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Abstract

We initiate a quantitative study of measure equivalence (and orbit equivalence) between finitely generated groups that extends the classical setting of L^p measure equivalence. In particular we investigate quantitative versions of Orstein Weiss' theorem. Our first main result is a very general monotonicity property satisfied by the isoperimetric profile, which provides a "lower bound" on how integrable a measure coupling between $\mathbb{Z}$ and an amenable group Γ can be. As another application we get that the isoperimetric profile is invariant under L^1 measure equivalence and monotonous under coarse embedding among amenable groups. On the other hand, we introduce a notion of Følner tiling, which we use to construct explicit orbit equivalences between amenable groups satisfying certain integrability conditions. We show in a number of instances that the bound given by the isoperimetric profile are sharp up to a logarithmic error. In the context of non-amenable groups, we study the stability of hyperbolicity: Shalom proved that Gromov hyperbolicity is not invariant under L^p measure equivalence. Actually his proof yields a stronger statement, which we prove to be optimal in some sense. As a by-product, we show that if Γ and Λ admit an orbit equivalence coupling satisfying a certain integrability condition, and if Γ is hyperbolic, then so is Λ.

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1 Introduction

Gromov introduced measure equivalence between countable groups as a measured analogue of quasi-isometry. A classical instance of a pair of measure equivalent groups is given by lattices in a common locally compact group. Another source of examples is given by orbit equivalent groups, i.e. groups which admit free measure-preserving actions on a standard probability space which share the same orbits.

The notion of measure equivalence has been extensively studied over the past 20 years, and we refer the reader to [Gab05, Sec. 2] for an overview of its main properties as well as its tight connections with invariants such as cost or ℓ^2 Betti numbers. Various rigidity phenomenons have also been uncovered. A famous example is Furman's superrigidity results for lattices in higher rank semi-simple Lie groups [Fur99], which implies for instance that any countable group which is measure-equivalent to a lattice in $\mathrm{PSL}_3(\mathbb{R})$ is commensurable up to finite kernel to another lattice in $\mathrm{PSL}_3(\mathbb{R})$. Another very nice example is provided by Kida's work on mapping class groups of surfaces: he showed that most surfaces can be reconstructed from the measure equivalence class of their mapping class group [Kid08], and that every group which is measure equivalent to a mapping class group must actually be commensurable up to finite kernel to it [Kid10].

In the opposite direction of flexibility, a celebrated result of Ornstein and Weiss implies that all infinite countable amenable groups are orbit equivalent and hence measure equivalent [OW80]. So most coarse geometric invariants (such as volume growth) are not preserved under orbit equivalence. Also, it is known that the class of groups measure equivalent to lattices in $\mathrm{PSL}_2(\mathbb{R})$ is very diverse and contains groups that are not virtually isomorphic to lattices of the latter (for instance, all free products of infinite amenable groups belong to this class). But as we will now see, measure equivalence admits

natural refinements which capture meaningful coarse geometric invariants and for which similar rigidity phenomenons hold.

Assume for simplicity that we are given an *orbit equivalence coupling* of two finitely generated groups $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$ over a probability space (X, μ), i.e. two measure-preserving free actions of Γ and Λ on (X, μ) which share the same orbits. Then we can equip the space (X, μ) with the Schreier graph metrics d_{S_Γ} and d_{S_Λ}, and consider for each $\gamma \in \Gamma$ and each $\lambda \in \Lambda$ the following *distance maps*

$$x \mapsto d_{S_\Lambda}(x, \gamma \cdot x) \text{ and } x \mapsto d_{S_\Gamma}(x, \lambda \cdot x).$$

The fact that the two actions share the same orbits means that these functions do not take the value $+\infty$, in other words they belong to $\mathrm{L}^0(X, \mu, \mathbb{R})$. We then say that Γ and Λ are L^p orbit equivalent when all these functions are in $\mathrm{L}^p(X, \mu; \mathbb{R})$. A similar definition can be given for measure equivalence, yielding the notion of L^p measure equivalence (see Section 2.3).

In the past ten years, L^1 measure equivalence has been intensely investigated. In the context of non-amenable groups, Bader, Furman and Sauer showed that any group that is L^1 measure equivalent to a lattice in $\mathrm{SO}(n, 1)$ for a given $n \geqslant 2$ must be virtually isomorphic to another such lattice [BFS13]. In particular, there is a L^1-measure equivalence rigidity phenomenon for lattices in $\mathrm{PSL}_2(\mathbb{R})$, as opposed to the pure measure equivalence case. Somewhat more surprising is a result of Austin in the context of amenable groups: he showed that L^1 measure equivalent virtually nilpotent groups have bi-Lipschitz asymptotic cones [Aus16]. Another important result is due to Bowen, who showed in the appendix of Austin's aforementioned paper that volume growth is invariant under L^1 measure equivalence.

This paper is in the direct continuation of such results, but we also aim at a deeper understanding by going in two new directions: finer integrability notions and asymmetric versions of measure equivalence. For this introduction, let us however only give the definitions in the context of orbit equivalence (see Sec. 2.3 for the measure equivalence versions and its asymmetric counterparts).

Assume again that we are given two orbit equivalent actions of two finitely generated groups $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$. Given an two unbounded increasing positive function $\varphi, \psi : (0, \infty) \to (0, \infty)$, we say that that we have a (φ, ψ) integrable orbit equivalence coupling if for each $\gamma \in \Gamma$ and each $\lambda \in \Lambda$, there are constants $c_\gamma, c_\lambda > 0$ such that the associated distance functions satisfy the following conditions:

$$\int_X \varphi\left(\frac{d_{S_\Lambda}(x, \gamma \cdot x)}{c_\gamma}\right) d\mu(x) < \infty \text{ and } \int_X \psi\left(\frac{d_{S_\Gamma}(x, \lambda \cdot x)}{c_\lambda}\right) d\mu(x) < \infty.$$

Two remarks are in order: since the fundamental domains have finite measures, the φ-integrability of the distance functions is only sensitive to the speed at which φ tends to infinity. On the other hand, the constants $c_\gamma > 0$ are partly motivated by the fact that we want our notion of φ-integrability to be independent of the choice of generating subset. We address these technical points in Section 2.3.

For consistency with the literature, we write L^p to replace the map $t \mapsto t^p$ for $p \in (0, \infty)$ and L^∞ if the distance maps are essentially bounded. If no assumption is made on the distance maps we shall write L^0. Also we won't always use the adjective integrable for the sake of brevity.

Saying that two groups are L^p orbit equivalent for some $p \geqslant 1$ means in our terminology that there exists an $(\mathrm{L}^p, \mathrm{L}^p)$ orbit equivalence coupling between them, and the same is true for measure equivalence. Note that for $p < 1$, what one would like to call L^p orbit equivalence fails to be a transitive relation, so we will refrain from using such a terminology. Nevertheless, such a notion actually defines a very natural pseudo-distance on the space of all finitely generated groups (see Sec. 1.4), and we will see for instance that the Heisenberg group and $\mathbb{Z}^4$ are at distance zero, while the distance between $\mathbb{Z}^n$ and $\mathbb{Z}^m$ is equal to $|\log(n) - \log(m)|$.

The rest of this introduction contains a summarized description of our main results, and is organized in four subsections. The first two ones are about amenable groups: §1.1 contains a general rigidity result related to the isoperimetric profile, while §1.2 has a flexibility flavor as it deals with constructions

of concrete orbit equivalences. In §1.3, we state a rigidity result for hyperbolic groups. Finally, in 1.4 we briefly discuss a point of view that emerges from the notion of (φ, ψ)-integrable coupling (this can be seen as an introduction to the results of Section 2).

1.1 Isoperimetric profile

Since finitely generated solvable groups that are not virtually nilpotent are amenable of exponential growth, it turns out that none of the known results distinguishes them up to L^1 measure equivalence. By contrast the isoperimetric profile and the probability of return of random walks have been thoroughly investigated and their asymptotic behaviors are known for a wide class of solvable groups. Let Γ be a finitely generated group, and let S_Γ be a finite generating subset. Given a function $f\colon \Gamma \to \mathbb{R}$ and $p > 0$ we define the L^p-norm of its left gradient by the equation

$$\left\|\nabla^l_{S_\Gamma} f\right\|_p^p = \sum_{g\in\Gamma, s\in S} |f(sg) - f(g)|^p.$$

Given a non-empty finite subset $A \subset \Gamma$, we define

$$j_p(A) = \inf_f \frac{\|\nabla^l_{S_\Gamma} f\|^p}{\|f\|_p},$$

where the infimum is taken over non-zero functions supported on A. When there is no ambiguity, we shall simply denote $\nabla^l_{S_\Gamma} f = \nabla f$. For every $0 < p \leqslant \infty$, the ℓ^p-isoperimetric profile is the non-increasing function

$$I_{p,\Gamma}(n) = \inf_{|A|\leqslant n} j_p(A).$$

Of course this quantity depends on a choice of generating subset, but we will be interested in the asymptotic behavior of $I_{p,\Gamma}$ which does not.

Given two monotonous real-valued functions f and g we say that f is *asymptotically less* than g and write $f \preccurlyeq g$ if there exists a positive constant C such that $f(n) = O(g(Cn))$ as $n \to \infty$. We say that f and g are *asymptotically equivalent* and write $f \asymp g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$. The asymptotic behavior of f is its equivalence class modulo $\asymp$.

Recall that for $p = 1$, the isoperimetric profile has the following geometric interpretation [CS93]:

$$I_{1,\Gamma}(n) \asymp \inf_{|A|\leqslant n} \frac{|\partial A|}{|A|},$$

where $\partial A = S_\Gamma A \triangle A$.

For $p = 2$, the asymptotic behavior of $I_{2,\Gamma}$ is intimately related to the probability of return of the simple random walks on the Cayley graph (Γ, S_Γ) as described in[CS93].

Our first main contribution is to prove the following theorem.

Theorem 1. *Assume that the finitely generated groups Γ and Λ admit a (φ, L^0) measure equivalence coupling. Then*

- *if $\varphi(t) = t^p$ for some $p \geqslant 1$, then*
$$I_{p,\Gamma} \preccurlyeq I_{p,\Lambda};$$
- *if φ and $t \mapsto t/\varphi(t)$ are increasing then*
$$I_{1,\Gamma} \preccurlyeq \frac{1}{\varphi(1/I_{1,\Lambda})}.$$

In particular, the ℓ^p-isoperimetric profile is stable under L^p measure equivalence for all $p \geqslant 1$. We deduce for instance that $\mathbb{Z} \wr \mathbb{Z}$ is not L^1 measure equivalence to the lamplighter groups, nor to any polycyclic group. In a recent groundbreaking work, Brieussel and Zheng managed to construct amenable groups with a prescribed isoperimetric profile [BZ15]. In particular, they show that for every increasing unbounded function φ, there exists an amenable group whose isoperimetric profile does not dominate φ. In combination with the previous theorem, we deduce that Orstein-Weiss' orbit equivalence coupling between amenable groups can be as poorly integrable as possible. More precisely, we obtain the following corollary.

Corollary 2. *For every amenable finitely generated group Γ and every increasing unbounded function φ, there exists an amenable finitely generated group Λ such that there are no (φ, L^0) measure equivalence coupling between them.*

This corollary triggers the question whether for all amenable finitely generated groups Γ there always exists a (φ, L^0) integrable orbit equivalence coupling from Γ to $\mathbb{Z}$ such that $\varphi(I_{1,\Gamma}(1/n)) \preccurlyeq 1/n$. We shall see in the next subsection that this holds up to a logarithmic error in various examples.

The following result turns out to be a special case of a more general monotonicity property of the ℓ^p-isoperimetric profile (Theorems 4.1 and 4.2) which encompasses its well-known monotonicity under subgroups and quotients. Writing down this statement motivated us to introduce and study natural "measured generalizations" of the notions of subgroup, quotient, subgroup of a quotient. Let us only mention here a striking application of this monotonicity result. Recall that a regular embedding $f : \Gamma \to \Lambda$ is a Lipschitz map whose pre-images have cardinality bounded by some $m \in \mathbb{N}$. Particular cases of regular embeddings are Lipchitz injective embeddings and coarse embeddings. Combining Theorem 4.1 with a trick essentially due to Shalom, we obtain the following rather unexpected fact.

Theorem 3 (see Cor. 5.6)**.** *For every $1 \leqslant p \leqslant \infty$, the ℓ^p-isoperimetric profile is monotonous under regular embedding between amenable groups.*

The mere monotonicity under coarse embedding is already very surprising and actually false if the embedded group is not assumed to be amenable: for instance the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ admits a quasi-isometrically embedded 4-regular tree. Hence, the free group on 2 generators quasi-isometrically (and hence coarsely) embeds into the lamplighter group. But the free group is non-amenable and therefore its isoperimetric profile is bounded below, while the isoperimetric profile of the lamplighter, which is amenable, tends to zero.

Very few geometric invariants are known to be monotonous under coarse embedding: the only known examples being the volume growth, the asymptotic dimension, the separation profile [BST12], and more recently the Poincaré profiles [HMT19]. All of these turn out to be monotonous under regular embedding as well (see [BST12] for the case of asymptotic dimension). For solvable groups of exponential growth, the asymptotic dimension is generally infinite, and the Poincaré profiles have only been computed in very specific examples. By contrast, our theorem provides us with a powerful obstruction. As a concrete application, it prevents the existence of a regular embedding from $\mathbb{Z} \wr \mathbb{Z}^a$ to $\mathbb{Z} \wr \mathbb{Z}^b$ as soon as $a > b$ (note that these groups have infinite asymptotic dimension). On the other hand, combining it with a construction of Erschler and Zheng [EZ17, Cor. 3.3] (see also [BZ15, Thm. 1.1]) we obtain the following result.

Theorem 4 (see Cor. 5.8)**.** *There exists an uncountable family of 3-step solvable groups Γ_i that do not pairwise regularly embed into one another, and that are pairwise non L^1 measure equivalent. Moreover, one can assume that these groups all have asymptotic dimension one.*

Using a monotonicity property of the volume growth essentially due to Bowen (see Theorem 3.2), one can similarly produce uncountable families of groups (of intermediate growth) that do not pairwise coarsely embed into one another, and that are pairwise non L^1 measure equivalent. Other examples of uncountable families of groups that pairwise do not coarsely embed into one another are due to Hume [Hum17, Thm. 1.2]: they involve (non-amenable) groups that contain isometrically embedded families of expanders.

1.2 Construction of measure couplings between amenable groups

Most of the known examples of L^p measure equivalence couplings come from lattices in semi-simple Lie groups. Here we introduce a general method for constructing orbit equivalent couplings with prescribed integrability conditions between amenable groups. Our main tool is a notion of Følner tiling, developed in §6.1. A Følner tiling sequence in a group Γ is a sequence of finite subsets T_n satisfying three conditions:

(i) (T_n) is a left Følner sequence for Γ,

(ii) each T_n is obtained as a union of right translates of T_{n-1}.

We note that Følner sequences T_n such that Γ can be obtained as a disjoint union of right translates of T_n do exist in all elementary amenable groups and all residually finite amenable groups thanks to the work of Weiss [Wei01]. Our second condition appears to be new, and is crucial in our construction. An easy example of a Følner tiling sequence is provided by $T_n = \{1, ..., 2^n\}$ in $\mathbb{Z}$.

Starting from a Følner tiling sequence for a group Γ, we construct a p.m.p. action of Γ on a certain infinite product space such that the equivalence relation generated by the action is the co-finite equivalence relation (in the above example the action is given by the dyadic odometer). It follows from our construction that if another group Λ admits a Følner tiling sequence whose basic tiles T'_n satisfy $|T_n| = |T'_n|$, then the corresponding action of Λ is orbit equivalent to that of Γ, and the degree of integrability of this orbit equivalence is controlled by the properties of the two Følner sequences (T_n) and (T'_n). In all cases considered in this paper, the actions constructed by means of Følner tiling sequences are profinite actions but it seems likely that non profinite actions can be built in this manner as well. To the best of our knowledge, this way to produce orbit equivalences between amenable groups is new, and quite different from the quasi-tiling machinery of Ornstein and Weiss [OW87]. Indeed, the latter allows one to prove that *all* measure-preserving ergodic actions of *all* amenable groups are orbit equivalent.

Theorem 3 provided us with an obstruction for finding φ-integrable couplings with certain functions φ between two amenable groups. We will now show that in many cases of interest (and especially in the case of couplings from certain amenable groups to $\mathbb{Z}$) this obstruction is close to being optimal.

Let us start with the case of groups with polynomial growth. We deduce from a straightforward extension of Bowen's theorem on volume growth (see Corollary 3.4), or alternatively from the monotonicity of the isoperimetric profile (Corollary 4.4), that $\mathbb{Z}^m$ and $\mathbb{Z}^n$ do not admit L^p measure equivalence couplings for $p > m/n$ if $n < m$. We show that this threshold is sharp:

Theorem 5 (see Thm. 6.9)**.** *For every positive integers $n < m$, there exists an orbit equivalence coupling between $\mathbb{Z}^m$ and $\mathbb{Z}^n$ that is L^p for every $p < n/m$.*

To be more precise the proof gives an orbit equivalence from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ which is $(L^p, L^{1/p})$ for all $p < n/m$. This asymmetry is not surprising given that Bowen's obstruction only concerns the cocycle from the large group to the smaller one. Note that this leaves open the following question (see also Question 6.11).

Question 1.1. Let $n < m$, do $\mathbb{Z}^m$ and $\mathbb{Z}^n$ admit an $(L^{n/m}, L^0)$ measure equivalence coupling?

Regarding groups with same degree of growth, Austin has proved that L^1 measure equivalent groups of polynomial growth have bi-Lipschitz asymptotic cones [Aus16]. Combined with a famous theorem of Pansu [Pan89, Thm. 3], this implies for instance that a non-virtually abelian nilpotent group cannot be L^1 measure equivalent to an abelian group. By contrast we show the following surprising converse for the Heisenberg group (see Theorem 6.13).

Theorem 6. *There exists an orbit equivalent coupling between $\mathbb{Z}^4$ and $\mathbb{H}(\mathbb{Z})$ that is L^p for every $p < 1$.*

Let us now turn to groups with exponential growth. First, by a straightforward application of Corollary 7.4, we obtain the following theorem.

Theorem 7. *For every $n < m$, and every non-trivial finite group F there exists an $(\mathrm{L}^p, \mathrm{L}^{1/p})$ orbit equivalence coupling from $F \wr \mathbb{Z}^m$ to $F \wr \mathbb{Z}^n$ for every $p < n/m$.*

On the other hand, we deduce from the monotocity of the isoperimetric profile that there exists no $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $F \wr \mathbb{Z}^m$ to $F \wr \mathbb{Z}^n$ for $p > 1 - 1/n + 1/m$. Note that this is sharp for $n = 1$.

We now focus on the existence of couplings with $\mathbb{Z}$. We first consider coupling between $\mathbb{Z}$ with the lamplighter group. Since the isoperimetric profile of the lamplighter group decays like $1/\log n$, we know that a (φ, L^0) coupling from the latter to $\mathbb{Z}$ must satisfy $\varphi(t) \preccurlyeq \log t$. This turns out to be sharp up to a logarithmic error.

Theorem 8. *For every finite group F, there exits an orbit equivalence coupling from $F \wr \mathbb{Z}$ to $\mathbb{Z}$ which is $(\varphi_\varepsilon, \exp)$-integrable for every $\varepsilon > 0$, where $\varphi_\varepsilon(x) = \frac{\log(x)}{\log(\log(x))^{1+\varepsilon}}$.*

Once again, we observe that this rather asymmetric condition is coherent with the fact that the main restriction should lie on the cocycle from the large group to the smaller one.

More generally, one can consider iterated wreath products as follows. Let $G_0 = \mathbb{Z}$, and for all $n \geqslant 1$, $G_n = F \wr G_{n-1}$. By [Ers03], we know that the isoperimetric profile of G_k is asymptotically equivalent to $\log^{\star n}$, namely the n-times iteration of log with itself. Similarly we show that this is almost sharp: see Corollary 7.6.

We end this subsection considering groups that have very similar geometric properties. The lamplighter group $(\mathbb{Z}/k\mathbb{Z}) \wr \mathbb{Z}$ and the Baumslag-Solitar group $\mathrm{BS}(1, k)$ have isometric asymptotic cones, and same isoperimetric profile. We therefore expect to be able to produce a "good" coupling between them.

Theorem 9. *For every $k \geqslant 2$, there exists an $(\mathrm{L}^\infty, \exp)$-integrable orbit equivalence coupling from $(\mathbb{Z}/k\mathbb{Z}) \wr \mathbb{Z}$ to $\mathrm{BS}(1, k)$.*

The asymmetry is not an artifact of the proof: indeed we could not have an $(\mathrm{L}^\infty, \exp)$-orbit equivalence coupling from $\mathrm{BS}(1, k)$ to $(\mathbb{Z}/k\mathbb{Z}) \wr \mathbb{Z}$ as the asymptotic dimension is monotonous under $(\mathrm{L}^\infty, \mathrm{L}^0)$ orbit equivalence (Corollary 5.5), and $\mathrm{BS}(1, k)$ has asymptotic dimension 2, while $(\mathbb{Z}/k\mathbb{Z}) \wr \mathbb{Z}$ has asymptotic dimension 1. Moreover, this shows that the asymptotic dimension is not preserved under L^p orbit equivalence for all $p < \infty$. Considering slightly more sophisticated examples, we similarly show that the finiteness of the asymptotic dimension is not preserved either. Finally, observe that $\mathrm{BS}(1, k)$ is finitely presented while $(\mathbb{Z}/k\mathbb{Z}) \wr \mathbb{Z}$ is not, hence finite presentability is also unstable.

1.3 Rigidity of hyperbolicity

Shalom proved that any two lattices in $\mathrm{SO}(n, 1)$ are L^p measure equivalent for all $p < n - 1$. More precisely, let $n \geqslant 2$, and let Γ (resp. Λ) be a uniform (resp. non-uniform) lattice in $\mathrm{SO}(n, 1)$. We consider the coupling associated to the action of Λ and Γ respectively by left and right-translations on the measure space $\mathrm{SO}(n, 1)$ equipped with an invariant Haar measure. Shalom showed that for a suitable fundamental domain X_Λ for Λ and any relatively compact fundamental domain X_Γ for Γ, the resulting coupling is an $(\mathrm{L}^p, \mathrm{L}^\infty)$ measure equivalence coupling from Γ to Λ for all $p < n - 1$. An additional property of this coupling is what we call *coboundedness*[1]: there is a finite subset $F \subset \Lambda$ such that $X_\Gamma \subset F X_\Lambda$. Here we show that this strong formulation of Shalom's result is in some sense sharp.

Theorem 10. *Let Γ be a finitely generated δ-hyperbolic group. There exists p only depending[2] on δ and $\mathrm{Ent}(\Gamma)$ such that if there exists a cobounded $(\mathrm{L}^p, \mathrm{L}^\infty)$ measure equivalence coupling from Γ to a finitely generated group Λ, then Λ is also hyperbolic.*

We immediately deduce the following corollary.

[1] Observe that every orbit equivalence coupling is cobounded.

[2] More precisely $p \geqslant 75\delta\,\mathrm{Ent}(\Gamma)$ suffices although we did not try to get the best possible bound.

Corollary 11. *Assume that Γ is a uniform lattice in center-free, real rank 1 simple Lie group G and Λ is another lattice of G. There exists p only depending on G such that if there exists a cobounded $(\mathrm{L}^p, \mathrm{L}^\infty)$ coupling from Γ to Λ, then Λ must be uniform as well.*

Actually, Theorem 10 is a special case of a more general statement where the two integrability conditions are linked via a complicated condition that we shall not state here. Let us only mention another application of this result.

Theorem 12. *Let Γ be a finitely generated hyperbolic group. For every $p > q > 0$ such that if there is a cobounded (φ, ψ) integrable measure equivalence coupling from Γ to any finitely generated group Λ where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is also hyperbolic.*

Note that this implies in particular that hyperbolicity is preserved under exponential orbit equivalence couplings (as the exponential integrability is much stronger than the one in Theorem 10, and coboundedness is implied by orbit equivalence). Both theorems should be compared with of a theorem of Bowen saying that if there exists an $(\mathrm{L}^1, \mathrm{L}^0)$ measure equivalence coupling from a finitely generated accessible group Λ to a virtually free group such that the cocycle from Λ to Γ is at most m-to-one[3] for some integer m, then Λ is virtually free.

Once again we deduce the following corollary for Lattices in rank 1 simple Lie groups.

Corollary 13. *Assume that Γ is a uniform lattice in a center-free, real rank 1 simple Lie group G and Λ is another lattice of G. For every $\eta > 0$ such that if there is a cobounded (φ, ψ) coupling from Γ to Λ, where $\varphi(t) = \exp(t^\eta)$ and $\psi(t) = t^{1-1/\eta}$, then Γ is uniform as well.*

In Corollaries 11 and 13, the case of $\mathrm{SL}(2, \mathbb{R})$ was already known and actually a much stronger conclusion holds in that case: Bader, Furman and Sauer have proved that non-uniform lattices and uniform ones are not L^1 measure equivalent.

As observed by Mikael de la Salle, Corollaries 11 and 13 are in sharp contrast with what happens for lattices in higher rank simple Lie groups: indeed if Γ and Λ are lattices in a simple Lie group Γ of rank $\geqslant 2$, then if X_Λ and X_Γ are Dirichlet fundamental domains for Λ and Γ, the resulting cocycles are exponentially integrable [dlS19, Lemme 5.6]. Hence if Γ is assumed to be uniform, this provides us with a $(\varphi, \mathrm{L}^\infty)$ measure equivalence coupling from Γ to Λ, where $\varphi(t) = \exp(ct)$ for some $c > 0$.

Question 1.2. Assume $n \geqslant 3$, and let Γ (resp. Λ) be a uniform (resp. non-uniform) lattice in $\mathrm{SO}(n, 1)$. Are Γ and Λ L^p measure equivalent for all $p < \infty$? Or at the opposite are they not L^p-measure equivalent as soon as $p > n - 1$ (or maybe even $p = n - 1$)?

The coboundedness condition seems to plays a crucial role in the proof of Theorem 10 as it enables us to combine the two cocycles together.

Remark 1.3. We pause here to mention that coboundedness was considered by Sauer in his PhD thesis [Sau02]. He proved that two amenable finitely generated groups are quasi-isometric if and only if they admit an L^∞ coupling such which is cobounded in both directions: namely there exist finite subsets $F_\Lambda \subset \Lambda$ and $F_\Gamma \subset \Gamma$ such that $X_\Gamma \subset F_\Lambda X_\Lambda$ and $X_\Lambda \subset F_\Gamma X_\Gamma$. While the L^∞ condition implies that both associated cocycle $\alpha(\cdot, x)$ and $\beta(\cdot, x)$ are almost surely C-Lipschitz (with C independent of x), this additional conditions implies that both cocycle $\alpha(\cdot, x)$ and $\beta(\cdot, x)$ are a.e. quasi-isometries (with constants that are uniform with respect to x).

This leads us to the following maybe surprising question.

Question 1.4. Is hyperbolicity invariant under L^∞ measure equivalence?

[3]This condition turns out to be weaker than coboundedness.

1.4 Remarks on the "space" of finitely generated groups

We may view the functions φ and ψ for which there exists a (φ, ψ) measurable coupling from a group Γ to a group Λ as an asymmetric way of quantifying how geometrically close these groups are from one another. This is of course only interesting within a class of groups that are measure equivalent, such as lattices (uniform and non-uniform ones) in a given locally compact group, or among amenable groups.

Let us illustrate this point of view in the following special case: given two finitely generated groups Γ and Λ, let $\alpha(\Gamma, \Lambda)$ be the infimum of $-\log p$ over all $p \leqslant 1$ such that there exists an $(\mathrm{L}^p, \mathrm{L}^p)$ integrable measure equivalence coupling between Γ and Λ. It follows from Proposition 2.27 that α satisfies the triangle inequality. Therefore, this can be viewed as a pseudo-distance between finitely generated groups. Note that Theorem 6 yields $\alpha(\mathbb{Z}^4, \mathrm{Heis}(\mathbb{Z})) = 0$, while Theorem 5 and the remark that precedes it yield $\alpha(\mathbb{Z}^n, \mathbb{Z}^m) = |\log n - \log m|$.

It is interesting to note that for $p \geqslant 1$ (and more generally for convex functions φ), things behave differently, as admitting an $(\mathrm{L}^p, \mathrm{L}^p)$ integrable measure equivalence coupling becomes an equivalence relation. This means that balls of radius 0 for α can be equipped with an ultrametric distance $\beta(\Gamma, \Lambda)$ defined for instance by the infimum of $1/p$ over all $p \geqslant 1$ such that the groups are L^p measure equivalent. In other words, we can distinguish two scales: a "large" scale measured by the pseudo distance α, and a finer scale, measured the ultrametric pseudo-distance β.

More generally, we will see in §2.3.2 that if the functions φ and ψ are concave, then (φ, ψ)-integrable measurable equivalence couplings satisfy a natural composition rule, which we may view as analogous to a "triangular inequality". Moreover, as for the case of L^p integrability, we shall see that for functions that grow faster than linear, e.g. $t \to t^p$ for $p \geqslant 1$, or $t \to \exp(t^a)$ for $a > 0$, one has to adopt a different point of view. Indeed, a stronger version of the integrability condition is required (which is automatic for L^p) in order for them to be well-behaved under composition. Actually under this stronger condition, we also get a stronger conclusion, which is analogous to an ultrametric property.

As a matter of fact, these composition rules are better stated in an even more asymmetric situation, where the measure coupling is no longer a measure *equivalence* coupling. If we drop on one side the finiteness of the measure of the fundamental domain, we obtain a natural notion of measured subgroup, while if we drop the freeness for one of the actions, we obtain a measurable notion of quotient. Combining them, we get a measurable notion of subquotient. In these situations, an integrability condition only makes sense in one direction, so that we obtain notions of φ-integrable measurable subgroups, quotients or subquotients, which we develop in the next section. To see how these notions can be useful, note that Theorem 3 is a corollary of the monotonicity of the isoperimetric profile under L^1 measure subquotient.

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2 Variations on measure equivalence

Convention. Throughout the paper, we allow metrics and pseudo-metrics to take the value $+\infty$.

