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ON PATH-DEPENDENT SDEs INVOLVING DISTRIBUTIONAL DRIFTS

ALBERTO OHASHI¹, FRANCESCO RUSSO², AND ALAN TEIXEIRA¹

ABSTRACT. In this paper, we study (strong and weak) existence and uniqueness of a class of non-Markovian SDEs whose drift contains the derivative in the sense of distributions of a continuous function.

Key words and phrases. SDEs with distributional drift; path-dependent stochastic differential equations.

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1. INTRODUCTION

This paper discusses in detail a framework of one-dimensional stochastic differential equations (henceforth abbreviated by SDEs) with distributional drift with possible path-dependency. Even though we could have worked in the multidimensional case, we have preferred to explore in a systematic way the real line case.

The main objective of this paper is to analyze the solution (existence and uniqueness) of the martingale problem associated with SDEs of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt, \quad X_0 \stackrel{d}{=} \delta_{x_0}, \quad (1.1)$$

where $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $\sigma > 0, x_0 \in \mathbb{R}$ and W is a standard Brownian motion. The assumptions on b , which will be formulated later, imply that b' is a Schwartz distribution. Concerning the path-dependent component of the drift, we consider a locally bounded functional

$$\Gamma : \Lambda \rightarrow \mathbb{R}, \quad (1.2)$$

where

$$\Lambda := \{(s, \eta) \in [0, T] \times C([0, T]); \eta = \eta^s\}$$

and

$$\eta^s(t) = \begin{cases} \eta(t), & \text{if } t \leq s \\ \eta(s), & \text{if } t > s. \end{cases}$$

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By convention, we extend Γ from Λ to $[0, T] \times C([0, T])$ by setting (in non-anticipating way)

$$\Gamma(t, \eta) := \Gamma(t, \eta^t), t \in [0, T], \eta \in C([0, T]).$$

Path-dependent SDEs were investigated under several aspects. Under standard Lipschitz regularity conditions on the coefficients, it is known, (see e.g Theorem 11.2 [20, chapter V]) that strong existence and uniqueness holds. In case the path-dependence takes the form of delayed stochastic equations, one-sided Lipschitz condition ensures strong existence and uniqueness, see e.g [23, 17]. Beyond Lipschitz regularity on the coefficients of the SDE, [14] shows uniqueness in law under structural conditions on an underlying approximating Markov process, where local-time and running maximum dependence are considered. Weak existence of infinite-dimensional SDEs with additive noise on the configuration space with path-dependent drift functionals with sublinear growth are studied by [8]. In all these works, the drift is a non-anticipative functional. Beyond Brownian motion based driving noises, [6] establishes existence of solutions for path-dependent Young differential equation.

The Markovian case ($\Gamma = 0$) with distributional drift has been intensively studied over the years. Diffusions in the generalized sense were first considered in the case when the solution is still a semimartingale, beginning with, at least in our knowledge [19]. Later on, many authors considered special cases of SDEs with generalized coefficients. It is difficult to quote them all; in the case when the drift b' is a measure and the solutions are semimartingales, we refer the reader to [3, 9, 21]. We also recall that [10] considered even special cases of non-semimartingales solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes, for which only existence is proved.

In the non-semimartingale case, time-independent SDEs in dimension one of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \quad t \in [0, T], \quad (1.3)$$

where σ is a strictly positive continuous function and b' is the derivative of a real continuous function was solved and analyzed carefully in [13] and [12], which treated well-posedness of the martingale problem, Itô's formula under weak conditions, semimartingale characterization and Lyons-Zheng decomposition. The only supplementary assumption was the existence of the function

$$\Sigma(x) := 2 \int_0^x \frac{b'}{\sigma^2}(y)dy, x \in \mathbb{R}, \quad (1.4)$$

considered as a suitable limit via regularizations. Those authors considered weak solutions. The SDE (1.3) was also investigated by [1], where the authors provided a well-stated framework when σ and b are γ -Hölder continuous, $\gamma > \frac{1}{2}$. In [22], the authors have also shown that in some cases strong solutions exist and pathwise uniqueness holds. More recently, in the time-dependent framework (but still one-dimensional), a significant contribution was done by [7]. As far as the multidimensional case is concerned, some important steps were done in [11] and more recently in [5], when the diffusion matrix is the identity and b is a time-dependent drift in some suitable negative Sobolev space. We also refer to [2], where the authors have focused on (1.1) in the case of a time independent drift b which is a measure of Kato class. To complement the list, we study existence and uniqueness of a class of path-dependent SDEs whose drift contains the derivative in the sense of distributions of a continuous function. To our best knowledge, this is the first paper which approaches a class of non-Markovian SDEs with distributional drifts.

In this work, equation (1.1) will be interpreted as a martingale problem with respect to some operator $\mathcal{L}f := Lf + \Gamma f'$, see (3.3), where L is the Markovian generator

$$Lf = \frac{\sigma^2}{2} f'' + b' f', \quad (1.5)$$

where we stress that b' is the derivative of some continuous function b , interpreted in the sense of distributions. If we denote Σ as in (1.4), then the operator L can be written as

$$Lf = (e^\Sigma f')' \frac{e^{-\Sigma} \sigma^2}{2}, \quad (1.6)$$

see [12]. We define a notion of martingale problem related to \mathcal{L} (see Definition 3.3) and a notion of strong martingale problem related to $\mathcal{D}_{\mathcal{L}}$ and a given Brownian motion W , see Definition 3.4; that definition has to be compared with the notion of strong existence and pathwise uniqueness of an SDE. In the Markovian case, the notion of *strong martingale problem* was introduced in [22] and it represents the corresponding notion to strong solution of SDEs in the framework of martingale problems.

As anticipated, we will concentrate on the case when b is continuous, the case of special discontinuous functions is investigated in [18]. When $\Gamma = 0$, this case was completely analyzed in [12] and [13]. Concerning the former case, under the existence of the function (1.4) and some boundedness or linear growth condition (see (4.16)), Theorem 4.23 presents existence for the martingale problem related to (1.1). Proposition 4.24 states uniqueness under more restrictive conditions. Under suitable Lipschitz

regularity conditions on a functional $\tilde{\Gamma}$, which is related to Γ via (4.13), Corollary 4.30 establishes well-posedness for the strong martingale problem associated to (1.1).

That type of process appears in some fields for instance, much work has been done on Markovian processes in a random environment. In this particular case, $b = B$ is a two-sided real-valued Brownian motion which is independent from W and (1.1) might be interpreted as the non-Markovian version

$$dX_t = -\frac{1}{2}\dot{B}(X_t)dt + \Gamma(t, X^t)dt + dW_t,$$

of the so-called Brox diffusion which is indeed obtained setting $\Gamma = 0$, see e.g [4, 15] and other references therein in the classical Markovian context.

