $V \in L^\kappa(\overline{A}^c)$. Then $V \in L^{3q/(3-q)}(\overline{A}^c)$ and $\|V\|_{3q/(3-q)} \leq \mathfrak{C}\|V\|_q$.*

Proof: This theorem may be deduced from [29, Theorem II.6.1]; see [17, Theorem 2.4] and its proof. \square

We mention some results about Bochner's integral. Our basic tool in this context is the following theorem (compatibility of bounded operators and Bochner integrals).

Theorem 2.4 *Let B_1, B_2 be Banach spaces, $A : B_1 \mapsto B_2$ a linear and bounded operator, $n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ an open set and $f : J \mapsto B_1$ a Bochner integrable mapping. Then $A \circ f : J \mapsto B_2$ is Bochner integrable, too, and $A(B_1 - \int_J f \, dx) = B_2 - \int_J A \circ f \, dx$.*

Proof: See [46, p. 134, Corollary 2], [32, Theorem 3.7.12 and the remark on p. 84]. \square

Next we indicate a compatibility result for Bochner integrals with values in L^p -spaces.

Lemma 2.2 *Let $J \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $B \subset \mathbb{R}^n$ and $A \subset B$ open sets, $q_1, q_2 \in [1, \infty)$ and $f : J \mapsto L^{q_1}(B)^3$ a Bochner integrable mapping with $f(t)|_A \in L^{q_2}(A)^3$ for $t \in J$ and $f|_A : J \mapsto L^{q_2}(A)^3$ Bochner integrable as well. Then $(L^{q_1}(B)^3 - \int_J f(s) \, ds)|_A = L^{q_2}(A)^3 - \int_J f(s)|_A \, ds$.*

Proof: According to [19, Lemma 2.3], for a. e. $x \in A$, the integral on the left-hand side of the equation at the end of the lemma taken at the point x equals the integral on the right-hand side taken at that point. \square

Theorem 2.5 *Let B be a Banach space, $n \in \mathbb{N}$, $p \in [1, \infty)$ and $J \subset \mathbb{R}^n$ measurable. Then the set of integrable functions from J into B only taking a finite number of values ("simple functions") is dense in $L^p(J, B)$.*

Proof: See [24, Section 8.18.1 and Exercise 8.29]. On the basis of Lebesgue's theorem, the proof can be done by first approximating $f \in L^p(J, B)$ by functions with bounded support, and then by functions with bounded support and such that the set of their values is bounded with respect to the norm of B . Functions of the latter kind belong to $L^1(J, B)$ and thus, by the definition of Bochner's integral, may be approximated in $L^1(J, B)$ by simple functions, which implies approximation in $L^p(J, B)$ in this situation, due to Lebesgue's theorem. \square

Corollary 2.1 *Let B be a Banach space, A a dense subset of B , $p \in [1, \infty)$ and $J \subset \mathbb{R}^n$ open. Then the set $\{\sum_{j=1}^k \varphi_j a_j : k \in \mathbb{N}, \varphi_j \in C_0^\infty(J), a_j \in A \text{ for } 1 \leq j \leq k\}$ is dense in $L^p(J, B)$. In particular the set of continuous functions $f : J \mapsto B$ with $\operatorname{supp}(f)$ compact is dense in $L^p(J, B)$.*

Proof: Use Theorem 2.5 and the density of $C_0^\infty(J)$ in $L^p(J)$. \square

In order to define Friedrich's mollifier for functions with values in Banach spaces, we fix a function $\rho \in C_0^\infty((-1, 1))$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho(s) ds = 1$, and put $\rho_\delta(r) := \delta^{-1} \rho(\delta^{-1} r)$ for $\delta \in (0, \infty)$, $r \in \mathbb{R}$. If B is a Banach space and $f \in L_{loc}^1(\mathbb{R}, B)$, define $f_\delta(t) := B - \int_{\mathbb{R}} \rho_\delta(t-s) f(s) ds$ for $t \in \mathbb{R}$, $\delta \in (0, \infty)$.

Key properties of Friedrich's mollifier of functions with values in \mathbb{R} carry over to functions with values in Banach spaces. Properties of this type needed in the work at hand are collected in the ensuing Theorem 2.6.

Theorem 2.6 *Let B be a Banach space and $f \in L_{loc}^1(\mathbb{R}, B)$. Then $f_\delta \in C^\infty(\mathbb{R}, B)$ and $f_\delta^{(n)}(t) = \int_{\mathbb{R}} \rho_\delta^{(n)}(t-s) f(s) ds$ ($n \in \mathbb{N}$, $t \in \mathbb{R}$, $\delta \in (0, \infty)$). Moreover, if $f \in W_{loc}^{1,1}(\mathbb{R}, B)$, then $(f_\delta)' = (f')_\delta$.*

Let $g \in L^p(\mathbb{R}, B)$. Then $\|g_\delta\|_{L^p(\mathbb{R}, B)} \leq \|g\|_{L^p(\mathbb{R}, B)}$ for $\delta \in (0, \infty)$ and $\|g_\delta - g\|_{L^p(\mathbb{R}, B)} \rightarrow 0$ ($\delta \downarrow 0$).

Let $h \in C^0(\mathbb{R}, B)$ and $t \in \mathbb{R}$. Then $\|(h_\delta - h)(t)\| \rightarrow 0$ ($\delta \downarrow 0$), where $\|\cdot\|$ denotes the norm of B .

Proof: The relation $f_\delta \in C^\infty(\mathbb{R}, B)$ and the equation for $f_\delta^{(n)}(t)$ follow from the assumption $\rho_\delta \in C_0^\infty((-\delta, \delta))$. If $f \in W_{loc}^{1,1}(\mathbb{R}, B)$ and $t \in \mathbb{R}$, the equation $(f_\delta)'(t) = (f')_\delta(t)$ holds due to the above equation for $f_\delta^{(n)}(t)$ with $n = 1$, and since the function $s \mapsto \rho_\delta(t-s)$ ($s \in \mathbb{R}$) belongs to $C_0^\infty(\mathbb{R})$. The inequality $\|g_\delta\|_{L^p(\mathbb{R}, B)} \leq \|g\|_{L^p(\mathbb{R}, B)}$ for $\delta > 0$ is an immediate consequence of Young's inequality and the choice of ρ_δ . We further note that for $\epsilon \in (0, \infty)$, Corollary 2.1 yields existence of a function $g^{(\epsilon)} \in C^0(\mathbb{R}, B)$ with compact support such that $\|g - g^{(\epsilon)}\|_{L^p(\mathbb{R}, B)} \leq \epsilon/2$. With this result available, the relation $\|g_\delta - g\|_{L^p(\mathbb{R}, B)} \rightarrow 0$ ($\delta \downarrow 0$) follows by the same arguments as in the case $B = \mathbb{R}$; see [1, p. 37-38] for example. The same reference yields the last claim of the theorem. \square

3 Some results on the Poisson equation and the stationary Oseen system.

In the ensuing theorem, we state some properties of the Newton potential. The proof of this theorem is well known (Use Hardy-Littlewood-Sobolev's inequality, Calderon-Zygmund's inequality and density arguments.), although we cannot give a reference.

Theorem 3.1 *Put $\mathfrak{N}(z) := (4\pi|z|)^{-1}$ for $z \in \mathbb{R}^3 \setminus \{0\}$ (fundamental solution of the Poisson equation). Let $q \in (1, 3/2)$, $F \in L^q(\mathbb{R}^3)$. Then $\int_{\mathbb{R}^3} \mathfrak{N}(x-y) |F(y)| dy < \infty$ for a. e. $x \in \mathbb{R}^3$. Put $(\mathfrak{N} * F)(x) := \int_{\mathbb{R}^3} \mathfrak{N}(x-y) F(y) dy$ for $x \in \mathbb{R}^3$ ("Newton potential"). Then $\mathfrak{N} * F \in W_{loc}^{2,q}(\mathbb{R}^3)$, $\Delta(\mathfrak{N} * F) = -F$ and $\nabla(\mathfrak{N} * F) \in L^{(1/q-1/3)^{-1}}(\mathbb{R}^3)$. If $F \in W^{1,q}(\mathbb{R}^3)$, then $\mathfrak{N} * F \in W_{loc}^{3,q}(\mathbb{R}^3)$ and $\|\partial_k \partial_l \partial_m (\mathfrak{N} * F)\|_q + \|\partial_l \partial_m (\mathfrak{N} * F)\|_q \leq C(q) \|F\|_{1,q}$ for $1 \leq k, l, m \leq 3$.*

In the next theorem, we introduce a pressure Π associated with the velocity part U of a weak solution to the Oseen system ($\lambda = 0$) or the Oseen resolvent system ($\lambda \neq 0$) in \mathbb{R}^3 . The case $\lambda \neq 0$ is included in view of an application in [20].

Theorem 3.2 *Let $A \subset \mathbb{R}^3$ be open, $\lambda \in \mathbb{C}$, $q \in (1, \infty)$, $U \in W_{loc}^{1,q}(A)^3$ with*

$$\int_A (\nabla U \cdot \nabla \vartheta + (\tau \partial_1 U + \lambda U - F) \cdot \vartheta) dx = 0 \text{ for } \vartheta \in C_{0,\sigma}^\infty(A), \text{ div } U = 0. \quad (3.1)$$

Then there is a function $\Pi \in L_{loc}^q(A)$ with

$$\int_A (\nabla U \cdot \nabla \vartheta + (\tau \partial_1 U + \lambda U - F) \cdot \vartheta - \Pi \operatorname{div} \vartheta) dx = 0 \quad (\vartheta \in C_0^\infty(A)^3), \quad \operatorname{div} U = 0. \quad (3.2)$$

If $B \subset \mathbb{R}^3$ is open, bounded, with Lipschitz boundary, $\overline{B} \subset A$ and $\int_B \Pi = 0$, then

$$\|\Pi\|_q \leq \mathfrak{C}(\|\nabla U\|_q + |\lambda| \|U\|_q + \|F\|_q). \quad (3.3)$$

Proof: Let $B \subset \mathbb{R}^3$ be open and bounded, with $\overline{B} \subset A$. Since $F \in L_{loc}^q(A)^3$, we have $F|_B \in L^q(B)^3$, hence $F|_B \in W_0^{-1,q}(B)$ by Poincaré's inequality. Obviously $-\tau \partial_1 U|_B \in W_0^{-1,q}(B)^3$. Thus, by [27, Lemma IV.1.1], there is a function $\Pi \in L_{loc}^q(A)$ such that (3.2) holds. The last statement of the theorem including (3.3) also follows from [27, Lemma IV.1.1] and Poincaré's inequality. \square

Theorem 3.3 *Let $A \subset \mathbb{R}^3$ be open, $\lambda \in \mathbb{C}$, $q, s \in (1, \infty)$, $F \in L_{loc}^q(A)^3$, $U \in W_{loc}^{1,1}(A)^3$ with $\nabla U \in L_{loc}^s(A)^9$ such that (3.1) holds. Then $U \in W_{loc}^{2,q}(A)^3$. Suppose in addition that $\Pi \in L_{loc}^s(A)$ is such that the pair (U, Π) satisfies (3.2). Then $\Pi \in W_{loc}^{1,q}(A)$ and*

$$-\Delta U + \tau \partial_1 U + \lambda U + \nabla \Pi = F, \quad \operatorname{div} U = 0. \quad (3.4)$$

Proof: The theorem is a consequence of interior regularity of solutions to the Stokes system; see [17, Theorem 3.2] and its proof. \square

Theorem 3.4 *Let $A \subset \mathbb{R}^3$ be open, $\lambda \in \mathbb{C}$, $F \in C^\infty(A)^3$, $q \in (1, \infty)$, $U \in W_{loc}^{1,1}(A)^3$ with $\nabla U \in L_{loc}^q(A)^9$, and $\Pi \in L_{loc}^q(A)$. Suppose that (3.2) holds. Then $U \in C^\infty(A)^3$, $\Pi \in C^\infty(A)$, and (3.4) (Oseen system if $\lambda = 0$) is valid.*

Proof: [17, Corollary 3.2]. The assumption $v \in L^q(A)^3$ in that reference should read $v \in L_{loc}^q(A)^3$. \square

We will need a solution theory for the Oseen system in weighted Sobolev spaces of functions defined in the whole space \mathbb{R}^3 . This theory is useful for us because it holds in an L^q -framework with any $q \in (1, \infty)$ being admitted.

Theorem 3.5 *Let $q \in (1, \infty)$. Then for any $F \in L^q(\mathbb{R}^3)^3$, there is a pair of functions $(U(F), \Pi(F))$ such that $U(F) \in W_2^{2,q}(\mathbb{R}^3)^3$, $\Pi(F) \in W_1^{1,q}(\mathbb{R}^3)$, the pair $(U(F), \Pi(F))$ solves (3.4) with $\lambda = 0$, $A = \mathbb{R}^3$, and $\|\partial_l \partial_k U(F)\|_q + \|\Pi\|_{W_1^{1,q}(\mathbb{R}^3)} + \|\partial_1 U(F)\|_q \leq C(q, \tau) \|F\|_q$ for $1 \leq k, l \leq 3$.*

Proof: The theorem reproduces some of the statements of [2, Theorem 3.3]. \square

In [17], we proved uniqueness of the velocity part U of a weak solution to the Oseen system (equation (3.4) with $\lambda = 0$) or to the Oseen resolvent problem (equation (3.4) with $\lambda \in \mathbb{C} \setminus \{0\}$, $\Re \lambda \geq 0$) in the whole space \mathbb{R}^3 , under the assumptions that $|\lambda| \leq (\tau/2)^2$ and $U|_{B_R^c} \in \sum_{j=1}^3 L^{r_j}(B_R^c)^3$, $\nabla U \in \sum_{j=1}^3 L^{q_j}(\mathbb{R}^3)^9$ for some $r_j, q_j \in (1, \infty)$ ($j \in \{1, 2, 3\}$) and some $R \in (0, \infty)$ ([17, Theorem 5.1]). In the following theorem, we generalize this result in the sense that we admit functions U growing polynomially for $|x| \rightarrow \infty$. However, such weak solutions need not vanish, but they turn out to be polynomials. The case $\lambda \neq 0$ will not be needed in what follows, but it is included because it can be handled without additional effort.

Theorem 3.6 Let $U \in W_{loc}^{1,1}(\mathbb{R}^3)^3$ satisfy (3.1) with $A = \mathbb{R}^3$, $F = 0$ and $\lambda \in \mathbb{C}$ with $\Re \lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$. Suppose there are numbers $R \in (0, \infty)$, $r \in [0, \infty)$, $k_0, m_0 \in \mathbb{N}$, $\varrho_k, q_m \in (1, \infty)$ for $1 \leq k \leq k_0$, $1 \leq m \leq m_0$, and functions $U^{(k)} \in L_{loc}^{\varrho_k}(\mathbb{R}^3)^3$, $V^{(m)} \in L_{loc}^{q_m}(\mathbb{R}^3)^{3 \times 3}$ for k, m as before such that $U = \sum_{k=1}^{k_0} U^{(k)}$, $\partial_\mu U_\sigma = \sum_{m=1}^{m_0} V_{\mu,\sigma}^{(m)}$ for $1 \leq \mu, \sigma \leq 3$, and $\int_{B_R^c} (|U^{(k)}(x)| |x|^{-r})^{\varrho_k} dx < \infty$, $\int_{B_R^c} (|V^{(m)}(x)| |x|^{-r})^{q_m} dx < \infty$ for $1 \leq k \leq k_0$, $1 \leq m \leq m_0$. Then U is a polynomial.

Proof: The theorem follows by the standard theory of topological vector spaces ([38, Section 7.3, 7.11]), as used in the proof of [17, Theorem 5.1]. The only point that should still be checked in more detail is whether the operators $T, S : \mathcal{S}(\mathbb{R}^3)^3 \mapsto \mathbb{R}$ defined by $T(\phi) := \int_{\mathbb{R}^3} (\nabla U \cdot \nabla \phi + \tau \partial_1 U \cdot \phi) dx$ and $S(\phi) := \int_{\mathbb{R}^3} U \cdot \phi dx$ for $\phi \in \mathcal{S}(\mathbb{R}^3)^3$, respectively, are tempered distributions, where $\mathcal{S}(\mathbb{R}^3)$ denotes the set of rapidly decreasing functions in \mathbb{R}^3 , equipped in the usual way with a topology. So let us show that T is a tempered distribution. To that end, put $\bar{R} := \max\{1, R\}$, $\bar{r} := \min\{n \in \mathbb{N} : n \geq r\}$, $p_{\alpha,\beta}(\phi) := \sup\{|x^\alpha \partial^\beta \phi(x)| : x \in \mathbb{R}^3\}$, $G(\phi) := \sup\{|x|^{\bar{r}+3} \sum_{\gamma \in \mathbb{N}_0^3, |\gamma| \leq 1} |\partial^\gamma \phi(x)| : x \in B_{\bar{R}}^c\}$ for $\phi \in \mathcal{S}(\mathbb{R}^3)^3$, $\alpha, \beta \in \mathbb{N}_0^3$. Then we find for $\phi \in \mathcal{S}(\mathbb{R}^3)^3$ that

$$G(\phi) \leq \mathfrak{C} \sup \left\{ \sum_{\gamma \in \mathbb{N}_0^3, |\gamma| \leq 1} \sum_{m=1}^3 |x^{\bar{r}+3} \partial^\gamma \phi(x)| : x \in B_{\bar{R}}^c \right\},$$

and thus $G(\phi) \leq \mathfrak{C} \sum_{\gamma \in \mathbb{N}_0^3, |\gamma| \leq 1} \sum_{m=1}^3 p_{(\bar{r}+3)e_m, \gamma}(\phi)$. On the other hand,

$$\begin{aligned} \int_{B_{\bar{R}}^c} |x|^{-r-3} \sum_{\mu=1}^3 |\partial_\mu U(x)| dx &\leq C \sum_{m=1}^{m_0} \sum_{\mu, \sigma=1}^3 \int_{B_{\bar{R}}^c} |x|^{-r-3} |V_{\mu\sigma}^{(m)}(x)| dx \\ &\leq C \sum_{m=1}^{m_0} \sum_{\mu, \sigma=1}^3 \left(\int_{B_{\bar{R}}^c} |V_{\mu\sigma}^{(m)}(x)|^{q_m} |x|^{-r q_m} dx \right)^{1/q_m} \left(\int_{B_{\bar{R}}^c} |x|^{-3 q'_m} dx \right)^{1/q'_m}. \end{aligned}$$

Since $\int_{B_{\bar{R}}^c} |x|^{-3 q'_m} dx < \infty$ for $1 \leq m \leq m_0$, our assumptions on the functions $V^{(m)}$ imply that the right-hand side of the preceding estimate is finite. Thus we may conclude that the integral $\int_{B_{\bar{R}}^c} |x|^{-r-3} \sum_{\mu=1}^3 |\partial_\mu U(x)| dx$ is finite. As a consequence

$$\left| \int_{B_{\bar{R}}^c} (\nabla U \cdot \nabla \phi + \tau \partial_1 U \cdot \phi) dx \right| \leq \mathfrak{C} G(\phi) \int_{B_{\bar{R}}^c} |x|^{-r-3} \sum_{\mu=1}^3 |\partial_\mu U(x)| dx,$$

with the right-hand side being bounded, in turn, by $\mathfrak{C} \sum_{\alpha \in \mathbb{N}_0^3, |\alpha| \leq 1} \sum_{m=1}^3 p_{\bar{r}+3e_m, \alpha}(\phi)$, for $\phi \in \mathcal{S}(\mathbb{R}^3)^3$. Since $|\int_{B_{\bar{R}}} (\nabla U \cdot \nabla \phi + \tau \partial_1 U \cdot \phi) dx| \leq \mathfrak{C} \|U|_{B_{\bar{R}}}\|_{1,1} p_{0,0}(\phi)$, we have thus found that $|T(\phi)|$ is bounded by $\mathfrak{C} (p_{0,0}(\phi) + \sum_{\alpha \in \mathbb{N}_0^3, |\alpha| \leq 1} \sum_{m=1}^3 p_{\bar{r}+3e_m, \alpha}(\phi))$ for $\phi \in \mathcal{S}(\mathbb{R}^3)^3$. This shows our claim for T . A similar but simpler reasoning is valid for S . \square

In Section 5, when we exploit the preceding theorem, the next two lemmas will be useful.

Lemma 3.1 Let $m_0 \in \mathbb{N}$, $p_1, \dots, p_{m_0} \in (1, \infty)$, $\tilde{p} := \max\{p_m : 1 \leq m \leq m_0\}$, $\epsilon_0 \in [0, 1/\tilde{p})$, $R_1 \in (0, \infty)$, $W^{(1)}, \dots, W^{(m_0)} \in L_{loc}^1(\mathbb{R}^3)^3$ such that $W := \sum_{m=1}^{m_0} W^{(m)}$ is a polynomial and $\int_{B_{R_1}^c} (|W^{(m)}(x)| |x|^{-2-\epsilon_0})^{p_m} dx < \infty$ for $1 \leq m \leq m_0$. Then the degree of W is at most 1.