2.1 Smooth actions and fundamental domains

A **standard Borel measure space** (Ω, μ) is a Borel space $(\Omega, \mathcal{B}(\Omega))$, frequently also known as a measurable space, equipped with a σ-finite measure μ on the σ-algebra $\mathcal{B}(\Omega)$ consisting of the the Borel subsets of some Polish (separable and completely metrizable) topology on Ω. The elements of $\mathcal{B}(\Omega)$ are called **Borel**. A Borel subset $\Omega_0 \subseteq \Omega$ is said to be **conull** or of **full measure** if $\mu(\Omega \setminus \Omega_0) = 0$ and a property that holds for all $\omega \in \Omega_0$ is said to hold **almost surely** or **for almost every** $\omega \in \Omega$. The space (Ω, μ) is said to be a standard Borel probability space if $\mu(\Omega) = 1$.

A **measure-preserving action** of a discrete countable group Γ on (Ω, μ), for short $\Gamma \curvearrowright (\Omega, \mu)$, is a Γ-action on Ω such that the action map $(\gamma, x) \mapsto \gamma \cdot x$ is Borel and that $\mu(\gamma \cdot E) = \mu(E)$ for all $\gamma \in \Gamma$

and all Borel $E \subseteq \Omega$. A measure-preserving action on a standard Borel probability space is said to be probability measure-preserving.

Since the groups that we are dealing with are countable, if we are given $\Gamma \curvearrowright (\Omega, \mu)$, then every full measure Borel subset $A \subseteq \Omega$ contains a Γ-invariant full measure Borel subset, namely the set $\bigcap_{\gamma \in \Gamma} \gamma \cdot A$. Moreover, since any full measure Borel subset of a standard Borel measure space is a standard Borel measure space, there will be no harm in considering that some properties that hold almost everywhere actually hold everywhere, so we will often do so without explicitly restricting to a full measure Borel subset. For instance, one says that a measure-preserving Γ-action on (Ω, μ) is *free* if for almost every $x \in \Omega$, we have $\gamma \cdot x = x$ if and only if $\gamma = e_\Gamma$, but we may as well assume that the latter implication holds for *every* $x \in \Omega$ since this becomes true once we restrict the action to a full measure Γ-invariant Borel set.

Finally, given $\Gamma \curvearrowright (\Omega, \mu)$, the **full pseudo-group** of the action $\Gamma \curvearrowright \Omega$ is the set of all partially defined Borel bijections $\varphi : A \to B$, where A and B are Borel subsets of Ω, such that for all $x \in A$, we have $\varphi(x) \in \Gamma \cdot x$. Every such element is measure-preserving, see [KM04, Prop. 2.1].

Definition 2.1. A **fundamental domain** for $\Gamma \curvearrowright (\Omega, \mu)$ is a Borel set $X_\Gamma \subseteq \Omega$ which intersects almost every Γ-orbit at exactly one point: there is a full measure Γ-invariant Borel set $\Omega_0 \subseteq \Omega$ such that for all $x \in \Omega_0$ we have that the intersection of the Γ-orbit of x with X_Γ is a singleton

Equivalently, a Borel set $X_\Gamma \subseteq \Omega$ is a fundamental domain if and only if the quotient map $\pi : X_\Gamma \to \Omega/\Gamma$ is a bijection. Note that since a fundamental domain intersects almost every orbit, the union of its translates has full measure, so every fundamental domain must have positive measure. Also note that since a fundamental domain for $\Gamma \curvearrowright (\Omega, \mu)$ intersects almost every orbit, the union of its translates has full measure, so every fundamental domain must have strictly positive measure whenever this is the case with Ω.

Moreover, after possibly passing to a Γ-invariant Borel subset of full measure, the existence of a Borel set $X_\Gamma \subseteq \Omega$ that intersects every Γ-orbit exactly once is equivalent with the fact that Ω/Γ is standard Borel, or in other words that the quotient map $\pi : X_\Gamma \to \Omega/\Gamma$ is a Borel bijection between standard Borel spaces. In this case, the orbit equivalence relation is said to be smooth, and we make the following definition.

Definition 2.2. A measure-preserving action of a countable group Γ on a standard measured space (Ω, μ) is **smooth** if it admits a fundamental domain.

Given a smooth action $\Gamma \curvearrowright (\Omega, \mu)$, if X is a fundamental domain for the Γ-action, we denote by π_X the map which takes (almost) every $\omega \in \Omega$ to the unique element of the Γ-orbit of ω which belongs to X. Observe that by definition, if X_1 and X_2 are two measure-fundamental domains, then the restriction of π_{X_1} to X_2 is an element of the full pseudo-group of the Γ-action whose inverse is the restriction of π_{X_2} to X_1.

In particular, X_1 and X_2 have the same measure, and so given any smooth action $\Gamma \curvearrowright (\Omega, \mu)$, we can unambiguously endow the quotient space Ω/Γ with the measure obtained by identifying it with one of the fundamental domains. We will still denote this measure by μ.

Finally, given a fundamental domain X, we denote by ι_X the inverse of the projection map $X \to \Omega/\Gamma$.

Convention. We shall use the notation "$\gamma * x$" instead of "$\gamma \cdot x$" for smooth Γ-actions. This distinction will prove useful later on since we will also have induced actions on fundamental domains, which we will denote by $\cdot$.

2.2 Asymmetric couplings

We begin by introducing an asymmetric variation of what has been called a coupling by various authors.

Definition 2.3. Let Γ and Λ be countable groups, a **measure subquotient coupling** from Γ to Λ is a triple (Ω, X_Λ, μ), where (Ω, μ) is a standard Borel measure space equipped with commuting measure-preserving smooth Γ and Λ actions such that

(i) the Γ action is free;

(ii) X_Λ is a fixed fundamental domain for the Λ-action;

(iii) X_Λ has finite measure.

We say that Γ is a **measure subquotient** of Λ when there exists a subquotient coupling from Γ to Λ.

Note that the definition is asymmetric since we only specify the Λ fundamental domain and only require Γ to be acting freely. This definition is motivated by the following situation: if N is a normal subgroup of Λ and Γ is a a subgroup of Λ/N, then $(\Lambda/N, \{N\}, c)$ is a measure subquotient coupling from Γ to Λ, where c is the counting measure, Λ acts by right-translation and Γ by left translation.

Given a measure subquotient coupling (Ω, X_Λ, μ) from Γ to Λ, we denote their two commuting actions by $*$, and then we have a natural action of Γ on Ω/Λ which we call the **induced action** and denote by $\cdot$. Through the natural identification of Ω/Λ to X_Λ, this induced action is given by: for all $x \in X_\Lambda$, $\gamma \in \Gamma$,

$$\{\gamma \cdot x\} = \Lambda * \gamma * x \cap X_\Lambda.$$

Note that the induced Γ-action defines elements of the pseudo full group of the Λ-action on Ω, so it is a measure-preserving action. We can now present measure equivalence as follows.

Definition 2.4. Let Γ and Λ be countable groups, a **measure equivalence coupling** from Γ to Λ is a quadruple $(\Omega, X_\Gamma, X_\Lambda, \mu)$, where (Ω, μ) is a standard Borel measure space equipped with commuting measure-preserving smooth Γ and Λ actions such that (Ω, X_Λ, μ) is a measure subquotient coupling from Γ to Λ and (Ω, X_Γ, μ) is a measure subquotient coupling from Λ to Γ. When there exists such a coupling, we say that Γ and Λ are **measure equivalent**.

Remark 2.5. Although the above definition is symmetric, we still talk about measure equivalence coupling *from* a group *to* another one because we will soon put some asymmetric restrictions on the two corresponding measure subquotient couplings.

We finally introduce two natural intermediate notions.

Definition 2.6. A group Γ is a **measure subgroup** of Λ if there is a measure subquotient coupling (Ω, X_Λ, μ) from Γ to Λ such that the Λ-actions is also free. Such a coupling is called a **measure subgroup coupling** from Γ to Λ.

Note that every measure equivalence coupling is actually a measure subgroup coupling in both directions.

Definition 2.7. A group Γ is a **measure quotient** of Λ if there is a measure subquotient coupling (Ω, X_Λ, μ) from Γ to Λ such that the Γ-action also admits a fundamental domain of finite measure. Such a coupling is called a **measure quotient coupling**.

We now recall how to compose couplings; for this to work, we need the middle group to act freely in both couplings. Let us first give the definition and then check that it actually makes sense.

Definition 2.8. Let Γ, Λ and Σ be three countable groups let $(\Omega_1, X_{1,\Lambda}, \mu_1)$ be a subgroup coupling from Γ to Λ and let $(\Omega_2, X_{2,\Sigma}, \mu_2)$ be a measure subquotient coupling from Λ to Σ. The **composition** of these two couplings $(\Omega_3, X_{3,\Sigma}, \mu_3)$ is the measure subquotient coupling from Γ to Σ obtained as follows: we consider the diagonal action $\Lambda \curvearrowright (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ which is smooth and commutes with the Γ-action on the first coordinate and the Λ-action on the second coordinate. Then the measured space of our new coupling is $\Omega_3 := (\Omega_1 \times \Omega_2)/\Lambda$ equipped with the measure μ_3 obtained by identifying it with a Λ-fundamental domain, and equipped with the induced Γ and Λ actions, which are both smooth, and we let $X_{3,\Sigma} = \pi_{\Omega_3}(X_{1,\Lambda} \times X_{2,\Sigma})$.

Similarly, when we are given two measure equivalence or measure quotient couplings $(\Omega_1, X_{1,\Gamma}, X_{1,\Lambda}, \mu_1)$ and $(\Omega_2, X_{2,\Lambda}, X_{2,\Sigma}, \mu_2)$, their composition $(\Omega_3, X_{3,\Gamma}, X_{3,\Sigma}, \mu_3)$ is obtained by the same construction as above, letting furthermore $X_{3,\Gamma} = \pi_{\Omega_3}(X_{1,\Gamma} \times X_{2,\Lambda})$.

Proposition 2.9. *Let Γ, Λ and Σ be three countable groups, let $(\Omega_1, X_{1,\Lambda}, \mu_1)$ be a measure subgroup coupling from Γ to Λ and let $(\Omega_2, X_{2,\Sigma}, \mu_2)$ be a measure subquotient coupling from Λ to Σ. The composition of these two couplings is a measure subquotient coupling from Γ to Σ.*

If both couplings are measure subgroup (resp. measure equivalence) couplings, then their composition is also a measure subgroup (resp. measure equivalence) coupling.

Finally if the coupling from Γ to Λ is a measure equivalence coupling and the coupling from Λ to Σ is a measure quotient coupling, then their composition is a measure quotient coupling.

Proof. We first check that the diagonal Λ-action on $\Omega_1 \times \Omega_2$ is smooth: indeed $X_{1,\Lambda} \times \Omega_2$ is a fundamental domains for it. In particular, it induces the measure μ_3 on Ω_3. We denote by $\star$ the induced Γ-actions on $\Omega_3 := (\Omega_1 \times \Omega_2)/\Lambda$. at most m-to-one Through the identification of Ω_3 with the fundamental domain $X_{1,\Lambda} \times \Omega_2$ we see that the induced Σ-action is given by : for all $(x, \omega) \in X_{1,\Lambda} \times \Omega_2$ and all $\sigma \in \Sigma$, $\sigma \star (x, \omega) = (x, \sigma * \omega)$. In particular, we see that $X_{3,\Sigma} := \pi_{\Omega_3}(X_{1,\Lambda} \times X_{2,\Sigma})$ is a measured fundamental domain for it, that the Σ-action on Ω_2 is free then so is the Σ-action on Ω_3, and that $X_{3,\Sigma}$ has finite measure because $X_{2,\Sigma}$ does. Finally, if we fix a fundamental domain $X_{2,\Lambda}$ for the Λ-action on Ω_2, then we also have a natural identification of Ω_3 with $\Omega_1 \times X_{2,\Lambda}$ through which we see that the Γ-action on Ω_3 is free.

Now all the properties stated in the proposition can directly be inferred from the two previous paragraphs. □

Remark 2.10. Keeping the notation of the proof, we can also describe the Γ-action on Ω_3 when the latter is identified with $X_{1,\Lambda} \times \Omega_2$ using the cocycle $\alpha \colon \Gamma \times X_{1,\Lambda} \to \Lambda$ given by the equation $\alpha(\gamma, x) * \gamma * x = \gamma \cdot x$ as follows

$$\gamma \star (x, \omega) = (\gamma \cdot x, \alpha(\gamma, x) * \omega),$$

where $\cdot$ denotes the induced Γ-action on $X_{1,\Lambda}$. Note that the induced Γ-action (denoted by $\bullet$) on the fundamental domain $X_{3,\Sigma}$, identified to $X_{1,\Lambda} \times X_{2,\Sigma}$, is then given by:

$$\gamma \bullet (x, y) = (\gamma \cdot x, \alpha(\gamma, x) \cdot y)$$

This point of view will be useful when we explore how composition of couplings behaves when we put integrability conditions.

Definition 2.11. Given $m \in \mathbb{N}$, we say that a measure subquotient coupling (Ω, X_Λ, μ) from Γ to Λ is **at most m-to-one** if for every $x \in X_\Lambda$ the map $\gamma \mapsto \gamma^{-1} * (\gamma \cdot x) \in \Lambda * x$ has pre-images of size at most m.

When $m = 1$, we also say that we have an **injective** measure subquotient coupling. Such couplings can also be characterized as follows.

Proposition 2.12. *A measure subquotient coupling (Ω, X_Λ, μ) from Γ to Λ is injective if and only if X_Λ intersects every Γ-orbit at most once.*

Proof. Suppose first that X_Λ intersects each Γ-orbit at most once. Let $x \in X_\Lambda$, suppose that $\gamma_1^{-1} * \gamma_1 \cdot x = \gamma_2^{-1} * \gamma_2 \cdot x$. Then $\gamma_1 \cdot x$ and $\gamma_2 \cdot x$ are two elements of the same Γ-orbit which belong to X_Λ so by our assumption they are equal. Since the Γ-action on Ω is free, we conclude that $\gamma_1 = \gamma_1$ as wanted.

Conversely, suppose that the coupling is injective. Let $x \in X_\Lambda$, suppose that $\gamma * x \in X_\Lambda$ for some $\gamma \in \Gamma$. Then $\gamma \cdot x = \gamma * x$, so $\gamma^{-1} * \gamma \cdot x = x$, and so by injectivity $\gamma = e_\Gamma$. We conclude that $\gamma * x = x$, so X_Λ intersects each Γ-orbit at most once as announced. □

An important example of injective coupling will be provided by the notion of *orbit subquotient couplings* (see Section 2.4). In fact, we will see that every injective measure subquotient coupling can be turned into such a coupling after making the induced action free (see Prop. 2.42).

Remark 2.13. When the Λ-action is free, the map $\gamma \mapsto \gamma^{-1} * (\gamma \cdot x) \in \Lambda * x$ is at most m-to-one if and only if the cocycle $\alpha : \Gamma \times X_\Lambda \to \Lambda$ defined by $\alpha(\gamma, x) * (\gamma * x) \in X_\Lambda$ satisfies that for all $x \in X_\Lambda$, the map $\gamma \mapsto \alpha(\gamma, x)$ is at most m-to-one.

Definition 2.14. A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ from Γ to Λ is called **cobounded** when there is a finite subset $F \subseteq \Lambda$ such that $X_\Gamma \subseteq FX_\Lambda$.

Example 2.15. We will see in Section 2.4 that every orbit equivalence coupling is cobounded.

2.3 Integrability conditions

In all that follows, we only deal with finitely generated groups, and if Λ is such a group, we will systematically denote by S_Λ one of its finite symmetric generating sets.

2.3.1 φ-equivalence between fundamental domains

Let us introduce a family of natural equivalence relations on fundamental domains of measure-preserving actions of a finitely generated group.

Definition 2.16. Given a finitely generated group Λ acting smoothly on a standard measured space (Ω, μ) and a finite symmetric generating set S_Λ for Λ, we denote by d_{S_Λ} the **Schreier graph metric** on the Λ-orbits, namely for $y \in \Lambda * x$, we let

$$d_{S_\Lambda}(x, y) = \min\{n \in \mathbb{N} \colon \exists s_1, ..., s_n \in S_\Lambda, y = s_1 \cdots s_n * x\}.$$

Observe that if we are given $\Lambda \curvearrowright (\Omega, \mu)$ and two finite generating sets S_1 and S_2 for Λ, then there is $C > 0$ such that for all $x \in \Omega$ and all $y \in \Lambda * x$,

$$\frac{1}{C} d_{S_1}(x, y) \leqslant d_{S_2}(x, y) \leqslant C d_{S_1}(x, y). \tag{1}$$

In the following definition, we recall that given a smooth action $\Lambda \curvearrowright (\Omega, \mu)$ and a measure fundamental domain X, the map ι_X is the inverse of the (bijective) projection $\pi_{\Omega/\Lambda} : X \to \Omega/\Lambda$.

Definition 2.17. Given any non-decreasing map $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ and two fundamental domains X_1 and X_2 of a smooth Λ-action on (Ω, μ), we say that they are **φ-equivalent** if there is $c > 0$ such that

$$\int_{\Omega/\Lambda} \varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x), \iota_{X_2}(x))}{c}\right) d\mu(x) < +\infty.$$

Note that this does not depend on the choice of the symmetric generating set S_Λ by virtue of inequation (1).

Remark 2.18. If φ satisfies that for every $c > 0$, there is a constant $C > 0$ such that for all $x \geqslant 0$, $\varphi(cx) \leqslant C\varphi(x)$, then X_1 and X_2 are φ-equivalent if and only if

$$\int_{\Omega/\Lambda} \varphi\left(d_{S_\Lambda}(\iota_{X_1}(x), \iota_{X_2}(x))\right) d\mu(x) < +\infty,$$

which is then also equivalent to: for *every* $c > 0$ we have

$$\int_{\Omega/\Lambda} \varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x), \iota_{X_2}(x))}{c}\right) d\mu(x) < +\infty.$$

This is the case if $\varphi(x) = x^p$ for some $p > 0$, or if φ is concave, or more generally subadditive. A motivating example where this is not true is when φ is the exponential map.

In order to show that φ-equivalence is indeed an equivalence relation, we introduce the following quantity: given a smooth Λ action and two fundamental domains X_1 and X_2, we let

$$c_{\varphi,S_\Lambda}(X_1,X_2)=\inf\left\{c>0\colon \int_{\Omega/\Lambda}\varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_2}(x))}{c}\right)d\mu(x)<+\infty\right\}.$$

Proposition 2.19. *The map c_{φ,S_Λ} is a pseudo-metric on the set of fundamental domains up to measure zero, except of course it can take the value $+\infty$.*

The proof relies on the following elementary observation.

Lemma 2.20. *Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a non-decreasing function, let $a,b\geqslant 0$ and $c,d>0$. Then*

$$\varphi\left(\frac{a+b}{c+d}\right)\leqslant\varphi\left(\frac{a}{c}\right)+\varphi\left(\frac{b}{d}\right).$$

Proof. By symmetry, we may as well assume that $\frac{a}{c}\geqslant\frac{b}{d}$, in which case $\frac{a}{c}\geqslant\frac{a+b}{c+d}$, so we have $\varphi\left(\frac{a+b}{c+d}\right)\leqslant\varphi\left(\frac{a}{c}\right)\leqslant\varphi\left(\frac{a}{c}\right)+\varphi\left(\frac{b}{d}\right)$. □

Proof of Proposition 2.19. The map c_{φ,S_Λ} is clearly symmetric and satisfies $c_{\varphi,S_\Lambda}(X,X)=0$ for every fundamental domain X, so we only need to check that it satisfies the triangle inequality.

To this end, let X_1, X_2 and X_3 be fundamental domains, let $c_1>c_{\varphi,S_\Lambda}(X_1,X_2)$ and $c_2>c_{\varphi,S_\Lambda}(X_2,X_3)$. We have for every $x\in\Omega/\Lambda$ that

$$\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_3}(x))}{c_1+c_2}\leqslant\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_2}(x))+d_{S_\Lambda}(\iota_{X_2}(x),\iota_{X_3}(x))}{c_1+c_2}.$$

By the previous lemma, we thus have

$$\varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_3}(x))}{c_1+c_2}\right)\leqslant\varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_2}(x))}{c_1}\right)+\varphi\left(\frac{d_{S_\Lambda}(\iota_{X_2}(x),\iota_{X_3}(x))}{c_2}\right).$$

By integrating and using our assumptions on c_1 and c_2, we then deduce that the integral $\int_{\Omega/\Lambda}\varphi\left(\frac{d_{S_\Lambda}(\iota_{X_1}(x),\iota_{X_3}(x))}{c_1+c_2}\right)d\mu(x)$ is finite, and so $c_{\varphi,S_\Lambda}(X_1,X_3)\leqslant c_{\varphi,S_\Lambda}(X_1,X_2)+c_{\varphi,S_\Lambda}(X_2,X_3)$ as wanted. □

Corollary 2.21. *The notion of φ-equivalence is an equivalence relation between fundamental domains.*

Proof. Observe that two fundamental domains X_1 and X_2 are φ-equivalent if and only if $c_{\varphi,S_\Lambda}(X_1,X_2)<+\infty$, so the fact that φ-equivalence is an equivalence relation is a direct consequence of the previous result. □

Definition 2.22. Let $\varphi,\psi\colon\mathbb{R}^+\to\mathbb{R}^+$ be non-decreasing maps.

A measure subquotient coupling (Ω,X_Λ,μ) from Γ to Λ is a called **$\boldsymbol{\varphi}$-integrable** if for every $\gamma\in\Gamma$ we have that X_Λ and $\gamma*X_\Lambda$ are φ-equivalent as fundamental domains of the Λ-action.

A measure equivalence coupling $(\Omega,X_\Gamma,X_\Lambda,\mu)$ from Γ to Λ is called **$(\boldsymbol{\varphi},\boldsymbol{\psi})$-integrable** when the coupling (Ω,X_Λ,μ) from Γ to Λ is φ-integrable and the coupling (Ω,X_Γ,μ) from Λ to Γ is ψ-integrable.

Proposition 2.23. *A measure subquotient coupling (Ω,X_Λ,μ) from $\Gamma=\langle S_\Gamma\rangle$ to Λ is φ-integrable if and only if for every $s\in S_\Gamma$ we have that X_Λ and $s*X_\Lambda$ are φ-equivalent.*

Proof. Since the Γ action commutes with the Λ action, it must preserve the equivalence relation of φ-equivalence, so for every $\gamma\in\Gamma$ and every $s\in S_\Gamma$ we have that $\gamma*X_\Lambda$ and $\gamma s*X_\Lambda$ are φ-equivalent. From there, using the assumption of the proposition, an easy recurrence yields that $(\Omega,X_\Gamma,X_\Lambda,\mu)$ is indeed a φ-coupling from Γ to Λ. □

We then say that Γ is a **φ-** (resp. **(φ,ψ)-**) **measure subquotient**, **subgroup** or **quotient** of Λ if the corresponding coupling is φ-integrable (resp. (φ,ψ)-integrable). When dealing with measure equivalence, since our definition is asymmetric in nature and often non transitive, we will always talk about φ-integrable (resp. (φ,ψ)-integrable) measure equivalence *couplings* from Γ to Λ.

It is not hard to see that being a φ-integrable measure subquotient coupling only depends on the asymptotic behavior of φ. More precisely, if $\varphi \preccurlyeq \psi$, then every ψ-integrable measure subquotient coupling is also a φ-integrable.

Remark 2.24. By spelling out what φ-equivalence means, we see that a measure subquotient coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ from Γ to Λ is φ-integrable if and only if for every $\gamma \in \Gamma$, there is $c_\gamma > 0$ such that

$$\int_{X_\Lambda} \varphi\left(\frac{d_{S_\Lambda}(\gamma \cdot x, \gamma * x)}{c_\gamma}\right) d\mu(x) < +\infty,$$

where $*$ denotes the Γ-action on Ω and $\cdot$ denotes the induced Γ-action on X_Λ.

We deduce from the above remark that a measure subgroup coupling from Γ to Λ is a φ-coupling if and only if the cocycle $\alpha : \Gamma \times X_\Lambda \to \Lambda$ defined by the equation $\alpha(\gamma, x) * \gamma * x = \gamma \cdot x$ satisfies that for all $\gamma \in \Gamma$, there exists a $c_\gamma > 0$ such that the map $X_\Lambda \to \mathbb{R} \colon x \mapsto \varphi(|\alpha(\gamma, x)|_{S_\Lambda} / c_\gamma)$ is integrable.

For easy notation and consistency with the literature, for $p \in (0, \infty)$ we talk about L^p couplings instead of $x \mapsto x^p$-integrable couplings. We also say that we have an **$\mathbf{L^\infty}$ measure subquotient coupling** from Γ to Λ when the Γ-action satisfies for every $\gamma \in \Gamma$ that the map $\Omega/\Lambda \to \Lambda \colon x \mapsto d_{S_\Lambda}(\gamma \cdot x, \gamma * x)$ is essentially bounded. Note that every L^∞ measure subquotient coupling is φ-integrable for any increasing map $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$.

Let us now explain how various established notions fit into our asymmetric framework.

- Two finitely generated groups Γ and Λ are *L^p measure equivalent* in the sense of [BFS13] when there is an $(\mathrm{L}^p, \mathrm{L}^p)$ measure equivalence coupling from Γ to Λ.
- Two finitely generated groups Γ and Λ are *uniform measure equivalent* in the sense of [Sha04] when there is an $(\mathrm{L}^\infty, \mathrm{L}^\infty)$ measure equivalence coupling between Γ and Λ.
- Two finitely generated groups Γ and Λ are *bounded measure equivalent* in the sense of [Sau02] when there is an $(\mathrm{L}^\infty, \mathrm{L}^\infty)$ measure equivalence coupling from Γ to Λ which is cobounded in both directions.

2.3.2 Composition of φ-integrable couplings

Next we look at the integrability of the composition of coupling as presented in Definition 2.8. First we consider the case where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is subadditive, e.g. when φ is concave. Our arguments will follow closely those from [BFS13, Sec. A.2].

Lemma 2.25. *Let Γ and Λ be two finitely generated groups, let $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing subadditive maps and let $(\Omega, X_\Gamma, X_\Lambda, \mu)$ be a φ-measure subquotient coupling from Γ to Λ. Then there is a constant $C > 0$ such that for every $\gamma \in \Gamma$*

$$\int_{X_\Lambda} \varphi\left(d_{S_\Gamma}(\gamma \cdot x, \gamma * x)\right) d\mu(x) \leqslant C\,|\gamma|_{S_\Gamma}.$$

Proof. Assume $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is a φ-measure subquotient coupling from Γ to Λ. Given two fundamental domains X_1 and X_2 two fundamental domains for the Λ-action, we define their φ-distance $d_{\varphi, S_\Gamma}(X_1, X_2)$ by

$$d_{\varphi, S_\Lambda}(X_1, X_2) = \int_{\Omega/\Gamma} \varphi\left(d_{S_\Gamma}(\iota_{X_1}(x), \iota_{X_2}(x))\right) d\mu(x)$$

Note that this distance is symmetric and satisfies the triangle inequality, so it is a pseudometric as soon as $\varphi(0) = 0$. Moreover, using Remark 2.18, we have for every $\gamma \in \Gamma$ that

$$d_{\varphi,S_\Lambda}(X_\Lambda, \gamma * X_\Lambda) < +\infty,$$

while by Remark 2.24 we have $d_{\varphi,S_\Lambda}(X_\Lambda, \gamma * X_\Lambda) = \int_{X_\Lambda} \varphi\left(d_{S_\Gamma}(\gamma \cdot x, \gamma * x)\right) d\mu(x)$.

Let $C = \max_{\gamma \in S_\Gamma} d_{\varphi,S_\Lambda}(X_\Lambda, \gamma * X_\Lambda)$. The Γ-action on the set of Λ-fundamental domains preserves d_{φ,S_Λ} because it commutes with the Λ-action, so for every $s_1, ..., s_n \in S_\Gamma$ we have by the triangle inequality

$$\begin{aligned} d_{\varphi,S_\Lambda}(s_1 \cdots s_n * X_\Lambda, X_\Lambda) &\leqslant \sum_{i=1}^{n} d_{\varphi,S_\Lambda}(s_1 \cdots s_i * X_\Lambda, s_1 \cdots s_{i-1} * X_\Lambda) \\ &\leqslant \sum_{i=1}^{n} d_{\varphi,S_\Lambda}(s_i * X_\Lambda, X_\Lambda) \\ &\leqslant Cn, \end{aligned}$$

which yields the desired result. □

Remark 2.26. Note that when φ is subadditive and non-decreasing, we always have $\varphi(x) \leqslant x\varphi(1) + \varphi(1)$, in particular every L^1 measure subquotient coupling is φ-integrable.

Let us now study how couplings compose in the subadditive regime.

Proposition 2.27. *Let $\varphi, \psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing subadditive maps with φ moreover concave and let Γ, Λ and Σ be three finitely generated groups. Let $(\Omega_1, X_{1,\Gamma}, X_{1,\Lambda}, \mu_1)$ be a φ-integrable measure subgroup coupling from Γ to Λ and let $(\Omega_2, X_{2,\Lambda}, X_{2,\Sigma}, \mu_2)$ be a ψ-integrable measure subquotient coupling from Λ to Σ. Then the composition of these two couplings is a $\varphi \circ \psi$-integrable measure subquotient coupling from Γ to Σ.*

Proof. Thanks to Lemma 2.25 we find $C > 0$ such that

$$\int_{X_{2,\Sigma}} \psi\left(d_{S_\Sigma}(\lambda \cdot x, \lambda * x)\right)\right) d\mu(x) \leqslant C|\lambda|_{S_\Lambda} \text{ for every } \lambda \in \Lambda. \tag{2}$$

By scaling the measure μ_2 we may assume that $\mu_2(X_{2,\Sigma}) = 1$. Denote by $\alpha : \Gamma \times X_{1,\Lambda} \to \Lambda$ the cocycle defined by the equation $\alpha(\gamma, x) * \gamma * x = \gamma \cdot x$. By Remark 2.10 and the definition of the composition of our two couplings, we need to show that the following quantity is finite:

$$\int_{X_{1,\Lambda}} \int_{X_{2,\Sigma}} \varphi \circ \psi\left(d_{S_\Sigma}(\alpha(\gamma, x) \cdot y, \alpha(\gamma, x) * y)\right) d\mu_2(y) d\mu_1(x).$$

Now by Jensen's inequality, this is at most

$$\int_{X_{1,\Lambda}} \varphi\left(\int_{X_{2,\Sigma}} \psi\left(d_{S_\Sigma}(\alpha(\gamma, x) \cdot y, \alpha(\gamma, x) * y)\right) d\mu_2(y)\right) d\mu_1(x),$$

which by inequation (2) is bounded above by $\int_{X_{1,\Lambda}} \varphi(C\,|\alpha(\gamma, x)|_{S_\Lambda})$. The latter is indeed finite by our assumption on the first coupling and Remark 2.18. □

The above result can be combined with Proposition 2.9 to obtain a $\varphi \circ \psi$-measure subgroup, $\varphi \circ \psi$-measure quotient or (φ, ψ)-measure equivalence coupling by composition.