The paper is organized as follows. After this Introduction and after having fixed some preliminaries, in Section 3 we define the suitable concept of martingale problems for SDE with distributional drift with path-dependent perturbations. In Section 4 we investigate the case when b' is the derivative of a continuous function.

2. NOTATIONS AND PRELIMINARIES

2.1. General notations.

Let I be an interval of \mathbb{R} . $C^k(I)$ is the space of real functions defined on I having continuous derivatives till order k . Such space is endowed with the uniform convergence topology on compact sets for the functions and all derivatives. Generally, $I = \mathbb{R}$ or $[0, T]$ for some fixed positive real T . The space of continuous functions on I will be denoted by $C(I)$. Often, if there is no ambiguity $C^k(\mathbb{R})$ will be simply indicated by C^k . Given an a.e. bounded real function f , $|f|_\infty$ will denote the essential supremum.

We recall some notions from [12]. For us, all filtrations \mathfrak{F} fulfill the usual conditions. When no filtration is specified, we mean the canonical filtration of the underlying process. Otherwise the canonical filtration associated with a process X is denoted by \mathfrak{F}^X . An \mathfrak{F} -Dirichlet process X is the sum of an \mathfrak{F} -local martingale M^X with an \mathfrak{F} -adapted zero quadratic variation process A^X . We will fix by convention that $A_0^X = 0$ so that the decomposition is unique. A sequence (X^n) of continuous processes indexed by $[0, T]$ is said to converge u.c.p. to some process X whenever $\sup_{t \in [0, T]} |X_t^n - X_t|$ converges to zero in probability.

Remark 2.1.

- (1) An \mathfrak{F} -continuous semimartingale Y is always an \mathfrak{F} -Dirichlet process. The A^Y process coincides with the continuous bounded variation component. Moreover the quadratic variation $[Y]$ is the usual quadratic variation for semimartingales.
- (2) Any \mathfrak{F} -Dirichlet process is a finite quadratic variation process and its quadratic variation gives $[X] = [M^X]$.
- (3) If $f \in C^1(\mathbb{R})$ and $X = M^X + A^X$ is an \mathfrak{F} -Dirichlet process, then $Y := f(X)$ is again an \mathfrak{F} -Dirichlet process and $[Y] = \int_0^\cdot f'(X)^2 d[M^X]$.

3. NON-MARKOVIAN SDE: THE FUNCTION CASE.

3.1. General considerations.

Similarly as for the case of Markovian SDEs, it is possible to formulate the notions of strong existence, pathwise uniqueness, existence and uniqueness in law for path-dependent SDEs of the type (1.1), see e.g. Section 5.

Let us suppose for the moment that $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$ are Borel functions. We will consider solutions X of

$$\begin{cases} dX_t &= \sigma(X_t)dW_t + b'(X_t)dt + \Gamma(t, X^t)dt \\ X_0 &= \xi, \end{cases} \quad (3.1)$$

for some initial condition ξ . The previous equation will be denoted by $E(\sigma, b', \Gamma; \nu)$ (where ν is the law of ξ), or simply with $E(\sigma, b', \Gamma)$ if we omit the initial condition. For simplicity in this paper, ξ will always be considered as deterministic. Γ is defined in (1.2).

Definition 3.1. *Let ν be the Dirac probability measure on \mathbb{R} such that $\nu = \delta_{x_0}, x_0 \in \mathbb{R}$. A stochastic process X is called **(weak) solution** of $E(\sigma, b', \Gamma; \nu)$ with respect to a probability \mathbb{P} if there is a Brownian motion W on some filtered probability space, such that X solves (3.1) and $X_0 = x_0$. We also say that the couple (X, \mathbb{P}) solves $E(\sigma, b', \Gamma)$ with initial condition distributed according to ν .*

Suppose $\Gamma \equiv 0$. A very well-known result in [24], Corollary 8.1.7, concerns the equivalence between martingale problems and solution in law. Suppose for a moment that b' is a continuous function. According to [16, chapter 5], it is well-known that a process X and probability \mathbb{P} solve the classical martingale problem, if and only if, X is a (weak) solution of (1.1). The proof of the result mentioned above can be easily adapted to the path-dependent case, i.e. when $\Gamma \neq 0$. This provides the statement below.

Proposition 3.2. *A couple (X, \mathbb{P}) is a solution of $E(\sigma, b', \Gamma)$, if and only if, under \mathbb{P} ,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds \quad (3.2)$$

is a local martingale, where $Lf = \frac{1}{2}\sigma^2 f'' + b'f'$, for every $f \in C^2$.

3.2. Comments about the distributional case.

When b' is a distribution, it is not obvious to introduce the notion of SDE, except in the case when L is close to the divergence form, i.e. when $Lf = (\sigma^2 f')' + \beta f'$ and β is a Radon measure, see e.g. Proposition 3.1 of [12]. For this reason, we replace the notion of weak solution with the notion of martingale problem. Suppose for a moment that L is a second order PDE operator with possible generalized coefficients. In general, as it is shown in [12], C^2 is not included in the domain of operator L and, similarly to [12], we will replace C^2 with some domain \mathcal{D}_L . Suppose that $L : \mathcal{D}_L \subset C^1(\mathbb{R}) \rightarrow C(\mathbb{R})$.

Definition 3.3. (1) *We say that a continuous stochastic process X solves (with respect to a probability \mathbb{P} on some measurable space (Ω, \mathcal{F})) the martingale problem related to*

$$\mathcal{L}f := Lf + \Gamma f', \quad (3.3)$$

with initial condition $\nu = \delta_{x_0}$, $x_0 \in \mathbb{R}$, with respect to a domain \mathcal{D}_L if

$$M_t^f := f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds, \quad (3.4)$$

is a \mathbb{P} -local martingale for all $f \in \mathcal{D}_L$.

We will also say that the couple (X, \mathbb{P}) is a solution of (or (X, \mathbb{P}) solves) the martingale problem with respect to \mathcal{D}_L .

- (2) *If a solution exists, we say that the martingale problem above admits existence.*
 (3) *We say that the martingale problem above admits uniqueness if any two solutions (X^i, \mathbb{P}^i) , $i = 1, 2$ (on some measurable space (Ω, \mathcal{F})) have the same law.*

In the sequel, when the measurable space (Ω, \mathcal{F}) is self-explanatory it will be often omitted. As already observed in Proposition 3.2, the notion of martingale problem is (since the works of Stroock and Varadhan [24]) a concept related to solutions of SDEs in law. In the case when b' and σ are continuous functions (see [24]), \mathcal{D}_L corresponds to $C^2(\mathbb{R})$.

Below we introduce the analogous notion of strong existence and pathwise uniqueness for our martingale problem.

Definition 3.4.

