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# ON SOME PATH-DEPENDENT SDES INVOLVING DISTRIBUTIONAL DRIFTS

ALBERTO OHASHI<sup>1</sup>, FRANCESCO RUSSO<sup>2</sup>, AND ALAN TEIXEIRA<sup>1</sup>

ABSTRACT. This paper investigates some one-dimensional path-dependent SDEs, which includes an irregular (distributional) drift  $b'$  depending on the present position. We treat essentially two cases: the first one concerns the case when the drift  $b'$  is the derivative of a continuous function, the second one when  $b'$  is the derivative of a logarithmic or an Heaviside function. In the second framework, we characterize Bessel processes in low dimension as unique solutions to some suitable strong martingale problems and we consider then path-dependent extensions.

**Key words and phrases.** SDEs with distributional drift; Bessel processes; path-dependent stochastic differential equations.

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## 1. INTRODUCTION

This paper discusses in detail a framework of one-dimensional stochastic differential equations (henceforth abbreviated by SDEs) with distributional drift with possible path-dependency. Even though we could have worked in the multidimensional case, we have preferred to explore in a systematic way the real line case.

The main objective of this paper is to analyze the solution (existence and uniqueness when it is possible) of the martingale problem associated with SDEs of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \quad (1.1)$$

where  $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$  are functions and  $\sigma > 0, x_0 \in \mathbb{R}$  and  $W$  is a standard Brownian motion. The assumptions on  $b$ , which will be formulated later, imply that  $b'$  is a Schwartz distribution. Concerning the path-dependent component of the drift, we consider a locally bounded functional

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (1.2)$$

where

$$\Lambda := \{(s, \eta) \in [0, T] \times C([0, T]); \eta = \eta^s\}$$

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and

$$\eta^s(t) = \begin{cases} \eta(t), & \text{if } t \leq s \\ \eta(s), & \text{if } t > s. \end{cases}$$

By convention, we extend  $\Gamma$  from  $\Lambda$  to  $[0, T] \times C([0, T])$  by setting (in non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]).$$

Path-dependent SDEs were investigated under several aspects. Under standard Lipschitz regularity conditions on the coefficients, it is known (see e.g Theorem 11.2 [27, chapter V]) that strong existence and uniqueness holds. In case the path-dependence takes the form of delayed stochastic equations, one-sided Lipschitz condition ensures strong existence and uniqueness, see e.g [30, 24]. Beyond Lipschitz regularity on the coefficients of the SDE, [15] shows uniqueness in law under structural conditions on an underlying approximating Markov process, where local-time and running maximum dependence are considered. Weak existence of infinite-dimensional SDEs with additive noise on the configuration space with path-dependent drift functionals with sublinear growth are studied by [8]. In all these works, the drift is a non-anticipative functional. Beyond Brownian motion based driving noises, [6] establishes existence of solutions for path-dependent Young differential equation.

The case when  $\Gamma = 0$ , i.e. the Markovian case (but with distributional drift) has been intensively studied during the recent years. Diffusions in the generalized sense were first considered in the case when the solution is still a semimartingale, beginning with, at least in our knowledge [25]. Later on, many authors considered special cases of SDEs with generalized coefficients. It is difficult to quote them all; in the case when the drift  $b'$  is a measure and the solutions are semimartingales, we refer the reader to [3, 10, 28]. We also recall that [11] considered even special cases of non-semimartingales solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes, for which only existence is proved.

In the non-semimartingale case, time-independent SDEs in dimension one of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \quad t \in [0, T], \quad (1.3)$$

where  $\sigma$  is a strictly positive continuous function and  $b'$  is the derivative of a real continuous function was solved and analyzed carefully in [14] and [13], which treated well-posedness of the martingale problem, Itô's formula under weak conditions, semimartingale characterization and Lyons-Zheng decomposition. The only supplementary

assumption was the existence of the function

$$\Sigma(x) := 2 \int_0^x \frac{b'}{\sigma^2}(y) dy, x \in \mathbb{R}, \quad (1.4)$$

considered as a suitable limit via regularizations. Those authors considered weak solutions. The SDE (1.3) was also investigated by [1], where the authors provided a well-stated framework when  $\sigma$  and  $b$  are  $\gamma$ -Hölder continuous,  $\gamma > \frac{1}{2}$ . In [29], the authors have also shown that in some cases strong solutions exist and pathwise uniqueness holds. More recently, in the time-dependent framework, a significant contribution was done by [7].

As far as the multidimensional case is concerned, some important steps were done by [12] and more recently by [5], when the diffusion matrix is the identity and  $b$  is a time-dependent drift in some suitable negative Sobolev space. We also refer to previous work of [2], where the authors have focused on (1.1) in the case of a time independent drift  $b$  which is a measure of Kato class.

In this work, equation (1.1) will be interpreted as a martingale problem with respect to some operator  $\mathcal{L}f := Lf + \Gamma$ , see (3.3), where  $L$  is the Markovian generator

$$Lf = \frac{\sigma^2}{2} f'' + b' f', \quad (1.5)$$

where we stress that  $b'$  is the derivative of some function  $b$ , interpreted in the sense in the sense of distributions. If we formally denote  $\Sigma$  as in (1.4), then the operator  $L$  can be written as

$$Lf = (e^\Sigma f')' \frac{e^{-\Sigma} \sigma^2}{2}. \quad (1.6)$$

We define a notion of martingale problem related to  $\mathcal{L}$  (see Definition 3.3) and a notion of strong martingale problem related to  $\mathcal{D}_{\mathcal{L}}$  and a given Brownian motion  $W$ , see Definition 3.4; that definition has to be compared with the notion of strong existence and pathwise uniqueness of an SDE. In the Markovian case, the notion of *strong martingale problem* was introduced in [29] and it represents the corresponding notion to strong solution of SDEs in the framework of martingale problems.

We will concentrate on two specific cases.

- (1) The case when  $b$  is continuous. When  $\Gamma = 0$ , this case was completely analyzed in [13] and [14].
- (2) The case when  $\sigma \equiv 1$  and  $b(x) = \frac{\delta-1}{2} \log(|x|)$  for  $\delta \in [0, 1[$  and  $b$  is the Heaviside function when  $\delta = 1$ . This situation will concern a class of *path-dependent Bessel processes*. Indeed when  $\Gamma \equiv 0$  natural expected solutions to the corresponding

martingale problem are Bessel processes. However, at our knowledge, even in this Markovian case (at least for the case  $\delta < 1$ ), those processes have not been clearly identified as (unique) solutions of an SDE or of a martingale problem.

Concerning (1), under the existence of the function (1.4) and some boundedness or linear growth condition (see (4.15)), Theorem 4.22 presents existence for the martingale problem related to (1.1). Proposition 4.23 states uniqueness under more restrictive conditions. Under suitable Lipschitz regularity conditions on a functional  $\tilde{\Gamma}$ , which is related to  $\Gamma$  via (4.12), Corollary 4.28 establishes well-posedness for the strong martingale problem associated to (1.1).

The case (2) is connected to a non-Markovian version of the classical Bessel process. We refer the reader to e.g [23, 32, 26] (section 2.3, chapter 3 and chapter XI, respectively) for an overview on Bessel processes. Let  $x_0 \geq 0$ . We recall that a Bessel process  $X$  (with initial condition  $x_0$  and dimension  $\delta \geq 0$  and denoted  $\text{BES}^\delta(x_0)$ ) is defined as the square root of the so called *squared Bessel process* (with initial condition  $s_0 = x_0^2$  and dimension  $\delta \geq 0$  and denoted  $\text{BESQ}^\delta(x_0)$ ), which is characterized as the unique solution of the SDE

$$dS_t = 2\sqrt{|S_t|}dW_t + \delta t, S_0 = x_0^2.$$

Except maybe for large values of  $\delta$  (for which the Bessel process is transient), it is not possible to characterize  $X$  as solution of

$$dX_t = \frac{\delta - 1}{2}X_t^{-1}dt + dW_t. \quad (1.7)$$

Consequently, it seems difficult to investigate (1.7) with memory, i.e.

$$dX_t = \frac{\delta - 1}{2}X_t^{-1}dt + \Gamma(t, X^t)dt + dW_t, \quad (1.8)$$

for  $\Gamma \neq 0$ . In this work, we treat the case when  $\delta \in [0, 1]$ . In that situation, the Bessel process is recurrent (so that it visits the point zero very often) and it is not even a semimartingale, except for  $\delta = 1$  and  $\delta = 0$ , see [26, 18]; see Chapter XI Section 1 and Section 6.1, respectively. When  $0 < \delta < 1$ , we expect therefore (in the sense of [23, 32]) that the Bessel process solves an SDE of the type

$$X_t = x_0 + \frac{\delta - 1}{2} \text{p.v.} \int_0^t \frac{ds}{X_s} + W_t, t \geq 0, \quad (1.9)$$

where p.v. stands for principal value. Our point of view consists in considering (analytically)  $\text{p.v.} \frac{1}{x}$  as the derivative in the sense of Schwartz distributions of the function  $x \mapsto \log|x|$ . At this stage of the introduction, it must be stressed again that even in the classical Markovian case ( $\Gamma = 0$ ), unique solvability of the SDE (1.7) associated

with Bessel processes in the most critical case ( $0 < \delta < 1$ ) is still an open question. Orthogonal decomposition of a Bessel process (with dimension  $\delta$ ) was investigated by [23, 32] in terms of (1.9).

In the Markovian case ( $\Gamma = 0$ ), a series of results of existence and uniqueness for the strong martingale problem are provided in Sections 5.2, 5.3, 5.4. In this article, we establish existence and uniqueness of the martingale problem associated with the SDE (1.8) under the condition that  $\Gamma$  is bounded; see Propositions 6.2 and 6.8. Lemma 6.5 and Theorem 6.13 illustrate sufficient conditions on  $\Gamma$  to have well-posedness of the strong martingale problem. Cases (1) and (2) arise in many fields.

- (1) For instance, much work has been done on Markovian processes in a random environment. In this particular case,  $b = B$  is a two-sided real-valued Brownian motion which is independent from  $W$  and (1.1) might be interpreted as the non-Markovian version

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + \Gamma(t, X^t)dt + dW_t,$$

of the so-called Brox diffusion which is indeed obtained setting  $\Gamma = 0$ , see e.g [4, 17] and other references therein in the classical Markovian context.

- (2) Similarly to the classical Markovian case with integer dimension  $\delta \geq 2$ , the path-dependent SDE (1.8) should be interpreted as the radial dynamics of  $\delta$ -dimensional Brownian motion  $\beta$  with drift having a radial intensity, i.e. with magnitude for a non-anticipative one-dimensional functional  $\Gamma$ . In other words, if  $Y$  is a weak solution to

$$dY_t = d\beta_t + \Gamma(t, \|Y_t\|)\frac{Y_t}{\|Y_t\|}dt, \quad (1.10)$$

then  $X_t = \|Y_t\|$  solves (1.8) formally for an integer  $\delta \geq 2$ . Indeed, if  $Y$  is a solution of (1.10), then a formal application of Itô's formula on  $\rho_t := \|Y\|_{\mathbb{R}^\delta}^2$  and Lévy's characterization theorem for local martingales show that

$$d\rho_t = 2\sqrt{\rho_t}dW_t + 2\sqrt{\rho_t}\Gamma(t, \sqrt{\rho_t})dt + \delta dt. \quad (1.11)$$

A subsequent formal application of Itô's formula shows that  $X_t = \sqrt{\rho_t}$  solves (1.8). On the other hand, Bessel-type processes with dimension  $\delta \in \mathbb{R}$  play an important role in the theory of Schramm-Loewner evolution (see e.g [21]). In particular, the two-parameter family of Schramm-Loewner evolution  $SLE(\kappa, \kappa - 4)$  defined in [20] provides a source of examples of  $BES^\delta$  flows with very singular behavior  $\delta = 1 - \frac{4}{\kappa}, \kappa > 4$ , which are covered by the SDE (1.7). In fact, the final

right-boundary of  $\text{SLE}(\kappa, \kappa - 4)$  processes is described by the excursions of  $\delta$ -dimensional Bessel processes. We refer the reader to [9] for more details. In this case, equation (1.8) describes a non-Markovian version of those phenomena.

The paper is organized as follows. After this Introduction and after having fixed some preliminaries, in Section 3 we define the suitable concept of martingale problems for SDE with distributional drift with path-dependent perturbations. In Section 4 we investigate the case (1) when  $b'$  is the derivative of continuous function and Section 5 is devoted to the case of Bessel processes in low dimension, under the perspective of strong martingale problem. Section 6 discusses the case of non-Markov perturbations of Bessel processes.

## 2. NOTATIONS AND PRELIMINARIES

### 2.1. General notations.

Let  $I$  be an interval of  $\mathbb{R}$ .  $C^k(I)$  is the space of real functions defined on  $I$  having continuous derivatives till order  $k$ . Such space is endowed with the uniform convergence topology on compact sets for the functions and all derivatives. Generally  $I = \mathbb{R}$  or  $[0, T]$  for some fixed positive real  $T$ . If there is no ambiguity  $C^k(\mathbb{R})$  will be simply indicated by  $C^k$ . Given an a.e. bounded real function  $f$ ,  $|f|_\infty$  will denote the essential supremum.

We recall some notions from [13]. For us all filtrations  $\mathfrak{F}$  fulfill the usual conditions. When no filtration is specified, we mean the canonical filtration of the underlying process. Otherwise the canonical filtration associated with a process  $X$  is denoted by  $\mathfrak{F}^X$ . An  $\mathfrak{F}$ -Dirichlet process  $X$  is the sum of an  $\mathfrak{F}$ -local martingale  $M^X$  with a  $\mathfrak{F}$ -adapted zero quadratic variation process  $A^X$ . We will fix by convention that  $A_0^X = 0$  so that the decomposition is unique. A sequence  $(X^n)$  of continuous processes indexed by  $[0, T]$  is said to converge u.c.p. to some process  $X$  whenever  $\sup_{t \in [0, T]} |X_t^n - X_t|$  converges to zero in probability.

#### Remark 2.1.

- (1) *An  $\mathfrak{F}$ -continuous semimartingale  $Y$  is always an  $\mathfrak{F}$ -Dirichlet process. The  $A^Y$  process coincides with the continuous bounded variation component. Moreover the quadratic variation  $[Y]$  is the usual quadratic variation for semimartingales.*
- (2) *Any  $\mathfrak{F}$ -Dirichlet process is a finite quadratic variation process and its quadratic variation gives  $[X] = [M^X]$ .*
- (3) *If  $f \in C^1(\mathbb{R})$  and  $X = M^X + A^X$  is an  $\mathfrak{F}$ -Dirichlet process, then  $Y := f(X)$  is again an  $\mathfrak{F}$ -Dirichlet process and  $[Y] = \int_0^\cdot f'(X_s)^2 d[M^X]$ .*

## 3. NON-MARKOVIAN SDE: THE FUNCTION CASE.

## 3.1. General considerations.

Similarly as for the case of Markovian SDEs, it is possible to formulate the notions of strong existence, pathwise uniqueness, existence and uniqueness in law for path-dependent SDEs of the type (1.1), see e.g. Section 7.