Proof: Abbreviate $\mathfrak{A}(x) := \sum_{m=1}^{m_0} (|W^{(m)}(x)| |x|^{-2-\epsilon_0})^{p_m}$ for $x \in \mathbb{R}^3$. Suppose there is $\tilde{R} \in [R_1, \infty)$ such that $\int_{\partial B_R} \mathfrak{A}(x) do_x \geq 1/R$ for $R \in [\tilde{R}, \infty)$. Then $\int_{B_{\tilde{R}^c}} \mathfrak{A}(x) dx = \int_{\tilde{R}}^{\infty} \int_{\partial B_R} \mathfrak{A}(x) do_x dr \geq \int_{\tilde{R}}^{\infty} r^{-1} dr = \infty$. Since $\int_{B_{R_1}^c} \mathfrak{A}(x) dx < \infty$ by our assumptions, we have arrived at a contradiction. Thus there is a sequence (R_n) in $[\max\{R_1, 1\}, \infty)$ with $\int_{\partial B_R} \mathfrak{A}(x) do_x \leq R_n^{-1}$ for $n \in \mathbb{N}$ and $R_n \rightarrow \infty$. It follows that $\mathfrak{K}_n := \int_{\partial B_1} \mathfrak{A}(R_n x) do_x \leq R_n^{-3}$ for $n \in \mathbb{N}$. But $(R_n^{-2-\epsilon_0})^{p_m} = (R_n^{-2-\epsilon_0+1/\tilde{p}})^{p_m} R_n^{-p_m/\tilde{p}} \geq (R_n^{-2-\epsilon_0+1/\tilde{p}})^{p_m} R_n^{-1}$ for $n \in \mathbb{N}$, $1 \leq m \leq m_0$, where the last inequality holds because $R_n \geq 1$ and $p_m \leq \tilde{p}$. Setting $\tilde{\mathfrak{K}}_n := \int_{\partial B_1} \sum_{m=1}^{m_0} (|W^{(m)}(R_n y)| R_n^{-2-\epsilon_0+1/\tilde{p}})^{p_m} do_y$ for $n \in \mathbb{N}$, we get $\mathfrak{K}_n \leq \tilde{\mathfrak{K}}_n R_n \leq R_n^{-2}$ ($n \in \mathbb{N}$) by the preceding inequalities. On the other hand, for $m \in \{1, \dots, m_0\}$, $n \in \mathbb{N}$, we find with Hölder's inequality that $R_n^{-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |W^{(m)}(R_n y)| do_y \leq (4\pi)^{1/p'_m} \tilde{\mathfrak{K}}_n^{1/p'_m}$. It follows that $R_n^{-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} \sum_{m=1}^{m_0} |W^{(m)}(R_n y)| do_y \rightarrow 0$ ($n \rightarrow \infty$), so we have found that $R_n^{-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |W(R_n y)| do_y \rightarrow 0$ ($n \rightarrow \infty$).

Now suppose that the degree of W is larger than 1. Then there is $k_0 \in \mathbb{N}$ and for any $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq k_0$ a number $a_\alpha \in \mathbb{R}$ such that $k_0 \geq 2$, $W(x) = \sum_{\alpha \in \mathbb{N}_0^3, |\alpha| \leq k_0} a_\alpha x^\alpha$ for $x \in \mathbb{R}^3$, and $a_{\alpha_0} \neq 0$ for some $\alpha_0 \in \mathbb{N}_0^3$ with $|\alpha_0| = k_0$. Put $P(x) := \sum_{\alpha \in \mathbb{N}_0^3, |\alpha| = k_0} a_\alpha x^\alpha$ ($x \in \mathbb{R}^3$). We distinguish two cases. In the first, we suppose that $\int_{\partial B_1} |P(y)| do_y > 0$. Since for $S \in (0, \infty)$

$$\int_{\partial B_1} |W(Sy)| do_y \geq S^{k_0} \int_{\partial B_1} |P(y)| do_y - \sum_{l=0}^{k_0-1} S^l \sum_{\alpha \in \mathbb{N}_0^3, |\alpha|=l} |a_\alpha| \int_{\partial B_1} |y^\alpha| do_y,$$

we may conclude there is $\tilde{R} \in [R_1, \infty)$ such that $\int_{\partial B_1} |W(Sy)| do_y \geq S^{k_0} \int_{\partial B_1} |P(y)| do_y / 2$ for $S \in [\tilde{R}, \infty)$. Thus $R_n^{-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |W(R_n y)| do_y \geq R_n^{k_0-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |P(y)| do_y / 2$ for $n \in \mathbb{N}$ with $R_n \geq \tilde{R}$. On the other hand, since we assumed $k_0 \geq 2$ and $\epsilon_0 \in [0, 1/\tilde{p})$, and because $R_n \rightarrow \infty$ and $\int_{\partial B_1} |P(y)| do_y > 0$, we get $R_n^{k_0-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |P(y)| do_y / 2 \rightarrow \infty$ ($n \rightarrow \infty$). We may thus conclude that $R_n^{-2-\epsilon_0+1/\tilde{p}} \int_{\partial B_1} |W(R_n y)| do_y \rightarrow \infty$ for n tending to infinity, which is a contradiction to what was shown above.

In the second case, we assume that $\int_{\partial B_1} |P(y)| do_y = 0$. Then $P(y) = 0$ for a. e. $y \in \partial B_1$, and hence for any y on ∂B_1 by continuity. Since P is homogeneous, it follows that $P = 0$ in \mathbb{R}^3 . For $x_2, x_3 \in \mathbb{R}$, the function $P(\cdot, x_2, x_3)$ is a polynomial in one variable, so we may conclude that all coefficients of this polynomial vanish. But this polynomial may be written as $\sum_{l=0}^{k_0} P_l(x_2, x_3) r^l$ for $r \in \mathbb{R}$, with $P_l(x_2, x_3) := \sum_{m=0}^{k_0-l} a_{(l, m, k_0-l-m)} x_2^m x_3^{k_0-l-m}$ for $x_2, x_3 \in \mathbb{R}$, $0 \leq l \leq k_0$. The numbers $P_l(x_2, x_3)$ are the coefficients of $P(\cdot, x_2, x_3)$ and must therefore vanish ($x_2, x_3 \in \mathbb{R}$). But each function P_l is a polynomial, too. So, in the next step, the same sort of reasoning may be applied to these polynomials, implying that their coefficients must vanish as well. In the end we get that $a_\alpha = 0$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = k_0$, which is a contradiction to the fact that $a_{\alpha_0} \neq 0$ for some $\alpha_0 \in \mathbb{N}_0^3$ with $|\alpha| = k_0$. Thus we arrive at a contradiction in any case. This proves that the degree of W cannot exceed 1. \square

Lemma 3.2 *Let $m_0 \in \mathbb{N}$, $p_m \in (1, \infty)$, $V^{(m)} \in L^{p_m}(\mathbb{R}^3)^3$ for $1 \leq m \leq m_0$. Suppose that $V := \sum_{m=1}^{m_0} V^{(m)}$ is constant. Then $V = 0$.*

Proof: By an argument already used in the proof of [19, Lemma 5.2] and Lemma 3.1, we may choose a sequence (R_n) in $(0, \infty)$ with $\int_{\partial B_{R_n}} \sum_{m=1}^{m_0} |V^{(m)}(x)|^{p_m} d\sigma_x \leq R_n^{-1}$ for $n \in \mathbb{N}$ and with $R_n \rightarrow \infty$. Put $\mathfrak{A}_n := \int_{\partial B_1} \sum_{m=1}^{m_0} |V^{(m)}(R_n y)|^{p_m} d\sigma_y$ for $n \in \mathbb{N}$, so $\mathfrak{A}_n \leq R_n^{-3}$. But $\int_{\partial B_1} |V^{(m)}(R_n y)| d\sigma_y \leq (4\pi)^{1/p'_m} \mathfrak{A}_n^{1/p_m}$ for $n \in \mathbb{N}$, $1 \leq m \leq m_0$ by Hölder's inequality, hence $\int_{\partial B_1} \sum_{m=1}^{m_0} |V^{(m)}(R_n y)| d\sigma_y \rightarrow 0$ ($n \rightarrow \infty$). Therefore $\int_{\partial B_1} |V(R_n y)| d\sigma_y \rightarrow 0$ for n tending to infinity. Since V is constant, this means that $V = 0$. \square

4 Some fundamental solutions and potential functions.

We recall that the fundamental solution \mathfrak{N} of the Poisson equation ("Newton kernel") and a convolution ("Newton potential") with this fundamental solution was introduced in Theorem 3.1. We define the usual heat kernel in 3D by setting

$$\mathfrak{H}(z, t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)} \quad \text{for } z \in \mathbb{R}^3, t \in (0, \infty), \quad \mathfrak{H}(z, 0) := 0 \quad \text{for } z \in \mathbb{R}^3 \setminus \{0\}.$$

Thus, in our context, \mathfrak{H} is defined on $\mathfrak{B} := (\mathbb{R}^3 \times (0, \infty)) \cup ((\mathbb{R}^3 \setminus \{0\}) \times \{0\})$.

Theorem 4.1 *The relations $\mathfrak{H} \in C^\infty(\mathfrak{B})$ and $\int_{\mathbb{R}^3} \mathfrak{H}(z, t) dt = 1$ for $t \in (0, \infty)$ hold. If $\alpha \in \mathbb{N}_0^3$, $\sigma \in \mathbb{N}_0$, the inequality $|\partial_z^\alpha \partial_t^\sigma \mathfrak{H}(z, t)| \leq C(\alpha, \sigma) (|z|^2 + t)^{-(3+|\alpha|+2\sigma)/2}$ is valid for $z \in \mathbb{R}^3$, $t \in (0, \infty)$.*

Proof: See [41] for the preceding estimate. \square

The estimate in Theorem 4.1 in the case $|\alpha| = 2$, $\sigma = 0$ allows to define the velocity part Γ of a fundamental solution to the time-dependent Stokes system,

$$\Gamma_{jk}(z, t) := \mathfrak{H}(z, t) \delta_{jk} + \int_t^\infty \partial z_j \partial z_k \mathfrak{H}(z, s) ds \quad \text{for } (z, t) \in \mathfrak{B}, j, k \in \{1, 2, 3\},$$

as well as the velocity part Λ of a fundamental solution to the time-dependent Oseen system (1.1),

$$\Lambda_{jk}(z, t) := \Gamma_{jk}(z - \tau t e_1, t) \quad \text{for } (z, t) \in \mathfrak{B}, j, k \in \{1, 2, 3\}.$$

We will need the following properties of Λ .

Lemma 4.1 *The relations $\Lambda \in C^\infty(\mathfrak{B})^{3 \times 3}$ and $\sum_{k=1}^3 \partial z_k \Lambda_{jk}(z, t) = 0$ are valid for $1 \leq j \leq 3$, $z \in \mathbb{R}^3$, $t \in (0, \infty)$.*

Let $K > 0$. Then $|\partial_z^\alpha \partial_t^\sigma \Lambda(z, t)| \leq C(K, \tau) \sum_{\mu=1}^2 (|z| \nu(z) + t)^{-(3+|\alpha|+\mu\sigma)/2}$ for z, t, α, σ as before, with the additional assumption $|z| \geq K$.

The estimate $\int_{\mathbb{R}^3} |\partial_x^\alpha \partial_t^\sigma \Lambda(x, t)|^q dx \leq C(\tau, q) \sum_{\mu=1}^2 t^{-(3+|\alpha|+\mu\sigma)q/2+3/2}$ is valid for $q \in (1, \infty)$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$, $\sigma \in \{0, 1\}$ with $|\alpha| + \sigma \leq 1$. If $|\alpha| + \sigma = 1$, the case $q = 1$ is also admitted.

Proof: See [19, Lemma 3.3, Corollary 3.3, 3.4]. \square

We introduce the first of our potential functions.

Lemma 4.2 ([19, Corollary 3.5]) *Let $A \subset \mathbb{R}^3$ be measurable, $q \in [1, \infty)$, $V \in L^q(A)^3$, and let \tilde{V} denote the zero extension of V to \mathbb{R}^3 . Then $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y, t) \tilde{V}(y)| dy < \infty$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $x \in \mathbb{R}^3$, $t \in (0, \infty)$. Define the function $\mathfrak{J}^{(\tau)}(V) : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ by $\mathfrak{J}^{(\tau)}(V)(x, t) := \int_{\mathbb{R}^3} \Lambda(x-y, t) \cdot \tilde{V}(y) dy$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$.*

The derivative $\partial x_l \mathfrak{J}^{(\tau)}(V)(x, t)$ exists and equals $\int_{\mathbb{R}^3} \partial x_l \Lambda(x-y, t) \cdot \tilde{V}(y) dy$ for x, t as above, and for $l \in \{1, 2, 3\}$. The functions $\mathfrak{J}^{(\tau)}(V)$ and $\partial x_l \mathfrak{J}^{(\tau)}(V)$ are continuous in $\mathbb{R}^3 \times (0, \infty)$. If $q > 1$, then $\|\mathfrak{J}^{(\tau)}(V)(t)\|_q \leq C(q, \tau) \|V\|_q$ for $t \in (0, \infty)$.

If $m_0 \in \mathbb{N}$, $p_l \in [1, \infty)$, $V^{(l)} \in L^{p_l}(A)^3$ for $1 \leq l \leq m_0$, put $\mathfrak{J}^{(\tau)}(\sum_{l=1}^{m_0} V^{(l)}) := \sum_{l=1}^{m_0} \mathfrak{J}^{(\tau)}(V^{(l)})$.

We will need a variant of $\mathfrak{J}^{(\tau)}(V)$.

Lemma 4.3 *Let $q \in (1, \infty)$, $A \subset \mathbb{R}^3$ be measurable, $V \in L^q(A)^3$. Write \tilde{V} for the zero extension of V to \mathbb{R}^3 . Then $\int_{\mathbb{R}^3} |\partial_x^\alpha \mathfrak{H}(x-y-\tau t e_1, t) \tilde{V}(y)| dy < \infty$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Define $\mathcal{H}^{(\tau)}(V)(x, t) := \int_{\mathbb{R}^3} \mathfrak{H}(x-y-\tau t e_1, t) \tilde{V}(y) dy$, $\mathcal{H}^{(\tau)}(V)(x, 0) := \tilde{V}(x, 0)$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$. Then $\mathcal{H}^{(\tau)}(V)$ belongs to $C^0([0, \infty), L^2(\mathbb{R}^3)^3)$, and the estimate $\|\mathcal{H}^{(\tau)}(V)(t)\|_2 \leq C(\tau) \|V\|_2$ holds. Moreover, the derivative $\partial x_l \mathcal{H}^{(\tau)}(V)(x, t)$ exists and equals $\int_{\mathbb{R}^3} \partial_x^\alpha \mathfrak{H}(x-y-\tau t e_1, t) \tilde{V}(y) dy$, for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, $1 \leq l \leq 3$. In addition $\nabla_x \mathcal{H}^{(\tau)}(V) \in L^2(0, \infty, L^2(\mathbb{R}^3)^3)$.

Proof: All the claims of the lemma except the continuity of $\mathcal{H}^{(\tau)}(V)$ and the integrability of $|\nabla_x \mathcal{H}^{(\tau)}(V)|^2$ follow by the same arguments as used in [19, proof of Corollary 3.5] with respect to $\mathfrak{J}^{(\tau)}(V)$. The continuity of $\mathcal{H}^{(\tau)}(V)$ as a mapping from $[0, \infty)$ to $L^2(\mathbb{R}^3)^3$ at $t = 0$ holds according to [19, Theorem 3.3]. Continuity at $t > 0$ may be shown by the same reasoning as in [19, proof of Corollary 3.6]. As for L^2 -integrability of the spatial gradient, we use that the Fourier transform $[\mathfrak{H}(\cdot, t)]^\wedge$ of the function $x \mapsto \mathfrak{H}(x, t)$ ($x \in \mathbb{R}^3$) (heat kernel as a function of the space variables) is given by $[\mathfrak{H}(\cdot, t)]^\wedge(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2 t}$ for $\xi \in \mathbb{R}^3$, $t \in (0, \infty)$. Therefore by Parseval's formula $\|\partial x_l \mathcal{H}^{(\tau)}(V)\|_2^2 = \int_0^\infty \int_{\mathbb{R}^3} |(2\pi)^{-3/2} \xi_l e^{(i\tau \xi_1 - |\xi|^2)t} \widehat{V}(\xi)|^2 d\xi dt$. By applying Hölder's inequality with respect to the variable ξ , and then integrating with respect to t , we get $\|\partial x_l \mathcal{H}^{(\tau)}(V)\|_2^2 \leq \mathfrak{C} \|\widehat{V}\|_2^2$, so Parseval's formula yields $\|\partial x_l \mathcal{H}^{(\tau)}(V)\|_2^2 \leq \mathfrak{C} \|V\|_2^2$. This completes the proof of Lemma 4.3 \square

We turn to the definition of another potential function.

Lemma 4.4 *Let $T_0 \in (0, \infty]$, $A \subset \mathbb{R}^3$ measurable, $q \in [1, \infty)$ and f a function from $L_{loc}^1([0, T_0), L^q(A)^3)$. Let \tilde{f} denote the zero extension of f to $\mathbb{R}^3 \times (0, \infty)$. Then the integral $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma)| dy$ is finite for any $x \in \mathbb{R}^3$, $t \in (0, \infty)$, $\sigma \in (0, t)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Moreover, for a. e. $t \in (0, \infty)$ and for α as before, the integral $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma)| dy d\sigma$ is finite for a. e. $x \in \mathbb{R}^3$. Thus we may define*

$$\mathfrak{R}^{(\tau)}(f)(x, t) := \int_0^t \int_{\mathbb{R}^3} \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma) dy d\sigma$$

for such t and x . The relation $\mathfrak{R}^{(\tau)}(f)(t) \in W_{loc}^{1,1}(\mathbb{R}^3)^3$ holds for a. e. $t \in (0, \infty)$, and for such t $\partial x_l \mathfrak{R}^{(\tau)}(f)(t)(x) = \int_0^t \int_{\mathbb{R}^3} \partial x_l \Lambda(x-y, t-\sigma) \cdot \tilde{f}(y, \sigma) dy d\sigma$ for $l \in \{1, 2, 3\}$ and a.

e. $x \in \mathbb{R}^3$.

Moreover the integral $\int_0^t |\int_{\mathbb{R}^3} \Lambda(x-y, t-s) \cdot \tilde{f}(y, s) dy| ds$ is finite for any $t \in (0, \infty)$ and for a. e. $x \in \mathbb{R}^3$. Thus the function $\mathfrak{R}^{(\tau)}(f)$ is well defined even for any $t \in (0, \infty)$ (instead of only for a. e. $t \in (0, \infty)$) and for a. e. $x \in \mathbb{R}^3$. In addition the inequality $\|\mathfrak{R}^{(\tau)}(f)(t)\|_q \leq C(q) \|\tilde{f}\|_{\mathbb{R}^3 \times (0, t)} \|f\|_{q, 1; t}$ holds for $t > 0$.

Let $m_0 \in \mathbb{N}$, $p_l \in (1, \infty)$ and $f^{(l)} \in L_{loc}^1([0, T_0), L^{p_l}(A)^3)$ for $1 \leq l \leq m_0$. Then define $\mathfrak{R}^{(\tau)}(\sum_{j=1}^{m_0} f^{(j)}) := \sum_{j=1}^{m_0} \mathfrak{R}(f^{(j)})$.

Proof: [19, Lemma 3.8, Corollary 3.7]. \square

Lemma 4.5 Let $q \in (1, \infty)$ and $f \in L^2(0, \infty, L^q(\mathbb{R}^3)^3)$. Then the function $\mathfrak{R}^{(\tau)}(f)$ belongs to the space $C^0([0, \infty), L^q(\mathbb{R}^3)^3)$.

Let $B \subset \mathbb{R}^3$ be measurable and bounded. Suppose that $q < 3$. Then $\mathfrak{R}^{(\tau)}(f)|_{B \times (0, \infty)} \in L^\infty(0, \infty, L^q(B)^3)$ and $\nabla_x \mathfrak{R}^{(\tau)}(f)|_{B \times (0, \infty)} \in L^2(0, \infty, L^q(B)^3)$.

Proof: Put $\mathfrak{B}(x, t, h) := \int_t^{t+h} \mathfrak{J}^{(\tau)}(f(s))(x, t+h-s) ds$ for $t \in [0, \infty)$, $h \in (0, \infty)$, $x \in \mathbb{R}^3$. Then $\|\mathfrak{B}(\cdot, t, h)\|_q \leq \int_t^{t+h} \|\mathfrak{J}^{(\tau)}(f(s))(\cdot, t+h-s)\|_q ds \leq \mathfrak{C} \int_t^{t+h} \|f(s)\|_q ds \leq \mathfrak{C} h^{1/2} \|f\|_{q, 2; \infty}$ by Theorem 2.1 and Lemma 4.2. We may conclude that $\|\mathfrak{B}(\cdot, t, h)\|_q \rightarrow 0$ ($h \downarrow 0$). It follows in particular that $\mathfrak{R}^{(\tau)}(f)$ as a mapping from $[0, \infty)$ in $L^q(\mathbb{R}^3)^3$ is continuous in $t = 0$. Let $t \in (0, \infty)$. Then $\mathfrak{R}^{(\tau)}(f)(x, t+h) - \mathfrak{R}^{(\tau)}(f)(x, t) = \mathfrak{B}(x, t, h) + \mathcal{C}(x, t, h)$, with $\mathcal{C}(x, t, h) := \int_0^t [\mathfrak{J}^{(\tau)}(f(s))(x, t+h-s) - \mathfrak{J}^{(\tau)}(f(s))(x, t-s)] ds$, for $x \in \mathbb{R}^3$, $h > 0$. Define $\mathfrak{A}(s, h) := \|\mathfrak{J}^{(\tau)}(f(s))(t+h-s) - \mathfrak{J}^{(\tau)}(f(s))(t-s)\|_q$ for $s \in (0, t)$, $h > 0$. Then Lemma 4.2 yields that $\mathfrak{A}(s, h) \leq \sum_{r \in \{0, h\}} \|\mathfrak{J}^{(\tau)}(f(s))(t+r-s)\|_q \leq \mathfrak{C} \|f(s)\|_q$ for s, h as before. The function $s \mapsto \|f(s)\|_q$ ($s \in (0, t)$) is integrable. Moreover, for $s \in (0, t)$, $h > 0$, we get by the definition of $\mathfrak{J}^{(\tau)}(f(s))$, Young's inequality and the last inequality in Lemma 4.1 that

$$\begin{aligned} \mathfrak{A}(s, h) &\leq \left(\int_{\mathbb{R}^3} |\Lambda(z, t+h-s) - \Lambda(z, t-s)| dz \right) \|f(s)\|_q \\ &= \left(\int_{\mathbb{R}^3} \left| \int_0^1 \partial_r \Lambda(z, r) |_{r=t+\vartheta h-s} d\vartheta \right| dz \right) \|f(s)\|_q h \leq \mathfrak{C} \sum_{\mu=1}^2 (t-s)^{-\mu/2} \|f(s)\|_q h. \end{aligned}$$

Therefore $\mathfrak{A}(s, h) \rightarrow 0$ for $h \downarrow 0$. At this point it follows with Lebesgue's theorem that $\|\mathcal{C}(\cdot, t, h)\|_q \rightarrow 0$ ($h \downarrow 0$). Since we have already shown that $\|\mathfrak{B}(\cdot, t, h)\|_q \rightarrow 0$ ($h \downarrow 0$), we may conclude that $\|\mathfrak{R}^{(\tau)}(f)(t+h) - \mathfrak{R}^{(\tau)}(f)(t)\|_q \rightarrow 0$ ($h \downarrow 0$). Altogether it follows that $\mathfrak{R}^{(\tau)}(f)$ as a mapping from $[0, \infty)$ into $L^q(\bar{\Omega}^c)^3$ is continuous.