Remark 2.28. Given two finitely generated groups Γ and Λ, one could define

$$\alpha(\Gamma,\Lambda) = -\log\left(\sup\{p \leqslant 1 : \Gamma \text{ and } \Lambda \text{ are } \mathrm{L}^p \text{ measure equivalent}\}\right)$$

The previous proposition implies that this is a pseudo-metric on the space of isomorphism classes of finitely generated groups. It would be interesting to understand this pseudo-metric further. For instance, Theorem 6.9 implies that $d(\mathbb{Z}^n,\mathbb{Z}^m) = |\log n - \log m|$.

For non subadditive maps φ, we need a stronger notion of φ-integrability so that it behaves well with respect to composition.

Definition 2.29. Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing map. We say that a coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ from Γ to Λ is **strongly φ-integrable** or $\varphi^\diamond$-integrable if for every $\varepsilon > 0$ there are $\delta > 0$ and $C > 0$ such that for every $\gamma \in \Gamma$,

$$\int_{X_\Lambda} \varphi\left(\delta\, d_{S_\Lambda}(\gamma\cdot x, \gamma * x)\right) d\mu(x) \leqslant C\varphi\left(\varepsilon|\gamma|_{S_\Gamma}\right)$$

Note again that thanks to inequation (1) strong integrability does not depend on the choice of the finite generating set S_Γ. However, the above condition has to be checked on *every* element of Γ.

Proposition 2.30. *Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing map and let Γ, Λ and Σ be three finitely generated groups. Let $(\Omega_1, X_{1,\Gamma}, X_{1,\Lambda}, \mu_1)$ be an strongly φ-integrable measure subgroup coupling from Γ to Λ and let $(\Omega_2, X_{2,\Lambda}, X_{2,\Sigma}, \mu_2)$ be a strongly φ-integrable measure subquotient coupling from Λ to Σ. Then the composition of these two couplings is a strongly φ-integrable measure subquotient coupling from Γ to Σ.*

Proof. Let $\varepsilon > 0$. As $(\Omega_1, X_{1,\Gamma}, X_{1,\Lambda}, \mu_1)$ is strongly φ-integrable, there are $\delta_\Lambda > 0$ and $C_\Gamma > 0$ such that

$$\int_{X_{1,\Lambda}} \varphi\left(\delta_\Lambda d_{S_\Lambda}(\gamma\cdot x, \gamma * x)\right) d\mu_1(x) \leqslant C_\Gamma\, \varphi\left(\varepsilon\,|\gamma|_{S_\Gamma}\right) \text{ for every } \gamma \in \Gamma. \tag{3}$$

and as $(\Omega_2, X_{2,\Lambda}, X_{2,\Sigma}, \mu)$ is strongly φ-integrable, there exist $\delta_\Sigma > 0$ and $C_\Lambda > 0$ such that

$$\int_{X_{2,\Sigma}} \varphi\left(\delta_\Sigma d_{S_\Sigma}(\lambda\cdot x, \lambda * x)\right) d\mu_2(x) \leqslant C_\Lambda\, \varphi\left(\delta_\Lambda|\lambda|\right) \text{ for every } \lambda \in \Lambda. \tag{4}$$

By Remark 2.10 and the definition of the composition of our two couplings, we need to estimate the following quantity:

$$\int_{X_{1,\Lambda}} \int_{X_{2,\Sigma}} \varphi\left(\delta_\Sigma d_{S_\Sigma}(\alpha(\gamma,x)\cdot y, \alpha(\gamma,x) * y)\right) d\mu_2(y) d\mu_1(x).$$

By inequation (4), this is bounded above by

$$\int_{X_{1,\Lambda}} C_\Lambda \varphi(\delta_\Lambda\, |\alpha(\gamma,x)|_{S_\Lambda}) \leqslant C_\Lambda C_\Gamma\, \varphi\left(\varepsilon|\gamma|_{S_\Gamma}\right)$$

as wanted, where the last inequation is a consequence of inequation (3), and the fact that by definition $|\alpha(\gamma,x)|_{S_\Lambda} = d_{S_\Lambda}(\gamma\cdot x, \gamma * x)$. □

For some maps φ we can weaken the strong integrability condition. Most notably for L^p couplings, where $p \geqslant 1$.

Proposition 2.31. *Let $p \geqslant 1$. Every L^p measure subquotient coupling from Γ to Λ is actually a strongly L^p measure subquotient coupling.*

Proof. We follow an approach similar to that of Lemma 2.25. For two fundamental domains X_1 and X_2 for the Λ-action, we define their L^p distance by

$$d_{\mathrm{L}^p,S_\Lambda}(X_1,X_2)=\left(\int_{\Omega/\Gamma}\varphi\left(d_{S_\Gamma}(\iota_{X_1}(x),\iota_{X_2}(x))\right)d\mu(x)\right)^{1/p}$$

It is not hard to check that $d_{\mathrm{L}^p,S_\Lambda}$ is a metric, and since the Γ-action commutes with the Λ-action, the group Γ acts on the set of Λ-fundamental domains by isometries. In particular, as in the proof of Lemma 2.25 we find $C_\Gamma>0$ such that for every $\gamma\in\Gamma$ and every Λ-fundamental domain X_Λ, $d_{\mathrm{L}^p,S_\Lambda}(X_\Lambda,\gamma*X_\Lambda)\leqslant C_\Gamma\,|\gamma|_{S_\Gamma}$. This means that for all $\gamma\in\Gamma$, we have

$$\int_{X_\Lambda}d_{S_\Lambda}(\gamma\cdot x,\gamma*x)^p d\mu(x)\leqslant C_\Gamma^p|\gamma|_{S_\Gamma}^p,$$

from which the result easily follows: given $\varepsilon>0$ we take $\delta=\varepsilon$, $C=C_\Gamma^p$ and note that $\int_{X_\Lambda}\left(\delta d_{S_\Lambda}(\gamma\cdot x,\gamma*x)\right)^p d\mu(x)=\delta^p\int_{X_\Lambda}d_{S_\Lambda}(\gamma\cdot x,\gamma*x)^p d\mu(x)\leqslant C_\Gamma^p(\varepsilon|\gamma|_{S_\Gamma})^p$ as wanted. □

Say for $p\geqslant 1$ that two finitely generated groups are L^p **measure equivalent** when there is an $(\mathrm{L}^p,\mathrm{L}^p)$ measure equivalence coupling from Γ to Λ. The we deduce from the two previous results that L^p measure equivalence is an equivalence relation between finitely generated groups, as proven by Bader, Furman and Sauer in [BFS13]. Note however that this is not true anymore for $p<1$; counter-examples are provided by Theorem 6.9.

We can also weaken the condition for strongly exponential-integrable couplings.

Proposition 2.32. *Let Γ and Λ be two finitely generated groups and let (Ω,X_Λ,μ) be a measure subquotient coupling from Γ to Λ. If there are $\varepsilon'>0$, $\delta'>0$ and $C'>0$ such that*

$$\int_{\gamma*X_\Lambda}\exp\left(\delta'\,d_{S_\Lambda}(x,\pi_{X_\Lambda}(x))\right)d\mu(x)\leqslant C'\exp\left(\varepsilon'|\gamma|_{S_\Gamma}\right)\ \textit{for every } \gamma\in\Gamma,$$

then the coupling is strongly exp*-integrable.*

Proof. Let $\varepsilon>0$. If $\varepsilon\geqslant\varepsilon'$, then take $C=C'$ and $\delta=\delta'$. Then we have that

$$\int_{\gamma*X_\Lambda}\exp\left(\delta\,d_{S_\Lambda}(x,\pi_{X_\Lambda}(x))\right)d\mu(x)\leqslant C'\exp\left(\varepsilon'|\gamma|\right)\leqslant C\exp\left(\varepsilon|\gamma|\right)\ \text{for every } \gamma\in\Gamma.$$

If $\varepsilon<\varepsilon'$, then take $C=C'^{\frac{\varepsilon}{\varepsilon'}}$ and $\delta=\frac{\delta'\varepsilon}{\varepsilon'}$. Then we have for every $\gamma\in\Gamma$ that

$$\begin{aligned}\int_{\gamma*X_\Lambda}\exp\left(\delta\,d_{S_\Lambda}(x,\pi_{X_\Lambda}(x))\right)d\mu(x)&=\int_{\gamma*X_\Lambda}\exp\left(\delta'\,d_{S_\Lambda}(x,\pi_{X_\Lambda}(x))\right)^{\frac{\varepsilon}{\varepsilon'}}d\mu(x)\\&\leqslant\left(\int_{\gamma*X_\Lambda}\exp\left(\delta'\,d_{S_\Lambda}(x,\pi_{X_\Lambda}(x))\right)d\mu(x)\right)^{\frac{\varepsilon}{\varepsilon'}}\\&\leqslant\left(C'\exp(\varepsilon'|\gamma|)\right)^{\frac{\varepsilon}{\varepsilon'}}\\&=C\exp(\varepsilon|\gamma|).\end{aligned}$$

Thus, the coupling is strongly exp-integrable as wanted □

Interesting examples of strongly $(\exp,\exp)$-integrable measure equivalence couplings will be provided in Section 8.

2.4 Variations on orbit equivalence

We now turn our attention to orbit equivalence of groups, which implies measure equivalence. To make the connection, we need to introduce measure-preserving equivalence relations.

Definition 2.33. Given a measure-preserving action of a countable group Λ on a standard probability space (X, μ), we associate to it a **measure-preserving equivalence relation** $\mathcal{R}_\Lambda$ defined by

$$\mathcal{R}_\Lambda := \{(x, \lambda \cdot x) \colon x \in X, \lambda \in \Lambda\}.$$

A key property of measure-preserving equivalence relations is that they can be endowed with a natural σ-finite measure M. Namely, for $\mathcal{R}$ a measure-preserving equivalence relation, and A a Borel subset of $\mathcal{R}$, we let

$$M(A) = \int_X |A_x| \, d\mu(x),$$

where $A_x = \{y \in X : (x, y) \in A\}$. Such a measure is invariant under the flip map $(x, y) \mapsto (y, x)$ (see [KM04, p. 34]), which means that we also have $M(A) = \int_X |A^x| \, d\mu(x)$, where $A^x = \{y \in X : (x, y) \in A\}$.

Definition 2.34. Let Γ and Λ be two finitely generated groups. An **orbit subquotient coupling** from Γ to Λ is a triple (X, Y, μ) where (Y, μ) is a standard σ-finite space equipped with a measure-preserving Λ-action, X is a Borel subset of Y of measure 1 equipped with a free measure-preserving Γ-action and finally for almost every $x \in X$ we have that $\Gamma \cdot x \subseteq \Lambda \cdot x$.

Every orbit subquotient coupling is a measure subquotient coupling as follows: given a subquotient coupling (X, Y, μ) from Γ to Λ, our coupling space is $\Omega := \mathcal{R}_\Lambda \cap (X \times Y)$ equipped with the measure induced by M, acted upon as follows: for every $\Gamma \in \Gamma$, every $\lambda \in \Lambda$ and every $(x, y) \in \mathcal{R}_\Lambda$,

$$\gamma * (x, y) = (\gamma \cdot x, y) \text{ and } \lambda * (x, y) = (x, \lambda \cdot y).$$

Note that the Γ-action is measure-preserving as a direct consequence of the definition of M and of the fact that its action on (X, μ) is measure-preserving, while the Λ-action is also measure-preserving because M is flip-invariant. Finally, the chosen Λ-fundamental domain is the diagonal: $X_\Lambda = \{(x, x) \colon x \in X\}$, and the fact that the Γ-action is smooth follows from the fact that a Borel fundamental domain can be obtained as the intersection with $X \times Y$ of a disjoint reunion of graphs of *Borel choice functions* for the subequivalence relation $(\mathcal{R}_\Gamma \cup \Delta_Y) \leqslant \mathcal{R}_\Lambda$ (see [IKT09, Sec. 2.(A)]).

Note furthermore that (X_Λ, M) is naturally isomorphic to (X, μ) via $x \mapsto (x, x)$, and that the induced action on X_Λ is conjugate via the inverse of this map to the original action on (X, μ).

Definition 2.35. An orbit subquotient coupling from Γ to Λ is φ-integrable when it is φ-integrable as a measure subquotient coupling.

Note that φ-integrability for an orbit subquotient coupling as above means that for all $\gamma \in \Gamma$, there is $c_\gamma > 0$ such that

$$\int_X \varphi\left(\frac{d_{S_\Lambda}(x, \gamma \cdot x)}{c_\gamma}\right) < +\infty.$$

Remark 2.36. When $\varphi(x) = x^p$ for some $p \geqslant 1$, this means that Γ is contained in the L^p *full group* of the Λ-action, as defined in [LM18].

Definition 2.37. Let (X, Y, μ) be an orbit subquotient coupling from Γ to Λ.

- If the Λ-action is also free, we say that (X, Y, μ) is an **orbit subgroup** coupling of Γ with Λ.
- If $X = Y$ and for almost every $x \in X$ we have that $\Gamma \cdot x = \Lambda \cdot x$, we say that (X, Y, μ) is an **orbit quotient coupling**, which we then simply write as (X, μ).

- Finally, (X, μ) is an **orbit equivalence** coupling of Γ with Λ if it is both an orbit quotient and an orbit subgroup coupling.

Remark 2.38. When (X, Y, μ) is an orbit subquotient coupling from Γ to Λ such that for almost every $x \in X$ we have that $\Gamma \cdot x = \Lambda \cdot x$, the set X is also Λ-invariant and so (X, μ) is an orbit quotient coupling.

We note that admitting an orbit equivalence coupling agrees with the definitions of orbit equivalence between countable groups in the literature, and that every orbit equivalence (resp. quotient) coupling is a measure equivalence (resp. quotient) coupling. Also, every orbit subgroup coupling is a measure subgroup coupling. We can thus define (φ, ψ)-integrable orbit equivalence/quotient couplings as well as φ-integrable subgroup couplings. Finally, we can also define strong φ-integrability conditions for orbit couplings.

It is well known that orbit equivalence couplings are exactly those of the form $(\Omega, X_\Gamma, X_\Lambda, \mu)$ with $X_\Gamma = X_\Lambda := X$ of measure 1 and both induced actions on X are free. The freeness hypothesis is not a restriction, as we will see now.

Proposition 2.39. *Let (Ω, X_Λ, μ) be a measure subquotient coupling from Γ to Λ, let (Y, ν) be a standard probability space equipped with a free Γ-action. Then $(\Omega \times Y, X_\Lambda \times Y, \mu \otimes \lambda)$ is a measure subquotient coupling from Γ to Λ, where Γ acts diagonally and Λ acts on the first coordinate. Moreover, the induced Γ-action on $X_\Lambda \times Y$ is free, the coupling $(\Omega \times Y, X_\Lambda \times Y, \mu \otimes \lambda)$ is φ-integrable if and only if (Ω, X_Λ, μ) was, and if X_Γ was a fundamental domain for $\Gamma \curvearrowright \Omega$, then $X_\Gamma \times Y$ is a fundamental domain for $\Gamma \curvearrowright \Omega \times Y$.*

Proof. It is clear that $X_\Lambda \times Y$ is a fundamental domain for the Λ-action, and since the Γ-action on Ω was free, if X_Γ was a fundamental domain for it, then $X_\Gamma \times Y$ is a fundamental domain for the new diagonal action. Moreover, the induced Γ-action on $X_\Lambda \times Y$ is the diagonal action obtained from its induced action on X_Λ and its action on Y, in particular it is free. Finally, the statement about φ-integrability follows directly from the fact that for every $y \in Y$ and every $\omega, \omega' \in \Omega$, we have by construction $d_{S_\Lambda}((\omega, y), (\omega', y)) = d_{S_\Lambda}(\omega, \omega')$. □

Remark 2.40. Note that the above lemma can be applied to any countable group Γ: if Γ is infinite, one can take (Y, ν) as a Bernoulli shift of Γ, and if Γ is finite, one can take $Y = \Gamma$ acted upon by left translation, equipped with the normalized counting measure.

Proposition 2.41. *Let Γ and Λ be countable groups. If there is a (φ, ψ) measure equivalence (resp. measure quotient) coupling from Γ to Λ where the two fundamental domains coincide, then there is a (φ, ψ) orbit equivalence (resp. orbit quotient) coupling from Γ to Λ.*

Proof. Let $(\Omega, X_\Gamma, X_\Lambda, \mu)$ be a (φ, ψ)-integrable measure equivalence coupling from Γ to Λ, and denote by $X = X_\Gamma = X_\Lambda$ the common fundamental domain. Up to rescaling the measure (which does not impact the integrability conditions), we may as well assume that X has measure 1. Using the previous remark, we can apply the above proposition twice and see that without loss of generality, we can also assume that the induced Γ- and Λ-actions on X are free. We denote them by $\cdot$.

Now observe that for every $\gamma \in \Gamma$ and every $x \in X$, we have $\gamma \cdot x \in \Lambda * \gamma * x$ so there is $\lambda \in \Lambda$ such that $\gamma \cdot x = \lambda * \gamma * x = \gamma * \lambda * x$. In particular $\lambda \cdot x = \gamma \cdot x$, so we conclude that $\Gamma \cdot x \subseteq \Lambda \cdot x$. By the symmetry, we also have $\Lambda \cdot x \subseteq \Gamma \cdot x$, so we conclude that (X, μ) is an orbit equivalence coupling. Finally, the map $(\gamma \cdot x, x) \mapsto \gamma * x$ is a $\Gamma \times \Lambda$-equivariant bijection from $\mathcal{R}_\Gamma = \mathcal{R}_\Lambda$ to Ω which takes $\{(x, x) \colon x \in X\}$ to X, and thus the (φ, ψ)-integrability of the orbit coupling (X, μ) follows from that of $(\Omega, X_\Gamma, X_\Lambda, \mu)$.

The statement for orbit quotient couplings follows from the same argument, except we apply the above proposition only once so as to make the induced Γ-action free. □

Similar comparisons can be made for orbit subgroup and subquotient couplings. Observe that in this situation there is no canonical choice for a Γ fundamental domain X_Γ, but X_Λ intersects every Γ-orbit at most once.

Proposition 2.42. *Let Γ and Λ be two finitely generated groups. If there is a φ-integrable measure subquotient (resp. measure subgroup) coupling (Ω, X_Λ, μ) from Γ to Λ such that X_Λ intersects every Γ-orbit at most once, then there is a φ-integrable orbit subquotient (resp. orbit subgroup) coupling from Γ to Λ.*

Proof. We first assume that (Ω, X_Λ, μ) is a φ-integrable measure subquotient coupling from Γ to Λ such that X_Λ intersects every Γ-orbit at most once. Up to scaling the measure and using Proposition 2.39, we may as well assume that $\mu(X_\Lambda) = 1$ and that the induced Γ-action on X_Λ is free.

By assumption the restriction of $\pi_{\Omega/\Gamma}$ to X_Λ is injective, so we may as well view X_Λ as a subset of Ω/Γ. Consider the induced Λ-action on $Y := \Omega/\Gamma$, the induced Γ-action on $X := X_\Lambda \subseteq Y$ and denote them by $\cdot$. For every $\gamma \in \Gamma$ and $x \in X_\Lambda$ there is $\lambda \in \Lambda$ such that $\gamma \cdot x = \lambda * \gamma * x \in X_\Lambda$. Then $\gamma * (\lambda * x) \in X_\Lambda$, so $\lambda \cdot x = \gamma * (\lambda * x) = \gamma \cdot x$. Thus, $\Gamma \cdot x \subseteq \Lambda \cdot x$, and (X, Y, μ) is indeed an orbit subquotient coupling.

We now check that the map $\lambda * x \mapsto (x, \lambda \cdot x)$ from Ω to $\mathcal{R}_\Lambda \cap X \times Y$ is well-defined and bijective. First, if $\lambda * x = \lambda' * x'$ for some $x, x' \in X_\Lambda$ and $\lambda, \lambda' \in \Lambda$, we have $x = x'$ since X_Λ is a fundamental domain, and then by definition of the induced action we must have $\lambda \cdot x = \lambda' \cdot x'$: our map is well-defined. In order to show it is injective, assume that $\lambda \cdot x = \lambda' \cdot x$, then $\lambda^{-1}\lambda' \cdot x = x$, so there is $\gamma \in \Gamma$ such that $\lambda^{-1}\lambda' * x = \gamma * x$, so $\gamma \cdot x = x$. By freeness of the induced Γ-action, we deduce that $\gamma = e_\Gamma$, and so $\lambda * x = \lambda' * x$ as wanted. Finally, surjectivity is clear from the definition, and since this map is $\Gamma \times \Lambda$-equivariant, we conclude that the orbit subquotient coupling that we obtained is φ-integrable if and only if our original coupling was.

For the orbit subgroup case we proceed as before, except we first make sure that both induced actions are free by using Proposition 2.39 twice. □

Remark 2.43. The two previous propositions show that orbit couplings can be composed as in Proposition 2.9. This is shown by combining Proposition 2.9 with Proposition 2.41 and Proposition 2.42. The integrability of this composition coupling satisfies the same result as in Proposition 2.27 and Proposition 2.30.

3 Revisiting Bowen's monotonicity theorem of the volume growth

Following the original proof of Bowen in [Aus16, Theorem B2], we can deduce a stronger conclusion. Towards this, we recall the following lemma [Aus16, Lemma B.11] by Bowen:

Lemma 3.1 (Bowen)**.** *Let Γ be a finitely generated group. Let $\Gamma \curvearrowright (X, \mu)$ be an measure-preserving action, and $X_0 \subseteq X$ a set with positive measure, and $R_{X_0}(x) \colon = \{g \in \Gamma \colon g.x \in X_0\}$ its associated return time set. Then, for every $n \in \mathbb{N}$ we have*

$$\int_{X_0} \frac{|R_{X_0}(x) \cap B_\Gamma(e_\Gamma, n)|}{|B_\Gamma(e_\Gamma, n)|} d\mu(x) \geqslant 2\mu(X_0) - \mu(X).$$

In what follows, we denote by V_Γ the growth function of a finitely generated group Γ, more precisely the asymptotic equivalence class of the volume growth given by $V_\Gamma(n) = |B(e_\Gamma, n)|$ with respect to some (any) finite generating set and its associated word metric.

Theorem 3.2. *Let φ be a positive, increasing, concave function such that $\varphi(0) = 0$ and let Γ and Λ be finitely generated groups. If Γ is an L^p measured subquotient of Λ, then*

$$V_\Gamma(n) \preccurlyeq V_\Lambda(\varphi^{-1}(n)),$$

where φ^{-1} denotes the generalized inverse of φ.

Proof. Let $(\Omega, X_\Gamma, X_\Lambda, \mu)$ be a coupling of Γ and Λ such that X_Λ has finite measure, the Γ-action is free and such that for all $s \in S_\Gamma$,

$$\int_{X_\Lambda} \varphi(d_{S_\Lambda}(s * x, x)))d\mu(x) < \infty$$

As $\mu(X_\Lambda)$ is finite and $\Gamma * X_\Gamma = \Omega$ we can take $R > 0$ and $X_0 = X_\Lambda \cap B_\Gamma(e_\Gamma, R) * X_\Gamma$ such that $\mu(X_0) \geqslant \frac{9}{10}\mu(X_\Lambda)$.

Claim 3.3. Given $\lambda \in \Lambda$ and $x \in X_\Lambda$, the cardinality of the set of $g \in R_{X_0}(x)$ such that $\lambda(g * x) \in X_\Lambda$ is at most $C = |B_\Gamma(e_\Gamma, R)|$.

Proof of the claim. Let g and g' be in that set. We deduce from the fact that $g'g^{-1} * (g \cdot x) \in X_\Lambda$ and the freeness of the Γ-action that $|g'g| \leqslant R$. So we are done. $\square_{\text{claim}}$

We deduce from the triangular inequality and the properties of φ that there exists $\kappa > 0$ such that

$$\int_{X_H} \varphi(d_{S_\Gamma}(g * x, x)))d\mu(x) < \kappa |g|_{S_\Lambda}$$

for every $g \in G$. Next fix $n \in \mathbb{N}$ and define

$$X_1 = \left\{ x \in X_\Lambda \colon \sum_{g \in B_\Gamma(e_\Gamma, n)} \frac{\varphi(d_{S_\Lambda}(g * x, x)))}{|g|} < \frac{10\kappa |B_\Gamma(e_\Gamma, n)|}{\mu(X_\Lambda)} \right\}.$$

Note that

$$\mu(X_1^c) \leqslant \frac{\mu(X_\Lambda)}{10\kappa |B_\Gamma(e_\Gamma, n)|} \int_{X_H} \sum_{g \in B_\Gamma(e_\Gamma, n)} \frac{\varphi(d_{S_\Lambda}(g * x, x)))}{|g|} d\mu(x) \leqslant \frac{1}{10}\mu(X_\Lambda).$$

Next for every $x \in X_\Lambda$ take $\Gamma_x = \{g \in B_\Gamma(e_\Gamma, n) \colon \varphi(d_{S_\Lambda}(g * x, x))) \leqslant 60\kappa |g|\}$. Note that if $x \in X_1$, then $|\Gamma_x| \geqslant \frac{5}{6}|B_\Gamma(e_G, n)|$. Finally take $X_2 = X_1 \cap X_0$. Then $\mu(X_2) \geqslant \frac{8}{10}\mu(X_\Lambda)$ and by Lemma 3.1 we have that

$$\int_{X_2} \frac{|R_{X_2}(x) \cap B_\Gamma(e_\Gamma, n)|}{|B_\Gamma(e_\Gamma, n)|} \geqslant 2\mu(X_2) - \mu(X_H) \geqslant \frac{6}{10}\mu(X_H).$$

As $\Gamma_x \subseteq B_\Gamma(e_\Gamma, n)$ we have that

$$\begin{aligned}
\int_{X_2} |R_{X_2}(x) \cap \Gamma_x| d\mu(x) &\geqslant \int_{X_2} \big(|R_{X_2}(x) \cap B_\Gamma(e_\Gamma, n)| + |\Gamma_x| - |B_\Gamma(e_\Gamma, n)|\big) d\mu(x) \\
&\geqslant \frac{6}{10}\mu(X_\Lambda)|B_\Gamma(e_\Gamma, n)| + \frac{5}{6}|B_\Gamma(e_\Gamma, n)|\mu(X_2) - |B_\Gamma(e_\Gamma, n)|\mu(X_2) \\
&\geqslant \frac{6}{10}\mu(X_\Lambda)|B_\Gamma(e_\Gamma, n)| + \frac{5}{10}|B_\Gamma(e_\Gamma, n)|\mu(X_\Lambda) - |B_\Gamma(e_\Gamma, n)|\mu(X_\Lambda) \\
&\geqslant \frac{1}{10}\mu(X_\Lambda)|B_\Gamma(e_\Gamma, n)|.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{X_2} |R_{X_2}(x) \cap \Gamma_x| d\mu(x) &\leqslant \int_{X_2} |\{g \in R_{X_2}(x) \colon \varphi(d_{S_\Lambda}(g * x, x))) \leqslant 60\kappa n\}| d\mu(x) \\
&\leqslant \sum_{\lambda \in B_\Lambda(e_\Lambda, \varphi^{-1}(60\kappa n))} \int_{X_2} |\{g \in R_{X_2}(x) \colon \lambda(g * x) \in X_\Lambda\}| d\mu(x).
\end{aligned}$$

Since $X_2 \subset X_0$, we deduce from Claim 3.3 that

$$\frac{1}{10}\mu(X_\Lambda)|B_\Gamma(e_\Gamma, n)| \leqslant \left|B_\Lambda\left(e_\Lambda, \varphi^{-1}(60\kappa n)\right)\right| C\mu(X_2)$$

which concludes the proof of the theorem. $\square$

Corollary 3.4. *Assume that two groups Γ and Λ have polynomial growth with respective growth exponents m and n satisfying $m > n$. Then Γ is not an L^p measured subquotient of Λ if $p > n/m$.*

4 Monotonicity of the isoperimetric profile

In this section we state and prove a general monotonicity result satisfied by the isoperimetric profile. We obtain two different conclusions depending on whether the coupling is L^p for some $p \geqslant 1$, or if it is φ-integrable for a sublinear function φ. However the first statement for $p = 1$ and the second one for $\varphi(t) = t$ have the same content.

Theorem 4.1. *Let $p \geqslant 1$, and assume that we are in one of the following situations.*

(i) Γ is an L^p measure quotient of Λ;

(ii) Γ is a at most m-to-one L^p measure subquotient of Λ.

Then

$$I_{p,\Gamma} \preccurlyeq I_{p,\Lambda}.$$

Theorem 4.2. *Let $\varphi : (0,\infty) \to (0,\infty)$ be a function such that φ and $t \mapsto t/\varphi(t)$ are non-decreasing. Assume that we are in one of the following situations.*

(i) Γ is a φ-integrable measure quotient of Λ;

(ii) Γ is an at most m-to-one φ-integrable measure subquotient of Λ

Then

$$I_{1,\Gamma} \preccurlyeq \frac{1}{\varphi(1/I_{1,\Lambda})}.$$

Corollary 4.3. *If $p \leqslant 1$ and Γ is an L^1 orbit subgroup of Λ, or $p \geqslant 1$ and Γ is an L^p orbit subquotient of Λ, then $I_{p,\Gamma} \preccurlyeq I_{p,\Lambda}$.* □

We illustrate this by the following corollary.

Corollary 4.4. *Assume that Γ and Λ have polynomial growth of growth exponents m and n respectively, with $m > n$, and let F and K be non-trivial finite groups.*

- *then Γ is not an L^p measured quotient (nor an finite-to-one L^p measured subquotient) of Λ if $p > n/m$.*
- *$F \wr \Gamma$ is not an L^p measured quotient (nor an finite-to-one L^p measured subquotient) of $\mathbb{F} \wr \Lambda$ if $p > 1 - 1/n + 1/m$.*

Remark 4.5. The first item is slightly weaker than the conclusion of Corollary 3.4 obtained by comparing the volume growths (as the latter does not require the finite-to-one assumption).

4.1 Γ-gradients on Γ-spaces

It will be convenient to define a notion of gradient on functions defined on measure spaces equipped with a measure-preserving group action.