Let us suppose for the moment that  $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$  are Borel functions. We will consider solutions  $X$  of

$$\begin{cases} dX_t &= \sigma(X_t)dW_t + b'(X_t) + \Gamma(t, X^t)dt \\ X_0 &= \xi, \end{cases} \quad (3.1)$$

for some initial condition  $\xi$ , which, for simplicity in this paper, will always be considered as deterministic.  $\Gamma$  is defined in (1.2).

Previous equation will be denoted by  $E(\sigma, b', \Gamma; \nu)$  or simply with  $E(\sigma, b', \Gamma)$  if we omit the initial condition.

**Definition 3.1.** *Let be the Dirac probability measure on  $\mathbb{R}$  such that  $\nu = \delta_\xi, \xi \in \mathbb{R}$ . A stochastic process  $X$  is called **(weak) solution** of  $E(\sigma, b', \Gamma; \nu)$  with respect to a probability  $\mathbb{P}$  if there is a Brownian motion  $W$  on some filtered probability space, such that  $X$  solves (3.1) and  $X_0 = \xi$ . We also say that the couple  $(X, \mathbb{P})$  solves  $E(\sigma, b', \Gamma)$  with initial condition distributed according to  $\nu$ .*

Suppose  $\Gamma \equiv 0$ . A very well-known result in [31], Corollary 8.1.7, concerns to the equivalence between martingale problems and solution in law. Suppose for a moment that  $b'$  is a continuous function. According to [19, chapter 5], it is well-known that a process  $X$  and probability  $\mathbb{P}$  solve the classical martingale problem, if and only if,  $X$  is a (weak) solution of (1.1). The proof of the result mentioned above can be easily adapted to the path-dependent case, i.e. when  $\Gamma \neq 0$ . This provides the statement below.

**Proposition 3.2.** *A couple  $(X, \mathbb{P})$  is a solution of  $E(\sigma, b', \Gamma)$ , if and only if, under  $\mathbb{P}$ ,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds \quad (3.2)$$

*is a local martingale, where  $Lf = \frac{1}{2}\sigma^2 f'' + b'f'$ , for every  $f \in C^2$ .*

## 3.2. Comments about the distributional case.

When  $b'$  is a distribution, it is not obvious to introduce the notion of SDE, except in the case when  $L$  is close to the divergence form, i.e. when  $Lf = (\sigma^2 f')' + \beta f'$ , and



$\beta$  is a Radon measure, see e.g. Proposition 3.1 of [13]. For this reason we replace the notion of weak solution with the notion of martingale problem. Unfortunately  $C^2$  is not included in the domain of operator  $L$ , so similarly as in [13], we will replace  $C^2$  with some domain  $\mathcal{D}_L$ .

Suppose now that  $b'$  is the derivative of some cadlag function  $b : \mathbb{R} \rightarrow \mathbb{R}$  such that the function  $\Sigma$  in (1.4) is defined in a suitable sense, generalizing the case when  $b'$  is a continuous function. For instance if either  $\sigma^2$  or  $b$  are of locally bounded variation, see [13] for some methods via regularization to do it. Suppose that there is a natural linear subspace  $\mathcal{D}_L$  of  $C^1$ , for which, for every  $f \in \mathcal{D}_L$  the function  $Lf$  is defined according (1.6). We remark that, when  $\sigma$  and  $b'$  are continuous functions,  $L$  defined on (1.5) gives (1.6). When  $\Gamma \equiv 0$ , [13, 14] introduced the notion of martingale problem associated with a process (1.1) with respect to a domain  $\mathcal{D}_L$  and the probability  $\mathbb{P}$ .

Inspired by the definition of the martingale problem in [14], given  $\Gamma \equiv 0$ , we formulate the following.

**Definition 3.3.** (1) *We say that a continuous stochastic process  $X$  solves (with respect to a probability  $\mathbb{P}$  on some measurable space  $(\Omega, \mathcal{F})$ ) the martingale problem related to*

$$\mathcal{L}f := Lf + \Gamma f', \quad (3.3)$$

*with initial condition  $\nu = \delta_{x_0}$ ,  $x_0 \in \mathbb{R}$ , with respect to a domain  $\mathcal{D}_L$  if*

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (3.4)$$

*is a  $\mathbb{P}$ -local martingale for all  $f \in \mathcal{D}_L$ .*

*We will also say that the couple  $(X, \mathbb{P})$  is a solution of (or  $(X, \mathbb{P})$  solves) the martingale problem with respect to  $\mathcal{D}_L$ .*

- (2) *If a solution exists we say that the martingale problem above admits existence.*
- (3) *We will say that the martingale problem above admits uniqueness if any two solutions  $(X^i, \mathbb{P}^i)$ ,  $i = 1, 2$  (on some measurable space  $(\Omega, \mathcal{F})$ ) have the same law.*
- (4) *In the sequel, when the measurable space  $(\Omega, \mathcal{F})$  is self-explanatory it will be often omitted.*

As already observed in Proposition 3.2, the notion of martingale problem is (since the works of Stroock and Varadhan [31]) a concept related to solutions of SDEs in law. In the case when  $b'$  and  $\sigma$  are continuous functions (see [31]),  $\mathcal{D}_L$  corresponds to  $C^2(\mathbb{R})$ .

Below we introduce the analogous notion of strong existence and pathwise uniqueness for our martingale problem.

**Definition 3.4.** *Let  $E = \mathbb{R}$  or  $E = \mathbb{R}_+$ .*

- (1) *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $\mathfrak{F} = (\mathcal{F}_t)$  be the canonical filtration associated with a fixed Brownian motion  $W$ . Let  $x_0 \in E$  being a constant. We say that a continuous  $\mathfrak{F}$ -adapted  $E$ -valued process  $X$  such that  $X_0 = x_0$  is a **solution to the strong martingale problem** (related to (3.3)) with respect to  $\mathcal{D}_L(E)$  and  $W$  (with related filtered probability space), if*

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds = \int_0^t f'(X_s)\sigma(X_s)dW_s, \quad (3.5)$$

*is an  $\mathfrak{F}$ -local martingale for all  $f \in \mathcal{D}_L(E)$ .*

- (2) *We say that the martingale problem related to (3.3) with respect to  $\mathcal{D}_L(E)$  admits **strong existence** if for every  $x_0 \in E$ , given a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{F})$ , where  $\mathfrak{F} = (\mathcal{F}_t)$  is the canonical filtration associated with a Brownian motion  $W$ , there is a process  $X$  solving the strong martingale problem (related to (3.3)) with respect to  $\mathcal{D}_L(E)$  and  $W$  with  $X_0 = x_0$ . When  $E = \mathbb{R}$  we often simply write  $\mathcal{D}_L = \mathcal{D}_L(E)$ .*
- (3) *We say that the martingale problem (related to (3.3)) with respect to  $\mathcal{D}_L(E)$  (or simply  $\mathcal{D}_L$  if  $E = \mathbb{R}$ ) admits **pathwise uniqueness** if given  $(\Omega, \mathcal{F}, \mathbb{P})$  and a Brownian motion  $W$  and  $X^i, i = 1, 2$  are solutions to the strong martingale with respect to  $\mathcal{D}_L(E)$  and  $W$  and  $\mathbb{P}[X_0^1 = X_0^2] = 1$  then  $X^1$  and  $X^2$  are indistinguishable.*

#### 4. THE CASE WHEN THE DRIFT IS A DERIVATIVE OF A CONTINUOUS FUNCTION

Here we extend the Markovian framework of [13] to the non-Markovian case.

##### 4.1. The Markovian case.

Let  $\sigma$  and  $b$  are functions in  $C^0$  with  $\sigma > 0$ . The authors in [13] define by mollification methods

$$\Sigma(x) = 2 \lim_{n \rightarrow \infty} \int_0^x \frac{b'_n(y)}{\sigma_n^2} dy, \quad \forall x \in \mathbb{R}, \quad (4.1)$$

where the limit is intended to be in the  $C^0(\mathbb{R})$ , i.e. uniformly on each compact. Given a continuous function  $g$ , they also define (still by mollifications) the meaning of the equation  $Lf = g$  (in  $C^1$ -generalized sense) for some  $f \in C^1(\mathbb{R})$ .

Similarly as in [13],  $\mathcal{D}_L$  will be the linear space of  $f \in C^1(\mathbb{R})$  for which there exists  $l \in C^0(\mathbb{R})$  such that  $Lf = l$ . As in [13] we define the *harmonic* function  $h : \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$h(0) = 0, \quad h' = e^{-\Sigma}. \quad (4.2)$$

Below we only use the characterization of  $\mathcal{D}_L$  given by Lemma 2.9 and Lemma 2.6 in [13].

**Proposition 4.1.**

- (1) Let  $f \in C^1$ .  $f \in \mathcal{D}_L$  if and only if there exists  $\phi \in C^1$  such that  $f' = \exp(-\Sigma)\phi$ .
- (2) If  $f \in \mathcal{D}_L$  then  $l = Lf$  is given by (1.6); in particular we have

$$Lf = \phi' \exp(-\Sigma) \frac{\sigma^2}{2}. \quad (4.3)$$

**Corollary 4.2.**

- (1) If  $f \in \mathcal{D}_L$  then  $f^2 \in \mathcal{D}_L$  and  $Lf^2 = \sigma^2 f'^2 + 2fLf$ .
- (2) If  $h$  is defined as in (4.2) we have

$$Lh = 0, \quad Lh^2 = \sigma^2 h'^2.$$

**Proof.**

- (1)  $f^2 \in \mathcal{D}_L$  because  $(f^2)' = 2ff' = (2f\phi) \exp(-\Sigma)$  taking into account Proposition 4.1 (1) and the fact that  $\phi_2 := 2f\phi \in C^1$ . By (4.3),

$$\begin{aligned} Lf^2 &= \phi_2' \exp(-\Sigma) \frac{\sigma^2}{2} = (f\phi)' \exp(-\Sigma) \sigma^2 \\ &= f' \sigma^2 \exp(-\Sigma) \phi + f \phi' \exp(-\Sigma) \sigma^2 = f'^2 \sigma^2 + 2fLf. \end{aligned}$$

- (2) It follows by Proposition 4.1 setting  $\phi = 1$  and item (1). □

From now on, we will suppose that

$$\int_{-\infty}^0 e^{-\Sigma(x)} dx = \int_0^{+\infty} e^{-\Sigma(x)} dx = \infty. \quad (4.4)$$

**Remark 4.3.**

- (1) In this case the harmonic function  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined in (4.2) is a  $C^1(\mathbb{R})$  diffeomorphism. In particular  $h$  is surjective because of (4.4).
- (2) It is easy to verify that the non-explosion condition (3.16) in Proposition 3.13 in [13] is fulfilled.

We now summarize the important assumption which will be made in the sequel below.

**Assumption 4.4.**

- $\Sigma$  is well-defined, see (4.1).
- The non-explosion condition (4.4) holds.

**Remark 4.5.** When  $\sigma$  and  $b'$  are continuous functions, then  $\mathcal{D}_L = C^2$ . Indeed, in this manner,  $\Sigma \in C^1$  and then  $f' = \exp(-\Sigma)\phi \in C^1$ . In particular  $Lf$  corresponds with its classical definition.

In relation to the *harmonic* function  $h$  defined in (4.2), Proposition 2.13 a) in [13] states the following.

**Proposition 4.6.**  $f \in \mathcal{D}_L$  if and only if  $f \circ h^{-1} \in C^2$ .

More generally we can state the following.

**Proposition 4.7.** We suppose Assumption 4.4. Let  $h \in \mathcal{D}_L$  being a diffeomorphism of class  $C^1$  such that  $h' > 0$ . Let  $f$  be a  $C^1$  function,  $f \in \mathcal{D}_L$  if and only if  $f \circ h^{-1}$  belongs to  $\mathcal{D}_{L^Y}$  with  $L^Y g = \frac{1}{2}\sigma_0^2 g'' + ((Lh) \circ h^{-1})g', g \in \mathcal{D}_{L^Y}$ , where

$$\sigma_0 = (\sigma h') \circ h^{-1}. \tag{4.5}$$

Moreover

$$Lf \circ h^{-1} = L^Y(f \circ h^{-1}).$$

**Proof.** By Proposition 4.1 there exists  $\phi^h \in C^1$  such that

$$h' = \exp(-\Sigma)\phi^h. \tag{4.6}$$

Concerning the direct implication, if  $f \in \mathcal{D}_L$ , first we prove that  $f \circ h^{-1} \in \mathcal{D}_{L^Y}$ . Again by Proposition 4.1 there exists  $\phi^f \in C^1$  such that  $f' = \exp(-\Sigma)\phi^f$ . So,  $(f \circ h^{-1})' = \frac{f'}{h'} \circ h^{-1} = \frac{\phi^f}{\phi^h} \circ h^{-1} \in C^1$  because  $h^{-1} \in C^1$  and  $\phi^h > 0$ , so  $f \circ h^{-1} \in C^2$ . Note that, by Remark 4.5,  $\mathcal{D}_{L^Y} = C^2$  so  $f \circ h^{-1} \in \mathcal{D}_{L^Y}$ . Moreover, by Proposition 4.1 (2),

$$(\phi^f)' = \frac{2Lf}{\sigma^2} \exp(\Sigma), \quad (\phi^h)' = \frac{2Lh}{\sigma^2} \exp(\Sigma).$$

A direct computation gives

$$(f \circ h^{-1})'' = \left( \frac{\phi^f}{\phi^h} \circ h^{-1} \right)' = \left[ \frac{2Lf}{h'^2 \sigma^2} - \frac{2Lh f'}{\sigma^2 h'^3} \right] \circ h^{-1}.$$

Consequently,

$$\frac{\sigma_0^2}{2}(f \circ h^{-1})'' = \left[ Lf - \frac{Lh f'}{h'} \right] \circ h^{-1}. \quad (4.7)$$

By (4.7)  $Lf \circ h^{-1} = \frac{\sigma_0^2}{2}(f \circ h^{-1})'' + (Lh) \circ h^{-1}(f \circ h^{-1})' = L^Y(f \circ h^{-1})$ .

Let us discuss the converse implication. Suppose that  $f \circ h^{-1}$  belongs to  $\mathcal{D}_{L^Y} = C^2$ . By Proposition 4.1 we need to show that  $f' \exp(\Sigma) \in C^1$  which is equivalent to show that  $(f' \exp(\Sigma)) \circ h^{-1}$  belongs to  $C^1$ . If  $\phi^h \in C^1$  is such that  $h' = \exp(-\Sigma)\phi^h$  (see (4.6)) we have

$$(f' \exp(\Sigma)) \circ h^{-1} = \left( f' \frac{\phi^h}{h'} \right) \circ h^{-1} = (f \circ h^{-1})'(\phi^h \circ h^{-1}),$$

which obviously belongs to  $C^1$ . Therefore  $f \in \mathcal{D}_L$ .  $\square$

**Remark 4.8.** *In the case where  $h$  is defined by (4.2) we have*

$$Lf \circ h^{-1} = L^Y(f \circ h^{-1}) = \frac{1}{2}\sigma_0^2 f'',$$

where we recall that  $\sigma_0$  was defined in (4.5).

In [13], the authors also show that the existence and uniqueness of the solution of the martingale problem are conditioned to a non-explosion feature. An easy adaptation and consequence of Proposition 3.13 in [13] gives the following.