- (1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathfrak{F} = (\mathcal{F}_t)$ be the canonical filtration associated with a fixed Brownian motion W . Let $x_0 \in \mathbb{R}$ be a constant. We say that a continuous \mathfrak{F} -adapted real-valued process X such that $X_0 = x_0$ is a **solution to the strong martingale problem** (related to (3.3)) with respect to \mathcal{D}_L and W (with related filtered probability space), if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)ds - \int_0^t f'(X_s)\Gamma(s, X^s)ds = \int_0^t f'(X_s)\sigma(X_s)dW_s, \quad (3.5)$$

is an \mathfrak{F} -local martingale for all $f \in \mathcal{D}_L$.

- (2) We say that the martingale problem related to (3.3) with respect to \mathcal{D}_L admits **strong existence** if for every $x_0 \in \mathbb{R}$, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{F})$, where $\mathfrak{F} = (\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W , there is a process X solving the strong martingale problem (related to (3.3)) with respect to \mathcal{D}_L and W with $X_0 = x_0$.
- (3) We say that the martingale problem (related to (3.3)) with respect to \mathcal{D}_L , admits **pathwise uniqueness** if given $(\Omega, \mathcal{F}, \mathbb{P})$, a Brownian motion W on it, two solutions $X^i, i = 1, 2$, of the strong martingale with respect to \mathcal{D}_L and W and $\mathbb{P}[X_0^1 = X_0^2] = 1$, then X^1 and X^2 are indistinguishable.

4. THE CASE WHEN THE DRIFT IS A DERIVATIVE OF A CONTINUOUS FUNCTION

Here we extend the Markovian framework of [12] to the non-Markovian case.

4.1. The Markovian case.

Let σ and b be functions in $C(\mathbb{R})$ with $\sigma > 0$. In [12], the authors define, by mollification methods, the function

$$\Sigma(x) = 2 \lim_{n \rightarrow \infty} \int_0^x \frac{b'_n}{\sigma_n^2}(y)dy, \quad \forall x \in \mathbb{R}, \quad (4.1)$$

where the limit is intended to be in $C(\mathbb{R})$, i.e. uniformly on each compact. For concrete examples, we refer the reader to [12], for instance if either σ^2 or b are of locally bounded variation. The function Lf is defined according to (1.6).

Similarly as in [12], \mathcal{D}_L will be the linear space of $f \in C^1(\mathbb{R})$ for which there exists $l \in C(\mathbb{R})$ such that $Lf = l$ (still by mollifications). Thereby we use the terminology $Lf = l$ in the C^1 -generalized sense. As in [12], we define the L -harmonic function $h : \mathbb{R} \rightarrow \mathbb{R}$ as

$$h(0) = 0, \quad h' = e^{-\Sigma}, \quad (4.2)$$

in particular $Lh = 0$, as we will see in Corollary 4.2. In this case, $L : \mathcal{D}_L \subset C^1 \rightarrow \mathbb{R}$ can be written as in (1.6). Indeed, Proposition 4.1 below is a direct consequence of Lemma 2.9 and Lemma 2.6 in [12]. From now on h will be the function defined in (4.2).

Proposition 4.1.

- (1) Let $f \in C^1$. $f \in \mathcal{D}_L$ if, and only if, there exists $\phi \in C^1$ such that $f' = \exp(-\Sigma)\phi$.
(2) If $f \in \mathcal{D}_L$, then $l = Lf$ is given by (1.6); in particular we have

$$Lf = \phi' \exp(-\Sigma) \frac{\sigma^2}{2}, \quad (4.3)$$

where ϕ is the function given in item (1) above.

Corollary 4.2.

- (1) If $f \in \mathcal{D}_L$, then $f^2 \in \mathcal{D}_L$ and $Lf^2 = \sigma^2 f'^2 + 2fLf$.
(2)

$$Lh = 0, Lh^2 = \sigma^2 h'^2.$$

Proof (of Corollary 4.2).

- (1) $f^2 \in \mathcal{D}_L$ because $(f^2)' = 2ff' = (2f\phi) \exp(-\Sigma)$ taking into account Proposition 4.1 (1) and the fact that $\phi_2 := 2f\phi \in C^1$. By (4.3),

$$\begin{aligned} Lf^2 &= \phi_2' \exp(-\Sigma) \frac{\sigma^2}{2} = (f\phi)' \exp(-\Sigma) \sigma^2 \\ &= f' \sigma^2 \exp(-\Sigma) \phi + f \phi' \exp(-\Sigma) \sigma^2 = f'^2 \sigma^2 + 2fLf. \end{aligned}$$

- (2) It follows by Proposition 4.1, setting $\phi = 1$ and item (1). □

We now formulate a standing assumption.

Assumption 4.3.

- Σ is well-defined, see (4.1).
- We suppose the non-explosion condition

$$\int_{-\infty}^0 e^{-\Sigma(x)} dx = \int_0^{+\infty} e^{-\Sigma(x)} dx = \infty. \quad (4.4)$$

Remark 4.4.

- (1) Under Assumption 4.3, the L -harmonic function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined in (4.2) is a C^1 -diffeomorphism. In particular h is surjective.

(2) It is easy to verify that Assumption 4.3 implies the non-explosion condition (3.16) in Proposition 3.13 in [12] is fulfilled.

Remark 4.5. When σ and b' are continuous functions, then $\mathcal{D}_L = C^2$. Indeed, in this manner, $\Sigma \in C^1$ and then $f' = \exp(-\Sigma)\phi \in C^1$. In particular, Lf corresponds to its classical definition.

In relation to the *harmonic* function h defined in (4.2), Proposition 2.13 in [12] states the following.

Proposition 4.6. $f \in \mathcal{D}_L$ if and only if $f \circ h^{-1} \in C^2$.

In fact previous result can be generalized to the case when h is replaced by a bijection g of class C^1 .

Proposition 4.7. We suppose Assumption 4.3. Let $g \in \mathcal{D}_L$ be a diffeomorphism of class C^1 such that $g' > 0$ and let f be a C^1 function. Then, $f \in \mathcal{D}_L$ if and only if $f \circ g^{-1}$ belongs to \mathcal{D}_{L^Y} , where $L^Y v := \frac{1}{2}(\sigma_0^g)^2 v'' + ((Lg) \circ g^{-1})v'$, $v \in \mathcal{D}_{L^Y}$, and

$$\sigma_0^g := (\sigma g') \circ g^{-1}. \quad (4.5)$$

Moreover

$$Lf \circ g^{-1} = L^Y(f \circ g^{-1}).$$

Remark 4.8. By Remark 4.5, we get $\mathcal{D}_{L^Y} = C^2$, since L^Y has continuous coefficients.