In view of the second part of the lemma, we obtain with Lemma 4.4 that $\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t) = \mathfrak{A}_1^{(\alpha)} + \mathfrak{A}_2^{(\alpha)}$, with $\mathfrak{A}_1^{(\alpha)}(x, t) := \int_0^t \chi_{(0,1)}(t-s) \int_{\mathbb{R}^3} \partial_x^\alpha \Lambda(x-y, t-s) f(y, s) dy ds$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, and with $\mathfrak{A}_2^{(\alpha)}(x, t)$ defined in the same way as $\mathfrak{A}_1^{(\alpha)}(x, t)$, except that the term $\chi_{(0,1)}(t-s)$ is replaced by $\chi_{[1, \infty)}(t-s)$. Then for $t \in (0, \infty)$, we have $\|\mathfrak{A}_1^{(0)}(t)\|_q \leq \int_0^t \chi_{(0,1)}(t-s) \|\mathfrak{J}^{(\tau)}(f(s))(t-s)\|_q ds$, so we may conclude with Lemma 4.2 that $\|\mathfrak{A}_1^{(0)}(t)\|_q \leq \mathfrak{C} \int_0^t \chi_{(0,1)}(t-s) \|f(s)\|_q ds \leq \mathfrak{C} \|f\|_{q, 2; \infty}$, where the last estimate follows with Hölder's inequality. Hence we get $\|\mathfrak{A}_1^{(0)}\|_{q, \infty; \infty} \leq \|f\|_{q, 2; \infty}$. Moreover,

by Hölder's inequality and the last inequality in Lemma 4.1, for $x \in \mathbb{R}^3$, $t > 0$,

$$\begin{aligned} |\mathfrak{A}_2^{(0)}(x, t)| &\leq \int_0^t \chi_{[1, \infty)}(t-s) \left(\int_{\mathbb{R}^3} |\Lambda(x-y, t-s)|^{q'} dy \right)^{1/q'} \|f(s)\|_q ds \\ &\leq \mathfrak{C} \int_0^t \chi_{[1, \infty)}(t-s) (t-s)^{-3/(2q)} \|f(s)\|_q ds. \end{aligned}$$

Suppose that $q < 3$. Then $3/(2q) > 1/2$, so by Hölder's inequality $|\mathfrak{A}_2^{(0)}(x, t)| \leq \mathfrak{C} \|f\|_{q, 2; \infty}$. It follows that $\|\mathfrak{A}_2^{(0)}\|_{\infty, \infty; \infty} \leq \mathfrak{C} \|f\|_{q, 2; \infty}$. This estimate and the one given above for $\|\mathfrak{A}_1^{(0)}\|_{q, \infty; \infty}$ imply $\|\mathfrak{R}^{(\tau)}(f)|B \times (0, \infty)\|_{q, \infty; \infty} \leq \mathfrak{C} \|f\|_{q, 2; \infty}$. Now let $l \in \{1, 2, 3\}$. Young's inequality and the last estimate in Lemma 4.1 yield

$$\begin{aligned} \|\mathfrak{A}_1^{(e_l)}(t)\|_q &\leq \int_0^t \chi_{(0, 1)}(t-s) \left(\int_{\mathbb{R}^3} |\partial_{z_l} \Lambda(z, t-s)| dz \right) \|f(s)\|_q ds \\ &\leq \mathfrak{C} \int_0^t \chi_{(0, 1)}(t-s) (t-s)^{-1/2} \|f(s)\|_q ds. \end{aligned}$$

for $t \in (0, \infty)$, and thus $\|\mathfrak{A}_1^{(e_l)}\|_{q, 2; \infty} \leq \int_0^\infty \chi_{(0, 1)}(r) r^{-1/2} dr \|f\|_{q, 2; \infty} \leq \mathfrak{C} \|f\|_{q, 2; \infty}$ again by Young's inequality. Applying Hölder's inequality and the last estimate in Lemma 4.1 as in the estimate of $|\mathfrak{A}_2^{(0)}(x, t)|$, we obtain for $x \in \mathbb{R}^3$, $t > 0$ that $|\mathfrak{A}_2^{(e_l)}(x, t)| \leq \mathfrak{C} \int_0^t \chi_{[1, \infty)}(t-s) (t-s)^{-3/(2q)-1/2} \|f(s)\|_q ds$. But $\|\mathfrak{A}_2^{(e_l)}(t)|B\|_q \leq \mathfrak{C} \|\mathfrak{A}_2^{(e_l)}(t)|B\|_\infty$. Thus, combining the two preceding inequalities, and recalling the assumption $q < 3$, hence $3/(2q) + 1/2 > 1$, we may conclude by another application of Young's inequality that $\|\mathfrak{A}_2^{(e_l)}|B \times (0, \infty)\|_{q, 2; \infty} \leq \mathfrak{C} \int_0^\infty \chi_{[1, \infty)}(r) r^{-3/(2q)-1/2} dr \|f\|_{q, 2; \infty} \leq \mathfrak{C} \|f\|_{q, 2; \infty}$. Combining this estimate of $\|\mathfrak{A}_2^{(e_l)}|B \times (0, \infty)\|_{q, 2; \infty}$ with the one of $\|\mathfrak{A}_1^{(e_l)}\|_{q, 2; \infty}$ given above, we obtain that $\|\partial_{x_l} \mathfrak{R}^{(\tau)}(f)|B \times (0, \infty)\|_{q, 2; \infty} \leq \mathfrak{C} \|f\|_{q, 2; \infty}$. This completes the proof of Lemma 4.5. \square

The next lemma deals with still another potential function, this one defined on the surface of an open bounded set.

Lemma 4.6 *Let $q \in [1, \infty]$, $T_0 \in (0, \infty]$, $A \subset \mathbb{R}^3$ open and bounded, with Lipschitz boundary, $\phi \in L_{loc}^1([0, T_0], L^q(\partial A)^3)$, $\tilde{\phi}$ the zero extension of ϕ to $\partial A \times (0, \infty)$. For $t \in (0, \infty)$, $x \in \mathbb{R}^3 \setminus \partial A$, $\alpha \in \mathbb{N}_0^3$, the term $|\partial_x^\alpha \Lambda(x-y, t-s) \cdot \tilde{\phi}(y, s)|$ is integrable as a function of $(y, s) \in \partial A \times (0, t)$. Define $\mathfrak{V}^{(\tau)}(\phi) := \mathfrak{V}^{(\tau, A)}(\phi) : (\mathbb{R}^3 \setminus \partial A) \times (0, \infty) \mapsto \mathbb{R}^3$ by*

$$\mathfrak{V}^{(\tau)}(\phi)(x, t) := \int_0^t \int_{\partial A} \Lambda(x-y, t-s) \cdot \tilde{\phi}(y, s) do_y ds \quad \text{for } x \in \mathbb{R}^3 \setminus \partial A, t \in (0, \infty).$$

Then, for any $t \in (0, \infty)$, the integral $\int_0^t \int_{\partial A} \Lambda(x-y, t-s) \cdot \tilde{\phi}(y, s) do_y ds$ as a function of $x \in \mathbb{R}^3 \setminus A$ belongs to $C^\infty(\mathbb{R}^3 \setminus A)^3$, and $\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x, t) = \int_0^t \int_{\partial A} \partial_x^\alpha \Lambda(x-y, t-s) \cdot \tilde{\phi}(y, s) do_y ds$ for $\alpha \in \mathbb{N}_0^3$, $x \in \mathbb{R}^3 \setminus A$.

Proof: The function Λ is C^∞ on $\mathbb{R}^3 \times (0, \infty)$ (Lemma 4.1), so the lemma follows from Lebesgue's theorem. \square

Lemma 4.7 *Let $\phi \in L^2(\partial \Omega \times (0, \infty))^3$ and $S \in (0, \infty)$ with $\overline{\Omega} \subset B_S$. Then $\mathfrak{V}^{(\tau)}(\phi)|B_S^c \times [0, \infty) \in C^0([0, \infty), L^4(B_S^c)^3)$.*

Proof: Since the function Λ and its derivatives are continuous on $(\mathbb{R}^3 \setminus \{0\}) \times [0, \infty)$ (Lemma 4.1), and because $\text{dist}(\partial\Omega, B_S^c) > 0$ and $|x - y| \geq |x|/2$ for $y \in \partial\Omega$, $x \in B_{2S}^c$, we may conclude from the first estimate in Lemma 4.1 that

$$\begin{aligned} |\partial_r^\sigma \Lambda(x - y, r)| &\leq \mathfrak{C} \sum_{\mu=1}^2 (\chi_{A_{2S}, S}(x) + \chi_{B_{2S}^c}(x) |x|^{-3/2-\mu\sigma/2}) \\ &\leq \mathfrak{C} (\chi_{A_{2S}, S}(x) + \chi_{B_{2S}^c}(x) |x|^{-3/2-\sigma/2}) \end{aligned} \quad (4.5)$$

for $x \in B_S^c$, $y \in \partial\Omega$, $r \in [0, \infty)$, $\sigma \in \{0, 1\}$. As a consequence,

$$\begin{aligned} |\Lambda(x - y, t + h - s) - \Lambda(x - y, t - s)| &= \left| \int_0^1 \partial_r^\sigma \Lambda(x - y, r) \Big|_{r=t+\vartheta h-s} d\vartheta \right| h \\ &\leq \mathfrak{C} (\chi_{A_{2S}, S}(x) + \chi_{B_{2S}^c}(x) |x|^{-2}) h \quad \text{for } x \in B_S^c, y \in \partial\Omega, h, t \in (0, \infty), s \in (0, t). \end{aligned} \quad (4.6)$$

Put $\mathcal{B}(x, t, h) := \int_t^{t+h} \int_{\partial\Omega} \Lambda(x - y, t + h - s) \phi(y, s) d\sigma_y ds$ for $x \in B_S^c$, $t \in [0, \infty)$, $h \in (0, \infty)$, and $\mathcal{C}(x, t, h) := \int_0^t \int_{\partial\Omega} (\Lambda(x - y, t + h - s) - \Lambda(x - y, t - s)) \phi(y, s) d\sigma_y ds$ for x, h as before and $t \in (0, \infty)$. Then by Theorem 2.1 and (4.5),

$$\|\mathcal{B}(\cdot, t, h)\|_4 \leq \mathfrak{C} \int_t^{t+h} \int_{\partial\Omega} |\phi(y, s)| d\sigma_y ds \left(\int_{B_S^c} [\chi_{A_{2S}, S}(x) + \chi_{B_{2S}^c}(x) |x|^{-3/2}]^4 dx \right)^{1/4},$$

hence $\|\mathcal{B}(\cdot, t, h)\|_4 \leq \mathfrak{C} h^{1/2} \|\phi\|_2$ for $t \in [0, \infty)$, $h \in (0, \infty)$. Thus $\|\mathcal{B}(\cdot, t, h)\|_4 \rightarrow 0$ when $h \downarrow 0$. This means in particular that $\mathfrak{V}^{(\tau)}(\phi)|_{B_S^c \times [0, \infty)}$ as a mapping from $[0, \infty)$ into $L^4(B_S^c)^3$ is continuous in $t = 0$. Let $t \in (0, \infty)$. Theorem 2.1 and inequality (4.6) imply that $\|\mathcal{C}(\cdot, t, h)\|_4$ may be estimated in the same way as $\|\mathcal{B}(\cdot, t, h)\|_4$, but with the integral over $(t, t + h)$ replaced by an integral over $(0, t)$, and with an additional factor h . It follows that $\|\mathcal{C}(\cdot, t, h)\|_4 \leq \mathfrak{C} t^{1/2} \|\phi\|_2 h$. Therefore $\|\mathfrak{V}^{(\tau)}(\phi)(t + h) - \mathfrak{V}^{(\tau)}(\phi)(t)\|_4 = \|\mathcal{B}(\cdot, t, h) + \mathcal{C}(\cdot, t, h)\|_4 \leq \mathfrak{C} (1 + t^{1/2}) \|\phi\|_2 (h + h^{1/2})$ for $h > 0$. Thus $\mathfrak{V}^{(\tau)}(\phi)|_{B_S^c \times [0, \infty)}$ is continuous on $(0, \infty)$. Since continuity in $t = 0$ is already proved, the lemma follows. \square

In the rest of this section, we recapitulate results from [19] that will be needed in what follows.

We introduce another kernel function, for the definition of which we refer to [19]. This kernel is a truncated version of Λ . Its relevant properties of this kernel are collected in the ensuing theorem. Recall that the numbers S_0 , R_0 , R_1 and the function φ_0 are introduced in Section 1.

Theorem 4.2 *There is a function $\mathfrak{G} := \mathfrak{G}_{R_0, S_0, \varphi_0} : B_{R_0}^c \times B_{R_1} \times [0, \infty) \mapsto \mathbb{R}^{3 \times 3}$ with the following properties.*

Let $x \in B_{R_0}^c$, $r \in [0, \infty)$. Then $\mathfrak{G}(x, \cdot, r) \in C_0^\infty(B_{R_1})^{3 \times 3}$, $\sum_{k=1}^3 \partial y_k \mathfrak{G}_{jk}(x, y, r) = 0$ for $1 \leq j \leq 3$, $y \in B_{R_1}$, and $\mathfrak{G}(x, y, r) = \Lambda(x - y, r)$ for $y \in B_{S_0 + (R_0 - S_0)/4}$.

Let $x \in B_{R_0}^c$, $q \in (1, \infty)$. Then the mapping $r \mapsto \mathfrak{G}(x, \cdot, r)$ ($r \in [0, \infty)$) belongs to $C^1([0, \infty), W^{1, q}(B_{R_1})^{3 \times 3})$. Thus a function $G' \in C^0([0, \infty), W^{1, q}(B_{R_1})^{3 \times 3})$ may be defined by the condition $\|(\mathfrak{G}(x, \cdot, r + h) - \mathfrak{G}(x, \cdot, r))/h - G'(r)\|_{1, q} \rightarrow 0$ ($h \rightarrow 0$) for $r \in [0, \infty)$. We write $\partial_r \mathfrak{G}(x, y, r)$ instead of $G'(r)(y)$ ($r \in [0, \infty)$, $y \in B_{R_1}$).

Let $r \in [0, \infty)$, $q \in (1, \infty)$.

Let $\sigma \in \{0, 1\}$, and define $L(x) : B_{R_1} \mapsto \mathbb{R}^{3 \times 3}$ by $L(x)(y) := \partial_r^\sigma \mathfrak{G}(x, y, r)$ for $x \in B_{R_0}^c$, $y \in B_{R_1}$. Then $L(x) \in C_0^\infty(B_{R_1})^{3 \times 3} \cap W^{1,q}(B_{R_1})^{3 \times 3}$ for $x \in B_{R_0}^c$, and L as a mapping from $B_{R_0}^c$ into $W^{1,q}(B_{R_1})^{3 \times 3}$ is partially differentiable on $\overline{B_{R_0}^c}$. Thus we may define $D_m L : \overline{B_{R_0}^c} \mapsto W^{1,q}(B_{R_1})^{3 \times 3}$ by the condition $\|(L(x + h e_m) - L(x))/h - D_m L(x)\|_{1,q} \rightarrow 0$ ($h \rightarrow 0$), for $m \in \{1, 2, 3\}$, $x \in \overline{B_{R_0}^c}$. Instead of $D_m L(x)(y)$, we write $\partial x_m \partial_r^\sigma \mathfrak{G}(x, y, r)$.

Let $l \in \{1, 2, 3\}$ and define $\tilde{L}(x) : B_{R_1} \mapsto \mathbb{R}^{3 \times 3}$ by $\tilde{L}(x)(y) := \partial y_l \mathfrak{G}(x, y, r)$ for $x \in B_{R_0}^c$, $y \in B_{R_1}$. Then $\tilde{L}(x) \in C_0^\infty(B_{R_1})^{3 \times 3} \cap L^q(B_{R_1})^{3 \times 3}$ for $x \in B_{R_0}^c$, and \tilde{L} considered as an operator from $B_{R_0}^c$ into $L^q(B_{R_1})^{3 \times 3}$ is partially differentiable on $\overline{B_{R_0}^c}$. Thus we may define $D_m \tilde{L} : \overline{B_{R_0}^c} \mapsto L^q(B_{R_1})^{3 \times 3}$ by the condition $\|(\tilde{L}(x + h e_m) - \tilde{L}(x))/h - D_m \tilde{L}(x)\|_q \rightarrow 0$ ($h \rightarrow 0$) ($m \in \{1, 2, 3\}$, $x \in \overline{B_{R_0}^c}$). Instead of $D_m \tilde{L}(x)(y)$, we write $\partial x_m \partial y_l \mathfrak{G}(x, y, r)$.

Let $q \in (1, \infty)$, $p \in [1, \infty]$. Then

$$\int_{B_{R_1}} |\partial_x^\alpha \partial_t^\sigma \partial_y^\beta \mathfrak{G}(x, y, t) \cdot V(y)| dy \leq \mathfrak{C}(|x| \nu(x))^{-(3+|\alpha|+\sigma)/2} \|V\|_q \quad (4.7)$$

for $V \in L^q(B_{R_1})^3$, $t \in (0, \infty)$, $x \in B_{R_0}^c$, $\alpha, \beta \in \mathbb{N}_0^3$, $\sigma \in \{0, 1\}$ with $|\alpha| \leq 1$, $|\beta| + \sigma \leq 1$,

$$\int_0^t \int_{B_{R_1}} |\partial_x^\alpha \partial_t^\sigma \partial_y^\beta \mathfrak{G}(x, y, t-s) \cdot v(y, s)| dy ds \leq \mathfrak{C}(|x| \nu(x))^{-(3+|\alpha|+\sigma)/2+1/(2p')} \|v\|_{q,p;t} \quad (4.8)$$

for $t, x, \alpha, \beta, \sigma$ as in (4.7), and $v \in L^p(0, t, L^q(B_{R_1})^3)$.

Proof: [19, (3.13), Lemma 3.11, 3.12, 3.13, Theorem 4.2]. \square

We note a consequence of the preceding theorem.

Corollary 4.1 ([19, Corollary 4.2]) *Let $\beta \in \mathbb{N}_0^3$, $\sigma \in \{0, 1\}$ with $|\beta| + \sigma \leq 1$. Let $q \in (1, \infty)$, and let the function v belong to $L_{loc}^1([0, \infty), L^q(B_{R_1})^3)$ and the function V to $L^q(B_{R_1})^3$. Define*

$$F(x, t) := \int_0^t \int_{B_{R_1}} \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x, y, t-s) \cdot v(y, s) dy ds, \quad H(x, t) := \int_{B_{R_1}} \mathfrak{G}(x, y, t) \cdot V(y) dy$$

for $x \in \overline{B_{R_0}^c}$, $t \in [0, \infty)$. Take a number $l \in \{1, 2, 3\}$. Then the derivatives $\partial x_l F(x, t)$ and $\partial x_l H(x, t)$ exist pointwise, and they equal $\int_0^t \int_{B_{R_1}} \partial x_l \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x, y, t-s) \cdot v(y, s) dy ds$ and $\int_{B_{R_1}} \partial x_l \mathfrak{G}(x, y, t) \cdot V(y) dy$, respectively, for $x \in \overline{B_{R_0}^c}$, $t \in [0, \infty)$

It will be convenient to subsume a number of terms in a single operator, which we define here, and whose definition makes sense due to the preceding Corollary 4.1. The parameters T_0, S_0, R_0, R_1 and the set Ω appearing in the following were fixed at the beginning of Section 1.

Let $A \subset B_{S_0}$ be open and bounded, with Lipschitz boundary. Put $A_R := B_R \setminus \overline{A}$, $Z_{R,T} := A_R \times (0, T)$ for $R \in [S_0, \infty)$, $T \in (0, \infty]$. Let $\mathfrak{A} \subset \mathbb{R}^3 \times \mathbb{R}$ be such that $Z_{R_1, T_0} \subset \mathfrak{A}$. Let $q \in (1, \infty)$ and let $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|_{Z_{R_1, T_0}} \in C^0([0, T_0), L^q(A_{R_1})^3)$, $v(s)|_{A_{R_1}} \in W_{loc}^{1,1}(A_{R_1})^3$ for $s \in (0, T_0)$ and $\nabla_x v|_{Z_{R_1, T_0}} \in L_{loc}^1([0, T_0), L^q(A_{R_1})^9)$. Then, for $t \in (0, T_0)$

and $x \in B_{R_0}^c$, we define

$$\begin{aligned} \mathfrak{K}_{R_0, S_0, \varphi_0, A, T_0}(v)(x, t) &:= \int_0^t \int_{\Omega_{R_1}} \left(\sum_{l=1}^3 \partial_{y_l} \mathfrak{G}(x, y, t-s) \cdot \partial_{y_l} v(y, s) \right. \\ &\quad \left. - \tau \partial_{y_1} \mathfrak{G}(x, y, t-s) \cdot v(y, s) - \partial_s \mathfrak{G}(x, y, t-s) \cdot v(y, s) \right) dy ds + \int_{\Omega_{R_1}} \mathfrak{G}(x, y, 0) \cdot v(y, t) dy. \end{aligned} \quad (4.9)$$

Next we reproduce some decay estimates proved in [19], beginning with a decay estimate of $\mathfrak{K}_{R_0, S_0, \varphi_0, A, T_0}(v)$. We use the same notation as in (4.9).