**Proposition 4.9.** *Let  $\nu = \delta_{x_0}, x_0 \in \mathbb{R}$ . Suppose the validity of Assumption 4.4. Then the martingale problem related to  $L$  (i.e. with  $\Gamma = 0$ ) with respect to  $\mathcal{D}_L$  with initial condition  $\nu$  admits existence and uniqueness.*

**Remark 4.10.** *By Proposition 3.2 of [13], if  $\Gamma = 0$ , if  $(X, \mathbb{P})$  is a solution of the above-mentioned martingale problem then there exists a  $\mathbb{P}$ -Brownian motion  $W$  such that (3.4) equals*

$$\int_0^t (f' \sigma)(X_s) dW_s, t \in [0, T].$$

From now on  $h$  will be the function defined in (4.2).

#### 4.2. The path-dependent case.

Let  $\sigma$  and  $b$  be functions in  $C^0$  with  $\sigma > 0$  and  $\Gamma$  as defined in (1.2). Let suppose again Assumption 4.4 and let  $h$  as defined in (4.2). The first result explains how to reduce our path-dependent martingale problem to a path-dependent SDE.

**Proposition 4.11.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $X$  be a stochastic process, we denote  $Y = h(X)$ .*

- (1)  *$(X, \mathbb{P})$  solves the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  if and only if the process  $Y := h(X)$  is a solution (with respect to  $\mathbb{P}$ ) of*

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds, \quad (4.8)$$

for some  $\mathbb{P}$ -Brownian motion  $W$ .

- (2) *Let  $W$  be a Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $X$  is a solution to the strong martingale problem with respect to  $\mathcal{D}_L$  and  $W$  if and only if (4.8) holds.*

**Proof.**

- (1) We start proving the direct implication. According to (3.4) and the notations introduced therein

$$M_t^h = h(X_t) - h(X_0) - \int_0^t Lh(X_s) ds - \int_0^t h'(X_s) \Gamma(s, X^s) ds, \quad (4.9)$$

is a  $\mathbb{P}$ -local martingale on some probability space  $(\Omega, \mathcal{F})$ .

In particular, by Corollary 4.2

$$Y_t = Y_0 + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + M_t^h,$$

where  $M^h$  is a local martingale, so  $Y$  is a semimartingale. We need now evaluate  $[M^h]_t = [Y]_t$ . We apply (3.4) for  $f = h^2$  and again by Corollary 4.2 we get

$$Y_t^2 = Y_0^2 + \int_0^t \sigma_0^2(Y_s) ds + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + M_t^{h^2}, \quad (4.10)$$

where  $M^{h^2}$  is a local martingale and we recall that  $\sigma_0$  was defined in (4.5). By integration by parts,

$$[Y]_t = Y_t^2 - Y_0^2 - 2 \int_0^t Y_s dY_s = Y_t^2 - Y_0^2 + M_t - 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds,$$

where  $M_t = \int_0^t Y_s h'(h^{-1}(Y_s)) dM_s^h$ . Note that as  $(Y_t h'(h^{-1}(Y_t)))$  is a continuous adapted process, then  $(M_t)$  is local martingale on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Therefore

$$Y_t^2 = Y_0^2 - M_t + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + [Y]_t. \quad (4.11)$$

Now we can use the uniqueness of the decomposition of a semimartingale  $Y^2$  which admits the two expressions (4.10) and (4.11). This says  $-M = M^{h^2}$  and

$\int_0^t \sigma_0^2(Y_s) ds = [Y]_t$ . By (4.9)

$$[M^h]_t = [Y]_t = \int_0^t \sigma_0^2(Y_s) ds.$$

Setting

$$W_t := \int_0^t \frac{dM_s^h}{\sigma_0(Y_s)}, t \geq 0,$$

we have

$$[W]_t \equiv t.$$

Therefore, by the Lévy's characterization of Brownian motion,  $W$  is a standard Brownian motion. Since

$$M^h = \int_0^\cdot \sigma_0(Y_s) dW_s,$$

(4.9) shows that  $Y$  solves (4.8).

About the converse implication suppose now that  $Y = h(X)$  satisfies (4.8), for some  $\mathbb{P}$ -Brownian motion  $W$ . We take  $f \in \mathcal{D}_L$ . By Proposition 4.6  $g \equiv f \circ h^{-1} \in C^2$ . Using Itô's formula, Proposition 4.7 and Remark 4.8 we get

$$\begin{aligned} g(Y_t) &= g(Y_0) + \int_0^t g'(Y_s) dY_s + \frac{1}{2} \int_0^t g''(Y_s) d[Y]_s \\ &= f(X_0) + \int_0^t Lf(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s + \int_0^t f'(X_s) \Gamma(s, X^s) ds. \end{aligned}$$

Therefore

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds = \int_0^t f'(X_s) \sigma(X_s) dW_s$$

is a local martingale, which concludes the proof.

- (2) The converse implication follows in the same way as for item (1). The proof of the direct implication follows directly by Itô's formula.

□

**Corollary 4.12.** *Let  $(X, \mathbb{P})$  be a solution to the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$ . Then  $X$  is a Dirichlet process (with respect to its canonical filtration) and  $[Y]_t = \int_0^t \sigma^2(X_s) ds$ .*

**Proof.**

By Proposition 4.11,  $X = h^{-1}(Y)$ , where  $Y$  is obviously a semimartingale such that  $[Y]_t = \int_0^t \sigma_0^2(Y_s) ds, t \in [0, T]$ . Consequently, by Remark 2.1  $X$  is indeed a Dirichlet process and

$$[X]_t = \int_0^t ((h^{-1})')^2 \sigma_0^2(Y_s) ds = \int_0^t \frac{\sigma_0^2}{(h' \circ h^{-1})^2}(Y_s) ds = \int_0^t \sigma^2(X_s) ds, t \in [0, T].$$

□

**Remark 4.13.** *If  $X$  is a solution to the strong martingale problem with respect to  $\mathcal{D}_L$  and some Brownian motion  $W$ , then  $X$  is a Dirichlet process with respect to the canonical filtration of the Brownian motion.*

An immediate consequence of Proposition 4.11 is the following.

**Corollary 4.14.** *Suppose that  $\Gamma = 0$  and let  $(X, \mathbb{P})$  be a solution of the martingale problem related to  $L$  with respect to  $\mathcal{D}_L$ . Then  $Y = h(X)$  is an  $\mathfrak{F}$ -local martingale where  $\mathfrak{F}$  is the canonical filtration of  $X$  with quadratic variation  $[Y] = \int_0^\cdot \sigma_0^2(X_s) ds$ .*

**4.3. Existence.**

We make here the same conventions as in Section 4.2. In the sequel we introduce the map  $\tilde{\Gamma} : \Lambda \rightarrow \mathbb{R}$  defined by

$$\tilde{\Gamma}(s, \eta) = \frac{\Gamma(s, \eta)}{\sigma(\eta(s))}, (s, \eta) \in \Lambda. \tag{4.12}$$

At this point, we introduce the following assumption.

**Assumption 4.15.**

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, h^{-1} \circ \eta)| \leq K \left( 1 + \sup_{s \in [0, T]} |\eta(s)| \right), K > 0.$$

**Remark 4.16.** *Given a stochastic process  $X$ , setting  $Y = h(X)$ , we get*

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, X^s)| \leq K \left( 1 + \sup_{s \in [0, T]} |Y_s| \right). \tag{4.13}$$

*In particular  $\int_0^T \tilde{\Gamma}^2(s, X^s) ds < \infty$  a.s.*

The next result is a well-known extension of Novikov criterion, see e.g. Corollary 5.14 [19, Chapter 3].



**Proposition 4.17.** *Suppose Assumption 4.15. Let  $W$  be a Brownian motion and  $X$  a continuous and adapted process for which there exists a partition  $0 = t_0, t_1, \dots, t_n = T$  such that for  $i \in \{1, \dots, n\}$*

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \right) \right] < \infty.$$

*Then, the process*

$$N_t = \exp \left( \int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right),$$

*is a martingale.*

We will need a slight adaptation of the Dambis-Dubins-Schwartz theorem to the case of a finite interval.

**Proposition 4.18.** *Let  $M$  be a local martingale such that  $[M]_t = \int_0^t A_s ds$ ,  $t \in [0, T]$ . Then, on a possibly enlarged probability space, there exists a copy of  $M$  (still denoted by the same letter  $M$ ) with the same distribution law and a Brownian motion  $\beta$  such that*

$$M_t = \beta_{\int_0^t A_s ds}, t \in [0, T].$$

**Proof.**

Let us define

$$\tilde{M}_t = \begin{cases} M_t, & t \in [0, T] \\ M_T + B_t - B_T, & t > T, \end{cases}$$

where  $B$  is a Brownian motion independent of  $M$ . If the initial probability space is not rich enough, one considers an enlarged probability space containing a copy of  $M$  (still denoted by the same letter) with the same law and the independent Brownian motion  $B$ . Note that  $\tilde{M}$  is a local martingale and we have,

$$[\tilde{M}]_t = \begin{cases} [M]_t, & t \in [0, T] \\ t - T + [M]_T, & t > T. \end{cases}$$

Observe that  $\lim_{t \rightarrow \infty} [\tilde{M}]_t = \infty$ . By the classical Dambis, Dubins-Schwarz theorem there exists a standard Brownian motion  $\beta$  such that a.s.  $\tilde{M}_t = \beta_{\int_0^t A_s ds}$ ,  $t \geq 0$ . In particular

$$M_t = \beta_{\int_0^t A_s ds}, \quad 0 \leq t \leq T.$$

□

The proposition below is an adaptation of a well-known argument for Markov diffusions.

**Proposition 4.19.** *Suppose that  $\sigma_0$  is bounded. Let  $(X, \mathbb{P})$  be a solution of the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  with  $\Gamma = 0$ . Let  $M^X$  the martingale component of  $X$ . We set*

$$W_t := \int_0^t \frac{1}{\sigma(X_s)} dM^X, t \in [0, T],$$

which, by Proposition 3.2 of [13], is a Brownian motion.

Then

$$\exp \left( \int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right), \tag{4.14}$$

is a martingale.

**Remark 4.20.** *We recall that, by Corollary 4.12,  $X$  is an  $\mathfrak{F}$ -Dirichlet process ( $\mathfrak{F}$  be the canonical filtration) and  $[X] = [M^X] = \int_0^\cdot \sigma^2(X_s) ds$  so that by Lévy characterization theorem,  $W$  is a  $\mathfrak{F}$ -Brownian motion.*

**Proof** (of Proposition 4.19).

Let  $Y = h(X)$ . By Proposition 4.11,  $[Y] = \int_0^\cdot \sigma_0^2(Y_s) ds$ . Let  $k \geq |\sigma_0|_\infty^2 T$  and  $\{t_0 = 0, \dots, t_n = T\}$  a grid of  $[0, T]$  so that

$$c_i := 3(t_i - t_{i-1})K^2 k < \frac{1}{2}, \tag{4.15}$$

( $K$  coming from Assumption 4.15)  $i = \{1, \dots, n\}$ . By (4.13) we know that

$$\int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \leq (t_i - t_{i-1})K^2 \left( 1 + \sup_{s \in [0, T]} |Y_s| \right)^2. \tag{4.16}$$

We set  $M_t = Y_t - Y_0, t \in [0, T]$ . Note that

$$\left( 1 + \sup_{s \in [0, T]} |Y_s| \right)^2 \leq 3 \sup_{s \in [0, T]} |M_s|^2 + 3(1 + Y_0^2). \tag{4.17}$$

We recall that  $Y_0$  is deterministic. In view of applying Proposition 4.17, taking into account (4.16) and (4.17) we get

$$\begin{aligned} & \mathbb{E} \left( \exp \left( \frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right) \leq \\ & \mathbb{E} \left( \exp \left( \frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |M_s|^2 \right) \right) \exp \left( \frac{3}{2} (t_i - t_{i-1})K^2 (1 + Y_0^2) \right). \end{aligned} \tag{4.18}$$

By Proposition 4.18 there is a copy (with the same distribution) of  $M$  (still denoted by the same letter) on another probability space, a Brownian motion  $\beta$  such that previous expression gives

$$\begin{aligned} & \mathbb{E} \left( \exp \left( \frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |\beta_{[M]_s}|^2 \right) \right) \exp \left( \frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right) \\ & \leq \mathbb{E} \left( \exp \left( \frac{3(t_i - t_{i-1})K^2}{2} \sup_{\tau \in [0, k]} |\beta_\tau|^2 \right) \right) \exp \left( \frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right), \end{aligned} \quad (4.19)$$

the latter inequality being valid because  $[M]_t = \int_0^t \sigma_0^2(Y_s)ds$ . By (4.15) we get

$$\begin{aligned} \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right] & \leq \mathbb{E} \left[ \exp \left( \frac{c_i}{2k} \sup_{\tau \in [0, k]} |B_\tau|^2 \right) \right] \exp \left( \frac{c_i}{2k} (1 + Y_0^2) \right) \\ & \leq \mathbb{E} \left[ \sup_{\tau \in [0, k]} \exp \left( \frac{c_i}{2k} |B_\tau|^2 \right) \right] \exp \left( \frac{c_i}{2k} (1 + Y_0^2) \right). \end{aligned} \quad (4.20)$$

By Remark 4.21 below

$$\mathbb{E} \left[ \exp \left( \frac{c_i}{2k} |B_\tau|^2 \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{c_i}{2} G^2 \right) \right] < \infty.$$

Since  $x \mapsto \exp(\frac{c_i}{2k}x)$  is increasing and convex, and  $(|B_\tau|^2)$  is a non-negative square integrable submartingale, then  $(\exp(\frac{c_i}{2k}|B_\tau|^2))$  is also a non-negative submartingale. Consequently, by Doob's inequality (with  $p = 2$ ) the expectation on the right-hand side of (4.20) is finite. Finally by Proposition 4.17 (4.14) is a martingale.  $\square$

**Remark 4.21.** *Let  $G$  be a standard Gaussian r.v. If  $c < \frac{1}{2}$  then*

$$\mathbb{E}[\exp(cG^2)] < \infty.$$

This opens the way to the following existence result for our martingale problem.

**Theorem 4.22.** *Suppose the validity of Assumption 4.4 and that one of the two conditions below are fulfilled.*

- (1)  $\tilde{\Gamma}$  is bounded.
- (2)  $\tilde{\Gamma}$  fulfills Assumption 4.15 and  $\sigma_0$  is bounded.

*Then the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  admits existence.*

**Proof.**

By Proposition 4.9, we can consider a solution  $(X, \mathbb{P})$  to the above-mentioned martingale problem with  $\Gamma = 0$ . By Remark 4.10, there is a Brownian motion  $W$  such that

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds = \int_0^t (f'\sigma)(X_s)dW_s, \tag{4.21}$$

for every  $f \in \mathcal{D}_L$ . We define the process

$$V_t := \exp \left( \int_0^t \tilde{\Gamma}(s, X^s)dW_s - \frac{1}{2} \int_0^t \tilde{\Gamma}^2(s, X^s)ds \right).$$

Under item (1)  $V$  is a martingale by the Novikov condition. Under item (2), Proposition 4.19 says that  $V$  it is a martingale. We define

$$\tilde{W}_t := W_t - \int_0^t \tilde{\Gamma}(s, X^s)ds. \tag{4.22}$$

By Girsanov's theorem, (4.22) is a Brownian motion under the probability  $\mathbb{Q}$  such that

$$d\mathbb{Q} := \exp \left( \int_0^T \tilde{\Gamma}(s, X^s)dW_s - \frac{1}{2} \int_0^T \tilde{\Gamma}^2(s, X^s)ds \right) d\mathbb{P}.$$

Applying (4.22) in (4.21) we obtain

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds = \int_0^t (f'\sigma)(X_s)d\tilde{W}_s,$$

for every  $f \in \mathcal{D}_L$ . Since  $\int_0^t (f'\sigma)(X_s)d\tilde{W}_s$  is a local martingale under  $\mathbb{Q}$ ,  $(X, \mathbb{Q})$  is proved to be a solution to the martingale problem in the statement. □

**4.4. Uniqueness in law.**

We use here again the notation  $\tilde{\Gamma}$  introduced in (4.12).