Proof (of Proposition 4.7). By Proposition 4.1 there exists $\phi^g \in C^1$ such that

$$g' = \exp(-\Sigma)\phi^g. \quad (4.6)$$

Concerning the direct implication, if $f \in \mathcal{D}_L$, first we prove that $f \circ g^{-1} \in \mathcal{D}_{L^Y}$. Again by Proposition 4.1 there exists $\phi^f \in C^1$ such that $f' = \exp(-\Sigma)\phi^f$. So, $(f \circ g^{-1})' = \frac{f'}{g'} \circ g^{-1} = \frac{\phi^f}{\phi^g} \circ g^{-1} \in C^1$ because $g^{-1} \in C^1$ and $\phi^g > 0$, so $f \circ g^{-1} \in C^2$. Note that, by Remark 4.8, $\mathcal{D}_{L^Y} = C^2$ so $f \circ g^{-1} \in \mathcal{D}_{L^Y}$. Moreover, by Proposition 4.1 (2),

$$(\phi^f)' = \frac{2Lf}{\sigma^2} \exp(\Sigma), \quad (\phi^g)' = \frac{2Lg}{\sigma^2} \exp(\Sigma).$$

A direct computation gives

$$(f \circ g^{-1})'' = \left(\frac{\phi^f}{\phi^g} \circ g^{-1} \right)' = \left[\frac{2Lf}{g'^2 \sigma^2} - \frac{2Lg f'}{\sigma^2 g'^3} \right] \circ g^{-1}.$$

Consequently,

$$\frac{(\sigma_0^g)^2}{2} (f \circ g^{-1})'' = \left[Lf - \frac{Lg f'}{g'} \right] \circ g^{-1}. \quad (4.7)$$

By (4.7), $Lf \circ g^{-1} = \frac{(\sigma_0^g)^2}{2}(f \circ g^{-1})'' + (Lg) \circ g^{-1}(f \circ g^{-1})' = L^Y(f \circ g^{-1})$.

Let us discuss the converse implication. Suppose that $f \circ g^{-1}$ belongs to $\mathcal{D}_{L^Y} = C^2$. By Proposition 4.1 we need to show that $f' \exp(\Sigma) \in C^1$ which is equivalent to show that $(f' \exp(\Sigma)) \circ g^{-1}$ belongs to C^1 . If $\phi^g \in C^1$ is such that $g' = \exp(-\Sigma)\phi^g$ (see (4.6)) we have

$$(f' \exp(\Sigma)) \circ g^{-1} = (f' \frac{\phi^g}{g'}) \circ g^{-1} = (f \circ g^{-1})'(\phi^g \circ g^{-1}),$$

which obviously belongs to C^1 . Therefore $f \in \mathcal{D}_L$. \square

Remark 4.9. *If $f \in \mathcal{D}_L$, setting $\varphi = f \circ h^{-1}$ in Proposition 4.7 (with $g = h$), gives*

$$L(\varphi \circ h) \circ h^{-1} = L^Y(\varphi) = \frac{1}{2}\sigma_0^2\varphi'',$$

where

$$\sigma_0 = (\sigma h') \circ h^{-1}. \quad (4.8)$$

In [12], the authors also show that the existence and uniqueness of the solution of the martingale problem are conditioned to a non-explosion feature. An easy consequence of Proposition 3.13 in [12] gives the following.

Proposition 4.10. *Let $\nu = \delta_{x_0}$, $x_0 \in \mathbb{R}$. Suppose the validity of Assumption 4.3. Then the martingale problem related to L (i.e. with $\Gamma = 0$) with respect to \mathcal{D}_L with initial condition ν admits existence and uniqueness.*

Remark 4.11. *By Proposition 3.2 of [12], if $\Gamma = 0$ and (X, \mathbb{P}) is a solution of the above-mentioned martingale problem, then there exists a \mathbb{P} -Brownian motion W such that (3.4) equals*

$$\int_0^t (f'\sigma)(X_s)dW_s, t \in [0, T].$$

4.2. The path-dependent case.

Let σ and b be functions in $C(\mathbb{R})$ with $\sigma > 0$ and Γ as defined in (1.2). Let us suppose again Assumption 4.3 and let h be the function defined in (4.2). We recall that σ_0 was defined in (4.8).

The first result explains how to reduce our path-dependent martingale problem to a path-dependent SDE.

Proposition 4.12. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a stochastic process, we denote $Y = h(X)$.*

- (1) (X, \mathbb{P}) solves the martingale problem related to (3.3) with respect to \mathcal{D}_L if and only if the process $Y := h(X)$ is a solution (with respect to \mathbb{P}) of

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds, \quad (4.9)$$

for some \mathbb{P} -Brownian motion W .

- (2) Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. X is a solution to the strong martingale problem with respect to \mathcal{D}_L and W if and only if (4.9) holds.

Proof.

- (1) We start proving the direct implication. According to (3.4) and the notations introduced therein

$$M_t^h = h(X_t) - h(X_0) - \int_0^t Lh(X_s) ds - \int_0^t h'(X_s) \Gamma(s, X^s) ds, \quad (4.10)$$

is a \mathbb{P} -local martingale on some probability space (Ω, \mathcal{F}) .

In particular, by Corollary 4.2

$$Y_t = Y_0 + \int_0^t h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + M_t^h,$$

where M^h is a local martingale, so Y is a semimartingale. We need now evaluate $[M^h]_t = [Y]_t$. We apply (3.4) for $f = h^2$ and again by Corollary 4.2 we get

$$Y_t^2 = Y_0^2 + \int_0^t \sigma_0^2(Y_s) ds + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + M_t^{h^2}, \quad (4.11)$$

where M^{h^2} is a local martingale and we recall that σ_0 was defined in (4.5). By integration by parts,

$$[Y]_t = Y_t^2 - Y_0^2 - 2 \int_0^t Y_s dY_s = Y_t^2 - Y_0^2 + M_t - 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds,$$

where $M_t = -2 \int_0^t Y_s dM_s^h$. Therefore

$$Y_t^2 = Y_0^2 - M_t + 2 \int_0^t Y_s h'(h^{-1}(Y_s)) \Gamma(s, h^{-1}(Y^s)) ds + [Y]_t. \quad (4.12)$$

Now we can use the uniqueness of the decomposition of a semimartingale Y^2 which admits the two expressions (4.11) and (4.12). This says $-M = M^{h^2}$ and $\int_0^t \sigma_0^2(Y_s) ds = [Y]_t$. By (4.10)

$$[M^h]_t = [Y]_t = \int_0^t \sigma_0^2(Y_s) ds.$$

Setting

$$W_t := \int_0^t \frac{dM_s^h}{\sigma_0(Y_s)}, t \geq 0,$$

we have

$$[W]_t \equiv t.$$

Therefore, by Lévy's characterization of Brownian motion, W is a standard Brownian motion. Since

$$M^h = \int_0^\cdot \sigma_0(Y_s) dW_s,$$

(4.10) shows that Y solves (4.9).