Corollary 4.2 ([19, Corollary 4.3]) *Let $p_1, p_2 \in [1, \infty]$ and take A, \mathfrak{A}, q as in (4.9). Then, for $v : \mathfrak{A} \mapsto \mathbb{R}^3$ such that $v|_{Z_{R_1, T_0}} \in C^0([0, T_0], L^q(A_{R_1})^3)$, $v(s)|_{A_{R_1}} \in W_{loc}^{1,1}(A_{R_1})^3$ ($s \in (0, T_0)$) and $\nabla_x v|_{Z_{R_1, T_0}} \in L^{p_2}(0, T_0, L^q(A_{R_1})^9)$, and for $x \in \overline{B_{R_0}^c}$, $t \in (0, T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, the term $|\partial_x^\alpha \mathfrak{K}_{R_0, S_0, \varphi_0, A, T_0}(v)(x, t)|$ is bounded by*

$$\mathfrak{C}(\|v|_{Z_{R_1, t}}\|_{q, p_1; t} + \|\nabla_x v|_{Z_{R_1, t}}\|_{q, p_2; t} + \|v(t)|_{\Omega_{R_1}}\|_q) \max_{j \in \{1, 2\}} (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2p_j')}.$$

Lemma 4.8 ([19, Lemma 4.3]) *Let A, \mathfrak{A}, q be given as in (4.9), and let $p_1, p_2 \in [1, \infty]$. Then, for $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|_{Z_{R_1, T_0}} \in L^{p_1}(0, T_0, L^q(A_{R_1})^3)$, $v(s)|_{A_{R_1}} \in W_{loc}^{1,1}(A_{R_1})^3$ ($s \in (0, T_0)$) and $\nabla_x v|_{Z_{R_1, T_0}} \in L^{p_2}(0, T_0, L^q(A_{R_1})^9)$, as well as for $x \in B_{R_0}^c$, $t \in (0, T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$, $l \in \{1, 2, 3\}$, the term $|\partial_x^\alpha \mathfrak{V}^{(\tau)}(n_l^{(A)} v)(x, t)|$ is bounded by*

$$\mathfrak{C}(\|v|_{Z_{R_1, t}}\|_{q, p_1; t} + \|\nabla_x v|_{Z_{R_1, t}}\|_{q, p_2; t}) \max_{j \in \{1, 2\}} (|x| \nu(x))^{-(3+|\alpha|)/2+1/(2p_j')},$$

where $(n_l^{(A)} v)(y, s) := n_l^{(A)}(y) v(y, s)$ for $y \in \partial A$, $s \in (0, T_0)$.

Lemma 4.9 ([19, Lemma 4.4]) *Recall that the Newton kernel \mathfrak{N} was introduced in Theorem 3.1. Let $A \subset B_{S_0}$ open and bounded, with Lipschitz boundary. Let $q \in (1, \infty)$. Then the estimate $|\int_{\partial A} (\partial^\alpha \nabla \mathfrak{N})(x-y) (n^{(A)} \cdot V)(y) dy| \leq \mathfrak{C} |x|^{-2-|\alpha|} \|V\|_q$ holds for $V \in L^q(A_{R_1})^3 \cap W^{1,1}(A_{R_1})^3$ with $\operatorname{div} V = 0$, and for $t \in (0, \infty)$, $x \in B_{R_0}^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. If $\int_{\partial A} n^{(A)} \cdot V dy = 0$, the factor $|x|^{-2-|\alpha|}$ may be replaced by $|x|^{-3-|\alpha|}$.*

Finally we restate the representation formula proved in [19] for regular solutions of (1.1). This formula will play a crucial role in the ensuing sections.

Theorem 4.3 ([19, Corollary 5.2]) *Let $A \subset B_{S_0}$ be open and with Lipschitz boundary. Let $k_0 \in \mathbb{N}$, $\varrho_k \in (1, \infty)$, and let $u^{(k)}$ belong to $C^0([0, T_0], L^{\varrho_k}(\overline{A}^c)^3)$ and to $W_{loc}^{1,1}(0, T_0, L^{\varrho_k}(\overline{A}^c)^3)$, for $1 \leq k \leq k_0$. Put $u = \sum_{k=1}^{k_0} u^{(k)}$. (This means in particular that $u|_{A_R \times [0, \infty)} \in C^0([0, \infty), L^q(A_R)^3)$ for $R \in [S_0, \infty)$, $q \in (1, \infty)$ with $q \leq \varrho_k$ for $1 \leq k \leq k_0$.) Let $\pi : (0, T_0) \mapsto W_{loc}^{1,1}(\overline{A}^c)$, $n_0 \in \mathbb{N}$, $p_j \in (1, \infty)$ and $f^{(j)} \in L_{loc}^1(0, T_0, L^{p_j}(\overline{A}^c)^3)$ for $1 \leq j \leq n_0$. Suppose that $u(s) \in W_{loc}^{2,1}(\overline{A}^c)^3$ for a. e. $t \in (0, T_0)$, and*

$$u'(s) - \Delta_x u(s) + \tau \partial_{x_1} u(s) + \nabla_x \pi(s) = f(s), \quad \operatorname{div}_x u(s) = 0 \quad \text{for a. e. } s \in (0, T_0), \quad (4.10)$$

with $f := \sum_{j=1}^{n_0} f^{(j)}$, $u' := \sum_{k=1}^{k_0} (u^{(k)})'$.

In addition suppose there is $q_1 \in (1, \infty)$ such that $\nabla_y u(s)|_{B_{R_0}^c} \in L^{q_1}(B_{R_0}^c)^9$ for a. e. $s \in (0, T_0)$ and $\nabla_y u|_{A_{R_0} \times (0, T_0)} \in L_{loc}^1([0, T_0], L^{q_1}(A_{R_0})^9)$. Assume that $u|_{A_{R_0} \times (0, T_0)} \in$

$L_{loc}^1(0, T_0, W^{2,1}(A_{R_0})^3)$, $\pi|_{A_{R_0} \times (0, T_0)} \in L_{loc}^1(0, T_0, W^{1,1}(A_{R_0}))$, and there are numbers $a_0 \in \mathbb{N}$, $\gamma_j \in (1, \infty)$ as well as functions $\pi^{(j)} : (0, T_0) \mapsto L_{loc}^1(\overline{B_{R_0}}^c)$ for $1 \leq j \leq a_0$ such that

$$\pi(s)|_{\overline{B_{R_0}}^c} = \sum_{j=1}^{a_0} \pi^{(j)}(s), \quad \sum_{j=1}^{a_0} \int_{\overline{B_{R_0}}^c} \left(|\pi^{(j)}(y, s)| [(1 + |y|) \ln(2 + |y|)]^{-1} \right)^{\gamma_j} dy < \infty$$

for a. e. $s \in (0, T_0)$. Then there is a subset \mathfrak{T}_{T_0} of $(0, T_0)$ of measure zero such that for $t \in (0, T_0) \setminus \mathfrak{T}_{T_0}$, the equation

$$\begin{aligned} u(x, t) = & \mathfrak{R}^{(\tau)}(f)(x, t) + \mathfrak{J}^{(\tau)}(u(0))(x, t) - \sum_{l=1}^3 \partial_{x_l} \mathfrak{V}^{(\tau)}(n_l^{(A)} u)(x, t) \\ & - \int_{\partial A} (\nabla \mathfrak{N})(x - y) (n^{(A)}(y) \cdot u(y, t)) d\sigma_y + \mathfrak{K}_{R_0, S_0, \varphi_0, A, T_0}(u)(x, t) \\ & - \int_{A_{R_1}} \mathfrak{G}_{R_0, S_0, \varphi_0}(x, y, t) \cdot u(y, 0) dy - \int_0^t \int_{A_{R_1}} \mathfrak{G}_{R_0, S_0, \varphi_0}(x, y, t - s) \cdot f(y, s) dy ds \end{aligned} \quad (4.11)$$

holds for $x \in \overline{B_{R_0}}^c \setminus N_t$, where N_t is some subset of $\overline{B_{R_0}}^c$ with measure zero. The function $\mathfrak{K}_{R_0, S_0, \varphi_0, A, T_0}(u)$ is defined in (4.17), and the function $\mathfrak{G}_{R_0, S_0, \varphi_0}$ in Theorem 4.3.

Next we restate a tool which will help to extend the integral representation (4.11) to solutions of (4.10) that are more general than the ones in the preceding theorem.

Lemma 4.10 ([19, Lemma 5.3]) *Let $A \subset B_{S_0}$ be open and bounded, with Lipschitz boundary. Let $k_0 \in \mathbb{N}$, $\varrho_k \in (1, \infty)$, $u^{(k)} \in C^0([0, T_0], L^{\varrho_k}(\overline{A}^c)^3)$ for $1 \leq k \leq k_0$, and put $u = \sum_{k=1}^{k_0} u^{(k)}$. In particular $u|_{A_R \times [0, \infty)} \in C^0([0, \infty), L^q(A_R)^3)$ for $R \in [S_0, \infty)$, $q \in (1, \min\{\varrho_k : 1 \leq k \leq k_0\}]$. Further suppose that $u(s) \in W_{loc}^{1,1}(\overline{A}^c)^3$ for a. e. $s \in (0, T_0)$, $\operatorname{div}_x u = 0$, and $\nabla_x u|_{A_{R_1} \times (0, T_0)} \in L_{loc}^1([0, T_0], L^{q_1}(A_{R_1})^9)$ for some $q_1 \in (1, \infty)$. Furthermore let $n_0 \in \mathbb{N}$, $p_j \in (1, \infty)$, $f^{(j)} \in L_{loc}^1([0, T_0], L^{p_j}(\overline{A}^c)^3)$ for $1 \leq j \leq n_0$, and put $f := \sum_{j=1}^{n_0} f^{(j)}$.*

Fix some function $\zeta \in C^\infty(\mathbb{R})$ with $\zeta|_{(-\infty, 1]} = 0$, $\zeta|_{[2, \infty)} = 1$, $0 \leq \zeta \leq 1$ and $\zeta' \geq 0$. For $\epsilon \in (0, \infty)$, $r \in \mathbb{R}$, define $\zeta_\epsilon(r) := \zeta(r/\epsilon)$. Put $u_\epsilon^{(k)}(s) := \zeta_\epsilon(s) u^{(k)}(s)$, $u_\epsilon(s) := \zeta_\epsilon(s) u(s)$, $f_\epsilon(s) := \zeta_\epsilon(s) f(s)$ and $g_\epsilon(s) := f_\epsilon(s) + \zeta'_\epsilon(s) u(s)$ for $s \in (0, T_0)$, $\epsilon \in (0, \infty)$, $1 \leq k \leq k_0$.

Let $t \in (0, T_0)$. Suppose there is some $\epsilon_0 \in (0, \infty)$ such that for $\epsilon \in (0, \epsilon_0]$, equation (4.11) holds with u , f replaced by u_ϵ and g_ϵ , respectively, if $x \in \overline{B_{R_0}}^c \setminus N_{t, \epsilon}$ for some subset $N_{t, \epsilon}$ of $\overline{B_{R_0}}^c$ of measure zero. (This means in particular that the second from last term in (4.11) vanishes.) Then there is some zero-measure set $N_t \subset \overline{B_{R_0}}^c$ such that equation (4.11) remains valid for u and f if $x \in \overline{B_{R_0}}^c \setminus N_t$.

5 Weak solutions of the Oseen system: representation formula and pointwise spatial decay.

We begin by introducing a L^q -weak solution to (1.1), with assumptions chosen in such a way that it will be possible to prove the representation formula (4.11) for this solution.

In order to deduce a decay estimate from this formula, we will require somewhat stronger conditions (see Theorem 5.2). We again point out that $T_0, S_0, R_0, R_1, \varphi_0$ and Ω are introduced at the beginning of Section 2.

Fix parameters $n_0, m_0 \in \mathbb{N}$, $\tilde{p}, q_0, q_1, p_1, \dots, p_{n_0}, \varrho_1, \dots, \varrho_{m_0} \in (1, \infty)$, as well as functions $u : (0, T_0) \mapsto W_{loc}^{1,1}(\overline{\Omega}^c)^3$, $f^{(j)} \in L_{loc}^1([0, T_0), L^{p_j}(\overline{\Omega}^c)^3)$ for $1 \leq j \leq n_0$, $G^{(l)} \in C^0([0, T_0), L^{\varrho_l}(\overline{B_{S_0}^c})^3)$ for $1 \leq l \leq m_0$, $U_0 \in L^{\tilde{p}}(\overline{\Omega}^c)^3$ with the following properties:

$u|_{\Omega_{S_0} \times (0, T_0)} \in L_{loc}^1([0, T_0), L^{q_0}(\Omega_{S_0})^3)$, $\operatorname{div}_x u(t) = 0$ and $u(t)|_{\overline{B_{S_0}^c}} = \sum_{l=1}^{m_0} G^{(l)}(t)$ for $t \in (0, T_0)$, $\nabla_x u \in L_{loc}^1([0, T_0), L^{q_1}(\overline{\Omega}^c)^3)$,

$$\int_0^{T_0} \int_{\overline{\Omega}^c} \left(-\gamma'(t) u(t) \cdot \vartheta + \gamma(t) [\nabla_x u(t) \cdot \nabla \vartheta + \tau \partial_{x_1} u(t) \cdot \vartheta - f(t) \cdot \vartheta] \right) dx dt \quad (5.1)$$

$$-\gamma(0) \int_{\overline{\Omega}^c} U_0 \cdot \vartheta dx = 0 \quad \text{for } \gamma \in C_0^\infty([0, T_0)), \vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c), \quad \text{with } f := \sum_{j=1}^{n_0} f^{(j)}.$$

There will be no claim that the preceding assumptions imply $u(0) = U_0$. But as it will turn out, they are sufficient to derive a representation formula. The case $m_0 > 1$ or $n_0 > 1$ and the distinction between the functions $G^{(l)}$ on $\overline{B_{S_0}^c} \times (0, \infty)$ with $u(t)|_{\overline{B_{S_0}^c}} = \sum_{l=1}^{m_0} G^{(l)}(t)$ on the one hand and $u|_{\Omega_{S_0} \times (0, T_0)}$ on the other one are introduced in order to take account of some technical difficulties arising when the theory in this work is applied to a nonlinear problem in [20], to the proof of Theorem 6.2 below, and in order to avoid any assumptions on $\partial\Omega$ stronger than Lipschitz regularity. Any reader who wants to avoid these technicalities may consider the case that $\partial\Omega$ is smooth, $u \in C^0([0, T_0), L^{q_0}(\overline{\Omega}^c)^3)$, $m_0 = n_0 = 1$ and $p_1 = q_1$. All the main difficulties of our proofs will then still be present.

Until further notice, we suppose that $T_0 = \infty$. It is only at the end of the present section that we will turn to the case $T_0 < \infty$ (Corollary 5.2).

In the ensuing lemma, we cut off the functions $u, f^{(j)}$ and $G^{(l)}$ near the instant $t = 0$, and then present the version of (5.1) satisfied by the extension of these modified functions to the whole real axis. In the lemma after that, we apply Friedrich's mollifier to these extensions. Both lemmas mainly serve to introduce notation and collect obvious facts which constitute the basis for the rest of this section.

Lemma 5.1 *Fix some $q \in (1, \infty)$ with*

$$q \leq \min(\{q_0, q_1, 5/4\} \cup \{\varrho_l : 1 \leq l \leq m_0\} \cup \{p_j : 1 \leq j \leq n_0\}).$$

Then the function $u|_{\Omega_R \times (0, \infty)}$ belongs to $L_{loc}^1([0, \infty), L^q(\Omega_R)^3)$ and $u|_{A_{R,S_0} \times [0, \infty)}$ to $C^0([0, \infty), L^q(A_{R,S_0})^3)$, for $R \in [S_0, \infty)$.

Choose functions ζ_ϵ for $\epsilon \in (0, \infty)$ as in Lemma 4.10. For $s, \epsilon \in (0, \infty)$, define $u_\epsilon(s) := \zeta_\epsilon(s) u(s)$, $f_\epsilon^{(j)}(s) := \zeta_\epsilon(s) f^{(j)}(s)$ ($1 \leq j \leq n_0$) and $G_\epsilon^{(l)}(s) := \zeta_\epsilon(s) G^{(l)}(s)$ ($1 \leq l \leq m_0$), $p_{n_0+1} := q_0$, $f_\epsilon^{(n_0+1)} := \zeta_\epsilon(s) \chi_{\Omega_{S_0}} u(s)$, $p_{n_0+1+l} := \varrho_l$, $f_\epsilon^{(n_0+1+l)}(s) := \zeta_\epsilon(s) \tilde{G}^{(l)}(s)$ for $1 \leq l \leq m_0$, where $\tilde{G}^{(l)}$ denotes the zero extension of $G^{(l)} : \overline{B_{S_0}^c} \times [0, \infty) \mapsto \mathbb{R}^3$ to $\overline{\Omega}^c \times [0, \infty)$. Then $f_\epsilon^{(n_0+k)} \in L_{loc}^1([0, \infty), L^{p_{n_0+k}}(\overline{\Omega}^c)^3)$ for $1 \leq k \leq m_0 + 1$, and $\zeta'_\epsilon(s) u(s) = \sum_{k=1}^{m_0+1} f_\epsilon^{(n_0+k)}(s)$ ($s \in (0, \infty)$).

The functions $u_\epsilon(t)$, $f_\epsilon^{(j)}(t)$, $G_\epsilon^{(l)}(t)$, $\tilde{G}_\epsilon^{(l)}(t)$ ($t \in (0, \infty)$) are extended by zero to the real axis \mathbb{R} ($1 \leq j \leq n_0 + m_0 + 1$, $1 \leq l \leq m_0$), without change of notation. Then $f_\epsilon^{(j)} \in L_{loc}^1(\mathbb{R}, L^{p_j}(\overline{\Omega^c})^3)$ for $j \in \{1, \dots, n_0 + m_0 + 1\}$, $G_\epsilon^{(l)} \in C^0(\mathbb{R}, L^{q_l}(\overline{B_{S_0}^c})^3)$ and $\tilde{G}_\epsilon^{(l)} \in C^0(\mathbb{R}, L^{q_l}(\overline{\Omega^c})^3)$ for $1 \leq l \leq m_0$. In addition $u_\epsilon|_{\Omega_R \times \mathbb{R}} \in L_{loc}^1(\mathbb{R}, L^q(\Omega_R)^3)$ and $u_\epsilon|_{A_{R,S_0} \times \mathbb{R}} \in C^0(\mathbb{R}, L^q(A_{R,S_0})^3)$ for $R \in [S_0, \infty)$, $u_\epsilon = \chi_{\Omega_{S_0}} u_\epsilon + \sum_{j=1}^{m_0} \tilde{G}_\epsilon^{(j)}$ and $u_\epsilon|_{\overline{B_{S_0}^c} \times \mathbb{R}} = \sum_{l=1}^{m_0} G_\epsilon^{(l)}$, $u(s) \in W_{loc}^{1,1}(\overline{\Omega^c})^3$ for $s \in \mathbb{R}$, $\text{div}_x u_\epsilon = 0$, $\nabla_x u_\epsilon \in L_{loc}^1(\mathbb{R}, L^{q_1}(\overline{\Omega^c})^9)$,

$$\int_{\mathbb{R}} \int_{\overline{\Omega^c}} \left(-\gamma'(t) u_\epsilon(t) \cdot \vartheta + \gamma(t) [\nabla_x u_\epsilon(t) \cdot \nabla \vartheta + \tau \partial_{x_1} u_\epsilon(t) \cdot \vartheta - \sum_{j=1}^{n_0+m_0+1} f_\epsilon^{(j)}(t) \cdot \vartheta] \right) dx dt = 0 \quad \text{for } \gamma \in C_0^\infty(\mathbb{R}), \vartheta \in C_{0,\sigma}^\infty(\overline{\Omega^c}). \quad (5.2)$$

We remark that the functions $f_\epsilon^{(n_0+1)}$, ..., $f_\epsilon^{(n_0+m_0+1)}$ are perturbation terms which arise in (5.2) due to the cut-off function ζ_ϵ and the decomposition of $u(s)$ into the sum $\chi_{\Omega_{S_0}} u(s) + \sum_{l=1}^{m_0} \tilde{G}_\epsilon^{(l)}$. In the ensuing lemma, we use Friedrich's mollifier with respect to the time variable, as defined in the passage preceding Theorem 2.6. This mollifier is applied to functions with domain \mathbb{R} and values in Banach spaces, in this context spaces $L^r(A)^3$, for certain $r \in (1, \infty)$ and certain open subsets A of \mathbb{R}^3 . The functions with index ϵ are those introduced in Lemma 5.1.

Concerning u_ϵ , there is no Banach space B which we deem useful and which is such that $u_\epsilon \in L_{loc}^1(\mathbb{R}, B)$. So we cannot directly apply our definition of Friedrich's mollifier to u_ϵ . Instead we use the equation $u_\epsilon = \chi_{\Omega_{S_0}} u_\epsilon + \sum_{j=1}^{m_0} \tilde{G}_\epsilon^{(j)}$ in order to define a more regular version of u_ϵ , denoted by $u_{\epsilon,\delta}$. Note that due to the relation $\nabla_x u_\epsilon \in L_{loc}^1(\mathbb{R}, L^{q_1}(\overline{\Omega^c})^3)$, no such problem arises with $\nabla_x u_\epsilon$.