Definition 4.6. Let Γ be a finitely generated group equipped with a finite symmetric generating set S_Γ. Given any standard measured Γ-space (Ω, μ) and any function $f \in \mathrm{L}^p(\Omega, \mu)$, define the L^p norm of its S_Γ-gradient by the formula

$$\|\nabla_{S_\Gamma} f\|_p^p = \sum_{s \in S_\Gamma} \|f - s * f\|_p^p,$$

where $s * f(x) = f(s^{-1} * x)$.

Remark 4.7. Note that by the linearity of the integral and the symmetry of S_Γ, we have

$$\|\nabla_{S_\Gamma} f\|_p^p = \int_\Omega \sum_{s\in S_\Gamma} |f(\omega) - f(s*\omega)|^p \, d\mu(\omega)$$

There are two natural ways to view Γ as a Γ-space (action by left or right multiplication), so we define the L^p-norm of the *left* S_Γ-gradient f on Γ by

$$\left\|\nabla^l_{S_\Gamma} f\right\|_p^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} |f(\gamma) - f(s\gamma)|^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} \left|f(\gamma) - f(s^{-1}\gamma)\right|^p$$

and the L^p-norm of the *right* S_Γ-gradient f on Γ by

$$\left\|\nabla^r_{S_\Gamma} f\right\|_p^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} |f(\gamma) - f(\gamma s)|^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} \left|f(\gamma) - f(\gamma s^{-1})\right|^p .$$

The following lemma shows how the right-gradient can be used to control what happens when we push forward functions on Γ to a Γ-space using different basepoints. It is reminiscent of the fact that right Reiter functions on a group Γ can be pushed to Reiter functions on any equivalence relation induced by a Γ-action (see e.g. [KM04, Prop. 9.2]).

Lemma 4.8. *Let X be a transitive Γ-set and let $x_0 \in X$. Let $p \geqslant 1$. The map $\mathbb{R}^\Gamma \to \mathbb{R}^X$ which associates to every $f \in \mathbb{R}^\Gamma$ the function f_{x_0} given by*

$$f_{x_0}(y) = \left(\sum_{\gamma:\gamma\cdot x_0 = y} |f(g)|^p\right)^{1/p}$$

satisfies $\|f_{x_0}\|_p = \|f\|_p$. Moreover, if S_Γ is a finite generating set for Γ and if $f \in \ell^p(\Gamma)$ then for all $k \in \Gamma$, and $x_1 = k \cdot x_0$,

$$\|f_{x_0} - f_{x_1}\|_p \leqslant d_{S_\Gamma}(x_0, x_1) \left\|\nabla^r_{S_\Gamma} f\right\|_p . \tag{5}$$

Proof. Now assume that $p \geqslant 1$. The verification that $\|f_x\|_p = \|f\|_p$ is immediate. To obtain the inequality (5), note that for each $s \in S_\Gamma$ and $x_0 \in X$, we have by the triangle difference inequality

$$\begin{aligned}
\|f_{x_0} - f_{s\cdot x_0}\|_p^p &= \sum_{y\in X} \left| \left(\sum_{\gamma:\gamma\cdot x_0 = y} |f(\gamma)|^p\right)^{1/p} - \left(\sum_{\gamma:\gamma s\cdot x_0 = y} |f(\gamma)|^p\right)^{1/p} \right|^p \\
&= \sum_{y\in X} \left| \left(\sum_{\gamma:\gamma\cdot x_0 = y} |f(\gamma)|^p\right)^{1/p} - \left(\sum_{\gamma:\gamma\cdot x_0 = y} |f(\gamma s^{-1})|^p\right)^{1/p} \right|^p \\
&\leqslant \sum_{y\in X} \sum_{\gamma:\gamma\cdot x_0 = y} |f(\gamma) - f(\gamma s^{-1})|^p \\
&\leqslant \sum_{\gamma\in\Gamma} \left|f(\gamma) - f(\gamma s^{-1})\right|^p ,
\end{aligned}$$

so by the definition of the gradient we have $\|f_{x_0} - f_{s\cdot x_0}\|_p \leqslant \|\nabla_{S_\Gamma} f\|_p$. The conclusion now follows using the triangle inequality for the L^p-norm and the fact that Γ is acting by isometries on $\ell^p(X)$. □

We now state a variant of that lemma that will serve in the proof of Theorem 4.2.

Lemma 4.9. *Let $\varphi : (0,\infty) \to (0,\infty)$ be a function such that both φ and $t \mapsto t/\varphi(t)$ are non-decreasing. Given a transitive Γ-set X and a point $x_0 \in X$, the map $\mathbb{R}^\Gamma \to \mathbb{R}^X$ which associates to every $f \in \mathbb{R}^\Gamma$ the function f_x given by*

$$f_{x_0}(y) = \sum_{\gamma:\gamma\cdot x = y} |f(\gamma)|$$

satisfies $\|f_{x_0}\|_1 = \|f\|_1$. *Moreover, if* S_Γ *a a finite generating set for* Γ, *if* $x_0, x_1 \in X$ *and if* f *is a finitely supported function on* Γ *such that* $\left\|\nabla^r_{S_\Gamma}\right\|_1 = 1$, *then*

$$\frac{\|f_{x_0} - f_{x_1}\|_1}{\|f\|_1} \leqslant \frac{2\varphi(d_{S_\Gamma}(x_0, x_1))}{\varphi(\|f\|_1)}. \tag{6}$$

Proof. The main observation is that by triangular inequality

$$\|f_{x_0} - f_{x_1}\|_1 \leqslant 2 \max_{i=0,1} \|f_{x_i}\|_1 = 2\|f\|_1.$$

Using the monotonicity of $t/\varphi(t)$ we obtain

$$\|f_{x_0} - f_{x_1}\|_1 = \varphi(\|f_{x_0} - f_{x_1}\|_1)\frac{\|f_{x_0} - f_{x_1}\|_1}{\varphi(\|f_{x_0} - f_{x_1}\|_1)} \leqslant \varphi(\|f_{x_0} - f_{x_1}\|_1)\frac{2\|f\|_1}{\varphi(2\|f\|_1)}.$$

Now, since $\|\nabla f\|_1 = 1$, by applying Lemma 4.8 for $p = 1$ and using the monotonicity of φ we get

$$\|f_{x_0} - f_{x_1}\|_1 \leqslant 2\varphi(d_{S_\Gamma}(x_0, x_1))\frac{\|f\|_1}{\varphi(\|f\|_1)}$$

So the lemma follows. □

4.2 The induction technique

In this section, we shall prove that if $p \geqslant 1$ and Γ is an L^p measure *subquotient* of Λ, then every ℓ^p function on Λ induces a function on Ω whose S_Γ-gradient is well behaved (and a similar statement for φ measure subquotient). To deal with subquotients, we need a non-free analogue of the fact that if Λ is acting freely on Ω and X_Λ is a fundamental domain, the map $(x, \lambda) \mapsto \lambda * x$ is a measure-preserving bijection between $X_\Lambda \times \Lambda$ and Ω.

Lemma 4.10. *Suppose* X_Λ *is a fundamental domain for a measure-preserving* Λ*-action on a standard measured space* (Ω, μ). *Then for every Borel* $A \subseteq \Omega$, *we have*

$$\mu(A) = \int_{X_\Lambda} |(\Lambda * x) \cap A| \, d\mu(x).$$

In particular, for every measurable function $f : \Omega \to \mathbb{R}$, *we have*

$$\int_\Omega f \, d\mu = \int_{X_\Lambda} \sum_{y \in \Lambda * x} f(y) \, d\mu(x).$$

Proof. Since every subset of Ω can be written as a disjoint union of Λ-translates of Borel subsets of X_Λ, it suffices to show that the right term defines a Borel measure on Ω which is Λ-invariant and coincides with μ when restricted to X_Λ. The fact that the formula

$$m(A) = \int_{X_\Lambda} |(\Lambda * x) \cap A| \, d\mu(x).$$

defines a Borel measure follows from the fact that the map $x \mapsto |(\Lambda * x) \cap A|$ is Borel. Λ-invariance is clear, and the fact that the two measures coincide when restricted on X_Λ is also straightforward to check. □

Proposition 4.11. *(Monotonicity of the* ℓ^p*-gradient under* L^p *measured subquotient) Let* $p \geqslant 1$. *If* Γ *is an* L^p *measure subquotient of* Λ *via a coupling* $(\Omega, X_\Gamma, X_\Lambda, \mu)$, *and if* $f \in \ell^p(\Lambda)$, *then the induced function* $\tilde{f}$ *on* Ω *defined by* $\tilde{f}(\omega) = \left(\sum_{\lambda : \lambda^{-1} * \omega \in X_\Lambda} |f(\lambda)|^p\right)^{1/p}$ *satisfies* $\|\tilde{f}\|_p^p = \|f\|_p^p \, \mu(X_\Lambda)$ *and*

$$\left\|\nabla_{S_\Gamma} \tilde{f}\right\|_p^p \leqslant C \left\|\nabla^r_{S_\Lambda} f\right\|_p^p,$$

where $C = |S_\Gamma| \max_{s \in S_\Gamma} \int_{X_\Lambda} d_{S_\Lambda}(s \cdot x, s * x)^p d\mu(x)$.

Proof. As in the previous lemma, we think of Ω as a fibered space over X_Λ whose fibers are Λ-orbits. We may then view $\tilde{f}$ as a function of two variables $F(x,y)$ where $x \in X_\Lambda$ and $y \in \Lambda * x$ given by

$$F(x,y) = \sum_{\lambda : \lambda * x = y} f(\lambda).$$

For each $s \in S_\Gamma$, the function $s^{-1} * \tilde{f} : \omega \mapsto \tilde{f}(s * \omega)$ then corresponds to the function

$$(x,y) \mapsto F(s \cdot x, s * y),$$

by the definitions of our identifications. Now note that s induces an Λ-equivariant bijection between $h * x$ and $h * (s * x)$ which takes x to $s * x$, the latter being at distance $d_{S_\Lambda}(s \cdot x, s * x)$ from $s \cdot x \in X_\Lambda$, so by Lemma 4.8(i) and our definition of F we have

$$\|F(s \cdot x, s * \cdot) - F(x, \cdot)\|_p^p \leqslant d_{S_\Lambda}(s \cdot x, s * x) \left\|\nabla_{S_\Gamma}^r f\right\|_p^p$$

We can now use Lemma 4.10 to compute for all $s \in S_\Gamma$

$$\begin{aligned} \left\|\tilde{f} - s^{-1} * \tilde{f}\right\|_p^p &= \int_{X_\Lambda} \sum_{y \in \Lambda \cdot x} |F(x,y) - F(s \cdot x, s * y)|^p \\ &\leqslant \int_{X_\Lambda} d_{S_\Lambda}(s \cdot x, s * x) \left\|\nabla_{S_\Gamma}^r f\right\|_p^p \end{aligned}$$

The desired inequality now follows by the definition of $\left\|\nabla_{S_\Gamma} \tilde{f}\right\|_p^p$ and the symmetry of S_Γ. □

Proposition 4.12. *(Monotonicity of the ℓ^1-gradient under φ measured subquotient) Let $\varphi : (0,\infty) \to (0,\infty)$ be a function such that φ and $t/\varphi(t)$ are non-decreasing. If Γ is an φ measure subquotient of Λ via a coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$, and if f is a finitely supported function on Λ such that $\left\|\nabla_{S_\Lambda}^r f\right\|_1 = 1$, then the induced function $\tilde{f}$ on Ω defined by $\tilde{f}(\omega) = \sum_{h : h^{-1} * \omega \in X_\Lambda} f(h)$ satisfies*

$$\frac{\|\nabla_{S_\Gamma} \tilde{f}\|_1}{\|\tilde{f}\|_1} \leqslant \frac{2C}{\varphi(\|f\|_1)},$$

where $C = \dfrac{|S_\Gamma|}{\mu(X_\Lambda)} \max_{s \in S_\Gamma} \displaystyle\int_{X_\Lambda} \varphi(d_{S_\Lambda}(s \cdot x, s * x)) d\mu(x)$.

Proof. We view $\tilde{f}$ as a function of two variables $F(x,y)$ where $x \in X_\Lambda$ and $y \in \Lambda * x$ given by

$$F(x,y) = \sum_{\lambda : \lambda * x = y} |f(\lambda)|.$$

The proof is then similar to the previous case: using Lemma 4.9, we obtain

$$\frac{\|F(s \cdot x, s * \cdot) - F(x, \cdot)\|_1}{\|f\|_1} \leqslant \frac{2\varphi(d_{S_\Lambda}(s \cdot x, s * x))}{\varphi(\|f\|_1)},$$

which together with Lemma 4.10 and the fact that $\|\tilde{f}\|_1 = \mu(X_\Lambda) \|f\|_1$ yields the desired inequality. □

Remark 4.13. Using Fubini's theorem and the natural identification of $X_\Gamma \times \Gamma$ with Ω given by $(x,g) \mapsto g * x$, we may rewrite $\left\|\nabla_{S_\Gamma} \tilde{f}\right\|_p^p$ as

$$\left\|\nabla_{S_\Gamma} \tilde{f}\right\|_p^p = \int_{X_\Gamma} \sum_{s \in S_\Gamma} \sum_{\gamma \in \Gamma} \left|\tilde{f}(\gamma * x) - \tilde{f}(s\gamma * x)\right|^p d\mu(x)$$

Now for each $x \in X_\Gamma$, the function $\tilde{f}$ defines a function f^x on Γ given by $f^x(\gamma) = \tilde{f}(\gamma * x)$, and the previous equality may thus be rewritten as

$$\left\|\nabla_{S_\Gamma} \tilde{f}\right\|_p^p = \int_{X_\Gamma} \left\|\nabla^l_{S_\Gamma} f^x\right\|_p^p d\mu(x).$$

So the conclusion of the two previous propositions is that under the right assumption, any function f on Λ of small right ℓ^p-gradient induces functions f^x on Γ which have on average a small left ℓ^p gradient. From there on, as we will see in the next section, it is not hard to conclude the proof that the isoperimetric profile goes down for $p \geqslant 1$ under the assumption that the coupling is finite-to-one.

But without the finite-to-one assumption, we lose the uniform control on the size of the support. We will circumvent this by simultaneously controlling the size of the support of f^x and bounding its norm from below on a large portion of the fundamental domain of Γ. This will be done in Section 4.4.

A similar comment holds for $p < 1$ except that the behavior under quotient (even group quotient) of the L^p-isoperimetric profile is unclear. Besides, we need an L^1-integrability condition to induce it (see Proposition 4.2).

4.3 Monotonicity under at most m-to-one measure subquotients

In this section we shall prove the second items of Theorem 4.1 and Theorem 4.2.

The common feature between these statements is that the coupling is supposed to be at most m-to-one. This has the following consequence.

Lemma 4.14. *Let $(\Omega, X_\Gamma, X_\Lambda, \mu)$ be an at most m-to-one measure subquotient coupling between Γ and Λ. Let f be a function on Λ whose support has cardinality at most K, and let $(f^x)_{x \in X_\Gamma}$ be the family of functions on Γ defined by*

$$f^x(\gamma) = \left(\sum_{\lambda : \lambda * (\gamma * x) \in X_\Lambda} |f(\lambda)|^p \right)^{1/p},$$

for some $p > 0$. Then for each $x \in X_\Gamma$, the function f^x has support of cardinality at most mK.

Proof. Let $x \in X_\Gamma$, let $\lambda_0 \in \Lambda$ such that $\lambda_0 * x \in X_\Lambda$. By definition for every $\gamma \in \Gamma$ and $\lambda \in \Lambda$ we have $\lambda * (\gamma * x) \in X_\Lambda$ if and only if $\lambda * (\gamma * x) = \gamma \cdot (\lambda_0 * x)$, which since the Γ and Λ-actions on Ω commute is in turn equivalent to $\lambda * x = \gamma^{-1} * (\gamma \cdot (\lambda_0 * x))$. So we have

$$f^x(\gamma) = \sum_{\lambda : \lambda * x = \gamma^{-1} * (\gamma \cdot (\lambda_0 * x))} f(h).$$

The conclusion now follows from our assumption that the map $\gamma \mapsto \gamma^{-1} * (\gamma \cdot (\lambda_0 * x))$ is at most m-to-one. □

Proof of Theorem 4.1(ii). We start with a function f that realize the L^p-isoperimetric profile of Λ, and we consider the function $\tilde{f}$ on Ω defined in Proposition 4.2. By Proposition 4.2, there exists C' only depending on the coupling such that

$$\frac{\|\nabla_{S_\Gamma} \tilde{f}\|_p^p}{\|\tilde{f}\|_p^p} \leqslant C' \frac{\left\|\nabla^r_{S_\Lambda} f\right\|_p^p}{\|f\|_p^p}.$$

This implies that on a set of positive measure, the function f^x on Γ satisfies

$$\frac{\|\nabla^l_{S_\Gamma} f^x\|_p^p}{\|f^x\|_p^p} \leqslant 2C' \frac{\left\|\nabla^r_{S_\Lambda} f\right\|_p^p}{\|f\|_p^p}.$$

On the other hand, by Lemma 4.14, the support of f^x is almost surely less than $m|\operatorname{supp}(f)|$. Hence we deduce that $I_{p,\Gamma}(mn) \leqslant 2C' I_{p,\Lambda}(n)$, so we are done. □

Proof of Theorem 4.2(ii). As before we start with a function f that realize the L^1-isoperimetric profile of Λ. We then normalize f such that $\left\|\nabla^r_{S_\Lambda} f\right\|_1 = 1$ and consider the function $\tilde{f}$ on Ω defined in Proposition 4.2. We deduce from Proposition 4.2 that

$$\frac{\|\nabla_{S_\Gamma} \tilde{f}\|_1}{\|\tilde{f}\|_1} \leqslant \frac{C'}{\varphi(\|f\|_1)},$$

and the rest of the proof is identical. □

4.4 Monotonicity under measure quotients

In this section we prove the first items of Theorem 4.1 and Theorem 4.2. We start by fixing a symmetric generating subset W of Λ so that $Z := W.X_\Lambda \cap X_\Gamma$ has measure at least $3/4$.

We are then given a function f_1 which realizes the isoperimetric profile of Λ which we first change as follows. We define a function f_2 on Λ by $f_2(h) = \max_{w\in W} |f_1(wh)|$. The important fact is that f_2 "almost" realizes the isoperimetric profile of Λ: the support of f_2 has size at most $|W|$ times the support of f_1, $\|f_2\|_p \geqslant \|f_1\|_p$, and

$$\|\nabla f_2\|_p^p \leqslant K\|\nabla f_1\|_p^p, \tag{7}$$

where $K = |W|^2(2\max_{w\in W} |w|_{S_\Lambda} + 1)^p$. Indeed,

$$\begin{aligned}
\|\nabla f_2\|_p^p &\leqslant \sum_{w,w'\in W,\lambda\in\Lambda,s\in S_\Lambda} |f_1(sw'\lambda) - f_1(w\lambda)|^p \\
&= \sum_{w,w'\in W,\lambda\in\Lambda,s\in S_\Lambda} |f_1(w^{-1}sw'\lambda) - f_1(\lambda)|^p \\
&\leqslant \sum_{w,w'\in W} (|w^{-1}|_{S_\Lambda} + |w'|_{S_\Lambda} + 1)^p \sum_{\lambda\in\Lambda,s'\in S_\Lambda} |f_1(s'\lambda) - f_1(\lambda)|^p \\
&\leqslant |W|^2(2\max_{w\in W} |w|_{S_\Lambda} + 1)^p \sum_{\lambda\in\Lambda,s'\in S_\Lambda} |f_1(s'\lambda) - f_1(\lambda)|^p,
\end{aligned}$$

where the third inequality follows by triangular inequality, after writing $w^{-1}sw'$ as a word in S_Λ.

For all $p > 0$, we set

$$F_2(\omega) = \left(\sum_{\lambda: h*\omega\in X_\Lambda} f_2(\lambda)^p\right)^{1/p}.$$

As in the previous section, this provides a random function $\tilde{f}_2(x,\cdot)$ on Γ for $x \in X_\Gamma$ given by

$$\tilde{f}_2(x,\gamma) = F_2(\gamma * x) = \left(\sum_{\lambda:\lambda*\gamma*x\in X_\Lambda} f_2(\lambda)^p\right)^{1/p}.$$

If the Λ-action is assumed to be free, then

$$F_2(\omega) = \sum_{\lambda:\lambda*\omega\in X_\Lambda} f_2(\lambda).$$

Therefore, under the assumption that $p \geqslant 1$ (item (i)) or that the Λ action is free (item (iii)), Proposition 4.2 implies that we find V and V' of X_Γ of measure $9/10$, such that for all $x \in V$,

$$\left\|\nabla_{S_\Gamma} \tilde{f}_2(x,\cdot)\right\|_p^p \leqslant 10C\,\|\nabla_{S_\Gamma} f_2\|_p^p,$$

and for all $x \in V'$

$$|\operatorname{supp}(\tilde{f}_2(x,\cdot)| \leqslant 10\mu(X_\Lambda)|\operatorname{supp}(f_2)|.$$

For the proof of Theorem 4.2, we normalize f_2 such that $\|\nabla_{S_\Gamma} f_2\|_1 = 1$. Using Proposition 4.2 instead of Proposition 4.2, we find V and V' of X_Γ of measure $9/10$, such that for all $x \in V$,

$$\frac{\left\|\nabla_{S_\Gamma} \tilde{f}_2(x,\cdot)\right\|_1}{\|f_2\|_1} \leqslant \frac{10C}{\varphi(\|f_2\|_1)},$$

and for all $x \in V'$

$$|\operatorname{supp}(\tilde{f}_2(x,\cdot)| \leqslant 10\mu(X_\Lambda)|\operatorname{supp}(f_2)|.$$

We now need to find a subset of large measure on which the ℓ^p-norm of $\tilde{f}_2(x,\cdot)$ satisfies a uniform lower bound. This is where we use that W is a symmetric generating subset so that $Z := W * X_\Lambda \cap X_\Gamma$ has measure at least $3/4$. Items (i) of Theorem 4.1 and Theorem 4.2 result from the following lemma, whose proof occupies the rest of the section.

Lemma 4.15. *There exists a measurable subset $Y \subset X_\Gamma$ of measure $\geqslant 1/2$ such that for all $y \in Y$,*

$$\sum_{\gamma\in\Gamma} \left|\tilde{f}_2(y,\gamma)\right|^p \geqslant \frac{1}{4|W|}\|f_1\|_p^p.$$

Proof. We start with the following observation.

Claim 4.16. For all $x \in X_\Gamma$ and $\lambda \in \Lambda$ such that $\lambda \cdot x \in Z$, there is $\gamma \in \Gamma$ such that $\lambda * \gamma * x \in W * X_\Lambda$.

Proof of the claim. This follows by inspection of the definitions: $\lambda \cdot x \in Z$ means that there exists $\gamma \in \Gamma$ such that $\lambda * \gamma * x \in Z$, and therefore that $\lambda * \gamma * x \in W * X_\Lambda$. $\square_{\text{claim}}$

We now have the following crude inequality from the claim, which we then rewrite by exchanging orders of summation:

$$\begin{aligned}\sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p &\leqslant \sum_{\lambda} \sum_{\gamma: h*g*x\in W*X_\Lambda} |f_1(\lambda)|^p \\ &\leqslant \sum_{\gamma} \sum_{\lambda:\lambda*\gamma*x\in W*X_\Lambda} |f_1(\lambda)|^p.\end{aligned}$$

We can rewrite the above sum as

$$\begin{aligned}\sum_{\gamma}\sum_{w\in W}\sum_{\lambda: w\lambda*\gamma*x\in X_\Lambda} |f_1(\lambda)|^p &= \sum_{\gamma}\sum_{w\in W}\sum_{\lambda:\lambda*\gamma*x\in X_\Lambda} \left|f_1(w^{-1}\lambda)\right|^p \\ &\leqslant \sum_{\gamma}\sum_{w\in W}\sum_{\lambda:\lambda*\gamma*x\in X_\Lambda} f_2(\lambda)^p \\ &\leqslant |W| \left\|\tilde{f}_2(x,\cdot)\right\|_p^p,\end{aligned}$$

where we used that $f_2(\lambda) = \max_{w\in W}|f_1(w\lambda)|$ to go from the first to the second line. Putting this together with the previous inequation, we obtain that

$$\sum_{\lambda,\lambda\cdot x\in Z} |f_1(\lambda)|^p \leqslant |W| \left\|\tilde{f}_2(x,\cdot)\right\|_p^p,$$

which we finally rewrite as

$$\sum_{\gamma\in\Gamma} \left|\tilde{f}_2(x,\gamma)\right|^p \geqslant \frac{1}{|W|}\sum_{\lambda,\lambda\cdot x\in Z} |f_1(\lambda)|^p. \tag{8}$$

On the other hand, we have

$$\begin{aligned}\int_{X_\Gamma} \sum_{\lambda,\lambda\cdot x\in Z} |f_1(\lambda)|^p \, d\mu(x) &\geqslant \int_{X_\Gamma} \sum_{\lambda\in\Lambda} |f_1(\lambda)|^p \, 1_{\lambda^{-1}\cdot Z}(x) d\mu(x) \\ &\geqslant \sum_{\lambda\in\Lambda} |f_1(\lambda)|^p \int_{X_\Gamma} 1_{\lambda^{-1}\cdot Z}(x) d\mu(x) \\ &= \|f_1\|_p^p \, \mu(Z) \\ &\geqslant \frac{3}{4} \|f_1\|_p^p .\end{aligned}$$

We will now use this towards the following claim from which the result easily follows.

Claim 4.17. There is a subset $Y \subset X_\Gamma$ of measure at least $1/2$ such that for all $x \in Y$,

$$\sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p \geqslant \frac{1}{4} \|f_1\|_p^p .$$

Proof. Indeed, assume by contradiction that

$$\sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p < \frac{1}{4} \|f_1\|_p^p$$

for all x in a set $V \subset X_\Gamma$ of measure $> 1/2$. Then by the inequation just before this claim, we have

$$\begin{aligned}\frac{3}{4}\|f_1\|_p^p &\leqslant \int_V \sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p \, d\mu(x) + \int_{V^c} \sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p \, d\mu(x) \\ &< \frac{1}{4} \|f_1\|_p^p + \frac{1}{2} \|f_1\|_p^p \\ &< \frac{3}{4} \|f_1\|_p^p ,\end{aligned}$$

which is a contradiction. □

Putting the above claim together with inequation (8) we deduce that for all $x \in Y$,

$$\sum_{\gamma\in\Gamma} \left|\tilde{f}_2(x,\gamma)\right|^p \geqslant \frac{1}{|W|} \sum_{\lambda:\lambda\cdot x\in Z} |f_1(\lambda)|^p \geqslant \frac{1}{4\,|W|} \|f_1\|_p^p$$

as wanted. □

4.5 A quantitative version of the Ornstein-Weiss theorem

In this section, we first observe that any two amenable groups admit a (φ,φ)-integrable orbit equivalence coupling for some φ which grows slower than the logarithm, and then that there cannot be a universal such φ, thus proving Corollary 2 from the introduction.

Proposition 4.18. *For every orbit equivalence coupling between two finitely generated groups Γ and Λ, there exists a concave increasing unbounded function φ with $\varphi(0) = 0$ and a (φ,φ) orbit equivalence coupling between them. Moreover one can assume that $\varphi(t)/\log t$ is non-increasing on $[1,\infty)$.*

Proof. Let

$$k(t) = \mu\left(\left\{x \in X;\ \max_{s\in S_\Gamma} |\alpha(s,x)|_{S_\Lambda} \geqslant t\right\}\right) + \mu\left(\left\{x \in X;\ \max_{s\in S_\Lambda} |\beta(s,x)|_{S_\Gamma} \geqslant t\right\}\right).$$

Since k tends to zero at infinity, there exists an increasing sequence of positive integers (a_n) such that $a_0 = 0$, and $k(a_n) \leqslant 2^{-n}$. Up to taking a subsequence, that we can assume that $a_{n+1} - a_n$ and $\frac{\log a_n}{n}$ are non-decreasing. Let φ be the continuous piecewise linear function with breakpoints a_n satisfying $\varphi(a_n) = n$. Note that φ is increasing, concave and such that $\varphi(t)/\log t$ is non-increasing on $[1,\infty)$. Moreover we have for all $s \in S_\Gamma$,

$$\begin{aligned}\int \varphi(|\alpha(s,x)|_{S_\Lambda} d\mu(x) &\leqslant \sum_{n\geqslant 0} \varphi(a_n)\mu\left(\{x \in X;\ |\alpha(s,x)|_{S_\Lambda} \geqslant a_n\}\right)\\ &\leqslant \sum_{n\geqslant 0} nk(a_n)\\ &\leqslant \sum_{n\geqslant 0} 2^{-n}n < \infty,\end{aligned}$$

The same computation shows that for all $s \in S_\Lambda$,

$$\int \varphi(|\beta(s,x)|_{S_\Gamma}) d\mu(x) < \infty.$$

So we are done. □

We deduce the following quantitative version of Ornstein and Weiss's theorem.

Corollary 4.19. *Let Γ and Λ be infinite finitely generated amenable groups. There exists a concave increasing unbounded function φ with $\varphi(0) = 0$ and $\varphi(t)/\log t$ non-increasing on $[1,\infty)$ such that there is a (φ,φ) orbit equivalence coupling between them.*

Recall the following theorem from [BZ15].

Theorem 4.20. *Let F be a decreasing function such that $F(t)\log t$ is non-decreasing on $[1,\infty)$. Then there exists a finitely generated group Γ whose isoperimetric profile satisfies $I_{1,\Gamma} \asymp F$.*

We deduce the following corollary, which is in sharp contrast with the previous corollary.

Corollary 4.21. *For every concave increasing unbounded function ψ such that $\psi(t)/\log t$ is non-increasing on $[1,\infty)$ there exists a finitely generated amenable group Γ with the following property: for every concave function φ such that $\varphi(0) = 0$, if Γ is a φ-integrable measure subquotient of $\mathbb{Z}$, then $\varphi \preccurlyeq \psi$.*

Proof. Consider the function $F(t) = \frac{1}{\psi(t)}$, then $F(t)\log t = \log(t)/\psi(t)$, so it is non-decreasing on $[1,+\infty)$. We can then apply Theorem 4.20 and find a finitely generated group Γ whose isoperimetric profile satisfies $I_{1,\Gamma} \asymp F$.