About the converse implication suppose now that $Y = h(X)$ satisfies (4.9), for some \mathbb{P} -Brownian motion W . We take $f \in \mathcal{D}_L$. By Proposition 4.6 $g \equiv f \circ h^{-1} \in C^2$. Using Itô's formula, Proposition 4.7 and Remark 4.9 we get

$$\begin{aligned} g(Y_t) &= g(Y_0) + \int_0^t g'(Y_s) dY_s + \frac{1}{2} \int_0^t g''(Y_s) d[Y]_s \\ &= f(X_0) + \int_0^t Lf(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s + \int_0^t f'(X_s) \Gamma(s, X^s) ds. \end{aligned}$$

Therefore

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds = \int_0^t f'(X_s) \sigma(X_s) dW_s$$

is a local martingale, which concludes the proof.

- (2) The converse implication follows in the same way as for item (1). The proof of the direct implication follows directly by Itô's formula. □

Corollary 4.13. *Let (X, \mathbb{P}) be a solution to the martingale problem related to (3.3) with respect to \mathcal{D}_L . Then X is a Dirichlet process (with respect to its canonical filtration) and $[X]_t = \int_0^t \sigma^2(X_s) ds, t \in [0, T]$.*

Proof.

By Proposition 4.12, $X = h^{-1}(Y)$, where Y is obviously a semimartingale such that $[Y]_t = \int_0^t \sigma_0^2(Y_s) ds, t \in [0, T]$. Consequently, by Remark 2.1, X is indeed a Dirichlet process and

$$[X]_t = \int_0^t ((h^{-1})')^2 \sigma_0^2(Y_s) ds = \int_0^t \frac{\sigma_0^2}{(h' \circ h^{-1})^2}(Y_s) ds = \int_0^t \sigma^2(X_s) ds, t \in [0, T].$$

□

Remark 4.14. *If X is a solution to the strong martingale problem with respect to \mathcal{D}_L and some Brownian motion W , then X is a Dirichlet process with respect to the canonical filtration of the Brownian motion.*

An immediate consequence of Proposition 4.12 is the following.

Corollary 4.15. *Suppose that $\Gamma = 0$ and let (X, \mathbb{P}) be a solution of the martingale problem related to L with respect to \mathcal{D}_L . Then $Y = h(X)$ is an \mathfrak{F} -local martingale where \mathfrak{F} is the canonical filtration of X with quadratic variation $[Y] = \int_0^\cdot \sigma_0^2(Y_s) ds$.*

4.3. Existence.

We make here the same conventions as in Section 4.2. In the sequel, we introduce the map $\tilde{\Gamma} : \Lambda \rightarrow \mathbb{R}$ defined by

$$\tilde{\Gamma}(s, \eta) = \frac{\Gamma(s, \eta)}{\sigma(\eta(s))}, (s, \eta) \in \Lambda. \quad (4.13)$$

At this point, we introduce the following assumption.

Assumption 4.16. *There is $K > 0$ such that*

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, h^{-1} \circ \eta^s)| \leq K \left(1 + \sup_{s \in [0, T]} |\eta(s)| \right), \forall \eta \in C([0, T]).$$

Remark 4.17. *Given a stochastic process X , setting $Y = h(X)$, we get*

$$\sup_{s \in [0, T]} |\tilde{\Gamma}(s, X^s)| \leq K \left(1 + \sup_{s \in [0, T]} |Y_s| \right). \quad (4.14)$$

In particular $\int_0^T \tilde{\Gamma}^2(s, X^s) ds < \infty$ a.s.

Proposition 4.18 below is a well-known extension of Novikov's criterion. It is an easy consequence of Corollary 5.14 [16, Chapter 3].

Proposition 4.18. *Suppose Assumption 4.16. Let W be a Brownian motion and X a continuous and adapted process for which there exists a partition $0 = t_0, t_1, \dots, t_n = T$ such that for $i \in \{1, \dots, n\}$*

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \right) \right] < \infty.$$

Then, the process

$$N_t = \exp \left(\int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right),$$

is a martingale.

We will need a slight adaptation of the Dambis-Dubins-Schwarz theorem to the case of a finite interval.

Proposition 4.19. *Let M be a local martingale vanishing at zero such that $[M]_t = \int_0^t A_s ds$, $t \in [0, T]$. Then, on a possibly enlarged probability space, there exists a copy of M (still denoted by the same letter M) with the same law and a Brownian motion β such that*

$$M_t = \beta_{\int_0^t A_s ds}, t \in [0, T].$$

Proof.

Let us define

$$\tilde{M}_t = \begin{cases} M_t, & t \in [0, T] \\ M_T + B_t - B_T, & t > T, \end{cases}$$

where B is a Brownian motion independent of M . If the initial probability space is not rich enough, one considers an enlarged probability space containing a copy of M (still denoted by the same letter) with the same law and the independent Brownian motion B . Note that \tilde{M} is a local martingale and we have,

$$[\tilde{M}]_t = \begin{cases} [M]_t, & t \in [0, T] \\ t - T + [M]_T, & t > T. \end{cases}$$

Observe that $\lim_{t \rightarrow \infty} [\tilde{M}]_t = \infty$. By the classical Dambis, Dubins-Schwarz theorem there exists a standard Brownian motion β such that a.s. $\tilde{M}_t = \beta_{\int_0^t A_s ds}$, $t \geq 0$. In particular

$$M_t = \beta_{\int_0^t A_s ds}, 0 \leq t \leq T.$$

□

The proposition below is an adaptation of a well-known argument for Markov diffusions.

Proposition 4.20. *Suppose that σ_0 is bounded. Let (X, \mathbb{P}) be a solution of the martingale problem related to (3.3) with respect to \mathcal{D}_L with $\Gamma = 0$. Let M^X the martingale*

component of X . We set

$$W_t := \int_0^t \frac{1}{\sigma(X_s)} dM^X, t \in [0, T].$$

Then

$$\exp \left(\int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t |\tilde{\Gamma}(s, X^s)|^2 ds \right), t \in [0, T], \quad (4.15)$$

is a martingale.

Remark 4.21. We recall that, by Corollary 4.13, X is an \mathfrak{F} -Dirichlet process (\mathfrak{F} be the canonical filtration) and $[X] = [M^X] = \int_0^\cdot \sigma^2(X_s) ds$ so that by Lévy's characterization theorem, W is an \mathfrak{F} -Brownian motion.

Proof (of Proposition 4.20).