**Proposition 4.23.** *Suppose the validity of Assumption 4.4 and that  $\tilde{\Gamma}$  is bounded. Then the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  admits uniqueness.*

**Proof.**

Let  $(X^i, \mathbb{P}^i)$ ,  $i = 1, 2$ , be two solutions of the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$ . Let us fix  $i = 1, 2$ . By Corollary 4.12,  $X^i$   $\mathfrak{F}^{X^i}$ -Dirichlet process with respect to  $\mathbb{P}^i$ , such that  $[X^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$ . Let  $M^i$  be its respectively martingale component. Since  $[M^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$ , by Lévy characterization theorem, the process

$$W_t^i = \int_0^t \frac{dM_s^i}{\sigma(X_s^i)}, t \in [0, T], \tag{4.23}$$

is an  $\mathfrak{F}^{X^i}$ -Brownian motion. In particular  $W^i$  is a Borel functional of  $X^i$ . We define the r.v. (which is also a Borel functional of  $X^i$ )

$$V_t^i = \exp \left( - \int_0^t \tilde{\Gamma}(s, X^{i,s}) dW_s^i - \frac{1}{2} \int_0^t \left( \tilde{\Gamma}(s, X^{i,s}) \right)^2 ds \right),$$

which, by the Novikov condition, are  $\mathbb{P}^i$ -martingales. This allows us to define the probabilities  $d\mathbb{Q}^i = V_T^i d\mathbb{P}^i$ .

By Girsanov theorem, under  $\mathbb{Q}^i$ ,  $B_t^i := W_t^i + \int_0^t \tilde{\Gamma}(s, X^{i,s}) ds$  are Brownian motions. Therefore,  $(X^i, \mathbb{Q}^i)$  solves the martingale problem related to  $L$  ( $\Gamma = 0$ ) with respect to  $\mathcal{D}_L$ . By uniqueness of the martingale problem with respect to  $\mathcal{D}_L$  and  $\Gamma = 0$  (see Proposition 4.9)  $X^i$  (under  $\mathbb{Q}^i$ ),  $i = 1, 2$  have the same law. Hence, for every Borel set  $B \in \mathfrak{B}(C[0, T])$ , we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_{\Omega} \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} d\mathbb{Q}^1 = \int_{\Omega} \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} d\mathbb{Q}^2 = \mathbb{P}^2\{X^2 \in B\}.$$

Therefore,  $X^1$  under  $\mathbb{P}^1$  has the same law as  $X^2$  under  $\mathbb{P}^2$ . Finally the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  admits uniqueness.  $\square$

#### 4.5. Results on pathwise uniqueness.

Before exploring conditions for strong existence and uniqueness for the martingale problem, we state and prove Proposition 4.25, which constitutes a crucial preliminary step.

Let  $\bar{\Gamma} : \Lambda \rightarrow \mathbb{R}$  be a generic Borel functional. Related to it we formulate the following.

#### Assumption 4.24.

- (1) *There exists a function  $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\int_0^\epsilon l^{-2}(u) du = \infty$  for all  $\epsilon > 0$  and*

$$|\sigma_0(x) - \sigma_0(y)| \leq l(|x - y|).$$

- (2)  *$\sigma_0$  has at most linear growth.*  
(3) *there exists  $K > 0$  such that*

$$|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K \left( |\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)| dr \right),$$

*for all  $s \in [0, T], \eta^1, \eta^2 \in C([0, T])$ .*

- (4)  $\sup_{s \in [0, T]} |\bar{\Gamma}(s, 0)| < \infty$ .

**Proposition 4.25.** *Let  $y_0 \in \mathbb{R}$ . Suppose the validity of Assumption 4.24. Then  $E(\sigma_0, 0, \bar{\Gamma})$ , i.e.*

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t \bar{\Gamma}(s, Y^s) ds, \tag{4.24}$$

*admits pathwise uniqueness.*

**Proof.**

Now let  $Y^1, Y^2$  be two solutions on the same probability space with respect to the same Brownian motion  $W$  of (4.24) such that  $Y_0^1 = Y_0^2 = y_0$ . By Lemma 7.6 (in the Appendix) we have

$$\mathbb{E} \left( \sup_{t \in [0, T]} |Y_t^i|^2 \right) < \infty. \tag{4.25}$$

By the assumption on  $\sigma_0$ , this obviously gives

$$\mathbb{E} \left( \int_0^T |\sigma_0(Y_t^i)|^2 dt \right) < \infty. \tag{4.26}$$

We set  $\Delta_t = Y_t^1 - Y_t^2, t \in [0, T]$ , and this gives

$$\Delta_t = \int_0^t (\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})) ds + \int_0^t (\sigma_0(Y_s^1) - \sigma_0(Y_s^2)) dW_s, t \in [0, T]. \tag{4.27}$$

We recall from the proof of Proposition 2.13 in [19, Chapter 5] the existence of the functions

$$\Psi_n(x) = \int_0^{|x|} \int_0^y \rho_n(u) du dy,$$

such that for every  $x \in \mathbb{R}$

$$0 \leq \rho_n(x) \leq \frac{2}{nl^2(x)}, \quad |\Psi'_n(x)| \leq 1, \quad |\Psi_n(x)| \leq |x|, \quad \lim_{n \rightarrow \infty} \Psi_n(x) = |x|. \tag{4.28}$$

By (4.27), applying Itô's formula we get

$$\begin{aligned} \Psi_n(\Delta_t) &= \int_0^t \Psi'_n(\Delta_s) [\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})] ds + \frac{1}{2} \int_0^t \Psi''_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)]^2 ds \\ &+ \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s. \end{aligned} \tag{4.29}$$

Using Assumption 4.24 and (4.28) we get

$$\Psi_n(\Delta_t) \leq \int_0^t K \left( |Y_s^1 - Y_s^2| + \int_0^s |Y_r^1 - Y_r^2| dr \right) ds + \frac{t}{n} + M_t, \tag{4.30}$$

where  $M_t = \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s$  is a local martingale. Since  $\Psi'_n$  is bounded and by (4.26),  $M$  is a (even square integrable) martingale.

We now apply the expectation and the Fubini's theorem in (4.30) to get

$$\mathbb{E}\Psi_n(\Delta_t) \leq K \int_0^t \mathbb{E}|Y_s^1 - Y_s^2| ds + KT \int_0^t \mathbb{E}|Y_s^1 - Y_s^2| ds + \frac{t}{n}, \quad (4.31)$$

since  $\mathbb{E}M_t = 0$ . Passing to the limit when  $n \rightarrow \infty$ , by Lebesgue's dominated convergence theorem, we get

$$\mathbb{E}|\Delta_t| \leq (K + TK) \int_0^t \mathbb{E}|\Delta_s| ds, \quad (4.32)$$

so, by the Gronwall's inequality we obtain  $\mathbb{E}|\Delta_t| = 0$ . By the continuity of the sample paths of  $Y^1, Y^2$  we conclude that  $Y^1, Y^2$  are indistinguishable.  $\square$

We come back to the framework of the beginning of Section 4.1. We suppose again the validity of Assumption 4.4. We recall the definition of the harmonic function  $h$  defined by  $h(0) = 0, h'(x) = e^{-\Sigma}$ , see (4.2). We recall the notations  $\sigma_0 = (\sigma h') \circ h^{-1}$ . We define

$$\bar{\Gamma}(s, \eta) := h'(h^{-1}(\eta(s)))\Gamma(s, h^{-1}(\eta^s)), \quad s \in [0, T], \eta \in C([0, T]). \quad (4.33)$$

**Corollary 4.26.** *Under Assumptions 4.4 and 4.24 the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  admits pathwise uniqueness.*

**Proof.** This follows by Proposition 4.25 and Proposition 4.11, taking into account (4.33).  $\square$

**Theorem 4.27.** *Suppose the validity of Assumptions 4.4 and 4.24 related to  $\bar{\Gamma}$  introduced in (4.33). Suppose moreover that one of two hypotheses below are in force.*

- (1)  $\bar{\Gamma}$  defined in (4.12) is bounded.
- (2)  $\sigma_0, \frac{1}{\sigma_0}$  are bounded.

*Then (4.8) admits strong existence and pathwise uniqueness.*

**Proof.**

By Proposition 4.25, pathwise uniqueness holds. Indeed, by (4.33) equation (4.8) is a particular case of (4.24).

Concerning existence we have in view the application of Theorem 4.22. For this we need to verify that either Hypothesis (1) or (2) (in Theorem 4.22) hold. Hypothesis (1) in Theorem 4.27 coincides with Hypothesis (1) in Theorem 4.22. It remains to check

that the fact that  $\frac{1}{\sigma_0}$  is bounded implies the validity of Assumption 4.15 holds true. By (4.12) and the definition of  $\bar{\Gamma}$  in (4.33) we obtain

$$\sigma_0(\eta(s))\tilde{\Gamma}(s, (h^{-1} \circ \eta)) = \bar{\Gamma}(s, \eta). \tag{4.34}$$

By (3) in Assumption 4.24, being  $\frac{1}{\sigma_0}$  bounded there exist a constant  $K_1 > 0$  such that

$$|\tilde{\Gamma}(s, (h^{-1} \circ \eta))| \leq K_1 \left( |\eta(s)| + \int_0^s |\eta(r)| dr \right) + |\bar{\Gamma}(s, 0)|.$$

By (4) in Assumption 4.24, and the fact that the  $L^1$ -norm is bounded by the sup-norm, yields Assumption 4.15.

By Theorem 4.22 the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$  admits existence and by Proposition 4.11 (1) we have that (4.8) has a (weak) solution. At this point we can apply Yamada-Watanabe theorem to guarantee that the solution is actually strong. We remark that the Yamada-Watanabe theorem (in the path-dependent case) proof is the same as the one in the Markovian case, which is for instance stated in Proposition 3.20 [19, Chapter 5].  $\square$

As a consequence of Proposition 4.11 and Theorem 4.27 we obtain the following.

**Corollary 4.28.** *Under the same assumptions as in Theorem 4.27, the martingale problem related to (3.3) with respect to  $\mathcal{D}_L$ , admits strong existence and pathwise uniqueness.*

## 5. MARTINGALE PROBLEMS RELATED TO THE BESSEL PROCESS

### 5.1. Preliminary considerations.

In this section, we are going to introduce and investigate well-posedness for a martingale problem related to a Bessel process. We recall that the rigorous definition of the Bessel process is the following. A non-negative process  $X$  is said to be a Bessel process starting at  $x_0$  with dimension  $\delta \geq 0$  (notation  $BES^\delta(x_0)$ ) if  $S = X^2$  is a squared Bessel process starting at  $s_0 = x_0^2$  of dimension  $\delta$ . Such a process is denoted by  $BESQ^\delta(s_0)$ .

In particular  $S$  is the unique solution of

$$S = s_0 + 2 \int_0^t \sqrt{|S_s|} dW_s + \delta t, t \geq 0, \tag{5.1}$$

where  $(W_s)$  is a standard Brownian motion.

As is shown in Proposition 2.13 chapter 5 of [19] (see also [32, chapter 3]) (5.1) admits pathwise uniqueness. Since  $x \mapsto \sqrt{|x|}$  has linear growth it has weak existence and so by Yamada-Watanabe theorem it also admits strong existence.



**Remark 5.1.** For  $\delta > 2$  we know that the Bessel process  $X$  is transient, and it fulfills

$$X_t = x_0 + \frac{\delta - 1}{2} \int_0^t X_s^{-1} ds + W_t. \quad (5.2)$$

This is essentially possible since for that dimension the process is transient so it never touches zero (see [26, Chapter XI]).

On the other hand it is possible to show that for  $\delta \in [1, 2[$   $X$  is still a semimartingale. Unfortunately if  $0 < \delta < 1$  it is not yet the case, see Chapter 10 of [23], it is just a Dirichlet process.

Our point of view consists in rewriting (5.2) under the form

$$X_t = x_0 + \int_0^t b'(X_s) ds + W_t, \quad (5.3)$$

where  $W$  is a Brownian motion and  $b'$  is the derivative of the function  $b(x) = \frac{\delta-1}{2} \log|x|$ , at least when  $\delta \neq 1$  the derivative  $b'$  restricted on  $]0, +\infty[$  gives  $\frac{\delta-1}{2x}$ , which explains (5.2). Unfortunately, for small values of  $\delta$ ,  $X$  is recurrent, so it touches zero very often and so the derivative of  $b$  has to be considered on  $[0, \infty[$  or  $\mathbb{R}$ . In those sets it is only a Schwartz distribution. In the case  $\delta = 1$ , we will have  $b$  being an Heaviside function so that  $b'$  is the  $\delta$ -Dirac measure at zero.

As anticipated, the martingale problem will be considered in this section according to Definition 3.3, but first in the case  $\Gamma = 0$  so that the solution will be indeed the Bessel process  $X$ . After that, we construct the framework for the non-Markovian case. We are going to construct two settings: one for  $0 \leq \delta < 1$  and another one for  $\delta = 1$ .

## 5.2. The framework for $0 \leq \delta < 1$ .

As anticipated, we fix  $b : \mathbb{R} \rightarrow \mathbb{R}$ ,  $b(x) = \frac{\delta-1}{2} \log|x|$  and  $\sigma \equiv 1$ . We also set  $\Sigma(x) = 2b(x)$ , so

$$\exp(-\Sigma(x)) = |x|^{1-\delta}. \quad (5.4)$$

The idea is to continue the approach of the Section 3.2 which consists in expressing  $L$  in the form (1.6), i.e.

$$Lf(x) = \frac{e^{-\Sigma} \sigma^2}{2} (e^{\Sigma} f')'. \quad (5.5)$$

Let us now try to define a (natural) domain  $\mathcal{D}_{L^\delta}$  of  $L = L^\delta$ , taking into account the expression (5.5). For this, any element  $f \in \mathcal{D}_{L^\delta}$  should fulfill the following.

### Property 5.2.

- (1)  $f$  is absolutely continuous.

- (2) There exists  $g : \mathbb{R} \rightarrow \mathbb{R}$  absolutely continuous such that  $g = f'(x)|x|^{\delta-1}$  on  $\mathbb{R}^*$ ,
- (3) There is a version of  $x \mapsto g'(x)|x|^{1-\delta}$  which is continuous.