Now let φ be a concave non-decreasing function such that $\varphi(0) = 0$, then by concavity we have that $t/\varphi(t)$ is non-decreasing. If there is a φ-integrable measure subquotient coupling from Γ to $\mathbb{Z}$, then we can apply Theorem 4.2, and since $I_{1,\mathbb{Z}}(n) \asymp 1/n$ and $I_{1,\Gamma} \asymp \frac{1}{\psi}$ we get that $\frac{1}{\psi} \preccurlyeq \frac{1}{\varphi}$, hence $\varphi \preccurlyeq \psi$ as wanted. □

It is now easy to deduce Corollary 2 from Corollary 4.21 using the composition rule for couplings (see Proposition 2.27).

5 L^∞-measure subgroups

In this section we discuss the notion of at most m-to-one L^∞-measure subgroups. We shall see that for amenable groups this notion is equivalent to that of regular embedding, which will allow us to deduce various results stated in the introduction.

5.1 L^∞-measure subgroups and regular embeddings

The following proposition can be extracted from the proof of [Sha04, Thm. 2.1.2], and provides a useful way of building measured subgroup couplings from *Borel* subquotient couplings.

Proposition 5.1. *Let Ω be a standard Borel space, suppose that we have a $\Gamma \times \Lambda$-action on Ω which has a Borel Λ fundamental domain X_Λ, that Λ is acting freely and that we have a Γ-invariant probability measure μ for the induced action on X_Λ. Then μ has a unique Λ-invariant extension to a σ-finite invariant measure m on Ω which is Γ-invariant as well.*

In particular, if the Γ-action on Ω was free and had a Borel fundamental domain, then Γ is a measure subgroup of Λ.

Proof. Since every Borel subset of Ω is a countable disjoint union of Λ-translates of Borel subsets of X_Λ, the Λ-invariant extension of μ is unique. We build it by letting

$$m(A) = \sum_{\lambda \in \Lambda} \mu((\lambda * A) \cap X_\Lambda).$$

By definition, the measure m is Λ-invariant, let us show that it is Γ-invariant as well. Let $\gamma \in \Gamma$, if A is a Borel subset of X_Λ, by the definition of the induced action on X_Λ we have

$$\gamma \cdot A = \bigsqcup_{\lambda \in \Lambda} (\lambda * \gamma * A) \cap X_\Lambda$$

and so $m(\gamma * A) = \mu(\gamma \cdot A) = \mu(A) = m(A)$. Now for an arbitrary $\lambda \in \Lambda$ and a Borel subset $A \subseteq X_\Lambda$, we have

$$m(\gamma * (\lambda * A)) = m(\lambda * (\gamma * A)) = m(\gamma * A) = m(A) = m(\lambda * A),$$

and since every Borel subset of Ω is a countable disjoint union of Λ-translates of Borel subsets of X_Λ, we conclude that the Γ-action on Ω preserves m. □

Recall the following definition from [BST12].

Definition 5.2. Let Γ and Λ be two countable finitely generated groups, a map $f : \Gamma \to \Lambda$ is a **regular embedding** if it is Lipschitz and there exists $m \in \mathbb{N}$ such that the preimages of f have size at most m for some $m \in \mathbb{N}$. When there exists such a map, we say that Γ regularly embeds in Λ.

Remark 5.3. Coarse embeddings are special cases of regular embeddings.

The second part of the following theorem is a slightly generalized version of [Sha04, Thm. 2.1.2] (which was proved for coarse embeddings).

Theorem 5.4. *Let Γ and Λ be finitely generated groups.*

- *Assume that Γ is an L^∞-subgroup of Λ. Then Γ regularly embeds into Λ.*
- *Conversely, if Γ regularly embeds into Λ and Γ is amenable, then Γ is an at most m-to-one L^∞ measure subgroup of Λ.*

We deduce from the first statement of Theorem 5.4 and the fact that asymptotic dimension is monotonous under regular embedding (see [BST12]) the following corollary.

Corollary 5.5. *The asymptotic dimension is monotonous under taking at most m-to-one* L^∞ *measure subgroup.* □

And from the second part of the theorem, we deduce the following result that we announced in the introduction.

Corollary 5.6. *For every* $1 \leqslant p \leqslant \infty$*, the isoperimetric profile is monotonous under regular embedding between amenable groups.*

Proof. Let Γ and Λ be finitely generated amenable groups. By the previous theorem, the existence of a regular embedding from Γ to Λ implies that of an at most m-to-one L^∞ measure subgroup coupling from Γ to Λ. Such a coupling is in particular L^1, so the result follows from the second item in Theorem 4.1. □

The rest of this subsection is dedicated to the proof of the theorem.

Proof of Theorem 5.4. The first statement is clear as the assumptions imply the cocycle is almost surely Lipschitz and at most m-to-one. We now prove the second statement of the theorem. We fix K such that there is an m-to one K-Lipschitz maps $\Gamma \to \Lambda$. We then let F be a finite set of cardinality m, and denote by $\pi_\Lambda : \Lambda \times F \to \Lambda$ the projection on the first coordinate. Our coupling space is

$$\Omega := \{f : \Gamma \to \Lambda \times F : f \text{ is injective and } \pi_\Lambda \circ f \text{ is } K\text{-Lipschitz}\}$$

Note that since F has cardinality m, every at most m-to-one map $\Gamma \to \Lambda$ can be lifted to an injective map $\Gamma \to \Lambda \times F$, so Ω is not empty. The Lipschitz condition and the fact that balls in Λ are finite ensures that Ω is locally compact for the product topology. As before we have a $\Gamma \times \Lambda$-action given by $(\gamma, \lambda) \cdot f(g) = \lambda f(\gamma^{-1} g)$, and a compact fundamental for the Λ-action is given by $X_\Lambda = \{f \in \Omega : \pi_\Lambda(f(1_\Gamma)) = 1_\Lambda\}$. We also have a Borel fundamental domain for the Γ-action obtained as follows: we fix a well-order $<$ on Λ, and then the set of functions f which attain their $<$-minimum at 1_Γ is the desired Borel fundamental domain.

The cocycle $c : \Gamma \times X_\Lambda \to \Lambda$ is given by $c(\gamma, f) = \pi_\Lambda(f(\gamma^{-1}))^{-1}$, and the injectivity condition yields that Γ acts freely. Finally for each $\gamma \in \Gamma$, the Lipschitz condition implies that $c(\gamma, \cdot)$ is bounded, and since Γ is amenable we may find a Γ-invariant measure on X_Λ which we extend via Proposition 5.1 in order to get the desired L^∞ measure subgroup coupling. □

5.2 A continuum of 3-solvable groups

In this section we prove the following result, announced in the introduction.

Theorem 5.7. *There exists a uncountable family of groups* Γ_i*, such that*

(i) $\Gamma_i = N_i \rtimes \mathbb{Z}$*, where* N_i *is locally finite, and 2-step nilpotent;*

(ii) for any $i \neq j$ *and any* $m \geqslant 1$*,* Γ_i *is not an at most m-to-one* L^1*-measure subquotient (nor an* L^1*-measure quotient) of* Γ_j*.*

Note that (i) implies in that Γ_i is 3-step solvable and has asymptotic dimension 1. We deduce the result announced in the introduction.

Corollary 5.8. *There exists a uncountable family of groups* Γ_i*, such that*

(i) $\Gamma_i = N_i \rtimes \mathbb{Z}$*, where* N_i *is locally finite, and 2-step nilpotent;*

(ii) for any $i \neq j$ *and any* $m \geqslant 1$*,* Γ_i *does not regularly embed into* Γ_j*.*

Proof of Theorem 5.7. To prove the theorem, we shall use [EZ17, Corollary 3.3] which is phrased in terms of Følner function. Since the Følner function is asymptotically equivalent to the generalized inverse of the L^1-isoperimetric profile (see for instance [BZ15, §4.1]), Corollary 5.6 implies that it is monotonous under regular embedding.

Now by [EZ17, Corollary 3.3], for any non-decreasing function $\tau : [1, \infty) \to [1, \infty)$ such that $\tau(n+1) - \tau(n) \leqslant n$ and $\tau(n) \geqslant n$, there exists a group as in (i) whose Følner function is asymptotically equivalent to $\exp(\tau(n))$. Let us now explain how to get uncountably many asymptotically incomparable such functions.

Claim 5.9. For every $1 < a < b < 2$, there is a non-decreasing function $\tau_{a,b} : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have

$$\tau_{a,b}(n) \geqslant n \text{ and } \tau_{a,b}(n+1) - \tau_{a,b}(n) \leqslant n, \tag{*}$$

and the following three conditions are met:

(1) for all but finitely many $n \in \mathbb{N}$ we have $n^a \leqslant \tau_{a,b}(n) \leqslant n^b$;

(2) there are infinitely many $n \in \mathbb{N}$ such that $\tau_{a,b}(n) = n^b$;

(3) the set of $n \in \mathbb{N}$ such that $\tau_{a,b}(n) = n^a$ contains arbitrarily large intervals.

Moreover, for every $1 < a' < a < b < b' < 2$, the functions $\exp(\tau_{a,b})$ and $\exp(\tau_{a',b'})$ are asymptotically incomparable.

Proof of the claim. Let us start by proving the existence of $\tau_{a,b}$ as above for $1 < a < b < 2$. First, we fix $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $(n+1)^b - n^b \leqslant n$, so in particular $(n+1)^a - n^a \leqslant n$. Since $b < 2$, we may and do fix $N_0 > N$ such that

$$\sum_{n=0}^{N_0-1} n \geqslant N_0^b.$$

We then define $\tau_{a,b}$ by induction on the interval $[0, N_0]$ by $\tau_{a,b}(0) = 0$ and for all $n < N_0$,

$$\tau_{a,b}(n+1) = \min(\tau_{a,b}(n) + n, N_0^b).$$

Observe that by the previous inequation $\tau_{a,b}(N_0) = N_0^b$.

Suppose now by induction $\tau_{a,b}$ has been defined on an interval $[0, N_k]$, satisfies (*) and satisfies $\tau_{a,b}(N_k) = N_k^b$, then it suffices to explain how to extend its definition to a bigger interval $[0, N_{k+1}]$ so that it still satisfies (*) and

(1') for all $n \in (N_k, N_{k+1}]$ we have $n^a \leqslant \tau_{a,b}(n) \leqslant n^b$;

(2') $\tau_{a,b}(N_{k+1}) = N_{k+1}^b$;

(3') there is an interval I_k of size k such that for all $n \in I_k$, $\tau_{a,b}(n) = n^a$.

In order to do so, let M_k be the first integer such that $(M_k)^a \geqslant N_k^b$. We then let $\tau_{a,b}(n) = N_k^b$ for all $n \in (N_k, M_k)$, and then for all $n \in [M_k, M_k + k]$ we let $\tau_{a,b}(n) = n^a$, which takes care of condition (2').

Then, since $b < 2$, we may and do define N_{k+1} as the least integer such that

$$\tau_{a,b}(N_k + k) + \sum_{n=N_k+k}^{N_{k+1}-1} n \geqslant N_{k+1}^b.$$

We now define by induction on $n \in [N_k + k, N_{k+1})$

$$\tau_{a,b}(n+1) = \min(n + \tau_{a,b}(n), n^b),$$

which by the previous inequation guarantees that $\tau_{a,b}(N_{k+1}) = N_{k+1}^b$, so (3') is satisfied. Finally, since for all $n \geqslant N_k$, we have $(n+1)^a - n^a \leqslant n$, an inspection of the definition shows that condition (*) is still satisfied, and (1') is clearly satisfied. This finishes the construction of the desired $\tau_{a,b}$.

We now prove the incomparability statement. First note that we cannot have $\exp(\tau_{a',b'})$ asymptotically bounded by $\exp(\tau_{a,b})$ since we have $\tau_{a',b'}(n) = n^{b'}$ for infinitely many $n \in \mathbb{N}$ but $\tau_{a,b}(n) \leqslant n^b$ for all but finitely many $n \in \mathbb{N}$ and $b < b'$.

Conversely, suppose that there is $C \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have $\exp(\tau_{a,b}(n)) \leqslant C\exp(\tau_{a',b'}(Cn))$. Note that the set $\{Cn : n \in \mathbb{N}\}$ has to intersect infinitely many times any set which contains arbitrarily large intervals. Since $\tau_{a',b'}$ satisfies condition (3) we thus find infinitely may $n \in \mathbb{N}$ such that $\tau_{a',b'}(Cn) = (Cn)^{a'}$. On the other hand for all but finitely many $n \in \mathbb{N}$ we have $\tau_{a,b}(n) \geqslant n^a$. We conclude that there are infinitely many $n \in \mathbb{N}$ such that

$$\exp(n^a) \leqslant C\exp((Cn)^{a'}),$$

a contradiction. $\square_{\text{claim}}$

We now fix for every $1 < a < b$ a function $\tau_{a,b}$ as above. Then it satisfies the assumption of [EZ17, Corollary 3.3], so there is a group $\Gamma_{a,b}$ satisfying (i) whose Følner function is asymptotically equivalent to $\exp(\tau_{a,b})$.

Now, as explained right before the previous claim, the Følner function is monotonous under coarse embeddings. So by the claim for all $1 < a' < a < b < b' < 2$, the groups $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ are not coarsely embeddable one another. Therefore the family $(\Gamma_{1+\varepsilon,2-\varepsilon})_{0<\varepsilon<1/2}$ is the uncountable family we seeked. $\square$

6 Følner tilings

6.1 Følner tiling sequences and orbit equivalence

Definition 6.1. Let Γ be an amenable group, given a sequence (F_k) of finite subsets of Γ, we say that (F_k) is a (left) **Følner tiling sequence** if when we define by induction the sequence of finite subsets (T_k) (the associated sequence of tiles) by $T_0 = F_0$ and $T_{k+1} = T_k F_{k+1}$, then

(i) we actually can write this as a *disjoint* union;

$$T_{k+1} = \bigsqcup_{\gamma \in F_{k+1}} T_k\gamma;$$

(ii) (T_k) is a left Følner sequence: for all $\gamma \in \Gamma$, $\lim\limits_{k\to+\infty} \dfrac{|\gamma F_k \setminus F_k|}{|F_k|} = 0$.

If in addition there exists a decreasing sequence of finite index subgroups Γ_k such that the F_k's are left coset representatives of Γ_{k-1} modulo Γ_k, then we call (F_k) a **profinite Følner tiling sequence** associated to (Γ_k).

Remark 6.2. In some examples it will be more convenient to consider *right* Følner tiling sequences (F_k), i.e. sequences such that (F_k^{-1}) is a left Følner tiling sequence. Equivalently, these are right Følner sequences (F_k) such that if we define $T_k = F_k \cdots F_0$ then every element of T_k can uniquely be written as $f_k \cdots f_0$ where each f_i belongs to F_i. Nevertheless, every Følner tiling sequence will be a *left* Følner tiling sequence unless specified otherwise.

To every Følner tiling sequence (F_k), we associate a measure-preserving Γ-action constructed as follows. We consider the standard Borel probability space $(X = \prod_k F_k, \mu)$, where each factor is equipped

with the normalized counting measure, and μ is the product measure. Each element $x = (x_k)_{k\in\mathbb{N}}$ of X defines a sequence $(g_k(x))_{k\in\mathbb{N}}$ of elements of Γ given by

$$g_k(x) = x_0 \cdots x_k \in T_k$$

Observe that by condition (i), each g_k is an equidistributed random element of T_k. Since (T_k) is a left Følner sequence, we deduce that for every $\gamma \in \Gamma$ and almost every $x \in X$, there is n such that $\gamma g_n(x) \in T_n$. We can then write uniquely $\gamma g_n(x) = x'_0 \cdots x'_n$ where $x'_i \in F_i$, and we then define

$$\gamma \cdot (x_k)_{k\in\mathbb{N}} = (x'_0, ..., x'_n, x_{n+1}, x_{n+2}, ...).$$

Observe that this does not depend on the choice of n because we have $\gamma g_{n+1}(x) = \gamma g_n(x) x_{n+1} \in T_{n+1}$. It is then not hard to check that this defines a measure-preserving free action on (a full measure subset of) X.

Up to measure zero, this group action induces the equivalence relation of equality up to a finite set of indices, also called the **cofinite equivalence relation** E_{cof}. Indeed, we have just seen that for almost every $x \in X$ and all $\gamma \in \Gamma$, x and $\gamma \cdot x$ are equal up to a finite set of indices. Conversely, if x and x' are such that $x_j = x'_j$ for all $j \geqslant k+1$, the element $\gamma = g_k(x)g_k(x')^{-1} \in G$ satisfies $\gamma g_k(x) = g_k(x') \in T_k$, and hence $\gamma \cdot x = x'$.

For instance, if $\Gamma = \mathbb{Z}$, $F_k = \{0, 2^k\}$, the tiles are $T_k = \{0, ..., 2^{k+1} - 1\}$ and for this example we get the usual odometer, up to renaming each F_k as $\{0, 1\}$. More generally, when the tiles T_k are left coset representatives for finite index subgroups Γ_k of Γ, and if $F_k \subset \Gamma_{k-1}$, then we get the profinite action $\Gamma \curvearrowright \operatorname{proj\,lim} \Gamma/\Gamma_k$. We deduce from this discussion the following proposition.

Proposition 6.3. *Assume that (F_k) is a Følner tiling sequence for Γ. Then Γ has a measure-preserving action on the infinite product probability space $(X = \prod_k F_k, \mu)$, which almost surely generates the co-finite equivalence relation on this product. Moreover if (F_k) is a profinite Følner tiling sequence associated to (Γ_k), then this action is isomorphic to the profinite action of Γ on $\varprojlim \Gamma/\Gamma_k$.*

Consider the following (possibly infinite) measurable distance on X given by

$$\rho(x, x') = \inf\{n \in \mathbb{N} \colon \forall k \geqslant n, x_k = x'_k\}$$

Observe that x and x' are equal up to a finite set of indices if and only if $\rho(x, x') < +\infty$. Also, by the definition of our action, for every $\gamma \in \Gamma$ and almost every $x \in X$ we have that $\rho(\gamma \cdot x, x) > k$ if and only if $\gamma g_k(x) \notin T_k$. In particular,

$$\mu\left(\{x,\ \rho(\gamma \cdot x, x) > k\}\right) = \frac{|T_k \setminus \gamma^{-1} T_k|}{|T_k|} = \frac{|\gamma T_k \,\triangle\, T_k|}{2|T_k|}. \tag{9}$$

In order to obtain quantitative statements, we introduce the following parameters.

Definition 6.4. Let (ε_k) be a sequence of strictly positive numbers tending to 0, and (R_k) be a sequence of positive reals. Say that a Følner tiling sequence (F_k) of a finitely generated group Γ equipped with a finite generating set S_Γ is an (ε_k, R_k)**-Følner tiling sequence** when each tile T_k has d_{S_Γ}-diameter at most R_k and every $s \in S_\Gamma$ satisfies $|T_k \setminus sT_k| \leqslant \varepsilon_k |T_k|$ (or equivalently $|sT_k \setminus T_k| \leqslant \varepsilon_k |T_k|$).

Lemma 6.5. *Let (F_k) be (ε_k, R_k)-Følner tiling sequence of a finitely generated group Γ equipped with a finite generating set S_Γ. Then*

(i) for all $s \in S_\Gamma$, we have for all $k \geqslant 0$ that $\mu\left(\{x \in X \colon \rho(s \cdot x, x) > k\}\right) \leqslant \varepsilon_k$;

(ii) for almost every $x \in X$, if $|\gamma|_{S_\Gamma} > 2R_k$, then $\rho(\gamma \cdot x, x) > k$.

Proof. The first item follows from equation (9), and the second item by its first part since if $|\gamma|_{S_\Gamma} > 2R_k$, then $\gamma T_k \cap T_k = \emptyset$ as $\operatorname{diam}(T_k) \leqslant R_k$. □

Observe that if two amenable groups Γ and Γ' admit respective Følner tiling sequences (F_k) and (F'_k) such that $|F_k| = |F'_k|$, then there is a natural measure-preserving bijection Ψ between the corresponding product spaces (X, μ) and (X', μ') that preserves the cofinite equivalence relation, and Proposition 6.3 now provides us with an explicit orbit equivalence between them. In what follows, we identify X and X' via Ψ. The following proposition relates the parameters of the Følner tiling sequences with the integrability of the cocycles.

Proposition 6.6. *Suppose that (F_k) is an (ε_k, R_k) Følner tiling sequences for Γ and (F'_k) is an (ε'_k, R'_k) Følner tiling sequence for Γ'. Then for all $s \in S_\Gamma$ and all $k \geqslant 0$,*

$$\mu\left(\{x \in X \colon d_{S_{\Gamma'}}(x, s \cdot x) > 2R'_k\}\right) \leqslant \varepsilon_k$$

In particular, the groups are orbit equivalent, and if $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that the sequence $(\varphi(2R'_k)(\varepsilon_{k-1} - \varepsilon_k))_{k \in \mathbb{N}}$ is summable, then for all $s \in S_\Gamma$

$$\int_X \varphi(d_{S_{\Gamma'}}(x, s \cdot x)) d\mu(x) < \infty.$$

Proof. The first part follows from Lemma 6.5 as $|F_k| = |F'_k|$ for all $k \in \mathbb{N}$ and

$$\mu\left(\{x \in X \colon d_{S_{\Gamma'}}(x, s \cdot x) > 2R'_k\}\right) \leqslant \mu\left(\{x \in X \colon \rho(s \cdot x, x) > k\}\right) \leqslant \varepsilon_k.$$

The second part follows by noting that for $s \in S_\Gamma$ we have that

$$\begin{aligned}\int_X \varphi(d_{S_{\Gamma'}}(x, s \cdot x)) d\mu(x) &\leqslant \varphi(2R'_0) + \sum_{k=1}^{\infty} \varphi(2R'_k) \mu\left(\{x \in X \colon 2R'_{k-1} < d_{S_{\Gamma'}}(x, s \cdot x) \leqslant 2R'_k\}\right) \\ &\leqslant \varphi(R'_0) + \sum_{k=1}^{\infty} \varphi(2R'_k)(\varepsilon_{k-1} - \varepsilon_k),\end{aligned}$$

which is finite by assumption. □

6.2 Applications to groups with polynomial growth

We start applying Proposition 6.6 to torsion-free abelian groups.

Proposition 6.7. *Let n be a positive integer. The group $\mathbb{Z}^n$ (equipped with its standard generating set) admits a profinite (ε_k, R_k)-Følner tiling sequence (F_k), with $|F_k| = 2^{nk}$, $R_k = n2^{k+1}$ and $\varepsilon_k = 2^{-k-1}$ for any $k \geqslant 0$.*

Proof. We let $F_k = \{0, 2^k\}^n$ for any $k \geqslant 0$. One can check that $T_k = \{0, 1, \dots, 2^{k+1} - 1\}^n$, which is a coset representative for the finite index subgroup $\Gamma_k = (2^{k+1}\mathbb{Z})^n$. The diameter of T_k is bounded by $n2^{k+1}$ and its size equals $2^{n(k+1)}$. Finally take s a generator of $\mathbb{Z}^n$. Without loss of generality, we can assume that s is the first basis vector in $\mathbb{Z}^n$. Then, we have

$$T_k \setminus ((1, 0, \dots, 0) + T_k) = \{0\} \times \{0, 1, \dots, 2^{k+1} - 1\}^{n-1},$$

whose cardinality is 2^{k+1} smaller than that of T_k, so we are done. □

Corollary 6.8. *Let n and m be positive integers. The group $\mathbb{Z}^n$ (equipped with its standard generating set) admits a (ε_k, R_k)-Følner tiling sequence (F'_k), with $|F'_k| = 2^{nmk}$, $R_k = n2^{m(k+1)}$ and $\varepsilon_k = 2^{-m(k+1)}$ for any $k \geqslant 0$.*

Proof. Let $(F_k)_k$ be the Følner tilling sequence given in Proposition 6.7 and for any $k \geqslant 0$ let $F'_k = F_{mk} F_{mk+1} \dots F_{mk+m-1}$. Note that $F'_k = \{0, 2^{mk}, 2 \cdot 2^{mk}, \dots, (2^m - 1)2^{mk}\}^n$ and $T'_k = \{0, 1, \dots, 2^{mk+m} - 1\}^n$.

As T'_k is the set T_{mk+m-1} from Proposition 6.7 we have that the diameter of T'_k is at most $n2^{mk+m}$ and the set $T'_k \setminus (s + T'_k)$ has cardinality at most $2^{-mk-m}|T'_k|$ for any standard generator s of $\mathbb{Z}^n$. □

The following theorem is almost sharp since by our extensions of Bowen's theorem (Theorem 3.2), if $n < m$ and $p > \frac{n}{m}$, then there cannot be an L^p measure subquotient coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$.

Theorem 6.9. *For every n and m, there exists an orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ which is for every $\varepsilon > 0$ a $(\varphi_\varepsilon, \psi_\varepsilon)$-integrable coupling, where*

$$\varphi_\varepsilon(x) = \frac{x^{n/m}}{\log(x)^{1+\varepsilon}} \text{ and } \psi_\varepsilon(x) = \frac{x^{m/n}}{\log(x)^{1+\varepsilon}}.$$

In particular if $n < m$,then then there is an $(\mathrm{L}^p, \mathrm{L}^{1/p})$ orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ for all $p < \frac{n}{m}$.

Proof. Due to Corollary 6.8, we can take a (ε_k, R_k)-Følner tiling sequence $(F_k)_k$ for $\mathbb{Z}^m$ with $|F_k| = 2^{nm}$, $R_k = m2^{n(k+1)}$ and $\varepsilon_k = 2^{-n(k+1)+1}$ for any $k \geqslant 0$. Similarly, due to Corollary 6.8, we can take a (ε'_k, R'_k)-Følner tiling sequence $(F'_k)_k$ for $\mathbb{Z}^n$ with $|F'_k| = 2^{nm}$, $R'_k = n2^{m(k+1)}$ and $\varepsilon'_k = 2^{-m(k+1)+1}$ for any $k \geqslant 0$. By Proposition 6.6 the groups are orbit equivalent. Moreover our orbit equivalence coupling is φ_ε-integrable as

$$\sum_{k=1}^{\infty} \varphi_\varepsilon(R'_k)\varepsilon_{k-1} = \sum_{k=1}^{\infty} \frac{(2n2^{m(k+1)})^{n/m}}{(\log(2n) + m(k+1))^{1+\varepsilon}} 2^{-nk+1},$$

so we are summing over $k \geqslant 1$ a term which is asymptotically equivalent to $k^{-1-\varepsilon}$, hence the sum is finite. By exchanging the roles of n and m, we then see that our orbit equivalence coupling is actually $(\varphi_\varepsilon, \psi_\varepsilon)$-integrable. Finally if $n < m$ then for all $p < n/m$ we both have that $x^p = o(\varphi_\varepsilon(x))$ and $x^{1/p} = o(\psi_\varepsilon(x))$ as $x \to +\infty$ so our coupling is also $(\mathrm{L}^p, \mathrm{L}^p)$. □

Remark 6.10. The expert reader will recognize in the above proof an explicit orbit equivalence between the dyadic $\mathbb{Z}^n$ and $\mathbb{Z}^m$ odometers. Moreover, using this point of view it can be shown that this coupling is not ψ_0-integrable, so we ask the following refinements of question 1.1.

Question 6.11. Let $n < m$, is there a $(\varphi, \mathrm{L}^\infty)$ measure equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$, where $\varphi(x) = \frac{x^{n/m}}{\log(x)}$? What about a (φ, L^0) measure equivalence coupling ?

Next we prove a similar result for the Heisenberg group, recall that the Heisenberg group is the 2-step torsion-free nilpotent group that can be defined as the group of triples $(x, y, z) \in \mathbb{Z}^3$ equipped with the group operation

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx'),$$

which comes from its identification with the group of matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{bmatrix}$.

Proposition 6.12. *The Heisenberg group admits a profinite (ε_k, R_k)-Følner tiling (F_k) such that $|F_k| = 16$, $R_k = 14 \cdot 2^{k+2}$, and $\varepsilon_k = 2^{-k}$ for any $k \geqslant 0$.*

Proof. For every $k \geqslant 0$ let

$$F_k = \left\{(2^k x, 2^k y, 4^k z) \colon x, y \in \{0, 1\}, z \in \{0, 1, 2, 3\}\right\}.$$

We claim that (F_k) is a profinite Følner tiling. First note that it is profinite because for each k, the set F_k is a right coset representative of Γ_{k-1} modulo Γ_k, where Γ_{k-1} is the (non normal) finite index subgroup

$$\Gamma_{k-1} = \left\{(x, y, z) \colon x \equiv y \equiv 0 \mod 2^k, z \equiv 0 \mod 4^k\right\}.$$

Towards proving that (F_k) is a left Følner tiling, let $E_1 = (1, 0, 0)$ and $E_2 = (0, 1, 0)$ be the standard generators of the Heisenberg group, and $A = (x_A, y_A, z_A) \in T_k$. By our definition, we can write A as

$$A = (x_0, y_0, z_0)(2x_1, 2y_1, 4z_1) \cdots (2^k x_k, 2^k y_k, 4^k z_k)$$

with $x_i, y_i, \in \{0,1\}$, and $z_i \in \{0,1,2,3\}$, which yields

$$x_A = \sum_{i=0}^{k} 2^i x_i, \quad y_A = \sum_{i=0}^{k} 2^i y_i, \quad \text{and} \quad z_A = \sum_{i=0}^{k} 4^i z_i + \sum_{i=1}^{k} 2^i x_i \sum_{j=0}^{i-1} 2^j y_j \tag{10}$$

We thus see that we can recover the finite sequences $(x_i)_{i=0}^k$ and $(y_i)_{i=0}^k$ from the coefficients (x_A, y_A, z_A) of A as the binary expansions of x_A and y_A respectively, and then similarly $((z_i)_{i=0}^k)$ is also completely determined by (x_A, y_A, z_A). Therefore the above decomposition of A as an element of $F_0 \cdots F_k$ is unique for every element in T_k. Next note that $\operatorname{diam}(F_k) \leqslant 2 \cdot 2^k + 4 \cdot 2^{k+1}$, so $\operatorname{diam}(T_k) \leqslant 10 \cdot 2^{k+1}$. Let us now estimate the cardinality of the set $E_1 T_k \setminus T_k$.