Lemma 5.2 *Let $\epsilon, \delta \in (0, \infty)$. Put $f_{\epsilon,\delta}^{(j)} := (f_\epsilon^{(j)})_\delta$ ($1 \leq j \leq n_0 + m_0 + 1$), $G_{\epsilon,\delta}^{(l)} := (G_\epsilon^{(l)})_\delta$, $\tilde{G}_{\epsilon,\delta}^{(l)} := (\tilde{G}_\epsilon^{(l)})_\delta$ ($1 \leq l \leq m_0$). Then, for j, l as before, the relations $f_{\epsilon,\delta}^{(j)} \in C^\infty(\mathbb{R}, L^{p_j}(\overline{\Omega^c})^3)$, $G_{\epsilon,\delta}^{(l)} \in C^\infty(\mathbb{R}, L^{q_l}(\overline{B_{S_0}^c})^3)$ and $\tilde{G}_{\epsilon,\delta}^{(l)} \in C^\infty(\mathbb{R}, L^{q_l}(\overline{\Omega^c})^3)$ hold. Define $u_{\epsilon,\delta} := (\chi_{\Omega_{S_0}} u_\epsilon)_\delta + \sum_{l=1}^{m_0} \tilde{G}_{\epsilon,\delta}^{(l)}$. Then $u_{\epsilon,\delta}|_{\Omega_R \times \mathbb{R}} = (u_\epsilon|_{\Omega_R \times \mathbb{R}})_\delta \in C^\infty(\mathbb{R}, L^q(\Omega_R)^3)$ for $R \in [S_0, \infty)$, and $u_{\epsilon,\delta}|_{\overline{B_{S_0}^c} \times \mathbb{R}} = \sum_{l=1}^{m_0} G_{\epsilon,\delta}^{(l)}$. Moreover $u_{\epsilon,\delta}(t) \in W_{loc}^{1,1}(\overline{\Omega^c})^3$ for $t \in \mathbb{R}$, $\partial_{x_k} u_{\epsilon,\delta} = (\partial_{x_k} u_\epsilon)_\delta \in C^\infty(\mathbb{R}, L^{q_1}(\overline{\Omega^c})^3)$ for $1 \leq k \leq 3$, and $\text{div}_x u_{\epsilon,\delta} = 0$.*

Define $s_0 := n_0 + 2m_0 + 2$, $f_{\epsilon,\delta}^{(n_0+m_0+2)} := -(\chi_{\Omega_{S_0}} u_\epsilon)_\delta'$, $p_{n_0+m_0+2} := q$, $f_{\epsilon,\delta}^{(n_0+m_0+2+l)} := -(\tilde{G}_{\epsilon,\delta}^{(l)})'$ and $p_{n_0+m_0+2+l} := q_l$ for $1 \leq l \leq m_0$. Then $f_{\epsilon,\delta}^{(j)} \in C^\infty(\mathbb{R}, L^{p_j}(\overline{\Omega^c})^3)$ for $n_0 + m_0 + 2 \leq j \leq s_0$.

For $t \in \mathbb{R}$, the function $u_{\epsilon,\delta}(t)$ satisfies (3.1) with A, U, F replaced by $\overline{\Omega^c}$, $u_{\epsilon,\delta}(t)$ and $\sum_{j=1}^{s_0} f_{\epsilon,\delta}^{(j)}(t)$, respectively, and with $\lambda = 0$.

The last statement of the lemma is the crucial one. It means that $u_{\epsilon,\delta}(t)$ is a weak solution in $\overline{\Omega^c}$ of the stationary Oseen system, with right-hand side F as indicated.

Proof of Lemma 5.2: The regularity properties of $f_{\epsilon,\delta}^{(j)}$, $G_{\epsilon,\delta}^{(l)}$, $\tilde{G}_{\epsilon,\delta}^{(l)}$ and $(\chi_{\Omega_R} u_\epsilon)_\delta$ follow

with Theorem 2.6 from the properties of $f_\epsilon^{(j)}$, $G_\epsilon^{(l)}$, $\tilde{G}_\epsilon^{(l)}$ and $\chi_{\Omega_R} u_\epsilon$, respectively, as stated in Lemma 5.1 ($1 \leq j \leq s_0$, $1 \leq l \leq m_0$ and $R \in [S_0, \infty)$). Lemma 2.2 and the equation $u_\epsilon|_{\overline{B_{S_0}^c} \times \mathbb{R}} = \sum_{l=1}^{m_0} G_\epsilon^{(l)}(t)$ (Lemma 5.1) yield that $(u_\epsilon|_{\Omega_R \times \mathbb{R}})_\delta = u_{\epsilon,\delta}|_{\Omega_R \times \mathbb{R}}$ for $R \in [S_0, \infty)$ and $u_{\epsilon,\delta}|_{\overline{B_{S_0}^c} \times \mathbb{R}} = \sum_{l=1}^{m_0} G_{\epsilon,\delta}^{(l)}$.

In order to check existence of weak derivatives of $u_{\epsilon,\delta}$ with respect to the space variables, let $k \in \{1, 2, 3\}$, $t \in \mathbb{R}$, $\psi \in C_0^\infty(\mathbb{R}^3)^3$ and $R \in [S_0, \infty)$ with $\text{supp}(\psi) \subset \Omega_R$. Then $\int_{\overline{\Omega}^c} \partial_k \psi \cdot u_{\epsilon,\delta}(t) dx = \int_{\Omega_R} \partial_k \psi \cdot (u_\epsilon|_{\Omega_R \times \mathbb{R}})_\delta(t) dx$ by what was shown above. The operator $V \mapsto \int_{\Omega_R} \partial_k \psi \cdot V dx$ ($V \in L^q(\Omega_R)^3$) is linear and bounded, so Theorem 2.4 yields that $\int_{\Omega_R} \partial_k \psi \cdot (u_\epsilon|_{\Omega_R \times \mathbb{R}})_\delta(t) dx = \int_{\mathbb{R}} \rho_\delta(t-s) \int_{\Omega_R} \partial_k \psi \cdot u_\epsilon(s) dx ds$, with ρ_δ defined in the passage preceding Theorem 2.6. Since $\int_{\Omega_R} \partial_k \psi \cdot u_\epsilon(s) dx = - \int_{\Omega_R} \psi \cdot \partial_{x_k} u_\epsilon(s) dx$ for a. e. $s \in \mathbb{R}$, and by a similar reasoning as before, we further get that the right-hand side of the second from last equation equals $-\int_{\overline{\Omega}^c} \psi \cdot (\partial_{x_k} u_\epsilon)_\delta(t) dx$. Altogether we arrive at the equation $\int_{\overline{\Omega}^c} \partial_k \psi \cdot u_{\epsilon,\delta}(t) dx = - \int_{\overline{\Omega}^c} \psi \cdot (\partial_{x_k} u_\epsilon)_\delta(t) dx$. Thus we have shown that $u_{\epsilon,\delta}(t) \in W_{loc}^{1,1}(\overline{\Omega}^c)^3$ for $t \in \mathbb{R}$ and $\partial_{x_k} u_{\epsilon,\delta} = (\partial_{x_k} u_\epsilon)_\delta$ for $1 \leq k \leq 3$. It follows with Theorem 2.6 and Lemma 5.1 that $\partial_{x_k} u_{\epsilon,\delta} \in C^\infty(\mathbb{R}, L^{q_1}(\overline{\Omega}^c)^3)$. The same references yield $f_{\epsilon,\delta}^{(j)} \in C^\infty(\mathbb{R}, L^{p_j}(\overline{\Omega}^c)^3)$ for $n_0 + m_0 + 2 \leq j \leq s_0$.

In order to determine the PDE satisfied by $u_{\epsilon,\delta}$, take $t \in \mathbb{R}$ and $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$. Our starting point is equation (5.2) with $\gamma(s) = \rho_\delta(t-s)$ for $s \in \mathbb{R}$. (We recall again that ρ_δ is introduced in the passage preceding Theorem 2.6.) For $\sigma \in \{0, 1\}$, $k \in \{1, 2, 3\}$, $p \in \{q_1, q\} \cup \{p_j : 1 \leq j \leq n_0 + m_0 + 1\} \cup \{q_l : 1 \leq l \leq m_0\}$, the functional $V \mapsto \int_{\overline{\Omega}^c} V \cdot \partial_k^\sigma \vartheta dx$ ($V \in L^p(\overline{\Omega}^c)^3$) is linear and bounded. Therefore by Theorem 2.4 and because $(\partial_{x_k} u_\epsilon)_\delta = \partial_{x_k} u_{\epsilon,\delta}$,

$$\int_{\mathbb{R}} \rho_\delta(t-s) \int_{\overline{\Omega}^c} \partial_{x_k} u_\epsilon(s) \cdot \partial_k \vartheta dx ds = \int_{\overline{\Omega}^c} \partial_{x_k} u_{\epsilon,\delta}(t) \cdot \partial_k \vartheta dx, \quad (5.3)$$

$$\int_{\mathbb{R}} \rho_\delta(t-s) \int_{\overline{\Omega}^c} \partial_{x_1} u_\epsilon(s) \cdot \vartheta dx ds = \int_{\overline{\Omega}^c} \partial_{x_1} u_{\epsilon,\delta}(t) \cdot \vartheta dx, \quad (5.4)$$

$$\int_{\mathbb{R}} \rho_\delta(t-s) \int_{\overline{\Omega}^c} f_\epsilon^{(j)}(s) \cdot \vartheta dx ds = \int_{\overline{\Omega}^c} f_{\epsilon,\delta}^{(j)}(t) \cdot \vartheta dx, \quad (5.5)$$

for $k \in \{1, 2, 3\}$, $1 \leq j \leq n_0 + m_0 + 1$. Put $K^{(1)} := \chi_{\Omega_{S_0}} \cdot u_\epsilon$, $K^{(l)} := \tilde{G}_\epsilon^{(l)}$ for $2 \leq l \leq m_0 + 1$. Then, using Theorem 2.4, the equation $u_\epsilon = \sum_{l=1}^{m_0+1} K^{(l)}$ (Lemma 5.1) and Theorem 2.6, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\overline{\Omega}^c} \partial_s(\rho_\delta(t-s)) u_\epsilon(s) \cdot \vartheta dx ds &= - \int_{\overline{\Omega}^c} \left(\int_{\mathbb{R}} \rho'_\delta(t-s) u_\epsilon(s) ds \right) \cdot \vartheta dx \\ &= - \int_{\overline{\Omega}^c} \left(\int_{\mathbb{R}} \rho'_\delta(t-s) \sum_{l=1}^{m_0+1} K^{(l)}(s) ds \right) \cdot \vartheta dx = \sum_{j=m_0+n_0+2}^{s_0} \int_{\overline{\Omega}^c} f_{\epsilon,\delta}^{(j)}(t) \cdot \vartheta dx. \end{aligned} \quad (5.6)$$

We may deduce from (5.2) and (5.3) – (5.6) that $u_{\epsilon,\delta}(t)$ satisfies the variational problem in (3.1) with A, U, F replaced as indicated in the lemma, and with $\lambda = 0$. Again by the equation $\partial_{x_k} u_{\epsilon,\delta} = (\partial_{x_k} u_\epsilon)_\delta$, and because $\text{div}_x u_\epsilon = 0$ (Lemma 5.2), we obtain $\text{div}_x u_{\epsilon,\delta} = 0$. Thus $u_{\epsilon,\delta}$ satisfies (3.1) in full. \square

Now we are in a position to deal with the main difficulty of this section, that is, to show that $u_{\epsilon,\delta}$ satisfies the representation formula (4.11). The notation from Lemma 5.1 and 5.2 will be used without further notice.

Theorem 5.1 *Let $\epsilon, \delta \in (0, \infty)$. Then, for any $t \in (0, \infty)$, there is a set $N_t \subset \overline{B_{R_0}}^c$ of measure zero such that equation (4.11) holds for $x \in \overline{B_{R_0}}^c \setminus N_t$, with $A = B_{S_0}$ and u, f replaced by $u_{\epsilon, \delta}|_{\overline{B_{S_0}}^c \times (0, \infty)}$ and $\sum_{j=1}^{n_0+m_0+1} f_{\epsilon, \delta}^{(j)}|_{\overline{B_{S_0}}^c \times (0, \infty)}$, respectively.*

Proof: Since $\overline{\Omega} \subset B_{S_0}$ by our assumptions at the beginning of this section, we may fix $S_1 \in (0, S_0)$ with $\overline{\Omega} \subset S_{S_1}$. Let $t \in \mathbb{R}$. For brevity, we set $U := u_{\epsilon, \delta}(t)$, $F^{(j)} := f_{\epsilon, \delta}^{(j)}(t)$ for $1 \leq j \leq s_0$. Due to the choice of q in Lemma 5.1 and by the properties of $u_{\epsilon, \delta}$ and $f_{\epsilon, \delta}^{(j)}$ (Lemma 5.2), we have $U \in W_{loc}^{1,q}(\overline{\Omega}^c)^3$, $\nabla U \in L^q(\overline{\Omega}^c)^9$, $F^{(j)} \in L_{loc}^q(\overline{\Omega}^c)^3 \cap L^{p_j}(\overline{\Omega}^c)^3$ for $1 \leq j \leq s_0$, and the function U satisfies (3.1) with $\lambda = 0$ and with A, F replaced by $\overline{\Omega}^c$ and $\sum_{j=1}^{s_0} F^{(j)}$, respectively. Thus, by Theorem 3.2, there is $\Pi \in L_{loc}^p(\overline{\Omega}^c)^3$ such that (3.2) holds with $A = \overline{\Omega}^c$, $\lambda = 0$ and $F = \sum_{j=1}^{s_0} F^{(j)}$. Since the latter equation remains valid when we replace Π by $\Pi + c$, for any number $c \in \mathbb{R}$, we may assume in addition that $\int_{A_{S_0, S_1}} \Pi dx = 0$. Equation (3.2) is valid in particular for $\vartheta \in C_0^\infty(A_{S_0, S_1})^3$. Therefore, according to Theorem 3.2, this function Π satisfies the estimate

$$\|\Pi|_{A_{S_0, S_1}}\|_q \leq C(q, S_0, S_1) (\|\nabla U|_{A_{S_0, S_1}}\|_q + \|\sum_{j=1}^{s_0} F^{(j)}|_{A_{S_0, S_1}}\|_q). \quad (5.7)$$

Due to (3.2) and the regularity properties indicated above for $U, F^{(j)}$ and Π , Theorem 3.3 now yields that $U \in W_{loc}^{2,q}(\overline{\Omega}^c)^3$, $\Pi \in W_{loc}^{1,q}(\overline{\Omega}^c)$ and the pair (U, Π) solves (3.4) in $\overline{\Omega}^c$ with $\lambda = 0$ and $F = \sum_{j=1}^{s_0} F^{(j)}$. We may choose a function $\tilde{\varphi} \in C^\infty(\mathbb{R}^3)$ with $0 \leq \tilde{\varphi} \leq 1$, $\tilde{\varphi}|_{B_{S_1+(S_0-S_1)/4}} = 0$, $\tilde{\varphi}|_{B_{S_1+(S_0-S_1)/2}^c} = 1$. Since $\overline{\Omega} \subset B_{S_1}$ and $\text{supp}(\partial^\alpha \tilde{\varphi}) \subset A_{S_0, S_1}$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \geq 1$, and by the regularity properties of U and Π just derived, the function

$$\tilde{F} := -\Delta \tilde{\varphi} U - 2(\nabla \tilde{\varphi} \cdot \nabla U_j)_{1 \leq j \leq 3} + \tau \partial_1 \tilde{\varphi} U + \Pi \nabla \tilde{\varphi} \quad (5.8)$$

belongs to $W^{1,q}(\mathbb{R}^3)$, as does $\nabla \tilde{\varphi} \cdot U$. In addition $\text{supp}(\tilde{F}) \cup \text{supp}(\nabla \tilde{\varphi} \cdot U) \subset A_{S_0, S_1}$. Put $\tilde{W} := \mathfrak{N} * (\nabla \tilde{\varphi} \cdot U)$; see Theorem 3.1. Since $q < 3/2$ by the choice of q in Lemma 5.1, we know by Theorem 3.1 that \tilde{W} is well defined and

$$\begin{aligned} \tilde{W} &\in W_{loc}^{3,q}(\mathbb{R}^3), \quad \nabla \tilde{W} \in L^{(1/q-1/3)^{-1}}(\mathbb{R}^3)^3, \quad \partial_l \partial_m \tilde{W} \in W^{1,q}(\mathbb{R}^3)^9, \\ \Delta \tilde{W} &= -\nabla \tilde{\varphi} \cdot U, \quad \|\partial_k \partial_l \partial_m \tilde{W}\|_q + \|\partial_k \partial_l \tilde{W}\|_q \leq C(q) \|\nabla \tilde{\varphi} \cdot U\|_{1,q} \quad (1 \leq k, l, m \leq 3). \end{aligned} \quad (5.9)$$

We put

$$\begin{aligned} \tilde{U} &:= \tilde{\varphi} U + \nabla \tilde{W}, \quad \tilde{\Pi} := \tilde{\varphi} \Pi, \quad \tilde{F}^{(j)} := \tilde{\varphi} F^{(j)} \text{ for } 1 \leq j \leq s_0, \\ p_{s_0+1} &:= q, \quad \tilde{F}^{(s_0+1)} := \tilde{F} - \Delta \nabla \tilde{W} + \tau \partial_1 \nabla \tilde{W}. \end{aligned} \quad (5.10)$$

Note that $\text{div} \nabla \tilde{W} = \Delta \tilde{W} = -\nabla \tilde{\varphi} \cdot U$, hence $\text{div} \tilde{U} = 0$. As a consequence of these observations and definitions, we have $\tilde{U} \in W_{loc}^{2,q}(\mathbb{R}^3)^3$, $\tilde{\Pi} \in W_{loc}^{1,q}(\mathbb{R}^3)$, $\tilde{F}^{(j)} \in L^{p_j}(\mathbb{R}^3)^3$ for $1 \leq j \leq s_0 + 1$, $\tilde{\varphi} U|_{B_{S_0}^c} = U|_{B_{S_0}^c}$, $\tilde{\Pi}|_{B_{S_0}^c} = \Pi|_{B_{S_0}^c}$, and the pair $(\tilde{U}, \tilde{\Pi})$ solves (3.4) in \mathbb{R}^3 with $\lambda = 0$ (stationary Oseen system) and with $F = \sum_{j=1}^{s_0+1} \tilde{F}^{(j)}$. Since $\tilde{F}^{(j)} \in L^{p_j}(\mathbb{R}^3)^3$ for $1 \leq j \leq s_0 + 1$, Theorem 3.5 yields functions $W^{(j)} \in W_2^{2,p_j}(\mathbb{R}^3)^3$, $\Pi^{(j)} \in W_1^{1,p_j}(\mathbb{R}^3)$ such that $\partial_1 W^{(j)} \in L^{p_j}(\mathbb{R}^3)^3$, equation (3.4) is satisfied in \mathbb{R}^3 with U, Π, F replaced by $W^{(j)}, \Pi^{(j)}, \tilde{F}^{(j)}$, respectively, and the inequality

$$\|\partial_l \partial_m W^{(j)}\|_{p_j} + \|\Pi^{(j)}\|_{W_1^{1,p_j}(\mathbb{R}^3)} \leq C(\tau, p_j) \|\tilde{F}^{(j)}\|_{p_j} \quad (1 \leq l, m \leq 3, 1 \leq j \leq s_0 + 1) \quad (5.11)$$

holds. Then $\bar{U} := \tilde{U} - \sum_{j=1}^{s_0+1} W^{(j)} \in W_{loc}^{2,q}(\mathbb{R}^3)^3$, $\bar{\Pi} := \tilde{\Pi} - \sum_{j=1}^{s_0+1} \Pi^{(j)} \in W_{loc}^{1,q}(\mathbb{R}^3)$, and $-\Delta \bar{U} + \tau \partial_1 \bar{U} + \nabla \bar{\Pi} = 0$, $\operatorname{div} \bar{U} = 0$ in \mathbb{R}^3 . It follows with Theorem 3.4 that $\bar{\Pi}$ is a C^∞ -function. Let us specify in which way the function \bar{U} satisfies the assumptions of Theorem 3.6. Put $q_j(0) := q_j(1) := p_j$ for $j \in \{1, \dots, s_0 + 1\}$. Then the definition of the space $W_2^{2,p_j}(\mathbb{R}^3)$ implies that $\int_{B_{R_0}^c} (|\partial^\alpha W^{(j)}(x)| [(1 + |x|)^{2-|\alpha|} \ln(2 + |x|)]^{-1})^{q_j(|\alpha|)} dx < \infty$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$ and $1 \leq j \leq s_0 + 1$. Put

$$\tilde{q} := \max(\{p_j : 1 \leq j \leq n_0\} \cup \{(1/q - 1/3)^{-1}, q_0\} \cup \max(\{\varrho_l : 1 \leq l \leq m_0\}) \quad (5.12)$$

and $\epsilon_0 := 1/(2\tilde{q})$. By the choice of $p_{n_0+1}, \dots, p_{s_0+1}$ (Lemma 5.1, 5.2, (5.10)), this means in particular that $\tilde{q} \geq p_j$ for $1 \leq j \leq s_0 + 1$, and $\tilde{q} \geq q$. Obviously $1 + |x| \leq C(R_0)|x|$ and $\ln(2 + |x|) \leq C(R_0, \epsilon_0)|x|^{\epsilon_0}$ for $x \in B_{R_0}^c$. Thus we may conclude for $1 \leq j \leq s_0 + 1$ that

$$\int_{B_{R_0}^c} (|\partial^\alpha W^{(j)}(x)| |x|^{-2-\epsilon_0})^{q_j(|\alpha|)} dx < \infty \quad \text{for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \quad (5.13)$$

Next we put $W^{(s_0+2)} := -\nabla \tilde{W}$, $q_{s_0+2}(0) := (1/q - 1/3)^{-1}$, $q_{s_0+2}(1) := q$. Then, due to (5.9), the relation in (5.13) is valid for $j = s_0 + 2$ as well. Since $\nabla U \in L^{q_1}(\bar{\Omega}^c)^3$ as mentioned above, $\tilde{\varphi}|_{B_{S_0}^c} = 1$ and $R_0 > S_0$, we get $\int_{B_{R_0}^c} (|\nabla(\tilde{\varphi}U)(x)| |x|^{-2-\epsilon_0})^{q_1} dx < \infty$.