It is then natural to define

$$L^\delta f(x) = \frac{1}{2}g'(x)|x|^{1-\delta}. \tag{5.6}$$

For this purpose we set

$$\begin{aligned} \mathcal{D}_{L^\delta} &= \{f \in C^2(\mathbb{R}) \mid f'(0) = 0\}, \\ \mathcal{D}_{L^\delta}(\mathbb{R}_+) &= \{f \in C^2(\mathbb{R}_+) \mid f'(0) = 0\}. \end{aligned}$$

**Proposition 5.3.** *For all  $f \in \mathcal{D}_{L^\delta}$ , Property 5.2 holds and*

$$L^\delta f(x) = \begin{cases} \frac{f''(x)}{2} + \frac{(\delta-1)f'(x)}{2x} & : x \neq 0 \\ \frac{\delta f''(0)}{2} & : x = 0. \end{cases} \tag{5.7}$$

**Corollary 5.4.** *For all  $f \in \mathcal{D}_{L^\delta}$ , yields  $L^0 f(0) = 0$ .*

**Proof** (of Proposition 5.3).

Clearly statement (1) is satisfied. We set  $g(x) := f'(x)|x|^{\delta-1}, x \neq 0$  and  $g(0) := 0$ . By l'Hospital rule we can show that  $\lim_{x \rightarrow 0} g(x) = 0$  and so we extend  $g$  continuously to  $\mathbb{R}$ . Taking the derivative of  $g$  on  $\mathbb{R}^*$  we get

$$g'(x) = f''(x)|x|^{\delta-1} + (\delta-1)f'(x)\text{sign}(x)|x|^{\delta-2}. \tag{5.8}$$

On each real compact  $K$  that does not contain  $\{0\}$  the right side of (5.8) is integrable so for each  $\epsilon > 0$  we have

$$g(x) - g(\epsilon) = \int_\epsilon^x g'(y)dy.$$

Passing the limit with  $\epsilon \rightarrow 0$  we obtain  $g(x) = \int_0^x g'(y)dy$  proving that  $g$  is absolutely continuous.

Concerning statement (3), as  $g'(x)|x|^{1-\delta}$  is continuous on  $\mathbb{R}^*$  it is enough to show that  $\lim_{x \rightarrow 0} g'(x)|x|^{1-\delta}$  exists. By (5.8) we obtain

$$g'(x)|x|^{1-\delta} = f''(x) + (\delta-1)\text{sign}(x)\frac{1}{|x|}f'(x) = f''(x) + (\delta-1)\text{sign}(x)\frac{1}{|x|} \int_0^x f''(y)dy, x \neq 0.$$

Passing the limit with  $x \rightarrow 0$  it gives

$$\lim_{x \rightarrow 0} g'(x)|x|^{1-\delta} = \delta f''(0).$$

This concludes the proof of the three properties. Using (5.6) we conclude (5.7).

□

Starting from Section 5.3 we will make use of convergence properties for functions and processes according to the remark below.

**Remark 5.5.**

- (1) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous (therefore uniformly continuous on compacts) then  $f_n(x) = f\left(x + \frac{1}{n}\right)$  converges to  $f$  uniformly on compacts.
- (2) Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $X$  a continuous stochastic process on  $(\Omega, \mathcal{F}, \mathbb{P})$ . If  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of functions that converges uniformly on compacts of  $\mathbb{R}$  to a function  $f$  then  $f_n(X)$  converges to  $f(X)$  u.c.p.

**5.3. The martingale problem in the Markovian case: the  $\mathbb{R}$ -case.**

We start considering the classical Bessel process without path-dependent perturbation, in the sense of Definition 3.3 and 3.4.

**Proposition 5.6.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and a Brownian motion  $W$ . Let  $x_0 \geq 0, 0 \leq \delta < 1$ . Let  $S$  be the solution of (5.1) (necessarily non-negative by comparison theorem) with  $s_0 = x_0^2$ , so that  $X = \sqrt{S}$  is a  $BES^\delta(x_0)$  process.*

*Then  $X$  solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ . Moreover, for every  $f \in \mathcal{D}_{L^\delta}$*

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s. \quad (5.9)$$

**Remark 5.7.** *Suppose that  $S$  is a non-negative solution of an SDE of the type (5.1), where the Brownian motion  $W$  is replaced by a continuous semimartingale. Then (5.9) still holds for every  $f \in \mathcal{D}_{L^\delta}$ .*

**Proof** (of Proposition 5.6).

We consider immediately the case of Remark 5.7 and suppose  $W$  to be a semimartingale. For  $\delta = 0$  and  $x_0 = 0$ , as  $BESQ^0(0)$  is the null process. By Lemma 5.4  $L^0 f(0) = 0$  for all  $f \in \mathcal{D}_{L^0}$ , obviously  $f(0) - f(0) - \int_0^t L^\delta f(0) ds \equiv 0$  and (5.9) holds.

We consider now the case  $x_0 > 0$ . Let  $X = \sqrt{S}$ , where  $S$  is a  $BESQ^\delta(s_0)$ , let  $f \in \mathcal{D}_{L^\delta}(\mathbb{R})$  and define  $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}$  as  $f_n(y) = f\left(\sqrt{y + \frac{1}{n}}\right)$ . Clearly  $f_n \in C^2(\mathbb{R}_+)$ .

Applying Itô's formula we have

$$f_n(S_t) = f_n(S_0) + \int_0^t \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \delta \frac{f'\left(\sqrt{S_s + \frac{1}{n}}\right)}{2\sqrt{S_s + \frac{1}{n}}} ds$$

$$+ \int_0^t \left[ \frac{1}{2} f'' \left( \sqrt{S_s + \frac{1}{n}} \right) - \frac{1}{2} \frac{f' \left( \sqrt{S_s + \frac{1}{n}} \right)}{\sqrt{S_s + \frac{1}{n}}} \right] \left[ \frac{S_s}{S_s + \frac{1}{n}} \right] ds, \quad (5.10)$$

which can be rewritten as

$$\begin{aligned} f_n(S_t) = f_n(S_0) &+ \int_0^t \frac{f' \left( \sqrt{S_s + \frac{1}{n}} \right)}{\sqrt{S_s + \frac{1}{n}}} \sqrt{S_s} dW_s + \int_0^t \frac{1}{2} f'' \left( \sqrt{S_s + \frac{1}{n}} \right) \left[ \frac{S_s}{S_s + \frac{1}{n}} \right] ds \\ &+ \frac{1}{2} \int_0^t \frac{f' \left( \sqrt{S_s + \frac{1}{n}} \right)}{\sqrt{S_s + \frac{1}{n}}} \left[ \delta - \frac{S_s}{S_s + \frac{1}{n}} \right] ds. \end{aligned} \quad (5.11)$$

The first integral converges to

$$\int_0^t f' \left( \sqrt{S_s} \right) dW_s, \quad (5.12)$$

u.c.p. by Remark 5.5, since  $f'$  is continuous.

As  $y \mapsto f'' \left( \sqrt{y + \frac{1}{y}} \right)$  is continuous, by Remark 5.5 and Lebesgue's dominated convergence theorem the second integral in (5.11) converges u.c.p. to

$$\frac{1}{2} \int_0^t f'' \left( \sqrt{S_s} \right) ds. \quad (5.13)$$

We set  $\ell : \mathbb{R} \rightarrow \mathbb{R}$ , the continuous function defined by

$$\ell(x) = \begin{cases} \frac{f'(x)}{x} & : x \neq 0. \\ f''(0) & : x = 0. \end{cases}$$

The third integral can be rewritten as

$$\frac{1}{2} \int_0^t \ell \left( \sqrt{S_s + \frac{1}{n}} \right) \left[ \delta - \frac{S_s}{S_s + \frac{1}{n}} \right] ds.$$

By Remark 5.5 and Lebesgue's dominated convergence the previous expression converges u.c.p. to

$$\int_0^t \ell \left( \sqrt{S_s} \right) \left( \frac{\delta - 1}{2} \right) ds. \quad (5.14)$$

Finally (5.12), (5.13) and (5.14) allow to conclude the proof of (5.9).  $\square$

**Corollary 5.8.** *Let  $x_0 \in \mathbb{R}, 0 \leq \delta < 1$ . The martingale problem with respect to  $\mathcal{D}_{L^\delta}$ , with initial condition  $X_0 = x_0$  admits strong existence. More precisely we have the following.*

- (1) *If  $x_0 \geq 0$ , the  $BES^\delta(x_0)$  process, being the square root of a solution of (5.1) solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ .*
- (2) *If  $x_0 < 0$ , minus the  $BES^\delta(-x_0)$  of item (1) solves the same strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $-W$ .*

**Proof.**

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and a Brownian motion  $W$ . We set  $s_0 = x_0^2$ . We know that (5.1) admits a strong solution  $S$ . Then, by Proposition 5.6  $X = \sqrt{S}$  is a solution for the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$  with initial condition  $|x_0|$ .

So, if  $x_0 \geq 0$  then strong existence is established. If  $x_0 < 0$  then we show below that  $-X$  also solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ .

Let  $f \in \mathcal{D}_{L^\delta}$ . Then obviously  $f_-(x) := f(-x) \in \mathcal{D}_{L^\delta}$  and

$$L^\delta f_-(x) = L^\delta f(-x).$$

Therefore, since  $X$  solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ , for all  $f \in \mathcal{D}_{L^\delta}$  we have

$$f_-(X_t) - f_-(x_0) - \int_0^t Lf_-(X_s)ds = \int_0^t f'_-(X_s)dW_s,$$

which implies

$$f(-X_t) - f(-x_0) - \int_0^t L^\delta f(-X_s)ds = \int_0^t f'(-X_s)d(-W)_s.$$

Thus  $-X$  also solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $-W$ . □

**Proposition 5.9.** *Let us suppose  $0 < \delta < 1$ . The martingale problem with respect to  $\mathcal{D}_{L^\delta}$  does not admit (in general) uniqueness in law.*

**Proof.**

Let  $S$  be the  $BESQ^\delta(0)$ . By Corollary 5.8, we know that  $X^+ = \sqrt{S}$  (resp.  $X^- = -\sqrt{S}$ ) solves the martingale problem with respect to an underlying probability  $\mathbb{P}$ .

Obviously  $X$  does not have the same law since  $X$  is positive and  $-X$  is negative. □

**Remark 5.10.** *If the initial condition  $x_0$  is different from zero, for instance positive, then uniqueness also fails since we can exhibit two solutions. The first one is still the classical Bessel process, the second one behaving as the first one until it reaches zero and then it behaves like minus a Bessel. Such a stopping time always exists since the Bessel process is recurrent, see Chapter XI, considerations after Corollary (1.4).*

In order to show uniqueness among positive solutions we will need the following.

**Proposition 5.11.** *Let  $0 \leq \delta < 1$ . Let  $(X, \mathbb{P})$  be a solution (not necessarily positive) of the martingale problem with respect to  $\mathcal{D}_{L^\delta}$ . Then  $S = X^2$  is a squared Bessel process.*

**Proof.**

We first show that

$$M_t^1 := X_t^2 - x_0^2 - \delta t \quad (5.15)$$

is a local martingale and

$$X_t^4 = x_0^4 + 2(2 + \delta) \int_0^t X_s^2 ds + M_t^2, \quad (5.16)$$

where  $M^2$  is a local martingale. Clearly,  $f_1(x) := x^2 \in \mathcal{D}_{L^\delta}$  since  $f_1'(0) = 0$  and by Proposition 5.3  $L^\delta f_1(x) \equiv \delta$  so that (5.15) follows. On the other hand, obviously  $f_2(x) := x^4 \in \mathcal{D}_{L^\delta}$  and then, by Proposition 5.3,  $L^\delta f_2(x) = 2(2 + \delta)x^2$ , so (5.16) follows. Now, setting  $S := X^2$ , by integration by parts and using (5.15) we have

$$[M^1]_t = [S]_t = S_t^2 - s_0^2 - 2 \int_0^t S_s dS_s = X_t^4 - x_0^4 - 2\delta \int_0^t X_s^2 ds + M_t, \quad (5.17)$$

where  $M$  is a local martingale. This implies

$$X_t^4 = x_0^4 + 2\delta \int_0^t X_s^2 ds + [M^1]_t - M_t. \quad (5.18)$$

We remark that (5.18) and (5.16) provide two decompositions of the semimartingale  $X^4$ . By uniqueness of the semimartingale decomposition we can identify the bounded variation component so that

$$[M^1]_t = 4 \int_0^t X_s^2 ds, t \in [0, T]. \quad (5.19)$$

Consequently the process

$$W_t := \int_0^t \frac{dM_s^1}{2|X_s|},$$

is a Brownian motion taking into account the fact that  $[W]_t \equiv t$  together with Lévy's characterization of Brownian motion. Hence, the process  $S$  is a (weak) solution of the SDE

$$dS_s = \delta ds + 2\sqrt{|S_s|}dW_s, \quad (5.20)$$

which shows that  $S$  is a  $BESQ^\delta(s_0)$ ,  $s_0 = x_0^2$ .  $\square$

With very similar arguments to those in the proof of previous proposition, we can prove the following.

**Proposition 5.12.** *Let  $0 \leq \delta < 1$ . Let  $X$  be a solution (not necessarily positive) to the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and a Brownian motion  $W$ . Then  $S = X^2$  is a solution to (5.1).*

***Proof.***

Let us suppose that  $X$  is a solution of the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and a Brownian motion  $W$  with related canonical filtration  $\mathfrak{F}$ . The same arguments as in the first part of the proof of Proposition 5.11 until (5.19), allow us to establish that  $M^1$  defined in (5.15) is an  $\mathfrak{F}$ -local martingale with quadratic variation given by (5.19). By the Brownian martingale representation theorem, we can prove that

$$M_t^1 = 2 \int_0^t |X_s| dW_s, t \in [0, T].$$

This shows that  $S = X^2$  is a solution of

$$S_t = s_0 + 2 \int_0^t \sqrt{|S_s|} dW_s + \delta t, t \in [0, T], s_0 = x_0^2.$$

$\square$

Concerning uniqueness on the whole real line  $\mathbb{R}$ , we conclude the section with a remark.

**Remark 5.13.** *Suppose  $\delta = 0$ . Then pathwise uniqueness holds for the strong martingale problem.*

- (1) *If  $x_0 = 0$ , by Proposition 5.11 if  $X$  is a solution of the strong martingale problem then  $X^2$  is a  $BESQ^0(0)$  which is the null process; this fact shows uniqueness.*
- (2) *Suppose  $x_0$  different from zero (for instance strictly positive). If  $X$  is a solution to the strong martingale problem then  $S := X^2$  is a  $BESQ^0(x_0^2)$  by Proposition 5.11. More precisely, by Proposition 5.12,  $S$  is a solution to (5.1). By the fact*

that (5.1) admits pathwise uniqueness, the strong Markov property shows that, whenever  $S$  reaches zero it is forced to remain there.

**5.4. The martingale problem in the Markovian case:  $\mathbb{R}_+$ -case.**

We remain still with the case  $0 \leq \delta < 1$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and a Brownian motion  $W$  with canonical filtration  $\mathfrak{F}$ . We will be interested in non-negative solutions  $X$  for the martingale problem in the strong sense, with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$ , which means that

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s, \tag{5.21}$$

for all  $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$ . Existence is a consequence of the  $\mathbb{R}$ -case, see Proposition 5.6 as it is stated below.