Towards this, we will estimate how many elements $A \in T_k$ decomposed as above satisfy $E_1 A \in T_k$. Fix $m \in \{0, ..., k\}$, let $x_0 = \cdots = x_{m-1} = 1$, $x_m = 0$, and then fix arbitrary $x_{m+1}, ..., x_k \in \{0,1\}$, $y_0, ..., y_k \in \{0,1\}$. We will bound from below the number of possible parameters $z_0, ..., z_k \in \{0,1,2,3\}$ such that the element

$$A = (x_0, y_0, z_0)(2x_1, 2y_1, 4z_1) \cdots (2^k x_k, 2^k y_k, 4^k z_k) = (x_A, y_A, z_A)$$

satisfies $E_1 A \in T_k$. We have $E_1 A = (x_A + 1, y_A, x_A + z_A)$, so if the latter belongs to T_k, then by equation (10) its decomposition

$$E_1 A = (x_0', y_0', z_0')(2x_1', 2y_1', 4z_1') \cdots (2^k x_k', 2^k y_k', 4^k z_k')$$

is given by $x_i' = 0$ for $i < m$, $x_m' = 1$ and $x_i' = x_i$ for $i > m$, $y_i' = y_i$ for all $i \in \{0, ..., k\}$, and finally the z_i''s are subject to the equation

$$\sum_{i=0}^{k} 4^i z_i' + \sum_{i=1}^{k} 2^i x_i' \sum_{j=0}^{i-1} 2^j y_j = \sum_{i=0}^{k} 4^i z_i + \sum_{i=1}^{k} 2^i x_i \sum_{j=0}^{i-1} 2^j y_j.$$

We thus have:

$$\sum_{i=0}^{k} 4^i z_i' = \sum_{i=0}^{k} 4^i z_i + \sum_{i=1}^{k} 2^i (x_i - x_i') \sum_{j=0}^{i-1} 2^j y_j,$$

Now recall that $x_i' = 0$ for $i < m$, $x_m' = 1$ and $x_i' = x_i$ for $i > m$, so we can rewrite the last term as

$$\sum_{i=1}^{k} 2^i (x_i - x_i') \sum_{j=0}^{i-1} 2^j y_j = \sum_{i=1}^{m} 2^i \sum_{j=0}^{i-1} 2^j y_j - 2^m \sum_{j=0}^{m-1} 2^j y_j$$

We deduce that $\sum_{i=1}^{k} 2^i (x_i - x_i') \sum_{j=0}^{i-1} 2^j y_j$ is negative, and its absolute value is strictly less than 4^m. Since

$$\sum_{i=0}^{k} 4^i z_i' = \sum_{i=0}^{k} 4^i z_i + \sum_{i=1}^{k} 2^i (x_i - x_i') \sum_{j=0}^{i-1} 2^j y_j,$$

we see that as soon as $\sum_{i=0}^{m} 4^i z_i \geqslant 4^m$, the above equation has a solution, so that $E_1 A \in T_k$. Taking complements, this yields at most 4^m possibilities for the sequence $(z_i)_{i=0}^k$ so that $E_1 A \notin T_k$.

Now for a fixed m, we have 2^{k-m} possible x_A's, 2^{k+1} possible y_A's and at most 4^{m+1} possible z_A's such that $E_1 A \notin T_k$, which yields a total of $2^{k-m} 2^{k+1} 4^m = 2^{2k+2} 2^m$ possibilities. When $x_i = 1$ for every i, we never have $E_1 A \in T_k$, and then this adds $2^{k+1} 4^{k+1} = 2^{3k+3}$ possibilities.

We conclude that there are at most $2^{3k+3} + \sum_{m=0}^{k} 2^{2k+2} 2^m < 2^{3k+4}$ choices of A such that $E_1 A \notin T_k$, which means that $|E_1 T_k \setminus T_k| < 2^{3k+4}$.

Now we estimate the cardinality of $E_2T_k \setminus T_k$. This time we fix m and assume that $y_0 = \cdots = y_{m-1} = 1$, $y_m = 0$, and need to bound from below the number of A's of the form

$$A = (x_0, y_0, z_0)(2x_1, 2y_1, 4z_1)\cdots(2^k x_k, 2^k y_k, 4^k z_k) = (x_A, y_A, z_A)$$

such that $E_2A \in T_k$. Similarly as before, if $AE_2 \in T_k$ we may write it as

$$E_2A = (x_A, y_A + 1, x_A + z_A) = (x'_0, y'_0, z'_0)(2x'_1, 2y'_1, 4z'_1)\cdots(2^k x'_k, 2^k y'_k, 4^k z'_k),$$

so by equation (10) for all $i \in \{0, ..., k\}$ we have $x'_i = x_i$, $y'_0 = \cdots = y'_{m-1} = 0$, $y'_m = 1$ and $y'_i = y_i$ for all $i > m$. Finally the z'_i's satisfy

$$\sum_{i=0}^{k} 4^i z'_i + \sum_{i=1}^{k} 2^i x_i \sum_{j=0}^{i-1} 2^j y'_j = \sum_{i=0}^{k} 2^i x_i + \sum_{i=0}^{k} 4^i z_i + \sum_{i=1}^{k} 2^i x_i \sum_{j=0}^{i-1} 2^j y_j.$$

We may rewrite our previous equation as

$$\sum_{i=0}^{k} 4^i z'_i - \sum_{i=0}^{k} 4^i z_i = \sum_{i=1}^{k} 2^i x_i \sum_{j=0}^{i-1} 2^j (y_j - y'_j) + \sum_{i=0}^{k} 2^i x_i$$

We then decompose the first sum in the right term, noting that by construction of the y'_i's, for all $i > m$, we have $\sum_{j=0}^{i-1} 2^j (y_j - y'_j) = -1$:

$$\begin{aligned}
\sum_{i=0}^{k} 4^i z'_i - \sum_{i=0}^{k} 4^i z_i &= \sum_{i=1}^{m} 2^i x_i \sum_{j=0}^{i-1} 2^j - \sum_{i=m+1}^{k} 2^i x_i + \sum_{i=0}^{k} 2^i x_i \\
&= \sum_{i=1}^{m} 2^i x_i (2^i - 1) + \sum_{i=0}^{m} 2^i x_i \\
&= \sum_{i=1}^{m} 4^i x_i + x_0 \\
&< 2^{2m+1}
\end{aligned}$$

So for these x_A and y_A there exists at most 2^{2m+1} values of z_A such that $E_1A \notin T_k$ (namely those such that $\sum_{i=0}^{k} 4^i z_i + 2^{2m+1} \geqslant 4^{k+1}$) and as before we conclude that there exists at most

$$2^{k+1}4^{k+1} + \sum_{m=0}^{k} 2^{k+1}2^{k-m}2^{2m+1} < 2^{3k+4}$$

choices of A such that $E_1A \notin T_k$. Thus, altogether we have that $|E_iT_k \setminus T_k| \leqslant 2^{3k+4}$ for $i \in \{1, 2\}$, and since $|T_k| = 2^{4k+4}$ we conclude that

$$|E_iT_k \setminus T_k| \leqslant 2^{-k}|T_k|,$$

so that we can pick $\varepsilon_k = 2^{-k}$. □

Theorem 6.13. *There exists an orbit equivalence coupling between $\mathbb{Z}^4$ and $\mathrm{Heis}(\mathbb{Z})$ which is L^p in both directions for all $p < 1$.*

Proof. This follows from a straightforward computation, using the Følner tilings provided by Proposition 6.7 and Proposition 6.12. □

Remark 6.14. As explained in the introduction, there cannot be an L^1 measure equivalence coupling between $\mathbb{Z}^4$ and $\mathrm{Heis}(\mathbb{Z})$. A finer analysis reveals that our coupling is actually φ_ε-integrable for all $\varepsilon > 0$, where

$$\varphi_\varepsilon(x) = \frac{x}{\log(x)^{1+\varepsilon}}.$$

We thus ask the following question.

Question 6.15. Is there a φ_0-integrable measure equivalence coupling between $\mathbb{Z}^4$ and $\mathrm{Heis}(\mathbb{Z})$?

6.3 A coupling between $\mathbb{Z}$ and the lamplighter groups

In what follows, recall that given two countable groups Λ, Γ and a function $f : \Gamma \to \Lambda$, we defined the **support** of f by $\operatorname{supp} f = \{\gamma \in \Gamma \colon f(\gamma) \neq e_\Lambda\}$. Note that Λ^Γ is a group for pointwise multiplication. We then define $\bigoplus_{\gamma\in\Gamma} \Lambda$ as the subgroup of Λ^Γ consisting of all functions which have finite support. Finally, we define the **wreath product** $\Lambda \wr \Gamma$ as the semi-direct product

$$\Lambda \wr \Gamma := \bigoplus_{\gamma\in\Gamma} \Lambda \rtimes \Gamma,$$

where the group multiplication is given by $(f, \gamma) \cdot (f', \gamma') = (f\gamma \cdot f', \gamma\gamma')$ and $\gamma \cdot f(g) = f(\gamma^{-1}g)$. Such groups are also referred to as lamplighter groups, especially when the *acting group* Γ is $\mathbb{Z}$ and the *lamp group* Λ is a cyclic group. In the latter case, a standard generating set is provided by the pair $\{(0, 1), (\delta_0, 0)\}$ where $\delta_0(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

Proposition 6.16. *Let $m \geqslant 2$ be a positive integer. The group $\mathbb{Z}/m\mathbb{Z}\wr\mathbb{Z}$ (equipped with the generating set $S = \{(0, 1), (\delta_0, 0)\}$) admits a (ε_k, R_k)-Følner tiling sequence $(F_k)_k$, with $|F_0| = 2m^2$, and $|F_k| = 2 \cdot m^{2^k}$, $R_k = (m + 1)2^{k+1}$ and $\varepsilon_k = 2^{-(k+1)}$ for $k \geqslant 1$.*

Proof. In this proof we construct a right Følner tiling sequence (F_k) (recall that (F_k^{-1}) is then a left Følner tiling sequence). We take $F_0 = \{(f, n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 1], n \in [0, 1]\}$ and

$$\begin{aligned} F_k = &\left\{(f, 0) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [2^k, 2^{k+1} - 1]\right\} \\ &\cup \left\{(f, 2^k) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^k - 1]\right\}. \end{aligned}$$

By induction, we show that $T_k = \{(f, n) \in \mathbb{Z}/m\mathbb{Z}\wr\mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+1} - 1], n \in [0, 2^{k+1} - 1]\}$. Indeed, we have that

$$\begin{aligned} T_{k+1} &= F_{k+1}T_k \\ &= \{(f, n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+2} - 1], n \in [0, 2^{k+1} - 1]\} \\ &\quad \cup \{(f, n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+2} - 1], n \in [2^{k+1}, 2^{k+2} - 1]\} \\ &= \{(f, n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+2} - 1], k \in [0, 2^{k+2} - 1]\}. \end{aligned}$$

Next note that the diameter of T_k is equal to $\lfloor \frac{m}{2} \rfloor 2^{k+1} + 2(2^{k+1} - 1)$, which is less than $(m+1)2^{k+1}$. Finally, it suffices to show that $|T_k s \setminus T_k| \leqslant 2^{-(k+1)} |T_k|$ for all $s \in S$. If $s = (\delta_0, 0)$, then $T_k s \setminus T_k = \emptyset$. If $s = (0, 1)$, then

$$T_k s \setminus T_k = \{(f, 2^{k+1}) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+1} - 1]\}.$$

So either way, $|T_k s \setminus T_k| \leqslant m^{2^{k+1}} = 2^{-(k+1)}|T_k|$, which concludes the proposition. □

Proposition 6.17. *For any integer $k \geqslant 2$, there exists an orbit equivalence coupling from $\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}\wr\mathbb{Z}$ which is $(\exp, \varphi_\varepsilon)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) = \frac{\log(x)}{\log(\log(x))^{1+\varepsilon}}$.*

Proof. Let $(F_k)_k$ be the Følner tiling sequence of $\mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z}$ constructed in Proposition 6.16. And let $(F'_k)_k$ the Følner tiling sequence of $\mathbb{Z}$ defined by $F_0 = [0, 2m^2 - 1]$ and

$$F'_k = |T_{k-1}| \left[0, |F_k| - 1\right] = \{0, m^{2^k} 2^k, 2m^{2^k} 2^k, \dots, (2m^{2^k} - 1)m^{2^k} 2^k\}.$$

Note that $T'_k = [0, |T_k| - 1]$. So $(F'_k)_k$ is a (ε'_k, R'_k)-Følner tiling sequence with $R'_k = |T'_k| - 1 \leqslant m^{2^{k+1}} 2^{k+1}$ and $\varepsilon'_k = \frac{2}{|T'_k|} = \frac{1}{m^{2^{k+1}} 2^k}$. Proposition 6.6 provides us an explicit orbit equivalence coupling (X, μ) between the groups $\Gamma' = \mathbb{Z}$ and $\Gamma = \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z}$. Moreover, defining $z > 1$ as the solution of the equation $z^{2^3(m+1)}/m = 1$, we have

$$z^{2R_k} \varepsilon'_{k-1} \leqslant \frac{z^{2(m+1)2^{k+2}}}{m^{2^k} 2^{k-1}} = (z^{2^3(m+1)}/m)^{2^k} \frac{1}{2^{k-1}} = \frac{1}{2^{k-1}},$$

so we deduce from Proposition 6.6 that for every $s \in S_\Gamma$,

$$\int_X z^{d_{S_{\Gamma'}}(x, s \cdot x)} d\mu(x) < \infty.$$

On the other hand, we have

$$\varphi_\varepsilon(2R'_k)\varepsilon_{k-1} \leqslant \frac{2^{k+1} \log m + (k+2) \log 2}{\log(2^{k+1} \log m + (k+2) \log 2)^{1+\varepsilon}} \frac{1}{2^k} \preccurlyeq \frac{1}{k^{1+\varepsilon}}.$$

Hence we deduce from Proposition 6.6 that that for every $s \in S_{\Gamma'}$,

$$\int_X \varphi(d_{S_{\Gamma'}}(x, s \cdot x)) d\mu(x) < \infty.$$

This concludes the proof of the proposition. □

7 Almost optimal couplings between $\mathbb{Z}$ and iterated wreath products

In this section, we show that taking wreath products is well-behaved with respect to φ-integrable orbit equivalence, and use this to find almost optimal orbit equivalence couplings between $\mathbb{Z}$ and iterated wreath products. The main construction can be understood purely in terms of measure-preserving equivalence relations, and provides in particular a proof that if Γ_1, Γ_2 are orbit equivalent, and Λ_1, Λ_2 are orbit equivalent, then $\Lambda_1 \wr \Gamma_1$ is orbit equivalent to $\Lambda_2 \wr \Gamma_2$.

7.1 Wreath products of measure-preserving equivalence relations

Let $\mathcal{R}$ be a measure-preserving equivalence relation on (X, μ), let $\mathcal{S}$ be a measure-preserving equivalence relation on (L, ν). Consider the space $L^{\mathcal{R}}$ which consists of couples $(x, (l_y)_{y \in [x]_{\mathcal{R}}})$ where $l_y \in L$ for every $y \in [x]_{\mathcal{R}}$. This space can be equipped with a natural standard Borel space structure which we don't make explicit for now since in our concrete case it will be easy to describe. Moreover, one can endow it with the following probability measure η given by

$$\eta = \int_X \nu^{\otimes [x]_{\mathcal{R}}} \, d\mu(x).$$

We then equip the space $(L^{\mathcal{R}}, \eta)$ with the **wreath product** $\mathcal{S} \wr \mathcal{R}$ of $\mathcal{S}$ by $\mathcal{R}$ defined by saying two couples $(x, (l_y)_{y \in [x]_{\mathcal{R}}})$ and $(x', (l'_y)_{y \in [x]_{\mathcal{R}}})$ are $\mathcal{S} \wr \mathcal{R}$-equivalent as soon as $(x, x') \in \mathcal{R}$ and the following two conditions are satisfied:

- for all but finitely many $y \in [x]_{\mathcal{R}}$, we have $l_y = l'_y$;
- for all $y \in [x]_{\mathcal{R}}$, we have $(l_y, l'_y) \in \mathcal{S}$.

One can check that $\mathcal{S} \wr \mathcal{R}$ is indeed a measure-preserving equivalence relation.

Let us make all this completely explicit in the case of interest to us: we now suppose that $\mathcal{R}$ comes from a fixed *free* Γ-action. We then have a cocycle map $c : \mathcal{R} \to \Gamma$ which takes $(x, y) \in \mathcal{R}$ to the unique $\gamma \in \Gamma$ satisfying $\gamma \cdot x = y$. The standard Borel space structure on $L^{\mathcal{R}}$ then comes from its natural identification to the standard Borel space $X \times L^{\Gamma}$ given by the bijection $\Phi : X \times L^{\Gamma} \to L^{\mathcal{R}}$ defined by

$$\Phi(x, (l_\gamma)_{\gamma \in \Gamma}) = (x, (l_{c(x,y)})_{y \in [x]_{\mathcal{R}}}).$$

The inverse map is given by $\Phi^{-1}(x, (l_y)_{y \in [x]_{\mathcal{R}}}) = (x, (l_{\gamma \cdot x})_{\gamma \in \Gamma})$. Note that $\Phi_*(\mu \otimes \nu^{\otimes \Gamma}) = \eta$. We have a natural Γ-action on $L^{\mathcal{R}}$ given by $\gamma \cdot (x, l) = (\gamma \cdot x, l)$. The corresponding action on $X \times L^{\Gamma}$ is given by

$$\gamma \cdot (x, (l_g)_{g \in \Gamma}) = (\gamma \cdot x, (l_{g\gamma})_{g \in \Gamma}).$$

Finally, suppose that the equivalence relation $\mathcal{S}$ comes from a free Λ-action on L. We then also have a natural $\bigoplus_{\gamma \in \Gamma} \Lambda$-action on $X \times L^{\Gamma}$ given by: for all $f \in \bigoplus_{\gamma \in \Gamma} \Lambda$,

$$f \cdot (x, (l_\gamma)_{\gamma \in \Gamma}) = (x, (f(g^{-1}) \cdot l_g)_{g \in \Gamma}),$$

or from the $L^{\mathcal{R}}$ viewpoint, $f \cdot (x, (l_y)_{y \in [x]_{\mathcal{R}}}) = (x, (f(c(y, x)) \cdot l_y)_{y \in [x]_{\mathcal{R}}})$.

One can check that the Γ and $\bigoplus_{\gamma \in \Gamma} \Lambda$-actions actually extend to a free action of $\Lambda \wr \Gamma$ on $L^{\mathcal{R}}$ given by

$$(f, \gamma) \cdot \left(x, (l_y)_{y \in [x]_{\mathcal{R}}}\right) = \left(\gamma \cdot x, \left(f\left(c\left(y, \gamma \cdot x\right)\right) \cdot l_y\right)_{y \in [x]_{\mathcal{R}}}\right).$$

The equivalence relation induced by this action is clearly $\mathcal{S} \wr \mathcal{R}$. Since the action is measure-preserving, this proves that $\mathcal{S} \wr \mathcal{R}$ is measure-preserving. We have shown the following result.

Proposition 7.1. *Let $\Gamma \curvearrowright (X, \mu)$ freely, let $\Lambda \curvearrowright (L, \nu)$ freely, where (X, μ) and (L, ν) are two standard probability spaces and the actions are measure-preserving. Denote by $\mathcal{R}$ and $\mathcal{S}$ the respective associated equivalence relations, and by $c : \mathcal{R} \to \Gamma$ the cocycle defined by $c(x, \gamma \cdot x) = \gamma$. Then the measure-preserving free $\Gamma \wr \Lambda$-action on $L^{\mathcal{R}}$ given by*

$$(f, \gamma) \cdot \left(x, (l_y)_{y \in [x]_{\mathcal{R}}}\right) = \left(\gamma \cdot x, \left(f\left(c\left(y, \gamma \cdot x\right)\right) \cdot l_y\right)_{y \in [x]_{\mathcal{R}}}\right)$$

induces the equivalence relation $\mathcal{S} \wr \mathcal{R}$. □

Remark 7.2. One could remove the freeness assumption on the Γ-action by replacing $\mathcal{R}$ by the *action groupoid* $\Gamma \ltimes X$. When X is a single point, we would then recover the well-known action of $\Lambda \wr \Gamma$ on L^{Γ}, which will be a crucial tool in Section 8.

7.2 Wreath products and quantitative orbit equivalence

It follows directly from the previous proposition that when Γ_1, Γ_2 are orbit equivalent, and Λ_1, Λ_2 are orbit equivalent, then $\Lambda_1 \wr \Gamma_1$ is orbit equivalent to $\Lambda_2 \wr \Gamma_2$. We now give a quantitative version of this fact.

For this, it is useful to identify Γ and Λ to subgroups of $\Gamma \wr \Lambda$ as follows: first we embed Λ in $\bigoplus_{\gamma \in \Gamma} \Lambda$ by associating to every $\lambda \in \Lambda$ the function

$$\iota(\lambda) : \gamma \mapsto \begin{cases} \lambda & \text{if } \gamma = e_\Gamma \\ e_\Lambda & \text{otherwise.} \end{cases}$$

Then we embed Λ in $\Lambda \wr \Gamma$ via $\lambda \mapsto (e_\Gamma, \iota(\lambda))$, and we embed Γ into $\Lambda \wr \Gamma$ via $\gamma \mapsto (\gamma, \iota(e_\Lambda))$. It is well known that if S_Γ is a generating set for Γ and S_Λ is a generating set for Λ, then through the above identification $S_\Lambda \cup S_\Gamma$ is a generating set for the wreath product $\Lambda \wr \Gamma$.

Proposition 7.3. *Let (X, μ) be an orbit equivalence coupling between finitely generated groups $\Gamma_1 = \langle S_{\Gamma_1} \rangle$ and $\Gamma_2 = \langle S_{\Gamma_2} \rangle$, and let (L, ν) be an orbit equivalence coupling between $\Lambda_1 = \langle S_{\Lambda_1} \rangle$ and $\Lambda_2 = \langle S_{\Lambda_2} \rangle$. Denote by $\mathcal{R}$ the equivalence relation generated by the Γ_i-action.*

Then the orbit coupling $L^{\mathcal{R}}$ satisfies that for every $(x, l) \in L^{\mathcal{R}}$, every $\gamma_1 \in \Gamma_1$ and every $\lambda_1 \in \Lambda_1$,

$$d_{S_{\Lambda_2} \cup S_{\Gamma_2}}((x,l), \gamma_1 \cdot (x,l)) = d_{S_{\Gamma_2}}(x, \gamma \cdot x)$$
$$d_{S_{\Lambda_2} \cup S_{\Gamma_2}}((x,l), \lambda_1 \cdot (x,l)) = d_{S_{\Lambda_2}}(l_{e_\Gamma}, \lambda \cdot l_{e_\Gamma})$$

Proof. By definition for all $i \in \{1, 2\}$, all $\gamma_i \in \Gamma_i$, and all $(x, l) \in L^{\mathcal{R}}$, we have $\gamma_i \cdot (x, l) = (\gamma_i \cdot x, l)$, so the first equation is clear. The second equation follows similarly by noting that if $\lambda_i \in \Lambda_i$, then their action on (x, l) only changes the value at e_Γ of l according to the λ_i-action on L. □

Corollary 7.4. *If Γ_1 and Γ_2 admit a (φ, ψ)-integrable orbit equivalence coupling, and if Λ_1 and Λ_2 admit a (φ, ψ)-integrable orbit equivalence coupling, then the wreath products $\Lambda_1 \wr \Gamma_2$ and $\Lambda_2 \wr \Gamma_2$ also admit a (φ, ψ)-integrable orbit equivalence couplings.* □

In combination with Theorem 6.9 we find the following:

Corollary 7.5. *Let $n, m \in \mathbb{N}$ with $n < m$ and let Δ be any finitely generated group, then there is an $(\mathrm{L}^p, \mathrm{L}^{1/p})$-orbit equivalence coupling from $\Delta \wr \mathbb{Z}^m$ to $\Delta \wr \mathbb{Z}^n$ for every $p < \frac{n}{m}$.* □

We also obtain the following corollary by using compositions of couplings. Recall that we denote by $\log^{\star n}$ the function log composed with itself n times. We denote $\mathrm{L}^{<\infty}$ the intersection of L^p for all $1 \leqslant p < \infty$.

Corollary 7.6. *Let $m > 0$, and let G_n be defined inductively by $G_0 = \mathbb{Z}$, and $G_{n+1} = \mathbb{Z}/m\mathbb{Z} \wr G_n$. For any integer $n \geqslant 1$, there exists an orbit equivalence coupling from $\mathbb{Z}$ to G_n which is $(\mathrm{L}^{<\infty}, \varphi_{n,\varepsilon})$-integrable for every $\varepsilon > 0$, where $\varphi_{n,\varepsilon} = \log^{\star n} / (\log^{\star (n+1)})^{1+\varepsilon}$.*

Proof. We first claim that for all $n \geqslant 2$, there exists an orbit equivalence coupling from G_{n-1} to G_n which is $(\mathrm{L}^{<\infty}, \varphi_{1,\varepsilon})$-integrable for all $\varepsilon > 0$: this follows by induction using Proposition 7.3, the case $n = 2$ resulting from Proposition 6.17.

We now pass to the proof of the corollary, which we also do by induction on n. The base case $n = 1$ follows from the claim, so we take $n \geqslant 2$ and assume that we have a orbit equivalence coupling from $\mathbb{Z}$ to G_{n-1} which is $(\mathrm{L}^{<\infty}, \varphi_{n-1,\varepsilon})$-integrable for all $\varepsilon > 0$. Using Proposition 2.27, we compose this coupling with the coupling from the claim between G_{n-1} and G_n, and obtain a $(\varphi_{1,\varepsilon} \circ \varphi_{n-1,\varepsilon}, \mathrm{L}^{<\infty})$-coupling from $\mathbb{Z}$ to G_n. Finally one easily shows that $\varphi_{1,\varepsilon} \circ \varphi_{n-1,\varepsilon} \succcurlyeq \varphi_{n,\varepsilon}$, so we are done. □

8 Unstable properties under exponential couplings

8.1 Finite presentability is unstable

In this section we prove that being finitely presented is not preserved by strongly exponentially integrable orbit equivalence couplings (see Definition 2.29). We do this by constructing an explicit strongly exponentially integrable orbit coupling between the wreath product $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$ (also known as a lamplighter group) and the Baumslag-Solitar group $\mathrm{BS}(1, k)$, for every $k \geqslant 2$.

Recall from the previous section that $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z} \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts by a shift on $\bigoplus_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$, that is

$$((x_i)_{i \in \mathbb{Z}}, m_x)((y_i)_{i \in \mathbb{Z}}, m_y) = ((x_i + y_{i - m_x})_{i \in \mathbb{Z}}, m_x + m_y).$$

Also recall that $\mathrm{BS}(1, k) = \mathbb{Z}[1/k] \rtimes \mathbb{Z}$, where $\mathbb{Z}[1/k] = \{\frac{x}{k^m} : x, m \in \mathbb{Z}\}$ and $\mathbb{Z}$ acts on $\mathbb{Z}[1/k]$ by multiplication by $1/k$, that is

$$\left(\frac{x}{k^{m_x}}, n_x\right)\left(\frac{y}{k^{m_y}}, n_y\right) = \left(\frac{x}{k^{m_x}} + \frac{y}{k^{m_y + n_x}}, n_x + n_y\right).$$

It is well known that the second group is finitely presented, while the first one is not by a result of Baumslag [Bau61]. In particular these groups are not quasi-isometric.

We will consider actions of these two groups on the standard Borel space $X = \prod_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$ equipped with the measure μ defined as the infinite product of the normalized counting measure. The action of the lamplighter group $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$ is the standard one, that is

$$((x_i)_{i\in\mathbb{Z}}, m) \cdot (y_i)_{i\in\mathbb{Z}} = ((x_i + y_{i-m})_{i\in\mathbb{Z}}, m).$$

For the Baumslag-Solitar group $\mathrm{BS}(1,k)$ we first define the action of the subgroup $\mathbb{Z}[1/k] = \langle (k^m, 0) : m \in \mathbb{Z}\rangle$ as follows: for all $m \in \mathbb{Z}$, we decompose the space X as

$$X = \prod_{i<m} \mathbb{Z}/k\mathbb{Z} \times \prod_{i\geqslant m} \mathbb{Z}/k\mathbb{Z},$$

and then $(k^m, 0)$ acts trivially on the first factor, and as the k-adic odometer on the second factor. To be completely explicit, $(k^m, 0) \cdot (x_i)_{i\in\mathbb{Z}} = (y_i)_{i\in\mathbb{Z}}$, where $x_i = y_i$ for every $i < m$, $y_N = x_N + 1$ where $N \geqslant m$ the smallest such that $x_N \neq k-1$, $y_i = 0$ for $m \geqslant i < N$ and $y_i = x_i$ for $i > N$. If $x_i = k-1$ for every $i \geqslant m$, then we take $y_i = 0$ for every $i \geqslant m$. Note that this action is realized by letting $\mathbb{Z}[1/k]$ act on $\mathbb{Q}_k$ by addition and extending this action to the bi-infinite product $X = \prod_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$. Also note that the equivalence relation generated by the action of $\mathbb{Z}[1/k]$ is the cofinite relation up to measure zero.

As with the lamplighter group we then let $\mathbb{Z}$ act by $(0,1) \cdot (x_i)_{i\in\mathbb{Z}} = (x_{i-1})_{i\in\mathbb{Z}}$. This defines a free pmp action of the group $\mathbb{Z}[1/k] \rtimes \mathbb{Z}$ on X. Since the natural action of $\oplus_{\mathbb{Z}}\mathbb{Z}/k\mathbb{Z}$ is the cofinite one, we deduce that the two actions we have build yield an orbit equivalence coupling (X, μ) from the lamplighter group $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$ to the Baumslag-Solitar group $\mathrm{BS}(1,k)$.

Theorem 8.1. *The orbit equivalence coupling (X, μ) we just constructed is an $(\mathrm{L}^\infty, \exp^\diamond)$ orbit equivalence coupling from $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$ to $\mathrm{BS}(1,k)$.*

Proof. We equip $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$ with the generating subset $S = \{\delta_0, 1_{\mathbb{Z}}\}$ and $\mathrm{BS}(1,k)$ with the generating subset $T = \{1_{\mathbb{Z}[1/k]}, 1_{\mathbb{Z}}\}$. We denote by $S^{\pm} = S \cup S^{-1}$ and $T^{\pm} = T \cup T^{-1}$ the corresponding symmetric generating sets.

Note that the generator of the $\mathbb{Z}$ copy in the two groups acts exactly the same. Moreover, the two subgroups $\bigoplus_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$ and $\mathbb{Z}[1/k]$ have the same orbits.