We further note that by the definition of \tilde{U} (see (5.10)), \bar{U} and $W^{(s_0+2)}$, the equation $\nabla \bar{U} = \nabla(\tilde{\varphi}U) - \sum_{j=1}^{s_0+2} \nabla W^{(j)}$ holds. Thus, if in Theorem 3.6 we replace U , m_0 , $(V^{(m)})_{1 \leq m \leq m_0}$ by \bar{U} , $s_0 + 3$, $((\partial_\mu[\tilde{\varphi}U_\sigma])_{1 \leq \mu, \sigma \leq 3}, (\partial_\mu W_\sigma^{(1)})_{1 \leq \mu, \sigma \leq 3}, \dots, (\partial_\mu W_\sigma^{(s_0+2)})_{1 \leq \mu, \sigma \leq 3})$, respectively, the assumptions on ∇U are satisfied with $r = 2 + \epsilon_0$. Concerning those on U , we recall that by the definition of U and $u_{\epsilon, \delta}$, we have $\tilde{\varphi}U = -\sum_{j=s_0+3}^{s_0+3+m_0} W^{(j)}$, with $W^{(s_0+3)} := -\tilde{\varphi}(\chi_{\Omega_{S_0}} u_\epsilon)_\delta(t)$, $W^{(s_0+3+l)} := -\tilde{\varphi}\tilde{G}_{\epsilon, \delta}^{(l)}(t)$ for $1 \leq l \leq m_0$. Lemma 5.2 yields $W^{(j)} \in L^{p_j(0)}(\mathbb{R}^3)^3$ for $s_0 + 3 \leq j \leq s_0 + 3 + m_0$, with $p_{s_0+3}(0) := q$, $p_{s_0+3+l}(0) := \varrho_l$ for $1 \leq l \leq m_0$. It follows the relation in (5.13) holds for $s_0 + 3 \leq j \leq s_0 + 3 + m_0$ with $\alpha = 0$. We may conclude with the definition of \tilde{U} and $W^{(s_0+2)}$ that $\tilde{U} = -\sum_{j=s_0+2}^{s_0+3+m_0} W^{(j)}$. Now the definition of \bar{U} yields that $\bar{U} = -\sum_{j=1}^{s_0+m_0+3} W^{(j)}$. Therefore the assumptions on U in Theorem 3.6 are satisfied with \bar{U} , $s_0 + m_0 + 3$, $(-W^{(k)})_{1 \leq k \leq s_0+m_0+3}$ in the role of U , k_0 , $(U^{(k)})_{1 \leq k \leq k_0}$, respectively, and with $r = 2 + \epsilon_0$. At this point Theorem 3.6 yields that \bar{U} is a polynomial. By the definitions in (5.12) and the choice of the exponents $q_j(0)$, we have $\tilde{q} = \max\{q_j(0) : 1 \leq j \leq s_0 + 3 + m_0\}$ and $\epsilon_0 \in (0, 1/\tilde{q})$. Thus it follows from (5.13) with $\alpha = 0$, $1 \leq j \leq s_0 + m_0 + 3$ and from Lemma 3.1 that the degree of \bar{U} cannot exceed 1. As a first consequence of this result, we get $\partial^\alpha \bar{U} = 0$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$. Since $\bar{U} = \tilde{\varphi}U - \sum_{j=1}^{s_0+2} W^{(j)}$, $\tilde{\varphi}|_{B_{S_0}^c} = 1$ and $W^{(s_0+2)} = -\nabla \tilde{W}$, we thus find

$$\partial^\alpha U|_{B_{S_0}^c} = -\partial^\alpha(\nabla \tilde{W}) + \sum_{j=1}^{s_0+1} \partial^\alpha W^{(j)}|_{B_{S_0}^c} \quad \text{for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| = 2. \quad (5.14)$$

It further follows there is $c_1 \in \mathbb{R}$ with $\partial_1 \bar{U} = c_1$. On the other hand, $\tilde{\varphi}|_{B_{S_0}^c} = 1$ and $S_0 < R_0$, so $\partial_1 U|_{B_{R_0}^c} \in L^{q_1}(B_{R_0})^3$. Moreover $\partial_1 W^{(s_0+2)} = \partial_1 \nabla \tilde{W} \in L^q(\mathbb{R}^3)^3$ (see (5.9)), and $\partial_1 W^{(j)} \in L^{p_j}(\mathbb{R}^3)^3$ for $1 \leq j \leq s_0 + 1$ because $W^{(j)} \in W_2^{2,p_j}(\mathbb{R}^3)^3$. It follows from Lemma 3.2 and the equation $\bar{U} = \tilde{\varphi}U - \sum_{j=1}^{s_0+2} W^{(j)}$ that c_1 vanishes. Recalling that

$-\Delta \bar{U} + \tau \partial_1 \bar{U} + \nabla \bar{\Pi} = 0$, we thus obtain $\nabla \bar{\Pi} = 0$. Since $\bar{\Pi} \in C^\infty(\mathbb{R}^3)$, as shown above, and because $\bar{\Omega}^c$ is connected, we may conclude there is $c \in \mathbb{R}$ with $\bar{\Pi} = c$. But $\bar{\Pi} = \tilde{\varphi} \Pi - \sum_{j=1}^{s_0+1} \Pi^{(j)}$ by the definition of $\bar{\Pi}$, so we arrive at the equation

$$\Pi - c|\overline{B_{S_0}}|^c = \sum_{j=1}^{s_0+1} \Pi^{(j)}|\overline{B_{S_0}}|^c. \quad (5.15)$$

Hölder's inequality yields the estimate $\|\Pi^{(j)}|A_{R_0, S_0}\|_1 \leq \mathfrak{C} \|\Pi^{(j)}|A_{R_0, S_0}\|_{p_j}$, and it is obvious that $\|\Pi^{(j)}|A_{R_0, S_0}\|_{p_j} \leq \mathfrak{C} \left[\int_{A_{R_0, S_0}} (|\Pi^{(j)}(x)| [(1+|x|) \ln(2+|x|)]^{-1})^{p_j} dx \right]^{1/p_j}$, so we arrive at the inequality $\|\Pi^{(j)}|A_{R_0, S_0}\|_1 \leq \mathfrak{C} \|\Pi^{(j)}\|_{W_1^{1, p_j}(\mathbb{R}^3)}$, for $1 \leq j \leq s_0 + 1$. Obviously $\|\nabla \Pi^{(j)}|A_{R_0, S_0}\|_1 \leq \mathfrak{C} \|\nabla \Pi^{(j)}|A_{R_0, S_0}\|_{p_j} \leq \mathfrak{C} \|\nabla \Pi^{(j)}\|_{p_j} \leq \mathfrak{C} \|\Pi^{(j)}\|_{W_1^{1, p_j}(\mathbb{R}^3)}$. The constants in the estimates following (5.15) only depend on S_0, R_0 and p_j , with the relevant index j . These estimates and (5.11) yield $\|\Pi^{(j)}|A_{R_0, S_0}\|_{1,1} \leq C(S_0, R_0, p_j, \tau) \|\tilde{F}^{(j)}\|_{p_j}$ for $1 \leq j \leq s_0 + 1$. Due to (5.15), we thus get

$$\|\Pi - c|A_{R_0, S_0}\|_{1,1} \leq C(S_0, R_0, p_1, \dots, p_{s_0+1}, \tau) \sum_{j=1}^{s_0+1} \|\tilde{F}^{(j)}\|_{p_j}. \quad (5.16)$$

Once more starting with Hölder's inequality, then using (5.9) and the relations $q \leq q_1$ (Lemma 5.1) and $\partial_l U = (\partial x_l u_\epsilon)_\delta(t)$ (Lemma 5.2), we get

$$\begin{aligned} \|\partial_k \partial_l \partial_m \widetilde{W}|A_{R_0, S_0}\|_1 &\leq \mathfrak{C} \|\partial_k \partial_l \partial_m \widetilde{W}|A_{R_0, S_0}\|_q \leq \mathfrak{C} \|\partial_k \partial_l \partial_m \widetilde{W}\|_q \leq \mathfrak{C} \|\nabla \tilde{\varphi} \cdot U\|_{1,q} \\ &\leq \mathfrak{C} \|U|A_{S_0, S_1}\|_{1,q} \leq \mathfrak{C} (\|U|\Omega_{S_0}\|_q + \|\nabla U\|_{q_1}) \leq \mathfrak{C} (\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1}). \end{aligned} \quad (5.17)$$

Obviously with (5.11), $\|\partial^\alpha W^{(j)}|A_{R_0, S_0}\|_1 \leq \mathfrak{C} \|\partial^\alpha W^{(j)}|A_{R_0, S_0}\|_{p_j} \leq \mathfrak{C} \|\tilde{F}^{(j)}\|_{p_j}$ for $1 \leq j \leq s_0 + 1$, $\alpha \in \mathbb{N}_0^3$, $|\alpha| = 2$. Now we may conclude with (5.14) that

$$\|\partial^\alpha U|A_{R_0, S_0}\|_1 \leq \mathfrak{C} (\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1} + \sum_{j=1}^{s_0+1} \|\tilde{F}^{(j)}\|_{p_j}) \quad (5.18)$$

for α as before. As in the last two inequalities in (5.17), we obtain that $\|U|A_{R_0, S_0}\|_{1,1}$ is bounded by the right-hand side of (5.17). The constants in the inequalities from (5.16) onwards only depend on τ, S_0, R_0, q, q_1 or on the respective exponent p_j . This observation, (5.18) and (5.16) yield

$$\begin{aligned} \|U|A_{R_0, S_0}\|_{2,1} + \|\Pi - c|A_{R_0, S_0}\|_{1,1} \\ \leq C(\tau, S_0, R_0, q, q_1, p_1, \dots, p_{s_0+1}) (\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1} + \sum_{j=1}^{s_0+1} \|\tilde{F}^{(j)}\|_{p_j}). \end{aligned} \quad (5.19)$$

Let us estimate $\|\tilde{F}^{(s_0+1)}\|_{p_{s_0+1}}$. (See (5.10) for the definition of $\tilde{F}^{(s_0+1)}$ and p_{s_0+1} .) By (5.9) and the last four inequalities in (5.17), we get

$$\|\Delta \nabla \widetilde{W}\|_q + \|\partial_1 \nabla \widetilde{W}\|_q \leq C(q) \|\nabla \tilde{\varphi} \cdot U\|_{1,q} \leq \mathfrak{C} (\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1}). \quad (5.20)$$

With \tilde{F} defined in (5.8), we further find that $\|\tilde{F}\|_q \leq \mathfrak{C} (\|U|A_{S_0, S_1}\|_{1,q} + \|\Pi|A_{S_0, S_1}\|_q)$. The quantity $\|\Pi|A_{S_0, S_1}\|_q$ was evaluated in (5.7). In view of the right-hand side of this

inequality, and by an estimate of $\|U|A_{S_0,S_1}\|_{1,q}$ as in the last two inequalities in (5.17), we may conclude that $\|\tilde{F}\|_q \leq \mathfrak{C}(\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1} + \sum_{j=1}^{s_0} \|F^{(j)}|A_{S_0,S_1}\|_q)$. It follows with the preceding inequality, (5.20), (5.10) and the choice of q in Lemma 5.1 that

$$\|\tilde{F}^{(s_0+1)}\|_{p_{s_0+1}} = \|\tilde{F}^{(s_0+1)}\|_q \leq \mathfrak{C}(\|U|\Omega_{S_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1} + \sum_{j=1}^{s_0} \|F^{(j)}\|_{p_j}). \quad (5.21)$$

Since $0 \leq \tilde{\varphi} \leq 1$, we deduce from the definition of $\tilde{F}^{(j)}$ in (5.10) that $\|\tilde{F}^{(j)}\|_{p_j} \leq \|F^{(j)}\|_{p_j}$ for $1 \leq j \leq s_0$. Returning to the notation $u_{\epsilon,\delta}(t)$ and $f_{\epsilon,\delta}^{(j)}(t)$ introduced at the beginning of this proof to replace U and $F^{(j)}$, respectively ($1 \leq j \leq s_0$), and putting $\pi_{\epsilon,\delta}(t) := \Pi - c$, we thus obtain from (5.21) and (5.19) that

$$\begin{aligned} & \|u_{\epsilon,\delta}(t)|A_{R_0,S_0}\|_{2,1} + \|\pi_{\epsilon,\delta}(t)|A_{R_0,S_0}\|_{1,1} \\ & \leq \mathfrak{C}(\|u_{\epsilon,\delta}(t)|\Omega_{R_0}\|_q + \sum_{l=1}^3 \|(\partial x_l u_\epsilon)_\delta(t)\|_{q_1} + \sum_{j=1}^{s_0} \|f_{\epsilon,\delta}^{(j)}(t)\|_{p_j}). \end{aligned} \quad (5.22)$$

In addition, equation (5.15) takes the form $\pi_{\epsilon,\delta}(t)|\overline{B_{S_0}}^c = \sum_{j=1}^{s_0+1} \Pi^{(j)}|\overline{B_{S_0}}^c$. Since $\Pi^{(j)} \in W_1^{1,1}(\mathbb{R}^3)$, we have in particular that $\int_{\overline{B_{S_0}}^c} (|\Pi^{(j)}(y)| [(1 + |y|) \ln(2 + |y|)]^{-1})^{p_j} dy < \infty$ ($1 \leq j \leq s_0 + 1$). By what was shown for U and Π at the beginning of this proof, the relations $u_{\epsilon,\delta}(t) \in W_{loc}^{2,q}(\overline{\Omega}^c)^3$ and $\pi_{\epsilon,\delta}(t) \in W_{loc}^{1,q}(\overline{\Omega}^c)$ hold, and the pair $(u_{\epsilon,\delta}(t), \pi_{\epsilon,\delta}(t))$ solves (3.4) in $\overline{\Omega}^c$, with $\lambda = 0$ (stationary Oseen system) and $F = \sum_{j=1}^{s_0} f_{\epsilon,\delta}^{(j)}(t)$.

Recall that $G_{\epsilon,\delta}^{(l)} \in C^\infty(\mathbb{R}, L^{q_l}(\overline{B_{S_0}}^c)^3)$ for $1 \leq l \leq m_0$, $u_{\epsilon,\delta}(t)|\overline{B_{S_0}}^c = \sum_{j=1}^{m_0} G_{\epsilon,\delta}^{(l)}(t)$ and $f_{\epsilon,\delta}^{(j)} \in C^\infty(\mathbb{R}, L^{p_j}(\overline{\Omega}^c)^3)$ for $1 \leq j \leq n_0 + m_0 + 1$ (Lemma 5.2). Since $f_{\epsilon,\delta}^{(n_0+m_0+2)}(t)|\overline{B_{S_0}}^c = 0$ and $f_{\epsilon,\delta}^{(n_0+m_0+2+l)}(t)|\overline{B_{S_0}}^c = (G_{\epsilon,\delta}^{(l)})'(t)$ for $1 \leq l \leq m_0$ (Lemma 2.2, Theorem 2.6), we obtain that $(u_{\epsilon,\delta}|\overline{B_{S_0}}^c \times \mathbb{R})'(t) = \sum_{j=n_0+m_0+2}^{n_0+2m_0+2} f_{\epsilon,\delta}^{(j)}(t)|\overline{B_{S_0}}^c$. At this point the relation $s_0 = n_0 + 2m_0 + 2$ and the fact that the pair $(u_{\epsilon,\delta}(t), \pi_{\epsilon,\delta}(t))$ solves (3.4) in $\overline{\Omega}^c$ with $\lambda = 0$ and $F = \sum_{j=1}^{s_0} f_{\epsilon,\delta}^{(j)}(t)$ and with t arbitrary in \mathbb{R} , allow us to conclude that the pair $(u_{\epsilon,\delta}|\overline{B_{S_0}}^c \times (0, \infty), \pi_{\epsilon,\delta}|\overline{B_{S_0}}^c \times (0, \infty))$ solves the time-dependent Oseen system (4.10) with $A = B_{S_0}$ and $f = \sum_{j=1}^{n_0+m_0+1} f_{\epsilon,\delta}^{(j)}|\overline{B_{S_0}}^c \times (0, \infty)$, without any exceptional values of t . In particular the equation $\operatorname{div}_x u(t) = 0$ holds for any $t \in (0, \infty)$, without exceptional values.

Lemma 5.2 states in particular that the function $\nabla_x u_{\epsilon,\delta}$ belongs to $C^\infty(\mathbb{R}, L^{q_1}(\overline{\Omega}^c)^9)$ and $u_{\epsilon,\delta}|\Omega_{R_0} \times \mathbb{R}$ to $C^\infty(\mathbb{R}, L^q(\Omega_{R_0})^3)$. Thus the right-hand side of (5.22) is integrable with respect to $t \in J$ for any bounded interval $J \subset \mathbb{R}$. Therefore we may conclude from (5.22) that $u_{\epsilon,\delta}|A_{R_0,S_0} \times (0, \infty) \in L_{loc}^1(\mathbb{R}, W^{2,1}(A_{R_0,S_0})^3)$ and $\pi_{\epsilon,\delta}|A_{R_0,S_0} \times \mathbb{R} \in L_{loc}^1(\mathbb{R}, W^{1,1}(A_{R_0,S_0}))$.

Thus all assumptions of Theorem 4.3 are verified with obvious replacements, in particular with B_{S_0} in the role of A . The theorem now follows from Theorem 4.3. \square

Next we show that (4.11) remains valid when δ tends to zero.

Lemma 5.3 *Let $\epsilon, t \in (0, \infty)$. Then there is a set $N_{t,\epsilon} \subset \overline{B_{R_0}}^c$ of measure zero such that*

equation (4.11) holds for $x \in \overline{B_{R_0}}^c \setminus N_{t,\epsilon}$, with $A = B_{S_0}$ and u, f replaced by $u_\epsilon|_{\overline{B_{S_0}}^c} \times (0, \infty)$ and $\sum_{j=1}^{n_0+m_0+1} f_\epsilon^{(j)}|_{\overline{B_{S_0}}^c} \times (0, \infty)$, respectively.

Proof: Abbreviate $Z_r := \overline{B_{S_0}}^c \times (0, r)$ for $r \in (0, \infty]$, $\mathfrak{G} := \mathfrak{G}_{R_0, S_0, \varphi_0}$, with $\mathfrak{G}_{R_0, S_0, \varphi_0}$ defined in Theorem 4.2. Recall that $R_1 := (S_0 + R_0)/2$, as defined at the beginning of this section. Let $l \in \{1, \dots, m_0\}$, $k \in \{1, 2, 3\}$, $j \in \{1, \dots, n_0 + m_0 + 1\}$, $v \in \{G_\epsilon^{(l)}, \partial x_k u_\epsilon, f_\epsilon^{(j)}\}$. Put $p := \varrho_l$, $A := \overline{B_{S_0}}^c$ if $v = G_\epsilon^{(l)}$, $p := q_1$, $A := \overline{\Omega}^c$ if $v = \partial x_k u_\epsilon$, and $p := p_j$, $A := \overline{\Omega}^c$ if $v = f_\epsilon^{(j)}$. Due to the choice of p and A , and by Lemma 5.1, the relation $v \in L_{loc}^1(\mathbb{R}, L^p(A)^3)$ holds. As a consequence $\chi_{(-1, t+1)} v \in L^1(\mathbb{R}, L^p(A)^3)$, and thus by Theorem 2.6, $\|\chi_{(-1, t+1)} v - (\chi_{(-1, t+1)} v)_\delta\|_{p,1;\mathbb{R}} \rightarrow 0$ ($\delta \downarrow 0$). But for $\delta \in (0, 1]$, $s \in (0, t)$, we have $(\chi_{(-1, t+1)} v)_\delta(s) = v_\delta(s)$, so $\|v_\delta - v|_A \times (0, t)\|_{p,1;t} \rightarrow 0$ ($\delta \downarrow 0$). Thus we have shown that

$$\begin{aligned} \|G_{\epsilon,\delta}^{(l)} - G_\epsilon^{(l)}|_{\overline{B_{S_0}}^c} \times (0, t)\|_{\varrho_l,1;t} &\rightarrow 0, \quad \|(\partial x_k u_\epsilon)_\delta - \partial x_k u_\epsilon|_{\overline{\Omega}^c} \times (0, t)\|_{q_1,1;t} \rightarrow 0, \\ \|f_{\epsilon,\delta}^{(j)} - f_\epsilon^{(j)}|_{\overline{\Omega}^c} \times (0, t)\|_{p_j,1;t} &\rightarrow 0 \quad (\delta \downarrow 0) \quad \text{for } 1 \leq l \leq m_0, 1 \leq k \leq 3, 1 \leq j \leq n_0 + m_0 + 1. \end{aligned} \quad (5.23)$$

In particular, since $\partial x_k u_{\epsilon,\delta} = (\partial x_k u_\epsilon)_\delta$ according to Lemma 5.2, we deduce from (5.23) that $\|\partial x_k u_{\epsilon,\delta} - \partial x_k u_\epsilon|_{\overline{\Omega}^c} \times (0, t)\|_{q_1,1;t} \rightarrow 0$ ($\delta \downarrow 0$) for $1 \leq k \leq 3$. Recall that $u_\epsilon|_{\overline{B_{S_0}}^c} \times \mathbb{R} = \sum_{l=1}^{m_0} G_\epsilon^{(l)}$ (Lemma 5.1), and $u_{\epsilon,\delta}|_{\overline{B_{S_0}}^c} \times \mathbb{R} = \sum_{l=1}^{m_0} G_{\epsilon,\delta}^{(l)}$ (Lemma 5.2). Using the parameter q introduced in Lemma 5.1, and observing that $q \leq \varrho_l$ for $1 \leq l \leq m_0$, we may thus further conclude from (5.23) that $\|u_{\epsilon,\delta} - u_\epsilon|_{A_{R,S_0}} \times (0, t)\|_{q,1;t} \rightarrow 0$ ($\delta \downarrow 0$) for any $R \in [S_0, \infty)$. For $l \in \{1, \dots, m_0\}$, since $G_\epsilon^{(l)}(t) = 0$ if $t \in (-\infty, \epsilon]$, we have $G_{\epsilon,\delta}^{(l)}(t) = 0$ for $t \in (-\infty, \epsilon/2]$, $\delta \in (0, \epsilon/2]$, hence $(u_{\epsilon,\delta} - u_\epsilon)(y, 0) = 0$ for $y \in \overline{B_{S_0}}^c$ and thus $\mathfrak{I}^{(\tau)}([u_{\epsilon,\delta} - u_\epsilon|_{\overline{B_{S_0}}^c} \times [0, \infty)](0)) = 0$, for $\delta \in (0, \epsilon/2]$. Since $G^{(l)} \in C^0(\mathbb{R}, L^{\varrho_l}(\overline{B_{S_0}}^c)^3)$, the last statement in Theorem 2.6 yields that $\|(G_{\epsilon,\delta}^{(l)} - G_\epsilon^{(l)})(t)\|_{\varrho_l} \rightarrow 0$ ($\delta \downarrow 0$) for $1 \leq l \leq m_0$. Again because $q \leq \varrho_l$, it follows that $\|(G_{\epsilon,\delta}^{(l)} - G_\epsilon^{(l)})(t)|_{A_{R,S_0}}\|_q \rightarrow 0$ ($\delta \downarrow 0$) for $1 \leq l \leq m_0$, $R \in [S_0, \infty)$, and thus $\|(u_{\epsilon,\delta} - u_\epsilon|_{A_{R,S_0}} \times (0, \infty))(t)\|_q \rightarrow 0$ ($\delta \downarrow 0$) for the same range of R .