**Proposition 5.14.** *Let  $0 \leq \delta < 1$ . The process  $BES^\delta(x_0)$  as stated in Proposition 5.6 solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$ . In particular, the martingale problem related to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  admits strong existence.*

**Proposition 5.15.** *The martingale problem with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$  admits pathwise uniqueness.*

***Proof.***

Let us suppose that  $X$  is a strong solution of the martingale problem with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$ . This implies the same with respect to  $\mathcal{D}_{L^\delta}$ . By Proposition 5.12  $S = X^2$  is a solution of (5.1). The result follows by the pathwise uniqueness of the SDE (5.1) and the positivity of  $X$ . □

**Remark 5.16.** *The techniques of Section 4 are unsuitable in this section since a similar kind of transformation leads to an SDE on the real line which is not well-posed. Indeed, consider  $\delta \in [0, 1[$  and suppose that  $(X, \mathbb{P})$  is a solution to the martingale problem with respect to  $\mathcal{D}_{L^\delta}$ . So, the natural choice for the harmonic function  $h$  would be*

$$h(x) = \text{sign}(x) \frac{|x|^{2-\delta}}{2-\delta}, x \in \mathbb{R}.$$

*This would lead to*

$$\sigma_0(y) = \text{sign}(y)(2-\delta)^{\frac{1-\delta}{2-\delta}} |y|^{\frac{1-\delta}{2-\delta}}.$$



Formally speaking the transformed equation would be

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s) dW_s. \quad (5.22)$$

Engelbert-Schmidt criterion, see Theorem 5.7 in [19, Chapter 5], for uniqueness in law leads to check that the quantity below

$$\int_0^\epsilon \frac{1}{\sigma_0^2(y)} dy = \frac{1}{(2-\delta)^{\frac{2-2\delta}{2-\delta}}} \int_0^\epsilon y^{\frac{2\delta-2}{2-\delta}} dy \quad (5.23)$$

is infinite. Now, previous quantity is always finite for any  $\delta > 0$ . This confirms that (5.22) has no uniqueness in law on  $\mathbb{R}$  when  $\delta > 0$ . So, the non-uniqueness observed in Proposition 5.9 is not astonishing.

On the other hand, when  $\delta = 0$  then (5.23) is infinite which implies uniqueness in law. This agrees with Remark 5.13 which even states pathwise uniqueness.

### 5.5. The framework for $\delta = 1$ .

By definition, a Bessel process of dimension  $\delta = 1$  starting at  $x_0 \geq 0$  is a non-negative process  $X$  such that  $S^2$  is a  $BESQ^1(x_0)$ . On the other hand, in the literature such a Bessel process  $X$  is also characterized as a strong solution of

$$X_t = x_0 + W_t + L_t, t \in [0, T], \quad (5.24)$$

where  $L$  is a local time at zero which can also be characterized as an increasing process only acting when  $X = 0$ , i.e.

$$\int_{[0,T]} Y_s dL_s = \int_{[0,T]} Y_s 1_{\{X_t=0\}} dL_s.$$

In particular  $X$  is a semimartingale. Indeed, let  $X$  be a non-negative solution of (5.24), then by an easy application of Itô's formula for semimartingales, setting  $S := X^2$ , we have

$$\begin{aligned} S_t &= x_0^2 + 2 \int_0^t X_s dW_s + \int_0^t X_s dL_s + \frac{1}{2} 2t \\ &= x_0^2 + 2 \int_0^t \sqrt{S_s} dW_s + \int_0^t X_s 1_{\{X_s=0\}} dL_s + t \\ &= x_0^2 + 2 \int_0^t \sqrt{S_s} dW_s + t, \end{aligned}$$

which implies that  $S$  is a squared Bessel process of dimension  $\delta = 1$ , and so  $X$  is a Bessel process of dimension 1. This shows in particular that (5.24) admits pathwise uniqueness. Existence and uniqueness of (5.24) can be seen via the Skorohod problem, see [16].

In this section, we represent alternatively  $X$  as a solution of a (strong) martingale problem. As we mentioned at the beginning of Section 5, we have fixed

$$b(x) = H(x) = \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0. \end{cases}$$

Formally speaking we get

$$\Sigma(x) = 2 \int_0^x \delta'_0 dy = 2H(x),$$

where  $H$  is the Heaviside function. Coming back to the expression (1.6), it is natural to set

$$L^1 f = (\exp(2H)f')' \frac{\exp(-2H)}{2}. \quad (5.25)$$

It looks reasonable to require similar conditions as for  $\delta \in [0, 1[$ , for characterizing  $f \in \mathcal{D}_{L^1}$ . They would be the following.

- (1)  $f$  is absolutely continuous.
- (2) There exists  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g = f'$  on  $\mathbb{R}^*$  absolutely continuous,
- (3) There exists a continuous version of  $x \mapsto g'(x)$ .

As in the  $0 \leq \delta < 1$  case, we set  $\mathcal{D}_{L^1} = \{f \in C^2(\mathbb{R}) | f'(0) = 0\}$ , which clearly satisfy these properties. Compatibly with (5.25), we set

$$L^1 f(x) = \frac{1}{2} f''(x). \quad (5.26)$$

The PDE operator  $L^1$  appearing at (5.26) coincides with the generator of Brownian motion. However the domain of that generator is larger since it is  $C^2(\mathbb{R})$ .

**Remark 5.17.** *The same preliminary analysis of Section 5.3 about the martingale problem related to  $0 \leq \delta < 1$  in the  $\mathbb{R}$ -case extends to the case  $\delta = 1$ . More precisely, Proposition 5.6, Corollary 5.8, Proposition 5.9 and Remark 5.10 hold. This is stated below.*

**Proposition 5.18.**

- (1)  $BES^1(x_0)$  with respect to any underlying Brownian motion  $W$  and  $\mathcal{D}_{L^1}$  is a solution of the strong martingale problem.
- (2) The martingale problem related to  $L^1$  with respect to  $\mathcal{D}_{L^1}$  admits (in general) no uniqueness.

Similarly to Corollary 5.8, the processes  $BES^1(x_0)$  and minus  $BES^1(x_0)$  are solutions to the strong martingale problem with respect to  $\mathcal{D}_{L^1}$  and an underlying Brownian

motion  $W$ . Other solutions on the real line are the so called *skew Brownian motions* which will be investigated more in detail in a future work. For this last one, we can mention the works of Harrison and Shepp ([16]) and Le Gall ([22]).

Concerning the  $\mathbb{R}_+$ -case, let again  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space equipped with the canonical filtration  $\mathfrak{F}^W$  of a Brownian motion  $W$ .

By using the same arguments as for Propositions 5.14 and 5.15 we get the following result.

**Proposition 5.19.** *The process  $BES^1(x_0)$  as stated in Proposition 5.6 solves the strong martingale problem with respect to  $\mathcal{D}_{L^1}(\mathbb{R}_+)$  and  $W$ . Moreover, for every  $f \in \mathcal{D}_{L^1}(\mathbb{R}_+)$  we have*

$$f(X_t) - f(X_0) - \int_0^t L^1 f(X_s) ds = \int_0^t f'(X_s) dW_s.$$

## 6. NON-MARKOVIAN MARTINGALE PROBLEM RELATED TO THE BESSEL PROCESS

### 6.1. Generalities.

Now we are going to treat a non-Markovian martingale problem which is a perturbation of the Bessel process  $BES^\delta(x_0)$ ,  $0 \leq \delta \leq 1, x_0 \geq 0$ . More precisely, we want to analyze existence and uniqueness of solutions to the martingale problem related to the SDE

$$X_t = x_0 + W_t + \int_0^t b'(X_s) ds + \int_0^t \Gamma(s, X^s) ds, \quad (6.1)$$

where  $\Gamma$  is the same path-dependent functional as in (1.2), and  $b(x) = \frac{\delta-1}{2} \log(|x|)$  if  $0 \leq \delta < 1$  and  $b(x)$  is the Heaviside function if  $\delta = 1$ . Alternatively to the case of Section 4.2, here  $\sigma = 1$  but  $b$  is not a continuous function.

We remark (at least when  $\delta > 0$ ) that the martingale problem on the whole  $\mathbb{R}$  is not well-posed (in general) even for  $\Gamma = 0$ , see Proposition 5.9.

We start with a simple observation concerning the case  $\delta = 0, x_0 = 0$ . The result below comes directly from the definition.

**Proposition 6.1.** *Suppose  $\delta = 0, x_0 = 0$ . Let  $W$  be a standard Brownian motion. The null process is a solution to the strong martingale problem (in the sense of Definition 3.4) with respect to  $\mathcal{D}_{L^s}$  and  $W$ .*

In presence of a path-dependent drift  $\Gamma$ , under suitable conditions, Corollary 6.14 allows to show that the null process is still the unique solution of the corresponding strong martingale problem.

## 6.2. The martingale problem in the path-dependent case: existence in law.

We recall that a pair  $(X, \mathbb{P})$  is a solution for the martingale problem related to  $\mathcal{L}$  in the sense of Definition 3.3 with  $L = L^\delta$  with respect to  $\mathcal{D}_{L^\delta}$  (resp.  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ ),  $0 \leq \delta \leq 1$ , if for all  $f \in \mathcal{D}_{L^\delta}$  (resp.  $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$ ),

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds, \quad (6.2)$$

is a  $\mathbb{P}$ -local martingale.

A first criterion of existence can be stated if  $\Gamma$  is measurable and bounded.

**Proposition 6.2.** *Suppose that  $\Gamma$  is bounded. The martingale problem related to  $\mathcal{L}$  (defined in (3.3)) admits existence with respect to  $\mathcal{D}_{L^\delta}$ . Moreover we have the following.*

- (1) *If the initial condition is  $x_0 \geq 0$ , then the solution can be constructed to be non-negative.*
- (2) *If the initial condition is  $x_0 \leq 0$ , then the solution can be constructed to be non-positive.*

**Proof.** Let  $x_0 \geq 0$ . Given a Brownian motion  $W$ , by Propositions 5.14 and 5.18, there exists a solution  $X$  to the (even strong) martingale problem related to (3.3) (with  $\Gamma = 0$ ) with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$ . That solution is in fact a  $BES^\delta(x_0)$ . In particular, for all  $f \in \mathcal{D}_{L^\delta}(\mathbb{R}_+)$ ,

$$f(X_s) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) dW_s. \quad (6.3)$$

Since the Bessel process is non-negative, (6.3) also holds for  $f \in \mathcal{D}_{L^\delta}$ . As  $\Gamma$  is bounded then, by Novikov's condition

$$N_t = \exp \left( \int_0^t \Gamma(s, X^s) dW_s - \frac{1}{2} \int_0^t \Gamma^2(s, X^s) ds \right),$$

is a martingale. By Girsanov's Theorem

$$B_t := W_t - \int_0^t \Gamma(s, X^s) ds,$$

is a Brownian motion under the probability measure  $\mathbb{Q}$  such that  $d\mathbb{Q} = N_T d\mathbb{P}$ . Then, we can rewrite (6.3) as follows:

$$f(X_s) - f(X_0) - \int_0^t L^\delta f(X_s) ds - \int_0^t f'(X_s) dB_s - \int_0^t f'(X_s) \Gamma(s, X^s) ds = 0.$$

Since  $\int_0^t f'(X_s) dB_s$  is a  $\mathbb{Q}$ -local martingale,  $(X, \mathbb{Q})$  happens to be a solution to the martingale problem in the sense of Definition 3.3 with respect to  $\mathcal{D}_{L^\delta}$ .

Suppose now that  $x_0 \leq 0$ . The process  $X$  defined as minus  $BES^\delta(-x_0)$  is a solution of (6.3). Then the same procedure as for the case  $x_0 \geq 0$  works. This shows existence for the martingale problem on  $\mathcal{D}_{L^\delta}$ .

Let us discuss the sign of the solution. Suppose that  $x_0 \geq 0$  (resp.  $x_0 \leq 0$ ). Then, our construction starts with  $BES^\delta(x_0)$  (resp. minus  $BES^\delta(-x_0)$ ) which is clearly non-negative (resp. non-positive). The constructed solution is again non-negative (resp. non-positive) since it is supported by an equivalent probability measure.  $\square$

**Remark 6.3.** *As we have mentioned in Proposition 5.9 and its extension to  $\delta = 1$ , the martingale problem in the sense of Definition 3.3 admits no uniqueness in general, at least with respect to  $\mathcal{D}_{L^\delta}$ , i.e. on the whole line.*

Besides Proposition 6.2, Proposition 6.5 below and Proposition 6.6 provide a new class of solutions to the martingale problem related to  $\mathcal{L}$  with respect to  $\mathcal{D}_{L^\delta}$ . The map defined below is associated with  $\Gamma$ :

$$\bar{\Gamma}(s, \eta) := 2\sqrt{|\eta(s)|}\Gamma(s, \sqrt{|\eta^s|}) + \delta, \quad s \in [0, T] \quad \eta \in C([0, T]). \quad (6.4)$$

Next, we introduce here a growth assumption on  $\Gamma$ .

**Assumption 6.4.**  *$\Gamma$  is continuous and there exists a constant  $K$  such that, for every  $(s, \eta) \in \Lambda$  we have*

$$|\Gamma(s, \eta)| \leq K \left( 1 + \sup_{r \in [0, T]} \sqrt{|\eta(r)|} \right).$$

**Proposition 6.5.** *Suppose that  $\Gamma$  fulfills Assumption 6.4. Then, we have the following.*

(1) *Let  $s_0 \geq 0$ . The path-dependent SDE*

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|}dW_s + \int_0^t 2\sqrt{|S_s|}\Gamma\left(s, \sqrt{|S^s|}\right) ds, \quad \delta \geq 0, \quad (6.5)$$

*admits existence in law, see Definition 7.4.*

(2) *The constructed solution of (6.5) in item (1) is non-negative.*

(3) *Let  $x_0 \geq 0$ . The martingale problem related to  $\mathcal{L}f = L^\delta f + \Gamma f'$  (see Definition 3.3) admits existence with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ .*

*Proof.* See Appendix 7.3.  $\square$

### 6.3. Equivalence between martingale problem and SDE in the path-dependent case.

We state here an important result establishing the equivalence between the martingale problem and a path-dependent SDE of squared Bessel type. Let  $0 \leq \delta \leq 1$ .

**Proposition 6.6.** *Let  $(\Omega, \mathcal{F}, P)$  be a probability space. Let  $X$  be a stochastic process and we denote  $S = X^2$ .*

- (1)  *$(|X|, \mathbb{P})$  is a solution to the martingale problem related to (3.3) with respect  $\mathcal{D}_{L^\delta}$ , if and only if, the process  $S$  is a solution of (6.5) for some  $\mathfrak{F}^X$ -Brownian motion  $W$ .*
- (2) *Let  $W$  be a standard Brownian motion (with respect to  $\mathbb{P}$ ). Then  $|X|$  is a solution to the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ , if and only if,  $S$  is a solution of (6.5).*

**Remark 6.7.** *In the statement of Proposition 6.6,  $\mathcal{D}_{L^\delta}$  can be replaced with  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ , provided that  $|X|$  is replaced by  $X$ .*

**Proof** (of Proposition 6.6). We discuss item (1).

Concerning the direct implication, by choosing  $f_1(x) = x^2, f_2(x) = x^4$  we have  $L^\delta f_1(x) = \delta, L^\delta f_2(x) = 2(2 + \delta)x^2$ . By definition of the martingale problem, the two processes ( $t \in [0, T]$ )

$$M_t := X_t^2 - x_0 - \delta t - \int_0^t 2|X_s|\Gamma(s, |X^s|)ds \quad (6.6)$$

and

$$N_t := X_t^4 - X_0^4 - 2(2 + \delta) \int_0^t X_s^2 ds - 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds, \quad (6.7)$$

are  $\mathfrak{F}^X$ -local martingales.