Let us start by showing that our orbit equivalence coupling is L^∞ as a coupling from the lamplighter group to the Baumslag-Solitar group. For this, we only need to check that $d_{T^\pm}(s \cdot x, x)$ is uniformly bounded for all $s \in S$. Note that the generator of the $\mathbb{Z}$ copy in the two groups acts exactly the same. So for $s = 1_{\mathbb{Z}}$, we have $d_{T^\pm}(s \cdot x, x) = 1$. Then observe that the generator δ_0 changes only x_0, which is achieved by the action of s^m for some $m \in \{-k+1, \cdots, k-1\}$, where $s = 1_{\mathbb{Z}[1/k]}$. Hence $d_{T^\pm}(\delta_0 \cdot x, x) \leqslant k-1$, which proves that our coupling is L^∞ in one direction.

To prove that the coupling is strongly exponential in the other direction, we need to obtain estimates for every $g \in \mathrm{BS}(1,k)$. So let $g \in \mathrm{BS}(1,k)$, we will show that g satisfies the following estimate: for all $M \geqslant 0$,

$$\mu\Big(\{x \in X : d_{S^\pm}(g \cdot x, x) \geqslant (k+1)\,(2\,|g|_T + 2M + 3)\}\Big) \leqslant k^{-M+1} \tag{11}$$

Let $n = |g|_{T^\pm}$, write $g = (z, j)$, with $|j| \leqslant n$ and $z \in \mathbb{Z}[1/k]$. First note that by the triangle inequality and the fact that $d_{S^\pm}((0,j) \cdot x, x) = j$ for all $x \in X$, it suffices to show that

$$\mu\Big(\{x \in X : d_{S^\pm}(z \cdot x, x) \geqslant k(2|g|_T + 2M + 3)\}\Big) \leqslant k^{-M+1} \tag{12}$$

By symmetry we may assume that $z > 0$. A straightforward induction on n then shows that we can always write z as $z = \sum_{i=-n}^{n} a_i k^i$, where $a_i \in \{0, ..., k-1\}$.

Let $x \in X$. Observe that the coefficients of $z \cdot x$ that differ from those of x are contained in the interval $[-n, \infty)$. Moreover, consider the event $(z \cdot x)_{n+m+2} \neq x_{n+m+2}$ for some $m \geqslant M$, and observe

that its occurrence forces the coefficients of x_{n+i} for $i = 1, 2, \dots, m+1$ to be equal to $k-1$. Therefore this event has probability at most $\sum_{m \geqslant M} k^{-m} \leqslant k^{-M+1}$.

Now note that $|f|_{S^\pm} \leqslant k(2n+M+2)$ for every $f \in \bigoplus_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$ with $\mathrm{supp}(f) \subseteq [-n, n+M+1]$, so the estimate (12) holds, and hence so does (11).

Let us finally check that this estimate yields the desired result. For a given $\delta > 0$ and $g \in \mathrm{BS}(1,k)$, by the integration by slices inequality and (11) we have

$$\begin{aligned}\int_X \exp(\delta d_{S^\pm}(x, g\cdot x) d\mu(x) &\leqslant \sum_{M \geqslant 0} k^{-M+1} \exp\big(\delta k(2\,|g|_T + 2M + 3)\big) \\ &= \sum_{M \geqslant 0} k \exp\big(\delta k(2\,|g|_T + M(2 - \log k/\delta) + 3)\big)\end{aligned}$$

So given $\varepsilon > 0$, we take δ such that $2\delta k < \varepsilon$ and $2 - \log k/\delta < 0$, and then we will have a positive constant $C > 0$ such that for every $g \in \mathrm{BS}(1,k)$,

$$\int_X \exp(\delta d_{S^\pm}(x, g\cdot x)) d\mu(x) < C \exp(\varepsilon\,|g|_T),$$

which finishes the proof that the coupling is strongly exp-integrable from $\mathrm{BS}(1,k)$ to $\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}$. □

Corollary 8.2. *Finite presentability is unstable under the equivalence relation of strongly exponential orbit equivalence. In particular, it is unstable under* $\mathrm{L}^{<\infty}$ *orbit equivalence.*

8.2 Finite asymptotic dimension is unstable

We now consider the following group:

$$\Gamma_1 = \bigoplus_{\mathbb{Z}} \mathbb{Z}[1/k] \rtimes \mathbb{Z}^2,$$

where the first coordinate of $\mathbb{Z}^2$ acts by shift on the direct sum $\oplus_{\mathbb{Z}} \mathbb{Z}[1/k]$, and the second coordinate multiplies each factor by the corresponding power of k. Recall the action of $\mathbb{Z}[1/k]$ on $X = \prod_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$ as defined in section 8.1 induces the cofinite equivalence relation up to measure zero. For every $n \in \mathbb{Z}$, we let X_n be a copy of X. We deduce an action of the direct sum $\oplus_{n\in\mathbb{Z}} \mathbb{Z}[1/k]$ on $Y = \prod_{n\in\mathbb{Z}} X_n = \prod_{\mathbb{Z}^2} \mathbb{Z}/k\mathbb{Z}$ which induces the cofinite equivalence relation up to measure zero. The cofinite equivalence relation is also induced by the natural action of the group $\oplus_{\mathbb{Z}^2} \mathbb{Z}/k\mathbb{Z}$ on Y.

Combining these actions with the Bernoulli action of $\mathbb{Z}^2$ produces an orbit equivalence coupling between Γ_1 and the wreath product

$$\Gamma_2 = \mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}^2.$$

We equip Γ_1 with the generating subset $S_1 = \{\pm 1_{\mathbb{Z}[1/k]_0}\} \cup S_{\mathbb{Z}^2}$, and Γ_2 with $S_2 = \{\delta_0\} \cup S_{\mathbb{Z}^2}$, where $S_{\mathbb{Z}^2}$ is the canonical generating set of $\mathbb{Z}^2$.

Theorem 8.3. *The coupling we just defined is an* $(\mathrm{L}^\infty, \exp)$*-integrable orbit equivalence coupling from* Γ_2 *to* Γ_1*. Actually the exponential integrability from* Γ_1 *to* Γ_2 *is uniform in the following sense: there exists a constant* $c > 0$ *such that for all* $g \in \Gamma_1$

$$\int_X \exp(c d_{S_2}(x, g\cdot x)) d\mu(x) < \infty.$$

Proof. The proof that the coupling is L^∞ from Γ_2 to Γ_1 is exactly the same as in the proof of Theorem 8.1, so we don't write it down. Also similar to the proof of Theorem 8.1, we observe that the two respective copies of $\mathbb{Z}^2$ act the same. As a warm up, let us prove the exponential integrability from Γ_1 to Γ_2. Thanks to Proposition 2.23 only needs to consider the action of the generators $\pm 1_{\mathbb{Z}[1/k]_0}$.

Observe that the coupling induces on every X_n an orbit-equivalence coupling between $\mathbb{Z}[1/k]_0$ and $\oplus_{\mathbb{Z}\times\{0\}}\mathbb{Z}/k\mathbb{Z}$ that coincides with that of section 8.1. Hence the result follows from Theorem 8.1.

The uniform exponential integrability requires a bit more work. It will result from the following statement: given $g \in \Gamma_1$ and $M \geqslant 0$ we have that

$$\mu\left(\{y \in Y : d_{S_2}(y, g\cdot y) \geqslant (k+1)(2M + (2|g|+5)(2|g|+1)) + |g|\}\right) \leqslant \int_{t\geqslant M} t^{2|g|+1}k^{-t}dt, \qquad (13)$$

where we write $|g|$ instead of $|g|_{S_1}$ in order to simplify notation. Indeed, this implies that

$$\mu\big(\{y \in Y : d_{S_2}(y, g\cdot y) \geqslant (k+1)(2M + (2|g|+5)(2|g|+1)) + 3|g| + 1\}\big) \leqslant \frac{k^{-M}}{\log k}.$$

and therefore

$$\int_X \exp(cd_{S_2}(x, g\cdot x))d\mu(x) < \infty$$

for all $c < \log k$.

We will use the fact that the n^{th} copy $\mathbb{Z}[1/k]$ and $\oplus_{\{n\}\times\mathbb{Z}}\mathbb{Z}/k\mathbb{Z}$ have the same orbits. Moreover they only act non-trivially on the n^{th} factor X_n, and their actions are those defined in section 8.1. Let $y = (y_{i,j})$ and $g = ((x_i), (m, n))$. Note that $|n| + |m| \leqslant |g|$. Hence

$$d_{S_2}(y, g\cdot y) \leqslant |g| + \sum_{i=-|g|}^{|g|} d_T((y_{i-n,j-m})_{j\in\mathbb{Z}}, x_i \cdot (y_{i-n,j-m})_{j\in\mathbb{Z}}),$$

where T is the generating subset of $\mathbb{Z}/k\mathbb{Z}\wr\mathbb{Z}$ from section 8.1. Now for every $-|g| \leqslant i \leqslant |g|$, let $M_i \in \mathbb{R}$ be such that

$$d_T((y_{i-n,j-m})_{j\in\mathbb{Z}}, x_i \cdot (y_{i-n,j-m})_{j\in\mathbb{Z}}) = (k+1)(2|g| + 2M_i + 5).$$

We deduce that $\sum_{i=-|g|}^{|g|} M_i \geqslant M$. Note that $|g| \geqslant |(x_i)_{i\in\mathbb{Z}}|_S$, where S is the generating subset of $\mathrm{BS}(1, k)$ from section 8.1. On the other hand, by (11) we have that for all $(t_i) \in \mathbb{R}_+^{2|g|+1}$,

$$\mu\left(\{y \in Y : d_T((y_{i-n,j-m})_{j\in\mathbb{Z}}, x_i \cdot (y_{i-n,j-m})_{j\in\mathbb{Z}}) \geqslant (k+1)(2|g|_S + 2t_i + 5)\}\right) \leqslant k^{-t_i}.$$

For every $t \geqslant 0$ and $m \in \mathbb{N}$, denote by $v_m(t) = \mathrm{Vol}\{(t_i) \in \mathbb{R}_+^m,\ \sum_i t_i = t\}$, and observe that $v_m(t) \leqslant t^m$. Finally, integrating over all values for $t_i = \max\{M_i, 0\}$ for which $t = \sum_i t_i \geqslant M$, we obtain

$$\begin{aligned}\mu\left(\{y \in Y : d_{S_2}(y, g\cdot y) \geqslant (k+1)(2M + (2|g|+5)(2|g|+1)) + |g|\}\right) &\leqslant \int_{t\geqslant M} k^{-t}v_{2|g|+1}(t)dt \\ &\leqslant \int_{t\geqslant M} t^{2|g|+1}k^{-t}dt\end{aligned}$$

which ends the proof of (13) and therefore of the theorem. □

Observing that Γ_1 has infinite asymptotic dimension, while Γ_2 has asymptotic dimension 2 (e.g. as a consequence of the results from [DS06, Sec. 3]), we deduce the following corollary.

Corollary 8.4. *There exist a group with asymptotic dimension* 2*, and a group of infinite asymptotic dimension that admit an exponential-integrable orbit equivalence measure coupling.*

9 Hyperbolicity

In this section we prove the results of §1.3. Our main tool will be to characterize Gromov hyperbolicity in terms of distortion of embedded cycles. More precisely we shall confront the facts that an n-cycle in a hyperbolic graph must have distortion in $\Theta(n/\log n)$ (see Proposition 9.2), while any non-hyperbolic graph admits arbitrary large cycles with universally bounded distortion (see Proposition 9.8). A version of the first statement for real hyperbolic spaces is proved in [VS14, Lemma 3.3] and a weaker version of our second statement is obtained in [VS14, Theorem 4.1].

9.1 Geometric preliminaries

Let X be a graph, a **discrete path** in X of length $l \geqslant 1$ is a map $\alpha : \{0, ..., l\} \to X$ such that for all $i \in \{0, ..., l-1\}$, we have that $\alpha(i)$ and $\alpha(i+1)$ are connected by an edge. We will also say that α is a path from $\alpha(0)$ to $\alpha(l)$, and we will often identify a path to its range.

Every connected graph X is viewed as a metric space (X, d) equipped with the **discrete path metric**, defined by setting $d(x, y)$ as the minimum length of a path from x to y. Any discrete path which realizes the discrete path metric between two points is called a **discrete geodesic**, and it is then an isometric embedding from $\{0, ..., d(x, y)\}$ to (X, d).

Another important metric space that we can get out of a connected graph X is given by the (continuous) **path metric** which we define as in [Gro07, 1.15_+]. We first identify each edge to the interval $[0, 1]$ isometrically, thus obtaining a *length structure* on our graph. The metric associated to this length structure is denoted by d_l, and it is by definition the continuous path metric on X. It agrees with the discrete path metric on the vertices of X, and it is geodesic. Every geodesic between vertices defines a discrete geodesic, and every discrete geodesic can be lifted to a geodesic between vertices.

A (geodesic) **triangle** in a metric space (X, d) with **vertices** $a_1, a_2, a_3 \in X$ is the set $[a_1, a_2, a_3] \subseteq X$ obtained by taking the union of a choice of geodesics $[a_1, a_2]$, $[a_2, a_3]$, and $[a_3, a_1]$ between it vertices. In the same way, we define a (geodesic) n**-gon** with vertices $a_1, \dots, a_n \in X$, and denote it by $[a_1, \dots, a_n]$. Given an n-gon where $n \geqslant 3$, we will frequently call any of its defining geodesics a **side**.

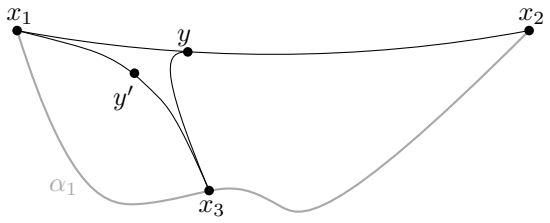
Now, recall that a geodesic space (X, d) is δ**-hyperbolic** in the sense of Rips if there exists a $\delta \geqslant 0$ such that for every geodesic triangle $[a_1, a_2, a_3]$ and for every $x \in [a_1, a_2]$, there exists an element y in either $[a_1, a_3]$ or $[a_2, a_3]$ such that $d(x, y) \leqslant \delta$; or equivalently, that the side $[a_1, a_2]$ is contained in the δ-neighborhood of $[a_1, a_3] \cup [a_2, a_3]$. Moreover, we say that a geodesic space (X, d) is **hyperbolic** if it is δ-hyperbolic for some δ, and that a finitely generated group Γ with generating set S is hyperbolic whenever its Cayley graph is hyperbolic when equipped with the continuous path metric.

We shall need the following well-known lemma.

Lemma 9.1. *Let X be a δ-hyperbolic geodesic space, let α be a path of length $\ell \geqslant 1$ between two points x_1 and x_2, and let y be a vertex belonging to a geodesic from x_1 to x_2. Then*

$$d(y, \alpha) \leqslant \delta \log_2(\ell) + 1.$$

Proof. Let us prove it by induction on $\lfloor \ell \rfloor$. The case $\lfloor \ell \rfloor = 1$ is clear. So assume $n \geqslant 2$ and suppose that the lemma is true for all paths of length $< n$. Let α be a path of length $\ell \in [n, n+1[$ from x_1 to x_2, represented as a gray path in the following figure. Let $[x_1, y]$ be a geodesic from x_1 to y. Let $x_3 \in \alpha$ be such that $d(y, \alpha) = d(y, x_3)$ and let $[y, x_3]$ be a geodesic from y to x_3. By exchanging x_1 and x_2 if necessary, we can assume that the portion α_1 of α from x_1 to x_3 has length $\ell_1 \leqslant \ell/2$.



Let $[x_1, x_3]$ be a geodesic from x_1 to x_3. In the geodesic triangle $[x_1, y, x_3]$, there is a point $y' \in [x_1, x_3]$ such that $d(y, y') \leqslant \delta$. Now applying the induction hypothesis to the path α_1 and the point $y' \in [x_1, x_3]$, we obtain

$$d(y', \alpha_1) \leqslant \delta \log_2(\ell_1) \leqslant \delta \log_2 \ell + 1 - \delta.$$

We deduce by the triangular inequality

$$d(y, \alpha) \leqslant d(y, y') + d(y', \alpha_1) \leqslant \delta \log_2(\ell) + 1.$$

So we are done. □

First we need an alternative definition of hyperbolicity in terms of embedded cycles. In what follows, the **cycle** C_n of length $n \geqslant 2$ is the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with respect to the generating set containing just the element 1 mod n which we view as a *discrete* metric space denoted by (C_n, d_{C_n}).

Proposition 9.2. *Let (X, d) be a δ-hyperbolic geodesic space, let n be a positive integer. Then for every $a \geqslant 0$ and every $b \geqslant 1$, if there is a map $\varphi : C_{2n} \to X$ such that for every $x, y \in C_{2n}$,*

$$a d_{C_{2n}}(x, y) \leqslant d(\varphi(x), \varphi(y)) \leqslant b d_{C_{2n}}(x, y)$$

then we have

$$a \leqslant \frac{4\delta \log_2(bn) + 4 + 2b}{n}$$

Before proving the above proposition, let us note the following straightforward corollary, using the estimate $\frac{1}{\log 2} < 3/2$.

Corollary 9.3. *Let $a \geqslant 0$, $b \geqslant 1$, and $\delta \geqslant 1$. There is an integer $n_0 = n_0(a, b) \geqslant 2$ such that the following holds. For all δ-hyperbolic geodesic space X, for all $n \geqslant n_0$, if there is a map $\varphi : C_n \to X$ such that for all $x, y \in C_n$*

$$a d_{C_n}(x, y) \leqslant d(\varphi(x), \varphi(y)) \leqslant b d_{C_n}(x, y)$$

then we have

$$a < 12\delta \cdot \frac{\log n}{n}. \tag{14}$$

In order to prove the proposition, we need the following additional notion. Given a discrete path β in a graph Y and a map $\varphi : Y \to (X, d)$ where (X, d) is geodesic, we say that a continuous path α in X is a **φ-direct image** of β if it is obtained by concatenating geodesics between $\varphi(\beta(i))$ and $\varphi(\beta(i+1))$ where i ranges from 0 to $\ell(\beta) - 1$.

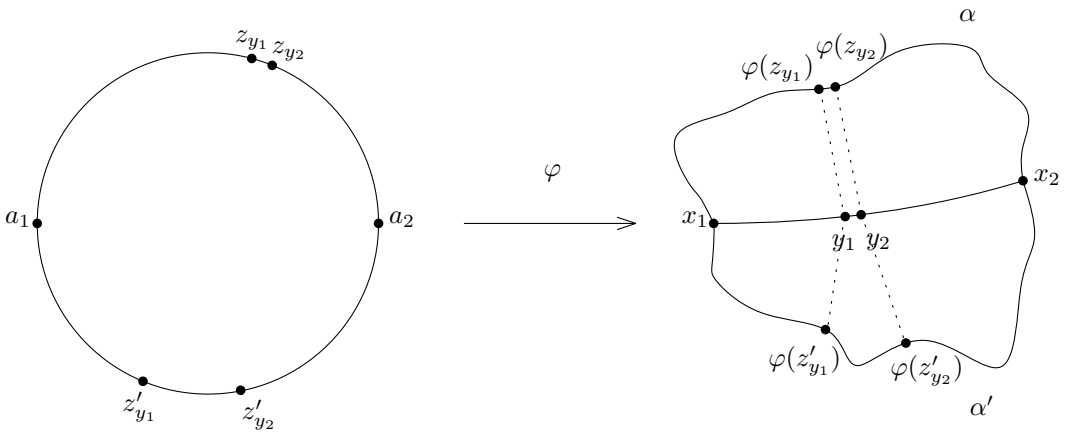
Proof of Proposition 9.2. Let $a_1, a_2 \in C_{2n}$ be such that $d_{C_{2n}}(a_1, a_2) = n$ and define $x_1 = \varphi(a_1)$, $x_2 = \varphi(a_2)$. Consider a geodesic $[x_1, x_2]$ from x_1 to x_2. In C_{2n} there are two discrete geodesic paths from a_1 to a_2, and they have length n. Denote by α and α' some respective φ-direct images of those paths in X, which by assumption have length at most bn. Let $y \in [x_1, x_2]$, we deduce from Lemma 9.1 that

$$d(y, \alpha) \leqslant \delta \log_2(\ell(\alpha)) + 1 \leqslant \delta \log_2(b\ell) + 1,$$

where $\ell(\alpha)$ is the length of α, and by the same argument $d(y, \alpha') \leqslant \delta \log_2(b\ell) + 1$. If we then pick z_y and z'_y points in C_{2n} such that their φ-images are in α and α' respectively and minimize the distance to y, we have

$$\max\left(d(y, \varphi(z_y)), d(y, \varphi(z'_y)\right) \leqslant \delta \log_2(b\ell) + 1 + \frac{b}{2}. \tag{15}$$

Note that for any $y \in [x_1, x_2]$, any geodesic from z_y to z'_y must pass through x_1 or through x_2. Moreover there are some $y \in [x_1, x_2]$ for which the first case occurs, and some for which the second case occurs. For all $\varepsilon > 0$, we may thus find $y_1, y_2 \in [x_1, x_2]$ such that $d(y_1, y_2) \leqslant \varepsilon$, the geodesic from z_{y_1} to z'_{y_1} passes through x_1 and the geodesic from z_{y_2} to z'_{y_2} passes through x_2.



Then we have that

$$d(z_{y_1}, z_{y_2}) + d(z_{y_2}, z'_{y_2}) + d(z'_{y_2}, z'_{y_1}) + d(z'_{y_1}, z_{y_1}) \geqslant 2n.$$

Hence one of these four distances is at least $\frac{n}{2}$. However, a straightforward case analysis using the bound (15) on the lengths of the dotted geodesics, the triangle inequality and the fact that $d(y_1, y_2) \leqslant \varepsilon$ shows that the distance between the corresponding images of the points in the previous inequation is always at most $2(\delta \log_2(bn) + 1 + \frac{b}{2}) + \varepsilon$. Using our assumption on φ, we thus have the following inequality: for all $\varepsilon > 0$,

$$\frac{an}{2} \leqslant 2\delta \log_2(bn) + 2 + b + \varepsilon.$$

So the proposition follows. □

In what follows, given a constant $a \geqslant 1$, we will say that a function $f\colon (X, d_X) \to (Y, d_Y)$ between two metric spaces **contracts distances by at most the factor** a if for every $x_1, x_2 \in X$ we have $d_X(x_1, x_2) \leqslant a d_Y(f(x_1), f(x_2))$.

Lemma 9.4. *Let $n \geqslant 4$ and $D \geqslant 1$, let (X, d) be a geodesic metric space containing an n-gon $[a_1, \ldots, a_n]$ such that the distance between points on non-adjacent sides is at least D. Let $R \geqslant D$ be the length of its longest side. Then there is a cycle C of integer length at least $\frac{Dn}{2}$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ which contracts distances by at most the factor $\frac{3nR}{D}$.*

Proof. Our cycle will be obtained via the following claim.

Claim 9.5. There is a $2n$-gon $C = [b_1, c_1, b_2, \ldots, b_n, c_n]$ such that for every $i \in \{1, \ldots, n\}$,

- $b_i \in [a_{i-1}, a_i]$ and $c_i \in [a_i, a_{i+1}]$, where we let $a_0 = a_n$;
- $d(b_i, c_i) = \frac{D}{8R}(d(b_i, a_i) + d(a_i, c_i))$;
- for each pair of points $x \in [c_{i-1}, b_i]$ and $y \in [c_i, b_{i+1}]$, we have $d(x, y) \geqslant \frac{D}{8R}(d(x, a_i) + d(a_i, y))$.

Proof of the claim. We start with the existence of b_1 and c_1. Note that for all points $u \in [a_n, a_1]$ and $v \in [a_1, a_2]$, we have $d(u, a_1) + d(a_1, v) \leqslant 2R$ and so we have $\frac{D}{8R}(d(u, a_1) + d(a_1, v)) \leqslant \frac{D}{4}$.

Taking $u = a_1 = v$, we see that there are $u \in [a_n, a_1]$ and $v \in [a_1, a_2]$ such that $d(u, v) = \frac{D}{8R}(d(u, a_1) + d(a_1, v))$. Also, for $u = a_n$ and $v = a_2$, we have $d(u, v) \geqslant D$ and so $d(u, v) > \frac{D}{8R}(d(u, a_1) + d(a_1, v))$.

Now by continuity we can take $u \in [a_n, a_1]$ and $v \in [a_1, a_2]$ such that $d(u, v) = \frac{D}{8R}(d(u, a_1) + d(a_1, v))$, but with $d(u, a_1) + d(a_1, v)$ maximal. Then for all $(u', v') \in [a_n, u] \times [v, a_2]$ such that $(u, v) \neq (u', v')$, we have $d(u', v') > \frac{D}{8R}(d(u', a_1) + d(a_1, v'))$. Indeed if not, by continuity since $d(a_n, a_2) > \frac{D}{8R}(d(a_n, a_1) + d(a_1, a_2))$ we could find $(u'', v'') \in [a_n, u'] \times [v', a_2]$ such that $d(u'', v'') = \frac{D}{8R}(d(u'', a_1) + d(a_1, v''))$, contradicting the maximality of $d(u, a_1) + d(a_1, v)$. Taking $b_1 = u$ and $c_1 = v$, we do have the desired properties for $i = 1$.

Similarly, for every $i \in \{2, \dots, n-1\}$ one can find $b_i \in [c_{i-1}, a_i]$ and $c_i \in [a_i, a_{i+1}]$ such that $d(b_i, c_i) = \frac{D}{8R}(d(b_i, a_i) + d(a_i, c_i))$, and such that $d(b_i, a_i) + d(a_i, c_i)$ is maximal in the sense that for every $x \in [a_{i-1}, b_i]$ and $y \in [c_i, a_{i+1}]$ we have that $d(x, y) \geqslant \frac{D}{8R}(d(x, a_i) + d(a_i, y))$. This can be done repeating the previous argument, and based on the fact that

$$d(c_{i-1}, a_{i+1}) \geqslant d(b_{i-1}, a_{i+1}) - d(b_{i-1}, c_{i-1}) \geqslant \frac{3D}{4}.$$

Finally, we take $b_n \in [c_{n-1}, a_n]$ and $c_n \in [a_n, b_1]$ as previously, completing the trip around $[a_1, \dots, a_n]$ (this time, this relies on the fact that $d(c_{n-1}, b_1) \geqslant D/2$). $\square_{\text{claim}}$

Now note that the length ℓ of C, which is the sum of its edge lengths, lies between $\frac{Dn}{2} \geqslant 2$ and nR since each side $[c_i, b_{i+1}]$ is of length at least $\frac{D}{2}$ and the sides of $[a_1, \dots, a_n]$ have length at most R. Denote the path metric on C by d_C. Then (C, d_C) is isometric to the cycle of real length ℓ, i.e. to $\mathbb{R}/\ell\mathbb{Z}$ equipped with the quotient metric (for $\ell \in \mathbb{N}$, this is isometric the continuous path metric on the graph C_ℓ). So we can find a 1-Lipschitz map $(C_{\lceil \ell \rceil}, d_{C_{\lceil l \rceil}}) \to (C, d_C)$ which contracts distances by at most the factor $\frac{\lceil \ell \rceil}{\ell} \leqslant \frac{3}{2}$. In order to prove the theorem, we will now show that the inclusion (C, d_C) in (X, d) contracts distances by at most the factor $\frac{2nR}{D}$, since by composing this will yield a 1-Lipschitz map $C_{\lceil l \rceil} \to (X, d)$ which contracts distances by at most the factor $\frac{3nR}{D}$ and the length of the cycle is at least $\frac{Dn}{2}$.

For each pair of points $x \in [c_{i-1}, b_i]$ and $y \in [c_i, b_{i+1}]$, we have $d(x, y) \geqslant \frac{D}{8R}(d(x, a_i) + d(a_i, y))$, which implies

$$d_C(x, y) \leqslant \frac{8R}{D} d(x, y). \tag{16}$$

We now give a similar bound for two points on adjacent sides of C. Towards this, let $x \in [b_i, c_i]$ and $y \in [c_i, b_{i+1}]$. Then,

$$\begin{aligned}
d(x, y) &\geqslant d(y, b_i) - d(x, b_i) \\
&\geqslant \frac{D}{8R} d(y, a_i) + \frac{D}{8R} d(b_i, a_i) - d(b_i, c_i) + d(x, c_i) \\
&= \frac{D}{8R} d(y, c_i) + \frac{D}{8R} d(a_i, c_i) + \frac{D}{8R} d(b_i, a_i) - d(b_i, c_i) + d(x, c_i) \\
&= \frac{D}{8R} d(y, c_i) + d(x, c_i) \\
&\geqslant \frac{D}{8R} (d(y, c_i) + d(x, c_i)) \\
&= \frac{D}{8R} d_C(x, y),
\end{aligned}$$

so (16) is also valid for such pairs (x, y). The case where $x \in [b_i, c_i]$ and $y \in [c_i, b_{i+1}]$ is analogous. Finally, for $x, y \in C$ not in any of the previous cases, there exist two different sides S_x and S_y in the

n-gon $[a_1, \dots, a_n]$ such that $d(x, S_x) \leqslant \frac{D}{4}$ and $d(y, S_y) \leqslant \frac{D}{4}$, and hence $d(x, y) \geqslant \frac{D}{2}$. As the perimeter of C is at most nR, it follows that

$$d_C(x, y) \leqslant \frac{2nR}{D} d(x, y). \tag{17}$$

The lemma now results from the combination of (16) and (17). □

Lemma 9.6. *Let (X, d) be a geodesic space and let $r > 0$. Assume that X contains a quadrilateral $[a_1, a_2, a_3, a_4]$ with the following properties:*

- $d(x, y) \geqslant \frac{r}{2}$ *for every* $x \in [a_2, a_3]$ *and* $y \in [a_1, a_4]$;
- $d(a_1, a_2) \leqslant r$;
- $d(a_2, a_3) \geqslant 4r$;
- $d(a_3, a_4) \leqslant 2r$.

Then there exist a cycle C of length at least $\lceil 2r \rceil$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances by at most a factor of 480 in X.

Proof. First note that by the triangular inequality, $d(x, y) \geqslant r$ for all $x \in [a_1, a_2]$ and $y \in [a_3, a_4]$.

Let us first assume that $d(a_2, a_3) \leqslant 12r$. We deduce that the distance between pairs of opposite sides are at least $r/2$. Assume first that $d(a_2, a_3) \leqslant 12r$, in which case we deduce that the longest side has length at most $15r$. By Lemma 9.4, it now follows that there exist a cycle C of length at least $\lceil 2r \rceil$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances at most by a factor of $(4 \cdot 4 \cdot 15r)/(r/2) = 480$ in X.