Let $Y = h(X)$. By Proposition 4.12, $[Y] = \int_0^\cdot \sigma_0^2(Y_s) ds$. Let $k \geq |\sigma_0|_\infty^2 T$ and $\{t_0 = 0, \dots, t_n = T\}$ a grid of $[0, T]$ so that

$$c_i := \frac{3}{2}(t_i - t_{i-1})K^2k < \frac{1}{2}, \quad (4.16)$$

(K coming from Assumption 4.16) $i = \{1, \dots, n\}$. By (4.14) we know that

$$\int_{t_{i-1}}^{t_i} |\tilde{\Gamma}(s, X^s)|^2 ds \leq (t_i - t_{i-1})K^2 \left(1 + \sup_{s \in [0, T]} |Y_s| \right)^2. \quad (4.17)$$

We set $M_t = Y_t - Y_0, t \in [0, T]$. Note that

$$\left(1 + \sup_{s \in [0, T]} |Y_s| \right)^2 \leq 3 \sup_{s \in [0, T]} |M_s|^2 + 3(1 + Y_0^2). \quad (4.18)$$

We recall that Y_0 is deterministic. In view of applying Proposition 4.18, taking into account (4.17) and (4.18) we get

$$\begin{aligned} & \mathbb{E} \left(\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right) \leq \\ & \mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |M_s|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right). \end{aligned} \quad (4.19)$$

Since M is a local martingale vanishing at zero, Proposition 4.19 states that there is a copy (with the same distribution) of M (still denoted by the same letter) on another probability space, a Brownian motion β such that previous expression gives

$$\mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{s \in [0, T]} |\beta_{[M]_s}|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right)$$

$$\leq \mathbb{E} \left(\exp \left(\frac{3(t_i - t_{i-1})K^2}{2} \sup_{\tau \in [0, k]} |\beta_\tau|^2 \right) \right) \exp \left(\frac{3}{2}(t_i - t_{i-1})K^2(1 + Y_0^2) \right), \quad (4.20)$$

the latter inequality being valid because $[M]_t = \int_0^t \sigma_0^2(Y_s)ds$. By (4.16) we get

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{i-1}}^{t_i} \tilde{\Gamma}^2(s, X^s) ds \right) \right] &\leq \mathbb{E} \left[\exp \left(\frac{c_i}{k} \sup_{\tau \in [0, k]} |B_\tau|^2 \right) \right] \exp \left(\frac{c_i}{k} (1 + Y_0^2) \right) \\ &\leq \mathbb{E} \left[\sup_{\tau \in [0, k]} \exp \left(\frac{c_i}{k} |B_\tau|^2 \right) \right] \exp \left(\frac{c_i}{k} (1 + Y_0^2) \right). \end{aligned} \quad (4.21)$$

By Remark 4.22 below

$$\mathbb{E} \left[\exp \left(\frac{c_i}{k} |B_\tau|^2 \right) \right] \leq \mathbb{E} [\exp(c_i G^2)] < \infty,$$

where G is a standard Gaussian random variable. Since $x \mapsto \exp(\frac{c_i}{2k}x)$ is increasing and convex, and $(|B_\tau|^2)_{\tau \geq 0}$ is a non-negative square integrable submartingale, then $(\exp(\frac{c_i}{2k}|B_\tau|^2))$ is also a non-negative submartingale. Consequently, by Doob's inequality (with $p = 2$) the expectation on the right-hand side of (4.21) is finite. Finally by Proposition 4.18, (4.15) is a martingale. \square

Remark 4.22. *Let G be a standard Gaussian r.v. If $c < \frac{1}{2}$ then*

$$\mathbb{E}[\exp(cG^2)] < \infty.$$

This opens the way to the following existence result for our martingale problem.

Theorem 4.23. *Suppose the validity of Assumption 4.3 and that one of the two conditions below are fulfilled.*

- (1) $\tilde{\Gamma}$ is bounded.
- (2) $\tilde{\Gamma}$ fulfills Assumption 4.16 and σ_0 is bounded.

Then the martingale problem related to (3.3) with respect to \mathcal{D}_L admits existence.

Proof.

By Proposition 4.10, we can consider a solution (X, \mathbb{P}) to the above-mentioned martingale problem with $\Gamma = 0$. By Remark 4.11, there is a Brownian motion W such that

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds = \int_0^t (f'\sigma)(X_s)dW_s, \quad (4.22)$$

for every $f \in \mathcal{D}_L$. We define the process

$$V_t := \exp \left(\int_0^t \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^t \tilde{\Gamma}^2(s, X^s) ds \right).$$

Under item (1), V is a martingale by the Novikov condition. Under item (2), Proposition 4.20 says that V is a martingale. We define

$$\tilde{W}_t := W_t - \int_0^t \tilde{\Gamma}(s, X^s) ds. \quad (4.23)$$

By Girsanov's theorem, (4.23) is a Brownian motion under the probability \mathbb{Q} such that

$$d\mathbb{Q} := \exp \left(\int_0^T \tilde{\Gamma}(s, X^s) dW_s - \frac{1}{2} \int_0^T \tilde{\Gamma}^2(s, X^s) ds \right) d\mathbb{P}.$$

Applying (4.23) in (4.22) we obtain

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds - \int_0^t f'(X_s) \Gamma(s, X^s) ds = \int_0^t (f' \sigma)(X_s) d\tilde{W}_s,$$

for every $f \in \mathcal{D}_L$. Since $\int_0^t (f' \sigma)(X_s) d\tilde{W}_s$ is a local martingale under \mathbb{Q} , (X, \mathbb{Q}) is proved to be a solution to the martingale problem in the statement. \square

4.4. Uniqueness in law.

We use here again the notation $\tilde{\Gamma}$ introduced in (4.13).

Proposition 4.24. *Suppose the validity of Assumption 4.3. Then the martingale problem related to (3.3) with respect to \mathcal{D}_L admits uniqueness.*

Proof.

Let (X^i, \mathbb{P}^i) , $i = 1, 2$, be two solutions of the martingale problem related to (3.3) with respect to \mathcal{D}_L . Let us fix $i = 1, 2$. By Corollary 4.13, X^i is a \mathfrak{F}^{X^i} -Dirichlet process with respect to \mathbb{P}^i , such that $[X^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$. Let M^i be its respective martingale component. Since $[M^i] \equiv \int_0^\cdot \sigma(X_s^i)^2 ds$, by Lévy's characterization theorem, the process

$$W_t^i = \int_0^t \frac{dM_s^i}{\sigma(X_s^i)}, t \in [0, T], \quad (4.24)$$

is an \mathfrak{F}^{X^i} -Brownian motion. In particular, W^i is a Borel functional of X^i .

By means of localization (similarly as in Proposition 5.3.10 of [16], without restriction of generality we can suppose $\tilde{\Gamma}$ to be bounded. We define the process (whose r.v. are

also Borel functionals of X^i)

$$V_t^i = \exp \left(- \int_0^t \tilde{\Gamma}(s, X^{i,s}) dW_s^i - \frac{1}{2} \int_0^t \left(\tilde{\Gamma}(s, X^{i,s}) \right)^2 ds \right),$$

which, by Novikov's condition, is a \mathbb{P}^i -martingale. This allows us to define the probability Q^i defined by $dQ^i = V_T^i d\mathbb{P}^i$.