Since  $S = X^2$ , by (6.6) we have  $[S] = [M]$ . By integration by parts and (6.6), we have

$$[M]_t = [X^2]_t = X_t^4 - X_0^4 - 2 \int_0^t X_s^2 dX_s^2 = X_t^4 - X_0^4 - 2\delta \int_0^t X_s^2 ds - 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds + M^1,$$

where  $M^1$  is a local martingale. Therefore

$$X_t^4 - X_0^4 = M^1 + 2\delta \int_0^t X_s^2 ds + 4 \int_0^t |X_s|^3 \Gamma(s, |X^s|) ds + [M]_t, \quad t \in [0, T]. \quad (6.8)$$

(6.8) and (6.7) give us two decomposition of the semimartingale  $X^4$ ; by the uniqueness of the semimartingale decomposition,  $[M]_t = 4 \int_0^t X_s^2 ds$ . We set

$$W_t = \int_0^t \frac{dM_s}{2|X_s|}, t \in [0, T]. \quad (6.9)$$

By Lévy characterization theorem,  $W$  is an  $\mathfrak{F}^X$ -Brownian motion and by (6.6), we conclude that  $S_t = s_0 + \delta t + \int_0^t 2\sqrt{S_s} dW_s + \int_0^t 2\sqrt{S_s} \Gamma(s, \sqrt{S_s}) ds, t \in [0, T]$ .

Concerning the converse implication, suppose that  $S = X^2$  solves (6.5) for some Brownian motion  $W$ . Then  $X$  solves

$$S_t = s_0 + \delta t + \int_0^t 2\sqrt{|S_s|} d\widetilde{W}_s, t \in [0, T], \quad (6.10)$$

where

$$\widetilde{W}_t := W_t + \int_0^t \Gamma(s, \sqrt{S_s}) ds, t \in [0, T].$$

Let  $f \in \mathcal{D}_{L^\delta}$ ; by Proposition 5.6 and Remark 5.7 we have

$$f(X_t) - f(X_0) - \int_0^t L^\delta f(X_s) ds = \int_0^t f'(X_s) d\widetilde{W}_s. \quad (6.11)$$

Consequently

$$\begin{aligned} M_t^f &:= f(|X_t|) - f(|x_0|) - \int_0^t L^\delta f(|X_s|) ds - \int_0^t f'(|X_s|) \Gamma(s, |X_s|) ds \\ &= \int_0^t f'(|X_s|) dW_s, \end{aligned}$$

is an  $\mathfrak{F}^X$ -local martingale. Then,  $(|X|, \mathbb{P})$  solve the martingale problem related to (3.3) with respect to  $\mathcal{D}_{L^\delta}$  in the sense of Definition 3.3. On the other hand,  $|X|$  also solves the strong martingale problem with respect to  $\mathcal{D}_{L^\delta}$  and  $W$ . This concludes the proof of item (1).

As far as item (2) is concerned, the converse implication argument has been given above. Concerning the direct implication, we define  $f_1$  as in the proof of item (1). By (3.5), (6.6) and the fact that

$$M = 2 \int_0^\cdot f_1(|X_s|) dW_s = 2 \int_0^\cdot \sqrt{S_s} dW_s,$$

we obtain (6.5). This concludes the proof.  $\square$

#### 6.4. The martingale problem in the path-dependent case: uniqueness in law.

A consequence of Girsanov theorem gives us the following.

**Proposition 6.8.** *Let  $0 \leq \delta \leq 1$ . Suppose that  $\Gamma$  is bounded. The martingale problem related to (3.3) with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  admits uniqueness.*

**Remark 6.9.** *Let  $x_0 \geq 0$  (resp.  $x_0 \leq 0$ ). By Proposition 6.2, every solution is non-negative (resp. non-positive).*

**Proof** (of Proposition 6.8).

The proof is similar to the one of Proposition 4.23 taking  $\sigma = 1, \tilde{\Gamma} = \Gamma$ .

Let  $(X^i, \mathbb{P}^i), i = 1, 2$  be a solution to the martingale problem with respect to  $\mathcal{L}f = Lf + \Gamma$  related to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ . By Proposition 6.6,  $S^i = (X^i)^2$  is a solution of (6.5), for some Brownian motion  $W^i$  and  $\mathbb{P}^i$ . Again by Proposition 6.6,  $X^i$  fulfills the strong martingale problem related to  $\mathcal{L}$  with respect to  $\mathcal{D}_{L^\delta}$  and  $W^i$ .

We define the random variable (which is also a Borel functional of  $X^i$ )

$$V_t^i = \exp\left(-\int_0^t \Gamma(s, X^i) dW_s^i - \frac{1}{2} \int_0^t (\Gamma(s, X^i))^2 ds\right).$$

By the Novikov condition, it is a  $\mathbb{P}^i$ -martingale. This allows us to define the probability  $d\mathbb{Q}^i = V_T^i d\mathbb{P}^i$ .

By Girsanov theorem, under  $\mathbb{Q}^i$ ,  $B_t^i := W_t^i + \int_0^t \Gamma(s, X^{i,s}) ds$  is a Brownian motion. Therefore,  $(X^i, \mathbb{Q}^i)$ , is a solution to the martingale problem (3.4) with respect to  $\mathcal{D}_L^\delta$  with  $\Gamma = 0$ . Propositions 5.15 and 5.19 say that the martingale problem with respect to  $\mathcal{D}_{L^\delta}$  with  $\Gamma = 0$  admits strong existence and pathwise uniqueness. We recall the statement *pathwise uniqueness implies uniqueness in law* of Yamada-Watanabe theorem extends to the property *pathwise uniqueness of the martingale problem implies uniqueness of the martingale problem*. So, with  $\Gamma = 0$ , pathwise uniqueness of the strong martingale problem implies uniqueness of the martingale problem related to (3.3) with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$ . Consequently  $X^i$  (under  $\mathbb{Q}^i$ ),  $i = 1, 2$  have the same law. Hence, for every Borel set  $B \in \mathfrak{B}(C[0, T])$  we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_{\Omega} \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} d\mathbb{Q}^1 = \int_{\Omega} \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} d\mathbb{Q}^2 = \mathbb{P}^2\{X^2 \in B\}.$$

So,  $X^1$  under  $\mathbb{P}^1$  has the same law as  $X^2$  under  $\mathbb{P}^2$ . Finally the martingale problem related to (3.3) with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  admits uniqueness.  $\square$



### 6.5. Path-dependent Bessel process: results on pathwise uniqueness.

In this section,  $\bar{\Gamma}$  is the same as the one defined in (6.4), i.e.

$$\bar{\Gamma}(s, \eta) := 2\sqrt{|\eta(s)|}\Gamma(s, \sqrt{|\eta^s|}) + \delta, s \in [0, T] \quad \eta \in C([0, T]).$$

At this point, we can state a pathwise uniqueness theorem. For this purpose, we state the following assumption.

#### Assumption 6.10.

- (1) *There exists a constant  $K > 0$  such that, for every  $t \in [0, T]$ ,  $\eta^1, \eta^2 \in C([0, T])$ , we have  $|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K (|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)| dr)$ .*
- (2)  $\sup_{t \in [0, T]} |\bar{\Gamma}(t, 0)| < \infty$ .

#### Remark 6.11.

- (1)  $\sigma_0(y) = 2\sqrt{|y|}$  has linear growth.
- (2) Defining  $l(x) = 2\sqrt{x}$ ,  $x \geq 0$ , we have  $\int_0^\epsilon l^{-2}(u) du = \infty$  for every  $\epsilon > 0$  and  $|l(x) - l(y)| \leq l(|x - y|)$ ,  $x, y \in \mathbb{R}_+$ .

**Remark 6.12.** *Note that Assumption 6.10 implies Assumption 4.24.*

We start the analysis by considering equation (6.5). For the definitions of strong existence and pathwise uniqueness, see Definition 7.3.

**Theorem 6.13.** *Suppose Assumptions 6.10 and 6.4.*

- (1) (6.5) admits pathwise uniqueness.
- (2) (6.5) admits strong existence.
- (3) Suppose  $x_0 \geq 0$ . Every solution of (6.5) with  $s_0 = x_0^2$  is non-negative.

**Proof** (of Theorem 6.13).

- (1) We remark that (6.5) is of the form (4.24). The result follows from Proposition 4.25 and Remark 6.11.
- (2) By Proposition 6.5 we have existence in law. By an extension of Yamada-Watanabe theorem to the path-dependent case, strong existence holds for (6.5).
- (3) Suppose  $x_0 \geq 0$ . By Proposition 6.5 (2), (6.5) admits even existence in law of a non-negative solution. By an extension of Yamada-Watanabe theorem pathwise uniqueness implies uniqueness in law, so that the above-mentioned solution has to be non-negative.

□

We are now able to state the following.

**Corollary 6.14.** *Suppose that  $\bar{\Gamma}$  (defined in (6.4)) fulfills Assumptions 6.10 and 6.4. Then the strong martingale problem related to (3.3) (see Definition 3.4) with respect to  $\mathcal{D}_{L^\delta}(\mathbb{R}_+)$  and  $W$  admits strong existence and pathwise uniqueness.*

**Proof.** By Theorem 6.13, the equation (6.5) admits a unique strong solution which is non-negative. Proposition 6.6 and Remark 6.7 allow us to conclude the proof. □

## 7. APPENDIX

### 7.1. Different notions of solutions when $b'$ is a function.

Let us suppose below that  $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$  are Borel functions and  $\Gamma$  as given in (1.2). As already mentioned, for simplicity we will only consider initial conditions  $\xi$  to be deterministic.

**Definition 7.1.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $(W_t)_{t \geq 0}$  a Brownian motion and  $\xi \in \mathbb{R}$ . A solution  $X$  of (3.1) (depending on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the Brownian motion  $W$  and an initial condition  $\xi$ ) is a progressively measurable process, with respect to  $\mathfrak{F}^W$ , fulfilling (3.1). That equation (3.1) will be denoted by  $E(a, b', \Gamma)$  (without specification of the initial condition).*

**Definition 7.2. (Strong existence).**

*We will say that equation  $E(a, b', \Gamma)$  admits **strong existence** if the following holds. Given any probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , a Brownian motion  $(W_t)_{t \geq 0}$  and  $\xi \in \mathbb{R}$ , there exists a process  $(X_t)_{t \geq 0}$  which is solution to  $E(a, b', \Gamma)$  with  $X_0 = \xi$  a.s.*

**Definition 7.3. (Pathwise uniqueness).** *We will say that equation  $E(a, b, \Gamma)$  admits pathwise uniqueness if the following property is fulfilled.*

*Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, carrying a Brownian motion  $(W_t)_{t \geq 0}$ . If two processes  $X, \tilde{X}$  are two solutions such that  $X_0 = \tilde{X}_0$  a.s., then  $X$  and  $\tilde{X}$  are indistinguishable.*

**Definition 7.4. (Existence in law or weak existence).** *Let  $\nu$  be a probability law on  $\mathbb{R}$ . We will say that  $E(a, b', \Gamma; \nu)$  admits weak existence if there exists a probability*

space  $(\Omega, \mathcal{F}, \mathbb{P})$ , carrying a Brownian motion  $(W_t)_{t \geq 0}$  and a process  $(X_t)_{t \geq 0}$  such that  $(X, \mathbb{P})$  is a (weak) solution of  $E(a, b', \Gamma; \nu)$ , see Definition 3.1.

We say that  $E(a, b', \Gamma)$  admits weak existence if  $E(a, b', \Gamma; \nu)$  admits weak existence for every  $\nu$ .

**Definition 7.5. (Uniqueness in law).** Let  $\nu$  be a probability law on  $\mathbb{R}$ . We say that  $E(a, b', \Gamma; \nu)$  has a **unique solution in law** if the following holds. Suppose we have a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  (respectively  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ ), carrying a Brownian motion  $(W_t)_{t \geq 0}$  (respectively  $(\tilde{W}_t)_{t \geq 0}$ ). We suppose that a process  $(X_t)_{t \geq 0}$  (resp. a process  $(\tilde{X}_t)_{t \geq 0}$ ) is a solution of  $E(a, b', \Gamma)$  such that both  $X_0$  and  $\tilde{X}_0$  are distributed according to  $\nu$ . Uniqueness in law means that  $X$  and  $\tilde{X}$  must have the same law as random elements taking values in  $C([0, T])$  or  $C(\mathbb{R}_+)$ .

We say that  $E(a, b', \Gamma)$  has a unique solution in law if  $E(a, b', \Gamma; \nu)$  has a unique solution in law for every  $\nu$ .