Let us now suppose $d(a_2, a_3) \geqslant 12r$ and let $m \in \mathbb{N}$ be such that $2mr \leqslant d(a_2, a_3) < 2(m+1)r$ and $m \geqslant 6$. We will prove the claim by induction with respect to $m \geqslant 6$. The initial step $m = 6$ is taken care of by the first part of the proof. Assume that the lemma is true for $d(a_2, a_3) < 2mr$. We claim that it is then also true for $d(a_2, a_3) < 2(m+1)r$ which concludes the proof by induction. Note there are two possibilities: either there exists $b \in [a_2, a_3]$ such that $d(a_2, b) \geqslant 2r$ and $d(a_3, b) \geqslant 2r$ that satisfies $d(b, [a_1, a_4]) < r$; or such a b does not exist. If such a b exists, fix $c \in [a_1, a_4]$ such that $d(b, c) < r$. Now, we have that $d(a_2, b) \leqslant 2mr$ and $d(b, a_3) \leqslant 2mr$, and at least of them must be more that $4r$ since $d(a_2, a_3) \geqslant 12r$. From this it follows that either $[a_1, a_2, b, c]$ or $[c, b, a_3, a_4]$ satisfies the induction hypothesis, which concludes the proof by induction under the assumption that b exists.

If b does not exists, let $b_1, b_2 \in [a_2, a_3]$ be such that $d(a_2, b_1) \leqslant 2r$ and $d(a_3, b_2) \leqslant 2r$ where $d(b_1, [a_1, a_4]) \leqslant 2r$ and $d(b_2, [a_1, a_4]) \leqslant 2r$, and $d(x, [a_1, a_4]) \geqslant r$ for every $x \in [b_1, b_2]$. Now, take $c_1, c_2 \in [a_1, a_4]$ such that $d(b_1, c_1) \leqslant 2r$ and $d(b_2, c_2) \leqslant 2r$. Finally, we can apply the induction hypothesis to the quadrilateral $[c_1, b_1, b_2, c_2]$, since $d(b_1, c_1) \leqslant 2r$, $d(b_1, b_2) \geqslant 8r$ as $d(a_2, a_3) \geqslant 12r$, $d(b_2, c_2) \leqslant 2r$, and $d(x, [c_1, c_2]) \geqslant r$ for every $x \in [b_1, b_2]$. So, there exist a cycle C of length at least $\lceil 4r \rceil$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances at most by a factor of 480 in X, which concludes the proof by induction. □

Corollary 9.7. *Let (X, d) be a geodesic space and let $R \geqslant r > 0$. Assume that X contains a quadrilateral $[a_1, a_2, a_3, a_4]$ with the following properties:*

- $d(x, y) \geqslant \frac{r}{2}$ *for every* $x \in [a_2, a_3]$ *and* $y \in [a_1, a_4]$;
- $d(a_1, a_2) \leqslant r$;
- $d(a_2, a_3) \geqslant \max\{12(R - r), 4r\}$;
- $d(a_3, a_4) \leqslant R$.

Then there exist a cycle C of length at least $\lceil 2r \rceil$ and a 1-Lipschitz-embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances by at most a factor of 480 in X.

Proof. By Lemma 9.6, the corollary is true for $R \leqslant 2r$, so suppose that $R \geqslant 2r$ and let $m \in \mathbb{N}$ be such that $2^m r < R \leqslant 2^{m+1} r$. We now prove the claim by induction with respect to m. If $m = 0$, then $R \leqslant 2r$ and the corollary is true. Next, suppose the corollary is true for $R \leqslant 2^m r$. We claim that it is then also true for $R \leqslant 2^{m+1} r$, which concludes the proof by induction.

Now we take $b \in [a_2, a_3]$ such that $d(b, [a_1, a_4]) = r$ and $d(a_2, b)$ is maximal. Also take $c \in [a_1, a_4]$ such that $d(b, c) = r$. If $d(a_2, b) \geqslant 4r$, then by Lemma 9.6 the quadrilateral $[a_1, a_2, b, c]$ satisfies the claim, which concludes the proof by induction. On the other hand, if $d(a_2, b) < 4r$, then $d(b, a_3) \geqslant d(a_2, a_3) - d(b, a_2) > 12(R - r) - 4r = 12(R - 2r) + 8r \geqslant \max\{12(R - 2r), 8r\}$, and we can apply the induction hypothesis to the quadrilateral $[c, b, a_3, a_4]$ as $d(c, b) \leqslant 2r$, $d(a_3, a_4) \leqslant R$ and $d(x, [a_1, a_4]) \geqslant r$ for every $x \in [a_2, a_3]$, from which it follows that there exist a cycle C of length at least $\lceil 4r \rceil$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances at most by a factor 480 in X, which concludes the proof by induction. □

Proposition 9.8. *Let (X, d) be a geodesic space that is not δ-hyperbolic. Then, there exist a cycle C of length at least $\lceil \frac{2\delta}{15} \rceil$ and a 1-Lipschitz embedding $\varphi\colon (C, d_C) \to (X, d)$ that contracts distances by at most a factor of 23760 in X.*

Proof. As (X, d) is not δ-hyperbolic, there exists a geodesic triangle $[a, b, c]$ such that the minimum $D > 0$ such that $[a, b]$ is contained in the D-neighborhood of the the union of $[a, c]$ and $[b, c]$ is $> \delta$. Let $x \in [a, b]$ be such that $d(x, [a, c] \cap [b, c]) = D$.

As the map $y \mapsto d(y, [a, c] \cup [b, c])$ is continuous, there exists $x_a \in [a, x]$ such that $d(x_a, [a, c] \cup [b, c]) = \frac{D}{15}$. Now, take $y_a \in [a, c] \cup [b, c]$ such that $d(x_a, y_a) = \frac{D}{15}$. Similarly, let $x_b \in [x, b]$ and $y_b \in [a, c] \cup [b, c]$ such that $d(x_b, [a, c] \cup [b, c]) = d(x_b, y_b) = \frac{D}{15}$. Moreover, take x_a and x_b above such that $d(x_a, x)$ and $d(x_b, x)$ are minimal.

Case 1 Suppose first that y_a and y_b are on the same side $[a, c]$ or $[b, c]$. To begin, note that $d(x_a, x_b) = d(x_a, x) + d(x, x_b) \geqslant \frac{28D}{15}$ since $D \leqslant d(x, y_a) \leqslant d(x, x_a) + \frac{D}{15}$ and $D \leqslant d(x, y_b) \leqslant d(x, x_b) + \frac{D}{15}$. Thus, we have for $[y_a, x_a, x_b, y_b]$ that $d(y_a, x_a) = \frac{D}{15} \leqslant \frac{2D}{15}$, $d(x_a, x_b) \geqslant \frac{28D}{15}$, $d(x_b, y_b) = \frac{D}{15} \leqslant \frac{2D}{15}$, and $d(x', [y_a, y_b]) \geqslant \frac{1}{2} \frac{2D}{15}$ for all $x' \in [x_a, x_b]$. Now, by Corollary 9.7 it follows by taking $r = \frac{2D}{15}$ and $R = r$ that there exist a cycle C of length at least $\lceil \frac{4D}{15} \rceil$ and a 1-Lipschitz embedding of C into X that contracts distances by at most a factor of 480 in X.

Case 2 Next, suppose that y_a and y_b are not on the same geodesic $[a, c]$ or $[b, c]$, but that $d(y_a, [y_b, c]) \leqslant \frac{3D}{15}$. Let $u \in [y_b, c]$ be such that $d(y_a, u) \leqslant \frac{3D}{15}$. Thus, we have for $[y_b, x_b, x_a, u]$ that $d(y_b, x_b) = \frac{D}{15} \leqslant \frac{2D}{15}$, $d(x_b, x_a) \geqslant \frac{28D}{15} \geqslant \max\left\{12(\frac{4D}{15} - \frac{2D}{15}), 4\frac{2D}{15}\right\}$, $d(x_a, u) \leqslant d(x_a, y_a) + d(y_a, u) \leqslant \frac{4D}{15}$, and $d(x', [y_b, u]) \geqslant \frac{1}{2} \frac{2D}{15}$ for every $x' \in [x_b, x_a]$. Thus, by Corollary 9.7 it follows by taking $r = \frac{2D}{15}$ and $R = 2r$ that there exist a cycle C of length at least $\lceil \frac{4D}{15} \rceil$ and a 1-Lipschitz embedding of C into X that contracts distances by at most by a factor of 480 in X.

Case 3 Finally, suppose that y_a and y_b are not on the same geodesic $[a, c]$ or $[b, c]$, and that $d(y_a, [y_b, c]) > \frac{3D}{15}$ and $d(y_b, [y_a, c]) > \frac{3D}{15}$. Take $z_a \in [y_a, c]$ and $z_b \in [y_b, c]$ such that $d(z_a, z_b) = \frac{D}{15}$ and $d(z_a, y_a)$ is minimal. Now consider the hexagon $[x_a, x_b, y_b, z_b, z_a, y_a]$. Recall that the sides $[x_a, x_b]$, $[y_a, z_a]$ and $[y_b, z_b]$ are $\frac{D}{15}$-disjoint and have length of at least $\frac{2D}{15}$, and that the length of the other sides is $\frac{D}{15}$.

Claim 9.9. We claim that all non-adjacent sides are $\frac{D}{45}$-disjoint.

Proof of the claim. Suppose w and w' are on non-adjacent sides. Recalling that the sides $[x_a, x_b]$, $[y_a, z_a]$ and $[y_b, z_b]$ are $\frac{D}{15}$-disjoint, we may assume without loss of generality that w is not on one of these sides, say $w \in [z_a, z_b]$. If $w' \in [x_a, x_b]$, then either $d(w, w') \geqslant d(w', [y_a, z_a]) - d(w, [y_a, z_a]) \geqslant \frac{D}{15} - \frac{D}{30} = \frac{D}{30}$ or $d(w, w') \geqslant d(w', [y_b, z_b]) - d(w, [y_b, z_b]) \geqslant \frac{D}{15} - \frac{D}{30} = \frac{D}{30}$ as it can not be that both $d(w, [y_a, z_a])$ and $d(w, [y_b, z_b])$ are strictly greater than $\frac{D}{30}$ because $d(z_a, z_b) = \frac{D}{15}$. So, suppose w' is either in $[x_a, y_a]$ or in $[x_b, y_b]$. Without loss of generality, we can assume that $w' \in [x_a, y_a]$. Now, if $d(x_a, w') + d(z_b, w) \leqslant \frac{2D}{45}$,

then $d(w,w') \geqslant d(z_b,x_a) - d(x_a,w') - d(w,z_b) \geqslant d(z_b,x_a) - \frac{2D}{45} \geqslant \frac{D}{45}$ since $d(z_b,x_a) \geqslant \frac{2D}{15}$, and we are done. On the other hand, if $d(x_a,w')+d(z_b,w) \geqslant \frac{2D}{45}$ then $d(y_a,w')+d(z_a,w) \leqslant \frac{4D}{45}$ since the side length of $[x_a,y_a]$ and $[z_a,z_b]$ is $\frac{D}{15}$, and thus $d(w,w') \geqslant d(w,y_a)-d(y_a,w') \geqslant d(z_a,y_a)-(d(w,z_a)+d(y_a,w')) \geqslant d(z_a,y_a) - \frac{2D}{45} = \frac{2D}{15} - \frac{2D}{45} = \frac{2D}{45}$, and we are done. In other words, all non-adjacent sides are $\frac{D}{45}$-disjoint, which completes the proof. $\square_{\text{claim}}$

We can now finish the proof of the third case. By Lemma 9.4 there exist a cycle C of length at least $\lceil \frac{6D}{45} \rceil = \lceil \frac{2D}{15} \rceil$ and a 1-Lipschitz embedding of C into X that contracts distances by at most a factor of $(4 \cdot 6 \cdot 22D)/(D/45) = 23760$ in X. On the other hand, if at least one of the sides has length more than $22D$, take u as the midpoint of that side, and $v \in [a,b,c]$ such that $d(u,v) \leqslant D$ where u and v are not on the same side of the triangle. Then, we find two adjacent corners w_u and w_v in $[x_a,x_b,y_b,z_b,z_a,y_a]$ such that $[u,v]$ splits this hexagon into the quadrilateral $[w_v,w_u,u,v]$ and a new hexagon, where $d(w_v,w_u) \leqslant \frac{2D}{15}$, $d(w_u,u) > \frac{22D}{2} = 11D \geqslant \max\left\{12(D-\frac{2D}{15}), 4\frac{2D}{15}\right\}$, $d(u,v) \leqslant D$ and $d(x',[w_v,v]) \geqslant \frac{1}{2}\frac{2D}{15}$ for each $x' \in [w_u,u]$. Thus, once again by Corollary 9.7 it follows by taking $r = \frac{2D}{15}$ and $R = D$ that there exist a cycle C of length at least $\lceil \frac{4D}{15} \rceil$ and a 1-Lipschitz embedding of C into X that contracts distances by at most by a factor of 480 in X.

$\square$

Corollary 9.10. *Let (X,d) be a geodesic space that is not hyperbolic. Then, there exist arbitrarily large integers n and embeddings $\varphi_n \colon (C_n, d_{C_n}) \to (X,d)$ such that*

$$\frac{1}{23760} d_{C_n}(x,y) \leqslant d(\varphi_n(x),\varphi_n(y)) \leqslant d_{C_n}(x,y),$$

for every $x,y \in C_n$.

9.2 Rigidity of hyperbolicity

We now prove a general rigidity result, that says that hyperbolicity is preserved under cobounded measure equivalence couplings satisfying certain integrability conditions. The following general statements are technical, so for that reason, we immediately provide striking corollaries.

However first we fix some notation: Given a group Γ equipped with a finitely generated subset S_Γ, we denote the growth function $\mathrm{Vol}_{S_\Gamma}(n) = |S_\Gamma^n|$ and its entropy $\mathrm{Ent}(S_\Gamma) = \limsup_{r\to\infty} \frac{\log(\mathrm{Vol}_{S_\Gamma}(r))}{r}$.

Theorem 9.11. *Let Γ and Λ be two finitely generated groups such that Γ is δ-hyperbolic. We let $L \geqslant 1$ and φ, ψ and r be increasing unbounded functions. Assume that the following conditions are satisfied:*

$$\lim_{n\to\infty} \frac{n^2 r(n) \mathrm{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))} \to 0; \tag{18}$$

and for all large enough n,

$$r(n) > M \log(n\psi^{-1}(3Ln)) + 8\psi^{-1}(3Ln), \tag{19}$$

with $M = \frac{48(\delta+1)}{\log 2}$. Assume that (Ω,μ) is a cobounded measure equivalence coupling from Γ to Λ, normalized so that $\mu(X_\Gamma) = 1$, and such that $\alpha \colon \Gamma \times X_\Lambda \to \Lambda$ and $\beta \colon \Lambda \times X_\Gamma \to \Gamma$ satisfy the following properties.

(i) for all $s \in S_\Gamma$,

$$\int_{X_\Lambda} \varphi(|\alpha(s,x)|_{S_\Lambda}) d\mu(x) < \infty;$$

(ii) for all $t \in S_\Lambda$

$$\int_{X_\Gamma} \psi(|\beta(t,x)|_{S_\Gamma}) d\mu(x) \leqslant L.$$

Then Λ is hyperbolic.

Assuming that β is bounded, we have the following variant.

Theorem 9.12. *Let Γ and Λ be two finitely generated groups such that Γ is δ-hyperbolic. We let $L \geqslant 1$ and φ, ψ and r be increasing unbounded functions. Assume that the following conditions are satisfied:*

$$\lim_{n\to\infty} \frac{n^2 r(n)\,\mathrm{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))} \to 0; \tag{20}$$

and for all large enough n,

$$r(n) \geqslant 75\delta \log n. \tag{21}$$

Assume that (Ω, μ) is a cobounded measure equivalence coupling from Γ to Λ, normalized so that $\mu(X_\Gamma) = 1$, and such that $\alpha\colon \Gamma \times X_\Lambda \to \Lambda$ and $\beta\colon \Lambda \times X_\Gamma \to \Gamma$ satisfy the following properties.

(i) for all $s \in S_\Gamma$,

$$\int_{X_\Lambda} \varphi(|\alpha(s,x)|_{S_\Lambda}) d\mu(x) < \infty;$$

(ii) for all $t \in S_\Lambda$

$$|\beta(t,\cdot)|_{S_\Gamma} \in \mathrm{L}^\infty(X_\Gamma).$$

Then Λ is hyperbolic.

Theorem 10 is an immediate consequence of the following corollary of Theorem 9.12.

Corollary 9.13. *Let Γ be a finitely generated δ-hyperbolic group and let $p > 75\delta\,\mathrm{Ent}(S_\Gamma) + 2$. Assume that there exists a cobounded $(\mathrm{L}^p, \mathrm{L}^\infty)$-integrable measure equivalence coupling from Γ to any finitely generated group Λ. Then Λ is hyperbolic.*

Proof. We apply Theorem 9.12 with $p = 75\delta(\mathrm{Ent}(S_\Gamma) + \varepsilon) + 2$ for some $\varepsilon > 0$. We let $r(n) = 75\delta \log n$. By definition of $\mathrm{Ent}(S_\Gamma)$, there exists have $\mathrm{Vol}_{S_\Gamma}(r(n)) = o(e^{(\mathrm{Ent}(S_\Gamma)+\varepsilon/2)r(n)})$. Hence we deduce that

$$\mathrm{Vol}_{S_\Gamma}(r(n)) = o(n^{75\delta(\mathrm{Ent}(S_\Gamma)+\varepsilon/2}),$$

which combined with the fact that $\phi(t) = t^{75\delta(\mathrm{Ent}(S_\Gamma)+\varepsilon)+2}$ implies that (20) is satisfied. Thus, Λ is hyperbolic. □

Corollary 9.14. *Let Γ be a finitely generated hyperbolic group. For every $p > q > 0$ such that if there is a cobounded (φ, ψ)-integrable measure equivalence coupling from Γ to any finitely generated group Λ where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is also hyperbolic.*

Proof. First we take the cobounded (φ, ψ)-integrable measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ and normalize such that $\mu(X_\Gamma) = 1$. We pick η strictly between q and p, take $r(n) = n^{\frac{\eta}{1+\eta}}$, and we let L be such that

$$\int_{X_\Gamma} \psi(|\beta(t,x)|_{S_\Gamma}) |d\mu(x) \leqslant L.$$

The latter is possible because $\psi(t)$ is a power of t, as noted in Remark 2.18. Note that $\psi^{-1}(t) = t^{\frac{q}{q+1}}$. So (19) follows from the fact that $\eta > q$. Finally, take $\varepsilon > 0$ and φ_ε such that $\varphi_\varepsilon(t) = \varphi(\varepsilon t)$ and

$$\int_{X_\Lambda} \varphi_\varepsilon(|\alpha(s,x)|) d\mu(x) < \infty$$

for every $s \in S_\Gamma$. One has $\varphi_\varepsilon(r(n)) \geqslant \exp(\varepsilon n^{\frac{p}{1+\eta}})$, while

$$\mathrm{Vol}_{S_\Gamma}(r(n)) \leqslant |S_\Gamma|^{n^{\frac{\eta}{1+\eta}}}$$

Hence since $\eta < p$, we have

$$\mathrm{Vol}_{S_\Gamma}(r(n))/\varphi_\varepsilon(r(n)) = O(n^{-k})$$

for any $k > 0$. So (18) is satisfied and we conclude by Theorem 9.11 that Λ is hyperbolic. □

Proof of Theorems 9.11 and 9.12. Let us start by strengthening the coboundedness condition.

Claim 9.15. On replacing M by $M/2$ and $\delta+1$ by δ in Theorems 9.11 and 9.12, we may assume that $X_\Gamma \subseteq X_\Lambda$.

Proof of the claim. Since this coupling is cobounded, we can take F be a finite subset of Λ such that $X_\Gamma \subseteq F * X_\Lambda$. Consider the new coupling space $\tilde{\Omega} := \Omega \times F$, let K be a finite group which acts simply transitively on F, and let $\tilde{\Gamma} = \Gamma \times K$ act on $\tilde{\Omega}$ by $(\gamma, k) * (\omega, f) = (\gamma * \omega, k \cdot f)$. This action is smooth, and we take as a fundamental domain the set

$$\tilde{X}_{\tilde{\Gamma}} := \bigsqcup_{f \in F} (X_\Gamma \cap f * X_\Lambda) \times \{f\}$$

The Λ-action on $\tilde{\Omega}$ is the action on the first coordinate; a fundamental domain is provided by

$$\tilde{X}_\Lambda = \bigsqcup_{f \in F} (f * X_\Lambda) \times \{f\}.$$

Viewing both Γ and K as subgroups of $\tilde{\Gamma}$, the latter has $S_{\tilde{\Gamma}} = S_\Gamma \cup K$ as a finite generating set. In fact, with this generating set $\tilde{\Gamma}$ is $(\delta+1)$-hyperbolic. Observe that the volume growth of $\tilde{\Gamma}$ is at least $|K|$ times that of Γ.
Next note that condition (i) is still met by this new generating set. Indeed $\tilde{X}_\Lambda$ is L^∞-equivalent to the fundamental domain $X_\Lambda \times F$, and for all $\gamma \in \Gamma$, $x \in X_\Lambda$ and $f \in F$ we have $d_{S_\Lambda}(\gamma * (x, f), \gamma \cdot (x, f)) = d_{S_\Lambda}(\gamma * x, \gamma \cdot x)$, so for all $\tilde{\gamma} \in S_{\tilde{\Gamma}}$ we have that $\tilde{\gamma} * X_\Lambda \times F$ is φ-equivalent to $X_\Lambda \times F$, so the same is true of $\tilde{X}_\Lambda$.

For condition (ii), we have, by construction, for all $x \in X_\Gamma$ and all $f, f' \in F$ that the distance $d_{S_{\tilde{\Gamma}}}((x, f), (x, f')) \leqslant 1$, so for every $\lambda \in \Lambda$ we have

$$d_{S_{\tilde{\Gamma}}}(\lambda \cdot (x, f), \lambda * (x, f)) \leqslant 1 + d_{S_\Gamma}(\lambda \cdot x, \lambda * x) = 1 + |\beta(\lambda, x)|,$$

hence the new coupling satisfies the same conditions on replacing $\psi(t)$ by $\psi'(t) = \psi(\max\{t-1, 0\})$ in Theorem 9.11. Note that for n large enough, $\psi^{-1}(3Ln) \geqslant 1$ and $\psi'(t) = \psi(t-1)$ for $t \geqslant 1$, from which we easily deduce that for large enough n,

$$r(n) > \frac{M \log\left(\psi'^{-1}(3Ln)n\right)}{2} + 8\psi'^{-1}(3Ln).$$

Therefore it is enough to prove Theorem 9.11 with $M/2$ instead of M for Γ'. So the claim is proved. $\square_{\text{claim}}$

Suppose that Λ is not hyperbolic. By Corollary 9.10 we can take a cycle C_n of arbitrary large length n, and a bi-Lipschitz embedding : $C_n \to \Lambda$ that are 1-Lipschitz and contracts distances at most by a factor 23760. In what follows we consider C_n as a subset of Λ.

Let M be such that $\int_{X_\Lambda} \varphi(|\alpha(s, x)|_{S_\Lambda}) d\mu(x) \leqslant K$ for all $s \in S_\Gamma$.

For every $x \in X_\Gamma$ we denote by $b_x : \Lambda \to \Gamma$ the map defined by $b_x(\lambda) = \beta(\lambda^{-1}, x)^{-1}$ for every $\lambda \in \Lambda$.

Upper estimates for the restriction of b_x to C_n. In the case of Theorem 9.12, we trivially have that b_x is a.e. L-Lipschitz. Under the assumption of Theorem 9.11, we claim that with probability at least $2/3$, the restriction of b_x to C_n is $\psi^{-1}(3Ln)$-Lipschitz. Here we use the integrability condition for β. For every u and v adjacent in C_n there exists an $s_{u,v} \in S_\Lambda$ such that $v = us_{u,v}$. Hence, $d_{S_\Lambda}(b_x(v), b_x(u)) = |b_x(v)^{-1} b_x(u)|_{S_\Lambda} = |\beta(s_{u,v}, u^{-1} \cdot x)|$. Next consider for any $u \in C_n$ the set of all $x \in X_\Gamma$ such that $\psi(|\beta(s_{u,v}, u^{-1} \cdot x)|_{S_\Gamma}) \geqslant 3Ln$. Recall that we assumed $\mu(X_\Gamma) = 1$. Hence, these sets have measure at most $\frac{1}{3n}$ and therefore the set A of all $x \in X_\Gamma$ such that b_x is $\psi^{-1}(3Ln)$-Lipschitz in restriction to C_n has measure at least $\frac{2}{3}$. So our claim follows.

Lower estimates for the restriction of b_x to C_n. Providing lower estimates on the quasi-isometric embedding constants is more involved as this requires to apply the cocycle α to C_n. We shall use the inverse relation between α and β: for all $x \in X_\Gamma$ and $\lambda \in \Lambda$

$$\alpha(\beta(\lambda, x), x) = \lambda \tag{22}$$

We claim that we have the following key inequality.

Claim 9.16. For every $R > 0$, $n \in \mathbb{N}$, and u and v in C_n , we have

$$\mu\left(\{x \in X_\Gamma \colon |b_x(v)^{-1} b_x(u)|_{S_\Gamma} \leqslant R\}\right) \leqslant KR \frac{\mathrm{Vol}_{S_\Gamma}(R)}{\varphi\left(\frac{|v^{-1}u|_{S_\Lambda}}{R}\right)},$$

where $K = \int_{X_\Lambda} \varphi(|\alpha(s, x)|_{S_\Lambda}) d\mu(x)$.

Proof of the claim. For any $n \in \mathbb{N}$, a and b in C_n and $\gamma \in \Gamma$, we define the set

$$A_\gamma = \{x \in X_\Gamma \colon b_x(v)^{-1} b_x(u) = \gamma\}.$$

Using (22), we have for every $x \in A_\gamma$ that $\left|\alpha\left(\gamma, u^{-1} \cdot x\right)\right)\Big|_{S_\Lambda} = |v^{-1}u|_{S_\Lambda}$. Let us start giving an upper bound of $\mu(A_\gamma)$ as a function of $|\gamma|$. Write $\gamma = s_1 \dots s_{|\gamma|_{S_\Gamma}}$ with $s_i \in S_\Gamma$. Then there exists an i such that the set

$$\left\{x \in A_\gamma \colon \left|\alpha\left(s_i, s_{i+1} \dots s_{|\gamma|_{S_\Gamma}} \cdot (u^{-1} \cdot x)\right)\right|_{S_\Lambda} \geqslant \frac{|v^{-1}u|_{S_\Lambda}}{|\gamma|_{S_\Gamma}}\right\}$$

has measure at least $\frac{\mu(A_\gamma)}{|\gamma|_{S_\Gamma}}$. Therefore, for $s = s_i$, we have that

$$\mu\left(\left\{y \in X_\Lambda \colon |\alpha(s, y)|_{S_\Lambda} \geqslant \frac{|v^{-1}u|_{S_\Lambda}}{|\gamma|_{S_\Gamma}}\right\}\right) \geqslant \frac{\mu(A_\gamma)}{|\gamma|_{S_\Gamma}},$$

from which we deduce the following upper bound on the measure of A_γ:

$$\begin{aligned}\mu(A_\gamma) &\leqslant |\gamma|_{S_\Gamma}\, \mu\left(\left\{y \in X_\Lambda \colon |\alpha(s, y)|_{S_\Lambda} \geqslant \frac{|v^{-1}u|_{S_\Lambda}}{|\gamma|_{S_\Gamma}}\right\}\right) \\ &\leqslant \frac{M|\gamma|_{S_\Gamma}}{\varphi\left(\frac{|v^{-1}u|_{S_\Lambda}}{|\gamma|_{S_\Gamma}}\right)}.\end{aligned}$$

Therefore we can conclude that for every $R > 0$,

$$\begin{aligned}\mu\left(\{x \in X_\Gamma \colon |b_x(v)^{-1} b_x(u)|_{S_\Gamma} \leqslant R\}\right) &= \sum_{\gamma \in B_\Gamma(e_\Gamma, R)} \mu(A_\gamma) \\ &\leqslant \sum_{\gamma \in B_\Gamma(e_\Gamma, R)} \frac{K|\gamma|_{S_\Gamma}}{\varphi\left(\frac{|v^{-1}u|_{S_\Lambda}}{R}\right)} \\ &\leqslant KR \frac{\mathrm{Vol}_{S_\Gamma}(R)}{\varphi\left(\frac{|v^{-1}u|_{S_\Lambda}}{R}\right)}.\end{aligned}$$

So the claim is proved. $\square_{\text{claim}}$

Applying Claim 9.16 with $R = \frac{r(n)}{n}|v^{-1}u|_{S_\Lambda}$, we obtain

$$\mu\left(\{x \in X_\Gamma \colon |b_x(v)^{-1} b_x(u)|_{S_\Gamma} \leqslant \frac{r(n)}{n}|v^{-1}u|_{S_\Lambda}\}\right) \leqslant \frac{Kr(n)\,\mathrm{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))},$$

since $\frac{|v^{-1}u|}{n} \leqslant 1$. As there are at most n^2 pairs (a,b) in C_n, we deduce that

$$\mu\left(\left(\{x \in X_\Gamma \colon \exists u, v \in C_n \colon |b_x(v)^{-1}b_x(u)|_{S_\Gamma} \leqslant \frac{r(n)}{n}|v^{-1}u|_{S_\Lambda}\right\}\right) \leqslant \frac{Kn^2 r(n)\operatorname{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))}.$$

Hence there exists n_0 such that for $n \geqslant n_0$, there exists a subset B of X_Γ of measure at least $2/3$ on which for all $u, v \in C_n$,

$$|b_x(v)^{-1}b_x(u)|_{S_\Gamma} \geqslant \frac{r(n)}{n}|v^{-1}u|_{S_\Lambda}.$$

Finally, for all x in the subset $A \cap B$ which has positive measure, we deduce for every $u, v \in C_n$ that

$$a|v^{-1}u|_{S_\Lambda} \leqslant |b_x(v)^{-1}b_x(u)|_{S_\Gamma} \leqslant b|v^{-1}u|_{S_\Lambda},$$

where in the case of Theorem 9.11, $a = \frac{r(n)}{n}$ and $b = \psi^{-1}(3Ln)$; and in the case of 9.12, $a = \frac{r(n)}{n}$ and $b = L$. In both cases we have

$$a > \frac{\log(bn)}{\varepsilon n} + \frac{8b}{n},$$

which contradicts Proposition 9.2 for some ε that only depends on δ. □

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