Let $x \in B_{R_0}^c$. By Theorem 4.2, the function $y \mapsto \mathfrak{G}(x, y, 0)$ ($y \in B_{R_1}$) belongs to $L^{q'}(B_{R_1})^3$, so we may conclude that $\int_{A_{R_1, S_0}} \mathfrak{G}(x, y, 0) \cdot (u_{\epsilon,\delta} - u_\epsilon)(y, t) dy \rightarrow 0$ for $\delta \downarrow 0$. Since $\|u_{\epsilon,\delta} - u_\epsilon|_{\Omega_R} \times (0, t)\|_{q,1;t} \rightarrow 0$ and $\|\partial y_k u_{\epsilon,\delta} - \partial y_k u_\epsilon|_{Z_t}\|_{q_1,1;t} \rightarrow 0$ for $\delta \downarrow 0$ if $1 \leq k \leq 3$, $R \in [S_0, \infty)$, as shown above, inequality (4.8) in Theorem 4.2 yields that $\int_0^t \int_{A_{R_1, S_0}} \partial y_k^\sigma \partial_s^{1-\sigma} \mathfrak{G}(x, y, t-s) \cdot (\partial y_k^\mu u_{\epsilon,\delta} - \partial y_k^\mu u_\epsilon)(y, s) dy ds \rightarrow 0$ ($\delta \downarrow 0$) for $\sigma, \mu \in \{0, 1\}$, $k \in \{1, 2, 3\}$. Altogether we arrive at the relation $\mathfrak{K}(u_{\epsilon,\delta} - u_\epsilon)(x, t) \rightarrow 0$ ($\delta \downarrow 0$), with $\mathfrak{K}(u_{\epsilon,\delta} - u_\epsilon) = \mathfrak{K}_{R_0, S_0, \varphi_0, B_{S_0}, T_0}(u_{\epsilon,\delta} - u_\epsilon)$ defined in (4.9). Recall that $(u_{\epsilon,\delta} - u_\epsilon)(y, 0) = 0$ for $\delta \in (0, \epsilon/2]$, $y \in \overline{B_{S_0}}^c$, so $\int_{A_{R_1, S_0}} \mathfrak{G}(x, y, t) \cdot (u_{\epsilon,\delta} - u_\epsilon)(y, 0) dy \rightarrow 0$ ($\delta \downarrow 0$). Since $\|f_{\epsilon,\delta}^{(j)} - f_\epsilon^{(j)}\|_{p_j,1;t} \rightarrow 0$ ($\delta \downarrow 0$) (see above), we may apply (4.8) again, to obtain that $\int_0^t \int_{A_{R_1, S_0}} \mathfrak{G}(x, y, t-s) \cdot (f_{\epsilon,\delta}^{(j)} - f_\epsilon^{(j)})(y, s) dy ds \rightarrow 0$ ($\delta \downarrow 0$) ($j \in \{1, \dots, n_0 + m_0 + 1\}$). Using the function $\varphi_0 \in C_0^\infty(B_{R_1})$ fixed at the beginning of this section, we set $\mathfrak{E}(y) := (-\partial_j \mathfrak{N})(x - y) \partial_k \varphi_0(y) + (\partial_k \partial_j \mathfrak{N})(x - y) \varphi_0(y)_{1 \leq j, k \leq 3}$ for $y \in \overline{B_{R_1}}$. Since $x \in B_{R_0}^c$, this function \mathfrak{E} is well defined and belongs to $C^1(\overline{B_{R_1}})^{3 \times 3}$. Hence, by the Divergence theorem

and because $\operatorname{div}_x(u_{\epsilon,\delta} - u_\epsilon) = 0$,

$$\int_{\partial S_0} (\nabla \mathfrak{N})(x - y) \cdot (S_0^{-1} y \cdot (u_{\epsilon,\delta} - u_\epsilon)(y, t)) \, do_y = \int_{A_{R_1, S_0}} \mathfrak{E}(y) \cdot (u_{\epsilon,\delta} - u_\epsilon)(y, t) \, dy.$$

Again referring to the relation $\|(u_{\epsilon,\delta} - u_\epsilon|_{A_{R_1, S_0} \times (0, \infty)})(t)\|_q \rightarrow 0$ ($\delta \downarrow 0$), it follows that $\int_{\partial S_0} (\nabla \mathfrak{N})(x - y) \cdot (S_0^{-1} y \cdot (u_{\epsilon,\delta} - u_\epsilon)(y, t)) \, do_y \rightarrow 0$ ($\delta \downarrow 0$). By Lemma 4.8, we find

$$\begin{aligned} K_\delta &:= \left| \sum_{l=1}^3 \partial x_l \mathfrak{V}^{(\tau, B_{S_0})} (n_l^{(B_{S_0})} (u_{\epsilon,\delta} - u_\epsilon))(x, t) \right| \\ &\leq \mathfrak{C} \left(\|u_{\epsilon,\delta} - u_\epsilon|_{A_{R_1, S_0} \times (0, t)}\|_{q, 1; t} + \|\nabla_y (u_{\epsilon,\delta} - u_\epsilon)|_{A_{R_1, S_0} \times (0, t)}\|_{q, 1; t} \right). \end{aligned}$$

But $\|\nabla_y (u_{\epsilon,\delta} - u_\epsilon)|_{A_{R_1, S_0} \times (0, t)}\|_{q, 1; t}$ is bounded by $\mathfrak{C} \|\nabla_y (u_{\epsilon,\delta} - u_\epsilon)|_{Z_t}\|_{q, 1; t}$ for any $\delta \in (0, \infty)$, so $K_\delta \rightarrow 0$ ($\delta \downarrow 0$) by what we have proved above for the convergence of $u_{\epsilon,\delta} - u_\epsilon$ and $\nabla_x (u_{\epsilon,\delta} - u_\epsilon)$.

Up to this point, x was arbitrary but fixed in $B_{R_0}^c$. Since $\|f_{\epsilon,\delta}^{(j)} - f_\epsilon^{(j)}|_{\overline{\Omega}^c \times (0, t)}\|_{p_j, 1; t} \rightarrow 0$ for $\delta \downarrow 0$ by (5.23), Lemma 4.4 yields $\|\mathfrak{R}^{(\tau)}(f_{\epsilon,\delta}^{(j)} - f_\epsilon^{(j)})(t)\|_{p_j} \rightarrow 0$ ($\delta \downarrow 0$), for $1 \leq j \leq n_0 + m_0 + 1$. Recalling that $\|(u_{\epsilon,\delta} - u_\epsilon|_{A_{R, S_0} \times (0, \infty)})(t)\|_q \rightarrow 0$ ($\delta \downarrow 0$) for $R \in [S_0, \infty)$, we see there is a zero-measure set $M_{t,\epsilon} \subset \mathbb{R}^3$ and a sequence (δ_n) in $(0, \infty)$ with $\delta_n \rightarrow 0$ such that $\mathfrak{R}^{(\tau)}(f_{\epsilon,\delta_n}^{(j)} - f_\epsilon^{(j)})(x, t) \rightarrow 0$ ($n \rightarrow \infty$) and $(u_{\epsilon,\delta} - u_\epsilon)(x, t) \rightarrow 0$ for $x \in \overline{B_{S_0}^c} \setminus M_{t,\epsilon}$, $1 \leq j \leq n_0 + m_0 + 1$. By Theorem 5.1, for any $n \in \mathbb{N}$, there is a set $N_{t,n} \subset \overline{B_{R_0}^c}$ of measure zero such that equation (4.11) holds for $x \in \overline{B_{R_0}^c} \setminus N_{t,n}$, with $A = B_{S_0}$, with $u_{\epsilon,\delta_n}|_{\overline{B_{S_0}^c} \times (0, \infty)}$, $\pi_{\epsilon,\delta_n}|_{\overline{B_{S_0}^c} \times (0, \infty)}$ in the role of u and π , respectively, with n_0 replaced by $n_0 + m_0 + 1$, and the functions $f^{(j)}$ ($1 \leq j \leq n_0$) by $f_{\epsilon,\delta_n}^{(j)}|_{\overline{B_{S_0}^c} \times (0, \infty)}$ ($1 \leq j \leq n_0 + m_0 + 1$). (The function π only appears in the assumptions of equation (4.11), not in the equation itself.) Letting n tend to zero in that equation, we may conclude by the preceding convergence results that (4.11) is satisfied by u_ϵ , π_ϵ , $f_\epsilon^{(j)}$ as well, in the way stated in the lemma. \square

Finally we let ϵ tend to zero in (4.11).

Corollary 5.1 *Define $n^{(S_0)}(y) := S_0^{-1} y$ for $y \in \partial B_{S_0}$. Let $t \in (0, \infty)$. Then there is a zero-measure set $N_t \subset \overline{B_{R_0}^c}$ such that the equation*

$$\begin{aligned} u(x, t) &= \mathfrak{R}^{(\tau)}(f)(x, t) + \mathfrak{I}^{(\tau)}(U_0|_{\overline{B_{S_0}^c}})(x, t) \\ &\quad - \sum_{l=1}^3 \partial x_l \mathfrak{V}^{(\tau, B_{S_0})} (n_l^{(S_0)} u)(x, t) - \int_{\partial B_{S_0}} (\nabla \mathfrak{N})(x - y) (n^{(S_0)}(y) \cdot u(y, t)) \, do_y \\ &\quad + \mathfrak{K}(u)(x, t) - \int_{A_{R_1, S_0}} \mathfrak{G}(x, y, t) \cdot U_0(y) \, dy - \int_0^t \int_{A_{R_1, S_0}} \mathfrak{G}(x, y, t - s) \cdot f(y, s) \, dy \, ds \end{aligned} \tag{5.24}$$

holds for $x \in \overline{B_{R_0}^c} \setminus N_t$, with $f = \sum_{j=1}^{n_0} f^{(j)}|_{\overline{B_{S_0}^c} \times (0, \infty)}$, where $\mathfrak{G} = \mathfrak{G}_{R_0, S_0, \varphi_0}$ is introduced in Theorem 4.2, $\mathfrak{K}(u) = \mathfrak{K}_{R_0, S_0, \varphi_0, B_{S_0}, T_0}(u)$ in (4.9), and R_1 at the beginning of this section.

Proof: By definitions in Lemma 5.1, the equation $\zeta'_\epsilon u(s)|\overline{B_{S_0}}^c = \sum_{j=n_0+1}^{n_0+m_0+1} f_\epsilon^{(j)}(s)|\overline{B_{S_0}}^c$ holds for $s \in (0, \infty)$. (Actually $f_\epsilon^{(n_0+1)}|\overline{B_{S_0}}^c \times \mathbb{R} = 0$, but this does not matter here.) Therefore Lemma 5.3 implies that for any $\epsilon \in (0, \infty)$, there is a set $N_{t,\epsilon} \subset \overline{B_{R_0}}^c$ of measure zero such that equation (4.11) holds for $x \in \overline{B_{R_0}}^c \setminus N_{t,\epsilon}$ with $A = B_{S_0}$ and with u, f replaced by $u_\epsilon|\overline{B_{S_0}} \times (0, \infty)$ and g_ϵ , respectively, where the function g_ϵ is defined by $g_\epsilon(s) := \sum_{j=1}^{n_0} f_\epsilon^{(j)}(s) + \zeta'_\epsilon(s) u(s)|\overline{B_{S_0}}^c$ for $s \in (0, \infty)$. We recall that $G^{(l)} \in C^0([0, \infty), L^{q_l}(\overline{B_{S_0}}^c)^3)$ for $1 \leq l \leq m_0$ and $u(s)|\overline{B_{S_0}}^c = \sum_{l=1}^{m_0} G^{(l)}(s)$ for $s \in (0, \infty)$. Thus we see that all assumptions of Lemma 4.10 are valid with $A, u, f^{(j)}$ ($1 \leq j \leq n_0$) replaced by $B_{S_0}, u|\overline{B_{S_0}} \times (0, \infty)$ and $f^{(j)}|\overline{B_{S_0}} \times (0, \infty)$ ($1 \leq j \leq n_0$), respectively, with m_0 in the role of k_0 and $G^{(l)}$ ($1 \leq l \leq m_0$) in that of $u^{(l)}$ ($1 \leq l \leq k_0$). Lemma 4.10 implies that there is a measurable set $N_t \subset \overline{B_{R_0}}^c$ with properties as stated in the corollary, provided the equation

$$\mathfrak{I}^{(\tau)}(u(0) - U_0|\overline{B_{S_0}}^c)(x, t) + \int_{A_{R_1, S_0}} \mathfrak{G}(x, y, t) \cdot (U_0(y) - u(y, 0)) dy = 0 \quad (5.25)$$

holds. (Note that in view of our assumptions at the beginning of this section, we do not claim that $u(0) = U_0$.) In order to show (5.25), fix some $T \in (0, \infty)$. Let $\vartheta \in C_{0,\sigma}^\infty(\overline{B_{S_0}}^c)$. In a first step, we prove that $\int_{\overline{B_{S_0}}^c} U_0 \cdot \vartheta dx = \int_{\overline{B_{S_0}}^c} u(0) \cdot \vartheta dx$. Since $\nabla_x u$ belongs to $L_{loc}^1([0, \infty), L^{q_1}(\overline{\Omega}^c)^9)$ and $f^{(j)}$ to $L_{loc}^1([0, \infty), L^{p_j}(\overline{\Omega}^c)^3)$ for $1 \leq j \leq n_0$, the function $K(s) := \int_{\overline{B_{S_0}}^c} (\nabla_x u(s) \cdot \nabla \vartheta + \tau \partial_{x_1} u(s) \cdot \vartheta - f(s) \cdot \vartheta) dx$ ($s \in (0, \infty)$) belongs to $L_{loc}^1([0, \infty))$, in particular $K|_{(0, T)} \in L^1((0, T))$. On the other hand, since $u|\overline{B_{S_0}} \times (0, \infty) = \sum_{l=1}^{m_0} G^{(l)}$ and $G^{(l)} \in C^0([0, \infty), L^{q_l}(\overline{B_{S_0}}^c)^3)$ ($1 \leq l \leq m_0$), the function $H(s) := \int_{\overline{B_{S_0}}^c} u(s) \cdot \vartheta dx$ with $s \in [0, \infty)$ belongs to $C^0([0, \infty))^3$. Moreover, due to equation (5.1), we get

$$\int_0^\infty (-\gamma'(s) H(s) + \gamma(s) K(s)) ds = \gamma(0) \int_{\overline{B_{S_0}}^c} U_0 \cdot \vartheta dx \quad (5.26)$$

for $\gamma \in C_0^\infty([0, \infty))$. This equation for $\gamma \in C_0^\infty((0, T))$ and the relations $H|_{[0, T]} \in C^0([0, T])$ and $K|_{(0, T)} \in L^1((0, T))$ yield that $H|_{(0, T)} \in W^{1,1}((0, T))$ and $H'(s) = -K(s)$ for a. e. $s \in (0, T)$. Let $\gamma \in C_0^\infty([0, T])$. It follows that $\gamma H \in C^0([0, T]) \cap W^{1,1}((0, T))$ and $(\gamma H)' = -\gamma K + \gamma' H$, hence $\int_0^T (\gamma K - \gamma' H) ds = \gamma(0) H(0)$. Comparing this equation with (5.26) and recalling that $H(0) = \int_{\overline{B_{S_0}}^c} u(0) \cdot \vartheta dx$, we arrive at the equation $\int_{\overline{B_{S_0}}^c} U_0 \cdot \vartheta dx = \int_{\overline{B_{S_0}}^c} u(0) \cdot \vartheta dx$ we wanted to show. Now let $t \in (0, \infty)$ and $x \in B_{R_0}^c$, and put $Y(y) := \Lambda(x - y, t) - \mathfrak{G}(x, y, t)$ for $y \in B_{R_1}$. Recall that $\mathfrak{G}(x, \cdot, t) \in C_0^\infty(B_{R_1})^{3 \times 3}$, $\mathfrak{G}(x, y, t) = \Lambda(x - y, t)$ for $y \in B_{S_0 + (R_0 - S_0)/4}$ (Theorem 4.2), and $\Lambda|\mathbb{R}^3 \times (0, \infty) \in C^\infty(\mathbb{R}^3 \times (0, \infty))$ (Lemma 4.1). Thus we get $\text{supp}(Y) \subset B_{S_0 + (R_0 - S_0)/4}^c$ and $Y \in C^\infty(\mathbb{R}^3)^{3 \times 3}$. Moreover $\sum_{k=1}^3 \partial_{y_k} Y_{jk}(y) = 0$ for $y \in \mathbb{R}^3$, $1 \leq j \leq 3$ and $Y \in W^{1,r}(\mathbb{R}^3)^{3 \times 3}$ for $r \in (1, \infty)$ also by Lemma 4.1. Therefore, for $1 \leq \mu \leq 3$, Theorem 2.2 provides a sequence $(\vartheta_n^{(\mu)})_{n \geq 1}$ in $C_{0,\sigma}^\infty(\overline{B_{S_0}}^c)$ such that in particular $\|(Y_{\mu k})_{1 \leq k \leq 3} - \vartheta_n^{(\mu)}\|_r \rightarrow 0$ ($n \rightarrow \infty$) for $r = \tilde{p}$ and $r \in \{\varrho_m : 1 \leq m \leq m_0\}$, with \tilde{p} introduced at the beginning of this section in the assumption $U_0 \in L^{\tilde{p}}(\overline{\Omega}^c)^3$. But $\int_{\overline{B_{S_0}}^c} \vartheta_n^{(\mu)} \cdot U_0 dx = \int_{\overline{B_{S_0}}^c} \vartheta_n^{(\mu)} \cdot u(0) dx$ ($1 \leq \mu \leq 3$, $n \in \mathbb{N}$) by what has already been proved, and $u(0)|\overline{B_{S_0}}^c = \sum_{l=1}^{m_0} G^{(l)}(0)$, $G^{(l)}(0) \in$

$L^{q_l}(\overline{B_{S_0}^c})^3$ ($1 \leq l \leq m_0$) by our assumptions. Hence, by letting n tend to infinity, we obtain $\int_{\overline{B_{S_0}^c}} Y \cdot U_0 dy = \int_{\overline{B_{S_0}^c}} Y \cdot u(0) dy$. Due to the definition of Y , this implies (5.25). So the proof of Corollary 5.1 is completed. \square

Up to now, we considered the case $T_0 = \infty$ in our assumptions at the beginning of this section. But in the present context, the transition from this case to the case $T_0 < \infty$ is easy to perform:

Corollary 5.2 *Suppose that $T_0 < \infty$. Then, for $t \in (0, T_0)$, there is a zero-measure set $N_t \subset \overline{B_{R_0}^c}$ such that equation (5.24) holds for $x \in \overline{B_{R_0}^c} \setminus N_t$ with $f = \sum_{j=1}^{n_0} f^{(j)}|_{\overline{B_{S_0}^c}} \times (0, T_0)$.*

Proof: Let $T' \in (0, T_0)$, and choose a function $\zeta \in C^\infty(\mathbb{R})$ such that ζ equals 1 on the interval $(-\infty, T' + (T_0 - T')/4)$ and vanishes on $(T' + (T_0 - T')/2, \infty)$. Define $\tilde{u}(s) := \zeta(s)u(s)$, $\tilde{f}^{(j)}(s) := \zeta(s)f^{(j)}(s)$, $\tilde{G}^{(l)}(s) := \zeta(s)G^{(l)}(s)$ for $1 \leq j \leq n_0$, $1 \leq l \leq k_0$, $s \in (0, T_0)$. On $[T_0, \infty)$, the value zero is assigned to these functions. Then all assumptions listed at the beginning of this section except equation (5.1) are valid with ∞ in the role of T_0 , and \tilde{u} , $\tilde{f}^{(j)}$, $\tilde{G}^{(l)}$ in that of u , $f^{(j)}$ and $G^{(l)}$, respectively ($1 \leq j \leq n_0$, $1 \leq l \leq m_0$). Define $\tilde{f}^{(n_0+1)}(s) := \zeta'(s)\tilde{u}(s)$ for $s \in (0, \infty)$. Let $\gamma \in C_0^\infty([0, \infty))$. Then $\gamma\zeta \in C_0^\infty([0, T_0])$, so we may replace γ by $\zeta\gamma$ in (5.1). It follows that (5.1) is valid with T_0 replaced by ∞ , u by \tilde{u} , n_0 by $n_0 + 1$, and $f^{(j)}$ ($1 \leq j \leq n_0$) by $\tilde{f}^{(j)}$ ($1 \leq j \leq n_0 + 1$). Let $t \in (0, \infty)$. Then Corollary 5.1 implies there is a set $N_t \subset \overline{B_{R_0}^c}$ of measure zero such that (5.24) holds for $x \in \overline{B_{R_0}^c} \setminus N_t$ with u replaced by \tilde{u} , and with $f = \sum_{j=1}^{n_0+1} \tilde{f}^{(j)}|_{\overline{B_{S_0}^c}} \times (0, \infty)$. Now suppose that $t \in (0, T')$. Then, for $s \in (0, t]$, the functions $\tilde{u}(s)$ and $u(s)$ coincide, as do $\tilde{f}^{(j)}(s)$ and $f^{(j)}(s)$ for $1 \leq j \leq n_0$, whereas $\tilde{f}^{(n_0+1)}(s) = 0$. By the definitions in Lemma 4.4, we get that $\Re^{(\tau)}(\sum_{j=1}^{n_0+1} \tilde{f}^{(j)}|_{\overline{B_{S_0}^c}} \times (0, \infty))(x, t) = \Re(\sum_{j=1}^{n_0} f^{(j)}|_{\overline{B_{S_0}^c}} \times (0, T_0))(x, t)$ for $x \in \mathbb{R}^3$. Therefore we may conclude that (5.24) is valid for $x \in \overline{B_{R_0}^c} \setminus N_t$ with $f = \sum_{j=1}^{n_0} f^{(j)}|_{\overline{B_{S_0}^c}} \times (0, T_0)$. Since T' was taken arbitrarily in $(0, T_0)$, the corollary is proved. \square

In order to exploit our representation formula (5.24) for the purpose of decay estimates, we need somewhat stronger assumptions on u . The ensuing theorem gives the details.