## 7.2. A technical lemma.

**Lemma 7.6.** Suppose the validity of the assumptions of Proposition 4.25. Let  $Y$  be a solution of (4.24) and  $m \geq 2$  an integer. Then there exists a constant  $C > 0$ , depending on the linear growth constant of  $\sigma_0$ ,  $Y_0$ ,  $K, T, m$  and the quantity (4) in Assumption 4.24 (which will be summarized by  $(*)$ ) such that

$$\mathbb{E} \left( \sup_{t \leq T} |Y_s|^m \right) \leq C.$$

### *Proof.*

We recall that for simplicity we have fixed the initial condition to be deterministic. Let  $N > 0$  such that  $|Y_0| \leq N$ . Consider the stopping time

$$\tau_N := \begin{cases} \inf\{s \in [0, T]; |Y_s| > N\}, & \text{if } \{s \in [0, T]; |Y_s| > N\} \neq \emptyset \\ +\infty, & \text{otherwise.} \end{cases}$$

We define  $Y_t^N := Y_{t \wedge \tau_N}$ , for  $0 \leq t \leq T$ . By (4.24) we have

$$Y_s^N = Y_0 + \int_0^{s \wedge \tau_N} \sigma_0(Y_r) dW_r + \int_0^{s \wedge \tau_N} \bar{\Gamma}(r, Y) dr.$$

By (3) of Assumption 4.24 there exists  $c_0 > 0$  depending on  $(*)$  such that

$$\begin{aligned} \mathbb{E} \left( \sup_{s \leq t} |Y_s^N|^m \right) &\leq c_0 \left[ Y_0^m + \mathbb{E} \left( \left( \sup_{s \leq t} \left| \int_0^s \sigma_0(Y_r^N) dW_r \right| \right)^m \right) \right. \\ &\quad \left. + \left( \sup_{s \leq t} \int_0^s |\bar{\Gamma}(r, 0)| dr \right)^m + \mathbb{E} \left( \sup_{s \leq t} \int_0^s |Y_r^N| dr + \sup_{s \leq t} \int_0^s \int_0^r |Y_z^N| dz dr \right)^m \right]. \end{aligned}$$

As for  $m \geq 2$  integer the function  $f(x) = |x|^m$  is convex then by the Burkholder-Davis-Gundy, Jensen's inequality and (2) of Assumption 4.24 there exists a constant  $c_1 > 0$  depending on (\*) such that,

$$\mathbb{E} \left( \sup_{s \leq t} |Y_s^N|^m \right) \leq c_1 \left[ 1 + \int_0^t \mathbb{E} \left( \sup_{r \leq s} |Y_r^N|^m \right) ds + \left( \sup_{s \leq t} |\bar{\Gamma}(s, 0)| \right)^m + \mathbb{E} \left( \int_0^t |Y_s^N|^m ds \right) \right].$$

By (4) of Assumptions 4.24 and Fubini's theorem there exists  $c_2 > 0$  depending on (\*) but not on  $N$  such that

$$\mathbb{E} \left( \sup_{s \leq t} |Y_s^N|^m \right) \leq c_2 \left( 1 + \int_0^t \mathbb{E} \left( \sup_{r \leq s} |Y_r^N|^m \right) ds \right).$$

We recall that the quantity  $\mathbb{E} \left( \sup_{s \leq t} |Y_s^N|^m \right)$  is bounded by  $N^m$  so it is finite. By Gronwall's Lemma there exists a constant  $C > 0$ , independent of  $N$ , such that

$$\mathbb{E} \left( \sup_{s \leq T} |Y_s^N|^m \right) \leq C. \quad (7.1)$$

Thus, as  $Y^N$  is continuous, by Fatou's Lemma and (7.1) we finally have

$$\mathbb{E} \left( \max_{s \leq T} |Y_s| \right) = \mathbb{E} \left( \lim_{N \rightarrow \infty} \max_{s \leq T} |Y_s^N|^m \right) \leq \liminf_{N \rightarrow \infty} \mathbb{E} \left( \max_{s \leq T} |Y_s^N|^m \right) \leq C. \quad (7.2)$$

□

### 7.3. Proof of Proposition 6.5.

We remark that the hypothesis on  $\Gamma$  implies that  $\bar{\Gamma}$  has linear growth, i.e. there is a constant  $K$  such that

$$\bar{\Gamma}(t, \eta^t) \leq K(1 + \sup_{s \in [0, T]} |\eta(s)|), \forall (t, \eta) \in [0, T] \times C([0, T]). \quad (7.3)$$

For item (1), we start truncating  $\Gamma$ . Let  $N > 0$ . Let us define, for  $s \in [0, T], \eta \in C([0, T])$ ,

$$\begin{aligned} \Gamma^N(s, \eta) &:= (\Gamma(s, \eta^s) \vee (-N)) \wedge N, \\ \bar{\Gamma}^N(s, \eta) &:= 2\sqrt{|\eta(s)|} \Gamma^N(s, \sqrt{|\eta|}) + \delta. \end{aligned}$$

We consider the SDE

$$\begin{cases} dS_t &= 2\sqrt{|S_t|} dW_t + \bar{\Gamma}^N(t, S) dt, \\ S_0 &= s_0. \end{cases} \quad (7.4)$$

We set  $x_0 := \sqrt{s_0}$ . Since  $\Gamma^N$  is bounded, by Proposition 6.2, the martingale problem related to  $\mathcal{L}$  (defined in (3.3)) with respect to  $\mathcal{D}_{L^\delta}$ , admits a solution  $(X, \mathbb{P})$  which is

non-negative. By Proposition 6.6 the SDE (7.4) admits existence in law and in particular there exists a solution  $S^N$  (which is necessarily non-negative) on some probability space  $(\Omega, \mathcal{F}, \bar{\mathbb{P}}^N)$ . By Itô's formula, this implies that (on the mentioned space),

$$M_t^N := f(S_t^N) - f(S_0^N) - \int_0^t f'(S_s^N) \bar{\Gamma}^N(s, S^N) ds - 2 \int_0^t f''(S_s^N) |S_s^N| ds, \quad (7.5)$$

is a martingale for all  $f \in C^2$  with compact support.

We want first to show that the family of laws  $(\bar{\mathbb{Q}}^N)$  of  $(S^N)$  is tight. For this we are going to use the Kolmogorov-Centsov Theorem. We denote by  $\bar{\mathbb{E}}^N$  the expectation related to  $\bar{\mathbb{P}}^N$ . According to Problem 4.11 in Section 2.4 of [19] it is enough to find constants  $\alpha, \beta > 0$  such that

$$\sup_N \bar{\mathbb{E}}^N(|S_t^N - S_s^N|^\alpha) \leq c|t - s|^{1+\beta}; s, t \in [0, T], \quad (7.6)$$

for some constant  $c > 0$ . Indeed we show (7.6) for  $\alpha = 6$  and  $\beta = 1$ . By (7.4) and Burkholder-Davis-Gundy inequality there exists a constant  $c_6$  such that, for  $0 \leq s \leq t \leq T$ ,

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \leq c_6 \left( \bar{\mathbb{E}}^N \left( \int_s^t (|S_r^N|) dr \right)^3 + \bar{\mathbb{E}}^N \left( \int_s^t \bar{\Gamma}^N(r, |S^N|) dr \right)^6 \right). \quad (7.7)$$

By (7.3), there exists a constant  $\mathcal{C}_1$  such that, for every  $(s, \eta) \in \Lambda$ , such that

$$|\bar{\Gamma}^N(s, \eta)| \leq 2\sqrt{|\eta(s)|} |\Gamma(s, \sqrt{|\eta|})| + \delta = |\bar{\Gamma}(s, \eta)| \leq \mathcal{C}_1 \left( 1 + \sup_{r \leq s} |\eta(r)| \right). \quad (7.8)$$

By Jensen's inequality and (7.8) there exists a constant  $\mathcal{C}_2 > 0$ , only depending on  $T$  and on  $\bar{\Gamma}$ , but not on  $N$ , such that

$$\bar{\mathbb{E}}^N(|S_t^N - S_s^N|^6) \leq \mathcal{C}_2 \left( (t - s)^2 \bar{\mathbb{E}}^N \left( \sup_{s \leq t} |S_s^N|^3 \right) + (t - s)^5 \bar{\mathbb{E}}^N \left( \sup_{s \leq t} |S_s^N|^6 \right) \right).$$

By Lemma 7.6, the quantity

$$\bar{\mathbb{E}}^N \left( \sup_{s \leq T} |S_s^N|^3 + \sup_{s \leq T} |S_s^N|^6 \right)$$

does not depend on  $N$  and therefore (7.6) holds. Consequently, the family of laws  $(\bar{\mathbb{Q}}^N)$  of  $(S^N)$  under  $(\bar{\mathbb{P}}^N)$  is tight. We can therefore extract a subsequence which, for simplicity, we will still call  $\bar{\mathbb{Q}}^N$  that converges weakly to a probability measure  $\bar{\mathbb{Q}}$  on  $(C[0, T], \mathfrak{B}(C[0, T]))$ .

We denote by  $\mathbb{E}^N$  the expectation with respect to  $\bar{\mathbb{Q}}^N$ . Let  $0 \leq s \leq t \leq T$  and let  $F : C([0, s]) \rightarrow \mathbb{R}$  bounded and continuous. By (7.5), if  $S$  is the canonical process we have

$$\mathbb{E}^N((\tilde{M}_t^N - \tilde{M}_s^N)F(S_r, 0 \leq r \leq s)) = 0, \quad (7.9)$$

where

$$\tilde{M}_t^N := f(S_t) - f(S_0) - \int_0^t f'(S_s)\bar{\Gamma}^N(s, S) ds - 2 \int_0^t f''(S_s)|S_s| ds. \quad (7.10)$$

By Skorokhod's convergence theorem there exist a sequence of processes  $(Y^N)$  and a process  $Y$  both on a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$ , such that, as random elements have  $(C[0, T], \mathfrak{B}(C[0, T]))$  as state space,  $Y^N$  converges  $\mathbb{Q}$ -almost surely (ucp) to  $Y$ . Moreover the law of  $Y^N$  is  $\bar{\mathbb{Q}}^N$ , so

$$\mathbb{E}^{\mathbb{Q}}((\bar{M}_t^N - \bar{M}_s^N)F(Y_r^N, 0 \leq r \leq s)) = 0, \quad (7.11)$$

and

$$\bar{M}_t^N := f(Y_t^N) - f(s_0) - \int_0^t f'(Y_r^N)\bar{\Gamma}^N(s, Y^N) ds - 2 \int_0^t f''(Y_r^N)|Y_r^N| dr. \quad (7.12)$$

In order to pass to the limit when  $N \rightarrow \infty$  using Lebesgue dominated convergence theorem and obtain

$$\mathbb{E}^{\mathbb{Q}}((\bar{M}_t - \bar{M}_s)F(Y_r, 0 \leq r \leq s)) = 0, \quad (7.13)$$

with

$$\bar{M}_t := f(Y_t) - f(s_0) - \int_0^t f'(Y_s)\bar{\Gamma}(s, Y) ds - 2 \int_0^t f''(Y_r)|Y_r| dr, \quad (7.14)$$

it remains to prove that, when  $N \rightarrow \infty$

$$\mathbb{E}^{\mathbb{Q}}\left(\int_s^t f'(Y_r^N)\bar{\Gamma}^N(r, Y^N) dr\right) \rightarrow \mathbb{E}^{\mathbb{Q}}\left(\int_s^t f'(Y)\bar{\Gamma}(r, Y) dr\right) \quad (7.15)$$

and

$$\mathbb{E}^{\mathbb{Q}}\left(\int_s^t f''(Y_r^N)|Y_r^N| dr\right) \rightarrow \mathbb{E}^{\mathbb{Q}}\left(\int_s^t f''(Y_r)|Y_r| dr\right). \quad (7.16)$$

We only prove below (7.15) since (7.16) follows similarly.

Note that (7.15) is true, if and only if,

$$\lim_{N \rightarrow \infty} I_1(N) = 0, \quad \lim_{N \rightarrow \infty} I_2(N) = 0,$$

where

$$I_1(N) := \mathbb{E}^{\mathbb{Q}}\left[\int_s^t f'(Y_r^N)(\bar{\Gamma}^N(r, Y^N) - \bar{\Gamma}(r, Y^N)) dr\right],$$

$$I_2(N) := \mathbb{E}^{\mathbb{Q}} \left[ \int_s^t f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y) dr \right].$$

By (7.3) together with the use of (7.8) we have

$$\begin{aligned} I_1(N) &\leq \|f'\|_{\infty} \mathbb{E}^{\mathbb{Q}} \left[ \mathbf{1}_{\{\sup_{s \in [0, T]} |\Gamma(r, Y^{N, r})| > N\}} \int_s^t |\bar{\Gamma}^N(r, Y^N) - \bar{\Gamma}(r, Y^N)| dr \right] \leq \\ &\leq 2KT \|f'\|_{\infty} \mathbb{E}^{\mathbb{Q}} \left[ \mathbf{1}_{\{\sup_{s \in [0, T]} |\Gamma(r, Y^N)| > N\}} \left( 1 + \sup_{r \in [0, T]} |Y_r^N| \right) \right]. \end{aligned}$$

By Cauchy-Schwarz there exists a non-negative constant  $C(f, T, K)$  such that

$$I_1(N)^2 \leq C(f, T, K) I_{11}(N) I_{12}(N), \quad (7.17)$$

where

$$\begin{aligned} I_{11}(N) &:= \mathbb{Q} \left( \sup_{s \in [0, T]} |\Gamma(r, Y^N)| > N \right), \\ I_{12}(N) &:= \mathbb{E}^{\mathbb{Q}} \left[ 1 + \sup_{r \in [0, T]} |Y_r^N|^2 \right]. \end{aligned}$$

By Chebyshev inequality we have

$$I_{11}(N) \leq \frac{1}{N^2} \mathbb{E}^{\mathbb{Q}} \left[ \sup_{r \in [0, T]} |\Gamma(r, Y^N)|^2 \right] \leq \frac{2K}{N^2} \mathbb{E}^{\mathbb{Q}} \left[ 1 + \sup_{r \in [0, T]} |Y_r^N|^2 \right].$$

Consequently,  $\lim_{N \rightarrow \infty} I_{11}(N) = 0$  because of Lemma 7.6. On the other hand, again by Lemma 7.6  $I_{12}(N)$  is bounded in  $N$  and so by (7.17) we get  $\lim_{N \rightarrow \infty} I_1(N) = 0$ .

Concerning  $I_2(N)$  we have

$$I_2(N)^2 \leq T \int_s^t \mathbb{E}^{\mathbb{Q}} \left[ |f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^2 \right] dr. \quad (7.18)$$

By Lemma 7.6 there exists a constant  $C$  not depending on  $N$  such that

$$\mathbb{E}^{\mathbb{Q}} \left[ \sup_{r \in [0, T]} |Y_r^N|^4 \right] \leq C,$$

and, consequently, by Fatou's Lemma

$$\mathbb{E}^{\mathbb{Q}} \left[ \sup_{r \in [0, T]} |Y_r|^4 \right] \leq C.$$

Let  $r \in [0, T]$  we have

$$\mathbb{E}^{\mathbb{Q}} [|f'(Y_r^N) \bar{\Gamma}(r, Y^N) - f'(Y_r) \bar{\Gamma}(r, Y)|^4] \quad (7.19)$$

$$\begin{aligned} &\leq 8\|f'\|_\infty^4 K^4 \left( 2 + \mathbb{E}^\mathbb{Q} \left[ \sup_{r \in [0, T]} |Y_r^N|^4 + \sup_{r \in [0, T]} |Y_r|^4 \right] \right) \\ &\leq 16\|f'\|_\infty^4 K^4 (1 + C). \end{aligned}$$

So the sequence

$$|f'(Y_r^N)\bar{\Gamma}(r, Y^N) - f'(Y_r)\bar{\Gamma}(r, Y)|^2$$

is uniformly integrable. We fix again  $r \in [0, T]$ . Since  $f'$  and  $\bar{\Gamma}$  are continuous it follows that

$$\mathbb{E}^\mathbb{Q} [ |f'(Y_r^N)\bar{\Gamma}(r, Y^N) - f'(Y_r)\bar{\Gamma}(r, Y)|^2 ] \longrightarrow 0. \quad (7.20)$$

Now (7.19) and Cauchy-Schwarz implies that

$$\mathbb{E}^\mathbb{Q} [ |f'(Y_r^N)\bar{\Gamma}(r, Y^N) - f'(Y_r)\bar{\Gamma}(r, Y)|^2 ] \leq 4\|f'\|_\infty^2 K^2 \sqrt{1 + C}. \quad (7.21)$$

This time (7.20), (7.21) and Lebesgue's dominated theorem shows that the entire Lebesgue integral of (7.20) on  $[s, t]$  converges to 0. Finally  $\lim_{N \rightarrow \infty} I_2(N) = 0$  so that we conclude to (7.15) and, consequently, (7.13). Consequently  $(Y, \mathbb{Q})$  solve the martingale problem of the type (3.2) as in Proposition 3.2 with

$$Lf(x) = 2|x|f''(x) + \delta f'(x)$$

and  $\bar{\Gamma}$  replacing  $\Gamma$ . By Proposition 3.2 this concludes the proof of item (1).

Concerning item (2), the previously constructed  $Y$  is a (weak) solution to (6.5) under the probability  $\mathbb{Q}$ . Since it is a limit of non-negative solutions, it will also be non-negative.

Item (3) follows from Proposition 6.6.

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