By Girsanov's theorem, under Q^i , $B_t^i := W_t^i + \int_0^t \tilde{\Gamma}(s, X^{i,s}) ds$ is a Brownian motion. Therefore, (X^i, Q^i) solves the martingale problem related to L ($\Gamma = 0$) with respect to \mathcal{D}_L . By uniqueness of the martingale problem with respect to \mathcal{D}_L and $\Gamma = 0$ (see Proposition 4.10), X^i (under Q^i), $i = 1, 2$ have the same law. Hence, for every Borel set $B \in \mathfrak{B}(C[0, T])$, we have

$$\mathbb{P}^1\{X^1 \in B\} = \int_{\Omega} \frac{1}{V_T^1(X^1)} 1_{\{X^1 \in B\}} dQ^1 = \int_{\Omega} \frac{1}{V_T^2(X^2)} 1_{\{X^2 \in B\}} dQ^2 = \mathbb{P}^2\{X^2 \in B\}.$$

Therefore, X^1 under \mathbb{P}^1 has the same law as X^2 under \mathbb{P}^2 . Finally the martingale problem related to (3.3) with respect to \mathcal{D}_L admits uniqueness. \square

4.5. Results on pathwise uniqueness.

Before exploring conditions for strong existence and uniqueness for the martingale problem, we state and prove Proposition 4.26, which constitutes a crucial preliminary step.

Let $\bar{\Gamma} : \Lambda \rightarrow \mathbb{R}$ be a generic Borel functional. Related to it, we formulate the following.

Assumption 4.25.

- (1) *There exists a function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\int_0^\epsilon l^{-2}(u) du = \infty$ for all $\epsilon > 0$ and*

$$|\sigma_0(x) - \sigma_0(y)| \leq l(|x - y|).$$

- (2) *σ_0 has at most linear growth.*
(3) *there exists $K > 0$ such that*

$$|\bar{\Gamma}(s, \eta^1) - \bar{\Gamma}(s, \eta^2)| \leq K \left(|\eta^1(s) - \eta^2(s)| + \int_0^s |\eta^1(r) - \eta^2(r)| dr \right),$$

for all $(s, \eta^1), (s, \eta^2) \in \Lambda$.

- (4) $\bar{\Gamma}_\infty := \sup_{s \in [0, T]} |\bar{\Gamma}(s, 0)| < \infty$.

Proposition 4.26. *Let $y_0 \in \mathbb{R}$. Suppose the validity of Assumption 4.25. Then $E(\sigma_0, 0, \bar{\Gamma})$, i.e.*

$$Y_t = y_0 + \int_0^t \sigma_0(Y_s) dW_s + \int_0^t \bar{\Gamma}(s, Y^s) ds, \quad (4.25)$$

admits pathwise uniqueness.

Before proceeding with the proof of the proposition, we state a lemma which is an easy consequence of Problem 5.3.15 of [16].

Lemma 4.27. *Suppose the validity of the assumptions in Proposition 4.26. Let Y be a solution of (4.25) and $m \geq 2$ an integer. Then there exists a constant $C > 0$, depending on the linear growth constant of σ_0 , Y_0 , T , m , and the quantities $K, \bar{\Gamma}_\infty$ respectively in Assumptions 4.25 (3)-(4), such that*

$$\mathbb{E} \left(\sup_{t \leq T} |Y_t|^m \right) \leq C.$$

Proof (of Proposition 4.26).

Now let Y^1, Y^2 be two solutions on the same probability space with respect to the same Brownian motion W of (4.25) such that $Y_0^1 = Y_0^2 = y_0$. By Lemma 4.27 we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} |Y_t^i|^2 \right) < \infty, \quad (4.26)$$

for $i = 1, 2$. By the assumption on σ_0 , this obviously gives

$$\mathbb{E} \left(\int_0^T |\sigma_0(Y_t^i)|^2 dt \right) < \infty, \quad (4.27)$$

for $i = 1, 2$. We set $\Delta_t = Y_t^1 - Y_t^2, t \in [0, T]$, and this gives

$$\Delta_t = \int_0^t (\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})) ds + \int_0^t (\sigma_0(Y_s^1) - \sigma_0(Y_s^2)) dW_s, t \in [0, T]. \quad (4.28)$$

We recall from the proof of Proposition 2.13 in [16, Chapter 5], the existence of the functions

$$\Psi_n(x) = \int_0^{|x|} \int_0^y \rho_n(u) du dy,$$

such that for every $x \in \mathbb{R}$

$$0 \leq \rho_n(x) \leq \frac{2}{nl^2(x)}, \quad |\Psi'_n(x)| \leq 1, \quad |\Psi_n(x)| \leq |x|, \quad \lim_{n \rightarrow \infty} \Psi_n(x) = |x|. \quad (4.29)$$

By (4.28), applying Itô's formula we get

$$\Psi_n(\Delta_t) = \int_0^t \Psi'_n(\Delta_s) [\bar{\Gamma}(s, Y^{1,s}) - \bar{\Gamma}(s, Y^{2,s})] ds + \frac{1}{2} \int_0^t \Psi''_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)]^2 ds$$

$$+ \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s. \quad (4.30)$$

Using Assumption 4.25 and (4.29) we get

$$\Psi_n(\Delta_t) \leq \int_0^t K \left(|Y_s^1 - Y_s^2| + \int_0^s |Y_r^1 - Y_r^2| dr \right) ds + \frac{t}{n} + M_t, \quad (4.31)$$

where $M_t = \int_0^t \Psi'_n(\Delta_s) [\sigma_0(Y_s^1) - \sigma_0(Y_s^2)] dW_s$ is a local martingale. Since Ψ'_n is bounded and by (4.27), M is a (even square integrable) martingale.

We now apply the expectation and the Fubini's theorem in (4.31) to get

$$\mathbb{E}\Psi_n(\Delta_t) \leq K \int_0^t \mathbb{E}|Y_s^1 - Y_s^2| ds + KT \int_0^t \mathbb{E}|Y_s^1 - Y_s^2| ds + \frac{t}{n}, \quad (4.32)$$

since $\mathbb{E}M_t = 0$. Passing to the limit when $n \rightarrow \infty$, by Lebesgue's dominated convergence theorem, we get

$$\mathbb{E}|\Delta_t| \leq (K + TK) \int_0^t \mathbb{E}|\Delta_s| ds, \quad (4.33)$$

so, by the Gronwall's inequality we obtain $\mathbb{E}|\Delta_t| = 0$. By the continuity of the sample paths of Y^1, Y^2 we conclude that Y^1, Y^2 are indistinguishable. \square

We come back to the framework of the beginning of Section 4.1. We suppose again the validity of Assumption 4.3. We recall the definition of the harmonic function h defined by $h(0) = 0, h'(x) = e^{-\Sigma}$, see (4.2). We recall the notations $\sigma_0 = (\sigma h') \circ h^{-1}$. We define

$$\bar{\Gamma}(s, \eta) := h'(h^{-1}(\eta(s)))\Gamma(s, h^{-1}(\eta^s)), \quad s \in [0, T], \eta \in C([0, T]). \quad (4.34)$$

Corollary 4.28. *Under Assumptions 4.3 and 4.25, the martingale problem related to (3.3) with respect to \mathcal{D}_L admits pathwise uniqueness.*

Proof. This follows by Proposition 4.26 and Proposition 4.12, taking into account (4.34). \square

Theorem 4.29. *Suppose the validity of Assumptions 4.3 and 4.25 related to $\bar{\Gamma}$ introduced in (4.34). Suppose moreover that one of two hypotheses below are in force.*

- (1) $\tilde{\Gamma}$ defined in (4.13) is bounded.
- (2) $\sigma_0, \frac{1}{\sigma_0}$ are bounded.