Theorem 5.2 *Abbreviate $\mathfrak{A} := A_{R_1, S_0} \times (0, T_0)$, $f := \sum_{j=1}^{n_0} f^{(j)}|_{\overline{B_{S_0}^c}} \times (0, T_0)$. In addition to the assumptions at the beginning of this section, suppose that the function $u|_{\mathfrak{A}}$ belongs to $L^\infty(0, T_0, L^q(A_{R_1, S_0})^3)$ and to $L^{\gamma_1}(0, T_0, L^q(A_{R_0, S_0})^3)$, the function $\nabla_x u|_{\mathfrak{A}}$ to $L^{\gamma_2}(0, T_0, L^q(A_{R_1, S_0})^9)$, and $f|_{\mathfrak{A}}$ is in the space $L^{\gamma_3}(0, T_0, L^q(A_{R_1, S_0})^3)$, for certain parameters $\gamma_1, \gamma_2, \gamma_3 \in [1, \infty]$, with q chosen in Lemma 5.1. Then there is a zero-measure set $\mathfrak{S}_{T_0} \subset (0, T_0)$ such that*

$$\begin{aligned} & |\partial_x^\alpha u(x, t) - \partial_x^\alpha \Re^{(\tau)}(f)(x, t) - \partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|_{B_{S_0}^c})(x, t)| \\ & \leq \mathfrak{C}(\|u|_{\mathfrak{A}}\|_{q, \infty; T_0} + \|u|_{\mathfrak{A}}\|_{q, \gamma_1; T_0} + \|\nabla_x u|_{\mathfrak{A}}\|_{q, \gamma_2; T_0} + \|f|_{\mathfrak{A}}\|_{q, \gamma_3; T_0} + \|U_0\|_{\tilde{p}}) \\ & \quad \left((|x| \nu(x))^{-(3+|\alpha|)/2+1/(2 \min\{\gamma_1', \gamma_2', \gamma_3'\})} + |x|^{-\gamma-|\alpha|} \right) \end{aligned} \quad (5.27)$$

for $t \in (0, T_0) \setminus \mathfrak{S}_{T_0}$, $x \in \overline{B_{R_0}^c} \setminus N_t$ with some set $N_t \subset \overline{B_{R_0}^c}$ of measure zero, and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, where $\gamma = 3$ if $\int_{\partial B_{S_0}} u(t) \cdot n^{(\Omega)} dx = 0$ for $t \in (0, T_0)$, and $\gamma = 2$ else.

If $\gamma = 3$, the last line in (5.27) may be replaced by $(|x| \nu(x))^{-(3+|\alpha|)/2+1/(2 \min\{\gamma_1', \gamma_2', \gamma_3'\})}$.

Proof: Since $q \leq \min\{\varrho_l : 1 \leq l \leq m_0\}$, $u|_{\overline{B_{S_0}^c}} \times (0, T_0) = \sum_{l=1}^{m_0} G^{(l)}$ and $G^{(l)} \in C^0([0, T_0), L^{\varrho_l}(\overline{B_{S_0}^c})^3)$ for $1 \leq l \leq m_0$, we have $u|_{\mathfrak{A}} \in C^0([0, T_0), L^q(A_{R_1, S_0})^3)$. As a consequence, we get $\|u(t)|_{A_{R_1, S_0}}\|_q \leq \|u|_{\mathfrak{A}}\|_{q, \infty; T_0}$ for $t \in [0, T_0]$. Thus inequality (5.27) follows from Corollary 5.1 and 5.2, (4.7), (4.8), Corollary 4.2, Lemma 4.8 and 4.9. Obviously $|x| \geq C(R)\nu(x)$ for $x \in B_R^c$, $R \in (0, \infty)$, so in the case $\gamma = 3$, inequality (5.27) holds without the term $|x|^{-\gamma-|\alpha|}$ in the last line. \square

6 Some comments and applications

In view of inequality (5.27), we remark that estimates of the terms $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)|$ and $|\partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|_{\overline{B_{S_0}^c}})(x, t)|$ may be found elsewhere. They are independent of the theory presented here, only depending on the decay properties of U_0 and f . An optimal decay bound, that is, a bound also valid for $|\partial_x^\alpha \Lambda(x, t)|$ if $|x|$ is large (see Lemma 4.1), is given by $\mathfrak{C}(|x|\nu(x))^{-(3+|\alpha|)/2}$. This bound is obtained if, for example, f and U_0 have compact support ([19, Lemma 4.1, 4.2]). For conditions on f and U_0 leading to the decay bound $\mathfrak{C}(|x|\nu(x))^{-(2+|\alpha|)/2}$, we refer to [14, Theorem 3.1] and [13, Theorem 1.1], respectively. These indications explain why inequality (1.3) follows from (5.27) under suitable assumptions on U_0 and f .

It should further be mentioned that the sum $\partial_x^\alpha \mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(U_0)$ constitutes a solution to (1.1) in the whole space \mathbb{R}^3 ($\Omega = \emptyset$), with initial data U_0 if U_0 is solenoidal ([19, (3.3), Lemma 3.7, Theorem 3.3]). So left-hand side of inequality (6.11) may be interpreted as the perturbation generated in the fluid by the presence of the rigid body. But it is precisely this perturbation which is of interest here. Thus inequality (6.11) may be considered as a decay estimate of that aspect of the flow which is relevant in the present context.

Let us compare the theory developed here with that in [14] and its predecessor paper [11]. The former paper extends the theory derived in the latter one, so we only consider the former one. Our general approach in the work at hand consists in taking existence of a solution to (1.1) for granted. We require conditions on the velocity u and on the right-hand side f in (1.1) as well as on the initial data U_0 in order to establish an integral representation of u (equation (5.24)), which, in turn, serves to derive a decay estimate of u (inequality (5.27)), under some additional assumptions on u and f . In contrast to that, the approach in [14] leads simultaneously to an existence result for the velocity part u of a solution to (1.1) and to an integral representation of u , and in a supplementary step to the estimate

$$|\partial_x^\alpha u(x, t) - \partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t) - \partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0)(x, t)| \leq \mathfrak{C}(|x|\nu(x))^{-1-|\alpha|/2} \quad (6.1)$$

for x, t, α as in (5.27) ([14, Corollary 2.28, Lemma 3.2, (3.7)]). No assumptions on the velocity u are needed because its existence is a result, not an assumption. The method of proof in [14] consists in solving an integral equation on $\partial\Omega \times (0, T_0)$, as proposed by Shen [39] for the time-dependent Stokes system in the case $f = 0$, $U_0 = 0$. The regularity of the solutions constructed in [39] corresponds to the regularity of the boundary data $b : \partial\Omega \times (0, T_0) \mapsto \mathbb{R}^3$. Thus the assumptions on b in [39] may be considered as natural. However, when in [14] this method is extended to the Oseen system with nonvanishing f

or U_0 , the assumptions on these latter functions are more restrictive than the ones in the work at hand. And if our theory here is applied to the solutions constructed in [14], under the same assumptions on the data as in [14], we obtain decay rates which are stronger than those in this reference. In fact, we are able to replace the factor $(|x| \nu(x))^{-1-|\alpha|/2}$ in (6.1) by $(|x| \nu(x))^{-5/4-|\alpha|/2}$. The details are given in the ensuing Theorem 6.1 ($f = 0$, $U_0 = 0$) and 6.2 ($b = 0$).

Theorem 6.1 *For $\varphi \in C_0^\infty(\mathbb{R}^4)^3$ with $\varphi|_{\mathbb{R}^3 \times (-\infty, 0]} = 0$, define the fractional derivative $\partial_t^{1/2} \varphi$ by $(\partial_t^{1/2} \varphi)(x, t) := \pi^{-1/2} \partial_t \left(\int_0^t (t-s)^{-1/2} \varphi(x, s) ds \right)$ for $t \in (0, \infty)$, $x \in \mathbb{R}^3$.*

Put $S_{T_0} := \partial\Omega \times (0, T_0)$. Let $b \in L^2(S_{T_0})^3$ with $\int_{\partial\Omega} b(t) \cdot n^{(\Omega)} d\sigma_x = 0$ for $t \in (0, T_0)$. Suppose there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^4)^3$ such that $\varphi_n|_{\mathbb{R}^3 \times (-\infty, 0]} = 0$ for $n \in \mathbb{N}$, $\|b - \varphi_n\|_2 \rightarrow 0$ ($n \rightarrow \infty$) and $\int_0^{T_0} \|(\varphi_n - \varphi_m)(t)|\partial\Omega\|_{H^1(\partial\Omega)^3}^2 dt \rightarrow 0$, $\int_0^{T_0} \|\partial_t^{1/2}(\varphi_n - \varphi_m)(t)|\partial\Omega\|_2^2 dt \rightarrow 0$, $\int_0^{T_0} \|\partial_t(\varphi_n - \varphi_m)(t) \cdot n^{(\Omega)}\|_{H^1(\partial\Omega)'}^2 dt \rightarrow 0$ for $m, n \rightarrow \infty$. Here the space $H^1(\partial\Omega)$ is to be defined in the usual way, and the symbol $\|\cdot\|_{H^1(\partial\Omega)^3}$ denotes the usual norm of $H^1(\partial\Omega)^3$ with respect to some local coordinates (see [28, Section III.6.7] for example). The symbol $\|\cdot\|_{H^1(\partial\Omega)'}$ stands for the canonical norm of the dual space of $H^1(\partial\Omega)$.

Then there is a unique function $u \in L_{loc}^2([0, T_0], H^1(\overline{\Omega}^c)^3)$ such that $\operatorname{div}_x u(t) = 0$ and $u(t)|_{\partial\Omega} = b(t)$ for $t \in (0, T_0)$, and such that equation (5.1) is satisfied with $f = 0$ and $U_0 = 0$. Moreover $u \in L^\infty(0, T_0, L^2(\overline{\Omega}^c)^3)$ and $\nabla_x u \in L^2(0, T_0, L^2(\overline{\Omega}^c)^9)$. In addition $|\partial_x^\alpha u(x, t)| \leq \mathfrak{C}(|x| \nu(x))^{-5/4-|\alpha|/2}$ for t, x, α as in (5.27).

Proof: The uniqueness statement follows as in the Stokes case; see [12, Theorem 3.7] and its proof. For the existence result we refer to [14, Theorem 2.26], which yields a function $u \in L_{loc}^2([0, T_0], H^1(\overline{\Omega}^c)^3)$ such that $\operatorname{div} u(t) = 0$, $u(t)|_{\partial\Omega} = b(t)$ for $t \in (0, T_0)$ and equation (5.1) is valid with $f = 0$, $U_0 = 0$. Due to [14, (2.13)] and [9, Theorem 2.3], this function u additionally belongs to $L^\infty(0, T_0, L^2(\overline{\Omega}^c)^3)$, and $\nabla_x u \in L^2(0, T_0, L^2(\overline{\Omega}^c)^9)$. This leaves us to consider the estimate stated at the end of this theorem. Theorem 2.3 implies that $u \in L^2(0, T_0, L^6(\overline{\Omega}^c)^3)$. In addition, by [14, (2.13)] and Lemma 4.7 we have $u|_{\overline{B_{S_0}^c} \times (0, T_0)} \in C^0([0, T_0], L^4(\overline{B_{S_0}^c})^3)$. Therefore the estimate in question follows from Theorem 5.2 with $U_0 = 0$, $n_0 = 1$, $p_1 = 2$, $f^{(1)} = 0$, $m_0 = 1$, $\varrho_1 = 4$, $G^{(1)} = u|_{\overline{B_{S_0}^c} \times (0, T_0)}$, $q_0 = q_1 = \tilde{p} = \gamma_j = 2$ for $j \in \{1, 2, 3\}$. \square

Theorem 6.2 *Let $\kappa \in (0, 1/2]$ and define $H_\sigma^{1/2+\kappa}(\overline{\Omega}^c)$ as the closure of $C_{0,\sigma}^\infty(\overline{\Omega}^c)$ with respect to the norm of the fractional-order Sobolev space $W^{1/2+\kappa, 2}(\overline{\Omega}^c)^3$. Suppose that $U_0 \in H_\sigma^{1/2+\kappa}(\overline{\Omega}^c)$. Let $\tilde{q} \in [1, 3/2)$ and let f belong to the spaces $L^2(0, \infty, L^{3/2}(\overline{\Omega}^c)^3)$ and $L^2(0, \infty, L^{\tilde{q}}(\overline{\Omega}^c)^3)$. Then there is a unique function $u \in L_{loc}^2([0, \infty), W^{1, 2}(\overline{\Omega}^c)^3)$ such that $u(t)|_{\partial\Omega} = 0$ and $\operatorname{div} u(t) = 0$ for $t \in (0, \infty)$, and such that u satisfies (5.1). In addition*

$$\begin{aligned} & |\partial_x^\alpha u(x, t) - \partial_x^\alpha \mathfrak{R}^{(\tau)}(f|_{\overline{B_{S_0}^c} \times (0, \infty)})(x, t) - \partial_x^\alpha \mathfrak{J}^{(\tau)}(U_0|_{\overline{B_{S_0}^c}})(x, t)| \\ & \leq \mathfrak{C}(|x| \nu(x))^{-5/4-|\alpha|/2} \quad \text{for } x, t, \alpha \text{ as in (5.27).} \end{aligned} \quad (6.2)$$

Proof: Concerning uniqueness, as in the proof of Theorem 6.1, we refer to [12, Theorem 3.7]. The assumptions on U_0 and f were chosen in such a way that the existence result in Theorem 6.2 holds according to [14, Theorem 2.26] with $T_0 = \infty$, $d = 0$, $c = U_0$, $h = f$.

The solution provided by this reference is given by

$$u = (\mathfrak{R}^{(\tau)}(f) + \mathcal{H}^{(\tau)}(U_0) + \mathfrak{V}^{(\tau)}(\varphi))|_{\overline{\Omega}^c} \times (0, \infty) \quad (6.3)$$

(see [14, (2.13)]), for a certain function $\varphi \in L^2(\partial\Omega \times (0, \infty))^3$ specified in [14, Theorem 2.26], and with $\mathcal{H}^{(\tau)}(U_0)$ defined in Lemma 4.3. We use this representation formula in order to check whether u satisfies the assumptions imposed at the beginning of Section 5 and in Theorem 5.2. To this end, we observe that Lemma 4.3 yields $\mathcal{H}^{(\tau)}(U_0) \in C^0([0, \infty), L^2(\mathbb{R}^3)^3) \cap L^\infty(0, \infty, L^2(\mathbb{R}^3)^3)$ and $\nabla_x \mathcal{H}^{(\tau)}(U_0) \in L^2(0, \infty, L^2(\mathbb{R}^3)^9)$. Moreover, by Lemma 4.5, we have $\mathfrak{R}^{(\tau)}(f) \in C^0([0, \infty), L^{3/2}(\mathbb{R}^3)^3)$, $\mathfrak{R}^{(\tau)}(f)|_{B_{R_0} \times (0, \infty)} \in L^\infty(0, \infty, L^{3/2}(B_{R_0})^3)$ and $\nabla_x \mathfrak{R}^{(\tau)}(f)|_{B_{R_0} \times (0, \infty)} \in L^2(0, \infty, L^{3/2}(B_{R_0})^9)$. According to [10, Corollary 2.17] with $s = 2$, $q = 3/2$, the function $\nabla_x \mathfrak{R}^{(\tau)}(f)$ belongs to $L^2_{loc}([0, \infty), L^2(\mathbb{R}^3)^9)$. Moreover [9, Theorem 2.3] yields that $\mathfrak{V}^{(\tau)}(\varphi)|_{\overline{\Omega}^c} \times (0, \infty)$ belongs to $L^\infty(0, \infty, L^2(\overline{\Omega}^c)^3)$, and $\nabla_x \mathfrak{V}^{(\tau)}(\varphi)|_{\overline{\Omega}^c} \times (0, \infty)$ to $L^2(0, \infty, L^2(\overline{\Omega}^c)^9)$. By Lemma 4.7 we have $\mathfrak{V}^{(\tau)}(\varphi)|_{\overline{B_{S_0}^c}} \times (0, \infty) \in C^0([0, \infty), L^4(\overline{B_{S_0}^c})^3)$. At this point Theorem 2.3 implies that $\mathcal{H}^{(\tau)}(U_0)|_{\overline{\Omega}^c} \times (0, \infty)$ and $\mathfrak{V}^{(\tau)}(\varphi)|_{\overline{\Omega}^c} \times (0, \infty)$ belong to $L^2(0, \infty, L^6(\overline{\Omega}^c)^3)$. Since $\nabla_x \mathfrak{R}^{(\tau)}(f)|_{B_{R_0} \times (0, \infty)} \in L^2(0, \infty, L^{3/2}(B_{R_0})^9)$, as mentioned above, and because of (6.3), we thus get that $\nabla_x u|_{\Omega_{R_0} \times (0, \infty)} \in L^2(0, \infty, L^{3/2}(\Omega_{R_0})^9)$. But $u(t)|_{\partial\Omega} = 0$ ($t \in (0, \infty)$), hence by Poincaré's lemma, $u|_{\Omega_{R_0} \times (0, \infty)} \in L^2(0, \infty, L^{3/2}(B_{R_0})^3)$. Again recalling (6.3), we thus see that u satisfies the assumptions imposed at the beginning of Section 5 and in Theorem 5.2 with $T_0 = \infty$, $\tilde{p} = q_1 = 2$, $q_0 = 3/2$, $n_0 = 1$, $p_1 = 3/2$, $\gamma_3 = 2$, $f^{(1)} = f$, $m_0 = 3$, $\varrho_1 = 2$, $G^{(1)} = \mathcal{H}^{(\tau)}(U_0)|_{\overline{B_{S_0}^c}} \times (0, \infty)$, $\varrho_2 = 3/2$, $G^{(2)} = \mathfrak{R}^{(\tau)}(f)|_{\overline{B_{S_0}^c}} \times (0, \infty)$, $\varrho_3 = 4$, $G^{(3)} = \mathfrak{V}^{(\tau)}(\varphi)|_{\overline{B_{S_0}^c}} \times (0, \infty)$, $\gamma_1 = \gamma_2 = 2$, $q = 5/4$. Now Theorem 5.2 yields (6.2). \square

As another application of Theorem 5.2, we can handle standard L^2 -weak solutions to (1.1) with side conditions $u(t)|_{\partial\Omega} = 0$ for $t \in (0, T_0)$ and $u(0) = U_0$, under assumptions that are only slightly more restrictive than the conditions needed for existence ($f \in L^2(0, T_0, L^2_\sigma(\overline{\Omega}^c))$ instead of $f \in L^2(0, T_0, \mathcal{V}')$, with \mathcal{V}' defined below). The theory in [13] does not cover this situation at all. The next theorem gives the details on existence and on our decay result for this type of solution.

Theorem 6.3 *Put $\mathcal{V} := \{V \in W_0^{1,2}(\overline{\Omega}^c)^3 : \operatorname{div} V = 0\}$, equip \mathcal{V} with the norm of $W^{1,2}(\overline{\Omega}^c)^3$, and let \mathcal{V}' denote the dual space to \mathcal{V} . Write $L^2_\sigma(\overline{\Omega}^c)$ for the closure of $C_{0,\sigma}^\infty(\overline{\Omega}^c)$ with respect to the norm of $L^2(\overline{\Omega}^c)^3$. Let $U_0 \in L^2_\sigma(\overline{\Omega}^c)$ and $f \in L^2(0, T_0, \mathcal{V}')$. Then there is a unique function $u : (0, T_0) \mapsto \mathcal{V}$ such that $u \in L^\infty(0, T_0, L^2(\overline{\Omega}^c)^3)$, $\nabla v \in L^2(0, T_0, L^2(\overline{\Omega}^c)^9)$, and equation (5.1) is fulfilled with the modification that the term $f(t) \cdot \vartheta$ is dropped, and instead the term $f(t)(\vartheta)$ is added outside the integral over $\overline{\Omega}^c$. This function u is in $C^0([0, T_0), L^2(\overline{\Omega}^c)^3)$.*

If $f \in L^2(0, T_0, L^2(\overline{\Omega}^c)^3)$, then inequality (6.2) holds for t, x, α as in (5.27).

Proof: As in the two preceding theorems, we refer to [14, Theorem 3.7] and its proof as regards uniqueness, which follows as in the Stokes case. Concerning existence, the argument is also the same as in the Stokes case. We refer to [44, p. 171-176 and p. 180], in particular [44, p. 175, (1.65)]. The equation $\int_{\overline{\Omega}^c} \partial_1 V \cdot V \, dx = 0$ valid for $V \in W_0^{1,2}(\overline{\Omega}^c)^3$ is the reason why the Oseen term does not generate a major problem. In order to show that

$u \in C^0([0, T_0), L^2(\overline{\Omega}^c)^3)$, we consider v as a weak solution of the time-dependent Stokes system, in an analogous way as stated in (5.1) for the Oseen system, but now with the right-hand side $f - \tau \partial x_1 u \in L^2(0, T_0, \mathcal{V}')$. Then [44, Theorem 3.1.1] yields continuity of u on $[0, T_0)$ with values in $L^2(\overline{\Omega}^c)^3$. Suppose that $f \in L^2(0, T_0, L^2_\sigma(\overline{\Omega}^c))$. Since Theorem 2.3 with $\kappa = q = 2$ implies $u \in L^2(0, T_0, L^6(\overline{\Omega}^c)^3)$, we thus get that u satisfies all the assumptions imposed on u at the beginning of Section 5 and in Theorem 5.2 if we take $n_0 = 1$, $p_1 = 2$, $f^{(1)} = f$, $m_0 = 1$, $q_1 = 2$, $G^{(1)} = u|_{\overline{B_{S_0}}^c} \times (0, T_0)$, $q_0 = q_1 = \tilde{p} = \gamma_j = 2$ for $1 \leq j \leq 3$, $q = 5/4$. Therefore inequality (6.2) follows from (5.27). \square

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