Then (4.9) admits strong existence and pathwise uniqueness.

Proof.

By Proposition 4.26, pathwise uniqueness holds. Indeed, by (4.34), the equation (4.9) is a particular case of (4.25).

To prove existence we wish to apply Theorem 4.23. For this we need to verify that either Hypothesis (1) or (2) (in Theorem 4.23) hold. Hypothesis (1) in the above statement coincides with Hypothesis (1) in Theorem 4.23. Suppose the validity of (2) in the above statement and we check that Assumption 4.16 holds true. By (4.13) and the definition of $\bar{\Gamma}$ in (4.34), we obtain

$$\sigma_0(\eta(s))\tilde{\Gamma}(s, (h^{-1} \circ \eta)) = \bar{\Gamma}(s, \eta). \quad (4.35)$$

By (3) in Assumption 4.25, $\frac{1}{\sigma_0}$ being bounded there exists a constant $K_1 > 0$ such that

$$|\tilde{\Gamma}(s, (h^{-1} \circ \eta))| \leq K_1 \left(|\eta(s)| + \int_0^s |\eta(r)| dr \right) + |\bar{\Gamma}(s, 0)|.$$

By (4) in Assumption 4.25 and the fact that, given a function $\gamma \in C([0, T])$,

$$\int_0^s |\gamma(r)| dr \leq s \sup_{r \in [0, s]} |\gamma(r)|,$$

Assumption 4.16 follows.

By Theorem 4.23, the martingale problem related to (3.3) with respect to \mathcal{D}_L admits existence and by Proposition 4.12 (1), we have that (4.9) has a (weak) solution. At this point, we can apply Yamada-Watanabe theorem to guarantee that the solution is actually strong. We remark that the Yamada-Watanabe theorem (in the path-dependent case) proof is the same as the one in the Markovian case, which is for instance stated in Proposition 3.20 [16, Chapter 5]. \square

As a consequence of Proposition 4.12 and Theorem 4.29 we obtain the following.

Corollary 4.30. *Under the same assumptions as in Theorem 4.29, the martingale problem related to (3.3) with respect to \mathcal{D}_L , admits strong existence and pathwise uniqueness.*

5. APPENDIX: DIFFERENT NOTIONS OF SOLUTIONS WHEN b' IS A FUNCTION

Let us suppose below that $\sigma, b' : \mathbb{R} \rightarrow \mathbb{R}$ are locally bounded Borel functions and Γ as given in (1.2). As already mentioned, for simplicity we will only consider initial conditions x_0 to be deterministic.

Definition 5.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \geq 0}$ a Brownian motion and $x_0 \in \mathbb{R}$. A solution X of (3.1) (depending on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the Brownian motion W and an initial condition x_0) is a progressively measurable process, with respect to \mathfrak{F}^W , fulfilling (3.1). That equation (3.1) will be denoted by $E(\sigma, b', \Gamma)$ (without specification of the initial condition).*

Definition 5.2. (Strong existence).

*We will say that equation $E(\sigma, b', \Gamma)$ admits **strong existence** if the following holds. Given any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Brownian motion $(W_t)_{t \geq 0}$ and $x_0 \in \mathbb{R}$, there exists a process $(X_t)_{t \geq 0}$ which is solution to $E(\sigma, b', \Gamma)$ with $X_0 = x_0$ a.s.*

Definition 5.3. (Pathwise uniqueness). *We will say that equation $E(\sigma, b', \Gamma)$ admits pathwise uniqueness if the following property is fulfilled.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a Brownian motion $(W_t)_{t \geq 0}$. If two processes X, \tilde{X} are two solutions to $E(\sigma, b', \Gamma)$ such that $X_0 = \tilde{X}_0$ a.s., then X and \tilde{X} are indistinguishable.

Definition 5.4. (Existence in law or weak existence). *Let ν be a probability law on \mathbb{R} . We will say that $E(\sigma, b', \Gamma; \nu)$ admits weak existence if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a Brownian motion $(W_t)_{t \geq 0}$ and a process $(X_t)_{t \geq 0}$ such that (X, \mathbb{P}) is a (weak) solution of $E(\sigma, b', \Gamma; \nu)$, see Definition 3.1.*

We say that $E(\sigma, b', \Gamma)$ admits weak existence if $E(\sigma, b', \Gamma; \nu)$ admits weak existence for every ν .

Definition 5.5. (Uniqueness in law). *Let ν be a probability law on \mathbb{R} . We say that $E(\sigma, b', \Gamma; \nu)$ has a **unique solution in law** if the following holds. Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (respectively $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$) carrying a Brownian motion $(W_t)_{t \geq 0}$ (respectively $(\tilde{W}_t)_{t \geq 0}$). We suppose that a process $(X_t)_{t \geq 0}$ (resp. a process $(\tilde{X}_t)_{t \geq 0}$) is a solution of $E(\sigma, b', \Gamma)$ such that both X_0 and \tilde{X}_0 are distributed according to ν . Uniqueness in law means that X and \tilde{X} must have the same law as random elements taking values in $C([0, T])$ or $C(\mathbb{R}_+)$.*

We say that $E(\sigma, b', \Gamma)$ has a unique solution in law if $E(\sigma, b', \Gamma; \nu)$ has a unique solution in law for every ν .

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1 DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, 70910-900, BRASÍLIA, BRAZIL.
Email address: amfohashi@gmail.com

2 ENSTA PARIS, INSTITUT POLYTECHNIQUE DE PARIS, UNITÉ DE MATHÉMATIQUES APPLIQUÉES,
828, BOULEVARD DES MARÉCHAUX, F-91120 PALAISEAU, FRANCE
Email address: francesco.russo@ensta-paris.fr

3 ENSTA PARIS, INSTITUT POLYTECHNIQUE DE PARIS, UNITÉ DE MATHÉMATIQUES APPLIQUÉES,
828, BOULEVARD DES MARÉCHAUX, F-91120 PALAISEAU, FRANCE
Email address: alan.teixeira@ensta-paris.fr